Lattice QCD and dense quark matter

Kim Splittorff

with: Maria Paola Lombardo

Jac Verbaarschot

KIAA, Beijing, 23 May 2009

- **Introduction:** What do we want
  What are we up against (*the sign problem*)

- **New:** Ensembles with θ fixed $re^{i\theta} = \det(D + \mu\gamma_0 + m)$

Matter antimatter asymmetry ($\mu \neq 0$)

$$n_q > 0$$

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Here: Fact which we adopt into QCD

Grand canonical approach: *Fix* μ *determine* n_q

$$n_q = \frac{1}{V} \partial_\mu \log Z(\mu)$$

How to include μ in Z

μ is conjugate variable to n_q

$$\mu n_q = \mu \langle q^\dagger q \rangle = \mu \langle \bar{q} \gamma_0 q \rangle$$

$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(D_\eta \gamma_\eta + \mu \gamma_0 + m) q + \mathrm{Gluons}$$

Hasenfratz, Karsch, PLB 125 (1983) 308

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μ enters as the 0th component of the gauge field

$$\mathcal{L}_{\mathrm{Lattice\ QCD}} = \ldots + e^{a\mu}\bar{q}_x\gamma_0 U_{x,x+\hat{0}} q_{x+\hat{0}} + e^{-a\mu}\bar{q}_{x+\hat{0}}\gamma_0 U^\dagger_{x,x+\hat{0}} q_x + \ldots$$

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Works fine for free quarks

Hasenfratz, Karsch, PLB 125 (1983) 308

The sign problem

$$Z_{1+1} = \int dA \, \det{}^2(D + \mu\gamma_0 + m) \, e^{-S_{\rm YM}}$$

Anti Hermitian

Hermitian

$$\det{}^2(D + \mu\gamma_0 + m) = |\det(D + \mu\gamma_0 + m)|^2 e^{2i\theta}$$

The measure is not real and positive

The sign problem

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Anti Hermitian Hermitian

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The measure is not real and positive

No Monte Carlo sampling of A_η at $\mu \neq 0$

Lattice methods for $\mu \neq 0$

Reweighting - absorb the sign in the observable

Taylor expansion - expand from $\mu = 0$

Imaginary μ - expand from imaginary values of μ

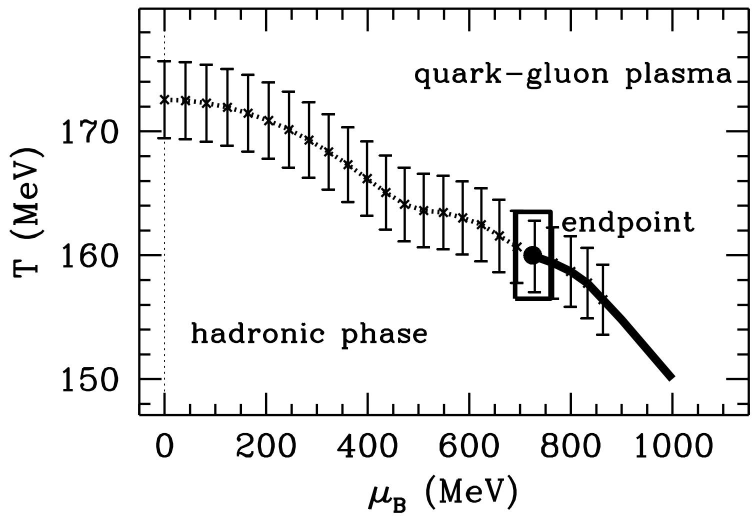
Density of states - determine the distribution of the phase

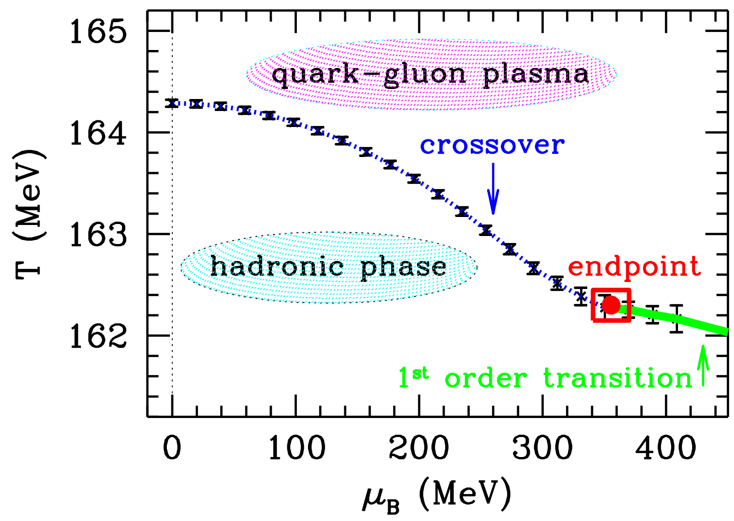
Canonical ensemble - fix n_B

Complex Langevin - stochastic quantization

Results obtained from the lattice (I)

Reweighting





Fodor Katz JHEP 0203 (2002) 014, JHEP 0404 (2004) 050

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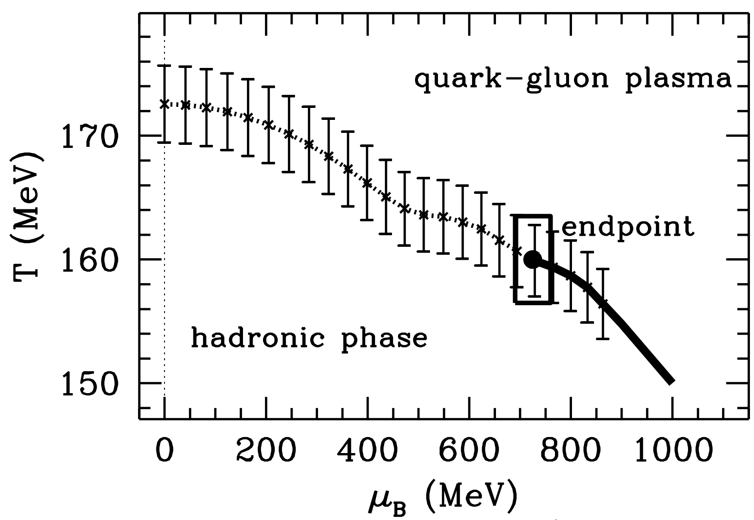
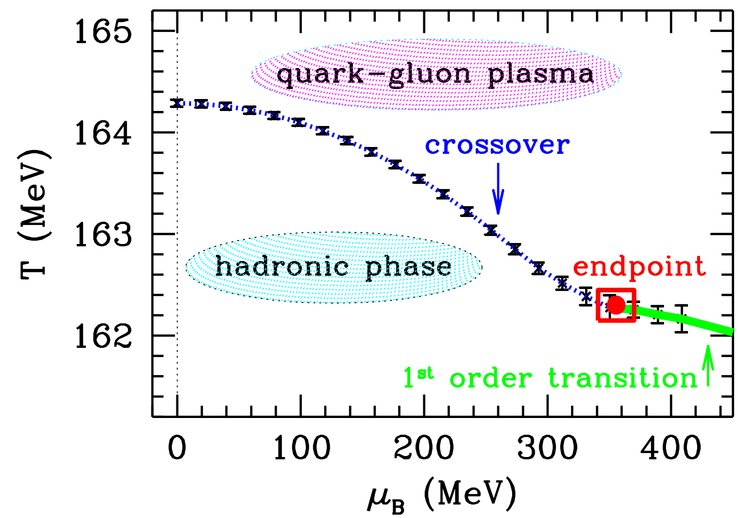




