The Renormalization Scale Problem

$$\rho = C_0 \alpha_s(Q) \left[1 + C_1(Q) \frac{\alpha_s(Q)}{\pi} + C_2(Q) \frac{\alpha_s{}^2(Q)}{\pi^2} + \cdots \right].$$

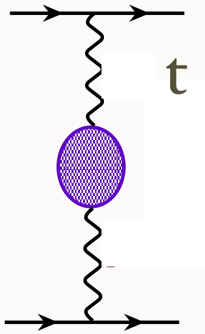
How does one
set scale Q?

with M. Binger

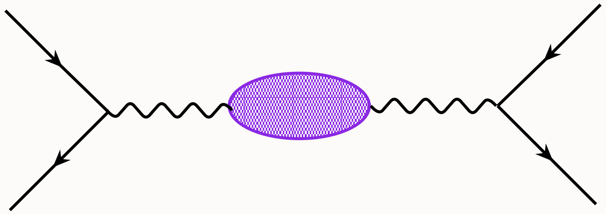
Electron-Electron Scattering in QED

$$\mathcal{M}_{ee \to ee}(++;++) = \frac{8\pi s}{t}\,\alpha(t) + \frac{8\pi s}{u}\,\alpha(u)$$

- No renormalization scale ambiguity!
- Two separate physical scales.
- Gauge Invariant. Dressed photon propagator
- Sums all vacuum polarization, non-zero beta terms into running coupling.
- If one chooses a different scale, one must sum an infinite number of graphs -- but then recover same result!
- Number of active leptons correctly set
- Analytic: reproduces correct behavior at lepton mass thresholds



$$e^+e^- \to \mu^+\mu^-$$



$$\mu^2 \equiv s$$

Scale of $\alpha_{QED}(\mu^2)$ unique!

The QED Effective Charge

- Complex
- Analytic through mass thresholds
- Distinguishes between timelike and spacelike momenta

Analyticity essential:
See C. Berger and L. Dixon talks

The Renormalization Scale Problem

M. Binger, sjb

- No renormalization scale ambiguity in QED
- Gell Mann-Low-Dyson QED Coupling defined from physical observable;
- Sums all Vacuum Polarization Contributions
- Renormalization Scale in QED scheme: Identical to Photon Virtuality
- Analytic: Reproduces lepton-pair thresholds
- Examples: muonic atoms, g-2, Lamb Shift
- Time-like and Space-like QED Coupling related by analyticity
- Uses Dressed Skeleton Expansion

Lessons from QED : Summary

- Effective couplings are complex analytic functions with the correct threshold structure expected from unitarity
- Multiple “renormalization” scales appear
- The scales are unambiguous since they are physical kinematic invariants
- Optimal improvement of perturbation theory

Features of BLM Scale Setting

On The Elimination Of Scale Ambiguities In Perturbative Quantum Chromodynamics.

Lepage, Mackenzie, sjb — **Phys.Rev.D28:228,1983**

- All terms associated with nonzero beta function summed into running coupling
- Resulting series identical to conformal series
- Renormalon n! growth of PQCD coefficients from beta function eliminated!
- In general, BLM scale depends on all invariants

BLM Scale Setting

$$\rho = C_0 \alpha_{\overline{MS}}(Q) \left[1 + \frac{\alpha_{\overline{MS}}(Q)}{\pi} \left(-\tfrac{3}{2}\beta_0 A_{VP} + \tfrac{33}{2} A_{VP} + B \right) + \cdots \right]$$

Use n_f dependence at NLO to identify A_{VP}

by

$$\rho = C_0 \alpha_{\overline{MS}}(Q^*) \left[1 + \frac{\alpha_{\overline{MS}}(Q^*)}{\pi} C_1^* + \cdots \right],$$

where

Conformal Coefficient

$$Q^* = Q \exp(3A_{VP}),$$

$$C_1^* = \tfrac{33}{2} A_{VP} + B .$$

The term $33A_{VP}/2$ in C_1^* serves to remove that part of the constant B which renormalizes the leading-order coupling. The ratio of these gluonic corrections to the light-quark corrections is fixed by $\beta_0 = 11 - \frac{2}{3} n_f$.

Use skeleton expansion
Gardi, Rathsman, sjb

Renormalization Scale Setting

$$R_{e^+e^-}(Q^2) \equiv 3 \sum_{\text{flavors}} e_q{}^2 \left[1 + \frac{\alpha_R(Q)}{\pi} \right] .$$

$$R_{e^+e^-}(Q^2) = 3 \sum_q e_q{}^2 \left[1 + \frac{\alpha_{\overline{\rm MS}}(Q)}{\pi} + \frac{\alpha_{\overline{\rm MS}}{}^2}{\pi^2}(1.98 - 0.115 n_f) + \cdots \right]$$

$$\to 3 \sum_q e_q{}^2 \left[1 + \frac{\alpha_{\overline{\rm MS}}(Q^*)}{\pi} + \frac{\alpha_{\overline{\rm MS}}{}^2(Q^*)}{\pi^2} 0.08 + \cdots \right],$$

$\boxed{Q^* = 0.710Q.}$ Notice that $\alpha_R(Q)$ differs from $\alpha_{\overline{\rm MS}}(Q^*)$ by only $0.08\alpha_{\overline{\rm MS}}/\pi$, so that $\alpha_R(Q)$ and $\alpha_{\overline{\rm MS}}(0.71Q)$ are effectively interchangeable (for any value of n_f).

Deep-inelastic scattering. The moments of the nonsinglet structure function $F_2(x,Q^2)$ obey the evolution equation

$$Q^2 \frac{d}{dQ^2} \ln M_n(Q^2)$$

$$= -\frac{\gamma_n^{(0)}}{8\pi} \alpha_{\overline{\mathrm{MS}}}(Q) \left[1 + \frac{\alpha_{\overline{\mathrm{MS}}}}{4\pi} \frac{2\beta_0\beta_n + \gamma_n^{(1)}}{\gamma_n^{(0)}} + \cdots \right]$$

$$\to -\frac{\gamma_n^{(0)}}{8\pi} \alpha_{\overline{\mathrm{MS}}}(Q_n^*) \left[1 - \frac{\alpha_{\overline{\mathrm{MS}}}(Q_n^*)}{\pi} C_n + \cdots \right],$$

