

Euclidean–Minkowskian Duality of Wilson–Loop Correlation Functions

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Abstract

We discuss the analyticity properties of the Wilson–loop correlation functions relevant to the problem of *soft* high–energy scattering, directly at the level of the functional integral, in a genuinely nonperturbative way.

1 Introduction

Since the seminal paper [1] by O. Nachtmann, much work has been done on the problem of *soft* high–energy scattering in strong interactions in the framework of nonperturbative QCD (for a review see Ref. [2]). In this approach, at high center–of–mass energy and small transferred momentum ($\sqrt{|t|} \leq 1 \text{ GeV} \ll \sqrt{s}$) hadronic scattering amplitudes are reconstructed from the elastic scattering amplitude of the corresponding system of constituent partons, after folding with the appropriate hadron wave functions. In particular, in the case of meson–meson elastic scattering, the corresponding amplitudes can be reconstructed from the correlation functions of two Wilson loops (which describe the scattering of two colour dipoles of fixed transverse size) running along the trajectories of the colliding hadrons [3] (see also [2]). As an infrared (IR) regularisation [4], the loops are taken to form a finite hyperbolic angle χ in Minkowski spacetime, and moreover they are taken to be of finite length $2T$; the physical amplitudes (which are expected to be IR finite [5]) are recovered in the limit $T \rightarrow \infty$ at large $\chi \simeq \log(s/m^2)$ (for $s \rightarrow \infty$).

It has been shown in [6, 7, 8, 9, 10, 11] that, under certain analyticity hypotheses, the relevant correlation functions can be reconstructed from the “corresponding” correlation functions of two Euclidean Wilson loops, of finite length $2T$, and forming an angle θ in Euclidean space, by means of the double analytic continuation $\theta \rightarrow -i\chi$, $T \rightarrow iT$. This *Euclidean–Minkowskian duality* of Wilson–loop correlation functions has made possible to approach the problem of *soft* high–energy scattering with the nonperturbative techniques of Euclidean Quantum Field Theory, such as

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the *Instanton Liquid Model* [12], the *Stochastic Vacuum Model* [13], the *AdS/CFT correspondence* [14], and *Lattice Gauge Theory* [15]. However, until recent times, the *analytic-continuation relations*, although expected to be an exact result, had only been verified in perturbation theory [6, 9, 10, 16], while a nonperturbative foundation was lacking (except in the case of *quenched QED*, where an exact calculation can be performed both in the Euclidean and Minkowskian theories [9]).

In Ref. [17] we have argued, on nonperturbative grounds, that the required analyticity hypotheses are indeed satisfied. The strategy we have used is to push the dependence on the relevant variables into the action by means of a field and coordinate transformation, and then to allow them to take complex values. In particular, we have determined the analyticity domain of the relevant Euclidean correlation function, and we have shown that the corresponding Minkowskian quantity is recovered with the usual double analytic continuation in θ and T inside this domain; moreover, the extra conditions that allow one to derive the *crossing-symmetry relations* found in Ref. [10] have been shown to be satisfied, and we have refined the argument given in Ref. [9] for the analytic continuation of the correlation function with the IR cutoff removed. The formal manipulations of the functional integral used to obtain these results have been justified making use of a lattice regularisation.

2 High-energy meson-meson scattering and Wilson-loop correlation functions

The elastic scattering amplitudes of two mesons (taken for simplicity with the same mass m) in the *soft* high-energy regime can be reconstructed, after folding with the appropriate wave functions, from the scattering amplitude $\mathcal{M}_{(dd)}$ of two dipoles of fixed transverse size $\vec{R}_{i\perp}$, and fixed longitudinal momentum f_i of the two quarks in the two dipoles, respectively ($i = 1, 2$) [3]:

$$\mathcal{M}_{(dd)}(s, t; 1, 2) \equiv -i 2s \int d^2 \vec{z}_\perp e^{i\vec{q}_\perp \cdot \vec{z}_\perp} \mathcal{C}_M(\chi; \vec{z}_\perp; 1, 2), \quad (1)$$

where the arguments “ i ” stand for “ $\vec{R}_{i\perp}, f_i$ ”, $t = -|\vec{q}_\perp|^2$ (\vec{q}_\perp being the transferred momentum) and $s = 2m^2(1 + \cosh \chi)$. The correlation function \mathcal{C}_M is defined as the limit $\mathcal{C}_M \equiv \lim_{T \rightarrow \infty} \mathcal{G}_M$ of the correlation function of two loops of finite length $2T$,