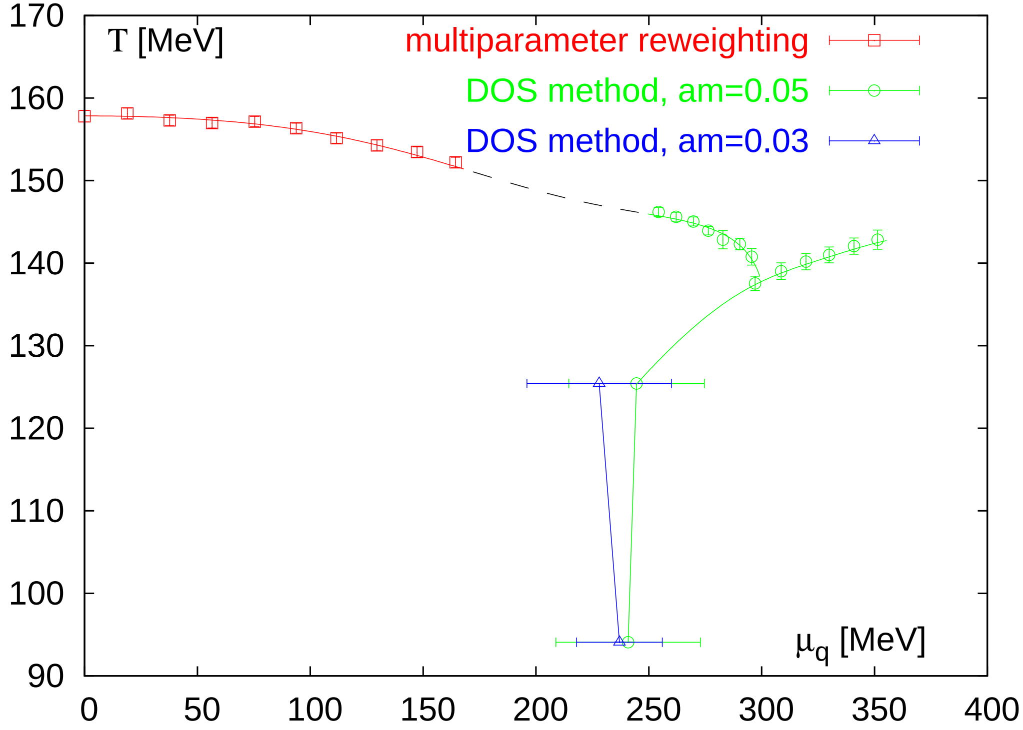
Light quarks

Heavy quarks

Fodor Katz JHEP 0203 (2002) 014, JHEP 0404 (2004) 050

Results obtained from the lattice (II)

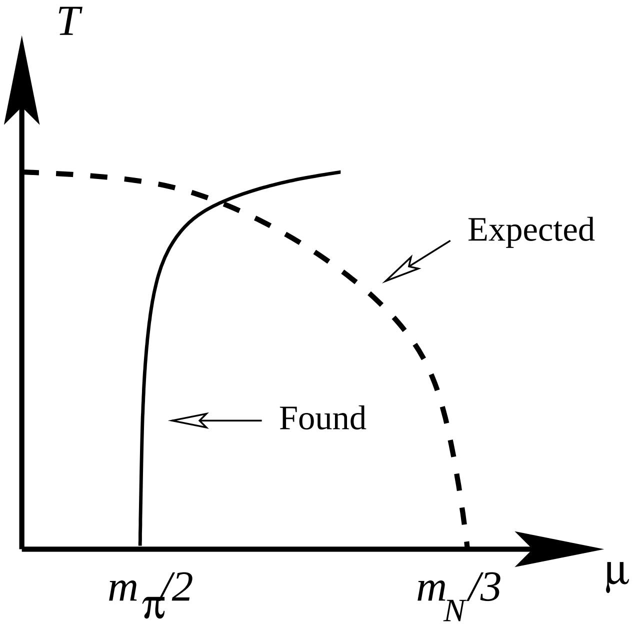
Density of states



Fodor Katz Schmidt JHEP0703:121,2007

What are we up against ?

What happens if we drop the phase ?



Alford Kapustin Wilczek PRD 59 (1999) 054502

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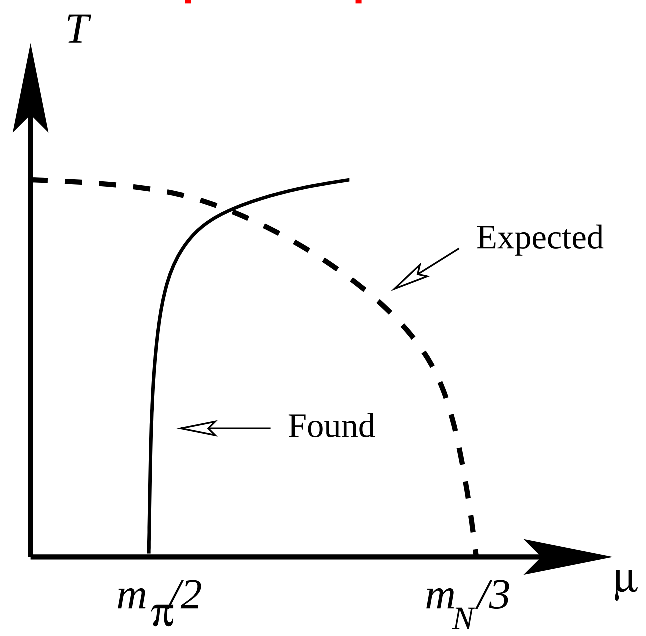

$$|\det(D + \mu\gamma_0 + m)|^2 = \det(D + \mu\gamma_0 + m)\det(D - \mu\gamma_0 + m)$$

μ becomes an isospin chemical potential

Alford Kapustin Wilczek PRD 59 (1999) 054502

How difficult is it to include the phase factor ?

Depends on how much the phase fluctuates

$$\langle e^{2i\theta} \rangle \sim \begin{cases} 1 & \text{mild sign problem} \\ 0 & \text{tough sign problem} \end{cases}$$

$$\langle e^{2i\theta} \rangle_{1+1^*} = \frac{Z_{1+1}(\mu_B = \mu)}{Z_{1+1^*}(\mu_I = \mu)} = e^{-V\Delta\Omega}$$

Average phase factor in Chiral Perturbation Theory

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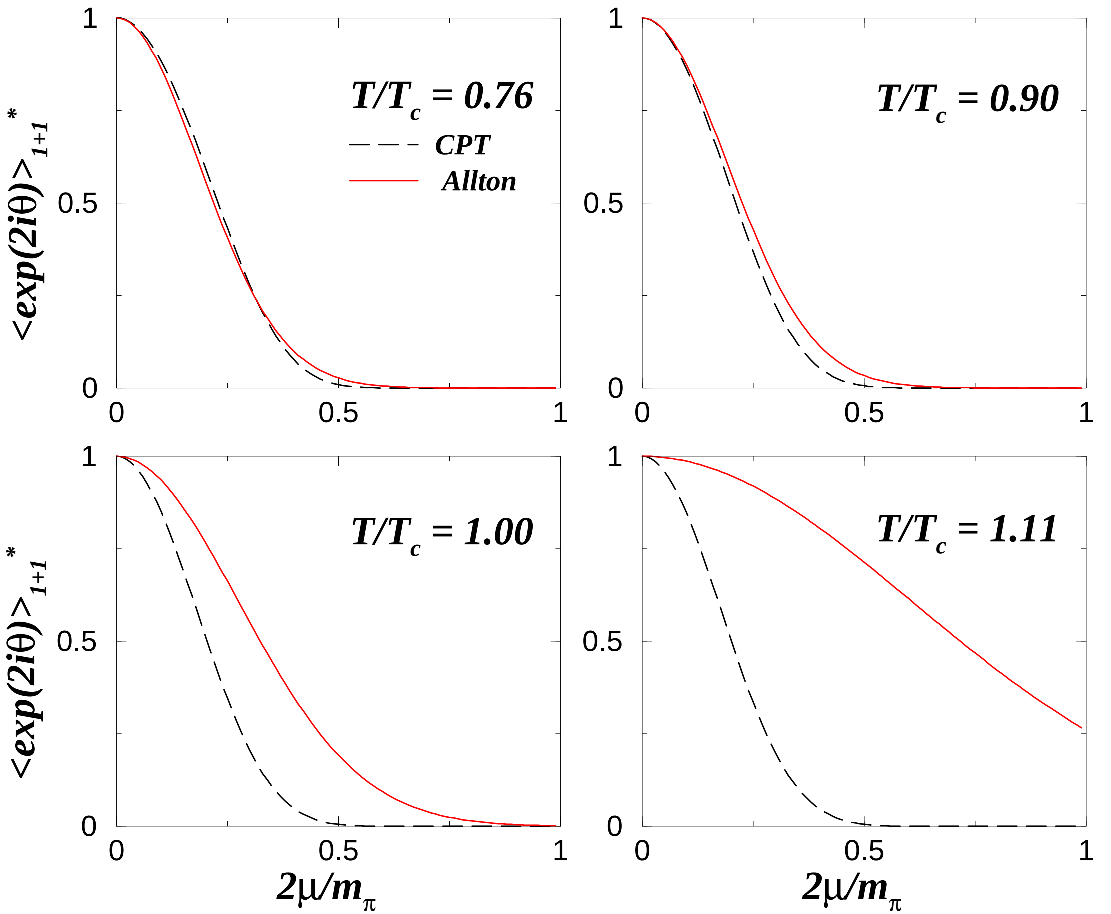
ΔG_0 *is independent of the cutoff*

