Lattice QCD and dense quark matter

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Abstract

This talk summarizes recent progress in lattice QCD for dense quark matter. The emphasis is on the insights obtained from analytical results derived within chiral perturbation theory.

1   Introduction

Despite the fact that all observed processes conserve baryon number, the total number of baryons present today is far greater than the number of anti-baryons. It is a long standing challenge to explain this excess of baryons. If we accept the baryon imbalance as a fact and ask:

What is the preferred ground state of strongly interacting matter given a specific baryon density and temperature ?

we are faced with an equally long standing challenge: The phases of strongly interacting matter must be determined by non-perturbative means, and the only known first principle method, lattice QCD, is extremely challenging to apply when there is an imbalance between baryons and anti-baryons.

Even though these two challenges are closely related their solutions will presumable be of a very different nature. To understand the generation of the baryon number most likely requires physics beyond the standard model. To understand the implications of a non zero baryon number on the ground state of strongly interacting matter requires the development of new non-perturbative tools for QCD.

This talk focuses on the latter challenge so let us explain in detail what we are up against when we seek to describe baryon asymmetric matter from first principles (even when we are modest, in that we do not aim to explain where this asymmetry originally came from, but simply want to adopt this into QCD). For simplicity we will also ignore electromagnetism.

Rather than fixing the density the natural approach is to start with the Grand Canonical partition function, Z, where we fix the quark chemical potential, μ, and then determine the quark number n (the baryon number is $3n$). The average quark number is

$$\langle n \rangle = \partial_\mu \log Z(\mu). \tag{1}$$

It is straightforward to include μ in the partition function. Since the chemical potential is conjugate to the quark number, the combination that appears in the Lagrangian is

$$\mu n = \mu q^\dagger q = \mu \overline{q} \gamma_0 q. \tag{2}$$

The simple extension of the Dirac operator in the Lagrangian of QCD

$$\mathcal{L}_{\text{QCD}} = \overline{q}(D_\eta \gamma_\eta + \mu \gamma_0 + m)q + \text{Gluons} \tag{3}$$

thus accommodates the chemical potential into QCD.

It is well known how to implement this extension on the lattice [1, 2]. Since μ enters as the zeroth component of a vector potential the choice with the correct continuum limit is [1]

$$\mathcal{L}_{\text{Lattice QCD}} = ... + e^{a\mu}\overline{q}_x \gamma_0 U_{x,x+\hat{0}} q_{x+\hat{0}} + e^{-a\mu}\overline{q}_{x+\hat{0}} \gamma_0 U^\dagger_{x,x+\hat{0}} q_x + ... \tag{4}$$

For two color QCD, when the fermion determinant is real, lattice simulations [3, 4, 5, 6, 7] based on this discretization give a rich phase diagram that agrees with theoretical expectations [8, 9, 10]. For three colors, however, the fermion determinant at non zero chemical potential is not real and positive

$$\det^2(D + \mu\gamma_0 + m) = |\det(D + \mu\gamma_0 + m)|^2 e^{2i\theta}. \tag{5}$$

While there is nothing wrong physically with a complex valued fermion determinant it is a major problem for lattice QCD. The Monte Carlo sampling of gauge field configurations, which is the back bone of lattice QCD, only works if the measure in the Euclidean partition function is real and positive. This is *the QCD sign problem*: at $\mu \neq 0$ direct Monte Carlo sampling is not possible and standard lattice QCD breaks down.

Several methods have been engineered to circumvent the sign problem, see table 1. Each have given important first principle insights into QCD at non zero μ, and a consistent picture emerges at small μ/T. All methods face serious challenges more or less directly induced by the sign problem. In order to deal with these challenges analytical results are of utmost importance. For example, in the method of Ejiri [28] the distribution of the phase of the fermion determinant is assumed to be Gaussian. Here we show that the Gaussian form comes out analytically for small chemical potentials, thus confirming the assumptions of this lattice method. However, as we also show, the distribution changes to a Lorentzian form for larger μ. This computation is carried out within chiral perturbation theory, which is the low energy effective theory for QCD in the chirally broken phase. The analytic results in this way both justifies the assumptions on which the method of Ejiri is based and show its limitations.

