

Stirling number identities and High energy String Scatterings

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Abstract We use Stirling number identities developed recently in number theory to show that ratios among high energy string scattering amplitudes in the fixed angle regime can be extracted from the Kummer function of the second kind. This result not only brings an interesting bridge between string theory and combinatoric number theory but also sheds light on the understanding of algebraic structure of high energy stringy symmetry.

1 Introduction

High energy behaviors of scattering amplitudes are of fundamental importance in quantum mechanics, quantum field theory and string theory. Not only can they be used to simplify the mathematical calculation of the amplitudes but also that one can use the high energy amplitudes to extract many fundamental characteristics of the physical theory. There are two fundamental regimes of high energy scattering amplitudes, namely, the fixed angle regime and the fixed momentum transfer regime. These two regimes represent two different high energy perturbation expansions of the scattering amplitudes, and contain complementary information of the underlying theory. In QCD, for example, the probe of high energy, fixed angle regime reveals the partonic structures of hadrons, quarks and gluons. On the other hand, the Regge behavior of high energy hadronic scattering amplitudes suggested a string model of hadrons with a linear relation between hadron spins and their mass squared. In string theory, the scattering amplitudes in the high energy, fixed angle regime [1, 2, 3], the Gross regime (GR), were recently reinvestigated for massive string states at arbitrary mass levels [4, 5, 6, 7, 8, 9, 10]. The calculations were carried out by using three different methods, the decoupling of zero-norm states (ZNS) [11, 12, 13],

the saddle-point method and the method of Virasoro constraint. All three methods gave the consistent results. In particular, an infinite number of linear relations, or stringy symmetries, among string scattering amplitudes of different string states were obtained. These linear relations can be solved for each fixed mass level $M^2 = 2(N-1)$, and ratios $T^{(N,2m,q)}/T^{(N,0,0)}$, $N \geq 2m + 2q$; $m, q \geq 0$ among the amplitudes at each fixed mass level can be obtained.

An important question one could ask then is the interpretation of these infinite number of ratios in terms of the concept of symmetry. For example, does there exist any algebraic structure (or group structure) of these ratios? Since so far little has been known for the full spacetime symmetry of 26D string theory (except ω_∞ for the case of toy 2D string theory [12]), mathematically the meaning of these infinite number of ratios remains mysterious. One of our recent research is to understand this issue from various directions. To our surprise, it turns out that some of the questions above can be answered by calculating high energy string scatterings in another regime, the fixed momentum transfer regime, or the Regge regime (RR).

There are two main results of this report. First, we discover that the leading order amplitudes at each fixed mass level in the RR can be expressed in terms of the Kummer function of the second kind. Second, the number of high energy scattering amplitudes for each fixed mass level in the RR is much more numerous than that of GR. For those leading order high energy amplitudes $A^{(N,2m,q)}$ in the RR with the same type of $(N, 2m, q)$ as those of GR, we can extract from them the above mentioned ratios $T^{(N,2m,q)}/T^{(N,0,0)}$ in the GR by using Kummer function of the second kind, which naturally shows up in the leading order of high energy string scattering amplitudes in the RR. The calculation brings a link between high energy string scattering amplitudes in the GR and the RR. On the other hand, it seems that both the saddle-point method and the method of decoupling of high energy ZNS adopted in the calculation of GR do not apply to the case of RR, and there is no linear relation anymore as in the case of scatterings in the GR. For more results on high energy string scatterings in the RR, see [14, 15, 16, 17, 18, 19].

We stress that, mathematically, the proof of the identification of the ratios in the GR from the Kummer function calculated in the RR turns out to be highly nontrivial. This is based on a summation algorithm for Stirling number identity derived by Mkauers in 2007 [21]. It is very interesting to see that the identity in Eq.(24) suggested by string theory calculation can be rigorously proved by a totally different mathematical method. Although this kind of coincidence is not unusual in the development of string theory, our results bring an interesting connection between string theory and combinatoric number theory. Moreover, the connection between Kummer function and high energy string scatterings may shed light on a deeper understanding of stringy symmetries.

