A Study of the Dirac-Sidharth Equation

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Abstract

The Dirac-Siddharth Equation has been constructed from the Siddharth hamiltonian by quantization of the energy and momentum in Pauli algebra. We have solved this equation by using tensor product of matrices.

Keywords: Dirac-Sidharth equation, Dirac equation.

1 Introduction

In the special relativity of Einstein (A.Einstein., 1905), from the energy-momentum relation

$$E^2 = c^2p^2 + m^2c^4 \tag{1}$$

we can deduce the Klein-Gordon equation and the Dirac equation. This theory use the concept of continuous spacetime.

Quantized spacetime was introduced at the first time by Snyder (H.S. Snyder., Phys Rev. 1947)[2, 3], which known as Snyder noncommutative geometry. That is because the commutation relations are modified and becomes [2, 3]

$$[x^\mu, x^\nu] = i\alpha \frac{\ell^2 c^2}{\hbar} \left(x^\mu p^\nu - x^\nu p^\mu\right), \tag{2}$$

$$[x^\mu, p_\nu] = i\hbar \left[\delta^\mu_\nu + i\alpha \frac{\ell^2 c^2}{\hbar^2} p^\mu p_\nu\right], \tag{3}$$

$$[p_\mu, p_\nu] = 0 \tag{4}$$

$$\epsilon = \frac{\hbar c}{\sqrt{\alpha}\ell} \tag{5}$$

is the energy due to the length scale ℓ, where α a dimensionless constant. As consequence the energy momentum relation gets modified and becomes [4]

$$E^2 = c^2p^2 + m^2c^4 + \alpha \left(\frac{c}{\hbar}\right)^2 \ell^2 p^4 \tag{6}$$

$\ell = \ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-33} cm$, Planck scale, the fundamental length scale, where G is the gravitational constant.

$\ell_c = \frac{e^2}{m_e c^2} \propto 10^{-12} cm$, Compton scale, where e is the electron charge and m_e the electron mass.

$\ell_{LHC} \approx 2 \times 10^{-18} cm$.
So,

$$\ell_p < \ell_{LHC} < \ell_c \tag{7}$$

From the above energy- momentum relation, Sidharth has deduced the so-called Dirac-Sidharth Equation [4, 5], a modified Dirac equation.
In the Section.2, we will derive the Dirac-Sidharth Equation by quantizing energy and momentum. In the Section.3, we will solve the Dirac-Sidharth Equation by using tensor product of matrices.

We think that using differents mathematical tools in physics will make to appear differents hidden mathematical or physical properties.

2 A derivation of the Dirac-Sidharth equation

For deriving the Dirac-Sidharth equation we use the method used by J.J. Sakurai [6] for deriving the Dirac equation.

The wave function of a spin-$\frac{1}{2}$ particle is two components. So, for quantizing the energy-momentum relation in order to have the modified Klein-Gordon equation [4, 5], or Klein-Gordon-Sidharth equation, of the spin-$\frac{1}{2}$ particle, the operators which take part in the quantization should be 2×2 matrices. So, let us take as quantization rules

$E \longrightarrow i\hbar\sigma^0 \frac{\partial}{\partial t} = i\hbar \frac{\partial}{\partial t}$

$\vec{p} \longrightarrow -i\hbar\sigma^1 \frac{\partial}{\partial x^1} - i\hbar\sigma^2 \frac{\partial}{\partial x^2} - i\hbar\sigma^3 \frac{\partial}{\partial x^3} = -i\hbar\vec{\sigma}\vec{\nabla} = \hat{p}_1\sigma^1 + \hat{p}_2\sigma^2 + \hat{p}_3\sigma^3$

where σ^1, σ^2, σ^3 are the Pauli matrices. Then we have, at first the Klein-Gordon-Sidharth equation

$$c^2\hbar^2 \left(\frac{\partial^2}{c^2 \partial t^2} - \Delta - m^2c^2 - \alpha \frac{\ell^2}{\hbar^2} \vec{\nabla}^4 \right) \phi = 0 \tag{8}$$

$$\begin{aligned} &\left(i\hbar\frac{\partial}{\partial t} + ic\hbar\vec{\sigma}\vec{\nabla} \right) \frac{1}{mc^2} \left\{ \sum_{k=0}^{+\infty} (-1)^k \left[\frac{i\sqrt{\alpha}}{mc\hbar} \ell \left(-i\hbar\vec{\sigma}\vec{\nabla} \right)^2 \right]^k \right\} \times \\ &\left(i\hbar\frac{\partial}{\partial t} - ic\hbar\vec{\sigma}\vec{\nabla} \right) \phi = \left[mc^2 + i\sqrt{\alpha}\frac{c}{\hbar}\ell \left(-i\hbar\vec{\sigma}\vec{\nabla} \right)^2 \right] \phi \end{aligned} \tag{9}$$

with application of the operator to two components wave function ϕ, which is solution of the Klein-Gordon-Sidharth equation. Let

$$\chi = \frac{1}{mc^2}\left\{\sum_{k=0}^{+\infty}(-1)^k\left[\frac{i\sqrt{\alpha}}{mc\hbar}\ell\left(-i\hbar\vec{\sigma}\vec{\nabla}\right)^2\right]^k\right\}\times\left(i\hbar\frac{\partial}{\partial t}-ic\hbar\vec{\sigma}\vec{\nabla}\right)\phi \tag{10}$$

then, we have the following system of partial differential equations

$$\begin{cases} i\hbar\frac{\partial}{c\partial t}\chi + i\hbar\vec{\sigma}\vec{\nabla}\chi = mc\phi + i\sqrt{\alpha}\frac{\ell}{\hbar}\left(i\hbar\vec{\sigma}\vec{\nabla}\right)^2\phi \\ i\hbar\frac{\partial}{c\partial t}\phi - i\hbar\vec{\sigma}\vec{\nabla}\phi = mc\chi - i\sqrt{\alpha}\frac{\ell}{\hbar}\left(i\hbar\vec{\sigma}\vec{\nabla}\right)^2\chi \end{cases} \tag{11}$$