Chiral perturbation theory also allows us to understand the distribution of the values which the Euclidean quark number operator assumes over the gauge fields. The results not only give us a direct insight in the way the total baryon number forms, it also shows how the Complex Langevin method can deal with the sign problem.

2   Fixed phase of the determinant

Perhaps the most direct way to understand the severity of the QCD sign problem is to consider the distribution of the phase of the fermion determinant

$$\langle \delta(\theta - \theta') \rangle_{1+1} d\theta = \frac{\int dA |\det(D + \mu\gamma_0 + m)|^2 e^{2i\theta'} \delta(\theta - \theta') e^{-S_{\mathrm{YM}}}}{\int dA |\det(D + \mu\gamma_0 + m)|^2 e^{2i\theta'} e^{-S_{\mathrm{YM}}}} d\theta. \qquad (6)$$

The δ function allows us to rewrite the unquenched distribution in terms of the phase quenched one times a phase factor and a normalization

$$\langle \delta(\theta - \theta') \rangle_{1+1} = e^{2i\theta} \frac{Z_{1+1^*}}{Z_{1+1}} \langle \delta(\theta - \theta') \rangle_{1+1^*}. \qquad (7)$$

The complex nature of the unquenched θ distribution is typical of the sign problem. Here it takes the simplest possible form. Nevertheless, the effect of the phase is dramatic: it leads to exponentially large cancellations in the volume. This follows directly since $Z_{1+1^*}/Z_{1+1} \sim \exp(V)$. These dramatic cancellations are what makes the sign problem severe.

To understand how the cancellations take part we have computed the θ distributions to leading order in chiral perturbation theory. For $\mu < m_\pi/2$ with the phase angle $\theta \in [-\pi, \pi]$ we have [38]

$$\langle \delta(\theta - \theta') \rangle_{1+1} = \frac{1}{\sqrt{\pi \Delta G_0}} e^{2i\theta + \Delta G_0} \sum_{n=-\infty}^{\infty} e^{-(\theta + 2n\pi)^2/\Delta G_0}, \qquad (8)$$

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Method	Idea	Challenge
Reweighting [11, 12, 13]	Absorb the sign in the observable	Exponential cancellations
Taylor expansion [14, 15, 16, 17, 18]	Expand at $\mu = 0$	Higher order terms
Imaginary μ [19, 20, 21, 22]	Determine the analytic function	Control the extrapolation
Density of states [23, 24, 25, 26, 27, 28]	Use the distribution of the phase	Determine the distribution
Canonical ensemble [29, 30, 31, 32, 33]	Work at fixed baryon number	Fix the baryon number
Complex Langevin [34, 35, 36, 37]	Stochastic flow in complex plane	Make it work for QCD

Table 1: Summary of the main methods used to circumvent the sign problem, the main idea used and the main challenge faced when using this method.

where

$$\Delta G_0 \;\; = \;\; \frac{V m_\pi^2 T^2}{\pi^2} \sum_{n=1}^{\infty} \frac{K_2(\frac{m_\pi n}{T})}{n^2} (\cosh(\frac{2\mu n}{T}) - 1). \tag{9}$$

Notice that ΔG_0 is the difference of the free energy in the full and the phase quenched theory

$$\frac{Z_{1+1}}{Z_{1+1^*}} = e^{-\Delta G_0}. \tag{10}$$

The θ distribution takes the form of a Gaussian modulo 2π. This is in agreement with observations on the lattice [28] and is the natural form suggested by the central limit theorem. However, at larger values of the chemical potential the quark mass enters the support of the Dirac spectrum [39, 40] and fluctuations of the phase will be far greater [41, 42, 43, 44].

To leading order in chiral perturbation theory for $\mu > m_\pi/2$ the θ distribution is a Lorentzian modulo 2π times the phase factor [43]:

$$\langle \delta(2\theta - 2\theta') \rangle_{1+1} = e^{2i\theta} \frac{e^{VL_B}}{\pi} \frac{\sinh(VL_B)}{\cosh(VL_B) - \cos(2\theta)}, \tag{11}$$
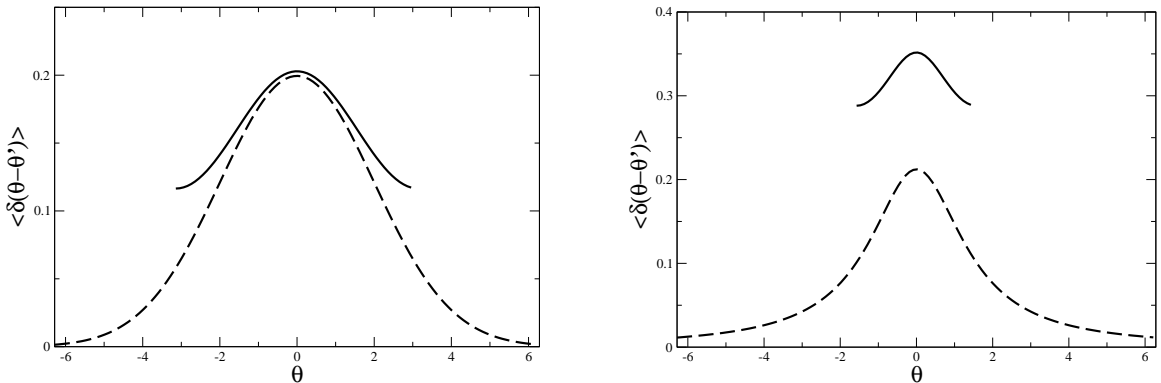
Figure 1: The quenched distribution of the phase of the fermion determinant. The dashed curves are standard Gaussian (left) and Lorentzian (right). The full curves are the corresponding distributions modulo 2π and π relevant for $\mu < m_\pi/2$ and $\mu > m_\pi/2$ respectively.

where

$$\frac{Z_{1+1}}{Z_{1+1^*}} = e^{-VL_B}. \tag{12}$$

This demonstrates that the assumptions of the central limit theorem are not satisfied and that the method of Ejiri must be examined carefully for $\mu > m_\pi/2$.

The Gaussian form (for $\mu < m_\pi/2$) and the Lorentzian form (for $\mu < m_\pi/2$) modulo 2π are illustrated in Fig. 1.

3   Distribution of the quark number over θ

Since chiral perturbation theory deals with pions alone, the quark number vanishes (we do not consider the effect of pion fields with non-trivial topological winding). However, the distribution of the quark number over θ is non trivial even when evaluated within chiral perturbation theory. The form it takes gives us direct insight into the noise produced by the pions in lattice QCD.

With $\mu < m_\pi/2$ we find to leading order in chiral perturbation theory [43]

$$\langle n\ \delta(\theta - \theta')\rangle_{1+1} = \nu_I \sum_{n=-\infty}^{\infty} (1 + i\frac{\theta + 2\pi n}{\Delta G_0})\frac{e^{2i\theta}}{\sqrt{\pi \Delta G_0}} e^{-(\theta+2\pi n)^2/\Delta G_0 + \Delta G_0}, \tag{13}$$

where

$$\nu_I = \left(\lim_{\tilde{\mu} \to \mu} \frac{d}{d\tilde{\mu}} \Delta G_0(-\mu, \tilde{\mu}) \right). \tag{14}$$

This distribution gives the quark number measured on a set of configurations for which the phase of the determinant is between θ and $\theta + d\theta$.

Note that the normalization is by the two flavor partition function, so that the total quark number is the integral over the phase angle.

For $\mu > m_\pi/2$ the distribution again changes its form drastically [43]

$$\langle n\ \delta(2\theta - 2\theta') \rangle_{1+1} = -\frac{2}{\pi}\left[\frac{VL_I}{\mu}\right]e^{2i\theta}e^{2VL_B}\frac{1}{e^{VL_B} - e^{2i\theta}}. \qquad (15)$$

Prior to discussing the importance of these results let us present the quark number distribution as predicted by chiral perturbation theory.

4 Distribution of the quark number

The distribution of the values which the quark number operator,

$$n(\mu) \equiv \text{Tr}\ \frac{\gamma_0}{D + \mu\gamma_0 + m}, \qquad (16)$$

assumes as the gauge fields vary is also accessible within chiral perturbation theory.

