Standard Cosmological Model: I
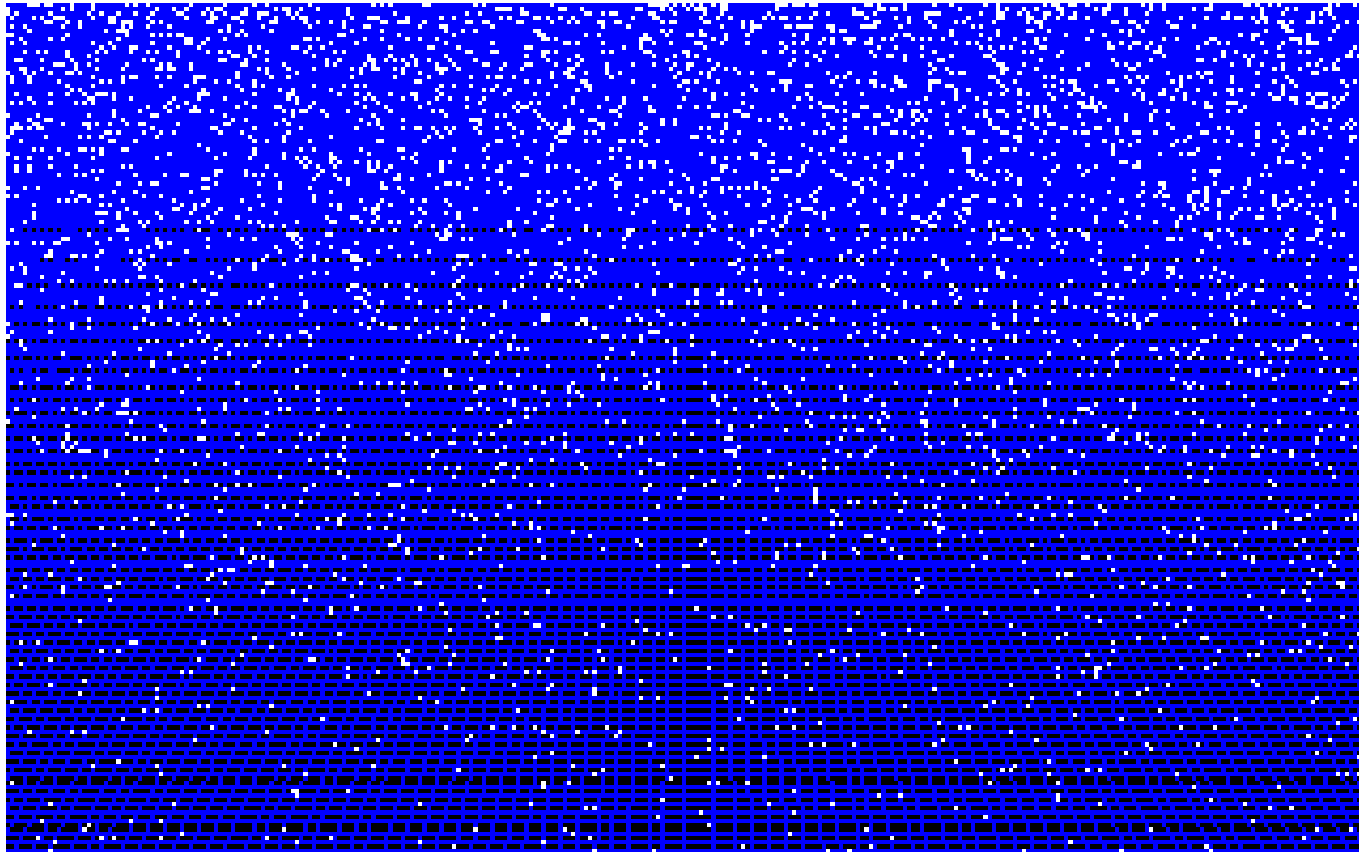
❑ Friedmann-Robertson-Walker (FRW) Metric /
    Friedmann Equation

❑ Constituents of the Universe

❑ Thermal History: Neutrino Abundance

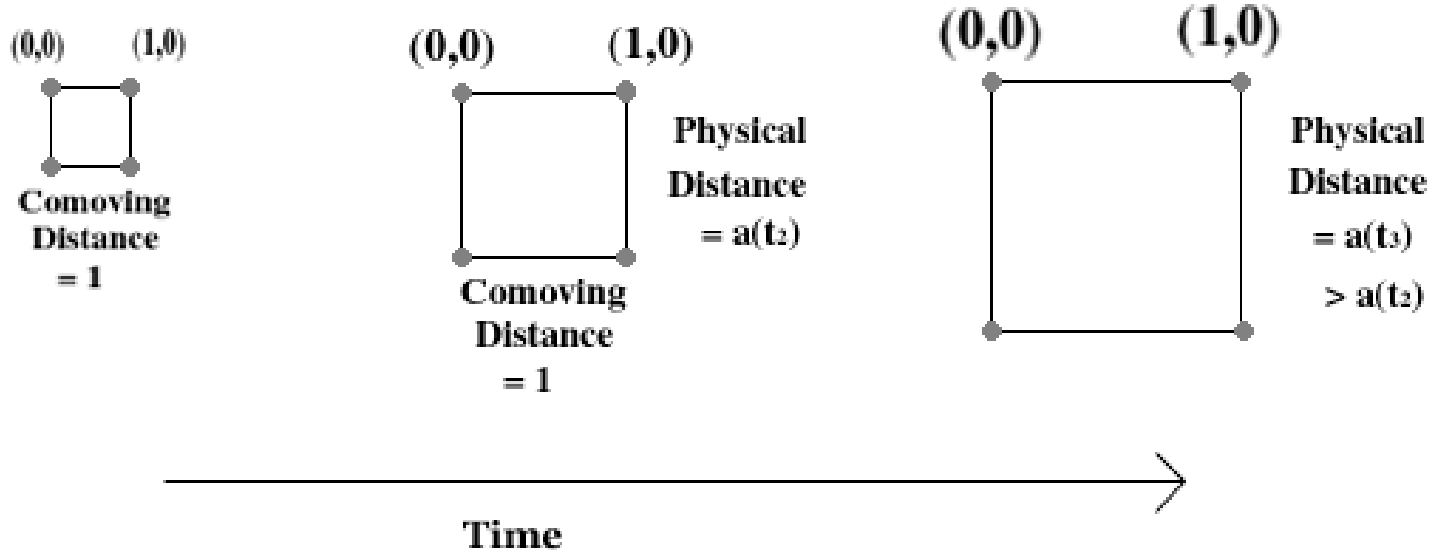❑ Thermal History: Weakly Interacting Massive Particles
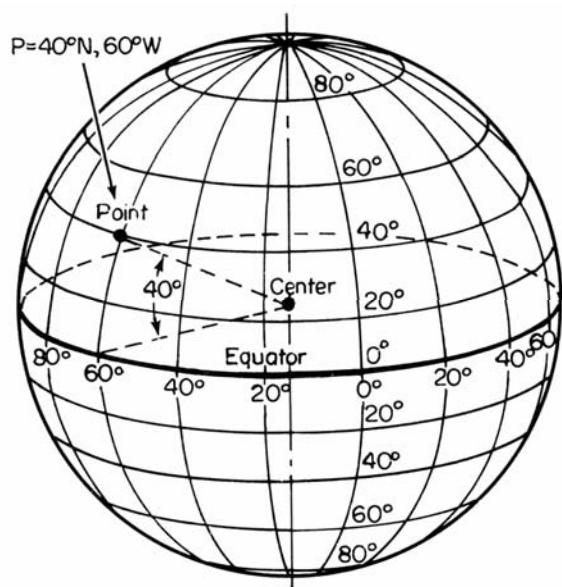   (WIMPs)

