# Evolution of Coupled Scalar and Spinor Particles in Classical Field Theory

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We study the evolution of mixed scalar as well as spinor fields within the context of the classical field theory. The initial condition problem is solved and the fields distributions, exactly accounting for the initial conditions, are obtained for both scalar and spinor fields. In the system of two coupled fields we consider the special case of the initial conditions which are rapidly oscillating functions. It is demonstrated that the energy densities of the scalar fields and the intensities of the spinor fields coincide with the usual transition and survival probabilities of neutrino flavor oscillations in vacuum.

## 1. INTRODUCTION

The mixing between different flavor eigenstates of quarks as well as leptons was theoretically predicted more than fifty years ago (see Refs. [1, 2]). Since then a considerable progress has been achieved in both experimental and theoretical studying of elementary particles systems with mixing. One should mention the experimental discoveries of $K$ and $B$ mesons oscillations (see, e.g., Refs. [3, 4]) as well as the recent achievements in solar, atmospheric and reactor neutrino experiments (see Refs. [5, 6, 7]).

The theoretical description of neutrino flavor oscillations as the quantum mechanical evolution of two mixed flavor eigenstates was carried out in Ref. [8]. More detailed quantum mechanical treatment of neutrino flavor oscillations was given in Ref. [9] where the wave packets approach was proposed. The quantum field theory was applied to the problem of neutrino flavor oscillations in Ref. [10] in which the process of oscillations was explained as the propagation of neutrino mass eigenstates described by internal lines of a Feynman diagram. The nonperturbative quantum field theory effects in neutrino flavor oscillations were discussed in Ref. [11] where the authors considered the Fock spaces flavor and mass eigenstates and reproduced the Pontecorvo formula for the transition probability as well as obtained the corrections to this expression. We analyzed neutrino spin and flavor oscillations on the basis of the classical field theory in Refs. [12, 13, 14]. It was shown that neutrino flavor oscillations in vacuum as well as in external fields can be described within the framework of the classical field theory. Finally it is worth noticing that many other important references on the considered issue are presented in the review [15] devoted to the theory of neutrino flavor oscillations.

In this paper we continue to study neutrino flavor oscillations in vacuum within the context of the classical field theory. *Classical particles* are represented with help of the *first quantized* fields in our approach. This terminology is borrowed from Ref. [16]. In Sec. 2 we discuss the evolution of $N$ coupled scalar fields. We solve the initial condition problem for this system and construct the fields distributions. The energy density of each field is calculated. It is shown that the obtained expressions are analogous to the formulae used in describing neutrino flavor oscillations in vacuum. In Sec. 3 the similar problem is solved for $N$ coupled spinor fields. We demonstrate that the intensities of the fermions are the same as the transition and survival probabilities of neutrino flavor oscillations in vacuum. We discuss our results in Sec. 4.

## 2. EVOLUTION OF SCALAR PARTICLES

Let us study the evolution of $N$ coupled scalar fields. The Lagrangian for this system has the form

$$\mathcal{L}(\boldsymbol{\varphi}) = \sum_{k=1}^{N} \mathcal{L}_0(\varphi_k) - \sum_{\substack{i,k=1 \\ i\neq k}}^{N} g_{ik}\varphi_i^\dagger \varphi_k, \tag{1}$$

where $g_{ik} = g_{ki}^*$ are the coupling constants responsible for vacuum mixing, $\boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_N)$, and

$$\mathcal{L}_0(\varphi_k) = \partial_\mu \varphi_k^\dagger \partial^\mu \varphi_k - \mathfrak{m}_k^2 |\varphi_k|^2, \tag{2}$$

is the free field Lagrangian for the field $\varphi_k$, $\mathfrak{m}_k$ is the mass corresponding to the field $\varphi_k$. In Eq. (1) we present the case of the complex fields $\varphi_k$ which can correspond to electrically charged particles.

Following the results of our previous work [13] we set the initial conditions problem to describe the evolution of our system (1) and (2). Supposing that the initial fields distributions

$$\varphi_i(\mathbf{r}, 0) = f_i(\mathbf{r}), \quad \dot{\varphi}_i(\mathbf{r}, 0) = g_i(\mathbf{r}), \tag{3}$$

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are the given functions, one should find the fields distributions $\varphi_k(\mathbf{r},t)$ for any subsequent moments of time. It should be noted that it is necessary to set two initial conditions since the corresponding wave equation – Klein-Gordon equation – is the second order differential equation.

In order to solve the initial conditions problem we introduce the new set of the fields $u_k(\mathbf{r},t)$, which diagonalize the Lagrangian (1). They are usually called the mass eigenstates. The fields $u_k$ are related to the fields $\varphi_k$ by the matrix transformation,

$$\varphi_i(\mathbf{r},t) = \sum_{a=1}^{N} M_{ia} u_a(\mathbf{r},t). \tag{4}$$

Note that the masses $\mathfrak{m}_i$ of the fields $\varphi_i$ are related to the masses $m_a$ of the fields $u_a$ by the following formula:

$$\mathfrak{m}_i^2 = \sum_{a=1}^{N} |M_{ia}|^2 m_a^2, \tag{5}$$

which results from Eqs. (1), (2) and (4).

