Many-body Lattice QCD

William Detmold
Department of Physics
College of William and Mary
Williamsburg, VA 23188, USA
and
Thomas Jefferson National Accelerator Facility
12000 Jefferson Ave, Newport News, VA 23606, USA

1 Introduction

Lattice QCD has made significant impacts in many aspects of particle physics phenomenology and in describing the spectra and structure of single hadrons. Computing resources and lattice algorithms have reached a stage where it is now possible to consider more complicated hadronic observables such as those in the baryon number, $B > 1$ sector — the domain of nuclear physics. Here there are many observables that are phenomenologically important to nuclear structure and interactions and to nuclear astrophysics about which very little (or nothing) is known experimentally or theoretically. Systems containing $n > 2$ mesons are also of interest in a number of areas from RHIC to neutron stars. These systems present a significant opportunity for contributions from lattice QCD. Recently, the first attempts to study systems of more than two hadrons have been made by the NPLQCD collaboration [1, 2, 3, 4]. The results of these studies, for both mesons and baryons, are summarised herein.

2 Multi-meson systems

It has been known for many years how to exploit the volume dependence of the eigenenergies of two hadron systems to extract infinite volume scattering phase shifts [5] provided that the effective range of the interaction, r is small compared to the box size L (since $r \sim m_\pi^{-1}$ for most interactions, this constraint is $m_\pi\, L \gg 1$). In recent works, this has been extended to systems involving $n > 2$ bosons [6, 7, 8] and $n = 3$ fermions [9] in the case when the relevant scattering length, a, is also small compared to the box size. The resulting shift in energy of n particles of mass M due to their interactions is

$$\Delta E_n \;=\; \frac{4\pi\,\overline{a}}{M\,L^3}{}^nC_2\Big\{1-\left(\frac{\overline{a}}{\pi\,L}\right)\mathcal{I}+\left(\frac{\overline{a}}{\pi\,L}\right)^2\Big[\mathcal{I}^2+(2n-5)\mathcal{J}\Big]$$

$$
\begin{aligned}
& - \left(\frac{\overline{a}}{\pi\, L}\right)^3 \left[\mathcal{I}^3 + (2n-7)\mathcal{I}\mathcal{J} + \left(5n^2 - 41n + 63\right)\mathcal{K} \right] \\
& + \left(\frac{\overline{a}}{\pi\, L}\right)^4 \left[\mathcal{I}^4 - 6\mathcal{I}^2\mathcal{J} + (4+n-n^2)\mathcal{J}^2 + 4(27-15n+n^2)\mathcal{I}\,\mathcal{K} \right. \\
& \qquad \left. \left. + (14n^3 - 227n^2 + 919n - 1043)\mathcal{L} \right] \right\} \\
& + {}^nC_3 \left[\frac{192\, \overline{a}^5}{M\pi^3 L^7} \left(\mathcal{T}_0 \ + \ \mathcal{T}_1\, n \right) \ + \ \frac{6\pi\overline{a}^3}{M^3 L^7}\, (n+3)\, \mathcal{I} \right] \\
& + {}^nC_3\, \frac{1}{L^6}\, \overline{\overline{\eta}}_3^L \quad + \ \mathcal{O}\left(L^{-8}\right) \qquad ,
\end{aligned}
\tag{1}
$$

where the parameter $\overline{a}$ is related to the scattering length, a, and the effective range, r, by

$$
a \ = \ \overline{a} \ - \ \frac{2\pi}{L^3}\overline{a}^3 r \left(1 \ - \ \left(\frac{\overline{a}}{\pi L}\right)\ \mathcal{I} \right) \quad . \tag{2}
$$

The geometric constants that enter into eq. (1) are defined in Ref.[8] and ${}^nC_m = n!/m!/(n-m)!$. The three-body contribution to the energy-shift given in eq. (1) is represented by the parameter $\overline{\overline{\eta}}_3^L$ (see Ref. [8]).

