An N-tropic Solution to the Cosmological Constant Problem

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Based on the assertion that the cosmological constant problem is essentially a quantum gravity problem, the framework which addresses the cosmological constant problem should also bear a picture for the "quantum space-time". In this talk in an attempt to address the cosmological constant problem I suggest to start with noncommutative fuzzy spheres as the toy model for the quantum space-time. In this setting, we show that the cosmological constant problem may be resolved due to the noncommutativity and "fuzziness" of the space and the fact that the smallest volume which could be measured in the a quantum space-time is much larger than the naively expected Planckian size. This talk is based on [1] which has appeared on the arXiv as hep-th/0605110.

I. INTRODUCTION

The Cosmological Constant (CC) problem, has been around since the early days of conception of the theory of General Relativity: The CC term is the only possibility which is generally covariant with less than two number of derivatives (of metric), and hence could be added to the gravity action. Explicitly, the action

$$S_{E.H.} = M_{Pl}^2 \int d^4x\ \sqrt{-g}(R + \frac{2}{M_{Pl}^2}\Lambda) \qquad (1)$$

where

$$M_{Pl}^2 \equiv \frac{1}{4\pi G_N}, \qquad (2)$$

describes the Einstein gravity plus the cosmological constant Λ whose value is *not* determined by any symmetry principle in the "standard physics"; its value should be determined through observation, or through a more fundamental theory. In the above conventions Λ is a given constant of dimension of energy density (*i.e.* energy^4).

For a long time, based on imprecise cosmological and astrophysical observations, it was believed that the value of Λ should be zero. The recent cosmic microwave background observations plus the supernovae data, however, have established that our Universe is now in an accelerating phase. The simplest, though not the only, possibility is to attribute this late-time cosmic acceleration to a non-vanishing CC which has a *positive* sign [2]:

$$\Lambda \simeq 3.6 \times 10^{-6} GeV\ cm^{-3} = 10^{-120} M_{Pl}^4. \qquad (3)$$

As we see the above value is not only non-zero, but also very small (basically in any physically relevant scale). This constitutes what is known as the cosmological constant problem.

To be more precise, there are two essentially different issues which come under the title of cosmological constant problem.

§ The first is a theoretical problem, "how can one stabilize the value of the cosmological constant (against quantum corrections) in a theoretical setting?"

§§ The second is regarding the value it has according to current observations, "why Λ has the value it has and why it is so small, actually smaller than any other physical quantity, in the scales natural to a pure gravitational problem, the Planck mass M_{Pl}. These two problems sometimes are also called the "technical naturalness" and the "naturalness" of the cosmological constant problem, respectively.

To just show why both of the CC problems are so non-trivial let us recall that in a physical model the action (1) should be added to the action governing the rest of the model. From *any* quantum field theory viewpoint the CC term, $\sqrt{-\det g}\Lambda$, which is proportional to *identity* operator, is obviously a relevant operator whose coefficient, the cosmological constant Λ, is receiving quantum corrections and, regardless of the details of the quantum field theory we are using, is UV divergent. If the UV cut-off of the theory is at Λ_{cutoff}, then the resulting CC is proportional to the zero point energy which is proportional to Λ_{cutoff}^4, the proportionality constant is positive for bosons and negative for fermions. (The supersymmetric theories use this fact to cancel the zero point energies, and hence the CC, by matching the number of bosonic and fermionic degrees of freedom.) On the other hand, from the gravity theory point of view the CC is an IR problem and has something to do with the large scale structure of the Universe. Therefore, there is not even a consensus whether the CC problem is a UV or an IR problem.

Here I advocate a different viewpoint that, as we'll see, reconciles the quantum field theory and cosmology viewpoints. Namely,

- Cosmological Constant is essentially a Quantum Gravity problem and
- Quantum Gravity should be formulated on a "Quantum Space-time".
- One of the features of quantum gravity, which is unlike standard quantum field theories, is presence of IR/UV mixing.

