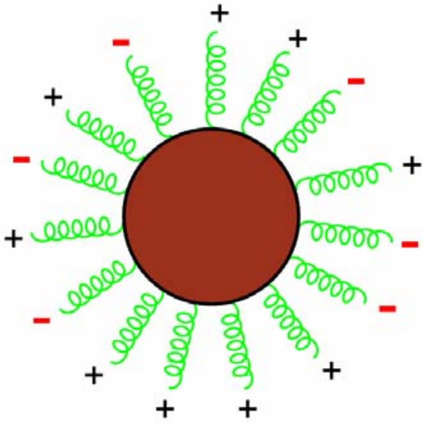
Higher Order QCD
Lecture 1



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SLAC Summer Institute
The Next Frontier: Exploring with the LHC
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Lecture 1 Outline

1. Levels of approximation
2. Modern color and helicity organization of amplitudes
3. Soft and collinear behavior

Levels of Approximation

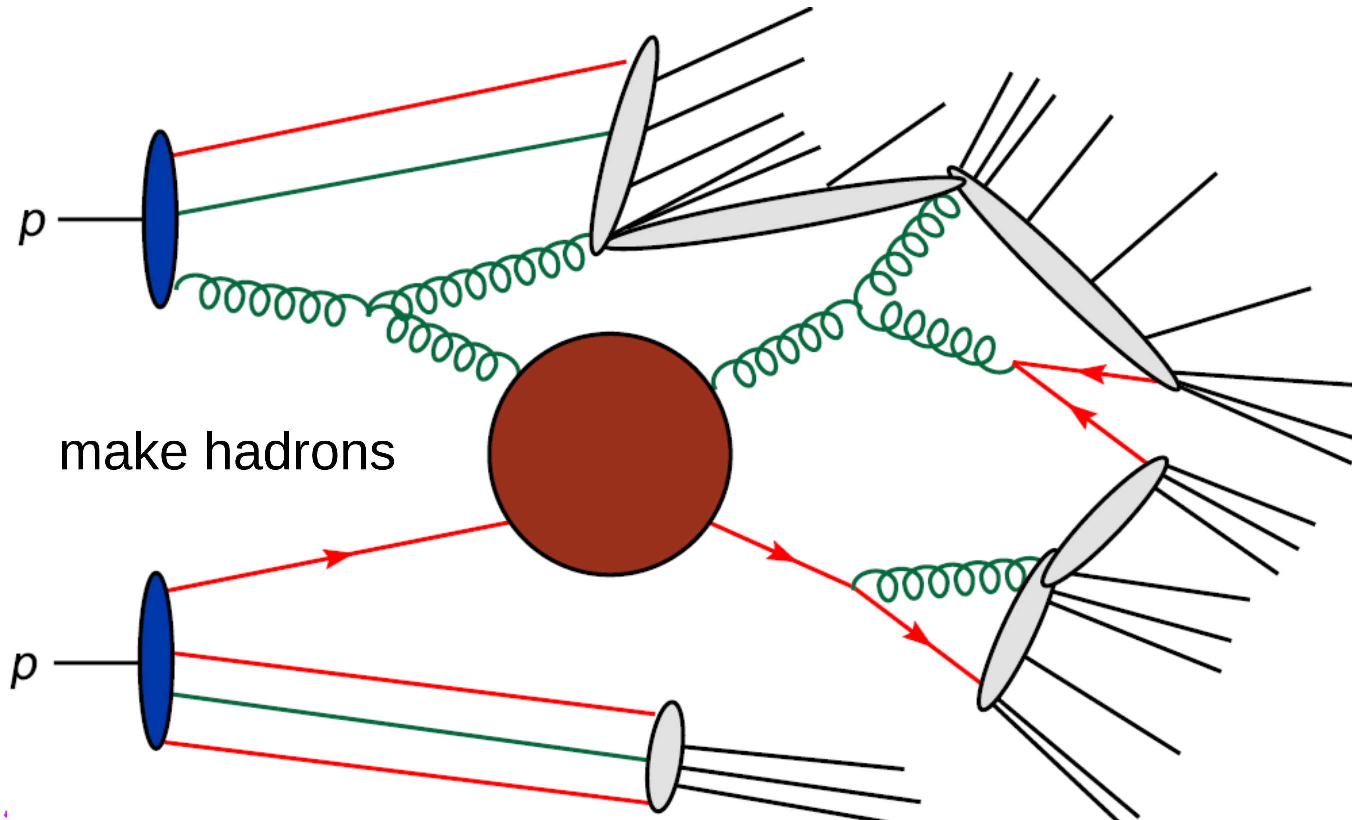
- Monte Carlos (PYTHIA, HERWIG,…)
- LO, fixed-order matrix elements (ALPGEN,…)
- LO MEs matched to parton showers
- NLO MEs (parton level)
- NLO MEs matched to showers (MC@NLO)
- NNLO MEs
- MC@NNLO?

Monte Carlos

- Based on properties of soft and collinear radiation in QCD
- Partons surrounded by "cloud" of soft and collinear partons
- Leading double logs of Q_{hard}/Q_{soft} exponentiate, can be generated probabilistically
- Shower starts with basic 2 → 2 parton scattering
  -- or basic production process for *W, Z, tt*, etc.
- Further radiation approximate, requires infrared cutoff
- Shower can be evolved down to very low Q_{soft} , where models for hadronization and spectator interactions can be applied
- Complete hadron-level event description attained
- Normalization of event rates unreliable
- Event "shapes" sometimes unreliable

Monte Carlos in pictures

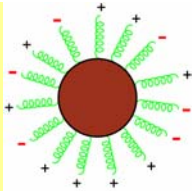
Splitting probability: $\mathrm{P}_g(q^2) = \int_0^1 dz \frac{\alpha_s(q^2)}{2\pi} \hat{P}_{gg}(z) \Theta(q^2 - q_0^2)$



Sudakov

$\Delta_g(Q^2, q^2) = \exp\left[- \int_{q^2}^{Q^2} \frac{d\tilde{q}^2}{\tilde{q}^2} \mathrm{P}_g(\tilde{q}^2)\right]$

Leading order matrix elements



- Based on sum of all tree-level Feynman diagrams in QCD
- Generates correct hard radiation pattern (at tree level)
- Event "shapes" often fairly reliable
- Event rates (normalization) still fairly unreliable, especially if:
  - more jets → more powers of $\alpha_s(\mu_{R,F})$
  - gluons in the initial state (lots of extra soft radiation)
  - cases where new subprocesses appear at NLO ($q\bar{q} \to \gamma\gamma$)
- Description is only at parton level
- Sophisticated programs can now rapidly produce tree-level cross sections for very high multiplicity
- Some use Feynman diagrams MadGraph; GRACE; CompHEP,…
- Other use recursive or iterative organization

  Berends, Giele, VECBOS, NJETS; HELAC; ALPHA → ALPGEN
- Recent techniques spun off from "twistor string theory":
  - MHV vertices; on-shell recursive; other scalar-type graphs

Cachazo, Svrcek, Witten; Britto, Cachazo, Feng (2004); Schwinn, Weinzierl (2005)

Leading order state of art

Number of Feynman diagrams grows very rapidly with number of legs!

ALPHA → ALPGEN

Process	$n = 7$	$n = 8$	$n = 9$	$n = 10$
$g\ g \to n\ g$	559,405	10,525,900	224,449,225	5,348,843,500
ME per minute	28000	9170	2870	870
$q\bar{q} \to n\ g$	231,280	4,016,775	79,603,720	1,773,172,275

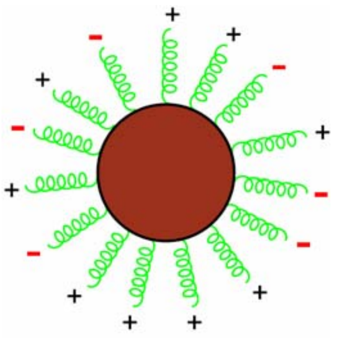


Table 1:

Number of Feynman diagrams corresponding to amplitudes with different numbers of quarks and gluons. CPU performance on a pentium III 850MH

(*notice*: $n = 10$ expected in some R-parity breaking scenarios)

Caravaglios, Moretti (1995); Caravaglios, Mangano, Moretti, Pittau (1999); Mangano, Moretti, Piccinini, Pittau, Polosa (2002)

Not just quarks and gluons

Shopping list ALPGEN

- $W^*Q\bar{Q} + n\text{-}jets,$ $W^* \equiv l\nu_l$ and $Q = b, t, (c)$.
- $W^* + n\text{-}jets$ $jets \equiv$ "light" quarks, gluons
- $Z^*/\gamma^* Q\bar{Q} + n\text{-}jets$
- $Z^*/\gamma^* + n\text{-}jets$
- $Q\bar{Q} + n\text{-}jets$
- $Q\bar{Q}Q\bar{Q} + n\text{-}jets$
- $Q\bar{Q} + H + n\text{-}jets$
- $n\text{-}W + m\text{-}Z + l\text{-}H + n\text{-}jets$
- $n\text{-}jets$
- $m\text{-}\gamma + n\text{-}jets$
- $t(+W,\ +b,\ +Wb) + n\text{-}jets$
- $H + n\text{-}jets$ ggH effective coupling $(m_t \to \infty)$
- $W^*(Z^*/\gamma^*) + m\text{-}\gamma + n\text{-}jets$ in progress
- $Q\bar{Q} + m\text{-}\gamma + n\text{-}jets$ in progress