$$\mathcal{G}_M(\chi; T; \vec{z}_\perp; 1, 2) \equiv \frac{\langle \mathcal{W}_1^{(T)} \mathcal{W}_2^{(T)} \rangle}{\langle \mathcal{W}_1^{(T)} \rangle \langle \mathcal{W}_2^{(T)} \rangle} - 1, \quad (2)$$

where $\langle \dots \rangle$ are averages in the sense of the QCD functional integral, and

$$\mathcal{W}_{1,2}^{(T)} \equiv \frac{1}{N_c} \text{tr} \left\{ \mathcal{P} \exp \left[-ig \oint_{c_{1,2}} A_\mu(x) dx^\mu \right] \right\} \quad (3)$$

are Wilson loops in the fundamental representation of $SU(N_c)$; the paths are made up of the classical trajectories of quarks and antiquarks,

$$\mathcal{C}_1 : X^{1q[\bar{q}]}(\tau) = z + \frac{p_1}{m}\tau + f_1^{q[\bar{q}]}R_1, \quad \mathcal{C}_2 : X^{2q[\bar{q}]}(\tau) = \frac{p_2}{m}\tau + f_2^{q[\bar{q}]}R_2, \quad (4)$$

with $\tau \in [-T, T]$, and closed by straight-line paths in the transverse plane at $\tau = \pm T$ in order to ensure gauge invariance. Here

$$p_1 = m \left(\cosh \frac{\chi}{2}, \sinh \frac{\chi}{2}, \vec{0}_\perp \right), \quad p_2 = m \left(\cosh \frac{\chi}{2}, -\sinh \frac{\chi}{2}, \vec{0}_\perp \right), \quad (5)$$

and moreover, $R_1 = (0, 0, \vec{R}_{1\perp})$, $R_2 = (0, 0, \vec{R}_{2\perp})$, $z = (0, 0, \vec{z}_\perp)$, and $f_i^q = 1 - f_i$, $f_i^{\bar{q}} = -f_i$ ($i = 1, 2$), with f_i the longitudinal momentum fraction of quark i , $f_i \in [0, 1]$. The Euclidean counterpart of Eq. (2) is

$$\mathcal{G}_E(\theta; T; \vec{z}_\perp; 1, 2) \equiv \frac{\langle \widetilde{\mathcal{W}}_1^{(T)} \widetilde{\mathcal{W}}_2^{(T)} \rangle_E}{\langle \widetilde{\mathcal{W}}_1^{(T)} \rangle_E \langle \widetilde{\mathcal{W}}_2^{(T)} \rangle_E} - 1, \quad (6)$$

where now $\langle \dots \rangle_E$ is the average in the sense of the Euclidean QCD functional integral, and the Euclidean Wilson loops

$$\widetilde{\mathcal{W}}_{1,2}^{(T)} \equiv \frac{1}{N_c} \text{tr} \left\{ \mathcal{P} \exp \left[-ig \oint_{\widetilde{\mathcal{C}}_{1,2}} A_{E\mu}(x_E) dx_{E\mu} \right] \right\} \quad (7)$$

are calculated on the following straight-line paths,

$$\widetilde{\mathcal{C}}_1 : X_E^{1q[\bar{q}]}(\tau) = z + \frac{p_{1E}}{m}\tau + f_1^{q[\bar{q}]}R_{1E}, \quad \widetilde{\mathcal{C}}_2 : X_E^{2q[\bar{q}]}(\tau) = \frac{p_{2E}}{m}\tau + f_2^{q[\bar{q}]}R_{2E}, \quad (8)$$

with $\tau \in [-T, T]$, and closed by straight-line paths in the transverse plane at $\tau = \pm T$. Here²

$$p_{1E} = m \left(\cos \frac{\theta}{2}, \sin \frac{\theta}{2}, \vec{0}_\perp \right), \quad p_{2E} = m \left(\cos \frac{\theta}{2}, -\sin \frac{\theta}{2}, \vec{0}_\perp \right), \quad (9)$$

and $R_{iE} = (0, 0, \vec{R}_{i\perp})$, $z_E = (0, 0, \vec{z}_\perp)$ (the transverse vectors are taken to be equal in the two cases). Again, we define the correlation function with the IR cutoff removed as $\mathcal{C}_E \equiv \lim_{T \rightarrow \infty} \mathcal{G}_E$.