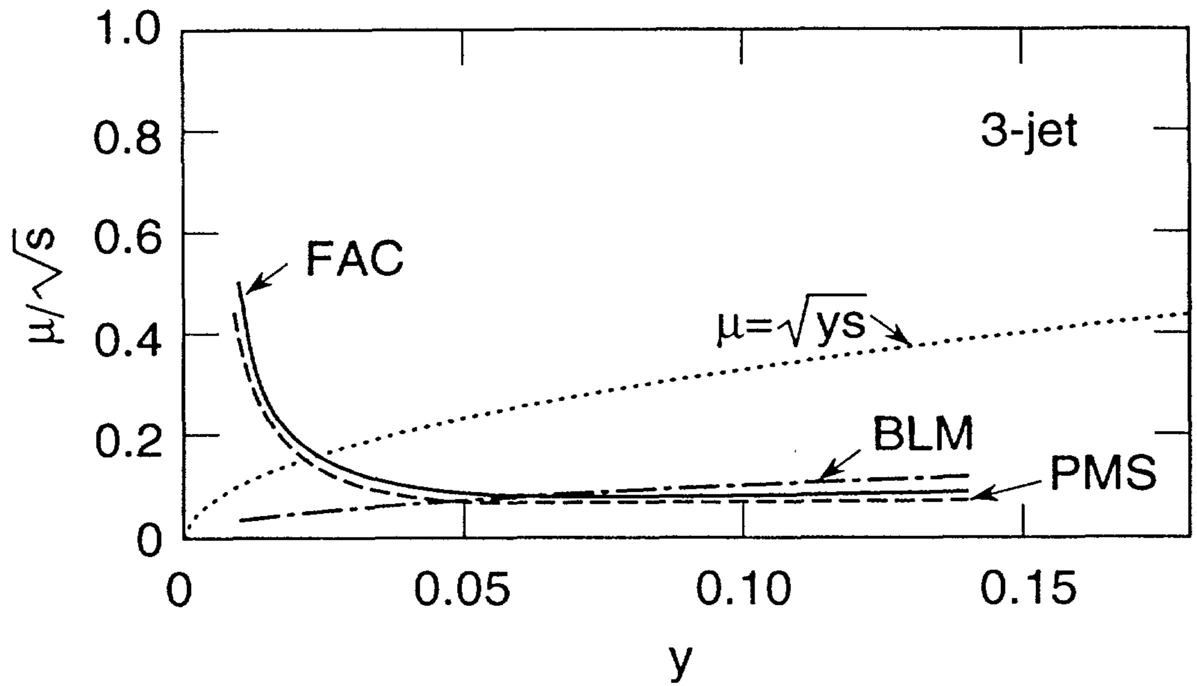
where, for example,

$$Q_2^* = 0.48Q, \quad C_2 = 0.27\,,$$

$$Q_{10}^* = 0.21Q, \quad C_{10} = 1.1\,.$$

For n very large, the effective scale here becomes $Q_n^* \sim Q/\sqrt{n}$

BLM scales for DIS moments



Kramer & Lampe

Three-Jet Rate

The scale $\mu/\sqrt{s}$ according to the BLM (dashed-dotted), PMS (dashed), FAC (full), and $\sqrt{y}$ (dotted) procedures for the three-jet rate in e^+e^- annihilation, as computed by Kramer and Lampe [10]. Notice the strikingly different behavior of the BLM scale from the PMS and FAC scales at low y. In particular, the latter two methods predict increasing values of μ as the jet invariant mass $\mathcal{M} < \sqrt{(ys)}$ decreases.

Rathsman

Other Jet Observables:

$$V(Q^2) = -\frac{C_F 4\pi\alpha_{\overline{MS}}(Q)}{Q^2}\left[1 + \frac{\alpha_{\overline{MS}}}{\pi}\left(\tfrac{5}{12}\beta_0 - 2\right) + \cdots\right]$$

$$\rightarrow -\frac{C_F 4\pi\alpha_{\overline{MS}}(Q^*)}{Q^2}\left[1 - \frac{\alpha_{\overline{MS}}(Q^*)}{\pi}2 + \cdots\right],$$

where $Q^* = e^{-5/6}$, $Q \cong 0.43Q$. This result shows that the effective scale of the $\overline{\text{MS}}$ scheme should generally be about half of the true momentum transfer occurring in the interaction. In parallel to QED, the effective potential $V(Q^2)$ gives a particularly intuitive scheme for defining the QCD coupling constant

$$V(Q^2) \equiv -\frac{4\pi C_F \alpha_v(Q)}{Q^2}$$

Application of BLM
to Multi-Scale Threshold Production

Hoang, Kuhn, Tuebner, SJB

Phys.Lett.B359:355-361,1995

Features of BLM Scale Setting

- All terms associated with nonzero beta function summed into running coupling
- Conformal series preserved
- BLM Scale Q* sets the number of active flavors
- Correct analytic dependence in the quark mass
- Only n_f dependence required to determine renormalization scale at NLO
- Result is scheme independent: Q* has exactly the correct dependence to compensate for change of scheme
- Correct Abelian limit!

$$\lim N_C \to 0 \text{ at fixed } \alpha = C_F \alpha_s, n_\ell = n_F / C_F$$

$$\text{QCD} \to \text{Abelian Gauge Theory}$$

Analytic Feature of SU(Nc) Gauge Theory

Huet, sjb

Relate Observables to Each Other

- Eliminate intermediate scheme
- No scale ambiguity
- Transitive!
- Commensurate Scale Relations
- Example: Generalized Crewther Relation

$$\frac{\alpha_R(Q)}{\pi} = \frac{\alpha_{\overline{MS}}(Q)}{\pi} + \left(\frac{\alpha_{\overline{MS}}(Q)}{\pi}\right)^2 \left[\left(\frac{41}{8} - \frac{11}{3}\zeta_3\right) C_A - \frac{1}{8} C_F + \left(-\frac{11}{12} + \frac{2}{3}\zeta_3\right) f\right]$$
$$+ \left(\frac{\alpha_{\overline{MS}}(Q)}{\pi}\right)^3 \left\{ \left(\frac{90445}{2592} - \frac{2737}{108}\zeta_3 - \frac{55}{18}\zeta_5 - \frac{121}{432}\pi^2\right) C_A^2 + \left(-\frac{127}{48} - \frac{143}{12}\zeta_3 + \frac{55}{3}\zeta_5\right) C_A C_F - \frac{23}{32} C_F^2 \right.$$
$$+ \left[\left(-\frac{970}{81} + \frac{224}{27}\zeta_3 + \frac{5}{9}\zeta_5 + \frac{11}{108}\pi^2\right) C_A + \left(-\frac{29}{96} + \frac{19}{6}\zeta_3 - \frac{10}{3}\zeta_5\right) C_F\right] f$$
$$\left. + \left(\frac{151}{162} - \frac{19}{27}\zeta_3 - \frac{1}{108}\pi^2\right) f^2 + \left(\frac{11}{144} - \frac{1}{6}\zeta_3\right) \frac{d^{abc}d^{abc}}{C_F d(R)} \frac{\left(\sum_f Q_f\right)^2}{\sum_f Q_f^2} \right\}.$$

$$\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_{\overline{MS}}(Q)}{\pi} + \left(\frac{\alpha_{\overline{MS}}(Q)}{\pi}\right)^2 \left[\frac{23}{12} C_A - \frac{7}{8} C_F - \frac{1}{3} f\right]$$
$$+ \left(\frac{\alpha_{\overline{MS}}(Q)}{\pi}\right)^3 \left\{ \left(\frac{5437}{648} - \frac{55}{18}\zeta_5\right) C_A^2 + \left(-\frac{1241}{432} + \frac{11}{9}\zeta_3\right) C_A C_F + \frac{1}{32} C_F^2 \right.$$
$$\left. + \left[\left(-\frac{3535}{1296} - \frac{1}{2}\zeta_3 + \frac{5}{9}\zeta_5\right) C_A + \left(\frac{133}{864} + \frac{5}{18}\zeta_3\right) C_F\right] f + \frac{115}{648} f^2 \right\}.$$

