Twin Paradox and Causality

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1 Introduction

The Twin Paradox is one of the longest standing scientific controversy of the twentieth century physics [1], and a one that pure speculations revealed unable to fully elucidate indeed. One often encounters the opinion that the problem is even trivial : the element of proper-time, $\mathrm{d}\tau$, is not an exact differential. As a matter of fact, the result of its integration along a given timelike continuous worldline depends on that worldline, and two twice intersecting worldlines will accordingly display the non-trivial differential ageing phenomenon, known under the spell of "clock" or "Langevin's twin paradox". It should appear clear, though, that the non-exact character of $\mathrm{d}\tau$ is nothing but the differential version of the non-trivial differential ageing phenomenon, certainly not its explanation. It is thus appropriate to look for the mechanism/principle that stands at the origin of such a remarquable phenomenon. In order to catch it, it is also appropriate to look for an explanation in the simplest context where the non-trivial differential ageing phenomenon is established, that is over a four dimensional Minkowski spacetime orientable manifold, M, endowed with a time arrow.

2 Matter of the Paradox

As well known, the matter of the paradox, is in its reciprocity. As (very roughly) illustrated by such a familiar formula as

$$\mathrm{d}t' = \mathrm{d}t/\sqrt{1 - v^2/c^2} \tag{1}$$

that can be expressed in either of any two inertial frames of reference with relative velocity v, differential ageing should manifest itself in exactly the same way for either twin : each twin has to be found younger/older than the other. Such a situation is of course impossible, but then, it remains to understand how an asymmetric ageing can possibly originate.

It is worth noticing that this difficulty was made the more acute as the Einstein's 1905 seminal paper explicitly referred to identical clocks and rods in all inertial frames of reference. This can certainly explain, at least partly, why major physicists and philosophers (H. Dingle, H. Bergson, etc..) [2] have long thought of relativistic effects in terms of appearances!

And indeed, a part of truth was contained in their opinion, provided one is careful enough to add that these relativistic appearances, or parallaxes effects, are so symmetric and real as are their euclidean 3-dimensional homologous ! .. and also, that they are not *proper* or Lorentz invariant.

A long known and textbook example is provided by π- meson beams. Their averaged lifetime is on the order of $10^{-8} sec$. Now, when brought to a speed close to c, the π- mesons can travel over distances of $300m$, corresponding to some $100\times$ their lifetimes. This is all real, measurable and persistent an effect (it is experimentally used!). But this has to do with the real measure of a real parallax effect, whereas the invariant measure of *life-duration*, the so-called lifetime, is the same in either frame of reference, beam or laboratory referred. In this case there is neither any untenable symmetry nor any mysterious asymmetry, nor any paradox, but solely a parallax effect, symmetric (the remaining sections should make appear the limits of this standard example, though!).

Since relativistic theories do not really discriminate spatial from temporal coordinates it is worth making contact with another related and remarquable phenomenon, known as Thomas-Wigner rotations.

At $d \geq 2$ spatial dimensions, two successive boosts may have non-collinear directions : $K_0 \to K(\vec{v}) \to K'(\vec{v'})$ with $\vec{v'}$ and $\vec{v}$ non-collinear, and the $K's$ standing for inertial reference frames. For example, with $\vec{v}\ //\ \vec{Ox}$ of K_0 and $\vec{v'}\ //\ \vec{Oy}$ of $K(\vec{v})$, the K_0 *-vs-* $K'(\vec{v'})$ relative velocity, expressed in $K'(\vec{v'})$ and in K_0 points to different directions, with an angular difference of

$$\mathrm{d}\theta = \frac{|\mathrm{d}\vec{v}_{Oy}|}{|\vec{v}_{Ox}|}\left(1 - \sqrt{1 - \frac{\vec{v}_{Ox}^2}{c^2}}\right) \tag{2}$$

From K_0, $K'(\vec{v'})$ is no longer perceived the same, still opposite way as $K'(\vec{v'})$ is perceived from K_0 : seen from K_0, the spatial axes of $K'(\vec{v'})$ have rotated an angle $\mathrm{d}\theta$. The elements of proper-orientation, $\mathrm{d}\theta$ and proper-time, $\mathrm{d}\tau$, define two invariant non-exact differential 1-forms which, in certain circumstances, are simply proportional, $\mathrm{d}\theta = \omega_{th}\, \mathrm{d}\tau$. As soon as discovered by L.H. Thomas in 1926, Thomas-Wigner rotations have resolved a discrepancy of a factor 2 between theoretical and experimental data (and not corrections of order of v^2/c^2 !) in a fine structure alcali doublet calculation. Today, they perfectly integrate the GPS high technology and have been noticed in new quantum mechanical phenomena. Indeed, as pure kinematical effects, Thomas-Wigner rotations are universal.