It has been shown in [6, 7, 8, 9] that the correlation functions in the two theories are connected by the *analytic-continuation relations*

$$\begin{aligned} \mathcal{G}_M(\chi; T; \vec{z}_\perp; 1, 2) &= \overline{\mathcal{G}}_E(-i\chi; iT; \vec{z}_\perp; 1, 2), & \forall \chi \in \mathcal{I}_M, \\ \mathcal{G}_E(\theta; T; \vec{z}_\perp; 1, 2) &= \overline{\mathcal{G}}_M(i\theta; -iT; \vec{z}_\perp; 1, 2), & \forall \theta \in \mathcal{I}_E. \end{aligned} \quad (10)$$

²For convenience, we take the Euclidean indices to run from 0 to 3, and we take the ‘‘Euclidean time’’ to be the zero-th Euclidean coordinate.

Here we denote with an overbar the analytic extensions of the Euclidean and Minkowskian correlation functions, starting from the real intervals $\mathcal{I}_E \equiv (0, \pi)$ and $\mathcal{I}_M \equiv (0, \infty)$ of the respective angular variables, with positive real T in both cases, into domains of the complex variables θ (resp. χ) and T in a two-dimensional complex space; these domains are assumed to contain the intervals $-i\mathcal{I}_M$ (at positive imaginary T)³ and $i\mathcal{I}_E$ (at negative imaginary T) in the two cases, respectively. Due to the symmetries of the two theories, the restriction to \mathcal{I}_M and \mathcal{I}_E does not imply any restriction on the physical content of the correlation functions (see Ref. [10]). Under certain analyticity hypotheses in the T variable, the following relations are obtained for the correlation functions with the IR cutoff T removed [9]:

$$\begin{aligned} \mathcal{C}_M(\chi; \vec{z}_\perp; 1, 2) &= \overline{\mathcal{C}}_E(-i\chi; \vec{z}_\perp; 1, 2), & \forall \chi \in \mathcal{I}_M, \\ \mathcal{C}_E(\theta; \vec{z}_\perp; 1, 2) &= \overline{\mathcal{C}}_M(i\theta; \vec{z}_\perp; 1, 2), & \forall \theta \in \mathcal{I}_E. \end{aligned} \quad (11)$$

Finally, we recall the *crossing-symmetry relations* [10]

$$\begin{aligned} \overline{\mathcal{G}}_M(i\pi - \chi; T; \vec{z}_\perp; 1, 2) &= \mathcal{G}_M(\chi; T; \vec{z}_\perp; 1, \overline{2}) = \mathcal{G}_M(\chi; T; \vec{z}_\perp; \overline{1}, 2), & \forall \chi \in \mathcal{I}_M, \\ \mathcal{G}_E(\pi - \theta; T; \vec{z}_\perp; 1, 2) &= \mathcal{G}_E(\theta; T; \vec{z}_\perp; 1, \overline{2}) = \mathcal{G}_E(\theta; T; \vec{z}_\perp; \overline{1}, 2), & \forall \theta \in \mathcal{I}_E, \end{aligned} \quad (12)$$

that hold for every positive real T , and thus also for the correlation functions with the IR cutoff removed; here the arguments “ \overline{i} ” stand for “ $-\vec{R}_{i\perp}, 1 - f_i$ ”. The Euclidean relation in (12) holds without any analyticity hypothesis, while in the Minkowskian case the analyticity domain for the analytic extension $\overline{\mathcal{G}}_M$ should include also the interval (in the complex- χ plane) $\mathcal{I}_M^{(c)} = i\pi - \mathcal{I}_M$ (for positive real T), where the physical amplitude for the “crossed” channel is then expected to lie.

3 Field and coordinate transformation

To address the issue of the analytic extension of the correlation functions to complex values of the angular variables and of T , we shall appropriately rescale the coordinates and fields, in order for the dependence on the relevant variables to drop from the Wilson-loop operators, and to move into the action. We consider here the pure-gauge case only; the inclusion of fermions (at the formal level) is discussed in [17].

We first rescale [7, 8, 9] $\tau \rightarrow \alpha\tau$ in the \mathcal{P} -exponentials corresponding to the longitudinal sides, setting $\alpha = T/T_0$ with T_0 some fixed time (length) scale: in this way one shows explicitly that the loops depend on T only through the combinations $(T/T_0)p_i/m$ and $(T/T_0)p_{Ei}/m$. Next, we rescale coordinates and fields as follows. To unify the treatment of the Euclidean and Minkowskian cases we use the same symbol ϕ_μ for the transformed gauge fields, and y^μ for the transformed coordinates

³We use here and in the following the notation $\alpha + \beta\mathcal{I} = \{\alpha + \beta z | z \in \mathcal{I}\}$.