Apply BLM, Eliminate MSbar, Find Amazing Simplification

Renormalization Scale Setting

$$\int_0^1 dx\,[g_1^{ep}(x,Q^2) - g_1^{en}(x,Q^2)] \equiv \frac{1}{3}\left|\frac{g_A}{g_V}\right|\left[1 - \frac{\alpha_{g_1}(Q)}{\pi}\right]$$

$$\frac{\alpha_{g_1}(Q)}{\pi} = \frac{\alpha_R(Q^*)}{\pi} - \left(\frac{\alpha_R(Q^{**})}{\pi}\right)^2 + \left(\frac{\alpha_R(Q^{***})}{\pi}\right)^3$$

Geometric Series in Conformal QCD

Generalized Crewther Relation

add Light-by-Light

Lu, Kataev, Gabadadze, Sjb

Generalized Crewther Relation

$$[1 + \frac{\alpha_R(s^*)}{\pi}][1 - \frac{\alpha_{g_1}(q^2)}{\pi}] = 1$$

$$\sqrt{s^*} \simeq 0.52Q$$

Conformal relation true to all orders in perturbation theory

Commensurate Scale Relation:

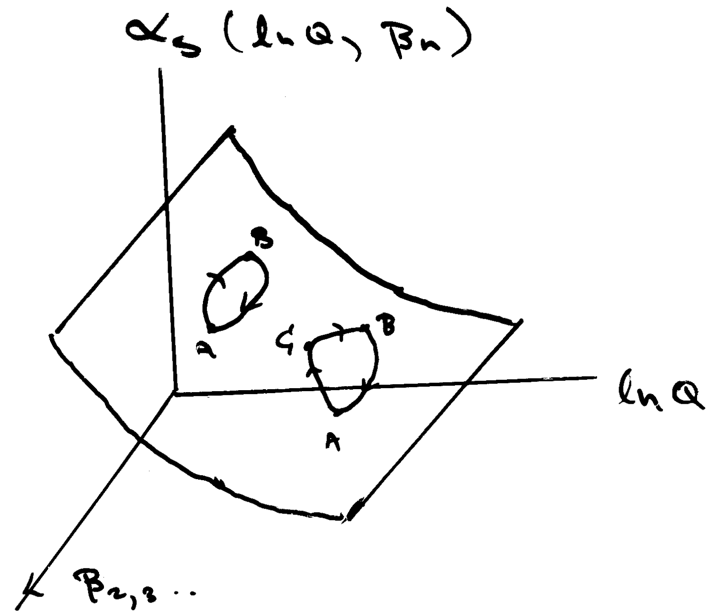
$$* \quad \alpha_B(Q_B) = \alpha_A(Q_A)\left[1 + C^{(1)}_{A/B}\,\frac{\alpha_A}{\pi} + \cdots\right]$$

conformal coeff.

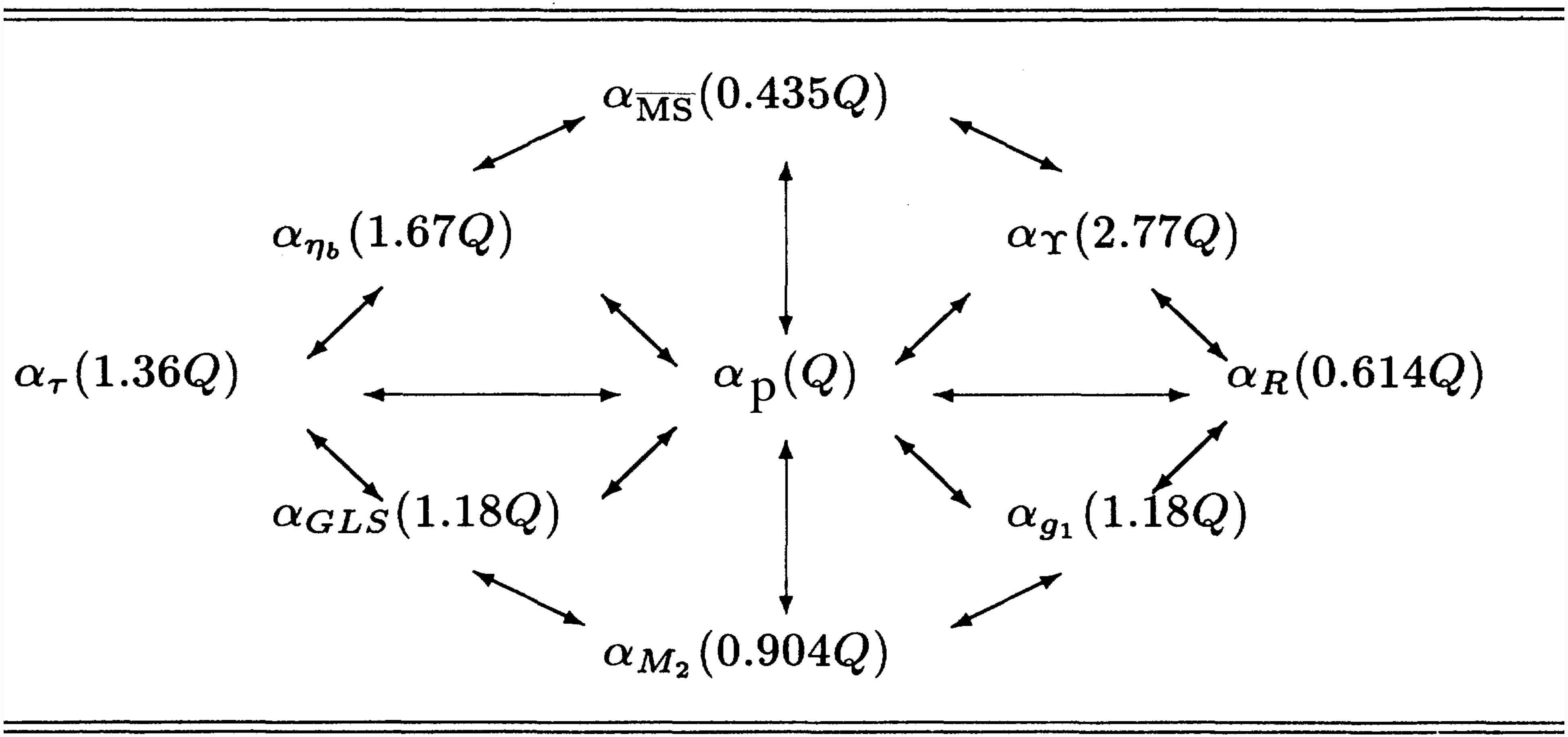
$$Q_B/Q_A = \lambda_{B/A}$$

Peterman Stückelberg Renormalization "Group"

$$\left\{\begin{array}{lll} \lambda_{B/A} = \lambda_{B/C}/\lambda_{A/C} & & \text{transitive} \\ \lambda_{B/A} = \lambda^{-1}_{A/B} & & \text{symmetry} \\ \lambda_{A/A} = I & & \text{identity} \end{array}\right.$$