The Universe is Expanding

Scale Factor *a* quantifies expansion

(0,0)   (1,0)

Comoving
Distance
= 1

(0,0)   (1,0)

Physical
Distance
= a(t₂)

Comoving
Distance
= 1

(0,0)   (1,0)

Physical
Distance
= a(t₃)

> a(t₂)

Time

Comoving Coordinates/Distances

The coordinate differences on the grid are called *comoving distances*. They are the equivalent of longitude & latitude.

To get a physical distance *dl* from a set of coordinate differences (*dθ*,*dφ*), use the *metric*.

$$dl^2 = R^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

$$g_{\theta\theta} = R^2 \quad ; \quad g_{\phi\phi} = R^2\sin^2\theta$$

SLAC Summer Institute:
Scott Dodelson

The scale factor *a(t)* is the key function in the Friedmann-Robertson-Walker metric

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \right)$$

In a flat universe (our universe) *k=0,* and the metric reduces to
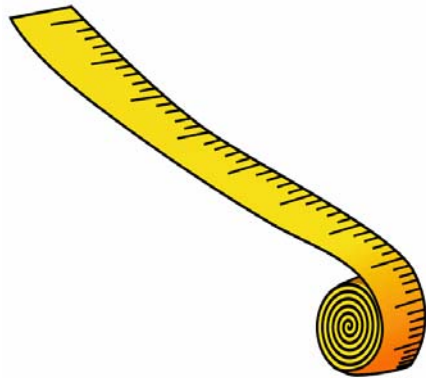
$$ds^2 = -dt^2 + a^2(t) dx^i dx^i$$

or

$\hbar = c = k_B = 1$

$$g_{00} = -1 \quad g_{ij} = \delta_{ij} a^2(t)$$

SLAC Summer Institute:
Scott Dodelson

Comoving Coordinates/Distances

Since we set $a_0=1$, the comoving distance between 2 objects is the physical distance one would get today if an infinitely long tape measure was placed between the objects.

A tape measure placed between the same 2 objects early on would find a physical distance of ($a(t)$ x comoving distance)

The evolution of *a* is determined by Einstein's Equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Ricci Tensor

Ricci Scalar

Newton's Constant

Energy Momentum Tensor

General Relativity in 1 Slide

Metric inverse

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$$

Raise/lower indices with metric/inverse

$$T^{\alpha}{}_{\beta} = g^{\alpha\mu} T_{\mu\beta}$$

Christoffel Symbol

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left[\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}} \right]$$

Ricci Tensor

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu,\alpha} - \Gamma^{\alpha}{}_{\mu\alpha,\nu} + \Gamma^{\alpha}{}_{\beta\alpha}\Gamma^{\beta}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\beta\nu}\Gamma^{\beta}{}_{\mu\alpha}$$

Ricci Scalar

$$R = g^{\mu\nu} R_{\mu\nu}$$

SLAC Summer Institute:
Scott Dodelson

Example: Christoffel Symbol

$$\Gamma^0_{ij} = \frac{1}{2} g^{0\alpha} \left[g_{\alpha i,j} + g_{\alpha j,i} - g_{ij,\alpha} \right]$$

$$= \frac{1}{2} g^{00} \left[g_{0i,j} + g_{0j,i} - g_{ij,0} \right]$$

But the metric has no spatial dependence, so ...

And $g^{00} = -1$ and $g_{ij} = \delta_{ij} a^2$, so

$$\Gamma^0_{ij} = \delta_{ij} \frac{1}{2} \frac{d}{dt} a^2 = \delta_{ij} \dot{a} a$$

SLAC Summer Institute:
Scott Dodelson

Time-Time Component of Einstein Equations

$$R_{00} + \frac{1}{2}R = 8\pi G T_{00} = 8\pi G \rho$$

Straightforward calculations lead to:

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]$$

Plug in to get the
Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H^2 = \frac{8\pi G}{3}\rho$$