(we can use upper indices for the new coordinates also in the Euclidean case without ambiguity). We then set

$$\begin{aligned} y^\mu &= M^\mu{}_\nu x^\nu, & A_\mu(x) &= \phi_\nu(y) M^\nu{}_\mu, \\ y^\mu &= M_{E\mu\nu} x_{E\nu}, & A_{E\mu}(x_E) &= \phi_\nu(y) M_{E\nu\mu}, \end{aligned} \quad (13)$$

in the Minkowskian and Euclidean cases, respectively, where M and M_E are the diagonal matrices

$$\begin{aligned} M^\mu{}_\nu &= \text{diag} \left(\frac{T_0}{T} \frac{1}{\sqrt{2} \cosh(\chi/2)}, \frac{T_0}{T} \frac{1}{\sqrt{2} \sinh(\chi/2)}, 1, 1 \right), \\ M_{E\mu\nu} &= \text{diag} \left(\frac{T_0}{T} \frac{1}{\sqrt{2} \cos(\theta/2)}, \frac{T_0}{T} \frac{1}{\sqrt{2} \sin(\theta/2)}, 1, 1 \right). \end{aligned} \quad (14)$$

The Wilson loops in the two theories are then changed into the same functionals of the new variables; however, the transformed actions are different in the two cases, and so are the expectation values. To make this clear, we introduce the notation

$$\langle \mathcal{O}[\phi] \rangle_S \equiv Z_S^{-1} \int [D\phi] \mathcal{O}[\phi] e^{-S[\phi]}, \quad Z_S = \int [D\phi] e^{-S[\phi]}, \quad (15)$$

and we denote with

$$W_{\Gamma_{1,2}}[\phi] \equiv \frac{1}{N_c} \text{tr} \left\{ \mathcal{P} \exp \left[-ig \oint_{\Gamma_{1,2}} \phi_\mu(y) dy^\mu \right] \right\} \quad (16)$$

the new Wilson loops. Here the new paths are given by

$$\begin{aligned} \Gamma_1 &: Y_{1q[\bar{q}]}^\mu(\tau) = z^\mu + \frac{\delta_0^\mu + \delta_1^\mu}{\sqrt{2}} \tau + f_1^{q[\bar{q}]} R_1^\mu, \\ \Gamma_2 &: Y_{2q[\bar{q}]}^\mu(\tau) = \frac{\delta_0^\mu - \delta_1^\mu}{\sqrt{2}} \tau + f_2^{q[\bar{q}]} R_2^\mu, \end{aligned} \quad (17)$$

with $\tau \in [-T_0, T_0]$, and closed by the usual transverse straight-line paths at $\tau = \pm T_0$. We then write for the correlation functions and expectation values in the two theories

$$\begin{aligned} \langle \mathcal{W}_1^{(T)} \mathcal{W}_2^{(T)} \rangle &= \langle W_{\Gamma_1} W_{\Gamma_2} \rangle_{-iS_M^{\text{Y.M.}}}, & \langle \mathcal{W}_i^{(T)} \rangle &= \langle W_{\Gamma_i} \rangle_{-iS_M^{\text{Y.M.}}}, \\ \langle \widetilde{\mathcal{W}}_1^{(T)} \widetilde{\mathcal{W}}_2^{(T)} \rangle_E &= \langle W_{\Gamma_1} W_{\Gamma_2} \rangle_{S_E^{\text{Y.M.}}}, & \langle \widetilde{\mathcal{W}}_i^{(T)} \rangle_E &= \langle W_{\Gamma_i} \rangle_{S_E^{\text{Y.M.}}}, \end{aligned} \quad (18)$$

where $S_M^{\text{Y.M.}}$ and $S_E^{\text{Y.M.}}$ are the transformed Minkowskian and Euclidean pure-gauge (Yang-Mills) actions:

$$S_M^{\text{Y.M.}} = - \sum_{\mu,\nu=0}^3 C_{M\mu\nu}(\chi, T) \frac{1}{2} \int d^4y \text{tr}(\Phi_{\mu\nu})^2 \quad (19)$$

$$S_E^{\text{Y.M.}} = \sum_{\mu,\nu=0}^3 C_{E\mu\nu}(\theta, T) \frac{1}{2} \int d^4y \text{tr}(\Phi_{\mu\nu})^2. \quad (20)$$