Since

$$n(\mu)^* = \left(\text{Tr}\frac{\gamma_0}{D + \mu\gamma_0 + m}\right)^* = \text{Tr}\frac{\gamma_0}{D - \mu\gamma_0 + m} = -n(-\mu), \qquad (17)$$

the quark number operator is in general complex (it is purely imaginary at $\mu = 0$). The fluctuations of the quark number thus occur in the complex plane. The distribution of the quark number in the complex plane is by definition

$$P_n^{1+1}(x, y) \equiv \left\langle \delta\left(x - \frac{1}{2}(n(\mu) - n(-\mu))\right) \delta\left(y + i\frac{1}{2}(n(\mu) + n(-\mu))\right) \right\rangle_{1+1}. \qquad (18)$$

In order to evaluate this average within chiral perturbation theory both δ functions are expanded in terms of the moments of the real and imaginary parts of the quark number. The first moment, i.e. the average quark number, vanishes. The higher moments are off-diagonal susceptibilities and are non-trivial even within chiral perturbation theory [44].

For $\mu < m_\pi/2$ we find to 1-loop order [44]

$$P_n^{1+1}(x, y) = \frac{1}{\pi\sqrt{(\chi_{ud}^I)^2 - (\chi_{ud}^B)^2}}e^{-(x-\nu_I)^2/(\chi_{ud}^I+\chi_{ud}^B)}e^{(iy+\nu_I)^2/(\chi_{ud}^I-\chi_{ud}^B)}, \qquad (19)$$

where

$$\chi_{ud}^B \equiv \left.\frac{d^2}{d\mu_1 d\mu_2}\Delta G_0(\mu_1, \mu_2)\right|_{\mu_1=\mu_2=\mu}, \qquad (20)$$

$$\chi_{ud}^I \equiv \left.\frac{d^2}{d\mu_1 d\mu_2}\Delta G_0(-\mu_1, \mu_2)\right|_{\mu_1=\mu_2=\mu}. \qquad (21)$$

Note that the distribution factorizes into the distribution of the real and imaginary part of the quark number.

For $\mu > m_\pi/2$ the higher moments diverge and lead to power law tails of the distribution of the quark number operator [44]. In the quenched case, for example, the distribution has a inverse cubic tail.

In the confined phase the pions dominate the free energy and the quark number is thus generated by subleading terms. A central aspect of the sign problem is to pick up this small baryon signal from the background produced by the pions. The results presented above give the analytical form of the pion background and thus the foundation for extracting the quark number.

5 Cancellations

The non trivial results for the distribution of the quark number with θ and the distribution of the quark number itself should of course be consistent with zero average quark number. Therefore, upon integration over θ and over the complex quark number plane respectively, we must find a vanishing average quark number. Here we show analytically how these cancellations occur.

5.1 Integration over θ

The total quark number is obtained from its distribution over θ by integration over the phase angle

$$\langle n \rangle_{1+1} = \int d\theta \langle n \ \delta(\theta - \theta') \rangle_{1+1}. \tag{22}$$

We now discuss how this becomes zero within chiral perturbation theory.

First we consider the case $\mu < m_\pi/2$. Here we can write the distribution of the quark number over θ as a total deriative of the θ distribution times ν_I

$$\langle n \ \delta(\theta - \theta') \rangle_{1+1} \ = \ \nu_I \sum_{n=-\infty}^{\infty} \frac{1}{2i} \frac{d}{d\theta} \frac{e^{2i\theta}}{\sqrt{\pi \Delta G_0}} e^{-(\theta + 2\pi n)^2/\Delta G_0 + \Delta G_0} \tag{23}$$

$$= \ \nu_I \frac{1}{2i} \frac{d}{d\theta} \langle \delta(\theta - \theta') \rangle_{1+1}. \tag{24}$$

From this we see that all phase angles, θ, are essential in order to obtain the desired vanishing value. Moreover, we see that these exact cancellations damp the exponentially large amplitude of the distribution completely (recall that ΔG_0 is extensive). Therefore, unless the θ integration is carried out exactly one will find an exponentially large contribution to the quark number from the pions.

Also for $\mu > m_\pi/2$ we can write the distribution of the baryon number over θ as a total derivative. However, this time it is *not* the derivative of the

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θ-distribution but rather

$$\langle n\ \delta(2\theta - 2\theta')\rangle_{1+1} = \frac{1}{\pi}\Big[\frac{VL_I}{\mu}\Big]e^{2VL_B}\frac{1}{i}\frac{d}{d\theta}\log(e^{VL_B} - e^{2i\theta}). \qquad (25)$$

Again this implies that all angles contribute in an essential way to the full quark number. The need for an exact integration over θ is even more urgent for $\mu > m_\pi/2$, since the amplitude now grows exponentially fast with the volume even at mean field level.