Leading Order Commensurate Scales



Translate between schemes at LO

Unification in Physical Schemes

"PHYSICAL RENORMALIZATION SCHEMES AND GRAND UNIFICATION"
M.B. and Stanley J. Brodsky. **Phys.Rev.D69:095007,2004**

$$\alpha_i(Q) = \frac{\alpha_i(Q_0)}{1+\hat{\Pi}_i(Q)-\hat{\Pi}_i(Q_0)}$$

i=1,2,3

$$\hat{\Pi}_i(Q) = \frac{\alpha_i}{4\pi}\sum_p \beta_i^{(p)}\left(L_{s(p)}(Q^2/m_p^2)+\cdots\right)$$

"log-like" function:

$$L_{s(p)} \approx \log(e^{\eta_p} + Q^2/m_p^2)$$

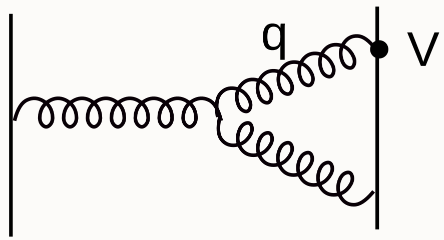
$$\eta_p = 8/3,\ 5/3,\ 40/21$$

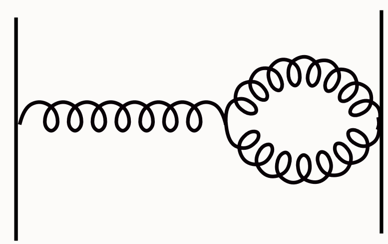
For spin s(p) = 0, ½, and 1

→ Elegant and natural formalism for all threshold effects

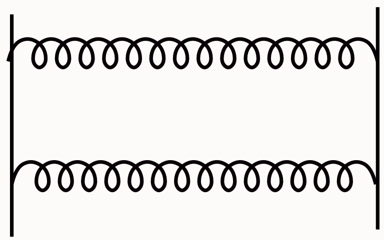
The Pinch Technique

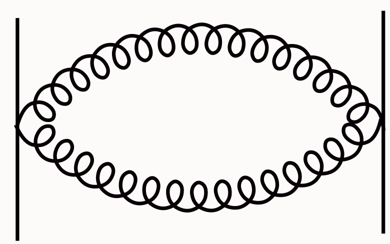
(Cornwall, Papavassiliou)