SLAC Summer Institute:
Scott Dodelson

Space-Space Component of Einstein Equations

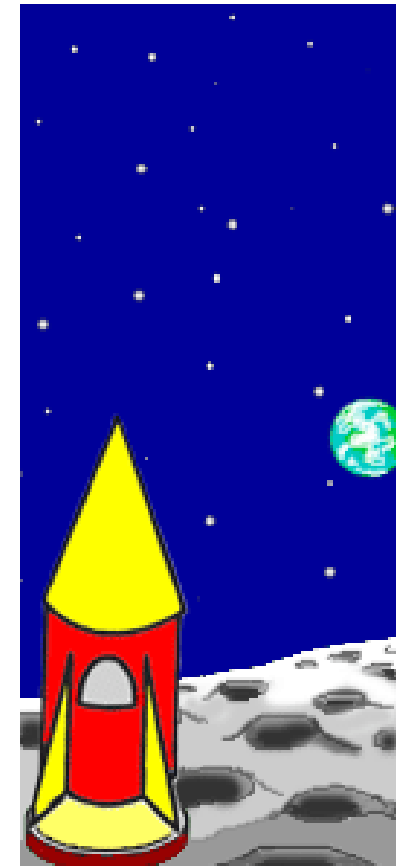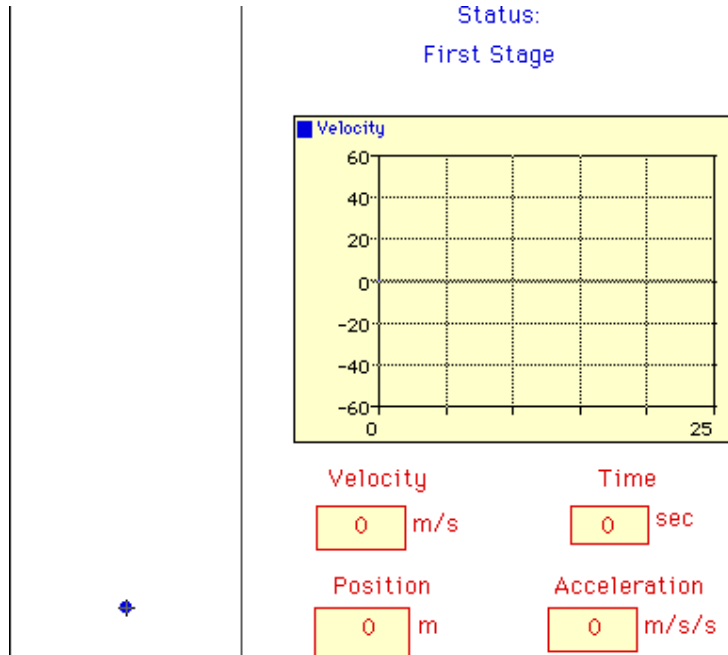$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi GP$$

where *P* is the diagonal space-space component of the energy momentum tensor.

Combine with the Friedmann equation to get:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right)$$
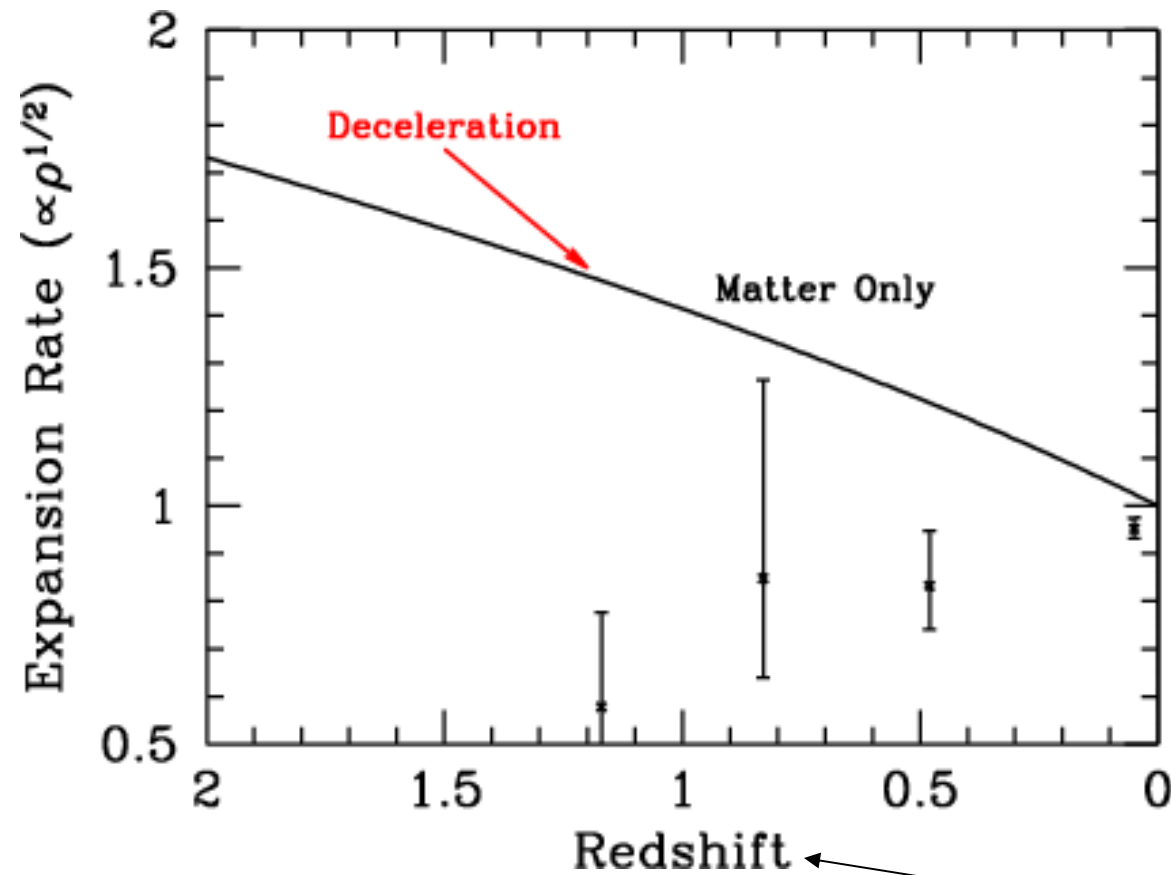
Deceleration unless *ρ+3P* is negative