2 Regge Scatterings

We begin with a brief review of high energy string scatterings in the GR. That is in the kinematic regime

$$s, -t \rightarrow \infty, t/s \approx -\sin^2 \frac{\theta}{2} = \text{fixed (but } \theta \neq 0) \quad (1)$$

where s, t and u are the Mandelstam variables and θ is the CM scattering angle. It was shown [7, 8] that for the 26D open bosonic string the only states that will survive the high-energy limit at mass level $M_2^2 = 2(N - 1)$ are of the form

$$|N, 2m, q\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q |0, k_2\rangle \quad (2)$$

where the polarizations of the 2nd particle with momentum k_2 on the scattering plane were defined to be $e^P = \frac{1}{M_2}(E_2, \mathbf{k}_2, 0) = \frac{k_2}{M_2}$ as the momentum polarization, $e^L = \frac{1}{M_2}(\mathbf{k}_2, E_2, 0)$ the longitudinal polarization and $e^T = (0, 0, 1)$ the transverse polarization. Note that e^P approaches to e^L in the GR, and the scattering plane is defined by the spatial components of e^L and e^T . Polarizations perpendicular to the scattering plane are ignored because they are kinematically suppressed for four point scatterings in the high-energy limit. One can then use the saddle-point method to calculate the high energy scattering amplitudes. For simplicity, we choose k_1, k_3 and k_4 to be tachyons and the final result of the ratios of high energy, fixed angle string scattering amplitude are [7, 8]

$$\frac{T^{(N, 2m, q)}}{T^{(N, 0, 0)}} = \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!! \quad (3)$$

The ratios in Eq.(3) can also be obtained by using the decoupling of two types of ZNS in the spectrum. As an example, for $M_2^2 = 4$ we get [4, 5]

$$T_{TTT} : T_{LLT} : T_{(LT)} : T_{[LT]} = 8 : 1 : -1 : -1. \quad (4)$$

To convince the readers that the infinite ratios in Eq.(3) are the symmetries or, at least, remnant of full spacetime symmetries of 26D string theory, it was shown that a set of 2D discrete ZNS Ω_{J_1, M_1}^+ carry ω_∞ symmetry charges [12]

$$\int \frac{dz}{2\pi i} \Omega_{J_1, M_1}^+(z) \cdot \Omega_{J_2, M_2}^+(0) = (J_2 M_1 - J_1 M_2) \Omega_{(J_1+J_2-1), (M_1+M_2)}^+(0). \quad (5)$$

A natural question arises. Is there any mathematical structure (e.g. group structure) of these infinite number of ratios? Let's consider a simple analogy from partial

physics. The ratios of the nucleon-nucleon scattering processes

$$\begin{aligned}
(a) \quad & p + p \rightarrow d + \pi^+, \\
(b) \quad & p + n \rightarrow d + \pi^0, \\
(c) \quad & n + n \rightarrow d + \pi^-
\end{aligned} \tag{6}$$

can be calculated to be

$$T_a : T_b : T_c = 1 : \frac{1}{\sqrt{2}} : 1 \tag{7}$$

from $SU(2)$ isospin symmetry. Similarly, as we will see in this report, the ratios in Eq.(3) can be extracted from Kummer function. The key is to study high energy string scatterings in the RR.

We now turn to the discussion on high energy string scatterings in the RR. That is in the kinematic regime

$$s \rightarrow \infty, \sqrt{-t} = \text{fixed (but } \sqrt{-t} \neq \infty). \tag{8}$$

The relevant kinematics in the RR are

$$e^P \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^P \cdot k_3 \simeq -\frac{\tilde{t}}{2M_2} = -\frac{t - M_2^2 - M_3^2}{2M_2}; \tag{9}$$

$$e^L \cdot k_1 \simeq -\frac{s}{2M_2}, \quad e^L \cdot k_3 \simeq -\frac{\tilde{t}'}{2M_2} = -\frac{t + M_2^2 - M_3^2}{2M_2}; \tag{10}$$

and

$$e^T \cdot k_1 = 0, \quad e^T \cdot k_3 \simeq -\sqrt{-t}. \tag{11}$$

Note that, unlike the case of GR, e^P does not approach to e^L in the RR. On the other hand, instead of states in Eq.(2) for the GR, one can argue that the most general string states one needs to consider at each fixed mass level $N = \sum_{n,m} nk_n + mq_m$ for the RR are

$$|k_n, q_m\rangle = \prod_{n>0} (\alpha_{-n}^T)^{k_n} \prod_{m>0} (\alpha_{-m}^L)^{q_m} |0\rangle. \tag{12}$$