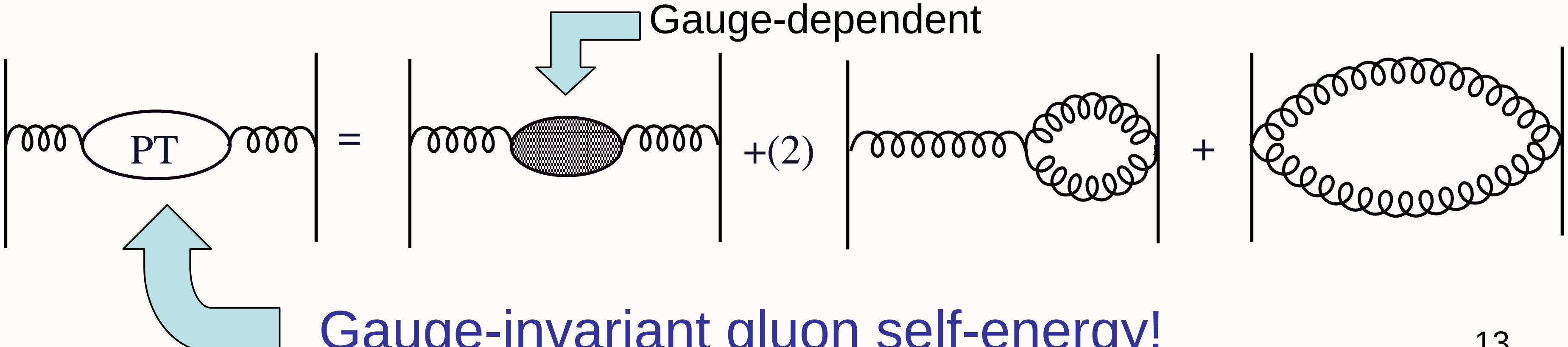




$$q \cdot V(p,k) = S^{-1}(p) - S^{-1}(k)$$





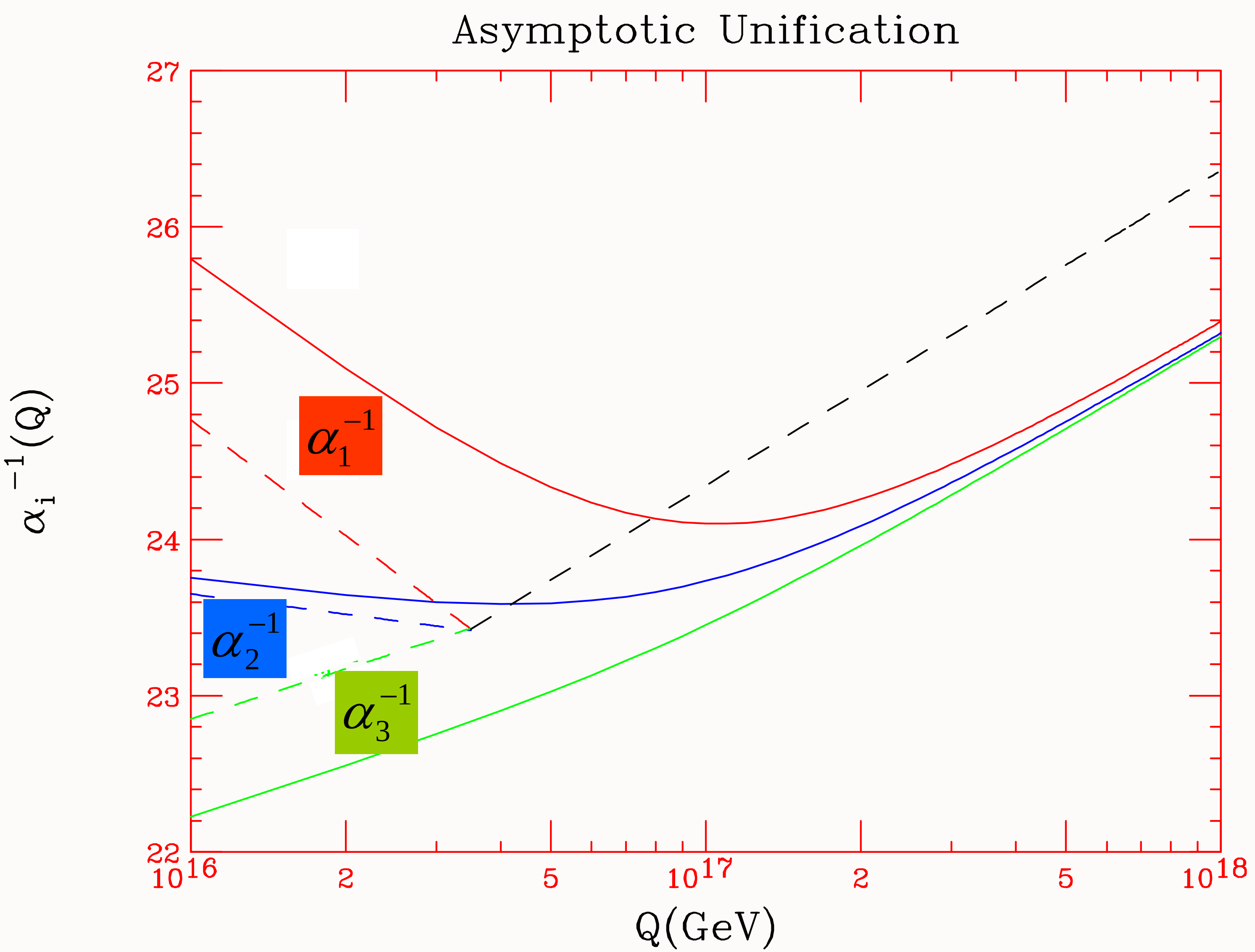


Gauge-invariant gluon self-energy!

13

natural generalization of QED charge

Renormalization Scale Setting



Binger, sjb

Renormalization Scale Setting

Analyticity and Mass Thresholds

$\overline{MS}$ does not have automatic decoupling of heavy particles

→ Must define a set of schemes in each desert region and match

$$\alpha_s^{(f)}(M_Q) = \alpha_s^{(f+1)}(M_Q)$$

- The coupling has discontinuous derivative at the matching point
- At higher orders the coupling itself becomes discontinuous!
- Does not distinguish between spacelike and timelike momenta

"AN ANALYTIC EXTENSION OF THE MS-BAR RENORMALIZATION SCHEME"
S. Brodsky, M. Gill, M. Melles, J. Rathsman. **Phys.Rev.D58:116006,1998**

Unification in Physical Schemes

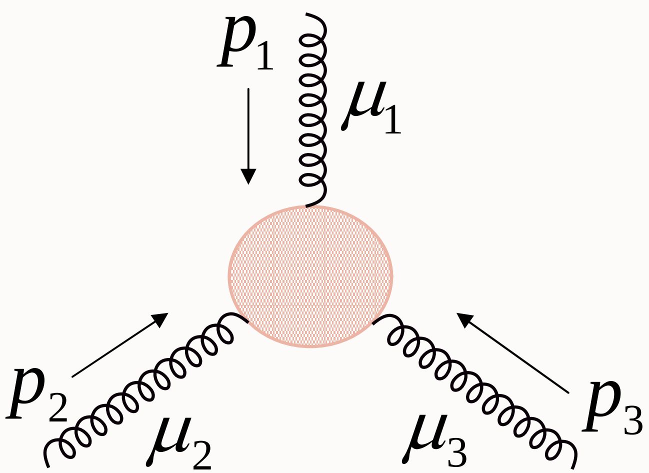
- Smooth analytic threshold behavior with automatic decoupling
- More directly reflects the unification of the forces
- Higher “unification” scale than usual

General Structure of the Three-Gluon Vertex

"THE FORM-FACTORS OF THE GAUGE-INVARIANT THREE-GLUON VERTEX"

Binger, sjb **hep-ph/0602199.**

$$\hat{\Gamma}_{\mu_1\mu_2\mu_3} =$$



Full calculation, general masses, spin

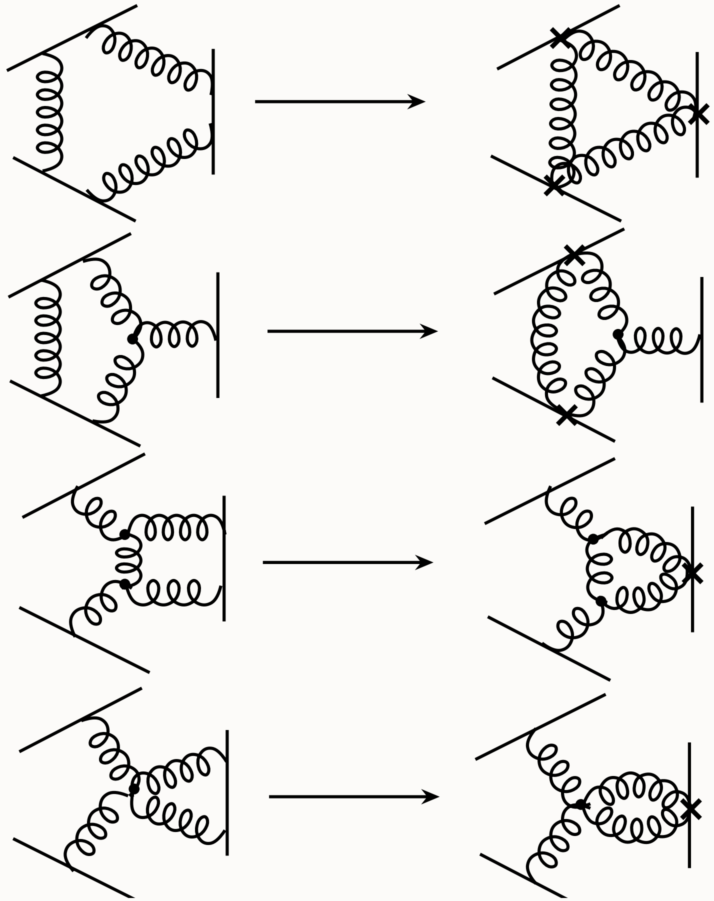
3 index tensor $\hat{\Gamma}_{\mu_1\mu_2\mu_3}$ built out of $g_{\mu\nu}$ and p_1, p_2, p_3

with $p_1 + p_2 + p_3 = 0$

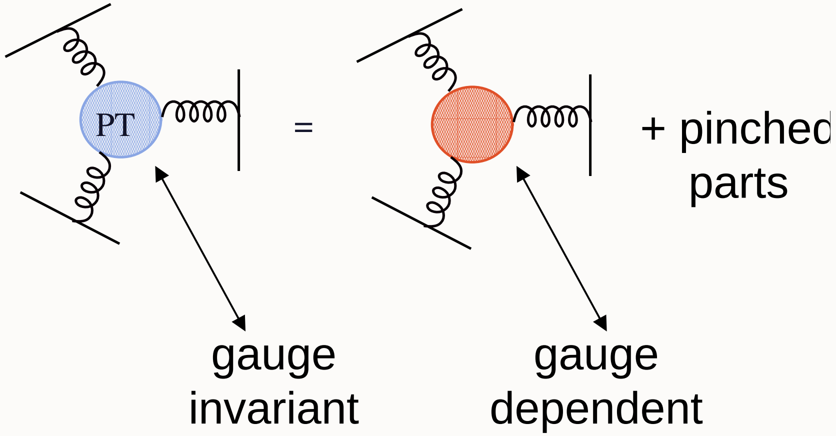
⇒ 14 basis tensors and form factors

The Gauge Invariant Three Gluon Vertex

Cornwall and Papavassiliou performed the PT construction :



The "pinched" parts are added to the "regular" 3 gluon vertex



Form Factors : Supersymmetric Relations (Massless)

....but certain linear sums are simple :

$$\Sigma_{QG}(F) \equiv \frac{d-2}{2} F_Q + F_G \longrightarrow 0$$

for 7 of the 13 FF's (in physical basis)

Simple N=1 SUSY contribution in d=4

$$F_G + 4F_Q + (10-d)F_S = 0$$

For all FF's !!