Scott Dodelson

That gravity typically decelerates is intuitive

SLAC Summer Institute:
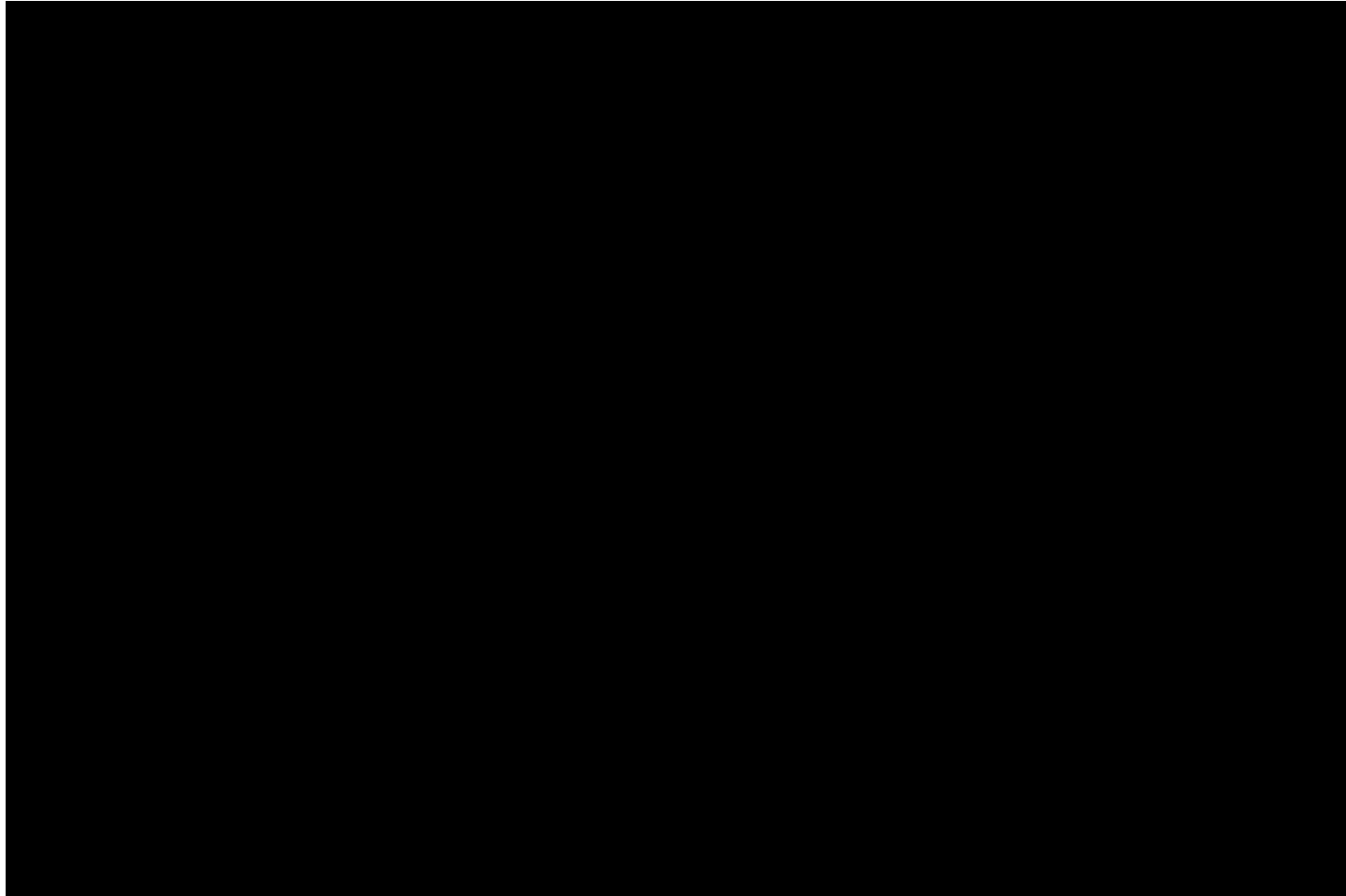Scott Dodelson

Ordinary Matter would induce deceleration

SLAC Summer Institute:
Scott Dodelson

$$z \equiv \frac{1}{a} - 1$$

So we must invoke ...

SLAC Summer Institute:
Scott Dodelson

Constituents of the Universe

Notation

Cosmologists measure densities in units of the *critical density*,

$$\rho_{cr} = 1.88 \, h^2 \times 10^{-29} \text{ g cm}^{-3}$$

where *h* defines the Hubble constant

$$H_0 = 100h \text{ km sec}^{-1} \text{ Mpc}^{-1}$$

The total density in the universe is equal to the critical density

Two popular ways of writing the baryon density are:

$$\Omega_b = \rho_b / \rho_{cr}$$

$$\omega_b = \Omega_b h^2$$

h=0.73+/-0.03

Constituents of the Universe Redux

- $\Omega_{\text{dark energy}} = 0.73 \pm 0.01$
- $\Omega_{\text{cdm}} = 0.23 \pm 0.01$
- $\Omega_b = 0.045 \pm 0.002$
- $0.001 < \Omega_v < 0.02$
- $\Omega_\gamma = (5.0 \pm 0.2) \times 10^{-5}$

SLAC Summer Institute:
Scott Dodelson

Photon Energy Density

The temperature of the cosmic microwave background (CMB) has been measured extremely well. Turn this into a measurement of the energy density.

SPECTRUM OF THE COSMIC MICROWAVE BACKGROUND

$T = 2.725 \pm 0.001°K$

$= 2.35 \times 10^{-4}$ eV

MAP990045
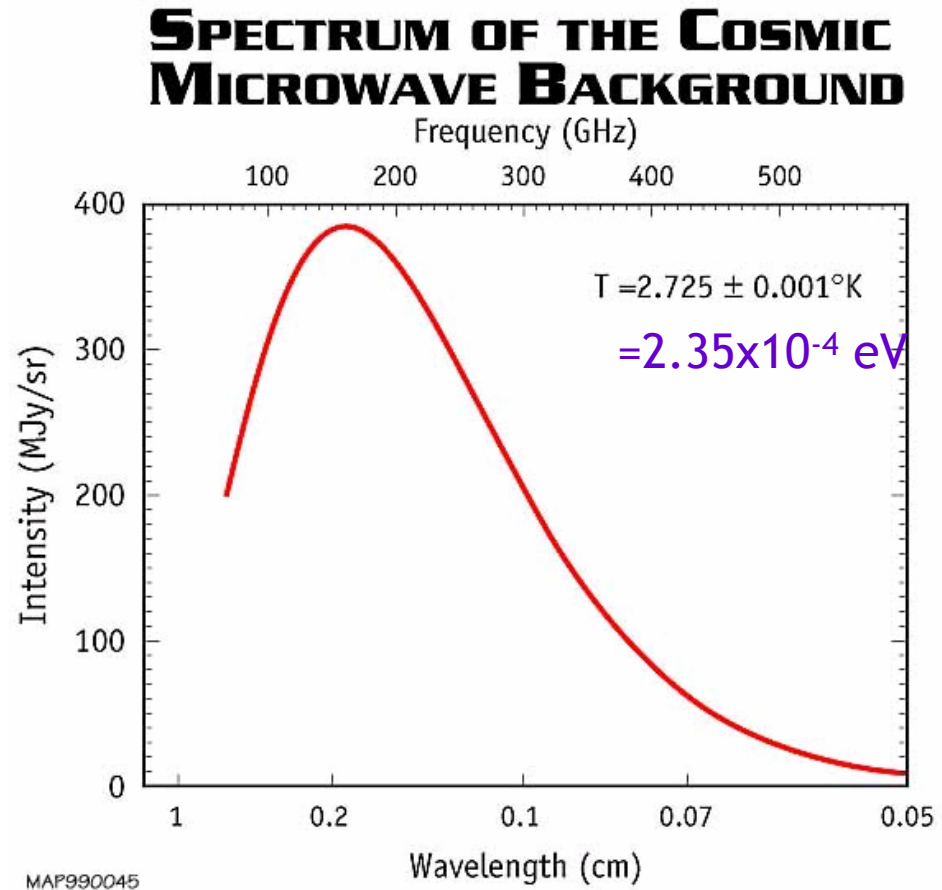
SLAC Summer Institute:
Scott Dodelson