3 The proper-time line functional

The proper-time lapses are defined by means of a stratagem first proposed by Einstein [3] and reformulated the same by other authors, like A. Schild. It consists in "*..imagining an infinity of inertial frames moving uniformly, relative to the laboratory frame, one of which instantaneously matching the velocity of the considered system, a twin, a clock, a particle..*". One writes

$$\Delta(\mathcal{C};\mathcal{R}) = \int \mathrm{d}\tau_{\mathcal{C}/\mathcal{R}}(p)\ , \qquad \forall p \in \mathcal{C} \tag{3}$$

where $\mathcal{C}$ is a continuous timelike worldline of M, whereas $\mathcal{R}$ stands for some inertial "laboratory" frame of reference. $\mathcal{R}$ is of course arbitrary, but conveniently choosen in practice. It is important to note that the proper-time line functional so introduced is a mathematically well defined quantity for any worldline $\mathcal{C}$. In particular, it is consistent, irrespective of the global spacetime manifold geometry. Gravitation/curvature, if any, must show up as an emergent or reconstructed effect, once admitted the equivalence principle if gravitation is accounted for by general relativity [4]. Formally, the stratagem finds a natural setting within the framework of differential geometry, in particular with the notions of tangent and cotangent fiber-bundles over a manifold.

Given $\mathcal{C}'$, another worldline with two points in common with $\mathcal{C}$, say 0 and ι, the functional of Eq.(3) will produce a differential ageing result of

$$\Delta(\mathcal{C},\mathcal{C}';\mathcal{R}) = \int_0^\iota \mathrm{d}\tau_{\mathcal{C}/\mathcal{R}}(p) - \int_0^\iota \mathrm{d}\tau_{\mathcal{C}'/\mathcal{R}}(p') \equiv \delta T \tag{4}$$

As we will see shortly, this special relativity result is non vanishing, because of the path 4-velocity and path 4-acceleration dependences which differentiate $\mathcal{C}$ from $\mathcal{C}'$ in the general case.

It will account for any acceptable twin paradox and its associated asymmetry, in the following sense. Exchanging $\mathcal{C}'$ and $\mathcal{C}$ just amounts, as it should, to change δT into $-\delta T$. This is in contradistinction to the symmetric parallax effect mentionned above, where such an exchange, leaving the (non-Lorentz invariant) measure of *life-duration* unaffected, had motivated the original idea of a paradox. The twin paradox must accordingly be re-qualified in terms of path-functional dependence of proper-time lapses.

4 Twin paradox and causality : a geometrical synthesis

The remarquable connection which on a local scale at least (extending over more than 40 orders of magnitude, though! [5]) relates the non-trivial differential ageing phenomenon to the principle of causality is worth exploring. A few geometrical tools are in order so as to realize that the whole paradox is but a selected aspect of a basic issue of differential geometry : how are connected to each others, the different tangent spaces to a given manifold ?

Let M be the $4D$ orientable and affine Minkowski spacetime manifold. For all $p \in M$, the tangent space to M at p, $\mathcal{T}_p M \equiv I\!\!M$ is the vectorial Minkowski spacetime. A partial ordering is assumed on M. Writing $x < y$, if an event at x can influence another event at y, we write

$$x < y \Longleftrightarrow Q_0(y-x) \equiv (y_0 - x_0)^2 - (\vec{y} - \vec{x})^2 > 0 \ , \qquad \& \quad y_0 - x_0 > 0 \tag{5}$$

This partial ordering relation on M just expresses a causality attached to both a finite velocity limit for energy/information transfers, and the existence of a time arrow on M. This *causal ordering* (written with the help of an implicit space and time resolution of M!) makes it possible to tell *histories* out of some points of M. Now, at any point and instant of his trip, a traveller twin is endowed with an instantaneous 4-velocity which decides of another space and time *canonical* resolution of M. The question is therefore : how are histories transformed along the different space and time resolutions of the traveller ?