Here $(\Phi_{\mu\nu})^2$ is understood as the square of the Hermitian matrix $\Phi_{\mu\nu} = \partial_\mu\phi_\nu - \partial_\nu\phi_\mu + ig[\phi_\mu, \phi_\nu]$, and the symmetric coefficients $C_{M\mu\nu}$ and $C_{E\mu\nu}$ are ($\nu_\perp = 2, 3$)

$$C_{M01} = -C_{M23}^{-1} = -\left(\frac{T_0}{T}\right)^2 \frac{1}{|\sinh \chi|}, \quad C_{M0\nu_\perp} = -C_{M1\nu_\perp}^{-1} = -\frac{|\sinh \chi|}{\cosh \chi + 1} \quad (21)$$

$$C_{E01} = C_{E23}^{-1} = \left(\frac{T_0}{T}\right)^2 \frac{1}{|\sin \theta|}, \quad C_{E0\nu_\perp} = C_{E1\nu_\perp}^{-1} = \frac{|\sin \theta|}{\cos \theta + 1}, \quad (22)$$

and $C_{M\mu\mu} = C_{E\mu\mu} = 0 \forall \mu$.

Restricting the angular variables to the intervals $\chi \in \mathcal{I}_M$ and $\theta \in \mathcal{I}_E$ (see remark above), we can drop the absolute values in Eqs. (21) and (22), obtaining coefficient functions which can be analytically extended throughout the respective complex planes in both variables, with the possible exception (depending on the specific coefficient) of the isolated singular points (poles) $T = 0, \infty$, and $\chi = 0, \infty$ in the Minkowskian case or $\theta = n\pi$, $n \in \mathbb{Z}$ in the Euclidean case. To avoid confusion, we will denote with an overbar the *analytic extensions* $\overline{C}_{M\mu\nu}$ and $\overline{C}_{E\mu\nu}$ obtained starting from \mathcal{I}_M and \mathcal{I}_E at real positive T , in the two cases respectively.

4 Analyticity domain of the Euclidean correlation function

A function of a complex variable is analytic if its derivative exists in complex sense. If we were allowed to bring the derivative under the sign of integral, we could infer the analytic properties of the correlation function \mathcal{G}_E directly from its functional–integral representation. Here we will give a formal argument, based on the following assumption: one can bring the derivative with respect to a parameter under the functional integral sign as long as the resulting integral is convergent, in analogy with the case of ordinary integrals⁴.

The functional integrals defined by means of Eq. (15) are expected to be convergent as long as the real part of the action is positive–definite; allowing for derivatives to pass under the sign of integral, it is easy to see that the convergence properties of the functional integral are left unchanged, and so we conclude (formally) that the correlation function \mathcal{G}_E can be analytically extended to complex values of θ and T for which the real part of the action $S_E^{Y.M.}$ is positive–definite, and this happens if and only if the *convergence conditions*

$$\text{Re} \overline{C}_{E\mu\nu}(\theta, T) > 0 \quad \forall \mu, \nu \quad (23)$$

for the (analytically extended) coefficients are satisfied. Note that singular points of the coefficients are artifacts of the field and coordinate transformation, and they are

⁴Strictly speaking, in the latter case a sufficient condition is *uniform* convergence.

not necessarily singular points of the correlation function. Indeed, while singularities are expected at the points $\theta = 0, \pi$ on the basis of the relation between the correlation function \mathcal{G}_E and the static dipole–dipole potential [18] (see also [10]), no singularity is expected at $T = 0$, where \mathcal{G}_E is expected to vanish.

Substituting θ with the complex variable $z \equiv \theta - i\chi$ (with real θ and χ) and writing for the complex variable T , $T = |T|e^{i\psi/2}$, one easily sees that the *convergence conditions* (23) are equivalent to

$$\begin{aligned} F(\theta, \chi, \psi) &\equiv e^\chi \sin(\theta + \psi) + e^{-\chi} \sin(\theta - \psi) > 0, \\ \sin \theta (\cosh \chi + \cos \theta) &> 0. \end{aligned} \quad (24)$$

It has been shown in [17] that the previous inequalities define a connected subset \mathcal{V} of the 3D real (θ, χ, ψ) –space; moreover, as the modulus $|T|$ never enters the previous equations, the section of the analyticity domain is the same irrespectively of $|T|$. No dependence on the arbitrary parameter T_0 is found, too, as expected. One then finds a connected analyticity domain \mathcal{D}_E ,

$$\mathcal{D}_E = \{(z, T) \in \mathbb{C}^2 \mid (\theta, \chi, \psi) \in \mathcal{V}\} \quad (25)$$

for the extension of the Euclidean correlation function from $\theta \in \mathcal{I}_E$ at positive real T . Note that the analyticity domain must be symmetric under $z \rightarrow z^*$ and under $z \rightarrow \pi - z$, as one can easily show taking into account the identities

$$F(\theta, \chi, \psi) = F(\theta, -\chi, -\psi) = F(\pi - \theta, -\chi, \psi). \quad (26)$$