Using our previous work [13] we obtain the field distribution of $\varphi_j(\mathbf{r},t)$ which is consistent with the initial conditions (3),

$$\begin{aligned}\varphi_j(\mathbf{r},t) = \sum_{ia=1}^{N} & M_{ja} M_{ai}^{-1} \\ & \times \int \mathrm{d}^3\mathbf{r}' \big[\dot{D}_a(\mathbf{r}-\mathbf{r}',t) f_i(\mathbf{r}') \\ & + D_a(\mathbf{r}-\mathbf{r}',t) g_i(\mathbf{r}')\big],\end{aligned} \tag{6}$$

where

$$D_a(\mathbf{r},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{\mathrm{i}\mathbf{p}\mathbf{r}} \frac{\sin \mathcal{E}_a t}{\mathcal{E}_a},$$

is the Pauli-Jordan function and $\mathcal{E}_a = \sqrt{p^2 + m_a^2}$.

It was demonstrated in Ref. [14] that it was more convenient to use the momentum representation rather than the coordinate one. Therefore we rewrite Eq. (6) in the form

$$\begin{aligned}\varphi_j(\mathbf{r},t) = \sum_{ia=1}^{N} & M_{ja} M_{ai}^{-1} \\ & \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{\mathrm{i}\mathbf{p}\mathbf{r}} \big[\dot{D}_a(\mathbf{p},t) f_i(\mathbf{p}) \\ & + D_a(\mathbf{p},t) g_i(\mathbf{p})\big],\end{aligned} \tag{7}$$

with help of the Fourier transforms of the Pauli-Jordan function,

$$\dot{D}_a(\mathbf{p},t) = \cos \mathcal{E}_a t, \quad D_k(\mathbf{p},t) = \frac{\sin \mathcal{E}_a t}{\mathcal{E}_a}, \tag{8}$$

and the initial conditions,

$$\begin{aligned} f_i(\mathbf{p}) &= \int \mathrm{d}^3\mathbf{r} e^{-\mathrm{i}\mathbf{p}\mathbf{r}} f_i(\mathbf{r}), \\ g_i(\mathbf{p}) &= \int \mathrm{d}^3\mathbf{r} e^{-\mathrm{i}\mathbf{p}\mathbf{r}} g_i(\mathbf{r}). \end{aligned} \tag{9}$$

It is interesting to demonstrate that the total energy is conserved in our system. Indeed, the total energy is given by the formula

$$\mathbb{E}(t) = \int \mathrm{d}^3\mathbf{r} T^{00}_{\text{total}}(\mathbf{r},t),$$

where $T^{00}_{\text{total}}$ is the time component of the energy-momentum tensor. Using the general expression for $T^{\mu\nu}_{\text{total}}$,

$$T^{\mu\nu}_{\text{total}} = \eta^{\nu\lambda} \sum_{k=1}^{N} \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_k)} \partial_\lambda \varphi_k + \text{h. c.} \right) - \mathcal{L}\eta^{\mu\nu},$$

where the Lagrangian $\mathcal{L}$ is presented in Eq. (1), we receive the formula for $\mathbb{E}(t)$ in the form

$$\begin{aligned}\mathbb{E}(t) = \int \mathrm{d}^3\mathbf{r} \Big\{ & \sum_{k=1}^{N} (|\dot{\varphi}_k(\mathbf{r},t)|^2 + |\boldsymbol{\nabla}\varphi_k(\mathbf{r},t)|^2 \\ & + \mathfrak{m}_k^2 |\varphi_k(\mathbf{r},t)|^2) + \sum_{\substack{i,k=1 \\ i\neq k}}^{N} g_{ik} \varphi_i^\dagger(\mathbf{r},t) \varphi_k(\mathbf{r},t) \Big\}.\end{aligned} \tag{10}$$

The initial value of the total energy can be obtained with help of Eqs. (3), (9) and (10),

$$\begin{aligned}\mathbb{E}(0) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \Big\{ & \sum_{k=1}^{N} (|g_k(\mathbf{p})|^2 + \mathbf{p}^2 |f_k(\mathbf{p})|^2 \\ & + \mathfrak{m}_k^2 |f_k(\mathbf{p})|^2) + \sum_{\substack{i,k=1 \\ i\neq k}}^{N} g_{ik} f_i^\dagger(\mathbf{p}) f_k(\mathbf{p}) \Big\}.\end{aligned} \tag{11}$$