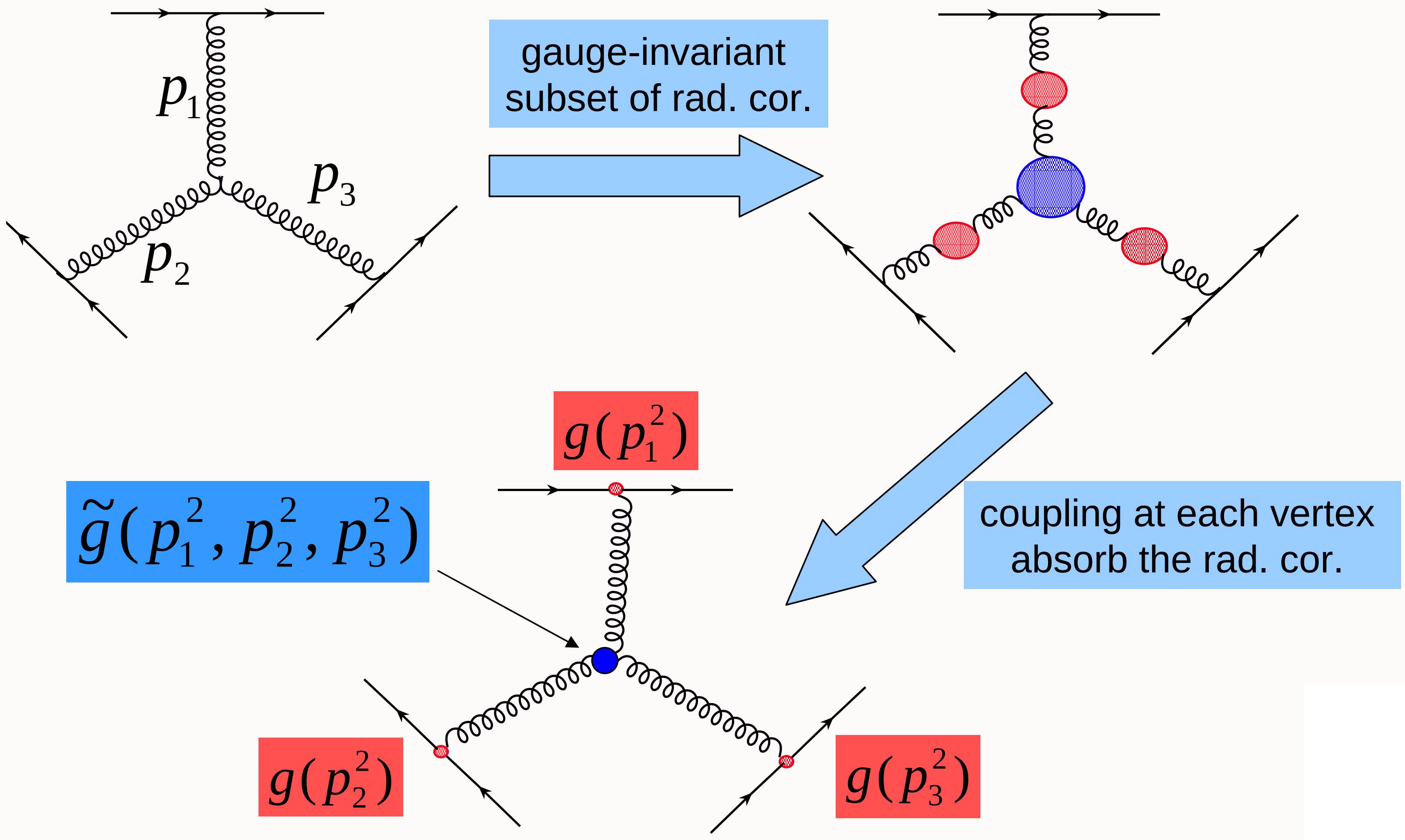
N=4 SUSY in d=4 gives 0

These are off-shell generalizations of relations found in SUSY scattering amplitudes by Z. Bern, L.J. Dixon, D.C. Dunbar, and D.A. Kosower (NPB 425,435)

Summary of Supersymmetric Relations

Massless	***Massive***
$F_G + 4F_Q + (10-d)F_S = 0$	$F_{MG} + 4F_{MQ} + (9-d)F_{MS} = 0$
$\Sigma_{QG}(F) \equiv \frac{d-2}{2} F_Q + F_G$	$\Sigma_{MQG}(F) \equiv \frac{d-1}{2} F_{MQ} + F_{MG}$
= simple	= simple

Multi-scale Renormalization of the Three-Gluon Vertex



3 Scale Effective Charge

$$\tilde{\alpha}(a,b,c) \equiv \frac{\tilde{g}^2(a,b,c)}{4\pi}$$

(First suggested by H.J. Lu)

$$\frac{1}{\tilde{\alpha}(a,b,c)} = \frac{1}{\alpha_{bare}} + \frac{1}{4\pi}\beta_0\left(L(a,b,c) - \frac{1}{\varepsilon} + \cdots\right)$$

$$\frac{1}{\tilde{\alpha}(a,b,c)} = \frac{1}{\tilde{\alpha}(a_0,b_0,c_0)} + \frac{1}{4\pi}\beta_0\left[L(a,b,c) - L(a_0,b_0,c_0)\right]$$

L(a,b,c) = 3-scale “log-like” function

L(a,a,a) = log(a)

3 Scale Effective Scale

$$L(a,b,c) \equiv \log\left(Q^2_{eff}(a,b,c)\right) + i\,\mathrm{Im}\,L(a,b,c)$$

Governs strength of the three-gluon vertex

$$\frac{1}{\tilde{\alpha}(a,b,c)} = \frac{1}{\tilde{\alpha}(a_0,b_0,c_0)} + \frac{1}{4\pi}\beta_0\left[L(a,b,c) - L(a_0,b_0,c_0)\right]$$

