When Quantum Mechanics Complies with Bell's Inequalities

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1 – Introduction

Violations of Bell's Inequalities are long known to provide us with a core and striking property of Quantum Mechanics, that is, that quantum correlations are stronger than those of any possible local and realistic theory of "hidden variables" [1]. This, by the way, complies with the remarkable intuition of E. Schrödinger who, long ago, regarded entanglement the key-property of Quantum Mechanics (it is in entangled states that Bell's Inequalities come out violated). The subject has been the matter of an impressive literature, with up today more than 90.000 published articles. Now, the preceeding sentences, the failure of local realism as it is widely accepted by now, stand at the level of the physical interpretations of Bell's Inequalities violations. But as inspection shows, an important point is the existence or non-existence of Joint Probability Distributions (J.P.D.s for short) for random variables (r.v.s) that are associated to quantum operators which may or may not all commute. A theorem can state that J.P.D.s do exist for any given set of n r.v.s associated to n quantum operators, provided the latter all commute, two by two [2]. This, though interesting

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in itself, is however a bit deceptive at the physical level because if all operators do commute two by two, then one is lead back to a classical-like situation where the constant $\hbar \to 0$. Not very surprisingly then, J.P.D.s for the r.v.s associated to these operators can be defined, and inequalities of the Bell's type are accordingly satisfied. Our analysis here will pay attention to less trivial situations, where:

1- not all of the quantum operators commute,
2- states are entangled (even maximally),
3- and still, Bell's inequalities are satisfied.

We define random variables associated to successives measurements of non commuting observables. Thus we obtain naturally the notion of statisticall insensivity.

2 – Statistical formalism of Probability theory

A random variable (r.v.) X is defined by:

- its possible values: $x_1, \ldots, x_n$,
- with probabilities: $p(X = x_1), \ldots, p(X = x_n)$.

These probabilities are positive numbers such that $\sum_i p(X = x_i) = 1$ and are interpreted as the frequencies of occurrence of the respective values over a large number of realizations.

The mean value of the r.v. X is by definition the number

$$E(X) = \sum_i x_i p(X = x_i) .$$

Definition 1 – *Let X and Y be two random variables, we define a* joint probability distribution of (X,Y) *as a family of positive numbers p_{ik} such that*

- $\sum_{i,k} p_{ik} = 1$
- $\sum_i p_{ik} = p(Y = y_k)$
- $\sum_k p_{ik} = p(X = x_i)$

We use the notation $p_{ik} = P(X = x_i\,, Y = y_k)$. Given a joint probability distribution of (X,Y), we say that X and Y are independant if $P(X = x_i\,, Y = y_k) = P(X = x_i)\,P(Y = y_k)$ for all i and k.

Note that if X and Y are two random variables then $X + Y$ or XY or any function $f(X, Y)$ is not necessarily defined. For exemple we can say that the values of the variable $X + Y$ are all the values $z = x_i + y_k$ but we must defie the probability of each of these values. Given a joint probability distribution (p_{ik}) of (X, Y) then we can define

$$P(X+Y=z)=\sum_{\substack{x_i+y_k=z\\ i,k}} P(X=x_i\ ,\ Y=y_k)\,.$$

Then we have $E(X+Y)=E(X)+E(Y)$ and if for the joint probability distribution of (X, Y), X and Y are independant then we have $E(XY)=E(X)E(Y)$.

Remark – The distribution $p_{ik}=P(X=x_i)P(Y=y_k)$ is always a JPD of (X,Y) and for this JPD the variables X and Y are independent.

3 – Kolmogorov formalism of Probability theory

In the Kolmogorov formalism of Probability theory the random variables are represented as functions on a set Ω. Any occurence of the experiment is represented by the choice of a value of the hidden variable $\omega\in\Omega$. The probability is defined by a positive function p on Ω such that $\sum_{\omega\in\Omega} p(\omega)=1$. If X is a random variable the values $X(\omega)$ are then determined for each occurence and the probability distibution of X is defined by

$$p(X=x_i)=\sum_{\omega,\,X(\omega)=x_i} p(\omega)\,.$$

If X and Y are random variables the values $(X(\omega), Y(\omega))$ are then determined for each occurence and in the Kolmogorov formalism there exist a natural joint probability distribution of (X,Y) defined by

$$p(X=x_i, Y=y_k)=\sum_{\substack{\omega,\,X(\omega)=x_i\\ Y(\omega)=y_k}} p(\omega)\,.$$

4 – Quantum formalism

The formalism of Quantum Mechanics can be developed in terms of the following basic objects and postulates (evolution, which would complete the quantum mechanical description, will not be dealt with in the sequel).

