

# Adler Function in the Analytic Approach to QCD

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**Abstract.** The low energy behavior of the Adler function  $D(Q^2)$  is studied by employing recently derived integral representation for the latter. This representation embodies the nonperturbative constraints on  $D(Q^2)$ , in particular, it retains the effects due to the nonvanishing pion mass. The Adler function is calculated within the developed approach by making use of its perturbative approximation as the only additional input. The obtained result agrees with the experimental prediction for the Adler function in the entire energy range and possesses remarkable stability with respect to the higher loop corrections.

## 1 Introduction

The Adler function  $D(Q^2)$  [1] plays a crucial role in various issues of elementary particle physics. Specifically, theoretical description of such strong interaction processes as  $e^+e^-$  annihilation into hadrons and inclusive  $\tau$  lepton decay is inherently based on this function. Moreover, Adler function is essential for confronting precise experimental measurements of some electroweak observables with their theoretical predictions. In turn, the latter represents a decisive test of the validity of the Standard Model and imposes strict restrictions on possible “new physics” beyond it.

The perturbation theory still remains the only reliable tool for calculating the Adler function at high energies. Namely, in the asymptotic ultraviolet region  $D(Q^2)$  can be approximated by the power series in the strong running coupling  $\alpha_s(Q^2)$ . However, spurious singularities of the latter, being the artifacts of perturbative calculations, invalidate this expansion at low energies. In turn, this significantly complicates theoretical description of the low-energy experimental data, and eventually forces one to resort to various nonperturbative approaches.

An important source of the nonperturbative information about the hadron dynamics at low energies is provided by the relevant dispersion relations. The latter, being based on the general principles of the local Quantum Field Theory, supply one with the definite analytic properties in a kinematic variable of a physical quantity at hand. The idea of employing this information together with perturbation theory and renormalization group (RG) method forms the underlying concept of the so-called

“analytic approach” to Quantum Chromodynamics (QCD) [2]. Some of the main advantages of this approach are the absence of unphysical singularities and a fairly good higher-loop and scheme stability of outgoing results. The analytic approach has been successfully employed in studies of the strong running coupling [2, 3], perturbative series for QCD observables (see paper [4] and references therein), meson spectrum [5], chiral symmetry breaking [6], and electromagnetic pion form factor [7].

The primary objective of this paper is to study the infrared behavior of the Adler function by employing recently obtained integral representation [8]. The latter has been derived in a general framework of the analytic approach to QCD, the effects due to the pion mass being retained. It is also of a particular interest to examine the stability of the calculated Adler function with respect to the higher loop corrections.

The layout of the paper is as follows. In Section 2 the dispersion relation for the Adler function and its interrelation with the  $R$ -ratio of electron-positron annihilation into hadrons are overviewed. In Section 3 a novel integral representation for  $D(Q^2)$  is discussed and the calculation of the Adler function within the developed approach is presented. In Section 4 the obtained results are summarized.

## 2 The Adler function

The Adler function  $D(Q^2)$  [1] naturally appears in the theoretical description of the process of electron-positron annihilation into hadrons. Specifically, the measurable ratio of two cross-sections is proportional to the discontinuity of the hadronic vacuum polarization function  $\Pi(q^2)$  across the physical cut:

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons}; s)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-; s)} = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0_+} [\Pi(s - i\varepsilon) - \Pi(s + i\varepsilon)] \quad (1)$$

with  $s = q^2 > 0$  being the center-of-mass energy squared. It is worth noting here that  $R(s)$  vanishes identically for the energies below the two-pion threshold due to the kinematic restrictions, see also Ref. [9]. The mathematical implementation of the latter condition consists in the fact that  $\Pi(q^2)$  has the only cut  $q^2 \geq 4m_\pi^2$  along the positive semiaxis of real  $q^2$  and satisfies the once-subtracted dispersion relation [1, 9]

$$\Pi(q^2) = \Pi(q_0^2) - (q^2 - q_0^2) \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s - q^2)(s - q_0^2)} ds, \quad (2)$$

where  $m_\pi \simeq 135$  MeV is the mass of the  $\pi^0$  meson.