$$\hat{\Gamma}_{\mu_1\mu_2\mu_3} \propto \sqrt{\tilde{\alpha}(a,b,c)}$$

Generalization of BLM Scale to 3-Gluon Vertex

Properties of the Effective Scale

$$Q^2_{eff}(a,b,c) = Q^2_{eff}(-a,-b,-c)$$

$$Q^2_{eff}(\lambda a, \lambda b, \lambda c) = |\lambda| Q^2_{eff}(a,b,c)$$

$$Q^2_{eff}(a,a,a) = |a|$$

$$Q^2_{eff}(a,-a,-a) \approx 5.54|a|$$

$$Q^2_{eff}(a,a,c) \approx 3.08|c| \quad \text{for} \quad |a| >> |c|$$

$$Q^2_{eff}(a,-a,c) \approx 22.8|c| \quad \text{for} \quad |a| >> |c|$$

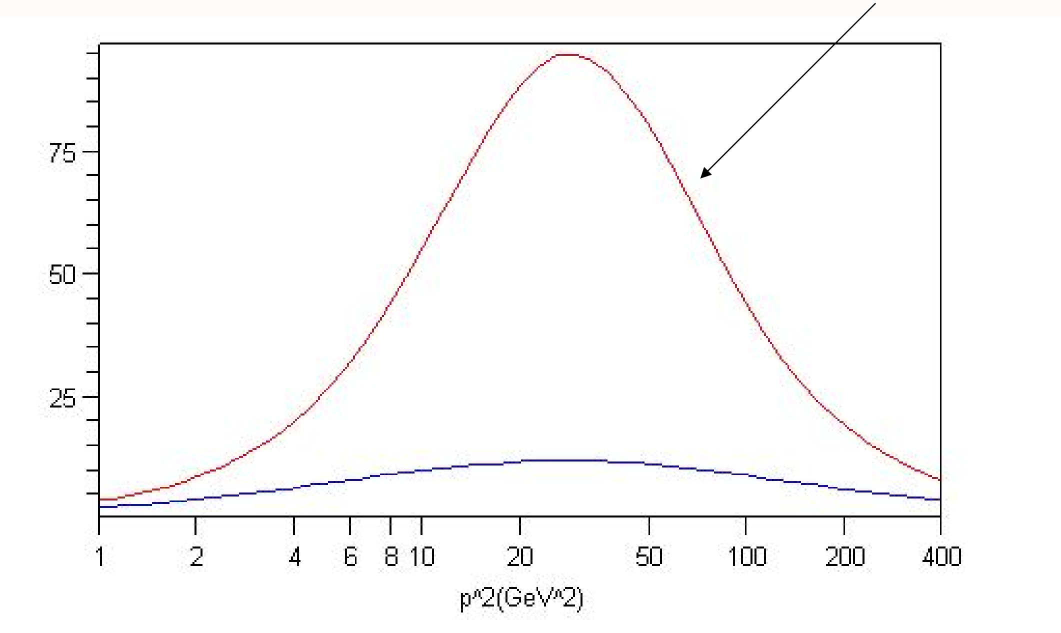
$$Q^2_{eff}(a,b,c) \approx 22.8\frac{|bc|}{|a|} \quad \text{for} \quad |a| >> |b|,|c|$$

Surprising dependence on Invariants

The Effective Scale

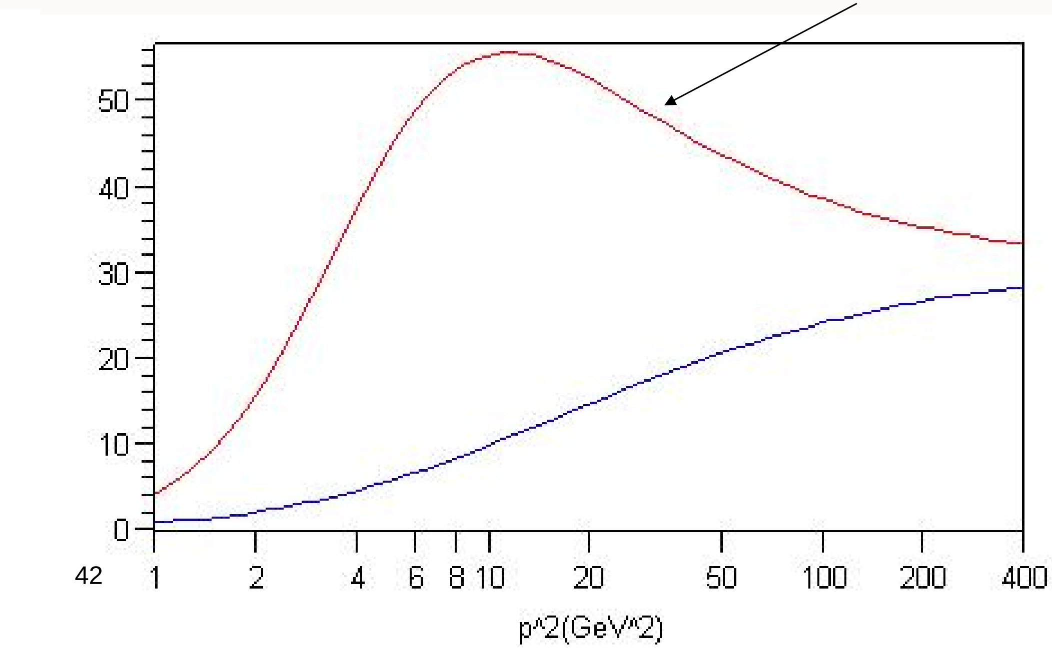
$Q^2_{eff}\left(10\,\mathrm{GeV}^2, 10\,\mathrm{GeV}^2, p^2\right)$

$Q^2_{eff}\left(-10\,\mathrm{GeV}^2, -10\,\mathrm{GeV}^2, p^2\right)$



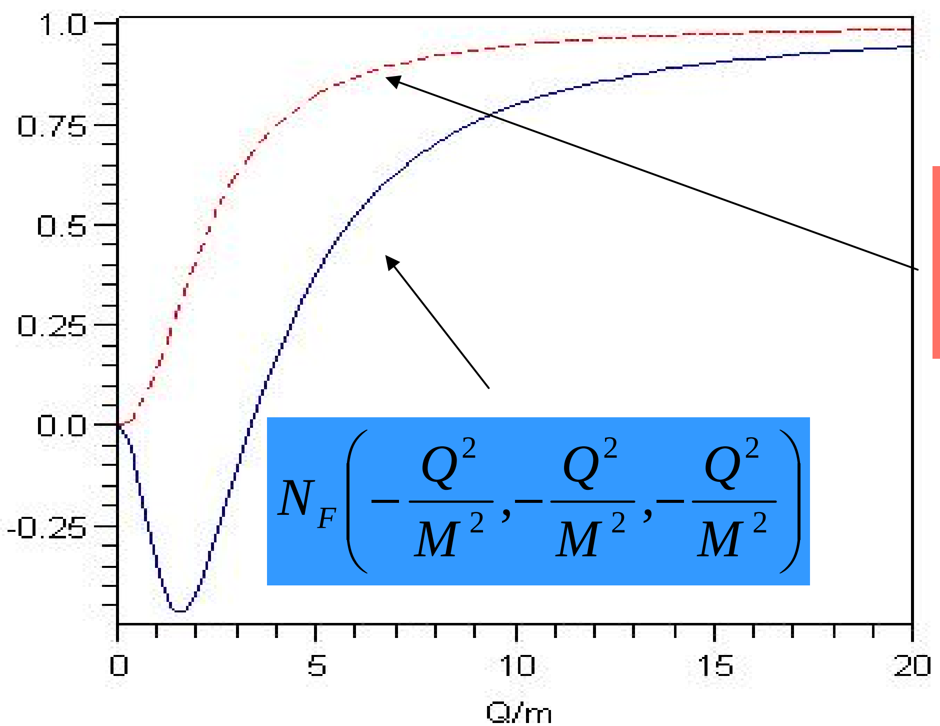
$Q^2_{eff}\left(10\,\mathrm{GeV}^2, p^2, p^2\right)$