The first statement has been previously alluded to in

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the literature, *e.g.* see [3] and one may hope that a theory of quantum gravity, for example string theory, should bear the solution to the CC problem. Within string theory, however, this has remained a challenging puzzle. Therefore, here I try to provide an alternative route utilizing the second statement, starting from baby examples of the quantum space-times and discuss a gravity theory which has these "quantum space-times" as its solutions.

In this talk we will address the technical naturalness problem of the cosmological constant in a *Euclidean* setting. Our line of logic will be as follows:

I) We introduce the "quantum" Euclidean de Sitter space which is a "quantum" sphere, the *fuzzy sphere*, S^4_F. Fuzzy spheres are described by $N \times N$ Hermitian matrices and their radii (in units of Planck length) is completely determined in terms of the *size of matrices* N.
II) We present a *Euclidean* quantum gravity theory which is a Matrix theory with Matrix-valued vierbein and spin connection as its degrees freedom.
III) We show that in this setting the vacuum solution is a *fuzzy four sphere*, S^4_F. The radius of the vacuum S^4_F solution determines the (Euclidean) cosmological constant. In our setting, therefore, the CC is integer-valued and is hence stable against perturbative quantum corrections.

As we will argue IR/UV mixing is an inherent feature of the gravity theory we use. Although it has not been shown explicitly, one expects that if we add matter fields to our matrix gravity theory, the matrix theory will tame and control the zero point energy contributions. Note, however, that all the other fields must be also added in the form of a matrix theory. This matrix theory is to be constructed such that in the large matrices (continuum) limit it goes over to the corresponding standard field theories.

II. A SHORT INTRODUCTION TO FUZZY SPHERES

The idea we use to quantize a given space-time parallels the steps of moving from classical mechanics to quantum mechanics, where we replace classical phase space by quantum phase space and Poisson brackets with commutators. We also note that structure of Poisson bracket is related to the isometries of the phase space, precisely the volume preserving diffeomorphisms in the phase space.

Let us focus on the case of our interest, the spheres (or Euclidean de Sitter spaces). First we recall that geometric S^d, in an algebraic setting, is equivalent to the quotient $SO(d+1)/SO(d)$ and everything could be put in the representations of the isometry group $SO(d+1)$. For the round commutative sphere we are dealing with the *infinite* dimensional representation of the $SO(d+1)$. This could be seen, e.g. when we expand any given function on the S^d in terms of $SO(d+1)$ spherical harmonics. The latter are generalization of Y_{lm}'s, the $SO(3)$ spherical harmonics. Explicitly, for a given function $\Phi(x)$,

$$\Phi(x) = \sum_{j=0}^{\infty} \Phi_{i_1 i_2 \cdots i_j} x^{i_1} x^{i_2} \cdots x^{i_j}\,, \tag{4}$$

this is reflected in the fact that the sum goes to infinity. In (4) i_k is ranging from one to $d+1$ and

$$\sum_{i=1}^{d+1} x_i^2 = R^2, \tag{5}$$

where R is the radius of the sphere.

The S^d_F is the *quantized* or "fuzzified" version of d-sphere in such a way that the $SO(d+1)$ invariance remains intact. This can be achieved noting the fact that $SO(d+1)$ is a compact group and has finite dimensional unitary representations. That is, we replace the continuous coordinates x_i with $N \times N$ matrix coordinates X_i, such that

$$\sum_{i=1}^{d+1} X_i^2 = R^2 \mathbf{1}_{N\times N}. \tag{6}$$

Explicit form of the matrices X^i could be worked out using representation theory of $SO(d+1)$ group, e.g. see [4, 5], with the generic result that

$$R^{d-1} \sim N \tag{7}$$

for large N. Any function (field) on the S^d_F is an $N\times N$ matrix. One may expand any field $\Phi(X)$ in terms of "truncated" $SO(d+1)$ spherical harmonics:

$$\Phi(X) = \sum_{j=0}^{J_{max}} \Phi_{i_1 i_2 \cdots i_j} X^{i_1} X^{i_2} \cdots X^{i_j}\,, \tag{8}$$

where $i_k = 1, 2, \cdots, d+1,\ 0 \le k \le J_{max}$ and J_{max} is related to the size of matrices N as

$$N \sim J^{d-1}_{max}. \tag{9}$$

Therefore, on the fuzzy spheres there is a *natural, inherent, cut-off* on the maximum possible angular momentum.