4.1 – A Statistical Operator

Over E, a $\mathbb{C}$-Hilbert space endowed with the inner product $\langle\ |\ \rangle$, and $\mathcal{H}(E)$, the set of Hermitian operators on E, the system is described by a statistical operator: $\rho\in\mathcal{H}(E)$ (density matrix) satisfying:

- $Tr(\rho)=1$

- ρ positive, *i.e.*, $\forall \psi \in E$, $\langle \psi | \rho \, \psi \rangle \geq 0$.

4.2 – Postulate 1

For any obsevable $A \in \mathcal{H}(E)$, one has $A = \sum_i a_i \, P_{A_i}$, where $a_i \in \mathbb{R}$, and where the P_{A_i} are projectors on the proper subspaces spanned by the $a_i s$, including the possible vanishing eigenvalue, i.e., the relation $\sum_i P_{A_i} = Id$ is satisfied in the definition above for A. Then, to any state ρ and any observable A one can associate the r.v. $X_A(\rho)$ representing the result of the measure of A in state ρ.

By definition, the random variable $X_A(\rho)$ has:

- the a_i as its values, appearing with.
- probabilities $p(X_A(\rho) = a_i) = Tr(\rho \, P_{A_i})$.

4.3 – Postulate 2

The Wave Packet Reduction postulate. Though this postulate is no longer considered as such nowadays, and seems to be reducible to a more intuitive statement [3], it can still be used as a convenient way to put things [1].

After the measure of A in state ρ, the latter gets reduced into the state $\Gamma_A(\rho)$, where Γ_A is the *"intertwining operator"* defined fon $\mathcal{H}(E)$ by

$$B \mapsto \Gamma_A(B) = \sum_i P_{A_i} \, B \, P_{A_i} \, ,$$

where $\sum_i P_{A_i} = Id$.

Remark – $\forall \, A, \, B, \, C \in \mathcal{H}(E)$, one has $Tr(\Gamma_C(A) \, B) = Tr(A \, \Gamma_C(B))$ that is, Γ_C is self-adjoint for the scalar product on $\mathcal{H}(E)$ defined by $(A|B) \equiv Tr(A \, B)$.

5 – Succession of measurements

Let us consider $X_{A \to B}(\rho)$, the random variable associated to a sequence where a measure of an observable B, in a given quantum system, is performed after another observable A has been measured on the same system (here, "after" is to be understood in a non-relativistic acceptation; a relativistic one would require the use of a $C^\star$-algebra analysis).

Definition 2 – *Let $A, \, B \in \mathcal{H}(E)$. One can define $X_{A \to B}(\rho)$, the r.v. with values*

- *$b_1, \, \ldots, \, b_n$, the eigenvalues of B,*
- *with associated probabilities $p(X_{A \to B} = b_j) = Tr(\Gamma_A(\rho) \, P_{B_j})$.*

By definition of the intertwinning operator we have $X_{A\to B}(\rho) = X_B(\Gamma_A(\rho))$.

Theorem 1 – *Let $X_A(\rho)$ and $X_{A\to B}(\rho)$, be two random variables. The family*

$$p_{A\to B,\rho}(a_i, b_k) = Tr(P_{A_i}\,\rho\, P_{A_i} P_{B_k})$$

satisfies the conditions

- $\sum_k p_{A\to B,\rho}(a_i, b_k) = p(X_A(\rho) = a_i)$
- $\sum_i p_{A\to B,\rho}(a_i, b_k) = p(X_{A\to B}(\rho) = b_k)$

that identify it as the natural joint probability distribution of $X_A(\rho)$ and $X_{A\to B}(\rho)$. We note $p(X_A(\rho) = a_i, X_{A\to B}(\rho) = b_k)$ this JPD.

Postulate 3 – The natural JPD of f$X_A(\rho)$ and $X_{A\to B}(\rho)$, noted $p(X_A(\rho) = a_i, X_{A\to B}(\rho) = b_k)$ represent the frequency of the obtention of the succesives results ($X_A(\rho) = a_i$ followed by $X_{A\to B}(\rho) = b_k$), over a large number of occurrences.