Splittorff Svetitsky PRD 75 (2007) 114504

Conradi D'Elia PRD 76 (2007) 074501

Average phase factor on the lattice



Allton+... Phys.Rev. D71 (2005) 054508

Splittorff Verbaarschot PRD 77 (2008) 014514

D'Elia Sanfi lippo arXiv:0904.1400

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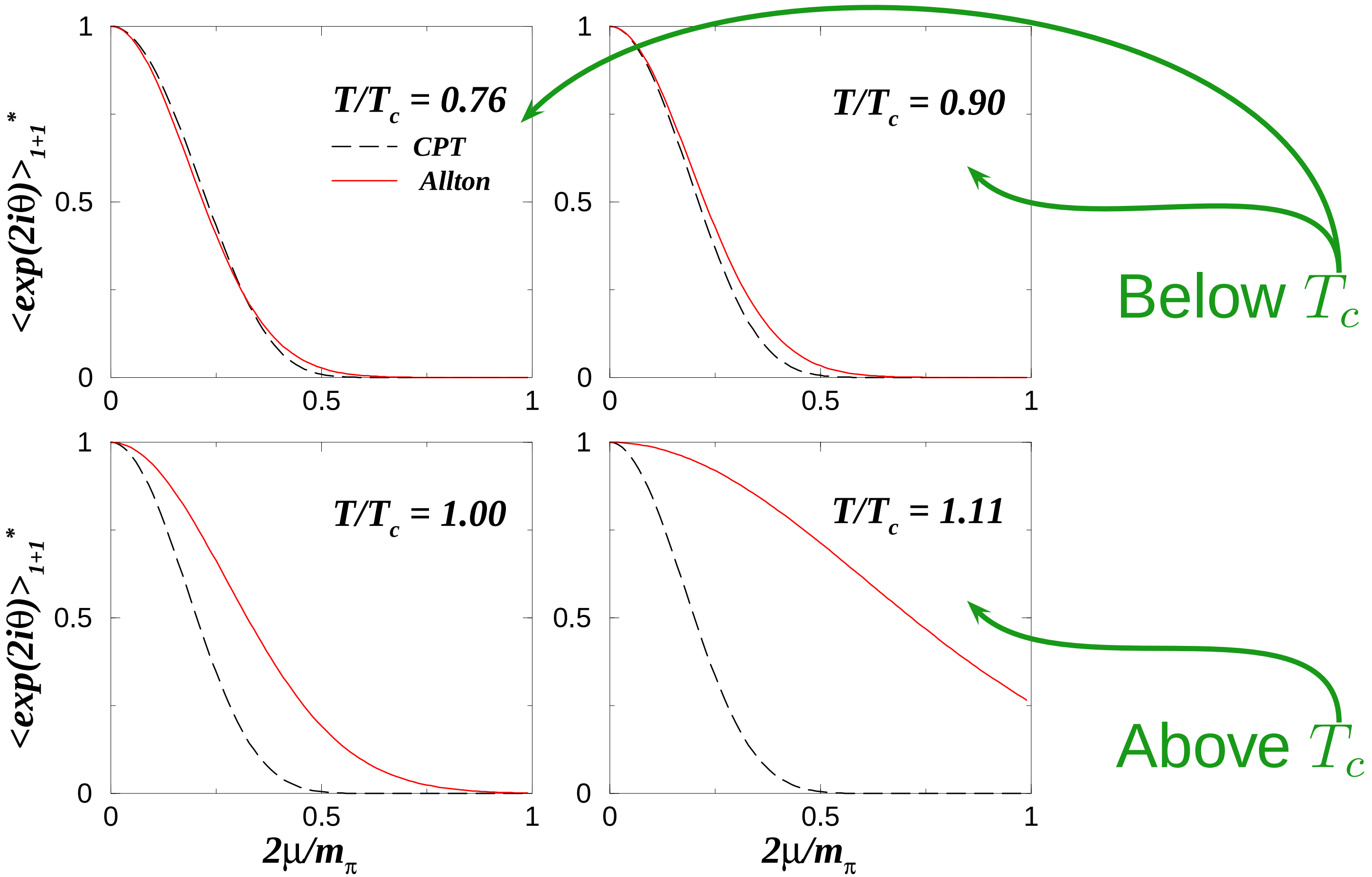

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Assumption: No Bose condensation of pions !

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$$\mu < m_\pi/2 \qquad \text{for } T = 0$$

Average phase factor in Chiral Perturbation Theory

Bose condensed phase ($\mu > m_\pi/2$)

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The free energies are different at mean fi eld level

$$\Delta\Omega = 2\mu^2 F^2 + \frac{\Sigma^2 m^2}{2\mu^2 F^2} - 4m\Sigma$$

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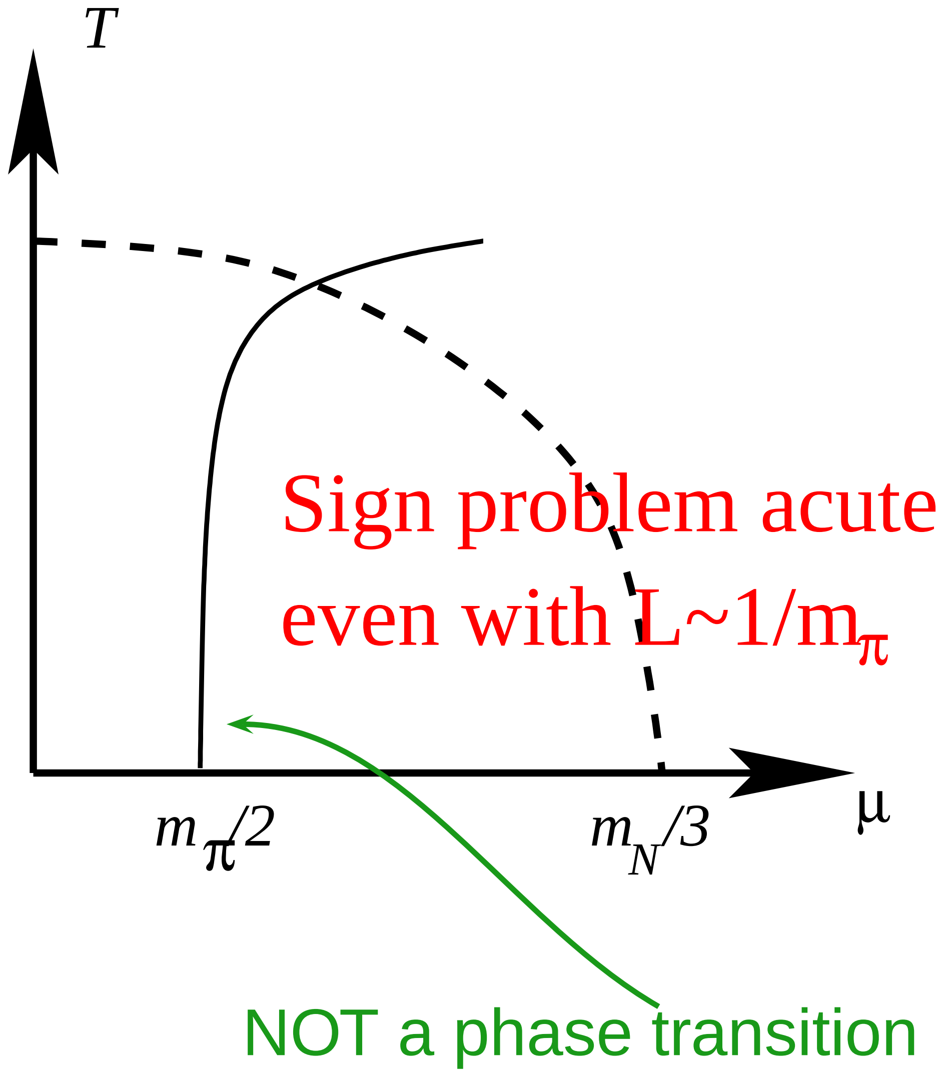
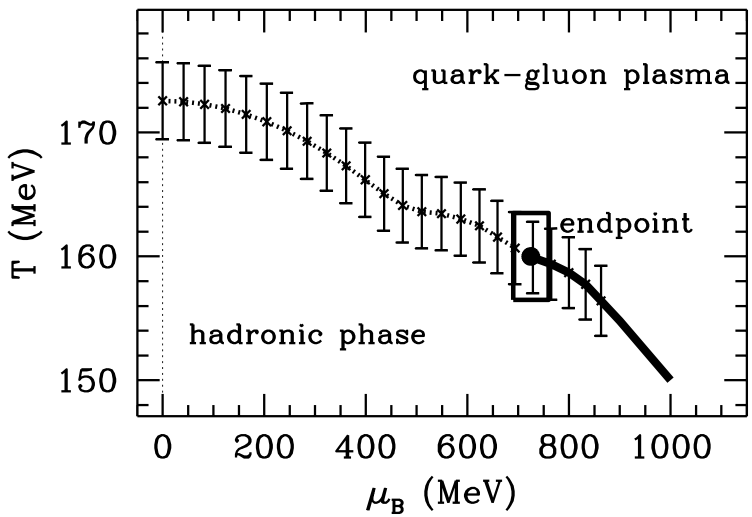
- Severe sign problem even in a small volume