One may define a short length scale on the sphere using the R-N relation:

$$l^{d-1} \equiv R^{d-1}/N \tag{10}$$

or written in a more suggestive way,

$$R \sim l J_{max} \quad \text{or} \quad J_{max} \sim X_{max} P_{max} \sim R \times \frac{1}{l}$$

where X_{max} and P_{max} are respectively the maximum value X or momentum P can take on the fuzzy sphere of radius R.

Finally, we would like to point out that the *smallest observable volume* on S_F^d is *not* l^d, but L^d where

$$L^d = (lR)^{d/2}. \tag{11}$$

More details on derivation and on discussion of the above can be found in [1].

III. THE MATRIX QUANTUM GRAVITY THEORY

We now present a model for quantum gravity, though a Euclidean one, which has fuzzy spheres among its vacuum solutions. In this theory the gravitational degrees of freedom, as well as the spacetime coordinates, are $N \times N$ hermitian matrices. That is, it is a Matrix *Euclidean* quantum gravity theory.

This gravity theory, which has been discussed in some detail in [6] and reviewed in [1], is based on the Mansouri-Cheng "gravity as gauge theory"[7]: In the ordinary (Einstein) gravity theory which has a group manifold G/H as its vacuum solutions, the coordinates $x_i,\ i = 1, 2, \cdots, d \equiv dimG - dimH$ that parameterize G/H space, are in the (infinite dimensional unitary) representation of the Lie algebra of G, g. In this setting the gravitational degrees are the "vierbein" $e_i^a(x),\ a = 1, 2, \cdots, d$ and the connection $\Omega_i^\alpha(x)$, $\alpha = 1, 2, \cdots, dimH$.[11] They appear through the covariant derivative $\mathcal{D}_i$

$$\mathcal{D}_i = \partial_i + e_i^a(x)T^a + \Omega_i^\alpha(x)I^\alpha \tag{12}$$

where $I^\alpha \in h$ form a complete basis for the fundamental representation of the Lie algebra of H, h, T^a the basis for $g - h$, and hence (T^a, I^α) form a complete basis for g.[12] As such they are $d \times d$ unitary matrices. The gravity action is then constructed from gauge invariant powers (of commutators) of $\mathcal{D}_i$. In our example $G = SO(5)$ and $H = SO(4)$ and since the enveloping algebra of G is other than G, it is $U(4)$, a is running from one to four and α from one to 12. In our example the most natural from for the action is the Chern-Simons gravity [6].

Using the above setup it is (in a straightforward way) possible to construct the gravity theory on a noncommutative "fuzzified" geometry. In order that we need to have some knowledge about the representation theory of the groups G and H. In particular, it is important to note that *if G and H are compact groups*, there exist finite dimensional unitary $N \times N$ representations. This representation is naturally embedded in $u(N)$. To formulate our gravity theory we pick this representation and take the coordinates x_i, $e_i^a(x)$ and $\Omega_i^\alpha(x)$ all to be in this representation. Note that these are in general non-commuting. $\mathcal{D}_i$ are then taking values in $U(d) \otimes U(N)$. ($U(d)$ is the enveloping algebra of G.) The curvature two-form in the non-commuting case can again be defined as $\mathcal{F}_{ij} = [\mathcal{D}_i, \mathcal{D}_j]$.

As the x_i's, and hence the derivatives ∂_i, are non-commuting, $\mathcal{F}_{ij}$ has a constant piece. That is this part which leads to the cosmological constant term in the gravity action. $\mathcal{F}_{ij}$ has also a piece which is proportional to I^α. This part contains the Riemann curvature two form $\mathcal{R}_{ij}$ and a part proportional to T^a which is the torsion [6].