The natural JPD of f $X_A(\rho)$ and $X_{A\to B}(\rho)$, allows us to define other random variables like $X_A(\rho) \pm X_{A\to B}(\rho)$ and $X_A(\rho)\, X_{A\to B}(\rho)$. A succession of measurements is represented by the action of the intertwinning operator

$$\begin{array}{ccccc}
\textit{observable} & \textit{state} & \textit{r.v.} & & \textit{mean value} \\
 & \rho & & & \\
A & \downarrow & X_{A,\rho} & \to & (\rho|A) \\
 & \Gamma_A\rho & & & \\
B & \downarrow & X_{B,\Gamma_A\rho} & \to & (\rho|\Gamma_A B) \\
 & \Gamma_B\Gamma_A\rho & & & \\
C & \downarrow & X_{C,\Gamma_B\Gamma_A\rho} & \to & (\rho|\Gamma_A\Gamma_B C) \\
 & \Gamma_C\Gamma_B\Gamma_A\rho & & &
\end{array}$$

6 – Statistical insensitivity

Definition 3 – *In a state ρ, B will be said to be* (statistically) insensitive to A, *if one has $X_{A\to B}(\rho) = X_B(\rho)$. That is, in a given state ρ, B is (statistically) insensitive to A, if*

$$\forall j,\ Tr(\Gamma_A(\rho) P_{B_j}) = Tr(\rho\, P_{B_j}).$$

In other words, a measure of A in state ρ does not affect the statistical behavior of X_B, when B is measured after A.

Definition 4 – *In a state ρ, A and B are* mutually insensitive *if $X_{A\to B}(\rho) = X_B(\rho)$ and $X_{B\to A}(\rho) = X_A(\rho)$. That is, if the following conditions are met:*

- $\forall\, k,\ Tr(\Gamma_A(\rho)\, P_{B_k}) = Tr(\rho\, P_{B_k})$.
- $\forall\, j,\ Tr(\Gamma_B(\rho)\, P_{A_j}) = Tr(\rho\, P_{A_j})$.

At this level, one may note that the more intuitive basis from which Postulate 2 can be deduced can be stated as the condition $X_{A\to A}(\rho) = X_A(\rho)$, when the two measures of A are performed at very close instants in time (W. H. Zurek [3]).

As a first result, it is elementary to check that the following result holds.

Theorem 2 – *If* $[A,B]=0$*, then* $\forall\,\rho$*:*

$$\begin{aligned} X_{A\to B}(\rho) &= X_B(\rho)\,, \\ X_{B\to A}(\rho) &= X_A(\rho)\,, \end{aligned}$$

that is, A and B are mutually insensitive in all states ρ.

In the case of commuting observables, of course, things simplify to a large extent. If $[A,B]=0$, then the natural J.P.D. for X_A and X_B is simply

$$p_{A\to B,\rho}(a_i,b_k) = p_{B\to A,\rho}(a_i,b_k) = Tr(\rho\, P_{B_k} P_{A_i})\,,$$

and the following result holds:

Theorem 3 – *If* $[A,B]=0$*, then, for all state* ρ*, one has, with the natural J.P.D.*

$$X_A(\rho) + X_B(\rho) = X_{A+B}(\rho)\,, \qquad X_A(\rho)\, X_B(\rho) = X_{AB}(\rho)\,.$$

In this situation, a simple mapping of the algebra generated by a complete set of commuting observables (CSCO), into the corresponding r.v.s one is realized.

7 – J.P.D.'s associated to statistical insensitivity

In view of the preceding section, one may look for J.P.D.s that, given a state ρ, could be canonically associated with r.v.s related to observables that are mutually insensitive in that state.

If B is insensitive to A in state ρ, that is if $X_B(\rho) = X_{A\to B}(\rho)$, then a natural J.P.D. of $(X_A(\rho), X_B(\rho))$ is provided by

$$p_{A\to B,\rho}(a_i,b_k) \;=\; p(X_A(\rho)=a_i,\ X_{A\to B}(\rho)=b_k) = Tr(P_{A_i}\rho P_{A_i} P_{B_k})\,.$$