An important result is the following. Let f be a one-to-one mapping (not necessarily linear and/or continuous!) of M into itself, satisfying

$$x < y \Longrightarrow f(x) < f(y) \ \ \& \ \ f^{-1}(x) < f^{-1}(y) \ , \qquad \forall x, y \in M \tag{6}$$

f is said to be a *causal automorphism* of M. The set of causal automorphisms form the "causality group of M", $\mathcal{G}$. Then, at $d = 3$ spatial dimensions a theorem [6] states that the Minkowski space causality group $\mathcal{G}$, coïncides with the inhomogeneous orthochronous Lorentz group, $\mathcal{L}_+^\uparrow$, augmented with dilatations of M (multiplication by a scalar).

Though the traveller successive inertial spacetimes are Minkowskian and isomorphic, they are different *physical spaces* attached, each, to a given 4-velocity vector u, at a given point p. It matters to know, thus, how are these different spacetimes connected to each others, that is, in which relations stand their different space and time coordinate maps. If $I\!\!M_u$ and $I\!\!M_{u'}$ are two such tangent spaces to M at p and p', a most *natural* connection is provided by the boost without rotation from u to u'

$$I\!\!B(u', u) = 1 - \frac{(u' + u) \otimes (u' + u)}{1 + u' \cdot u} + 2u' \otimes u \tag{7}$$

with metric, $g_0 = diag\ (+1, -1, -1, -1)$, and where the symbol $u' \otimes u$ stands for the linear mapping of $I\!\!M$ into itself : $\forall x, u, u' \in I\!\!M \ , \quad u' \otimes u \ : x \longmapsto u'(u \cdot x)$. The form (7) is a geometrical, coordinate-independent expression. Though *natural*, (7) does not provide a *canonical* connection between $I\!\!M_u$ and $I\!\!M_{u'}$ spaces because it is based on a particular convention of synchronization, namely, the Einstein convention. However it has been argued recently [7] that the Einstein convention should be regarded as inherent to the special relativity theory itself. This connection has several interesting properties. For example the composition of 3 successive boosts without rotation is not a boost without rotation.. but a rotation without boost

$$\mathbb{B}(u,u'') \circ \mathbb{B}(u'',u') \circ \mathbb{B}(u',u) = \mathbb{R}_u(u',u'') \tag{8}$$

Passing from the three vectors, u, u' and u'', to the two relative velocities, $\vec{v}_{Ox}$ and $\mathrm{d}\vec{v}_{Oy}$, the above expression just reproduces the Thomas-Wigner Rotation of Eq.(2). The $\mathbb{B}(u',u)$ also preserve the scalar product, the orientation, the time-arrow and causal ordering of M.

The travelling twin history is accounted for by a twice-differentiable mapping $r(s)$ of an interval $[s_i, s_f]$ into M, whose range is the twin's worldline $\mathcal{C} = \{r(s) \mid s \in [s_i, s_f]\}$. The *absolute* [5] Minkowski spacetime M is reported to the stayed home twin space and time resolution, of unit 4-velocity $\dot{r}(s_i)$. One can prove that the traveller co-moving resolutions, of instantaneous 4-velocity $\dot{r}(s)$, preserve the causal ordering relations (6) of $M_r(s_i)$. That is, between $M_r(s_i)$ and $M_r(s)$, for all s, there exists a causal isomorphism $\varphi_{s,s_i} : M_{r(s_i)} \to M_{r(s)}$ mapping the (O- and $\dot{r}(s_i)$ -referred) partial ordering (6) of $M_{r(s_i)}$, onto the ($r(s)$- and $\dot{r}(s)$ -referred) partial ordering (6) of $M_{r(s)}$,

$$\forall q \in M_{r(s_i)}\ , \qquad \varphi_{s,s_i}(q) = r(s) + \mathbb{L}_{s,s_i}\,(q - r(s_i))\ \in M_{r(s)} \tag{9}$$

where (in view of the previous theorem) $\mathbb{L}_{s,s_i}$ is an element of $\mathcal{L}_+^{\uparrow}$, the homogeneous orthochronous Lorentz group, or a dilatation, which, both, act on vectors of $\mathbb{M}_{\dot{r}(s_i)}(\equiv \mathbb{M})$.