Photon Energy Density

Energy density of a gas of bosons in equilibrium:

$$\rho = 2 \int \frac{d^3 p}{(2\pi)^3} E \frac{1}{e^{E/T} - 1}$$

Spin states

Bose-Einstein distribution

Sum over phase space

For massless particles, *E=p,* so

$$\rho = \frac{8\pi}{(2\pi)^3} \int_0^\infty \frac{dp \, p^3}{e^{p/T} - 1}$$

$$= \frac{T^4}{\pi^2} \int_0^\infty \frac{dx \, x^3}{e^x - 1} \qquad \pi^4/15 \qquad \Rightarrow \rho_\gamma = \frac{\pi^2}{15} T^4$$

SLAC Summer Institute:
Scott Dodelson

Evolution of the energy density

This is a snapshot of energy budget today. What did this look like in the past?

Evolution of energy density

Impose conservation of the energy-momentum tensor

$$\rho(a) = \rho_0 \exp\left\{3\int_a^1 \frac{da'}{a'}\left[1 + w(a')\right]\right\}$$

Energy density today

Equation of state:
$w=P/\rho$

SLAC Summer Institute:
Scott Dodelson

Example 1: Non-relativistic matter

$$\rho(a) = \rho_0 \exp\left\{3\int_a^1 \frac{da'}{a'}\left[1 + w(a')\right]\right\}$$

The pressure of non-relativistic matter is very small compared to the energy density (*T<<m*), so *w=0*.

$$\rho_m = \rho_{m,0}\, a^{-3}$$ Consistent with simple dilution by volume expansion

Example 2: Relativistic particles

$$\rho(a) = \rho_0 \exp\left\{3\int_a^1 \frac{da'}{a'}[1+w(a')]\right\}$$

To determine *w*, recall that the pressure is:

$$P = g\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{3E}f(E)$$

For relativistic particles, *E=p, so P=ρ/3, or w=1/3.*

$$\rho_r = \rho_{r,0}a^{-4}$$

❑ Volume dilution PLUS wavelength stretching
❑ T~1/a

Example 3: Cosmological Constant

$$\rho(a) = \rho_0 \exp\left\{ 3\int_a^1 \frac{da'}{a'} [1 + w(a')] \right\}$$

One possibility is that the *dark energy* is a cosmological constant with *w=-1*.

$$\rho_\Lambda = \rho_{\Lambda,0}$$ Empty space contains energy

Thermal History of the Universe

☐ The equation of state of dark energy is -1 to within 20%

☐ Structure begins to grow when the universe becomes matter dominated

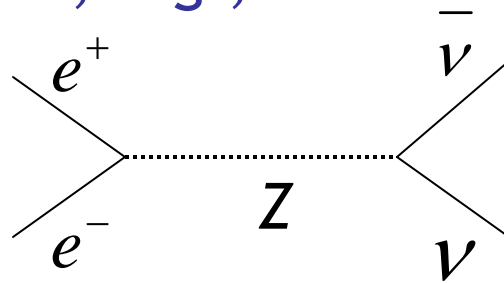☐ a_{EQ} is extremely well-determined; pins down matter density
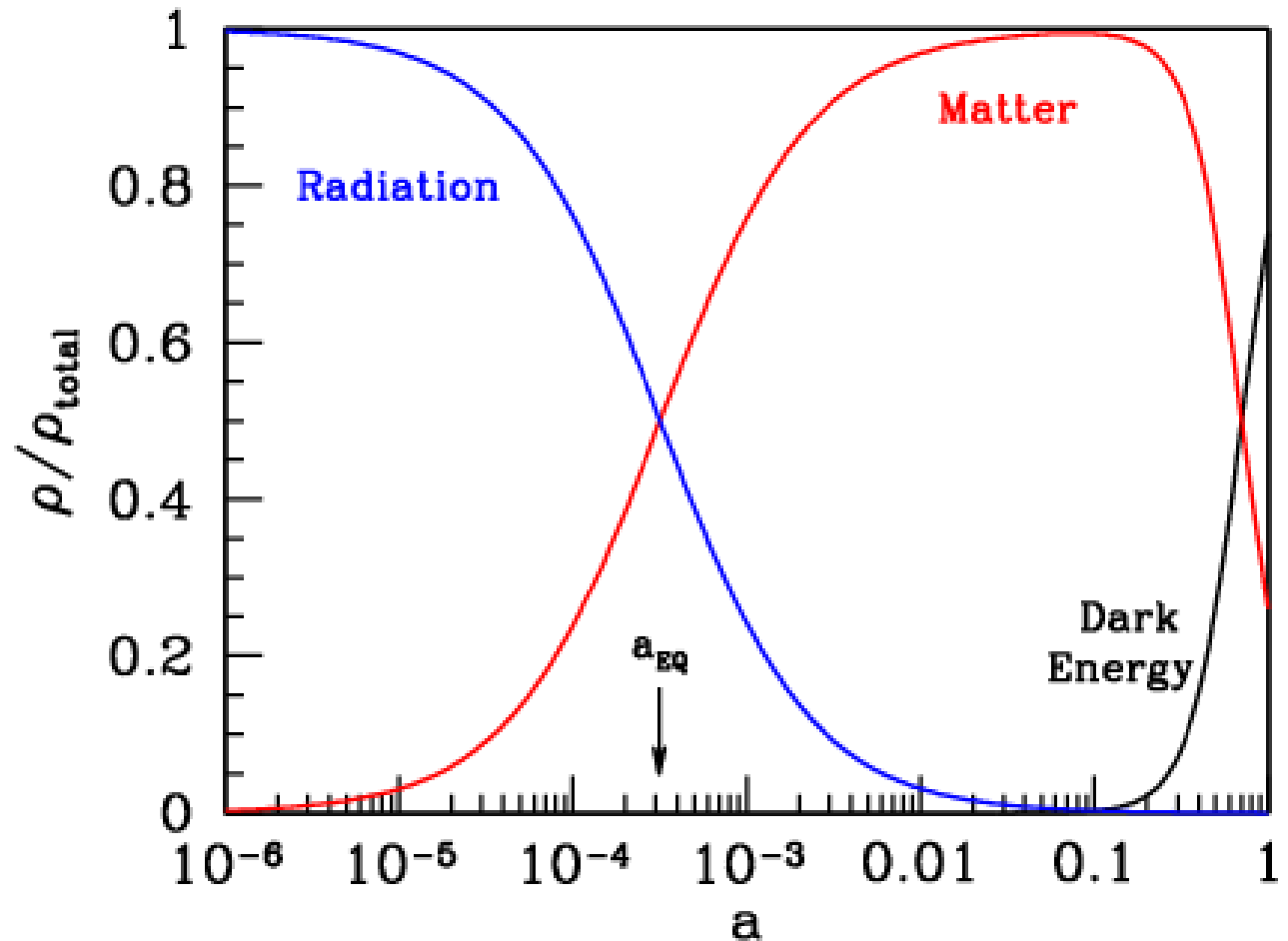
Neutrinos are produced in the early universe