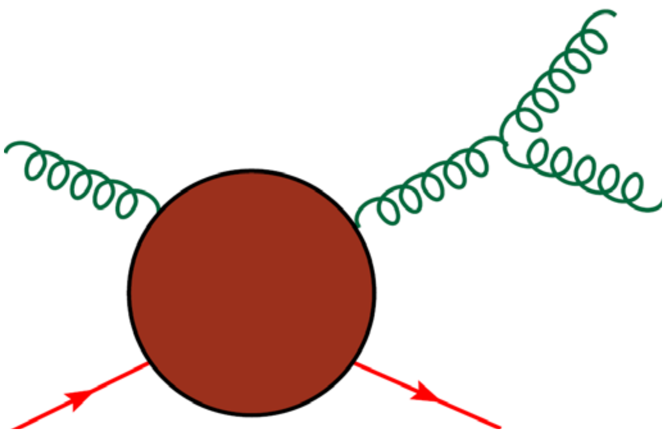
Matching MEs to showers

- Would like to have both:
  - accurate hard radiation pattern of MEs
  - hadron-level event description of parton-shower MCs
- Why not just use 2 → 3,4,… parton processes as starting point for the shower?
- Problem of double-counting:

  When does radiation "belong"

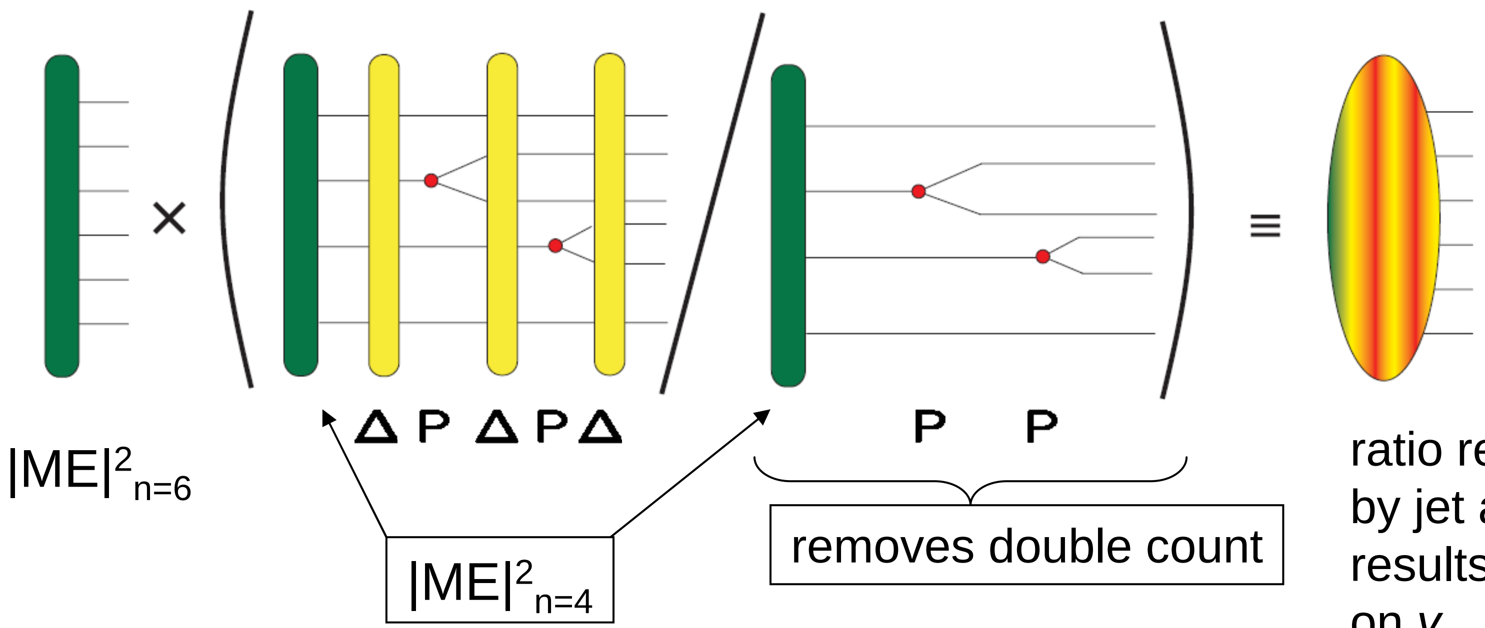
  to the shower, and

  when to the hard matrix element?



ME/shower matching

- CKKW matching: Catani, Kuhn, Krauss, Webber, hep-ph/0109231
  - separate ME and shower domains using a common jet cluster algorithm variable (k_T algorithm with $y = y_{ini}$)
- an example in pictures: Nagy, Soper, hep-ph/0607046



ratio regulated by jet algorithm; results depend on y_{ini}

ME/shower matching (cont.)

Several other general matching schemes available or in the works, e.g,:

MLM scheme (ALPGEN)

Lonnblad, hep-ph/0112284 (Ariadne)

CKKW (Sherpa)

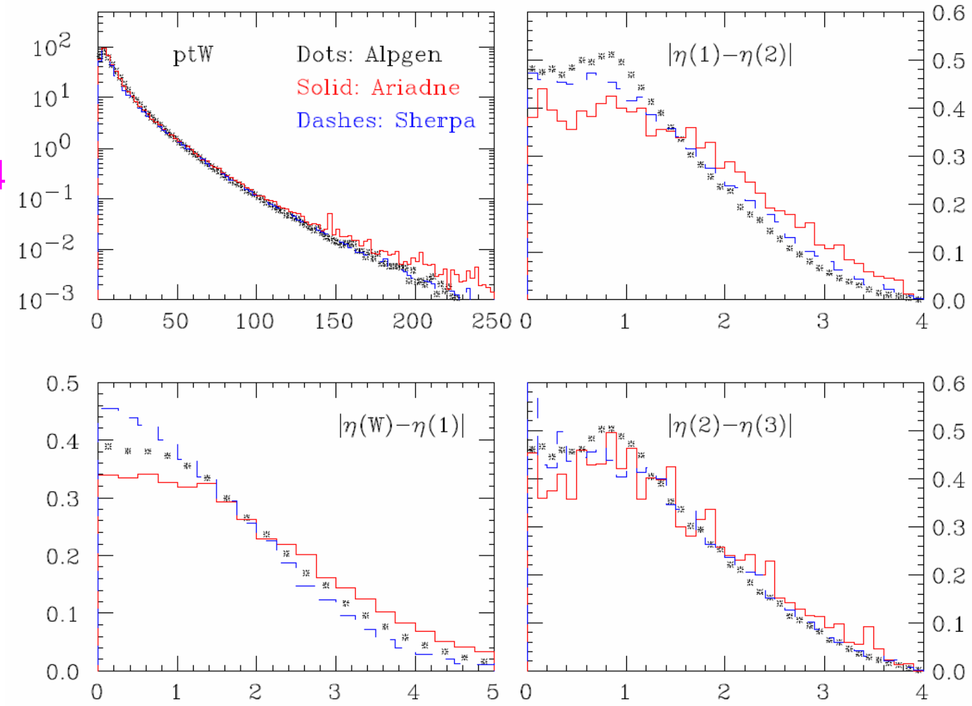
Mrenna, Richardson, hep-ph/0312274

Nagy, Soper, hep-ph/0601021

Skands, Giele, Kosower

ALPGEN, Ariadne, Sherpa compared in Hoche et al., hep-ph/0602031

$p\bar{p} \to W + 4$ jets at Tevatron

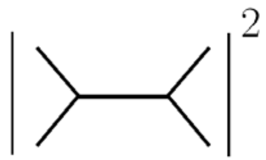


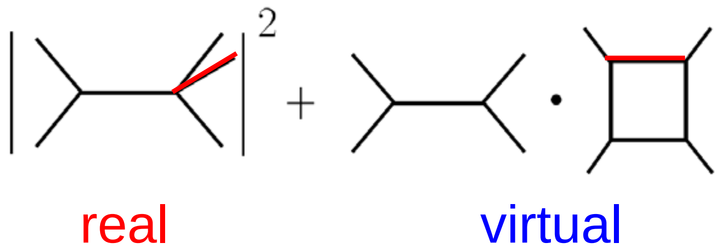
reasonable agreement between different schemes

NLO ME calculations

- Based on sum of all one-loop QCD Feynman diagrams for a given *n*-parton process (plus any "electroweak" particles)
- Also need to square tree amplitudes for *(n+1)*-parton process
  - these contribute at same order in α_s
  - infrared singularities cancel between virtual and real terms
- Event "shapes" usually quite reliable
  - except near kinematic boundaries (e.g. $p_T(W) \to 0$)
- Normalization of event rates usually pretty reliable (10% level)
- Description is only at parton level
- One-loop amplitudes are still generally hand-crafted
  - often with agonizing care taken over the finished product!
- NLO programs scattered about
  - many at http://www.cedar.ac.uk/hepcode/ , http://mcfm.fnal.gov/
- Feynman diagrams very often used
- Techniques spun off from "twistor string theory" – MHV vertices, on-shell recursive bootstrap – now almost ready for phenomenology