At base-point $r(s_i) = O$, one may attach a *tetrad* of orthonormal basis vectors spanning the vectorial Minkowski space $\mathbb{M}_{\dot{r}(s_i)} = \mathbb{M}$, the set $\{e_0(s_i) \equiv \dot{r}(s_i),\ e_j(s_i)\ ;\ j = 1,2,3\}$. Then, the tetrad $\{\varphi_{s,s_i*}\, e_0(s_i),\ \varphi_{s,s_i*}\, e_j(s_i)\ ;\ j =\ 1,2,3\}$ will span the vectorial Minkowski space $\mathbb{M}_{\dot{r}(s)}$, where φ_{s,s_i*} is the differential of the application φ_{s,s_i} at $r(s_i)$. That is, $\varphi_{s,s_i*} = \mathbb{L}_{s,s_i}$. The Einstein stratagem requires that at any proper instant s of the travelling twin worldline, the co-moving tetrad complies with the identity $e_0(s) \equiv \dot{r}(s)$, whereas the other three spacelike vectors, $e_j(s), j = 1,2,3$, are formally thought of as *gyrovectors* [8] and can be physically realized as gyroscopes. The tetrad is said to be *Fermi-Walker transported* along $\mathcal{C}$, and the tangent mapping φ_{s,s_i*} results of a composition of an infinite series of infinitesimal boosts ranging from proper-instants s_i to s, along $\mathcal{C}$: $\mathrm{d}\mathbb{L}_{s,s_i} = \lim_{\ \mathrm{d}s\to 0}\ \mathbb{B}\,(\dot{r}(s), \dot{r}(s-\mathrm{d}s)) \circ\ ..\ \circ \mathbb{B}\,(\dot{r}(s_i+\mathrm{d}s), \dot{r}(s_i))$. One has

$$\mathrm{d}\mathbb{L}_{s,s_i} = T \prod_{s'=s_i}^{s} \circ\ \left(\ \mathbb{I} + \mathrm{d}s'\ (\dot{r}(s') \wedge \ddot{r}(s'))\ \right) \tag{10}$$

where T stands for a prescription of time ordering, $\dot{r}(s') \wedge \ddot{r}(s')$ is the Fermi-Walker operator, and the symbol $a \wedge b$ is introduced as a shortand notation for the antisymmetric product: $\forall z \in I\!\!M, \quad (\dot{r} \wedge \ddot{r})z = \dot{r}\,(\ddot{r} \cdot z) - \ddot{r}\,(\dot{r} \cdot z)$. Taken at different times, Fermi-Walker operators do not commute, so that the following integral representation of $I\!\!L_{s,s_i}$ may be proposed

$$I\!\!L_{s,s_i} = T \exp \int_{s_i}^{s} \mathrm{d}s' \ \dot{r}(s') \wedge \ddot{r}(s') \tag{11}$$

Actually, to be read in $I\!\!M_{\dot{r}(s_i)}$, a vector of $I\!\!M_{\dot{r}(s)}$ must undergo the pure boost transformation of $I\!\!B(\dot{r}(s_i), \dot{r}(s))$ [9]. This gives rise to $M_{r(s_i)}$ causal automorphisms, elements of $\mathcal{G}$, the M-causality group, such as

$$f_{s,s_i} : M \to M, \qquad f_{s,s_i}(q) = r(s) + I\!\!B\left(\dot{r}(s_i), \dot{r}(s)\right) \circ I\!\!L_{s,s_i}\left(q - r(s_i)\right) \tag{12}$$

and this series of *causal automorphisms* encodes everything :

- the continuous timelike worldline itself,

$$\{f_{s,s_i}\left(r(s_i)\right) /\ s \in [s_i, s_f]\} = \{r(s)/\ s \in [s_i, s_f]\} = \mathcal{C} \tag{13}$$

and all of the local (differential) and global (integrated) characteristics of the travelling twin journey :

- the instantaneous Thomas-Wigner rotation of the spatial coordinate axes, the $e_j(s)$, with respect to the stayed home twin axes, provided that one has $e_0(s) = \dot{r}(s_i)$ (this condition, in effect, is mandatory in order to have identical 3-dimensional spaces, and define meaningful rotations),

$$e_j(s) = f_{s,s_i *}\left(e_j(s_i)\right) \ , \qquad s \in [s_i, s_f] \ , \qquad j = 1, 2, 3. \tag{14}$$