Alpher, Herman, & Gamow 1953

Assume there are no neutrinos initially when the tepmerature is much larger than m_e. The rate for producing them via, e.g.,

$$e^+ \quad \bar{\nu}$$
$$Z$$
$$e^- \quad \nu$$

is of order

$$n\sigma \sim T^3 \frac{\alpha^2 T^2}{m_Z^{\,4}}$$

SLAC Summer Institute:
Scott Dodelson

Compare this to the expansion rate

$$H = \sqrt{\frac{8\pi G\rho}{3}} \sim \frac{T^2}{m_{Planck}}$$

$$\frac{\Gamma_\nu}{H} \sim \left(\frac{T}{5 MeV}\right)^3$$

The ratio of th... ...ate to the
expansion rate... on

$$\frac{\alpha^2 T^5}{m_Z{}^4}\frac{m_{Planck}}{T^2} \sim \frac{10^{-4}(1GeV)^3(10^{19}GeV)}{(100GeV)^4}\left(\frac{T}{1GeV}\right)^3$$

SLAC Summer Institute:
Scott Dodelson

Above 5 MeV, neutrinos are in equilibrium with the rest of the cosmic plasma

Fermi-Dirac distribution with temperature *T* equal to the electron/photon temperature.
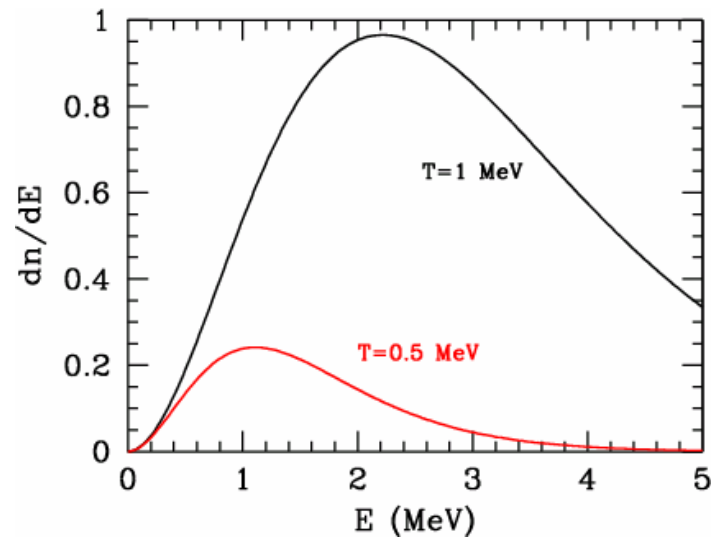
After the neutrinos decouple from the rest of the plasma (*T*<MeV), they still maintain Fermi-Dirac distribution with *T*~1/*a*.

SLAC Summer Institute:
Scott Dodelson

You might think neutrinos and photons have the same temperature today ...

But photons gained energy from electron/positron annihilation when $T \sim m_e$

Use entropy conservation: $sa^3 = constant$

Together with the fact that the neutrino temperature does scale as a^{-1}

$$\frac{s}{T_\nu^{\ 3}} = const$$

SLAC Summer Institute:
Scott Dodelson

Compute this ratio before and after electron/positron annihilation

photons electrons/positrons neutrinos

Initially:

$$\frac{s}{T_\nu^{\ 3}} = \frac{cT^3(2+(7/8)[4+6])}{T^3} = c\,\frac{43}{4}$$

Finally:

$$\frac{s}{T_\nu^{\ 3}} = \frac{c\left(2T_\gamma^{\ 3}+(7/8)6T_\nu^{\ 3}\right)}{T_\nu^{\ 3}} = c\left(2[T_\gamma/T_\nu]^3 + 21/4\right)$$

Equate the two to get:

$$\frac{T_\gamma}{T_\nu} = \left(\frac{11}{4}\right)^{1/3}$$

SLAC Summer Institute:
Scott Dodelson

Calibrate off the well-known photon temperature to get the prediction

$$n_{\bar{\nu}\nu} = 115 N_\nu \, cm^{-3}$$

Number of species of weakly interacting neutrinos

There are ~ a hundred quadrillion cosmic neutrinos (flux of 115x3c = 10^{13} cm^{-2} sec^{-1}) passing through this screen (~10^4 cm^2) every second.