Now, this implies that if A, B are mutually insensitive in state ρ, then two natural J.P.D.s of $(X_A(\rho), X_B(\rho))$ can be devised,

$$\begin{aligned} p_{A\to B,\rho}(a_i,b_k) &= Tr(P_{A_i}\rho P_{A_i} P_{B_k})\,, \\ p_{B\to A,\rho}(a_i,b_k) &= Tr(P_{B_k}\rho P_{B_k} P_{A_i})\,, \end{aligned}$$

which are not necessarily the same. But the following theorem holds:

Theorem 4 – Theorem of the "Joint Distributions" – *Two observables A and B are mutually insensitive in a state ρ iff for all $\alpha \in]0,1[$*

$$p_{i,j} = \alpha\, Tr(\rho\, \Gamma_{P_{A_i}}(P_{B_j})) + (1-\alpha)\, Tr(\rho\, \Gamma_{P_{B_j}}(P_{A_i}))\,,$$

is a joint distribution for $(X_A(\rho), X_B(\rho))$.

8 – Insensitivity and independence

a) It is interesting to compare independence and insensitivity. Having a J.P.D for $(X_A(\rho), X_{A\to B}(\rho))$, one may analyze their statistical independence. Indeed, one can see that these variables can be independent in a context where a statistical insensitivity does not hold. In other words, a measure of A modifies the law of X_B, while $X_A(\rho)$ et $X_{A\to B}(\rho)$ display no correlation.

Example – Consider $E = \mathbb{C}^2$ and

$$\rho = \begin{pmatrix} \alpha & \beta \\ \beta & 1-\alpha \end{pmatrix},$$

with $0 < \alpha^2 + \beta^2 \leq \alpha < 1$, and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose eigenvalues are $a_1 = 1$ et $a_{-1} = -1$. The law of $X_A(\rho)$ reads as

$$P(X_A(\rho) = 1) = \beta + \frac{1}{2}, \qquad P(X_A(\rho) = -1) = -\beta + \frac{1}{2}\,.$$

Also, consider

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

whose eigenvalues are $b_1 = 1$ et $b_{-1} = -1$. The law of $X_B(\rho)$ reads as

$$P(X_B(\rho) = 1) = \alpha, \qquad P(X_B(\rho) = -1) = 1 - \alpha\,.$$

One has

$$\Gamma_A(\rho) = \begin{pmatrix} \frac{1}{2} & \beta \\ \beta & \frac{1}{2} \end{pmatrix},$$

and the law of $X_B(\Gamma_A(\rho))$ is given by

$$P(X_B(\Gamma_A(\rho)) = 1) = P(X_B(\Gamma_A(\rho)) = -1) = \frac{1}{2}\,.$$

The natural J.P.D for $X_A(\rho)$ and $X_B(\Gamma_A(\rho))$ reads

$$\begin{aligned}
p_{A\to B,\rho}\,(1,1) &= \frac{1}{2}\beta+\frac{1}{4},\\
p_{A\to B,\rho}\,(1,-1) &= \frac{1}{2}\beta+\frac{1}{4},\\
p_{A\to B,\rho}\,(-1,-1) &= -\frac{1}{2}\beta+\frac{1}{4},\\
p_{A\to B,\rho}\,(-1,1) &= -\frac{1}{2}\beta+\frac{1}{4}.
\end{aligned}$$

One then observes that $X_A(\rho)$ et $X_B(\Gamma_A(\rho))$ are stochastically independent. Still, if $\alpha < 1/2$ one has

$$X_B(\Gamma_A(\rho)) \neq X_B(\rho)\,.$$

b) On the other hand, if one takes $B = A$, then $[A, A] = 0$, but $X_A(\rho)$ and $X_A(\Gamma_A(\rho)) = X_A(\rho)$ are not stochastically independent if $X_A(\rho)$ is not a constant r.v., that is if $A \neq cste\,.\,Id$. That is, commutativity does not imply independence.

c) Insensivity does not involve independence (as commutativity does not involve independence).