For practical purposes it proves to be convenient to deal with the so-called Adler function [1], which is defined as

$$D(Q^2) = \frac{d\Pi(-Q^2)}{d \ln Q^2} \quad (3)$$

and, therefore, does not depend on the choice of subtraction point  $q_0^2$  in the dispersion relation (2). In Eq. (3)  $Q^2 = -q^2 \geq 0$  denotes a spacelike momentum. In addition to the relevance to the strong interaction processes mentioned in the Introduction, the Adler function (3) plays a crucial role for the congruous analysis of hadron dynamics in spacelike and timelike domains. In particular, the required link between the experimentally measurable  $R$ -ratio (1) and theoretically computable Adler function (3) is represented by the dispersion relation [1]

$$D(Q^2) = Q^2 \int_{4m_\pi^2}^{\infty} \frac{R(s)}{(s + Q^2)^2} ds. \quad (4)$$

At the same time, one is also able to continue an explicit theoretical expression for the Adler function (3) into timelike domain by making use of the inverse relation

$$R(s) = \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_{s+i\varepsilon}^{s-i\varepsilon} D(-\zeta) \frac{d\zeta}{\zeta}, \quad (5)$$

where the integration contour lies in the region of analyticity of the integrand [10].

Although there are no direct measurements of the Adler function (3), it can be restored by employing the data on  $R$ -ratio (1). Specifically, in the integrand of the dispersion relation (4) one usually approximates  $R(s)$  by its experimental measurements at low and intermediate energies, and by its theoretical prediction at high energies. For the energies below the mass of the  $\tau$  lepton the  $R$ -ratio (which possesses rather large systematic uncertainties in the infrared domain) can be substituted (up to the isospin breaking effects) by precise spectral function of the vector current  $R_{\text{exp}}^V(s)$  [11] extracted from the hadronic  $\tau$  decays. Thus, the  $R$ -ratio in Eq. (4) can be parameterized by  $R(s) = R_{\text{exp}}^V(s) \theta(s_0 - s) + R_{\text{theor}}^{(3)}(s) \theta(s - s_0)$ , where  $\theta(x)$  is the Heaviside step-function,  $R_{\text{theor}}^{(3)}(s)$  stands for the theoretical prediction of  $R(s)$  at three-loop level,  $n_f = 3$  is assumed, and  $s_0 = 2.1 \text{ GeV}^2$ . The overall factor  $N_c \sum_f Q_f^2$  is omitted throughout, where  $N_c = 3$  is the number of colors and  $Q_f$  denotes the charge of the quark of the  $f$ th flavor. Computed in this way experimental prediction for the Adler function is presented in Fig. 1 by shaded band, see Ref. [12] for the details.

As it has been mentioned above, the high-energy behavior of the Adler function (3) can be approximated by the power series in the strong running coupling  $\alpha_s(Q^2)$  in the framework of the perturbative approach. Specifically, at the  $\ell$ -loop level

$$D_{\text{pert}}^{(\ell)}(Q^2) = 1 + \sum_{j=1}^{\ell} d_j [\alpha_s^{(\ell)}(Q^2)]^j, \quad Q^2 \rightarrow \infty, \quad (6)$$

where  $\alpha_s^{(\ell)}(Q^2)$  is the  $\ell$ -loop perturbative invariant charge. The expansion coefficients  $d_j$  are known up to the three-loop level, in particular,  $d_1 = 1/\pi$ . The numerical estimation [13] of the uncalculated yet four-loop coefficient  $d_4$  is adopted in what

follows. However, as one may infer from Fig. 1, the perturbative approximation (6) is reliable for the energies  $Q \gtrsim 1.5 \text{ GeV}$  only. Besides, expansion (6) is incompatible with the dispersion relation (4) due to unphysical singularities of the running coupling  $\alpha_s(Q^2)$  in the infrared domain. The latter also causes certain difficulties for processing the low-energy experimental data.

### 3 Novel integral representation for $D(Q^2)$

The aforementioned integral relations (4) and (5) express the Adler function (3) and  $R$ -ratio (1) in terms of each other. For practical purposes it proves to be convenient to express both these quantities in terms of the common spectral function. This objective can be achieved by employing Eqs. (4) and (5), the parton model prediction  $R_0(s) = \theta(s - 4m_\pi^2)$  [9], and the fact that the strong correction to the Adler function vanishes in the asymptotic ultraviolet limit  $Q^2 \rightarrow \infty$ . Eventually one arrives at (see Refs. [8, 12] for the details)

$$D(Q^2) = \frac{Q^2}{Q^2 + 4m_\pi^2} \left[ 1 + \int_{4m_\pi^2}^{\infty} \rho_D(\sigma) \frac{\sigma - 4m_\pi^2}{\sigma + Q^2} \frac{d\sigma}{\sigma} \right], \quad (7)$$

$$R(s) = \theta(s - 4m_\pi^2) \left[ 1 + \int_s^{\infty} \rho_D(\sigma) \frac{d\sigma}{\sigma} \right]. \quad (8)$$

Here the spectral function  $\rho_D(\sigma)$  can be determined either as the discontinuity of the theoretical expression for the Adler function across the physical cut  $\rho_D(\sigma) = \text{Im} D(-\sigma + i0_+)/\pi$  or as the numerical derivative of the experimental data on  $R$ -ratio  $\rho_D(\sigma) = -dR(\sigma)/d \ln \sigma$ . It is worth noting that Eq. (7) embodies the nonperturbative constraints on Adler function arising from the dispersion relation (4). Besides, Eq. (8) by construction properly accounts for the effects due to the analytic continuation of spacelike theoretical results into timelike domain.