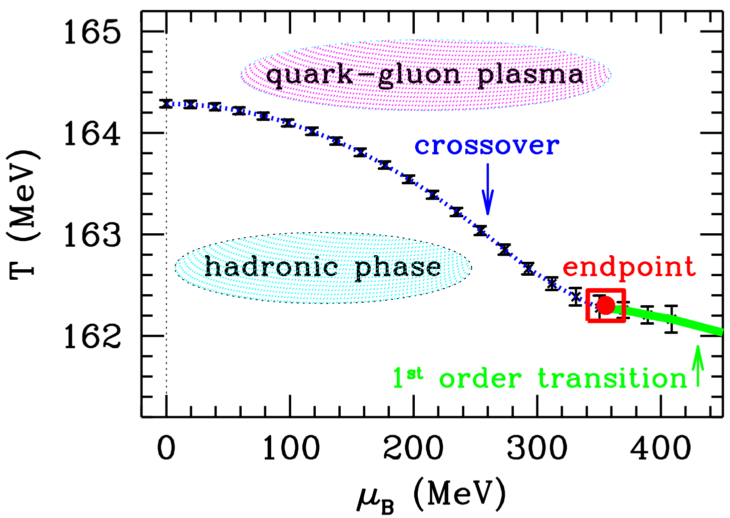
Conclude: $m_\pi/2$ relevant for the sign problem



Lattice measurement of endpoint

Reweighting



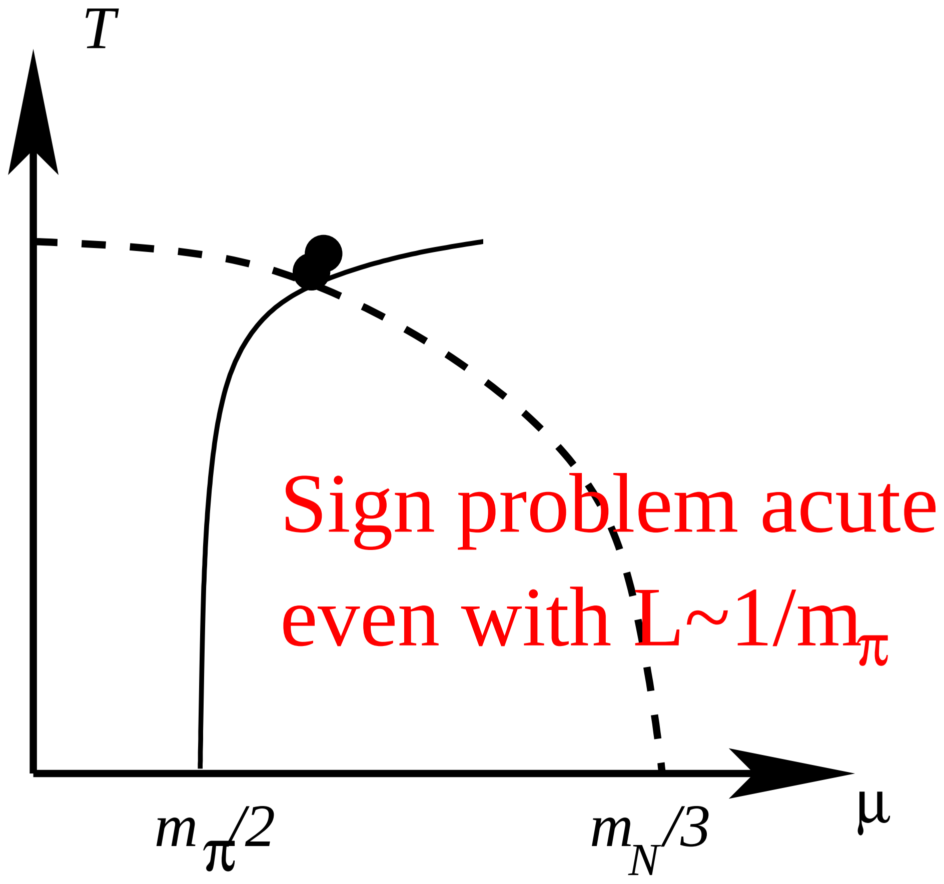


Fodor Katz JHEP 0203:014,2002; JHEP 0404:050,2004
Splittorff, hep-lat/0505001, PoS LAT2006:023,2006

Philipsen 0710.1217

Lattice measurement of endpoint

Same endpoints with rescaled axis

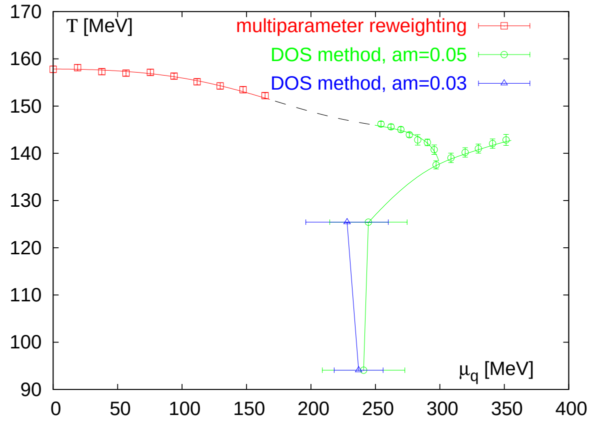


Fodor Katz JHEP 0203:014,2002; JHEP 0404:050,2004
Splittorff, hep-lat/0505001, PoS LAT2006:023,2006

