ANPAEQED II : a Mechanism for Finite Charge Renormalization

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Our second paper of the ANPAEQED (Analytic, Non-Perturbative, Almost Exact QED) series is concerned with manifestly gauge invariant calculations of the dressed photon propagator in several functional approximations of increasing complexity, which lead in a natural way to the extraction of the leading logarithmic divergences of every perturbative order, and to an explicit demonstration of the cancellation of all such divergences in the calculation of the (inverse of the) photon's wavefunction renormalization constant Z_3. This functional analysis leads to equations whose unique solution is to yield the numerical value of the renormalized fine-structure constant; and we display an approximate solution to these equations which generates a small band of such values about 1/137.

1 Introduction

We here build upon the basic, functional formalism of ANPAEQED I [1], to give a new, functional representation of the sum of all Feynman Graphs (FGs) corresponding to the sum of all radiative corrections to the photon propagator, using as our fundamental, calculational tool a convenient version of the manifestly gauge-invariant Fradkin representation [2] for the functional fermion determinant. The latter represents the sum of all, single fermion loop graphs which contain all possible (even) numbers of attached photon lines; it is, for this problem, the relevant part of Schwinger's original functional solution to QED [3].

In addition to the advantage of automatic gauge-invariance, the functional formalism we adopt displays, in every order of the calculation, cancellations which ordinarily appear between different FGs of that order, cancellations which have always required tedious computation to achieve; but here, in this formulation, those cancellations appear and may be realized initially, before any computation is required, and to all orders of the coupling.

The exact functional solution for the dressed photon propagator is given by :

$$D'_{c,\mu\nu}(x-y) = D_{c,\mu\nu}(x-y) + \iint D_{c,\mu\lambda}(x-u) K_{\lambda\sigma}(u-w) D_{c,\sigma\nu}(w-y) du\, dw$$

$$iK_{\mu\nu}(x-y) = e^{\mathcal{D}_A} \frac{\delta}{\delta A_\mu(x)} \frac{\delta}{\delta A_\nu(y)} e^{L[A]} / < S > \Big|_{A=0}$$

$$= e^{\mathcal{D}_A} \Big[\frac{\delta^2 L}{\delta A_\mu(x) \delta A_\nu(y)} + \frac{\delta L}{\delta A_\mu(x)} \frac{\delta L}{\delta A_\nu(y)} \Big] e^{L[A]} / < S > \Big|_{A=0} \tag{1}$$

with $\mathcal{D}_A = -\frac{i}{2} \int \frac{\delta}{\delta A_\mu} D_{c,\mu\nu} \frac{\delta}{\delta A_\nu}$, and the " quenched " version of (1.1) considered in ANPAEQED I was simply :

$$iK_{\mu\nu}(x-y) = e^{\mathcal{D}_A} \frac{\delta^2 L}{\delta A_\mu(x) \delta A_\nu(y)} \Big|_{A=0} \tag{2}$$

t is most important to note that $L[A]$ depends only upon $F_{\mu\nu}$, and can be easily be written as such. This property immediately carries the consequence that currents induced in the vacuum, $< j_\mu(x) >= ig\, \delta L / \delta A_\mu(x)$, are to satisfy charge conservation : $\partial_\mu < j_\mu(x) >= 0$. In terms of the $K_{\mu\nu}$ of (1.1) or (1.2), this means that $\partial_\mu K_{\mu\nu} = \partial_\nu K_{\mu\nu} = 0$, so that the $\tilde{K}_{\mu\nu}(k)$ are expected to have the gauge invariant form $(k_\mu k_\nu - k^2 \delta_{\mu\nu}) \Pi(k^2)$. The simplest, order $\alpha = g^2/4\pi$, Feynman graph corresponding to a single closed fermion loop does not display this property; and in the past, special, ad hoc maneuvers were invented to restore gauge invariance. In the Fradkin representation for $L[A]$ used in this paper, gauge invariance to all orders is automatically satisfied. The fermion determinant has an exact Fradkin representation of the form

$$L[A] = -\frac{1}{2} \int_0^\infty \frac{ds}{s}\, e^{-ism_0^2}\, e^{i \int_0^s ds' \frac{\delta^2}{\delta v_\mu^2(s')}}\, \delta^{(4)}\Big(\int_0^s ds'\, v(s') \Big)$$

$$\times \int d^4x'\, e^{-ig_0 \int_0^s ds'\, v_\mu(s')\, A_\mu(x' - \int_0^{s'} v)}\, \mathrm{tr} \Big(e^{g_0 \int_0^s ds'\, \sigma_{\mu\nu}\, F_{\mu\nu}(x' - \int_0^{s'} v)} \Big)_+ \tag{3}$$

and we find it useful to adopt the change of variable$u_\mu(s') = \int_0^{s'} ds'' v_\mu(s'')$, which, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\sigma_{\mu\nu}$ denotes the gamma matrix combination $\frac{1}{4}[\gamma_\mu, \gamma_\nu]$, and $v_\mu(s')$ denotes the four velocity of the fermion with an instantaneous proper time s', yields :

$$L[A] = -\frac{1}{2} \int_0^\infty \frac{ds}{s}\, e^{-ism_0^2} \int d^4x'\, N \int d[u]\, e^{\frac{i}{2} \int u (2h)^{-1} u}\, \delta^{(4)}\Big(u(s) \Big)$$

$$\times\, e^{-ig_0 \int_0^s ds'\, u'_\mu(s')\, A_\mu(x' - u(s'))}\, \mathrm{tr} \Big(e^{g_0 \int_0^s ds'\, \sigma_{\mu\nu}\, F_{\mu\nu}(x' - u(s'))} \Big)_+ \tag{4}$$