Lattice QCD measurements of these energy shifts allow one to extract the parameters $\overline{a}$ and $\overline{\overline{\eta}}_3^L$. To determine the energy shifts, we study the correlators (specifying to the multi-pion system)

$$
C_n(t) \ \ \propto \ \ \left\langle \left(\sum_{\mathbf{x}} \pi^-(\mathbf{x},t) \right)^n \left(\pi^+(\mathbf{0},0) \right)^n \right\rangle \quad , \tag{3}
$$

where $\pi^+(\mathbf{x},t) = \overline{u}(\mathbf{x},t)\gamma_5 d(\mathbf{x},t)$. On a lattice of infinite temporal extent,[1] the combination

$$
G_n(t) \ \equiv \ \frac{C_n(t)}{[\ C_1(t)\ ]^n} \ \overset{t\to\infty}{\longrightarrow} \ \mathcal{B}_0^{(n)}\ e^{-\Delta E_n\ t} \quad , \tag{4}
$$

where ΔE_n is the energy shift appearing in Eq. (1).

To compute the $(n!)^2$ Wick contractions in Eq. (3), we note that this correlation function can be written as

$$
C_n(t) \ \ \propto \ \ \langle\ (\ \overline{\eta}\Pi\eta\ )^n\ \rangle \ \ , \tag{5}
$$

where

$$
\Pi \ \ = \ \ \sum_{\mathbf{x}}\ S(\mathbf{x},t;0,0)\ S^\dagger(\mathbf{x},t;0,0) \quad , \tag{6}
$$

[1] Effects of temporal (anti-)periodicity are discussed in Ref. [3].

and $S(\mathbf{x}, t; 0, 0)$ is a light-quark propagator. The object (block) Π is a 12×12 (4-spin and 3 color) bosonic time-dependent matrix, and η_α is a twelve component Grassmann variable. The required n-meson contractions can be expressed as products of traces of the this matrix. As an example, the contractions for the 3-π^+ system are

$$C_3(t) \propto \mathrm{tr}_{\mathrm{C,S}}[\Pi]^3 - 3\,\mathrm{tr}_{\mathrm{C,S}}\left[\Pi^2\right]\mathrm{tr}_{\mathrm{C,S}}[\Pi] + 2\,\mathrm{tr}_{\mathrm{C,S}}\left[\Pi^3\right] \quad , \tag{7}$$

where the traces, $\mathrm{tr}_{\mathrm{C,S}}$, are over color and spin indices. Contractions for $n \leq 12$ mesons are given explicitly in Ref. [2].

3 Two- and three- body interactions

The NPLQCD collaboration have computed the n pion and kaon correlators in the previous section using domain wall fermion [10, 11] propagators on various ensembles of MILC 2+1 flavour rooted staggered gauge configurations [12] (quark masses, lattice spacings and volumes and further details are given in Refs. [1, 2, 3]). In order to correctly calculate these correlators for large n, very high numerical precision is necessary (our calculations use the `arprec` library [13]). By performing a correlated fit to the effective energy differences extracted from these measurements, we have determined the two- and three-body interactions. The two body interactions extracted from this analysis agree with those extracted from the two-body sector alone [14]. The resulting three body interactions are displayed in Fig. 1. The three pion interaction is found to be repulsive with a magnitude consistent with the expectation from naive dimensional analysis. In contrast, the three K^+ interaction is consistent with zero within somewhat larger uncertainties.

4 Pion and kaon condensation

The ground state of the n meson systems that are being studied is a Bose-Einstein condensate of fixed z component of isospin (and strangeness in the case of kaons). It is of great interest to investigate the properties of such systems. Theoretical efforts have used leading order chiral perturbation theory to investigate the phase diagram at low chemical potential [15] and it is important to assess the extent to which these results agree with QCD. Our numerical calculations allow us to probe the dependence of the energy on the pion (kaon) density, and thereby extract the chemical potential via a finite difference. The results using the coarse MILC lattice are shown for kaon systems in Fig. 2. Also shown is the prediction from tree-level chiral perturbation theory, with which we find surprisingly good agreement. This is encouraging for studies of kaon condensation in neutron stars where, typically, tree level chiral perturbation theory interactions are assumed amongst kaons and between kaons and baryons [16].