5.2   Integration over the complex n plane

Given the distribution (19) of the baryon number, $P_n^{1+1}(x, y)$, we obtain the average quark number upon integration over the complex quark number plane

$$\langle n\rangle_{1+1} = \int dxdy\ (x + iy)P_n^{1+1}(x, y). \qquad (26)$$

The analytical result for the distribution shows precisely how the vanishing expectation value of the quark number is realized

$$\langle n\rangle_{1+1} = \int dx\ xP_{\mathrm{Re}[n]}^{1+1}(x) + i\int dy\ yP_{\mathrm{Im}[n]}^{1+1}(y) = \nu_I + ii\nu_I = 0. \qquad (27)$$

We see that the total quark number is obtained only after a detailed cancellation between the contribution from the real part and the imaginary part. The challenge for unquenched lattice QCD is to identify the baryon signal within these massive cancellations due to the pions. The analytical result from chiral perturbation theory shows precisely how this may be realized within the framework of the Complex Langevin method, for a detailed discussion see [44].

6   Summary

The description of the macroscopic phases of strongly interacting matter based directly on the microscopic QCD dynamics is hampered by the QCD sign problem.

Despite the complex nature of the sign problem substantial progress has been made recently. We have derived the distribution of the complex phase of the Dirac operator analytically and the distribution of the quark number over the phase, as well as the distribution of the quark number itself. These results give direct analytical insights into the numerical challenges faced by the density of states method and the Complex Langevin method when applied to QCD at non zero chemical potential.

The new results form a novel branch of applications of chiral perturbation theory. This branch has its root in the evaluation of partially quenched averages by the replica method [45].

References

[1] P. Hasenfratz and F. Karsch, Phys. Lett. B **125**, 308 (1983).

[2] J. B. Kogut, H. Matsuoka, M. Stone, H. W. Wyld, S. H. Shenker, J. Shigemitsu and D. K. Sinclair, Nucl. Phys. B **225** (1983) 93.

[3] S. Hands, J. B. Kogut, M. P. Lombardo and S. E. Morrison, Nucl. Phys. B **558** (1999) 327 [arXiv:hep-lat/9902034].

[4] J. B. Kogut, D. Toublan and D. K. Sinclair, Nucl. Phys. B **642**, 181 (2002) [arXiv:hep-lat/0205019].

[5] J. B. Kogut, D. Toublan and D. K. Sinclair, Phys. Rev. D **68**, 054507 (2003) [arXiv:hep-lat/0305003].

[6] J. B. Kogut, D. K. Sinclair, S. J. Hands and S. E. Morrison, Phys. Rev. D **64**, 094505 (2001) [arXiv:hep-lat/0105026].

[7] S. Hands, S. Kim and J. I. Skullerud, Eur. Phys. J. C **48**, 193 (2006) [arXiv:hep-lat/0604004].

[8] J. B. Kogut, M. A. Stephanov and D. Toublan, Phys. Lett. B **464**, 183 (1999) [arXiv:hep-ph/9906346].

[9] J. B. Kogut, M. A. Stephanov, D. Toublan, J. J. M. Verbaarschot and A. Zhitnitsky, Nucl. Phys. B **582**, 477 (2000) [arXiv:hep-ph/0001171].

[10] J. T. Lenaghan, F. Sannino and K. Splittorff, Phys. Rev. D **65**, 054002 (2002) [arXiv:hep-ph/0107099].

[11] I. M. Barbour, S. E. Morrison, E. G. Klepfish, J. B. Kogut and M. P. Lombardo, Nucl. Phys. Proc. Suppl. **60** A (1998) 220.

[12] Z. Fodor and S. D. Katz, JHEP **0203**, 014 (2002) [arXiv:hep-lat/0106002].

[13] Z. Fodor and S. D. Katz, JHEP **0404**, 050 (2004) [arXiv:hep-lat/0402006].

[14] R. V. Gavai and S. Gupta, Phys. Rev. D **68**, 034506 (2003) [arXiv:hep-lat/0303013].

[15] C. R. Allton *et al.*, Phys. Rev. D **66**, 074507 (2002) [arXiv:hep-lat/0204010].

