Yang-Mills Amplitudes and Twistor String Theory

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1 Introduction

Gluon scattering amplitudes in QCD and supersymmetric gauge theories are very difficult to compute, so this is a fertile ground for new insights and methods. Over the past several years we have learned a lot about remarkably rich mathematical structures in Yang-Mills theories. On the one hand, in the context of AdS/CFT a lot of work has been done exploring Yang-Mills integrable structures and computing anomalous dimensions. On the other hand, following the discovery of twistor string theory [1], we have seen a lot of progress in Yang-Mills scattering amplitudes. These advances have culminated recently in the conjectured exact answer for planar $\mathcal{N} = 4$ Yang-Mills theory [2].

2 Gluon Scattering Amplitudes

Formulas for scattering amplitudes in gauge theory exhibit simplicity that is completely obscure in the underlying Feynman diagrams. Witten uncovered several new layers of previously hidden mathematical richness in gluon scattering amplitudes and argued that the unexpected simplicity could be understood in terms of twistor string theory [1]. Today twistor string theory has blossomed into a very diverse and active community, which produces an impressive array of results.

Why is it so hard to compute gluon scattering amplitudes? We've known the rules for covariant perturbation theory for decades, they can be found in any textbook. One would think that to calculate any amplitude, all one has to do is to simply write down all Feynman diagrams and sum them up. However, there are two complications. First of all, the number of diagrams grows faster than factorially: for 4 gluons there are 4 diagrams, for 5 gluons there are 25, for 6 there are 220, for 7 there are 2485, for 8 there are 34300 and for 9 there are 559405, etc. Second of all, even a contribution of a single diagram is very complicated.

It turns out that there is an alternative choice of variables which drastically simplifies the calculations. The idea is very simple. Instead of labeling the gluons with momentum p_μ and polarization ϵ_μ one labels them using two-dimensional commuting spinors λ^a and $\tilde{\lambda}^{\dot{a}}$ and helicity plus or minus. The momentum is related to spinors by the following formula

$$p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}. \tag{1}$$

This notation allows compact expressions for gluon amplitudes. The simplest nonzero amplitudes are called maximally helicity violating (MHV). The Parke-Taylor formula for MHV n-point amplitude reads

$$A_{\mathrm{MHV}} = ig^{n-2}\delta^4(p_1 + \cdots + p_n)\left[\langle r, s\rangle^4 \prod_{i=1}^{n} \frac{1}{\langle i, i+1\rangle}\right], \tag{2}$$

where gluons r and s have negative helicity and the rest have positive helicity and we defined the inner products $\langle i, j\rangle = \lambda_i^1\lambda_j^2 - \lambda_i^2\lambda_j^1$.

Clearly, if one finds that a huge number of Feynman diagrams add up to a simple expression which fits on one line, one becomes suspicious that something important might be going on. It turns out that this can be explained by going to twistor space.

An amplitude can be expressed in (the naïvest version of) twistor space by "$\frac{1}{2}$-Fourier transform" with respect to $\widetilde{\lambda}$

$$\widetilde{A}(\lambda_i, \mu_i, \pm_i) = \int d^{2n}\widetilde{\lambda}_i^a \ \exp\left[i\sum_{i=1}^{n} \mu_{\dot{a}i}\widetilde{\lambda}_i^{\dot{a}}\right] A(\lambda_i^a, \widetilde{\lambda}_i^{\dot{a}}, \pm_i). \tag{3}$$

Witten observed that the structure of gluon scattering amplitudes is very simple in twistor space [1]. A tree-level n-point amplitude with q negative helicity gluons is zero unless it lies on a curve of degree $= q-1$. These observations motivated Witten to try to construct some kind of string theory where the calculation of amplitudes would involve curves in twistor space, so that these geometric properties would be manifest.

3 Twistor String Theory

Now let us describe the ingredients which enter into the construction of twistor string theory [1]. We want a theory whose spectrum is precisely that of (supersymmetric) Yang-Mills theory, without the infinite tower of massive string excitations that one usually has in string theory. This suggests that one should consider a topological string theory. Supersymmetric twistor space, $\mathbb{CP}^{3|4}$, is actually a Calabi-Yau manifold, so it makes sense to consider the B-model on this space. Indeed the spectrum of open strings in this theory precisely corresponds to the field content of $\mathcal{N} = 4$ super-Yang-Mills theory. Open strings in the topological B-model on supertwistor space are the gluons in Yang-Mills theory. However, these 'gluons' are essentially free—their interactions constitute only self-dual Yang-Mills theory. So we need additional ingredients which contribute to the effective action for the gluons, completing it to the full Yang-Mills theory. We have to introduce 'instantonic' D-branes, which can wrap any curve inside supertwistor space. They are associated with new degrees of freedom. Integrating out these degrees of freedom produces an effective action for the gluons which, it turns out, is exactly $\mathcal{N} = 4$ Yang-Mills theory [1].