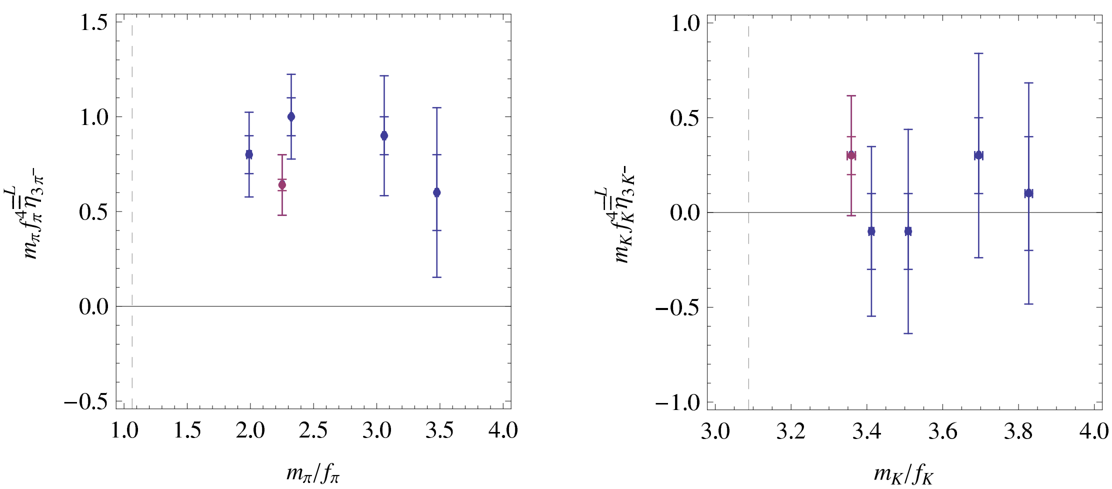

Figure 1: Three pion (left) and kaon (right) interactions determined from the MILC coarse (blue) and fine (magenta) lattices plotted as a function of the dimensionless ratio m_π/f_π.