In order to compute the Adler function in the framework of the approach at hand, one first has to determine the spectral function  $\rho_D(\sigma)$ . In what follows we restrict ourselves to the study of only perturbative contributions to the latter, namely

$$\rho_{\text{pert}}^{(\ell)}(\sigma) = \text{Im} D_{\text{pert}}^{(\ell)}(-\sigma + i0_+)/\pi, \quad (9)$$

where  $D_{\text{pert}}^{(\ell)}(Q^2)$  is given by Eq. (6). It is worthwhile to mention also that in the limit of massless pion ( $m_\pi = 0$ ) the obtained expressions for the Adler function (7) and  $R$ -ratio (8) become identical to those of the so-called Analytic perturbation theory (APT) [4], the definition (9) being assumed.

For the illustration of the significance of the pion mass within the approach at hand, it is worth presenting the Adler function (7) computed by making use of the spectral function (9) for both, massless and massive cases. The obtained results

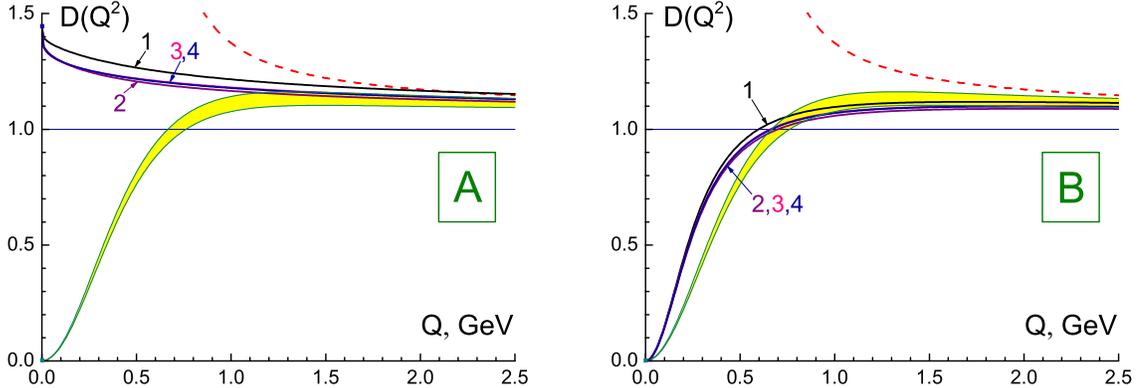


Figure 1: The Adler function (7) (solid curves) calculated by making use of the spectral function (9) in the massless (A) and massive (B) cases. Numerical labels correspond to the loop level considered. The experimental prediction for  $D(Q^2)$  is shown by the shaded band, whereas its perturbative approximation is denoted by the dashed curve.

are presented in Fig. 1 by solid curves. In the case of the massless pion (which is identical to the APT [4]), one arrives at the result, which is free of infrared unphysical singularities, but fails to describe the Adler function for the energies  $Q \lesssim 1.0$  GeV, see Fig. 1 A. It is worth noting here that in the framework of the massless APT the infrared behavior of  $D(Q^2)$  can be further improved by additionally invoking into consideration relativistic quark mass threshold resummation [14] or vector meson dominance assumption [15].

At the same time, as one may infer from Fig. 1 B, for the case of the nonvanishing pion mass the representation (7) is capable of providing an output for the Adler function, which agrees with its experimental prediction in the entire energy range [8]. Moreover, the Adler function (7) is remarkably stable with respect to the higher loop corrections. Namely, the relative difference between the  $\ell$ -loop and  $(\ell + 1)$ -loop expressions for  $D(Q^2)$  (7) is less than 4.9%, 1.5%, and 0.3% for  $\ell = 1$ ,  $\ell = 2$ , and  $\ell = 3$ , respectively, for  $0 \leq Q^2 < \infty$ , see Ref. [12] for the details. It is worthwhile to mention also that the obtained results are supported by recent studies of meson spectrum in the framework of the Bethe–Salpeter formalism [5].

## 4 Summary

The infrared behavior of the Adler function is examined by employing representation (7), which accounts for the pion mass effects. The approach at hand possesses all the appealing features of the massless APT [4]. Namely, it supplies a self-consistent analysis of spacelike and timelike experimental data; additional parameters are not introduced into the theory; the outcoming results possess no unphysical singulari-

ties and display enhanced higher loop stability. In addition, the developed approach provides a reliable description of the Adler function in the entire energy range.

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