There actually exist two very distinct recipes for calculating the effective action in Witten's twistor string theory, depending on what kinds of curves one considers. Specifically, one can consider either connected or disconnected curves, Calculations based on both kinds of curves separately reproduce the complete tree-level gluon S-matrix. The former leads to a mysterious formula, derived from string theory, which recasts the problem of calculating any tree-level n-gluon scattering amplitude into the problem of solving some polynomial equations [3, 4, 5]. The disconnected prescription leads to a computationally useful formula which expresses an arbitrary tree-level amplitude in terms of all possible decompositions into MHV subamplitudes (which must be continued off-shell in a suitable way) [6].

There are many interesting directions to explore in order to understand twistor string theory better. It would be interesting to construct one-loop twistor string theory, to relate twistor string theory to AdS string theory, to understand other formulations of twistor string theory and to construct a twistor string theory for $\mathcal{N} = 8$ supergravity.

4 Applications of Twistor-Inspired Methods

It turns out that tree level gluon amplitudes have structures even richer then those indicated by twistor string theory. To illustrate this, lets consider the six-particle NMHV amplitude originally calculated by summing 220 Feynman diagrams. Today we know a very simple formula for this amplitude, involving only 3 terms [7]. Similarly, the eight-particle NNMHV amplitude would require 34,300 Feynman diagrams (probably never seriously attempted), or 44 MHV diagrams. In this case there is a simpler formula [7]

$$A_8 = \frac{[5|4+3+2|1\rangle^3}{(p_2+p_3+p_4+p_5)^2[2\,3][3\,4][4\,5]\langle 6\,7\rangle\langle 7\,8\rangle\langle 8\,1\rangle[2|3+4+5|6\rangle} + \text{ 5 terms} \quad (4)$$

Where do these simple formulas come from? Their discoveries were 'accidents', but in hindsight we can observe that these compact formulas all seem to come out naturally from the on-shell recursion [7, 8, 9]

$$A_n = \sum_{r=2}^{n-2} A_{r+1} \frac{1}{p_r^2} A_{n+1-r} \quad (5)$$

The recursion relations admit closed form, analytic solutions for 'split helicity' amplitudes [10].

Amplitudes which were previously impossible to compute, or could only be evaluated numerically, can now be written down in closed form with no effort. The tree-level techniques have been widely applied to include processes with fermions and scalars.

5 Higher Loops

The tree-level MHV rules can be sewn together to evaluate one-loop amplitudes [11]. This is like having a 'disconnected prescription' at one loop, so in some sense this is the closest we have to a 'twistor string' construction which works at one loop.

One can use generalized unitarity to determine any one-loop amplitude in $\mathcal{N} = 4$ [13, 14, 15]. Scalar box integrals provide a complete basis for all one-loop gluon amplitudes in $\mathcal{N} = 4$. Each scalar box integral has has a unique leading singularity, the discontinuity of any desired amplitude across this singularity is given by a quadruple cut. These results have been extended to less symmetric cases such as $\mathcal{N} = 1$ Yang-Mills and finally taken all the way to QCD. The on-shell bootstrap has been succesully used to derive analytic formulas for several new one-loop multi-parton amplitudes in QCD [12].

Unfortunately, complete basis of integrals is not known for an arbitrary two-loop amplitude. Nevertheless, planar two loop MHV amplitudes are believed to possess the remarkable iterative structure [16]

$$M_n^{(2)}(\epsilon) = \frac{1}{2}(M_n^{(1)}(\epsilon))^2 + f(\epsilon)M_n^{(1)}(2\epsilon) - \frac{5}{4}\zeta_4 + \mathcal{O}(\epsilon) \tag{6}$$

where $f(\epsilon) = (\psi(1-\epsilon) - \psi(1))/\epsilon$. They were verified explicitly for $n = 4, 5$ [16, 17, 18, 19].

It has been conjectured that similar iterative relations hold to all loop orders [20]

$$M_n^{(L)}(\epsilon) = P^{(L)}(M_n^{(1)}(\epsilon), \ldots, M_n^{(L-1)}(\epsilon)) + f^{(L)}(\epsilon)M_n^{(1)}(L\epsilon) + C^{(L)} + \mathcal{O}(\epsilon). \tag{7}$$

They have been explicitly verified perturbatively for $(L, n) = (2, 4), (3, 4), (4, 4), (2, 5)$ and recently for strong coupling for $n = 4$ [21, 22, 23, 24]. The function $f^{(L)}(\epsilon)$ is related to the cusp anomalous dimension.

The anomalous dimension of a twist-2 operator $\mathrm{Tr}ZD^JZ$ for large J behaves like

$$\Delta - J = f(\lambda)\ln J + \mathcal{O}(J^0). \tag{8}$$

The function $f(\lambda)$ has been conjectured to obey an integral equation from which it could be determined to all orders in the coupling constant λ [2]. Remarkably, direct calculations support the conjecture for $f(\lambda)$ up to four loops [21, 22]!

6 Conclusions

Formulas for scattering amplitudes in gauge theory exhibit simplicity that is completely obscure in the underlying Feynman diagrams. Some of this simplicity can be made manifest by thinking about the structure of amplitudes expressed in twistor space, and can be explained (at least at tree level) in terms of a corresponding twistor string theory. New insights into the structure of amplitudes (in particular, generalized analyticity) have led to great progress in our ability to calculate amplitudes which were previously out of reach. Prospects are great for continued progress, both in supersymmetric gauge theories as well as QCD.

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