It seems that both the saddle-point method and the method of decoupling of high energy ZNS adopted in the calculation of GR do not apply to the case of RR. However the calculation is still manageable, and the general formula for the high energy scattering amplitudes in the RR can be written down explicitly. The $s - t$ channel scattering amplitudes of this state with three other tachyonic states can be calculated

to be

$$\begin{aligned}
A^{(k_n, q_m)} &= \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \left[\frac{ie^L \cdot k_1}{-x} + \frac{ie^L \cdot k_3}{1-x} \right]^{q_1} \\
&\cdot \prod_{n=1} \left[\frac{ie^T \cdot k_3 (n-1)!}{(1-x)^n} \right]^{k_n} \prod_{m=2} \left[\frac{ie^L \cdot k_3 (m-1)!}{(1-x)^m} \right]^{q_m} \\
&= \left(\frac{-i\tilde{t}'}{2M_2} \right)^{q_1} \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{s}{-\tilde{t}} \right)^j \int_0^1 dx x^{k_1 \cdot k_2 - j} (1-x)^{k_2 \cdot k_3 + j - \sum_{n,m} (nk_n + mq_m)} \\
&\cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2} \right) \right]^{q_m} \\
&= \left(\frac{-i\tilde{t}'}{2M_2} \right)^{q_1} \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{s}{-\tilde{t}} \right)^j B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 + j - N + 1) \\
&\cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2} \right) \right]^{q_m}. \tag{13}
\end{aligned}$$

The Beta function above can be approximated in the large s , but fixed t limit as follows

$$\begin{aligned}
&B(k_1 \cdot k_2 - j + 1, k_2 \cdot k_3 + j - N + 1) \\
&= B\left(-1 - \frac{s}{2} + N - j, -1 - \frac{t}{2} + j\right) \\
&= \frac{\Gamma(-1 - \frac{s}{2} + N - j)\Gamma(-1 - \frac{t}{2} + j)}{\Gamma(\frac{u}{2} + 2)} \\
&\approx B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \left(-1 - \frac{s}{2}\right)^{N-j} \left(\frac{u}{2} + 2\right)^{-N} \left(-1 - \frac{t}{2}\right)_j \\
&\approx B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \left(-\frac{s}{2}\right)^{-j} \left(-1 - \frac{t}{2}\right)_j \tag{14}
\end{aligned}$$

where

$$(a)_j = a(a+1)(a+2)\dots(a+j-1) \tag{15}$$

is the Pochhammer symbol. The leading order amplitude in the RR can then be written as

$$\begin{aligned}
A^{(k_n, q_m)} &= \left(\frac{-i\tilde{t}'}{2M_2} \right)^{q_1} B\left(-1 - \frac{1}{2}s, -1 - \frac{t}{2}\right) \sum_{j=0}^{q_1} \binom{q_1}{j} \left(\frac{2}{\tilde{t}'} \right)^j \left(-1 - \frac{t}{2}\right)_j \\
&\cdot \prod_{n=1} [i\sqrt{-t}(n-1)!]^{k_n} \prod_{m=2} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2} \right) \right]^{q_m}, \tag{16}
\end{aligned}$$

which is UV power-law behaved as expected. The summation in eq. (16) can be represented by the Kummer function of the second kind U as follows,

$$\sum_{j=0}^p \binom{p}{j} \left(\frac{2}{\tilde{t}'}\right)^j \left(-1 - \frac{t}{2}\right)_j = 2^p (\tilde{t}')^{-p} U\left(-p, \frac{t}{2} + 2 - p, \frac{\tilde{t}'}{2}\right) .. \quad (17)$$

Finally, the amplitudes can be written as

$$A^{(k_n, q_m)} = \left(-\frac{i}{M_2}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}'}{2}\right) B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \cdot \prod_{n=1}^{k_n} [i\sqrt{-t}(n-1)!] \prod_{m=2}^{q_m} \left[i\tilde{t}'(m-1)! \left(-\frac{1}{2M_2}\right)\right]^{q_m}. \quad (18)$$

In the above, U is the Kummer function of the second kind and is defined to be

$$U(a, c, x) = \frac{\pi}{\sin \pi c} \left[\frac{M(a, c, x)}{(a-c)!(c-1)!} - \frac{x^{1-c} M(a+1-c, 2-c, x)}{(a-1)!(1-c)!} \right] \quad (c \neq 2, 3, 4...) \quad (19)$$

where $M(a, c, x) = \sum_{j=0}^{\infty} \frac{(a)_j x^j}{(c)_j j!}$ is the Kummer function of the first kind. U and M are the two solutions of the Kummer Equation

$$xy''(x) + (c-x)y'(x) - ay(x) = 0. \quad (20)$$

It is crucial to note that $c = \frac{t}{2} + 2 - q_1$, and is not a constant as in the usual case, so U in Eq.(18) is not a solution of the Kummer equation. This will make our analysis in the next section more complicated as we will see soon. On the contrary, since $a = -q_1$ is an integer, the Kummer function in Eq.(17) terminated to be a finite sum. This will simplify the manipulation of Kummer function used in this report.