It is important to note that there are two restrictions on the $u(s')$ variables, the first an implicit condition $u(0) = 0$, which arises from the definition of $u(s')$; and the second condition, $u(s) = 0$, explicitly stated by the delta function of (1.4).

The first simplification to be noted was proven in Appendix B of ANPAEQED I[1],

$$e^{\mathcal{D}_A} \,\mathrm{tr}\left(e^{g_0 \int_0^s ds' \, \sigma.F} \right)_+ \Big|_{A\to 0} = \mathrm{tr}\, 1 = 4$$

and can be trivially generalized to the more relevant statement :

$$e^{\mathcal{D}_A} \,\mathrm{tr}\left(e^{g_0 \int_0^s ds' \, \sigma.F} \right)_+ = \mathrm{tr}\left(e^{g_0 \int_0^s ds' \, \sigma.F} \right)_+$$

so that the self linkages acting on this OE factor exactly cancel, to all orders in the coupling. In Feynman graph language, this would correspond to momentum space cancellations occurring in every higher order; in the Fradkin representation, one sees them immediately.

Our analysis continues in terms of a " DP Model " which, before any functional integration is performed, extracts the " Dominant (and divergent) Part " of each order's functional radiative corrections to Z_3^{-1}, and sums them into a single integral which converges when and if the parameter p satisfies $2 > p > 1$, where $p\pi = \alpha_0 = g_0^2/4\pi$. Our final result for the inverse of Z_3 takes the form:

$$Z_3^{-1} = 1 + \left(\frac{p}{3}\right) e^{i\pi p/2} \int_\varepsilon^\infty \frac{ds}{s} \, e^{-ism^2} \left(\frac{s}{\varepsilon}\right)^p e^{T(s/\varepsilon)} \tag{5}$$

where we are keeping to the conventional perturbative usage (although in configuration, rather than momentum space) of cutting off all proper time integrals with a lower limit of ε, which will shortly be set equal to zero. The exponential factor T results from the infinite number of closed-fermion-loops coupling to that closed-fermion-loop which provides the definition of the inverse of Z_3; without those linkages, the expression is divergent in the limit as ε vanishes, but for $2 > p > 1$ that limit may be taken, and the integrals converge:

$$Z_3^{-1} = 1 + \left(\frac{p}{3}\right) e^{i\pi p/2} \int_1^\infty dx \, x^{p-1} \, e^{-ixm^2\varepsilon} \, e^{T(x)} \tag{6}$$

where:

$$T(x) = i\left(\frac{\xi}{2}\right)^4 e^{i\pi p} \, x^p \int_1^\infty \frac{dy}{y^3} \, e^{-iym^2\varepsilon} \, y^p \left(\frac{y - i}{y - x - i}\right)^p$$

Z_3^{-1} has both a real and imaginary part; and the condition that its imaginary part vanishes defines the unique, and finite, value of p. Substitution of that value of p into the integral which now defines the ReZ_3^{-1}, and multiplication of the resulting Z_3 by α_0 then produces the renormalized fine-structure constant, α. And, it should be noted, the value of Z_3 so obtained is independent of the fermion mass, in agreement with experiment (in the sense that any fermion satisfying QED - but not simultaneously QCD - has the same value of its renormalized electric charge).

In the evaluation of these terms, we have employed three distinct approximations, the first in defining the contributions coming from (the infinite number) of closed-fermion-loops, corresponding to the factor $\exp L[A]$ of the functional formulation; the second approximation upon defining the extraction of the divergent contributions of each of these closed-fermion-loops to that first closed-fermion-loop whose $K_{\mu\nu}$ provides the framework for the calculation of Z_3^{-1}; and the third and crudest approximation when evaluating the real and imaginary parts of our final, convergent expression for Z_3^{-1}. We find that if $p = 3/2 - \delta$, where δ is sufficiently small, that our final answer for α can be made very close to $1/137$. It should be emphasized that the value of p is quite sensitive to the removal of all phase dependence from Z_3^{-1}, and our result $p \simeq 3/2$ may well be subject to change; a precise evaluation of the real and imaginary parts of our integrals is certainly needed.

In summary, we emphasize that our evaluations of the results of this functional analysis are only approximate; but we strongly believe that this functional approach is the right and proper way to attempt gauge-invariant sums over relevant FGs. With this type of analysis, which most certainly can be extended to QCD, we predict that QFT will enter a new phase, from an Adolescence defined in terms of FGs, to a new Maturity given in terms of Schwinger's original functional solutions made newly accessible by the use of relevant Fradkin representations.

References

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