Example – $E = \mathbb{C}^2$ and

$$\rho=\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad A=\begin{pmatrix}a_1&0\\0&a_2\end{pmatrix},\quad B=\begin{pmatrix}b_1&0\\0&b_2\end{pmatrix}.$$

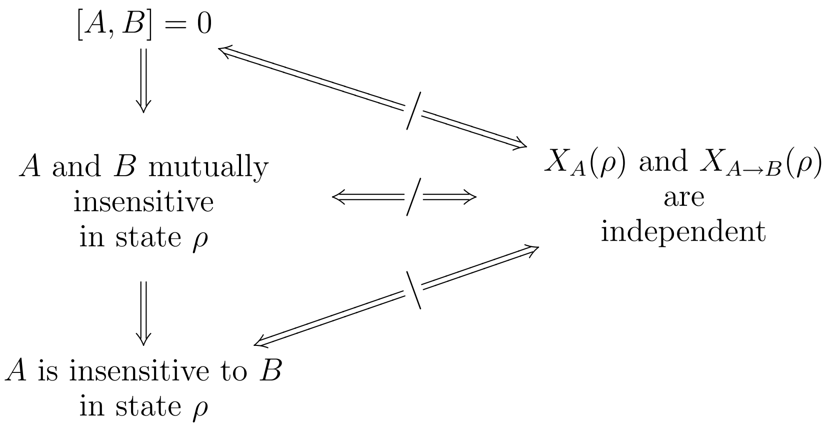
One has $AB = BA$ and

$$\begin{aligned}
P(X_A(\rho) &= a_1) = Tr(\rho p_{A_1}) = \frac{1}{2},\\
P(X_B(\rho) &= b_1) = Tr(\rho p_{B_1}) = \frac{1}{2}.
\end{aligned}$$

Now, it can be checked that

$$p_\rho(a_1,b_1) = Tr(\rho p_{A_1}p_{B_1}) = \frac{1}{2} \neq Tr(\rho p_{A_1})Tr(\rho p_{B_1})\,.$$

The following diagram summarizes some of the implications we have just dealt with.



9 – Bell's Inequalities

9.1 –A simple inequality

Let us consider the case of four bivalent (± 1) random variables: X, X', Y and Y', and assume that we can apply the Kolmogorov formalism, that is for each occurrence ω of the experiment, the numbers $X(\omega), X'(\omega), Y(\omega), Y'(\omega)$ are determined. This implies that $Z = (X + X')Y + (X - X')Y'$ is well defined as a random variable by

$$Z(\omega) = (X(\omega) + X'(\omega))Y(\omega) + (X(\omega) - X'(\omega))Y'(\omega).$$

By inspection of all the possible values of $X(\omega), X'(\omega), Y(\omega), Y'(\omega)$, we check that $|Z(\omega)| \leq 2$. Thus in the Kolmogorv formalism, the following Bell's Inequality obtains trivially

$$|E(Z)| = |E(XY) + E(X'Y) + E(XY') - E(X'Y')| \leq 2\,.$$

9.2 – Quantum mechanics violation of Bell's inequality

Let us consider the quantum mechanical situation where one has:

- Four observables A, A', B and B',
- The "As" commute only with the "Bs" but $[A, A'] \neq 0$ and $[B, B'] \neq 0$.

By the commutation of 'As' with the 'Bs' there are JPD's for (X_A, X_B), $(X_A, X_{B'})$, $(X_{A'}, X_B)$ and $(X_{A'}, X_{B'})$. In view of A. Fine's Theorem, however, [2], no joint distribution can be defined for the whole set of associated r.v.s, $X = X_A$, $X' = X_{A'}$, $Y = X_B$ and $Y' = X_{B'}$ compatible with all four probability distributions for (X_A, X_B) , $(X_A, X_{B'})$, $(X_{A'}, X_B)$ and $(X_{A'}, X_{B'})$. In consequence a combination like

$Z = (X_A + X_{A'})X_B + (X_A - X_{A'})X_{B'}$ *does not define* a random variable. Still, the 4 products $X_A\, X_B$, $X_{A'}\, X_B$, etc... define 4 r.v.s, separately, so that an expression like

$$|E(XY) + E(X'Y) + E(XY') - E(X'Y')|$$

is meaningful and can certainly be measured experimentally, but is not bound to comply with the Bell's inequality. And effectively, in some entangled states the absolute value written above is found to display values greater than 2 (up to $2\sqrt{2}$ indeed!). It is this surprising result of Quantum Mechanics that has been verified experimentally and has given rise to the notions of *failure of the local realism hypothesis* and of *quantum non-separability.*

9.3 – Inequality for a succession of measurements

Intuitively it seems plausible that the Kolmogorv formalism can be applied to the variables obtained when we perform a succession of measurements:

$$X_A \quad ; \quad X_{A\to A'} \quad ; \quad X_{A\to A'\to B} \quad ; \quad X_{A\to A'\to B\to B'}$$

Thus it seems that Bell's inequalities remain valid whithin the context of a succession of measurements. We prove this in the following theorem.