The distribution of the phase

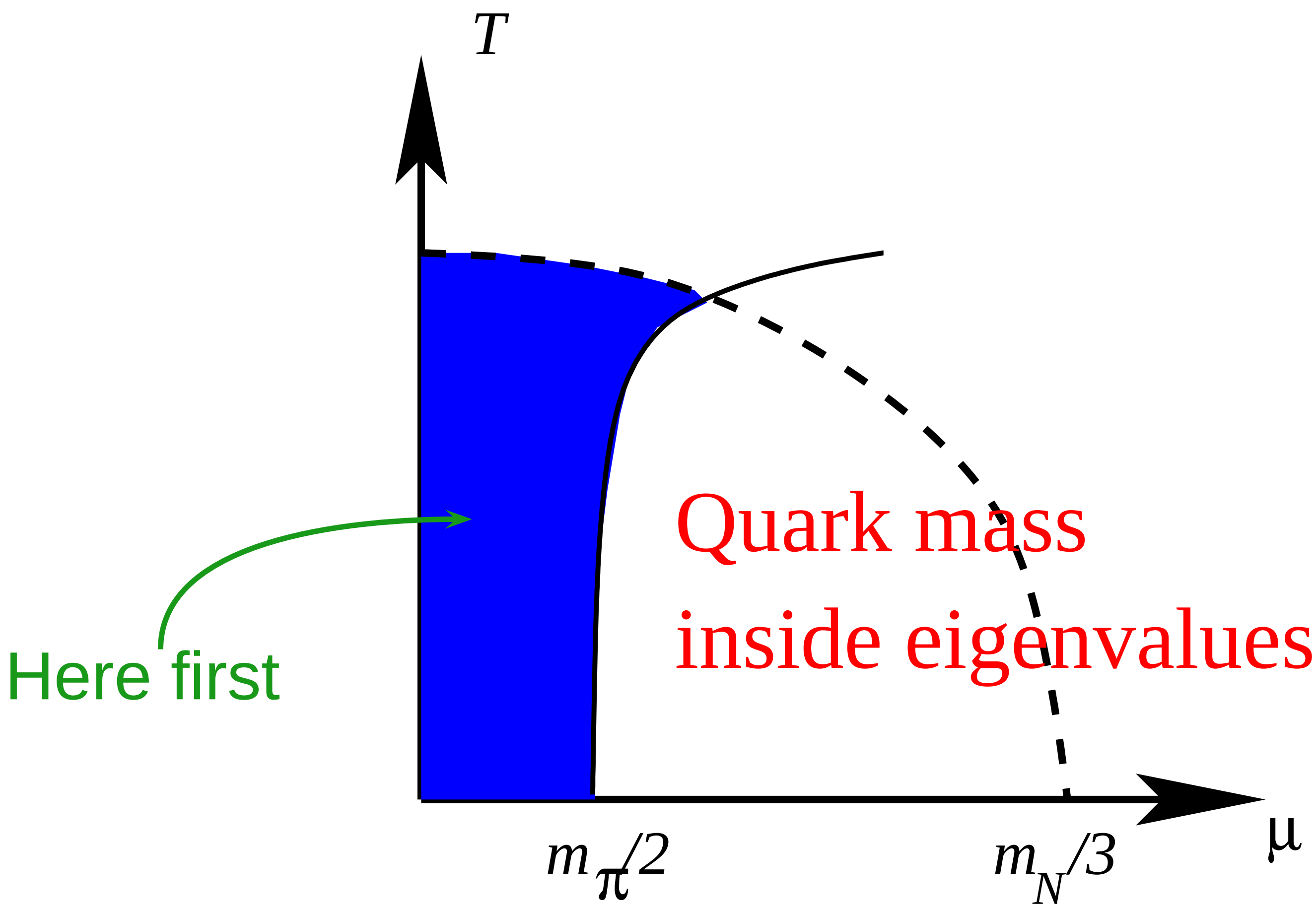
The distribution of the phase

Density of states method



- **Step 1:** Determine the distribution of θ
- **Step 2:** Obtain Z, n_q, $\langle\bar{\psi}\psi\rangle$ as $\int d\theta\ldots$
- **Here:** Analytically in Chiral Perturbation Theory

Step 1: The distribution of θ


T
Quark mass
inside eigenvalues
Here first
$m_\pi/2$
$m_N/3$
μ

The delta function $\&$ the moments

$$\langle \delta(2\theta - 2\theta') \rangle_{1+1} \equiv \frac{1}{Z_{1+1}} \int dA \, \delta(2\theta - 2\theta') \det{}^2(D+\mu\gamma_0+m) e^{-S_{YM}}$$

The delta function & the moments

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The moments of the phase factor

$$\langle e^{2ip\theta'} \rangle_{N_f} \equiv \left\langle \frac{\det^{p}(D + \mu\gamma_0 + m)}{\det^{p}(D - \mu\gamma_0 + m)} \right\rangle_{N_f}$$

is a ratio of two partition functions

Phase transition at $\mu = m_\pi/2$

$$\langle e^{2ip\theta'} \rangle_{N_f} = \frac{Z_{N_f+p|p^*}}{Z_{N_f}} = e^{-V\Delta\Omega_p}$$

In CPT at 1-loop

Number of charged pions

$$\langle e^{2ip\theta'} \rangle_{N_f} = e^{-p(N_f+p)V\Delta G_0}$$

ΔG_0 is the difference between charged and neutral pions

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The θ-distribution from CPT

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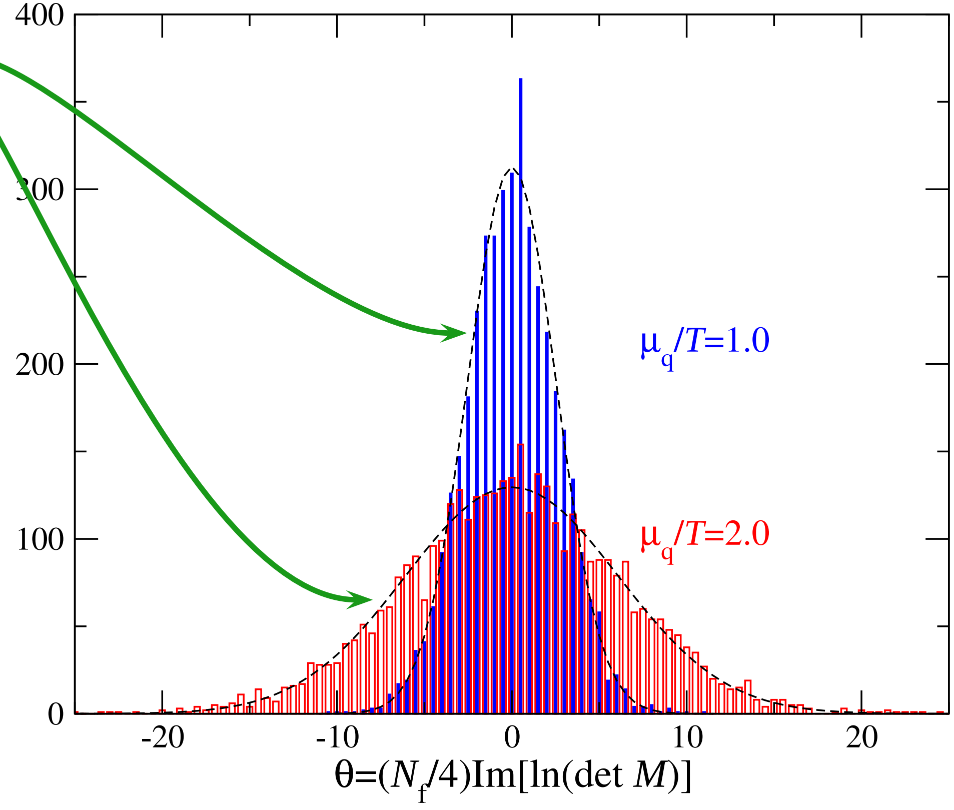
Prediction: Gaussian folded onto $[-\pi : \pi] \times$ phase

$$\langle \delta(2\theta - 2\theta') \rangle_{1+1} = \frac{e^{V\Delta G_0}}{\sqrt{\pi V \Delta G_0}} e^{2i\theta} \sum_{n=-\infty}^{\infty} e^{-(\theta+2\pi n)^2/V\Delta G_0}$$