Infrared cancellations at NLO

LO 

NLO 

Use dimensional regularization,
$D = 4 - 2\epsilon$

$d^4k \rightarrow d^{4-2\epsilon}k$
in all phase-space and loop integrals

soft singularities: $k_s \rightarrow 0$ $\quad \sigma^{\rm real} \sim \int \frac{dk_s^2}{k_s^{2(1+\epsilon)}} \sigma^{\rm LO}(k_s = 0)$

collinear singularities: $k_{ab}^2 \rightarrow 0 \quad (k_a \mid\mid k_b)$ $\quad \sim \int \frac{dk_{ab}^2}{k_{ab}^{2(1+\epsilon)}} \sigma^{\rm LO}(k_P)$

virtual soft/collinear singularities: $\quad \sigma^{\rm virt} \sim \Big[-\frac{1}{\epsilon^2}\sum_i C_i - \frac{1}{\epsilon}\sum_{i,j} D_{ij} \ln\Big(\frac{\mu^2}{-s_{ij}}\Big)\Big] \sigma^{\rm LO}$

• Virtual corrections cancel real singularities, but only for quantities insensitive to soft/collinear radiation → **infrared-safe observables *O***

Infrared safety

infrared-safe observables *O*:
- Behave smoothly in soft limit as any parton momentum → 0
- Behave smoothly in collinear limit as any pair of partons → parallel (||)

$$O_n(\ldots, k_s, \ldots) \;\to\; O_{n-1}(\ldots, \not{k_s}, \ldots) \qquad k_s \to 0$$
$$O_n(\ldots, k_a, k_b, \ldots) \;\to\; O_{n-1}(\ldots, k_P, \ldots) \qquad k_a \,||\, k_b$$

• Cannot predict perturbatively any infrared-unsafe quantity, such as:
- the number of partons (hadrons) in an event
- observables requiring no radiation in some region (rapidity gaps or overly strong isolation cuts)
- $p_T(W)$ precisely at $p_T = 0$

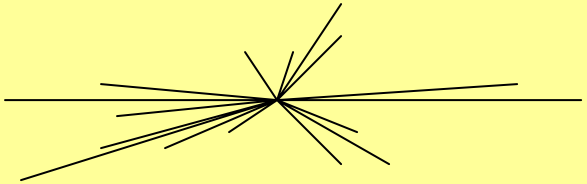
Infrared safety (cont.)

Examples of IR safe quantities:

- jets, defined by cluster or (suitable) cone algorithm
- most kinematic distributions of "electroweak" objects, *W, Z, Higgs* (photons tricky because they can come from fragmentation)

k_T jet cluster algorithm:

- Construct a list of objects, starting with particles *i* (or maybe calorimeter towers), plus "the beam" *b*
- Define a "distance" between objects, which vanishes in soft/collinear limits: $d_{ij} = 2\,\mathrm{min}\{k_T^{(i)}, k_T^{(j)}\}^2\,[\cosh(\eta^{(i)} - \eta^{(j)}) - \cos(\phi^{(i)} - \phi^{(j)})]$
- Cluster together the 2 objects with smallest distance; $d_{ib} = k_T^2$ combine their 4-momenta into one.
- Repeat until all $d_{ij} > d_{ij}^{\mathrm{cut}}$
- The remaining objects are jets



MC@NLO

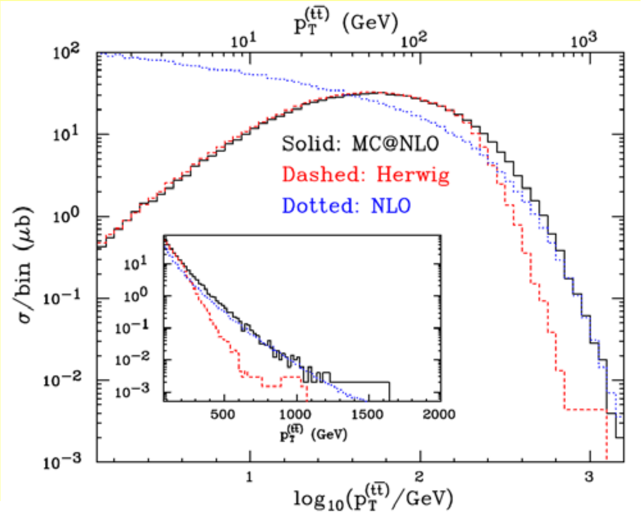
- As with LO matching of MEs to MCs, goal is to combine best features of two approaches: more accurate normalization of event rates (NLO) and hadron-level event descriptions (MC).
- More intricate than LO matching – must perform an exact NLO subtraction, then correct it to remove the parton-shower double-count

Working example: **MC@NLO** Frixione, Webber, hep-ph/0204244

Based on HERWIG MC

LHC processes available to date:

- single vector and Higgs bosons
- vector boson pairs
- heavy quark pairs →
- single top
- lepton pairs
- Higgs bosons in association with a W or Z.



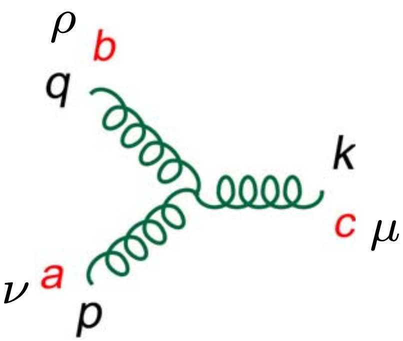
NNLO

- Based on sum of 2-loop *n*-parton process, plus 1-loop (*n*+1)-parton process, tree-level (*n*+2)-parton processes
- Required for high precision at LHC, because NLO results often have 10% or more residual uncertainties
- Where is high precision warranted?
- parton distributions
  - evolution (NNLO DGLAP kernels)
  - fits to DIS, Drell-Yan, and jet data
- LHC production of single *W*s and *Z*s
  - “partonic” luminosity monitor
  - precision m_W
- Higgs production via gluon fusion and extraction of Higgs couplings
- Inclusive jets? Vector boson pairs? Not yet available
- Use NNLO studies to reweight MC[@NLO]

Davatz et al.
hep-ph/0604077

How to organize pQCD amplitudes

- Avoid tangled algebra of color and Lorentz indices generated by Feynman rules



$$= i g f^{abc} [\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho]$$

structure constants

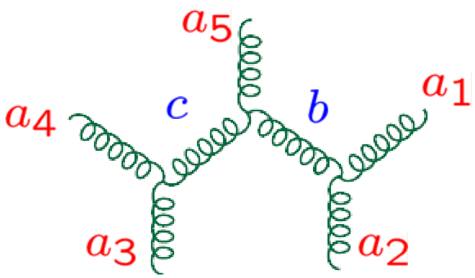
- Take advantage of physical properties of amplitudes

- Basic tools:
  - dual (trace-based) color decompositions
  - spinor helicity formalism

review: LD, hep-ph/9601359

Color

Standard color factor for a QCD graph has lots of structure constants contracted in various orders; for example:



$$\propto \; f^{a_1 a_2 b} f^{a_3 a_4 c} f^{b c a_5}$$

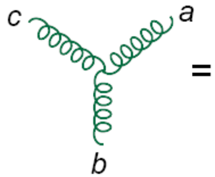
We can write every *n*-gluon tree graph color factor as a sum of traces of matrices T^a in the fundamental (defining) representation of $SU(N_c)$:

$$\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) \qquad \text{+ all non-cyclic permutations}$$

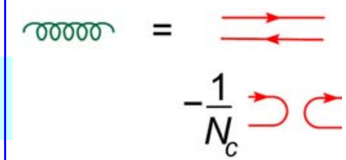
Use definition: $[T^a, T^b] = i f^{abc} T^c$

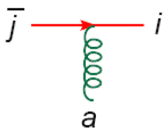
+ normalization: $\mathrm{Tr}(T^a T^b) = \delta^{ab}$ → $\boxed{f^{abc} = -i\,\mathrm{Tr}([T^a, T^b]\, T^c)}$

Color in pictures

Insert 

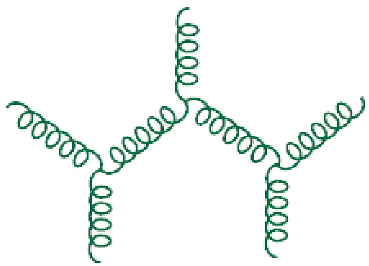
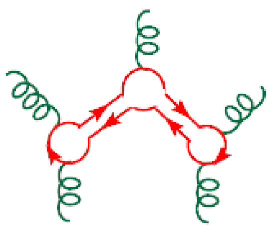
$= f^{abc} = Tr([T^a, T^b]\ T^c) =$  - 

and 

where 

$= (T^a)_i^{\bar{j}}$ is color factor for *qqg* vertex

into typical string of f^{abc} structure constants for a Feynman diagram:

 =  + *permutations* $= \mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$ + permutations

- Always single traces (at tree level)
- $\mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n})$ comes only from those planar diagrams with cyclic ordering of external legs fixed to 1,2,…,n