In additionning and in subtracting these equations, and in transforming the obtained equation under matricial form, we have the Dirac-Sidharth equation

$$i\hbar\gamma_D^\mu\partial_\mu\psi_D - mc\psi_D - i\sqrt{\alpha}\ell\hbar\gamma_D^5\Delta\psi_D = 0 \tag{12}$$

in the Dirac (or "Standard") representation of the γ-matrices, where
$\gamma_D^0 = \begin{pmatrix}\sigma^0 & 0\\ 0 & -\sigma^0\end{pmatrix} = \sigma^3\otimes\sigma^0$, $\gamma_D^j = \begin{pmatrix}0 & \sigma^j\\ -\sigma^j & 0\end{pmatrix} = i\sigma^2\otimes\sigma^j$, $j = 1,2,3$,
$\gamma_D^5 = i\gamma_D^0\gamma_D^1\gamma_D^2\gamma_D^3 = \begin{pmatrix}0 & \sigma^0\\ \sigma^0 & 0\end{pmatrix} = \sigma^1\otimes\sigma^0$, and $\psi_D = \begin{pmatrix}\chi+\phi\\ \chi-\phi\end{pmatrix}$,
$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.
We know that (J.D. Bjorken and S.D. Drell., 1964)[7]

$$P\gamma^5 = -\gamma^5 P \tag{13}$$

It follows that the Dirac-Sidharth equation is not invariant under reflections (B.G. Sidharth., Mass of the Neutrinos, 2009). The equation

$$i\hbar\gamma_W^\mu\partial_\mu\psi_W - mc\psi_W - i\sqrt{\alpha}\ell\hbar\gamma_W^5\Delta\psi_W = 0 \tag{14}$$

is the Dirac-Sidharth equation in the Weyl(or "chiral") representation, where $\psi_W = \begin{pmatrix}\chi\\ \phi\end{pmatrix}$.
So, χ is the left-handed two components spinor and ϕ the right-handed one. This method makes to appear that the right-handed two components spinor is solution of the Klein-Gordon-Sidharth equation.

3 Resolution of the Dirac-Sidharth equation

In this section we will use the tensor product of matrices for solving the Dirac-Sidharth equation. We had used this method, suggested by Raoelina Andriambololona for solving the Dirac equation (C. Rakotonirina., Thesis, 2003).
Let us look for a solution of the form

$$\psi_D = U(p)e^{\frac{i}{\hbar}(\vec{p}\vec{x}-Et)} \tag{15}$$

Let Ψ a four components spinor which is eigenstate both of $\hat{p}_j = -i\hbar\frac{\partial}{\partial x^j}$ and $\hat{E} = i\hbar\frac{\partial}{\partial t}$, $\vec{p} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}$, and $\vec{n} = \frac{\vec{p}}{p} = \begin{pmatrix} n^1 \\ n^2 \\ n^3 \end{pmatrix}$.
The Dirac-Sidharth equation becomes

$$\sigma^0 \otimes \sigma^0 U(p) - \frac{2}{\hbar} cp\sigma^1 \otimes \left(\frac{\hbar}{2}\vec{\sigma}\vec{n}\right) U(p) - mc^2\sigma^3 \otimes \sigma^0 U(p) + c\sqrt{\alpha}p^2\frac{\ell}{\hbar}\sigma^2 \otimes \sigma^0 U(p) = 0 \tag{16}$$

Let us take $U(p)$ of the form

$$U(p) = \varphi \otimes u \tag{17}$$

where u is the eigeinvector of the spin operator $\left(\frac{\hbar}{2}\vec{\sigma}\vec{n}\right)$. $\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}$ and u are two components.
Since $u \neq 0$, so

$$\left(\epsilon cp\sigma^1 + mc^2\sigma^3 - c\sqrt{\alpha}p^2\frac{\ell}{\hbar}\sigma^2\right)\varphi = E\varphi \tag{18}$$

with $\epsilon = \begin{cases} +1 & \text{spin up} \\ -1 & \text{spin down} \end{cases}$
Solving this equation with respect to φ^1 and φ^2, we have

$$\Psi_+ = \sqrt{\frac{E+mc^2}{2E}} \begin{pmatrix} 1 \\ \frac{\epsilon cp - i\frac{c}{\hbar}\sqrt{\alpha}p^2\ell}{mc^2+E} \end{pmatrix} \otimes s e^{\frac{i}{\hbar}(\vec{p}\vec{x} - Et)} \tag{19}$$

the solution with positive energy, where $s = \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1 + in^2 \\ 1 + n^3 \end{pmatrix}$ spin up, $s = \frac{1}{\sqrt{2(1+n_3)}} \begin{pmatrix} 1 + n^3 \\ n^1 + in^2 \end{pmatrix}$ spin down.

This method makes to appear the 2×2 matrix $h = \epsilon cp\sigma^1 - c\sqrt{\alpha}p^2\frac{\ell}{\hbar}\sigma^2 + mc^2\sigma^3$ whose eigeinvalues are the positive and the negative energies. h is like a vector in Pauli algebra. So, energy of the spin-$\frac{1}{2}$ particle can be associated to a vector in Pauli algebra, whose length or intensity is given by the energy-momentum relation.

$$h^2 = E^2 \tag{20}$$

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