The θ-distribution from the lattice

Gaussian fits



CPT $\Rightarrow$ width $\sqrt{V \Delta G_0}$

Ejiri PRD 77 (2008) 014508

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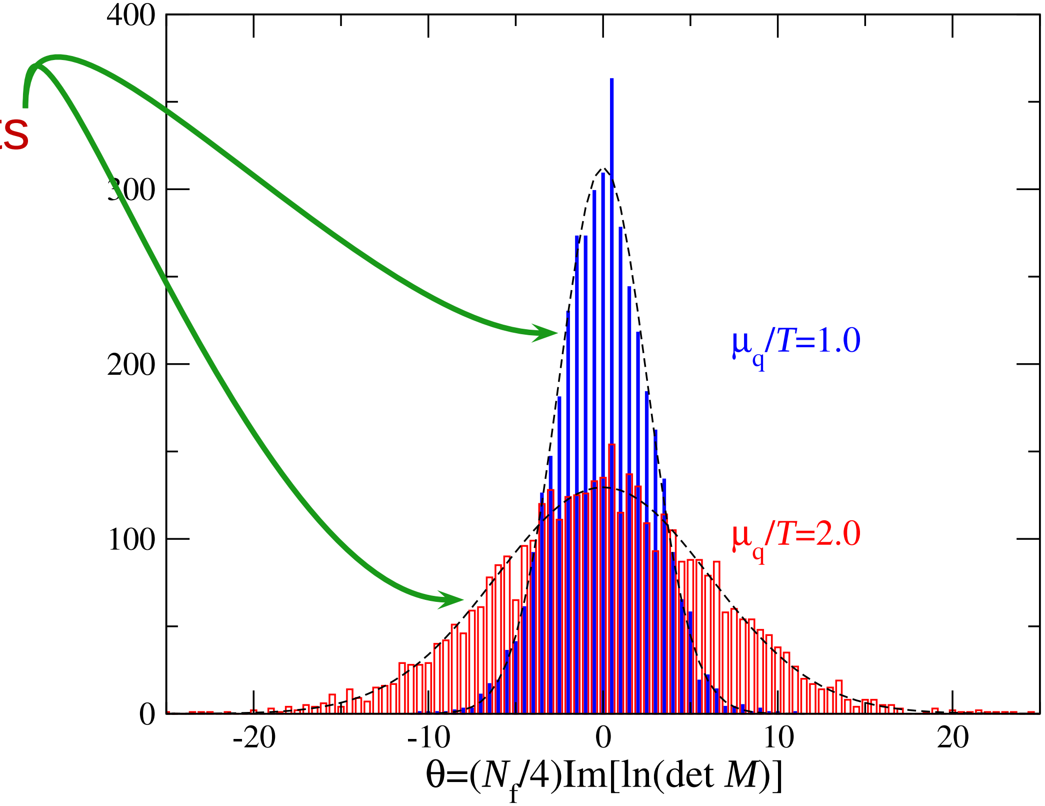

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Expected from the central limit theorem

Ejiri PRD 77 (2008) 014508

Step 2: Obtain n_q as $\int d\theta$

The distribution of n_q with θ

$$\langle n_q \delta(\theta - \theta') \rangle_{1+1}$$

$$\equiv \quad \frac{1}{Z_{1+1}} \lim_{\tilde{\mu} \to \mu} \frac{d}{d\tilde{\mu}} \int dA \; \delta(\theta - \theta'(\mu)) \det{}^2 (D + \tilde{\mu}\gamma_0 + m) e^{-S_{YM}}$$

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From CPT at 1-loop

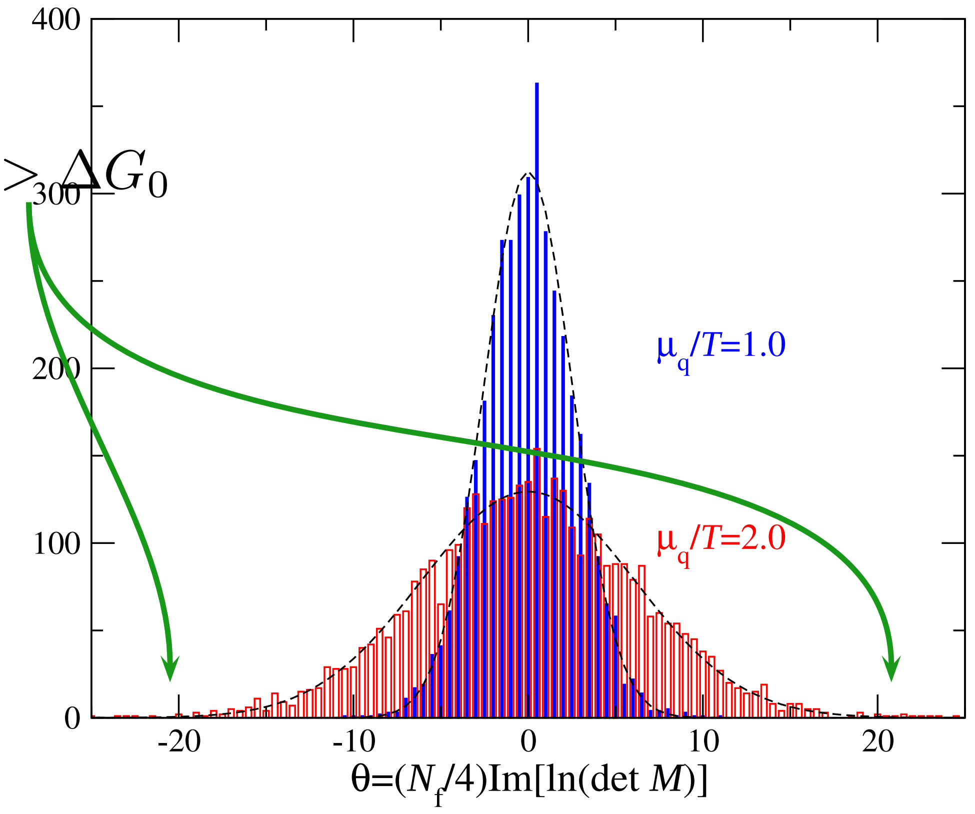
$$\langle {\color{red}n_q} \delta(\theta - \theta')\rangle_{1+1}$$

$$= \quad [\lim_{\tilde{\mu}\to\mu} \frac{d}{d\tilde{\mu}} \Delta G_0(-\mu, \tilde{\mu})](1 + i\frac{\theta}{\Delta G_0}) \frac{e^{\Delta G_0}}{\sqrt{\pi \Delta G_0}} e^{2i\theta} e^{-\theta^2/\Delta G_0}$$