Trace-based (dual) color decomposition

Similarly

$$q\bar{q}gg\cdots g \text{ amplitudes } \Rightarrow (T^{a_1}T^{a_2}\cdots T^{a_n})_i^{\bar{j}} + \text{permutations}$$

In summary, for the *n*-gluon trees, the color decomposition is

$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g^{n-2}\text{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) + \text{non-cyclic perm's}$$

momenta

color

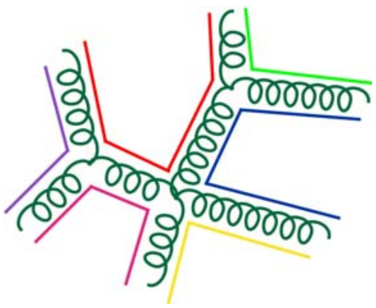
helicities $h_i = \pm 1$

color-ordered subamplitude only depends on momenta. Compute separately for each cyclicly inequivalent helicity configuration $(h_1, h_2, \ldots, h_n)$

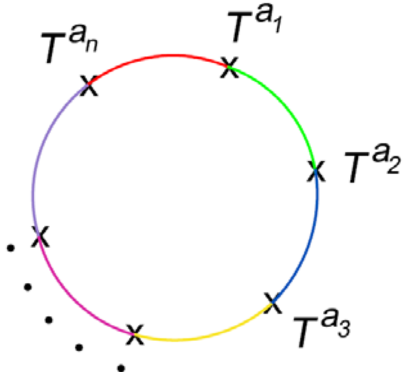
- Because $A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$ comes from planar diagrams with cyclic ordering of external legs fixed to 1,2,…,n, it only has singularities in cyclicly-adjacent channels $s_{i,i+1}$, …

Aside: Strings and Color

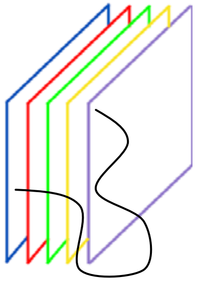
• The "trace" color basis for QCD is also called the "dual" basis because it first arose, as Chan-Paton factors in the description of SU(N) symmetric dual models (string theories)
– initially it was describing flavor!
• A modern string theorist would say that a string end moves from one of N D-branes to another by emitting a green-antiblue gluon
• Also related to 't Hooft double-line formalism



't Hooft



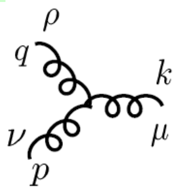
dual



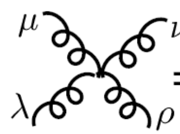
D-brane

Color-ordered Feynman rules

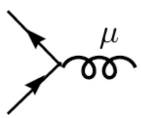
In Feynman gauge, use these "color-stripped" rules



$$= \frac{i}{\sqrt{2}}\left(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho\right)$$

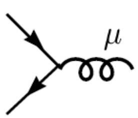


$$= i\eta_{\mu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho})$$

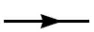


$$= \frac{i}{\sqrt{2}}\gamma_\mu$$

$$= -i\frac{\eta_{\mu\nu}}{p^2}$$



$$= -\frac{i}{\sqrt{2}}\gamma_\mu$$



$$= \frac{i}{\not p}$$

in the (1,2,…,n)-ordered planar diagrams, to compute

$$A_n^{\rm tree}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \qquad A_n^{\rm tree}(1_{\bar q}^-, 2_q^+, 3^{h_3}, \ldots, n^{h_n})$$

$$gg\cdots g \qquad q\bar{q}gg\cdots g$$

etc.

Color sums

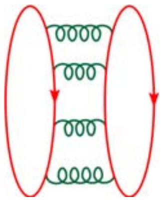
In the end, we want to sum/average over final/initial colors (as well as helicities): $d\sigma^{\text{tree}} \propto \sum_{a_i} \sum_{h_i} |\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\})|^2$

Inserting:

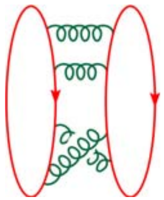
$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) = g^{n-2} \text{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) \, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) + \text{non-cyclic perm's}$$

and doing the color sums diagrammatically:

we get:



$= N_c^n$



$= N_c^n \times \frac{1}{N_c^2}$

$$d\sigma^{\text{tree}} \propto N_c^n \sum_{\sigma \in S_n/Z_n} \sum_{h_i} |A_n^{\text{tree}}(\sigma(1^{h_1}), \sigma(2^{h_2}), \ldots, \sigma(n^{h_n}))|^2 + \mathcal{O}(N_c^{-2})$$

→ Up to $1/N_c^2$ suppressed effects, squared subamplitudes have definite color flow – important for handoff to parton shower programs

Spinor helicity formalism

Scattering amplitudes for massless plane waves of definite momentum:
Lorentz 4-vectors k_i^μ $k_i^2=0$

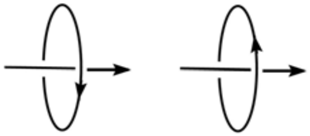
Natural to use Lorentz-invariant products (invariant masses): $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

But for elementary particles with **spin** (*e.g.* all observed ones!) **there is a better way:**

Take "square root" of 4-vectors k_i^μ (spin 1)
use Dirac (Weyl) spinors $u_\alpha(k_i)$ (spin ½)

right-handed: $(\lambda_i)_\alpha = u_+(k_i)$ left-handed: $(\tilde{\lambda}_i)_{\dot{\alpha}} = u_-(k_i)$

q,g,γ, all have 2 helicity states, $h = \pm$



Spinor products

Instead of Lorentz products: $s_{ij} = 2k_i \cdot k_j = (k_i + k_j)^2$

Use spinor products: $\bar{u}_-(k_i) u_+(k_j) = \varepsilon^{\alpha\beta} (\lambda_i)_\alpha (\lambda_j)_\beta = \langle i\, j \rangle$

$$\bar{u}_+(k_i) u_-(k_j) = \varepsilon^{\dot\alpha\dot\beta} (\tilde\lambda_i)_{\dot\alpha} (\tilde\lambda_j)_{\dot\beta} = [i\, j]$$

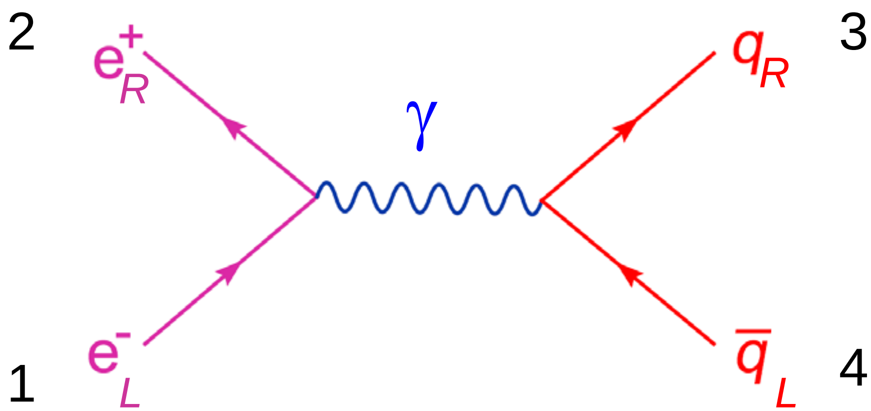
Identity $k_i^\mu (\sigma_\mu)_{\alpha\dot\alpha} = (\not{k}_i)_{\alpha\dot\alpha} = u_+(k_i) \bar{u}_+(k_i) = (\lambda_i)_\alpha (\tilde\lambda_i)_{\dot\alpha}$

$\Rightarrow$ These are **complex square roots** of Lorentz products:

$$\langle i\, j \rangle [j\, i] = \frac{1}{2} \mathrm{Tr} \left[\not{k}_i \; \not{k}_j \right] = 2 k_i \cdot k_j = s_{ij}$$

$$\langle i\, j \rangle = \sqrt{s_{ij}}\, e^{i\phi_{ij}} \qquad [j\, i] = \sqrt{s_{ij}}\, e^{-i\phi_{ij}}$$

Most famous (simplest) Feynman diagram



add helicity information, numeric labels

$$\mathcal{A}_4 = 2ie^2 Q_e Q_q \delta^{\bar{\imath}_4}_{i_3} A_4$$

$$A_4 = \frac{1}{2\,s_{12}} \overline{v_-}(k_2)\gamma^\mu u_-(k_1)\ \overline{u_+}(k_3)\gamma_\mu v_+(k_4)$$

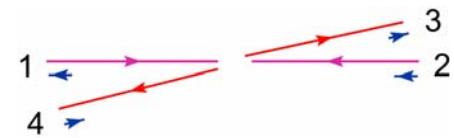
$$= \frac{1}{2\,s_{12}} (\sigma^\mu)_{\alpha\dot\alpha}(\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}\ (\sigma_\mu)^{\dot\beta\beta}(\tilde\lambda_3)_{\dot\beta}(\lambda_4)_\beta$$

$$= \frac{1}{s_{12}} (\lambda_2)^\alpha(\tilde\lambda_1)^{\dot\alpha}(\lambda_4)_\alpha(\tilde\lambda_3)_{\dot\alpha}$$