3 Applying Stirling number identities

As an important application of Eq.(18), the leading order amplitudes of string states in the RR, which share the same structure as Eq.(2) in the GR can be written as

$$A^{(N, 2m, q)} = B\left(-1 - \frac{s}{2}, -1 - \frac{t}{2}\right) \sqrt{-t}^{N-2m-2q} \left(\frac{1}{2M_2}\right)^{2m+q} 2^{2m} (\tilde{t}')^q U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'}{2}\right). \quad (21)$$

It is important to note that there is no linear relation among high energy string scattering amplitudes of different string states for each fixed mass level in the RR as

can be seen from Eq.(21). This is very different from the result in the GR in Eq.(3). In other words, the ratios $A^{(N,2m,q)}/A^{(N,0,0)}$ are t -dependent functions. In particular, we can extract the coefficients of the highest power of t in $A^{(N,2m,q)}/A^{(N,0,0)}$. We can use the identity of the Kummer function

$$\begin{aligned}
& 2^{2m} (\tilde{t}')^{-2m} U \left(-2m, \frac{t}{2} + 2 - 2m, \frac{\tilde{t}'}{2} \right) \\
&= {}_2F_0 \left(-2m, -1 - \frac{t}{2}, -\frac{2}{\tilde{t}'} \right) \\
&\equiv \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2} \right)_j \frac{\left(-\frac{2}{\tilde{t}'} \right)^j}{j!} \\
&= \sum_{j=0}^{2m} \binom{2m}{j} \left(-1 - \frac{t}{2} \right)_j \left(\frac{2}{\tilde{t}'} \right)^j
\end{aligned} \tag{22}$$

to calculate

$$\frac{A^{(N,2m,q)}}{A^{(N,0,0)}} = (-1)^q \left(\frac{1}{2M_2} \right)^{2m+q} (-t)^m \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2} \right)_j \frac{(-2/t)^j}{j!} + O \left\{ \left(\frac{1}{t} \right)^{m+1} \right\} \tag{23}$$

where we have replaced \tilde{t}' by t as t is large. If the leading order coefficients in Eq.(23) extracted from the high energy string scattering amplitudes in the RR are to be identified with the ratios calculated previously among high energy string scattering amplitudes in the GR in Eq.(3), we need the following identity

$$\begin{aligned}
& \sum_{j=0}^{2m} (-2m)_j \left(-1 - \frac{t}{2} \right)_j \frac{(-2/t)^j}{j!} \\
&= 0(-t)^0 + 0(-t)^{-1} + \dots + 0(-t)^{-m+1} + \frac{(2m)!}{m!} (-t)^{-m} + O \left\{ \left(\frac{1}{t} \right)^{m+1} \right\}. \tag{24}
\end{aligned}$$

The coefficient of the term $O \left\{ (1/t)^{m+1} \right\}$ in Eq.(24) is irrelevant for our discussion. The proof of Eq.(24) turns out to be nontrivial. The standard approach by using integral representation of the Kummer function seems not applicable here. Presumably, the difficulty of the rigorous proof of Eq.(24) is associated with the unusual non-constant c in the argument of Kummer function in Eqs.(21) and (22) as mentioned above. It is a nontrivial task to do the proof compared to the usual cases where the argument c of the Kummer function is a constant. Here we will adopt another approach to prove Eq.(24). This approach strongly relies on the algorithm for Stirling number identity derived by Mkauers [21] in 2007, and is highly nontrivial

either. The leading order identity of Eq.(24) can be written as

$$f(m) \equiv \sum_{j=0}^m (-1)^j \binom{2m}{j+m} [s(j+m-1, j-1) + s(j+m-1, j)] = (2m-1)!! \quad (25)$$

where the signed first Stirling number $s(n, k)$ is defined to be

$$(x)_n = \sum_{k=0}^n (-1)^{n-k} s(n, k) x^k. \quad (26)$$

The authors had verified the validity of Eq.(25) for $m = 1, 2, \dots, 2000$ before they carried out the exact proof to be discussed below. To prove Eq.(25) we define

$$f(u, m) \equiv \sum_{j=0}^{m+u} (-1)^j \binom{2m+u}{j+m} [s(j+m-1, j-1) + s(j+m-1, j)] \quad (27)$$

with $f(0, m) = f(m)$. By using the result of [21], one can prove that $f(u, m)$ satisfies the following recurrence relation

$$-(1+2m+u)f(u, m) + (2m+u)f(u+1, m) + f(u, m+1) = 0. \quad (28)$$

Eq.(28) is the most nontrivial step to prove Eq.(25). Finally by taking $u = 0$, it can be shown that the second term of Eq.(28) vanishes [20]. Eq.(25) is then proved by mathematical induction. The vanishing of the coefficients of $(-t)^0, (-t)^{-1}, \dots, (-t)^{-m+1}$ terms on the LHS of Eq.(24) means, for $1 \leq i \leq m$,

$$g(m, i) \equiv \sum_{j=0}^{m+i} (-1)^{j-i} \binom{2m}{j+m-i} [s(j+m-1-i, j) + s(j+m-1-i, j-1)] = 0. \quad (29)$$