The θ-distribution from the lattice

n_q & $\langle\bar{\psi}\psi\rangle$

sensitive to $|\theta| > \Delta G_0$



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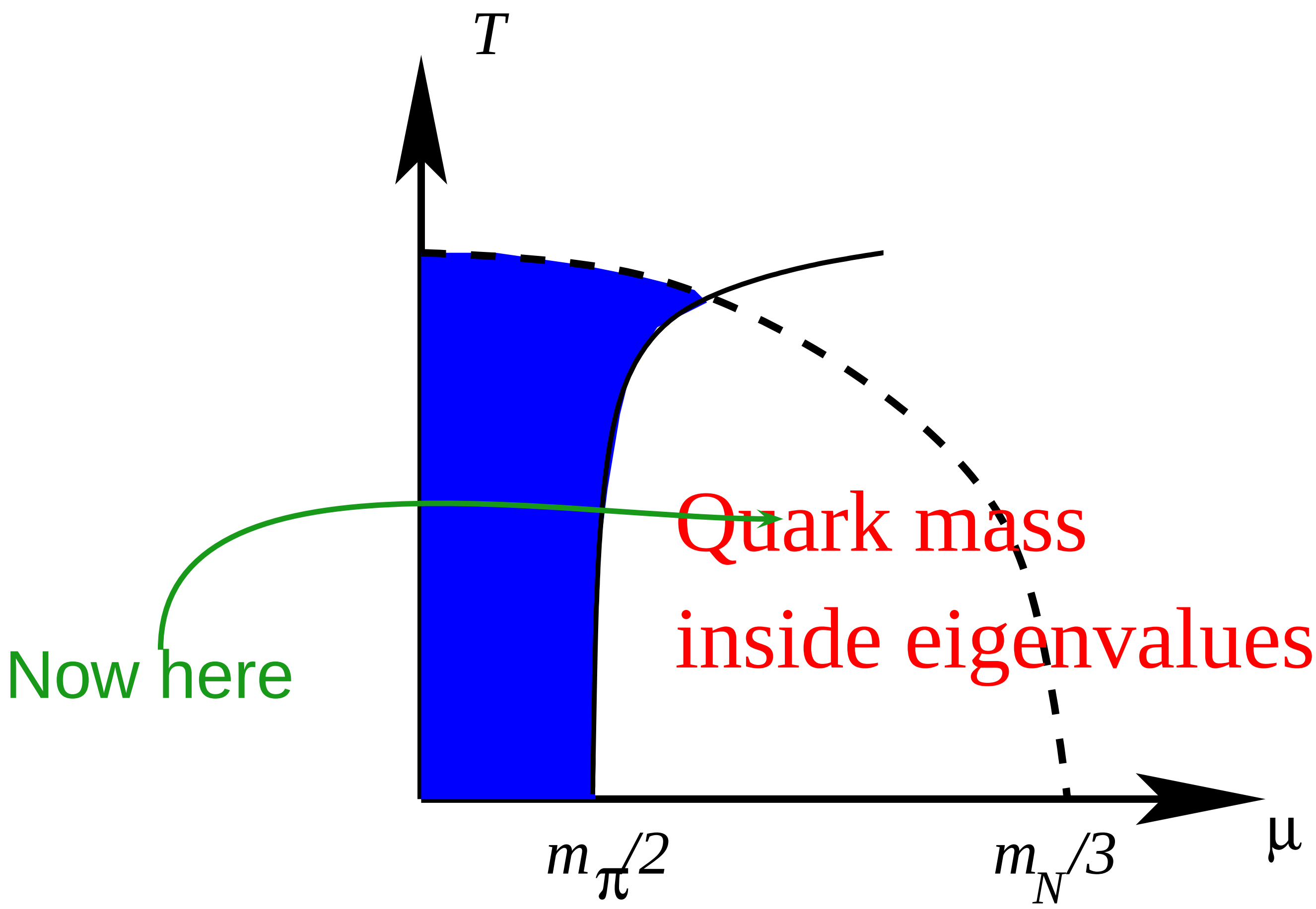
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In CPT at mean field level

Bosonic mean field rules

$$\langle e^{2ip\theta'} \rangle_{N_f} = e^{-(|p+N_f/2|-N_f/2)V\Delta\Omega}$$

$\Delta\Omega$ is the difference between the mean field terms

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Lombardo Splittorff Verbaarschot arXiv:0904.2122

The θ-distribution ($\mu > m_\pi/2$)

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Lorentzian (on $[-\pi : \pi]$)

$$\langle \delta(2\theta - 2\theta') \rangle_{1+1} = e^{2i\theta} \frac{e^{V\Delta\Omega}}{2\pi} \frac{\sinh(V\Delta\Omega)}{\cosh(V\Delta\Omega) - \cos(2\theta)}$$

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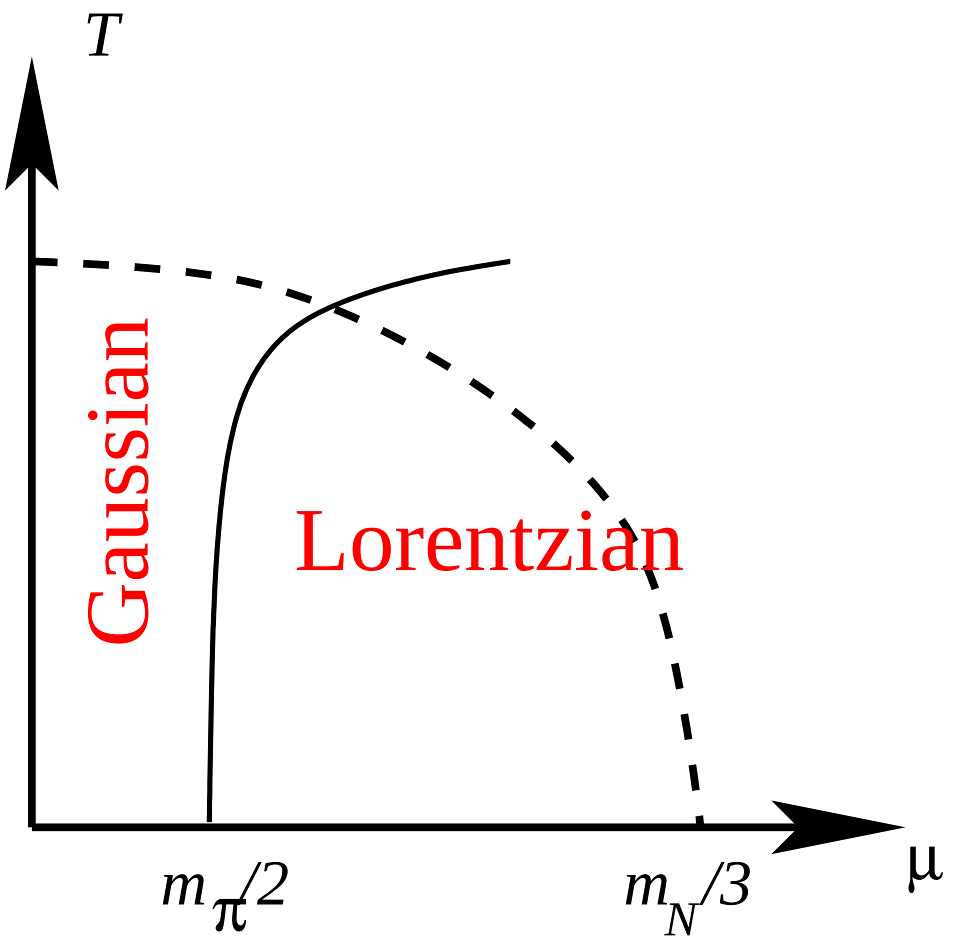
Central limit theorem fails!