Fierz identity

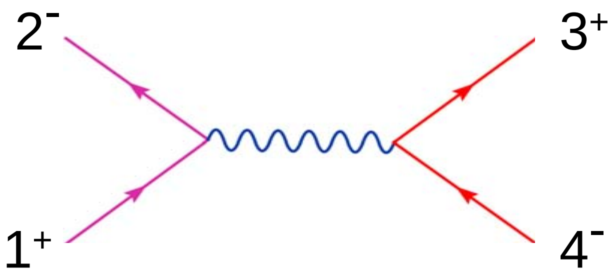
$$(\sigma^\mu)_{\alpha\dot\alpha}\,(\sigma_\mu)^{\dot\beta\beta} = 2\ \delta_\alpha^\beta\,\delta_{\dot\alpha}^{\dot\beta}$$

$$A_4 = \frac{\langle 2\,4\rangle\,[1\,3]}{s_{12}} = e^{i\phi}\frac{s_{13}}{s_{12}} = \frac{-e^{i\phi}}{2}(1-\cos\theta)$$



helicity suppressed as 1 || 3 or 2 || 4

Sometimes useful to rewrite answer



Crossing symmetry more manifest
if we switch to all-outgoing helicity labels
(flip signs of incoming helicities)

$$A_4 = \frac{\langle 2\,4\rangle\,[1\,3]}{s_{12}}$$

$$= \frac{\langle 2\,4\rangle\,[1\,3]}{\langle 1\,2\rangle[2\,1]}\frac{\langle 1\,3\rangle}{\langle 1\,3\rangle}$$

$$= -\frac{\langle 2\,4\rangle\,[2\,4]\langle 2\,4\rangle}{\langle 1\,2\rangle[2\,4]\langle 4\,3\rangle}$$

$$A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

"holomorphic"

or $$A_4 = \frac{[1\,3]^2}{[1\,2][3\,4]}$$

"antiholomorphic"

useful identities

$$\langle i\,j\rangle = -\langle j\,i\rangle$$

$$[i\,j] = -[j\,i]$$

$$\langle i\,i\rangle = [i\,i] = 0$$

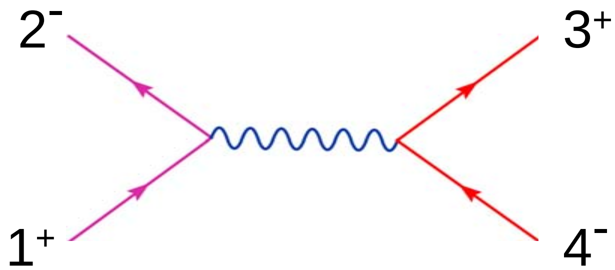
$$\langle i\,j\rangle\,[j\,i] = s_{ij}$$

$$\sum_{j=1}^{4}\langle i\,j\rangle\,[j\,k] = 0$$

$$s_{12} = s_{34}$$

$$s_{13} = s_{24}$$

Symmetries for all other helicity config's

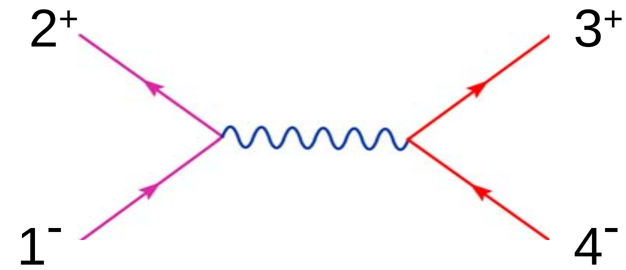


$$A_4 = \frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

C

$1 \leftrightarrow 2$

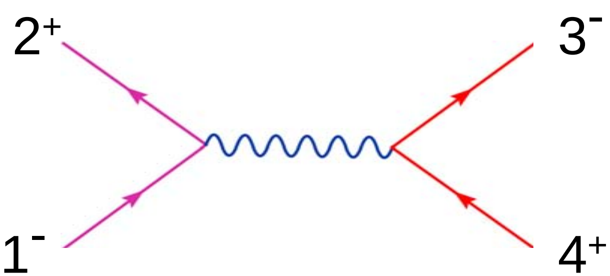
$\Rightarrow$



$$A_4 = -\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}$$

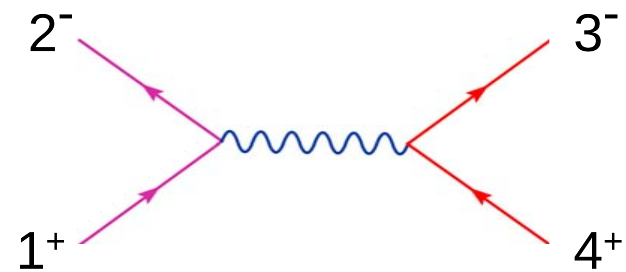
P $\Downarrow$ $\langle\,\rangle \leftrightarrow [\,]$

$\Downarrow$



$$A_4 = \frac{[2\,4]^2}{[1\,2][3\,4]}$$

$\Rightarrow$



$$A_4 = -\frac{[1\,4]^2}{[1\,2][3\,4]}$$

Unpolarized, helicity-summed cross sections

(the norm in QCD)

$$\frac{d\sigma(e^+e^- \to q\bar{q})}{d\cos\theta} \propto \sum_{\text{hel.}} |A_4|^2 = 2\Big\{\Big|\frac{\langle 2\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\Big|^2 + \Big|\frac{\langle 1\,4\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle}\Big|^2\Big\}$$

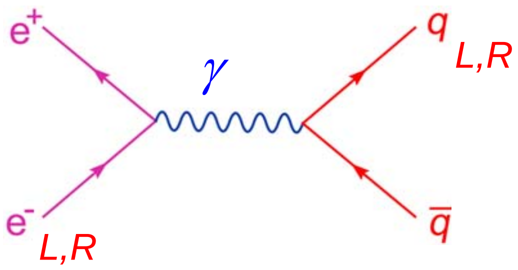
$$= 2\,\frac{s_{24}^2 + s_{14}^2}{s_{12}^2}$$

$$= \frac{1}{2}[(1-\cos\theta)^2 + (1+\cos\theta)^2]$$

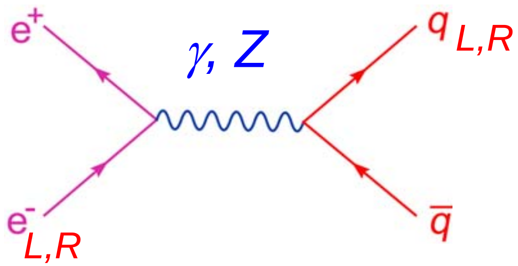
$$= 1 + \cos^2\theta$$

Reweight helicity amplitudes → electroweak/QCD processes

For example, *Z* exchange



$\Rightarrow$

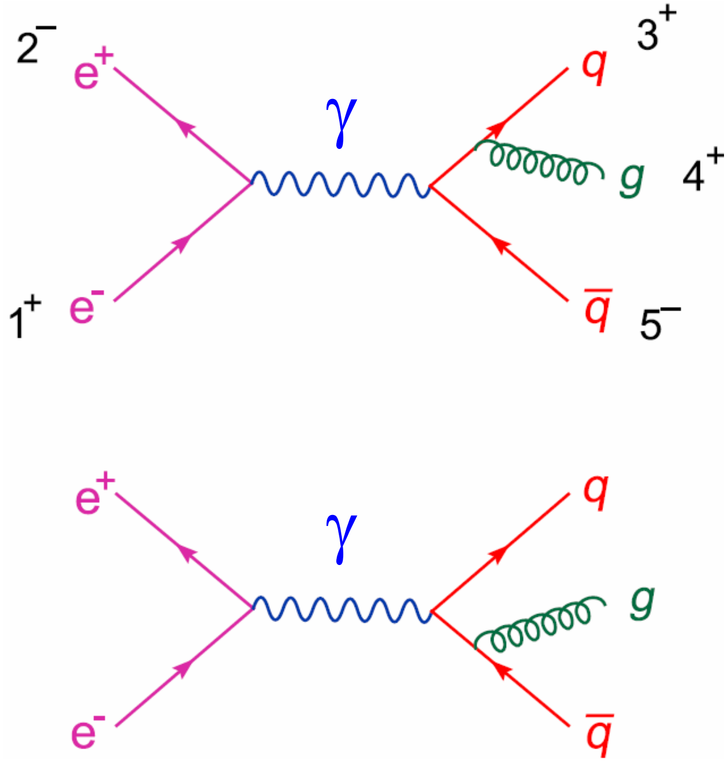


$$Q_e Q_q \quad \Rightarrow \quad Q_e Q_q + \frac{v^e_{L,R} v^q_{L,R} \; s}{s - M_Z^2 + i\Gamma_Z M_Z}$$

$$v_L^f = \frac{2I_3^f - 2Q_f \sin^2\theta_W}{\sin 2\theta_W} \qquad v_R^f = -\frac{2Q_f \sin^2\theta_W}{\sin 2\theta_W}$$

Next most famous pair of Feynman diagrams

(to a higher-order QCD person)



$$\mathcal{A}_5 = 2ie^2 g Q_e Q_q (T^{a_4})_{i_3}^{\bar{\imath}_5} A_5$$

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\, s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\, s_{45}}$$