Sections of this subset at fixed χ are shown in Fig. 1 (left). Note that the domain “thins out” as one tends toward the “physical” edges E^{dir} and E^{cross} ,

$$E^{\text{dir/cross}} = \{(z, T) \in \mathbb{C}^2 \mid \theta = 0/\pi, \chi \in \mathbb{R}^{+/-}, \psi = \pi\}, \quad (27)$$

and also towards the other two edges $E^{\text{dir}*}$ and $E^{\text{cross}*}$ [here $E^* = \{(z, T) \mid (z^*, T^*) \in E\}$]. Sections of the same analyticity domain at fixed ψ are shown in Fig. 1 (right): the whole “strip” $\mathcal{S}_E \equiv \{z = \theta - i\chi \mid \theta \in (0, \pi), \chi \in \mathbb{R}\}$ (at $\psi = 0$) reduces to disjoint regions near the edges of the domain (at $\psi \simeq \pm\pi$).

As we approach E^{dir} from the inside, the coefficients $\overline{C}_{E\mu\nu}$ become imaginary, and

$$\overline{C}_{E\mu\nu}(-i\chi, iT) = iC_{M\mu\nu}(\chi, T) \quad (28)$$

so that $S_E^{\text{Y.M.}} \xrightarrow{\theta \rightarrow -i\chi, T \rightarrow iT} -iS_M^{\text{Y.M.}}$, i.e., according to Eq. (18),

$$\mathcal{G}_M(\chi; T; \vec{z}_\perp; 1, 2) = \overline{\mathcal{G}}_E(-i\chi; iT; \vec{z}_\perp; 1, 2), \quad \forall \chi \in \mathcal{I}_M, T \in \mathbb{R}^+. \quad (29)$$

We thus find that Minkowskian and Euclidean correlation functions are connected by the expected analytic continuation [6, 7, 8, 9], of which we have given here an

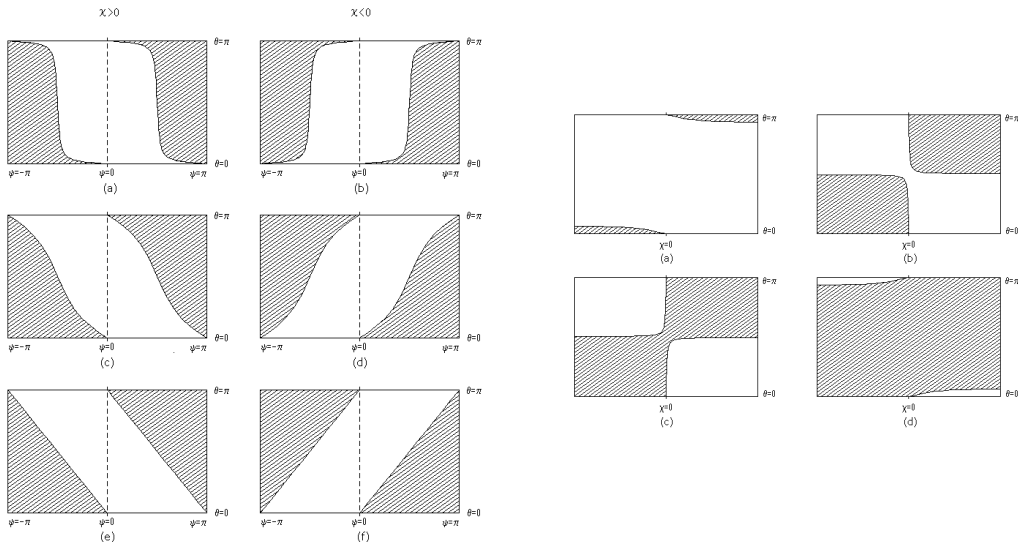


Figure 1: Sections of the analyticity domain (white area) at fixed χ (left) and at fixed ψ (right).

alternative derivation. Note that the Minkowskian correlation function is approached from above in the complex plane of the hyperbolic angle χ , in agreement with the usual “ $-i\epsilon$ ” prescription [11].