SLAC Summer Institute:
Scott Dodelson

Unfortunately, we can't detect these because the cross-section is too small
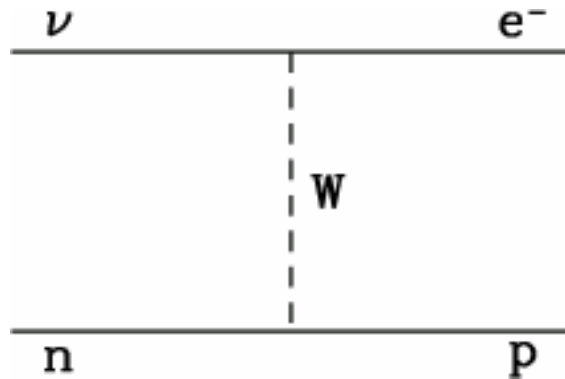
$$\sigma \sim 10^{-63} \, cm^2 \left(\frac{E_\nu}{10^{-4} \, eV} \right)^2$$

So the detection rate is of order

$$\Gamma \sim 10^{-50} \times 10^{27} \left(\frac{M_{detector}}{1 \, kg} \right) \times 3 \times 10^7 \, yr^{-1}$$
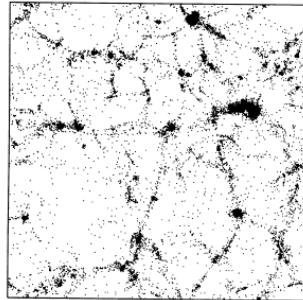
But ... these neutrinos do contribute to the energy density of the universe.
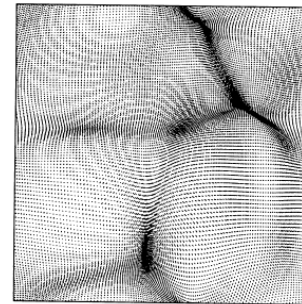
The energy density of massive neutrinos is:

$$\Omega_\nu = \frac{\rho_\nu}{\rho_{cr}} = \frac{n_\nu \sum m_\nu}{\rho_{cr}} = \frac{0.01}{h^2}\left(\frac{\sum m_\nu}{1eV}\right)$$

This could be as large as 10% of the total matter, so affects large scale structure.
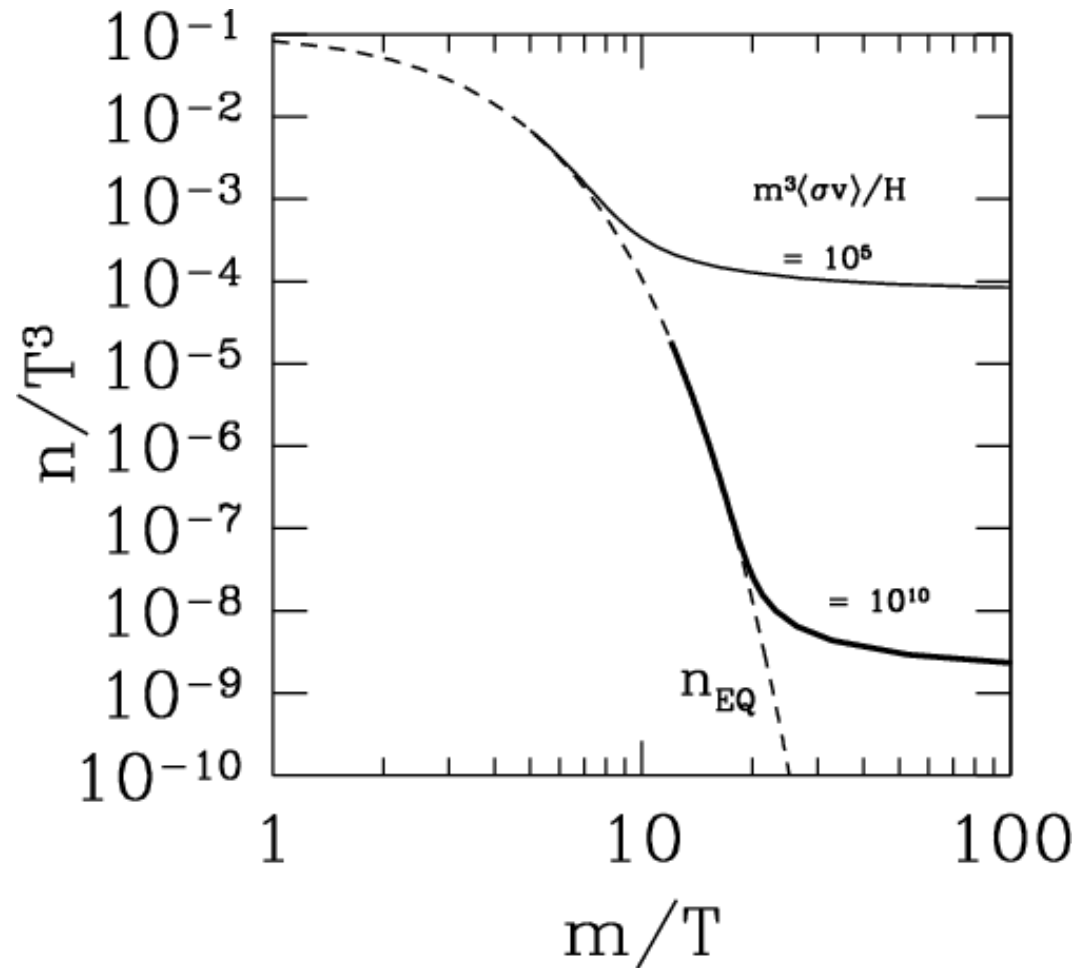

Cold Dark Matter

Cold+ Hot Dark Matter

SLAC Summer Institute:
Scott Dodelson

# Weakly interacting stable massive particles could be the dark matter

❑ All particles start with density roughly equal to photon density
❑ When temperature drops beneath mass, annihilations deplete density …
❑ until freeze-out