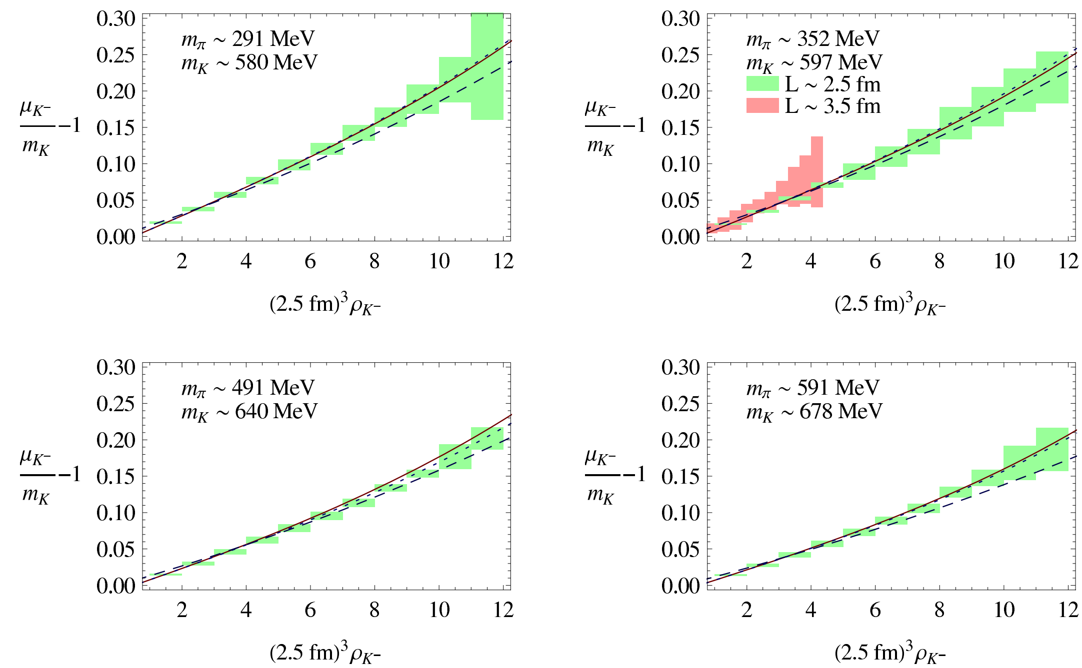

Figure 2: Dependence of the strangeness chemical potential on the kaon density, calculated on the coarse MILC ensembles. The curves correspond to the predictions of tree level chiral perturbation theory (dashed) [15], the energy shift of Eq. (2.1) (solid) and with the three-body interaction removed (dotted).

5 Multi-baryon systems

In Refs. [4], the first calculation of three baryon systems were presented based on extremely high statistics measurements on anisotropic lattice configurations. These calculations focused primarily on a strangeness 4, $I_z = 1/2$ system "$\Xi 60\Xi^0 n$" with preliminary results for the triton, pnn. The particular details of the energies and energy shifts that were extracted in these calculations are given in Ref. [4]. An important result of this study was the realisation that the exponential growth of statistical noise expected from simple arguments can be suppressed. The effective energy plot for the $\Xi^0\Xi^0 n$ state is shown in Fig. 3. The salient point is that for a large range of time-slices, the uncertainties it the measured correlation function remain essentially constant. In this "golden window", fits can be performed to the correlator and the energies can be cleanly extracted. This region appears to shrink as baryon number, B, increases (heuristic arguments indicate this happens logarithmically with B), but it seems likely that four- and five- baryon systems are resolvable a the current level of statistics.

6 Summary

Multi-meson systems (and in general multi-hadron systems) have been investigated using lattice QCD. The calculations presented here provide a first insight into the nature of these systems, but much remains to be studied. Recently, we have started to explore the effects of these condensed systems on other observables, looking at how the pion condensate screens the potential between a static quark–anti-quark pair [17]. Multi-baryon systems are numerically more challenging, but significant progress has been made in this regard, with the first computations of the $\Xi^0\Xi^0 n$ and triton systems.

I am grateful to the organisers for a stimulating and very enjoyable workshop. I gratefully acknowledge my collaborators in NPLQCD and R. Edwards and B. Joo for help with the `chroma` software environment [18].

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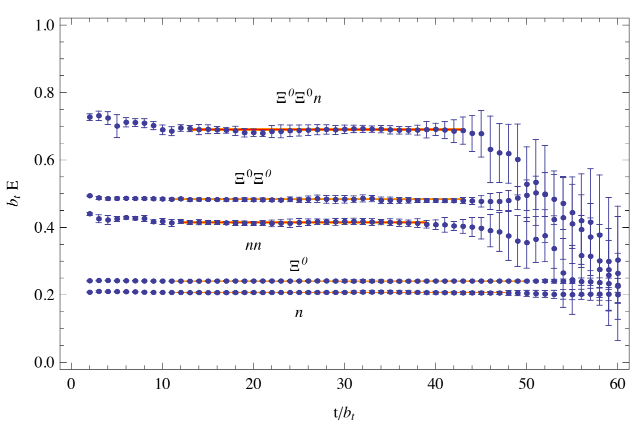

Figure 3: The effective energy plot for the one-, two- and three- body systems involving the Ξ^0 and n.

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Discussion

S. Brodsky (SLAC): What are the possibilities for lattice QCD to investigate different components of the deuteron wave-function?

W. Detmold: Once we see a bound two-nucleon system, we can address this question by looking at overlaps onto different types of interpolating operators or using external currents.