Theorem 5 – *If measures are performed sequentially, then the 4-uplet* $(X_A, X_{A\to A'}, X_{A\to A'\to B}, X_{A\to A'\to B\to B'})$ *admit a J.P.D. The random variable*

$$\begin{aligned} X_Z \;=\; & X_A\, X_{A\to A'\to B} \;+\; X_{A\to A'}\, X_{A\to A'\to B} \\ & + X_A\, X_{A\to A'\to B\to B'} \;-\; X_{A\to A'}\, X_{A\to A'\to B\to B'} \end{aligned}$$

is well defined; its mean value is given by

$$E(X_Z) = Tr(\rho\, K)\,,$$

where

$$K = A(B + \Gamma_B(B')) + \Gamma_A(A')(B - \Gamma_B(B'))\,.$$

And we have

$$|E(X_Z)| \le 2\,.$$

9.4 – When Bell's Inequalities Complies with Quantum Mechanics

As an illustration of the preceding theorem, we here provide a simple, the particular case of the four operators:

$$A = \sigma_z \otimes \mathbb{I}d\,, \qquad A' = \sigma_x \otimes \mathbb{I}d\,,$$
$$B = -\frac{1}{\sqrt{2}}\,\mathbb{I}d \otimes (\sigma_z + \sigma_x)\,, \qquad B' = \frac{1}{\sqrt{2}}\,\mathbb{I}d \otimes (\sigma_z - \sigma_x)\,,$$

with $[A, A'] \neq 0$ and $[B, B'] \neq 0$ and the $A's$ commute with the $B's$. Then, there exists a family of states, $\mathcal{F} = \{\rho_\bullet\}$, which are such that, in them, A, A', B and B' are all mutually insensitive. Elements of $\mathcal{F}$ can be parametrized as follows:

$$\rho_\bullet = \begin{pmatrix} \frac{1}{4}+a & u & v^* & w \\ u^* & \frac{1}{4}-a & h & k^* \\ v & h^* & \frac{1}{4}-a & p \\ w^* & k & p^* & \frac{1}{4}+a \end{pmatrix},$$

where $|a| \leq 1/4$, $\Re e(k+v) = 0$ and $\Re e(p+u) = 0$. The entangled (spin zero singlet) state peculiar to Bell's states reads

$$\rho_s = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and turns out to be an element of $\mathcal{F}$. Even in maximally entangled states, such as ρ_s, and even though not all of the 4 operators do commute two by two, one should accordingly expect the Bell's inequality to be satisfied. And effectively, explicit calculation shows that one has

$$E(X_Z) = Tr(\rho_s\, K) = -2\sqrt{2}\, a = \frac{\sqrt{2}}{2} \leq 2\,.$$

that is Bell's inequality is clearly satisfied in this example (this result is indeed more general in the sense that it specifies more accurately the upper bound of 2, appearing in the previous theorem).

10 – Conclusion

Quantum Non-Separability, an unavoidable consequence of the Bell's inequality violation is undoubtly one of the most striking aspects of Quantum Mechanics, as advocated by E. Schrödinger long before any experimental check could be performed. Since then, several experiments have provided this amazing phenomenon with enough support, even though solely 5% of the entangled produced pairs are effectively measured [1]. Now, *Quantum Non-Separability* as well as the failure of *Local Realism* stand at the level of physical interpretations; the essential formal point being the existence or non existence of J.P.D.*s*. It matters to specify the *environment* of such an important phenomenon as the one under consideration; one can realize thus that this *environment* is essentially contextual depending on the experimental protocol at play: It is it which (in line with the basics of the Copenhagen interpretation of Quantum Mechanics) decides of the existence or non-existence of J.P.D.*s*. In this article we have proven that J.P.D.s can be defined for observables under more general circumstances than prescribed in A. Fine's theorem, provided measures are performed sequentially.

This preliminary analyses need to be extended and this could be done in several different directions. Also, what is the general determination concerning the observables to satisfy the Bell's inequalities within "sequential protocols"? Also, it would be interesting to investigate more closely the so-called "statistical insensivity" introduced here, in particular in its possible relations to relativistic and non-relativistic causal independence. Eventually, an extension to Quantum Field Theories and $C^{\star}$-algebras would allow us to take proper account of a relativistic treatment of the matter.

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