SLAC Summer Institute:
Scott Dodelson

Annihilation rate compared to expansion rate

$$\Gamma \sim n_X \sigma v \sim (mT)^{3/2} e^{-m/T} \sigma \left(\frac{T}{m}\right)^{1/2}$$

$$H \sim \frac{T^2}{m_{Planck}}$$

SLAC Summer Institute:
Scott Dodelson
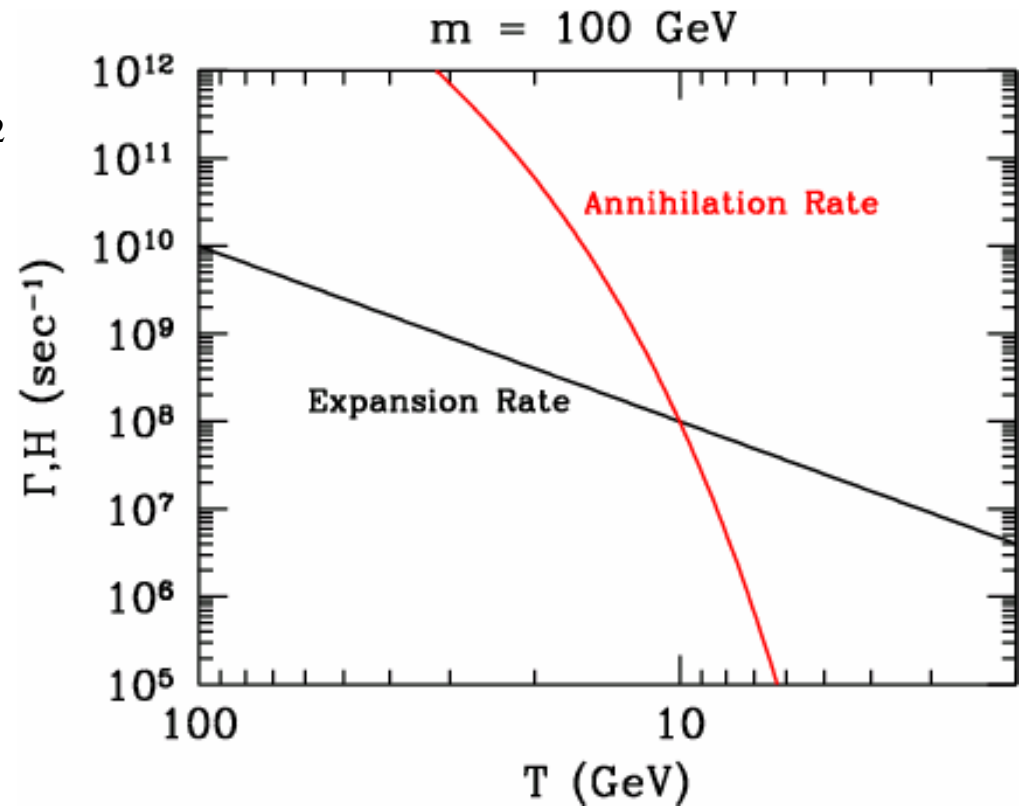
Freeze-out takes place when the two rates are equal

$$e^{-m/T_{fo}} \sigma m \sim m_{Planck}^{-1}$$

After freeze-out, WIMP number scales as photon number density

$$\frac{n_X}{n_\gamma} \sim \left(\frac{m}{T_{fo}}\right)^{3/2} e^{-m/T_{fo}} \sim \frac{1}{m\sigma m_{Planck}}\left(\frac{m}{T_{fo}}\right)^{3/2}$$

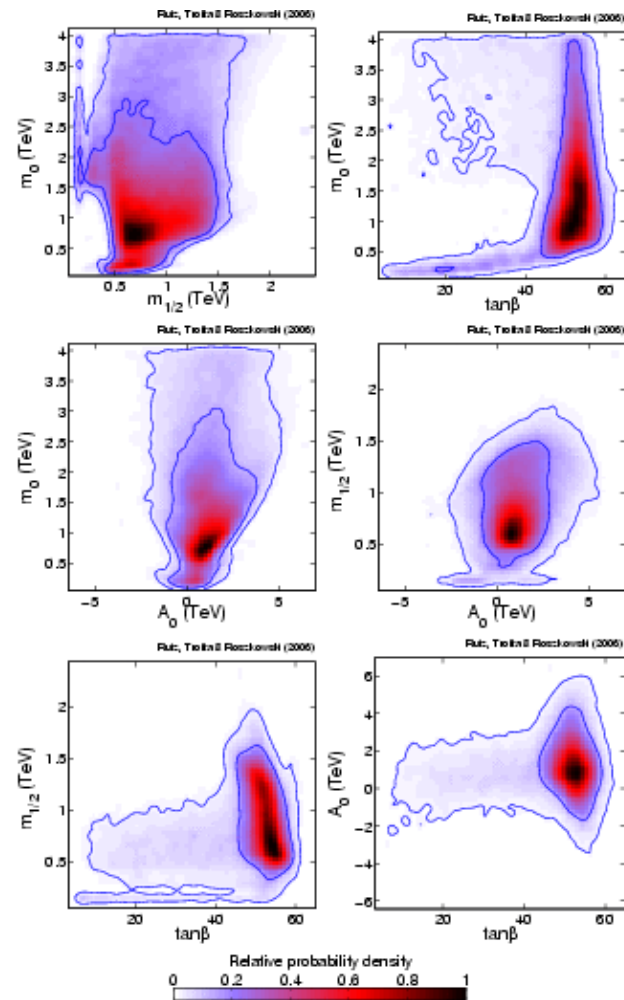# Multiply by mass to estimate the contribution to the energy density today

$$\Omega_x \sim \frac{T_0^{\,3}}{\rho_{cr}\sigma m_{Planck}} \left(\frac{m}{T_{fo}}\right)^{3/2}$$

Plug in numbers

$$\Omega_x = 0.3 h^{-2} \left(\frac{m}{10 T_{fo}}\right)^{3/2} \frac{10^{-37}\, cm^2}{\sigma}$$

Mild mass dependence, but mostly depends on cross-section only.

Easy to get correct dark matter abundance in suppersymmetric models

De Austri, Trotta, & Roszkowski 2006

SLAC Summer Institute:
Scott Dodelson

Final Slide

If you want to get your hands dirty check out ...

```
http://www.physto.se/~edsjo/darksusy/
```

Meet me at the bar tonight if you have a good idea about ...

```
Models which simultaneously produce dark matter and
dark energy
```