Helicity formalism for massless vectors

Berends, Kleiss, De Causmaecker, Gastmans, Wu (1981); De Causmaecker, Gastmans, Troost, Wu (1982); Xu, Zhang, Chang (1984); Kleiss, Stirling (1985); Gunion, Kunszt (1985)

$$(\varepsilon_i^+)_\mu = \varepsilon_\mu^+(k_i, q) = \frac{\langle i^+|\gamma_\mu|q^+\rangle}{\sqrt{2}\,\langle i\,q\rangle}$$

$$(\not{\varepsilon}_i^+)_{\alpha\dot{\alpha}} = \not{\varepsilon}_{\alpha\dot{\alpha}}^+(k_i, q) = \frac{\sqrt{2}\,\tilde{\lambda}_i^{\dot{\alpha}}\lambda_q^{\alpha}}{\langle i\,q\rangle}$$

reference vector q^μ is null, $q^2 = 0$

$\not{q}|q^\pm\rangle = 0$

obeys $\varepsilon_i^+ \cdot k_i = 0$ (required transversality)

$\varepsilon_i^+ \cdot q = 0$ (bonus)

under azimuthal rotation about k_i axis, helicity +1/2 $\tilde{\lambda}_i^{\dot{\alpha}} \to e^{i\phi/2}\tilde{\lambda}_i^{\dot{\alpha}}$

helicity -1/2 $\lambda_i^{\alpha} \to e^{-i\phi/2}\lambda_i^{\alpha}$

so $\not{\varepsilon}_i^+ \propto \frac{\tilde{\lambda}_i^{\dot{\alpha}}}{\lambda_i^{\alpha}} \to e^{i\phi}\,\not{\varepsilon}_i^+$ as required for helicity +1

$e^+e^- \to qg\bar{q}$ (cont.)

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\, s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\, s_{45}}$$

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|q^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)|4^-\rangle\langle q\,5\rangle}{s_{45}\langle 4\,5\rangle}$$

Choose $q = k_5$ to remove 2nd graph

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|5^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle}$$

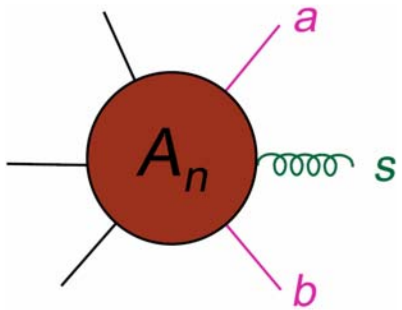
$$= -\frac{\langle 2\,5\rangle[1\,2]\langle 2\,5\rangle[4\,3]}{\langle 1\,2\rangle[2\,1]\langle 3\,4\rangle[4\,3]\langle 4\,5\rangle}$$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle}$$

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$

1. Soft gluon behavior $\quad k_4 \to 0$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} = \frac{\langle 3\,5\rangle}{\langle 3\,4\rangle\langle 4\,5\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,5\rangle}$$
$$\to \mathcal{S}(3,4^+,5) \times A_4(1^+,2^-,3^+,5^-)$$



Universal "eikonal" factors
for emission of soft gluon *s*
between two hard partons *a* and *b*

$$\mathcal{S}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$
$$\mathcal{S}(a,s^-,b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

Soft emission is from the classical chromoelectric current:
independent of parton type (*q* *vs.* *g*) and helicity
– only depends on momenta of *a*,*b*, and color charge

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$ (cont.)

2. Collinear behavior

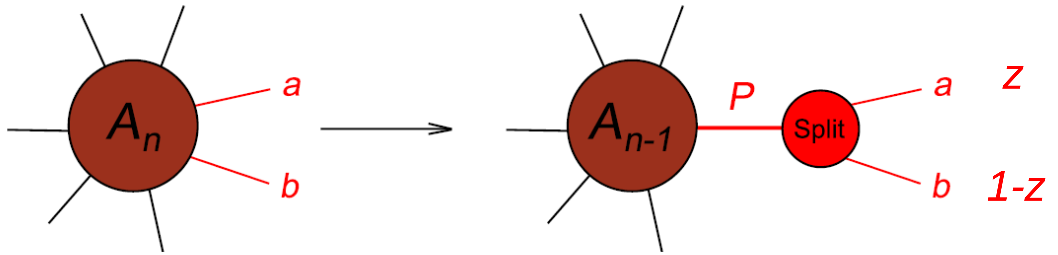
$k_3 \,||\, k_4: \qquad k_3 = z\, k_P, \quad k_4 = (1-z)\, k_P$

$k_P \equiv k_3 + k_4, \qquad k_P^2 \to 0$

$\lambda_3 \approx \sqrt{z}\lambda_P, \quad \lambda_4 \approx \sqrt{1-z}\lambda_P, \quad$ etc.

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{1}{\sqrt{1-z}\,\langle 3\,4\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle P\,5\rangle}$$

$$\to \text{Split}_-(3_q^+, 4_g^+) \times A_4(1^+, 2^-, P^+, 5^-)$$



Time-like kinematics (fragmentation).
Space-like (parton evolution)
related by crossing

Universal collinear factors, or splitting amplitudes
$\text{Split}_{-h_P}(a^{h_a}, b^{h_b})$ depend on parton type and helicity h

Collinear limits (cont.)

We found, from $k_3 \,||\, k_4$:

$$\text{Split}_-(a_q^+, b_g^+) = \frac{1}{\sqrt{1-z}\,\langle a\, b\rangle}$$

Similarly, from $k_4 \,||\, k_5$:

$$\text{Split}_+(a_g^+, b_{\bar{q}}^-) = \frac{1-z}{\sqrt{z}\,\langle a\, b\rangle}$$

$$\Downarrow$$

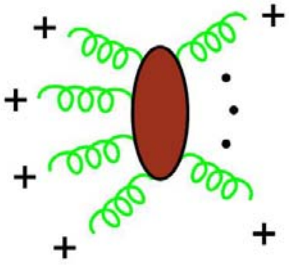
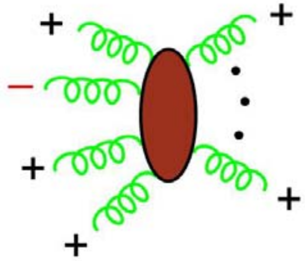
Applying C and P:

$$\text{Split}_-(a_q^+, b_g^-) = -\frac{z}{\sqrt{1-z}\,[a\, b]}$$

Simplest pure-gluonic amplitudes

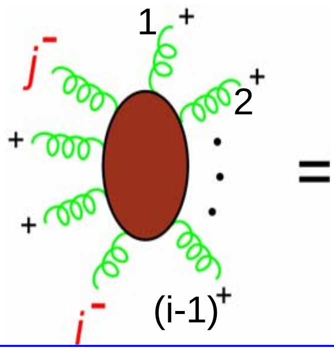
Note: helicity label assumes particle is outgoing; reverse if it's incoming

Strikingly, many vanish:

$A_n^{\text{tree}}(1^{\pm}, 2^{+}, \ldots, n^{+}) \;=$  $=$  $= \; 0$

Maximally helicity-violating (MHV) amplitudes:

$$A_n^{ij,\ \text{MHV}} \;=\; A_n^{\text{tree}}(1^{+}, 2^{+}, \ldots, i^{-}, \ldots, j^{-}, \ldots, n^{+})$$

$=$ 

$$= \frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

Parke-Taylor formula (1986)

Remarkable simplicity – has inspired many formal developments

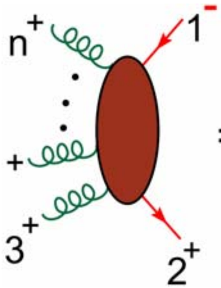
MHV amplitudes with massless quarks

Helicity conservation on fermion line →

$$A_n^{\text{tree}}(1_{\bar{q}}^{\pm}, 2_q^{\pm}, 3^{h_3}, \ldots, n^{h_n}) \equiv 0$$

more vanishing ones:

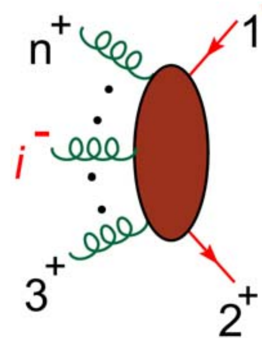
$$A_n^{\text{tree}}(1_{\bar{q}}^{-}, 2_q^{+}, 3^{+}, \ldots, n^{+}) =$$



$= 0$

the MHV amplitudes:

$$A_n^{\text{tree}}(1_{\bar{q}}^{-}, 2_q^{+}, \ldots, i^{-}, \ldots, n^{+}) =$$



$$= \frac{\langle 1\, i\rangle^3 \langle 2\, i\rangle}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

Related to pure-gluon MHV amplitudes by a secret supersymmetry: after stripping off color factors, massless quarks ~ gluinos

Grisaru, Pendleton, van Nieuwenhuizen (1977); Parke, Taylor (1985); Kunszt (1986)