[16] C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann and C. Schmidt, Phys. Rev. D **68**, 014507 (2003) [arXiv:hep-lat/0305007].

[17] C. R. Allton *et al.*, Phys. Rev. D **71**, 054508 (2005) [arXiv:hep-lat/0501030].

[18] G. Endrodi, Z. Fodor, S. D. Katz and K. K. Szabo, PoS **LATTICE2008**, 205 (2008) [arXiv:0901.3018 [hep-lat]].

[19] P. de Forcrand and O. Philipsen, Nucl. Phys. B **642**, 290 (2002) [arXiv:hep-lat/0205016].

[20] M. D'Elia and M. P. Lombardo, Phys. Rev. D **67**, 014505 (2003) [arXiv:hep-lat/0209146].

[21] P. de Forcrand and O. Philipsen, Nucl. Phys. B **673**, 170 (2003) [arXiv:hep-lat/0307020].

[22] M. D'Elia, F. Di Renzo and M. P. Lombardo, Phys. Rev. D **76**, 114509 (2007) [arXiv:0705.3814 [hep-lat]].

[23] A. Gocksch, Phys. Rev. Lett. **61**, 2054 (1988).

[24] V. Azcoiti, G. Di Carlo, A. Galante and V. Laliena, Phys. Rev. Lett. **89**, 141601 (2002) [arXiv:hep-lat/0203017].

[25] K. N. Anagnostopoulos and J. Nishimura, Phys. Rev. D **66**, 106008 (2002) [arXiv:hep-th/0108041].

[26] J. Ambjorn, K. N. Anagnostopoulos, J. Nishimura and J. J. M. Verbaarschot, JHEP **0210**, 062 (2002) [arXiv:hep-lat/0208025].

[27] Z. Fodor, S. D. Katz and C. Schmidt, JHEP **0703**, 121 (2007) [arXiv:hep-lat/0701022].

[28] S. Ejiri, Phys. Rev. D **77**, 014508 (2008) [arXiv:0706.3549 [hep-lat]].

[29] D. E. Miller and K. Redlich, Phys. Rev. D **35**, 2524 (1987).

[30] A. Hasenfratz and D. Toussaint, Nucl. Phys. B **371**, 539 (1992).

[31] J. Engels, O. Kaczmarek, F. Karsch and E. Laermann, Nucl. Phys. B **558**, 307 (1999) [arXiv:hep-lat/9903030].

[32] S. Kratochvila and P. de Forcrand, PoS **LAT2005**, 167 (2006) [arXiv:hep-lat/0509143].

[33] A. Li, X. Meng, A. Alexandru and K. F. Liu, PoS **LATTICE2008**, 178 (2008) [arXiv:0810.2349 [hep-lat]].

[34] F. Karsch and H. W. Wyld, Phys. Rev. Lett. **55**, 2242 (1985).

[35] J. Flower, S. W. Otto and S. Callahan, Phys. Rev. D **34**, 598 (1986).

[36] J. Ambjorn, M. Flensburg and C. Peterson, Nucl. Phys. B **275**, 375 (1986).

[37] G. Aarts and I. O. Stamatescu, JHEP **0809**, 018 (2008) [arXiv:0807.1597 [hep-lat]]; G. Aarts, arXiv:0810.2089 [hep-lat]; arXiv:0902.4686 [hep-lat].

[38] K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. D **77**, 014514 (2008) [arXiv:0709.2218 [hep-lat]].

[39] P. E. Gibbs, Glasgow preprint 86-0389 (unpublished).

[40] D. Toublan and J. J. M. Verbaarschot, Int. J. Mod. Phys. B **15**, 1404 (2001) [hep-th/0001110].

[41] K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. Lett. **98**, 031601 (2007) [arXiv:hep-lat/0609076].

[42] K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. D **75**, 116003 (2007) [arXiv:hep-lat/0702011].

[43] M. P. Lombardo, K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. D **80**, 054509 (2009) [arXiv:0904.2122 [hep-lat]].

[44] M. P. Lombardo, K. Splittorff and J. J. M. Verbaarschot, arXiv:0910.5482 [hep-lat].

[45] P. H. Damgaard and K. Splittorff, Phys. Rev. D **62**, 054509 (2000) [arXiv:hep-lat/0003017].