$Q^2_{eff}\left(-10\,\mathrm{GeV}^2, p^2, p^2\right)$



Effective Number of Flavors

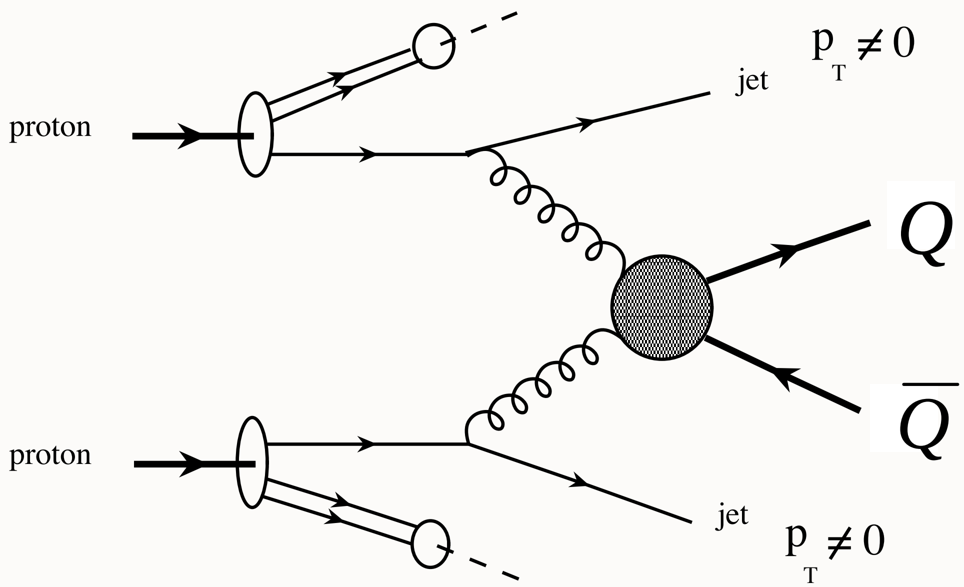
$$N_F\left(\frac{a}{M^2},\frac{b}{M^2},\frac{c}{M^2}\right)=-\frac{d}{d\log M^2}L_{MQ}\left(\frac{a}{M^2},\frac{b}{M^2},\frac{c}{M^2}\right)$$

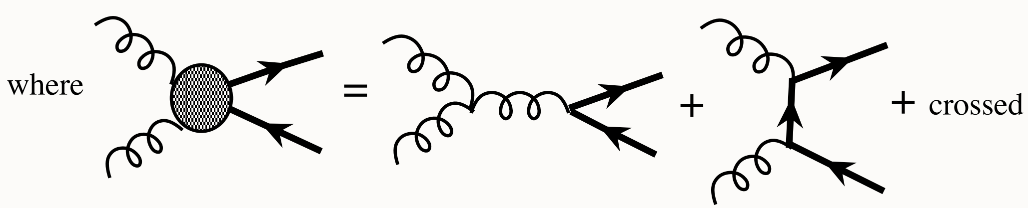


$$n_f\left(\frac{Q^2}{M^2}\right)=-\frac{d}{d\log M^2}L_{1/2}\left(\frac{Q^2}{M^2}\right)$$

$$\approx\frac{1}{1+\frac{M^2}{Q^2}e^{5/3}}$$

Heavy Quark Hadro-production





- Preliminary calculation using (massless) results for tree level form factor
- Very low effective scale

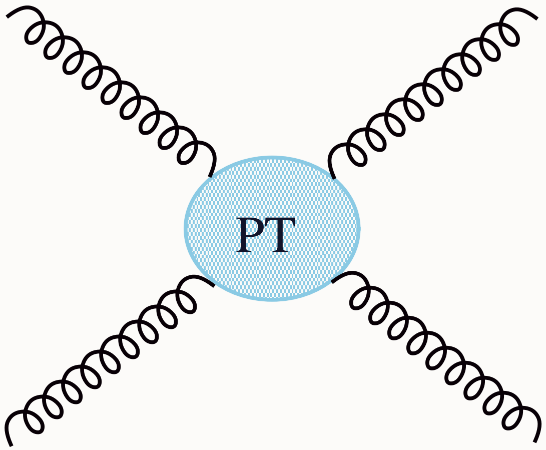
$\Rightarrow$ much larger cross section than $\overline{MS}$ with scale $\mu_R = M_{Q\overline{Q}}$ or M_Q

- Future : repeat analysis using the full mass-dependent results and include all form factors

Expect that this approach accounts for most of the one-loop corrections

Future Directions

Gauge-invariant four gluon vertex



$$L_4(p_1, p_2, p_3, p_4)$$

$$Q^2_{4\,eff}(p_1, p_2, p_3, p_4)$$

Hundreds of form factors!

Summary and Future

- ***Multi-scale analytic*** renormalization based on ***physical, gauge-invariant*** Green's functions

- ***Optimal*** improvement of perturbation theory with ***no scale-ambiguity*** since physical kinematic invariants are the arguments of the (multi-scale) couplings

Conventional renormalization scale-setting method:

- Guess arbitrary renormalization scale and take arbitrary range. Wrong for QED and Precision Electroweak.
- Prediction depends on choice of renormalization scheme
- Variation of result with respect to renormalization scale only sensitive to nonconformal terms; no information on genuine (conformal) higher order terms
- Conventional procedure has no scientific basis.
- FAC and PMS give unphysical results.
- Renormalization scale not arbitrary: Analytic constraint from flavor thresholds

Use Physical Scheme to Characterize QCD Coupling

- Use Observable to define QCD coupling or Pinch Scheme
- Analytic: Smooth behavior as one crosses new quark threshold
- New perspective on grand unification

Binger, Sjb

Factorization scale

$$\mu_{\text{factorization}} \neq \mu_{\text{renormalization}}$$

- Arbitrary separation of soft and hard physics
- Dependence on factorization scale not associated with beta function - present even in conformal theory
- Keep factorization scale separate from renormalization scale $\frac{d\mathcal{O}}{d\mu_{\text{factorization}}} = 0$
- Residual dependence when one works in fixed order in perturbation theory.

Use BLM!

- Satisfies Transitivity, all aspects of Renormalization Group; scheme independent
- Analytic at Flavor Thresholds
- Preserves Underlying Conformal Template
- Physical Interpretation of Scales; Multiple Scales
- Correct Abelian Limit ($N_C = 0$)
- Eliminates unnecessary source of imprecision of PQCD predictions
- Commensurate Scale Relations: Fundamental Tests of QCD free of renormalization scale and scheme ambiguities
- BLM used in many applications, QED, LGTH, BFKL, ...