To prove this identity, we need the recurrence relation [21]

$$\begin{aligned} & -2(1+m)^2(1+2m)g(m, i) + (2+7m+4m^2)g(m+1, i) \\ & -2m(1+m)(1+2m)g(m+1, i+1) - m \times g(m+2, i) = 0. \end{aligned} \quad (30)$$

Putting $i = 0, 1, 2, \dots$, and using the fact we have just proved, i.e. $g(m+1, 0) = (2m+1)g(m, 0)$, one can prove Eq.(29). Eq.(24) is thus finally proved. It is very interesting to see that the identity in Eq.(24) suggested by string scattering amplitude calculation can be rigorously proved by a totally different but sophisticated mathematical method. In conclusion, ratios in Eq.(3) can be extracted from Kummer function of the second kind

$$\frac{T^{(N, 2m, q)}}{T^{(N, 0, 0)}} = \left(-\frac{1}{2M}\right)^{2m+q} 2^{2m} \lim_{t \rightarrow \infty} (-t)^{-m} U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{t}{2}\right). \quad (31)$$

In view of Eq.(7), this result may help to uncover the fundamental symmetry of string theory. At last, we give an explicit calculation of the high energy string scattering amplitudes to subleading orders in the RR for $M_2^2 = 4$ [20]

$$A_{TTT} \sim \frac{1}{8}\sqrt{-t}s^3 + \frac{3}{16}\sqrt{-t}(t+6)s^2 + \frac{3t^3 + 84t^2 - 68t - 864}{64}\sqrt{-t}s + O(1), \quad (32)$$

$$A_{LLT} \sim \frac{1}{64}\sqrt{-t}(t-6)s^3 + \frac{3}{128}\sqrt{-t}(t^2 - 20t - 12)s^2 + \frac{3t^3 - 342t^2 - 92t + 5016 + 1728(-t)^{-1/2}}{512}\sqrt{-t}s + O(1), \quad (33)$$

$$A_{(LT)} \sim -\frac{1}{64}\sqrt{-t}(t+10)s^3 - \frac{1}{128}\sqrt{-t}(3t^2 + 52t + 60)s^2 - \frac{3[t^3 + 30t^2 + 76t - 1080 - 960(-t)^{-1/2}]}{512}\sqrt{-t}s + O(1), \quad (34)$$

$$A_{[LT]} \sim -\frac{1}{64}\sqrt{-t}(t+2)s^3 - \frac{3}{128}\sqrt{-t}(t+2)^2s^2 - \frac{(3t-8)(t+6)^2[1 - 2(-t)^{-1/2}]}{512}\sqrt{-t}s + O(1). \quad (35)$$

We have ignored an overall irrelevant factors in the above amplitudes. Note that the calculation of Eq.(34) and Eq.(35) involves amplitude of the state $(\alpha_{-2}^T)(\alpha_{-1}^L)|0, k_2\rangle$ which can be shown to be of leading order in the RR [20], but is of subleading order in the GR as it is not in the form of Eq.(2). However, the contribution of the amplitude calculated from this state will not affect the ratios $8 : 1 : -1 : -1$ in the RR [20]. One can now easily see that the ratios of the coefficients of the highest power of t in these leading order coefficient functions $\frac{1}{8} : \frac{1}{64} : -\frac{1}{64} : -\frac{1}{64}$ agree with the ratios in the GR calculated in Eq.(4) as expected. Moreover, one further observation is that these ratios remain the same for the coefficients of the highest power of t in the subleading orders (s^2) $\frac{3}{16} : \frac{3}{128} : -\frac{3}{128} : -\frac{3}{128}$ and (s) $\frac{3}{64} : \frac{3}{512} : -\frac{3}{512} : -\frac{3}{512}$. More examples will be given in [20]. We thus conjecture that the existence of these GR ratios of Eq.(3) in the RR persists to the subleading orders in the Regge expansion of high energy string scattering amplitudes.

4 Conclusion

In conclusion, physically, the connection between Kummer function and high energy string scattering amplitudes derived in this report may shed light on a deeper understanding of stringy symmetries. Mathematically, the proof of identity in Eq.(24) brings an interesting bridge between string theory and combinatoric number theory.

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