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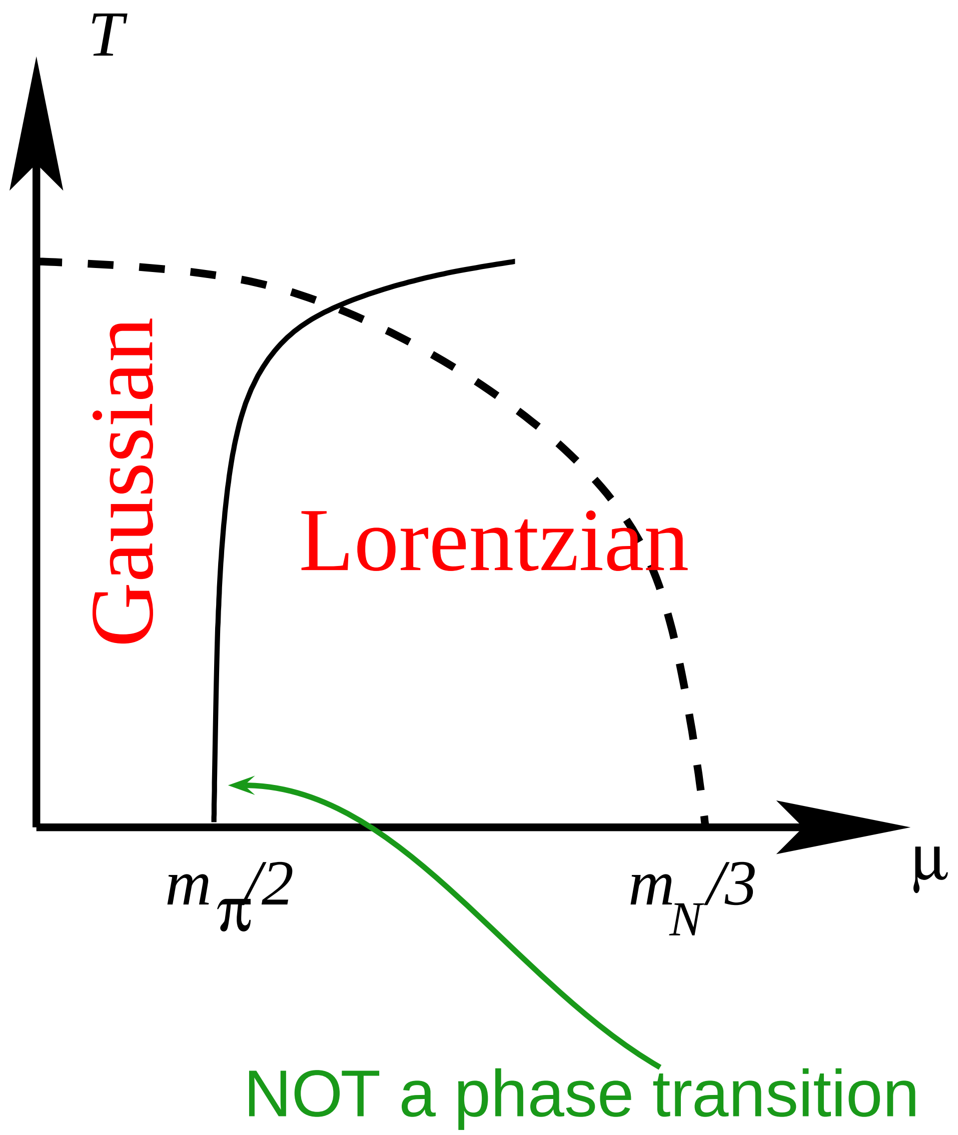
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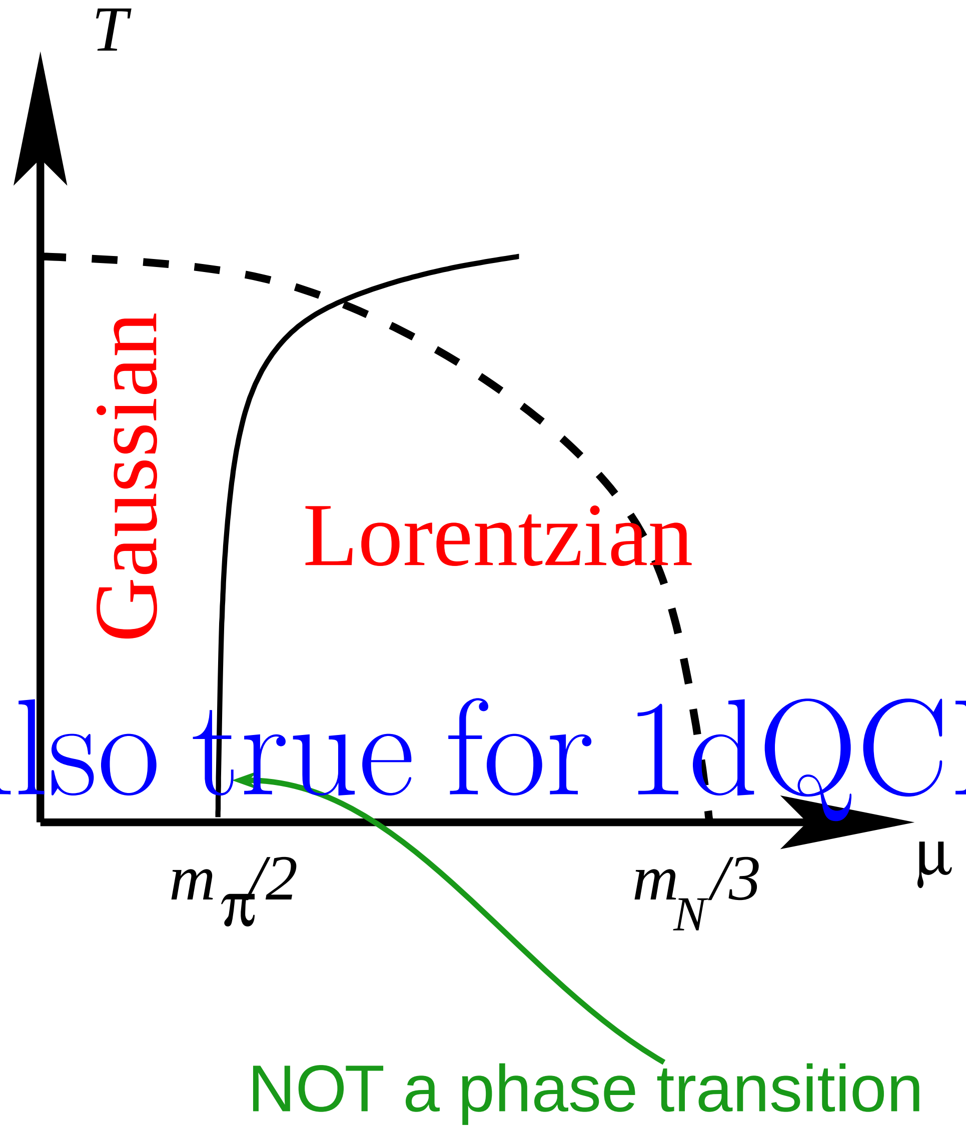
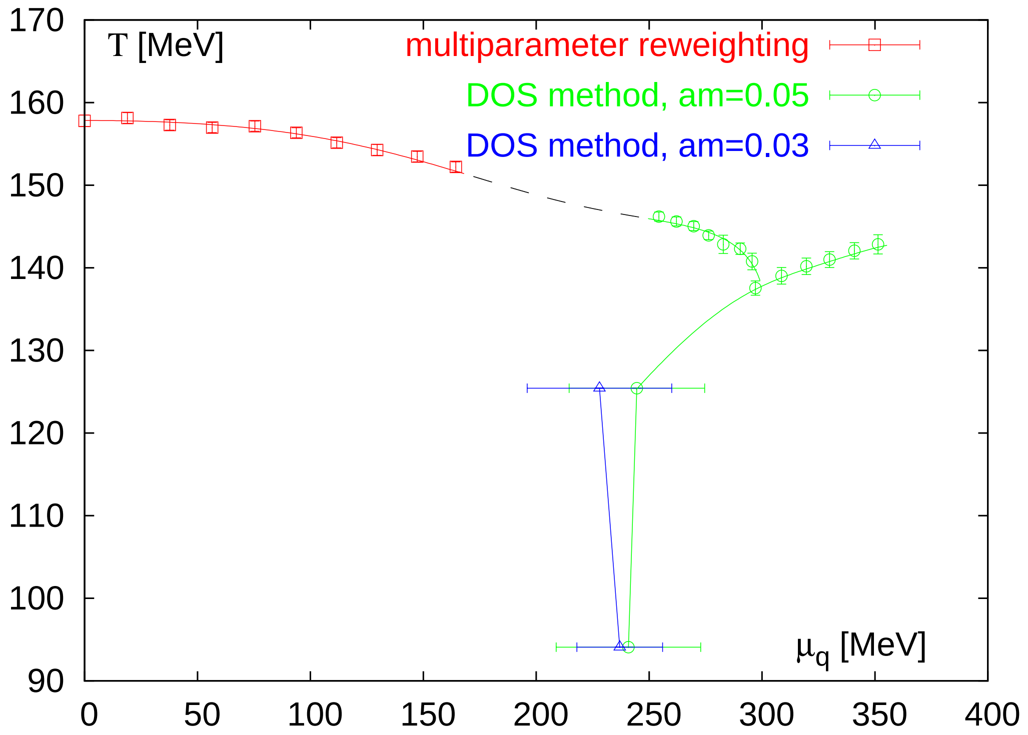
NOT a phase transition



Also true for 1dQCD

Then what are the results obtained from the lattice (II)

Density of states



Fodor Katz Schmidt JHEP0703:121,2007

Conclusions

Used chiral perturbation theory to examine results from lattice QCD at $\mu \neq 0$

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Main messages:

- Sign problem tractable for $\mu < m_\pi/2$
- Delicate cancellations for $\mu > m_\pi/2$

New progress

Complex Langevin dynamics at finite chemical potential

Aarts Phys.Rev.Lett. 102 (2009) 131601

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QCD with 2 colors does not have a sign problem

Hands Sitch Skullerud Phys.Lett.B662:405,2008

Additional slides