In our case, where $d = 4$ and $G/H = SO(5)/SO(4)$, we choose $I^\alpha = \{i\gamma^5, \gamma^a\gamma^5, \gamma^{ab}, i\mathbf{1}\}$, $T^a = i\gamma^a$ and for the action we take the Chern-Simons action

$$S = \kappa \frac{1}{N} \mathrm{Tr}(\gamma^5 \epsilon_{ijkl} \mathcal{F}_{ij} \mathcal{F}_{kl}) \tag{13}$$

where the Tr is over both the 4×4 and $N \times N$ matrices. After expanding the above action in terms of the Riemann curvature and the torsion, what we find is an Einstein-Hilbert gravity action plus a cosmological constant and some torsional terms [6]. The torsional terms are proportional to the fuzziness and hence go away in the continuum limit. The demand that in the continuum (large N) limit, and after proper scaling of the gauge fields and coordinates, we should recover the usual Einstein gravity, upon assumption $\ell = l_P$, fixes κ as $\kappa^{-1} = R^2 l^2$ which is equal to the cosmological constant.

The vacuum solutions to the above gravity theory, by construction, include the fuzzy four sphere the volume of which and the cosmological constant Λ, are related as $\Lambda^{-1} = R^4 N^{-2/3} = L^4$. Therefore, the value of the cosmological constant is tied to the number of degrees of freedom (or the size of the matrices), and being quantized is protected against perturbative quantum corrections. That is, in our model we have a way to solve the technical naturalness of the CC problem.

IV. DISCUSSIONS AND CONCLUDING REMARKS

In this talk we put forward the idea that the solution to the cosmological constant problem lies in finding a setup in which cosmological constant is quantized and is hence stable against perturbative (continuous) quantum corrections. We did this, in a Euclidean setting, by first noting that the radius of a fuzzy sphere is quantized (in Planck units), and second, introduced a gravity theory which has the fuzzy sphere as its vacuum solution.

One should, however, note that quantization of the CC (in Planck units) in itself is not enough to solve the CC problem. For example, within the string theory setup of flux compactifications [8] the value of the four dimensional CC is proportional to the fluxes and hence quantized. In that case, unlike ours, there are extremely large number of possibilities which leads to the string theory "landscape" [9]. The CC problem hence re-appears as how/why one of these possibilities is realized in the real world.

In our model we do not face the above problem as we start with a Matrix gravity theory with a given size of matrices, N. The value of N is among the parameters which is defining our theory. As we discussed N eventually turns out to be determining the value of the cosmological constant. In this sense, in our model the cosmological constant (in Planck units) is a constant of nature (like $\hbar$, c and l_P) and is *not* determined dynamically.

Within our approach, however, it is not explicitly seen how the gravity theory given by (13) manages to overcome the usual problem about the contribution of the tadpole diagrams and zero point energies to the CC. The answer should definitely lie in the fact that in our gravity theory both UV and IR dynamics of the gravity are modified due to the noncommutativity which in part forces us to add some other terms, *e.g.* torsion, to the Einstein gravity. Exploring this line is postponed to future works.

One of the interesting features of the fuzzy spheres and hence our model, on which we did not elaborate here, is that the smallest physically observable volume on the fuzzy spheres is not Planckian, but is much larger (*cf.* (11)). One would then expect that the inverse of the smallest volume L^{-4}, which can also be viewed as the uncertainty in measurement of the energy density, is related to the vacuum energy density. As we showed, this is indeed the case in our model. If we take the radius of the four dimensional fuzzy sphere to be equal to the Hubble radius today and l to be the Planck length, we find that $L = \sqrt{lR} \simeq submilimeter$.

Finally, we would like to stress that here we only discussed the Euclidean case. Our discussions on the cosmological constant does not go through for the Minkowski signature, *i.e.* the fuzzy de Sitter space dS^4_F case, as for this case we should take $G/H = SO(4,1)/SO(3,1)$ and $SO(4,1)$ is non-compact which has no finite dimensional unitary representation. The formulation developed in [10] may, however, help to extend our arguments and results to the more realistic Lorentzian signature.

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[11] To be more general and more precise, the α index is running from one to $dim\,\mathrm{Env}G-d$, where Env$G$ is the enveloping algebra for the fundamental representation of g.

[12] In more general cases the set of I^α should be extended, so that (T^a, I^α) covers the Enveloping algebra for the fundamental representation of g, Env$G$.