- the instantaneous relation of M-time to $M_{r(s)}$-time

$$\mathrm{d}s = \dot{r}(s_i) \cdot f_{s,s_i *}\left(\dot{r}(s_i)\right) \ \mathrm{d}t \tag{15}$$

- the non-trivial differential ageing phenomenon, by integration along $\mathcal{C}$ of the non-exact differential proper-time 1-form. An equality of proper-time lapses, $s_f - s_i$ and $t_f - t_i$ does not hold in the general case, where one has,

$$t_f - t_i = \oint_{s_i}^{s_f} \mathrm{d}s \ \left(\dot{r}(s_i) \cdot f_{s,s_i *}\left(\dot{r}(s_i)\right)\right)^{-1} \ \leq \ s_f - s_i \tag{16}$$

for those of the tangent automorphisms, $f_{s,s_i *}$ that are in $\mathcal{L}_+^\uparrow$. If, instead, $f_{s,s_i *}$ is a *contraction*, i.e., a global dilatation by a factor smaller than 1, then the opposite relation obviously results, of $t_f - t_i > s_f - s_i$, indicating that causality alone does not tell whom, of either twin, is ageing faster.

Note that (16) is made more familiar if we keep in mind that the integrand is ordinarily thought of as the usual γ^{-1} factor of $\sqrt{1-\vec{v}^2(s)/c^2} \leq 1$. However, this sends us back to the controversial *paradigm* of inertial frames of reference and their relative uniform velocities, whereas, in the present geometrical context, it matters to realize that (16) is not bound to that interpretation and/or derivation. Rather, it is worth emphasizing that it is again a pure consequence of causality, without it being necessary to call for anything else. This can be phrased as follows. Causality is implemented on M through the partial ordering relation (6), that is, through the Lorentzian non-degenerate quadratic form (5). Now, because $\dot{r}(s_i)$ is timelike, the inverted Cauchy-Schwarz inequality holds

$$\forall v \in I\!\!M \ , \qquad g_0\left(\dot{r}(s_i), \dot{r}(s_i)\right) \times g_0\left(v, v\right) \ \leq \ g_0^2\left(\dot{r}(s_i), v\right)) \tag{17}$$

where there is equality whenever v and $\dot{r}(s_i)$ are linearly dependent. Furthermore, since $\dot{r}(s_i)$ is of unit length ($\dot{r}^2(s_i) = 1$), for those $f_{s,s_{i*}}$ that are in $\mathcal{L}_+^\uparrow$, one has $\dot{r}(s_i) \cdot f_{s,s_{i*}}(\dot{r}(s_i)) \geq 1$, in view of (17), and (16) follows.

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Discussion

Since it was launched by P. Langevin in 1911 (and was indeed explicit in the Einstein's 1905 famous article), the twin paradox has motivated some 25.000 articles up today. Because of accelerations that are unavoidable in a flat spacetime manifold, the paradox proper context has long been thought to be the one of general relativity theory. Unduly, because accelerations should no way be mistaken for gravitation, and are consistently dealt with in a Minkowski spacetime manifold, M. It is in that context that we have been looking for the mechanism/principle at the origin of a so counter-intuitive fact as the twin paradox, first re-qualified in terms of path-functional dependence of proper-time lapses. It turns out, then, that it is the preservation of the Minkowski spacetime causal order along continuous timelike worldlines, that is responsible for the path functional dependence of proper-time lapses.

In particular, one and the same series of M-causal automorphisms, which, at $d = 3$ spatial dimensions correspond to inhomogeneous orthochronous Lorentz transformations (plus dilatations), encodes all of the local and global characteristics of the travelling twin journey : not solely the twin paradox itself, but also the somewhat related Thomas-Wigner rotations.

The causal (partial) order of M just expresses the existence of a finite velocity limit to energy and information transfers in M. And in the end, it is such an intuitively clear principle as causality which comprises the twin paradox, as an unexpected, still necessary outcome ! Similar situations have long been noticed of mathematical theories whose axioms, even when unquestionable, are sometimes able to generate counter intuitive .. if not "paradoxical" consequences [10].