According to the *crossing-symmetry relations* (12) (derived in [10]), we should find the physical amplitude in the “crossed” channel at the edge E^{cross} of the analyticity domain. Here we find

$$\overline{C}_{E\mu\nu}(\pi - i\chi, iT) = \sum_{\alpha,\beta=0}^3 iS_{\mu\alpha}S_{\nu\beta}C_{M\alpha\beta}(-\chi, T), \quad (30)$$

where S is a matrix which simply interchanges the 0 and 1 components of fields and coordinates, and which can be reabsorbed into the loops with a further transformation of fields and coordinates, with the only effect of reversing the orientation of W_{Γ_2} , so that $W_{\Gamma_2} \rightarrow W_{\Gamma_2}^*$. We thus find that the Euclidean correlation function is analytically continued to the physical correlation function (with positive hyperbolic angle $-\chi$) of a loop and an antiloop, as expected [10]; as a by-product, we reobtain the crossing-symmetry relation for the loops, Eq. (12),

$$\mathcal{G}_M(i\pi - \chi; T; \vec{z}_\perp; 1, 2) = \overline{\mathcal{G}}_M(\chi; T; \vec{z}_\perp; 1, \bar{2}) = \overline{\mathcal{G}}_M(\chi; T; \vec{z}_\perp; \bar{1}, 2), \quad \forall \chi \in \mathcal{I}_M, T \in \mathbb{R}^+. \quad (31)$$

To see what happens at the other two edges of the analyticity domain it suffices to recall that \mathcal{D}_E possesses the symmetry $\mathcal{D}_E = \mathcal{D}_E^*$, and that the coefficients $\overline{C}_{E\mu\nu}$

satisfy the *reflection relation* $\overline{\mathcal{C}}_{E\mu\nu}(z^*, T^*) = \overline{\mathcal{C}}_{E\mu\nu}(z, T)^*$. Exploiting this relation, C -invariance and reality of the integration measure, one easily shows that

$$\overline{\mathcal{G}}_E(z^*; T^*; \vec{z}_\perp; 1, 2) = \overline{\mathcal{G}}_E(z; T; \vec{z}_\perp; 1, 2)^*. \quad (32)$$

In particular, this means that at $\psi = -\pi$ we find the complex conjugate of the physical correlation functions, respectively at $E^{\text{dir}*}$ ($\chi < 0$) for the “direct channel” and at $E^{\text{cross}*}$ ($\chi > 0$) for the “crossed channel”. Moreover, from the previous relation we find that the Euclidean correlation function at $\chi = 0, \psi = 0$ is a real function (see also [15]).

4.1 Analyticity properties of the correlation function with the IR cutoff removed

As the physically relevant quantities are the correlation functions with the IR cutoff removed [5, 9], i.e., $\mathcal{C}_{M,E}$, we will discuss now what can be inferred about their analyticity properties from the properties of \mathcal{G}_E . As a function of the complex variable T at fixed $z = \theta - i\chi$, $\overline{\mathcal{G}}_E$ is analytic in the sector $-\pi/2 + \Delta < \arg T = \psi/2 < \Delta$, where $\Delta = \Delta(z) \in (0, \pi/2)$; one can then define $I_\Delta \equiv (-\pi + 2\Delta, 2\Delta)$, and rewrite \mathcal{D}_E as $\mathcal{D}_E = \{(z, T) \mid z \in \mathcal{S}_E, \psi \in I_\Delta(z)\}$.

Moreover, the normalised correlation function is expected to be IR finite: in a non-Abelian gauge theory, the short-range nature of the interactions implies that those parts of the partons’ trajectories which lie too far aside with respect to the “vacuum correlation length” [19] do not affect each other. There should then be a “critical” length T_c , beyond which the normalised correlation function becomes independent of T : this is confirmed by lattice calculations [15]. As the existence of a “vacuum correlation length” is usually ascribed to the non-trivial dynamics dictated by non-Abelian gauge invariance, the previous argument is expected to apply also for the analytically-extended correlation functions, substituting the real variable T with the modulus of the complex variable $|T|$.

In conclusion, the analytically extended correlation function is expected to satisfy the hypotheses of the Phragmén–Lindelöf theorem (see theorem 5.64 of Ref. [20]), which implies that $\overline{\mathcal{G}}_E$ converges uniformly to a unique value in the whole sector as $|T| \rightarrow \infty$. We can then define unambiguously the function

$$\overline{\mathcal{C}}_E(z; \vec{z}_\perp; 1, 2) \equiv \lim_{|T| \rightarrow \infty} \overline{\mathcal{G}}_E(z; T; \vec{z}_\perp; 1, 2), \quad \forall z \in \mathcal{S}_E \quad (33)$$

since the limit on the right-hand side does not depend on the particular direction in which one performs it. One easily sees that $\overline{\mathcal{C}}_E$ is the analytic extensions of \mathcal{C}_E , and if we now take the limit $|T| \rightarrow \infty$ in the analytic continuation relation, Eq. (29), we obtain the analytic continuation relation with the IR cutoff removed [9],

$$\mathcal{C}_M(\chi; \vec{z}_\perp; 1, 2) = \overline{\mathcal{C}}_E(-i\chi; \vec{z}_\perp; 1, 2), \quad \forall \chi \in \mathcal{I}_M. \quad (34)$$

The crossing–symmetry relations are still valid for the analytic extensions $\bar{\mathcal{C}}_M$ and $\bar{\mathcal{C}}_E$ throughout the respective analyticity domains, as one can prove by taking the limit $|T| \rightarrow \infty$ in Eq. (31) (relying again on the Phragmén–Lindelöf theorem mentioned above). Note also that $\mathcal{C}_E(z^*) = \mathcal{C}_E(z)^*$ throughout the domain of analyticity, as one can easily see by taking $|T| \rightarrow \infty$ in Eq. (32).