Properties of MHV amplitudes

1. Verify soft limit $\quad k_s \to 0$

$$\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\cdots\langle a\,s\rangle\langle s\,b\rangle\cdots\langle n\,1\rangle} = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\cdots\langle a\,b\rangle\cdots\langle n\,1\rangle}$$

$$\to \mathrm{Soft}(a,s^+,b)\times A_{n-1}^{ij,\ \mathrm{MHV}}$$

2. Extract gluonic collinear limits: $\quad k_a \,||\, k_b \quad (b = a+1)$

$$\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\cdots\langle a-1,a\rangle\langle a\,b\rangle\langle b,b+1\rangle\cdots\langle n\,1\rangle} = \frac{1}{\sqrt{z(1-z)}\langle a\,b\rangle}\frac{\langle i\,j\rangle^4}{\langle 1\,2\rangle\cdots\langle a-1,P\rangle\langle P,b+1\rangle\cdots\langle n\,1\rangle}$$

$$\to \mathrm{Split}_-(a^+,b^+)\times A_{n-1}^{ij,\ \mathrm{MHV}}$$

So

$$\mathrm{Split}_-(a^+,b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

and

$$\mathrm{Split}_+(a^-,b^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

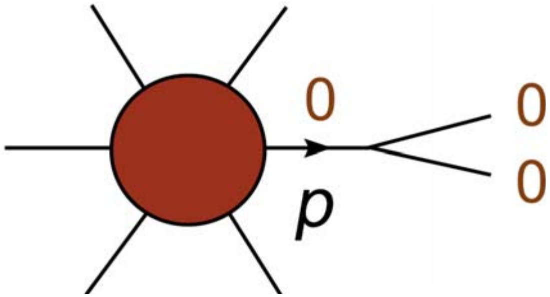
$$\mathrm{Split}_+(a^+,b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle a\,b\rangle}$$

plus parity conjugates

Spinor Magic

Spinor products precisely capture
square-root + phase behavior in **collinear limit**.
Excellent variables for **helicity amplitudes**

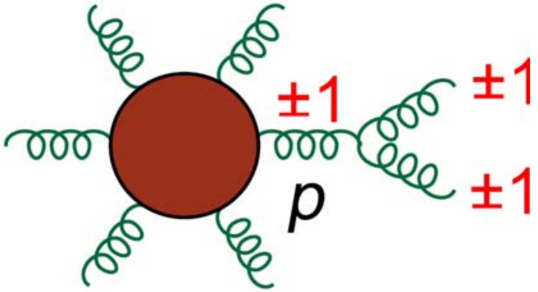
scalars



$$\sim \frac{1}{p^2} \sim \frac{1}{s_{ij}}$$

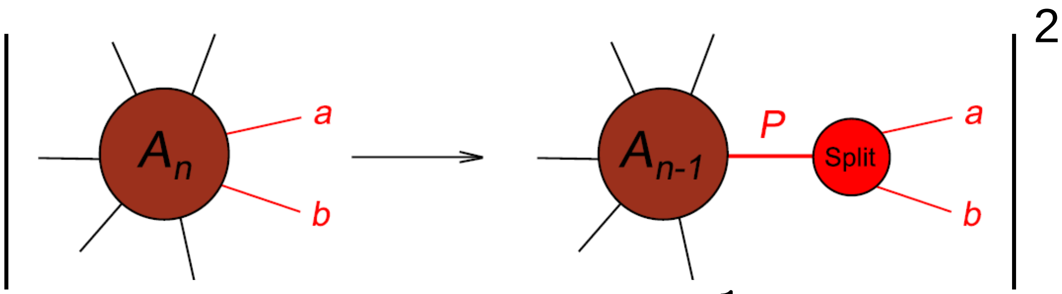
gauge theory

angular momentum mismatch



$$\sim \frac{e^{\pm i\phi}}{\sqrt{p^2}} \sim \frac{1}{\langle i\, j\rangle} \text{ or } \frac{1}{[i\, j]}$$

From splitting amplitudes to probabilities



$$d\sigma_n \sim d\sigma_{n-1} \times \frac{1}{s_{ab}} \times P(z)$$

$$P(z) \propto \sum_{h_P,h_a,h_b} |\text{Split}_{-h_P}(a^{h_a}, b^{h_b})|^2 \, s_{ab}$$

$q \to qg$:

$$P_{qq}(z) \propto C_F \left\{ \left| \frac{1}{\sqrt{1-z}} \right|^2 + \left| \frac{z}{\sqrt{1-z}} \right|^2 \right\}$$

$$= C_F \frac{1+z^2}{1-z} \qquad z < 1$$

$$C_F = \frac{N_c^2-1}{2N_c}$$

Note soft-gluon singularity as $z_g = 1 - z \to 0$

Similarly for gluons

$g \to gg$:

$$P_{gg}(z) \propto C_A \left\{ \left| \frac{1}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{z^2}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{(1-z)^2}{\sqrt{z(1-z)}} \right|^2 \right\}$$

$$= C_A \frac{1 + z^4 + (1-z)^4}{z(1-z)} \qquad C_A = N_c$$

$$= 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \qquad z < 1$$

Again a soft-gluon singularity. Gluon number not conserved.
But momentum is. Momentum conservation mixes $g \to gg$ with

$g \to q\bar{q}$:

$$P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right] \qquad T_R = \frac{1}{2}$$

(can deduce, up to color factors, by taking
$e^+ \,||\, e^-$ in $\mathcal{A}_5(e^+e^- \to qg\bar{q})$)

Gluon splitting (cont.)

$g \to gg$: Applying momentum conservation,

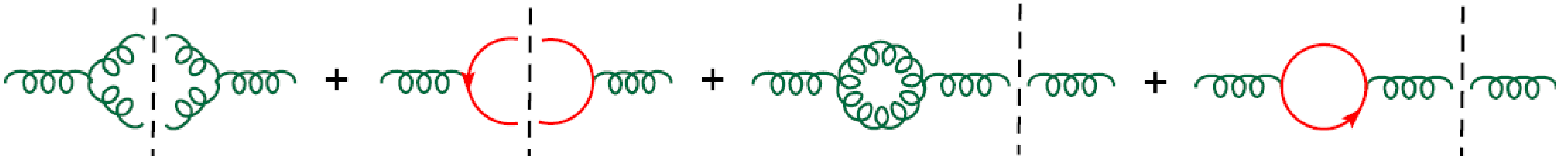
$$\int_0^1 dz\, z\, \Big[P_{gg}(z) + 2 n_f P_{qg}(z) \Big] \;=\; 0$$

gives

$$P_{gg}(z) \;=\; 2 C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] \;+\; b_0\, \delta(1-z)$$

$$b_0 \;=\; \frac{11 C_A - 4 n_f T_R}{6}$$

Amusing that first β-function coefficient enters,
since no loops were done, except implicitly via unitarity:

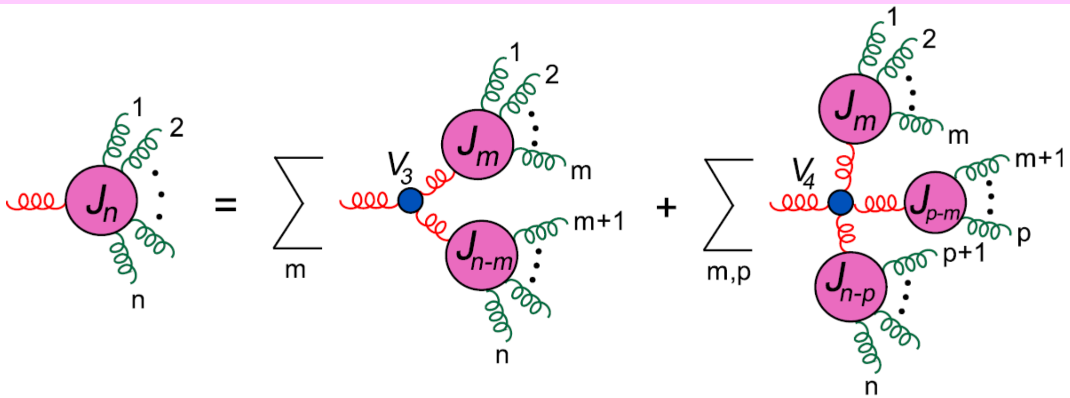


Recursive Tree Techniques

Illustrate with Berends-Giele (1987) [off-shell] recursion relations

Other [off-shell] recursive approaches underly HELAC; ALPHA

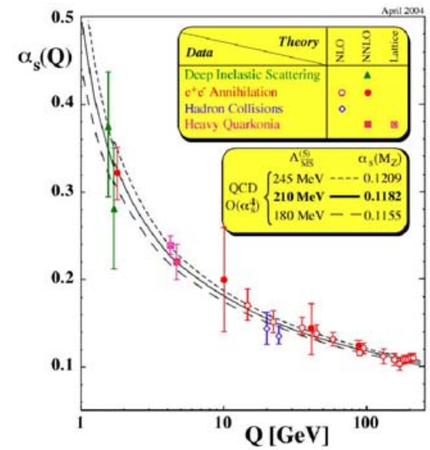
- Follow an off-shell gluon (or quark) line into “forest”
of color-ordered tree graphs
- All other legs on shell
- Trail forks into either 2 or 3 more lines, via 3- or 4-gluon vertex
- Each new path enters a forest with fewer on-shell legs
- Put last leg on shell to get A_n^{tree}



Extra Slides

QCD factorization & parton model

- Asymptotic freedom guarantees that at short distances (large transverse momenta), partons in the proton are almost free.
- They are sampled "one at a time" in hard collisions.
- Leads to QCD-improved parton model:



"suitable" final state

Parton distribution function: prob. of finding parton a in proton 1, carrying fraction x_1 of its momentum

factorization scale ("arbitrary")

$$\sigma^{pp\to X}(s;\alpha_s,\mu_R,\mu_F) = \sum_{a,b}\int_0^1 dx_1 \int_0^1 dx_2\, f_a(x_1,\alpha_s,\mu_F)\, f_b(x_2,\alpha_s,\mu_F) \times \hat{\sigma}^{ab\to X}(sx_1x_2;\alpha_s,\mu_R,\mu_F)$$

Partonic cross section, computable in perturbative QCD

partonic CM energy^2

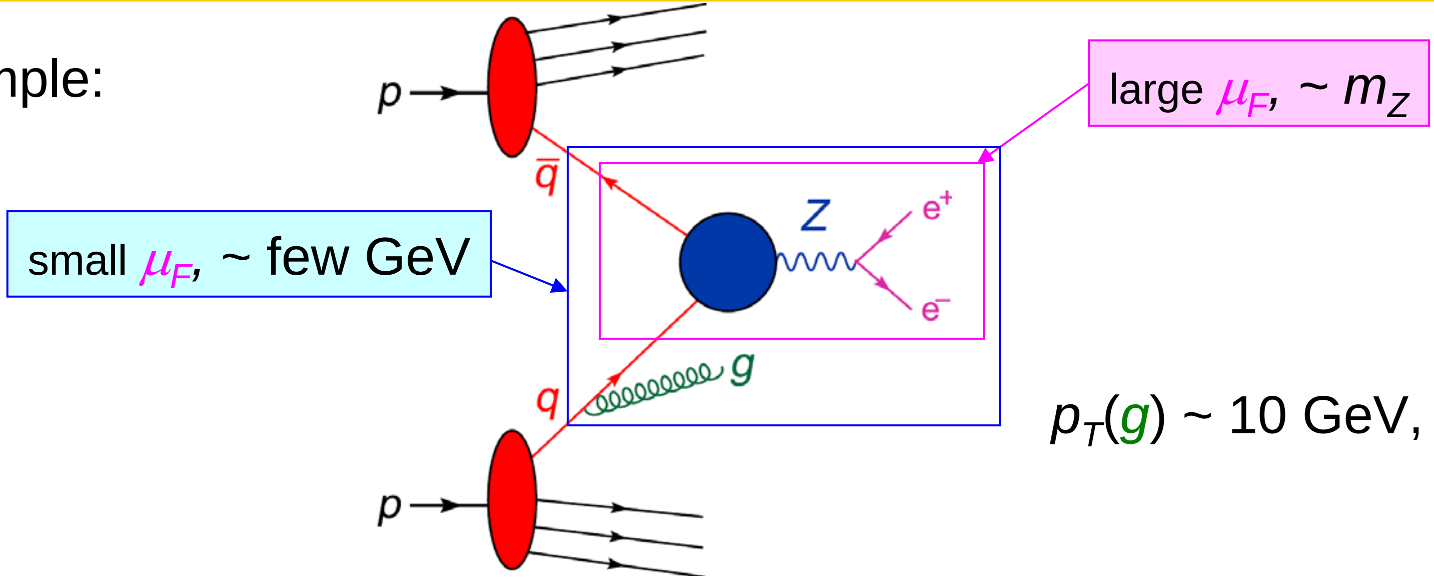
renormalization scale ("arbitrary")

Parton evolution

- partons in the proton are not quite free
- distributions $f_a(x,\mu_F)$ evolve as scale μ_F at which they are resolved varies

$$\sigma^{pp\to X}(s;\alpha_s,\mu_R,\mu_F) = \sum_{a,b}\int_0^1 dx_1 \int_0^1 dx_2\; f_a(x_1,\alpha_s,\mu_F)\, f_b(x_2,\alpha_s,\mu_F) \times \hat{\sigma}^{ab\to X}(sx_1x_2;\alpha_s,\mu_R,\mu_F)$$

Example:

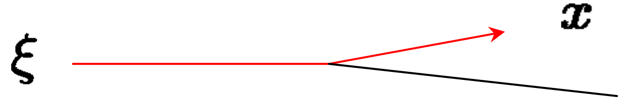


$p_T(g)$ ~ 10 GeV, say

Parton evolution (cont.)

- parton distributions are nonperturbative
- must be measured experimentally
- experimental data at much lower μ_F^2 than (100-1000 GeV)^2
- fortunately, evolution at $\mu_F > 1\text{-}2$ GeV is perturbative
- DGLAP equation (return to later)

$$\mu^2 \frac{\partial}{\partial \mu^2} f_a(x,\mu) = \frac{\alpha_s(\mu)}{2\pi} \sum_b \int_x^1 \frac{d\xi}{\xi} P_{ab}(x/\xi, \alpha_s(\mu)) \, f_b(\xi,\mu)$$



$$x = \frac{x}{\xi} \times \xi$$

$$P_{ab}(x,\alpha_s) = P_{ab}^{(0)}(x) + \frac{\alpha_s}{2\pi} P_{ab}^{(1)}(x) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ab}^{(2)}(x) + \cdots$$

LO (1974) NLO (1980) NNLO (2004)

Also expand partonic cross section:

$$\hat{\sigma}(\alpha_s, \mu_F, \mu_R) = [\alpha_s(\mu_R)]^{n_\alpha} \left[\hat{\sigma}^{(0)} + \frac{\alpha_s}{2\pi} \hat{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \hat{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots \right]$$

LO NLO NNLO

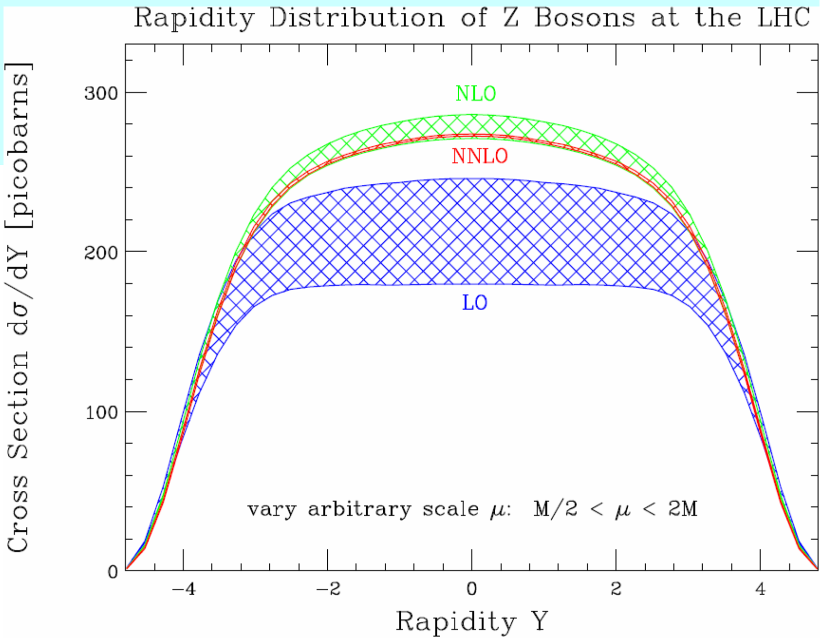
Problem: Leading-order, tree-level predictions often only qualitative due to **poor convergence** of expansion in $\alpha_s(\mu)$ (setting $\mu_R = \mu_F = \mu$)

Example: Z production at LHC. Predict distribution in rapidity

$$Y = \frac{1}{2} \ln\left(\frac{E + p_z}{E - p_z}\right)$$

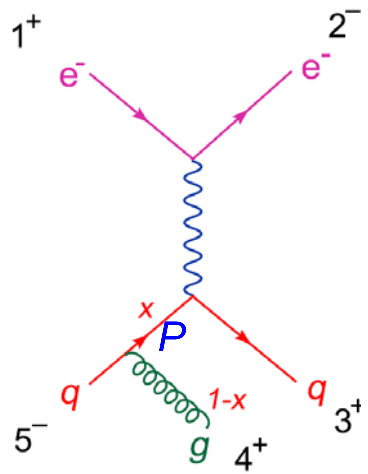
$\frac{d\sigma}{dY}$ has $n_\alpha = 0$

still 30% NLO corrections



Space-like splitting

- The case relevant for parton evolution
- Related by crossing to time-like case
- Have to watch out for flux factor, however



$$q \to qg: \quad k_P = x\, k_5, \quad k_4 = (1-x)\, k_5$$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{\frac{1}{x}}{\sqrt{\frac{1-x}{x}}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

$$= \frac{1}{\sqrt{x}} \frac{1}{\sqrt{1-x}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

absorb into flux factor:

$$d\sigma_5 \propto \frac{1}{s_{15}}$$

$$d\sigma_4 \propto \frac{1}{s_{1P}} = \frac{1}{x\, s_{15}}$$

When dust settles, get exactly the same splitting kernels (at LO)