4.2 Lattice regularisation

The functional integral must be regularised to become a well–defined mathematical object; here we justify the formal argument given above using a lattice regularisation. In this approach the ill–defined continuum functional integral is replaced with a well–defined (multidimensional) integral, which in the case of gauge theories can be chosen to be an integral on the gauge group manifold [21],

$$\langle \mathcal{O}[U] \rangle_{S_{\text{lat}}} \equiv \frac{\int [DU] \mathcal{O}[U] e^{-S_{\text{lat}}[U]}}{\int [DU] e^{-S_{\text{lat}}[U]}} \quad (35)$$

where DU is the invariant Haar measure. It is easy to see that in our case the action

$$S_{\text{lat}} = \beta \sum_{n, \mu < \nu} C_{E\mu\nu}(\theta, T) \left[1 - \frac{1}{N_c} \text{Re tr } U_{\mu\nu} \right], \quad (36)$$

where $U_{\mu\nu}$ is the usual plaquette variable (in the fundamental representation) [21] and $\beta = 2N_c/g^2$, gives back the action $S_E^{\text{Y.M.}}$ of Eq. (20) in the limit $a \rightarrow 0$, upon identification of the link variables with $U_\mu(n) = \exp\{iga\phi_\mu(na)\}$. For compact gauge groups, such as $SU(N_c)$, the integration range is compact, so that, as long as the volume and the lattice spacing are finite, the integral (35) with the action (36) is convergent and analytic in θ and T ; in the $V \rightarrow \infty$ limit, one has to impose positive–definiteness of the real part of the action in order for the integral to remain convergent, and this leads exactly to the *convergence conditions* (23).

The action (36) is correct at tree–level, but one has also to ensure that quantum effects do not modify its form. It is easy to see that Eq. (36) is also the correct tree–level action for an anisotropic lattice regularisation of the usual Euclidean Yang–Mills action, as one can directly check [see Eqs. (13) and (14)] by identifying $U_\mu(n) = \exp\{iga_\mu A_{E\mu}(na)\}$, with $a_\mu = a/M_{E\mu\mu}$. Showing that Eq. (36) is a good lattice action on an isotropic lattice for the modified action Eq. (20) is then equivalent to show that it is a good action on an anisotropic lattice for the usual Yang–Mills action.

Since the general anisotropic action is not guaranteed to belong to the same universality class as the isotropic lattice action [22], one has to enforce that rotation

invariance is restored in the continuum limit by properly tuning the coefficients of the various terms of the action, obtaining in our case

$$\tilde{S}_{\text{lat}} = \sum_{n,\mu<\nu} \beta_{\mu\nu} C_{E\mu\nu}(\theta, T) \left[1 - \frac{1}{N_c} \text{Re tr } U_{\mu\nu} \right], \quad (37)$$

with properly chosen functions $\beta_{\mu\nu} = \beta_{\mu\nu}(a, \theta, T)$. Due to the asymptotic freedom property of non-Abelian gauge theories, one can determine these functions analytically in perturbation theory for small lattice spacings. We have calculated $\beta_{\mu\nu}(a, \theta, T)$ to one-loop accuracy, and we have found that quantum effects do not impose further restrictions on the analyticity domain \mathcal{D}_E found above [23].

As a final remark, we notice that after the field and coordinate transformation, the longitudinal sides of the two continuum Wilson loops are at 45° with respect to the new axes, and have to be approximated by a broken line (see, e.g., Ref. [15]): this introduces approximation errors which have to be carefully considered, but which should vanish in the continuum limit, thus leaving unaltered our analysis. One could use on-axis Wilson loops, thus performing an “exact” calculation on the lattice, if one performs a further transformation of the action, choosing the new basis vectors along the directions of the longitudinal sides of the loops, but in this case one has to deal with the more complicated (“chair-like”) terms $\text{tr}[U_{0\alpha_\perp} U_{1\alpha_\perp}^\dagger]$.

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