Using the explicit form for the field distribution $\varphi_j(\mathbf{r},t)$ [Eq. (7)] and the Fourier transforms of the Pauli-Jordan function [Eq. (8)] one can rewrite Eq. (10) in the following way:

$$\begin{aligned}\mathbb{E}(t) = \sum_{ika=1}^{N} & M_{ia} M_{ak}^{-1} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \\ & \times \left[ g_i^\dagger(\mathbf{p}) g_k(\mathbf{p}) + \mathcal{E}_a^2 f_i^\dagger(\mathbf{p}) f_k(\mathbf{p}) \right].\end{aligned} \tag{12}$$

In deriving Eq. (12) we use the unitarity of the matrix $M_{ja}$, i.e. $(M^{-1})_{ai} = M_{ia}^*$, and the property of the masses eigenstates $u_a$,

$$m_a^2 \delta_{ab} = \sum_{k=1}^{N} \mathfrak{m}_k^2 M_{ka}^* M_{kb} + \sum_{\substack{i,k=1 \\ i\neq k}}^{N} g_{ik} M_{ia}^* M_{kb}.$$

With help of the identity

$$\sum_{a=1}^{N} M_{ia} M_{ak}^{-1} m_a^2 = \mathfrak{m}_i^2 \delta_{ik} + g_{ik}, \quad g_{ii} = 0,$$

which generalizes Eq. (5) we can transform Eq. (12) to the form which coincides with Eq. (11), i.e. we demonstrate that $\mathbb{E}(t) = \mathbb{E}(0)$. This explicit calculation shows that the total energy in the system of $N$ coupled scalar fields is conserved independently of initial conditions.

It is very difficult to analyze Eq. (7) for arbitrary functions $f_i(\mathbf{p})$ and $g_i(\mathbf{p})$. Thus we pick out the special form of the initial conditions (see also Refs. [13, 14]),

$$f_i(\mathbf{r}) = \frac{A_i}{\sqrt{\mathfrak{E}_i}} e^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}, \quad g_i(\mathbf{r}) = 0, \tag{13}$$

where $A_i$ is the "amplitude" of the function $f_i$, $\mathfrak{E}_i = \sqrt{\omega^2 + \mathfrak{m}_i^2}$ and $\boldsymbol{\omega}$ are the initial energy and momentum of the field $\varphi_i$. Note that $1/\sqrt{\mathfrak{E}_i}$ is the normalization factor; its value will be clarified below [see Eq. (16)]. Using Eq. (9) we obtain the Fourier transforms of the initial conditions,

$$f_i(\mathbf{p}) = \frac{A_i}{\sqrt{\mathfrak{E}_i}} (2\pi)^3 \delta^3(\boldsymbol{\omega} - \mathbf{p}), \quad g_i(\mathbf{p}) = 0. \tag{14}$$

The physical quantity measured in an experiment is *not* a field distribution. It was demonstrated above that the total energy of the system is conserved. Therefore we assume that one can detect the energy density of a *single* scalar field. Constructing the energy-momentum tensor for the *single* field $\varphi_i$ and taking its time component we obtain for the energy density,

$$\begin{aligned} \mathfrak{H}_i(\mathbf{r}, t) =& T^{00}[\varphi_i(\mathbf{r}, t)] \\ =& |\dot{\varphi}_i|^2 + |\boldsymbol{\nabla}\varphi_i|^2 + \mathfrak{m}_i^2 |\varphi_i|^2. \end{aligned} \tag{15}$$

Note that here we assume that there would be no mixing between different fields $\varphi_i$. It can be verified by means of direct calculations that the following expression:

$$\mathfrak{H}_i(\mathbf{r}, 0) = |A_i|^2 \mathfrak{E}_i, \tag{16}$$

results from Eqs. (13) and (15). Therefore the normalization factor in Eq. (13) was chosen to get the correct value of the initial energy density.

Without losing generality we can discuss now the evolution of two scalar fields. In this simple case the matrix $M_{ja}$ can be parameterized with help of only one mixing angle $\theta$,

$$M_{ja} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{17}$$

It is also necessary to fix the "amplitudes" $A_i$. They can be chosen in following form, $A_1 = 0$ and $A_2 = 1$. This choice of the "amplitudes" has rather simple physical meaning: the first field is absent initially and the second one has the plane wave field distribution.

Using Eqs. (7), (14) and (17) we get the field distributions of $\varphi_{1,2}$:

$$\begin{aligned} \varphi_1(\mathbf{r}, t) =& \frac{1}{\sqrt{\mathfrak{E}_2}} \cos\theta \sin\theta e^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}} \\ & \times (\cos[\mathcal{E}_1(\omega)t] - \cos[\mathcal{E}_2(\omega)t]), \\ \varphi_2(\mathbf{r}, t) =& \frac{1}{\sqrt{\mathfrak{E}_2}} e^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}} \\ & \times (\sin^2\theta \cos[\mathcal{E}_1(\omega)t] + \cos^2\theta \cos[\mathcal{E}_2(\omega)t]), \end{aligned} \tag{18}$$

where $\mathcal{E}_a(\omega) = \sqrt{\omega^2 + m_a^2}$. To analyze the obtained expressions we consider the limiting case of high initial frequency, $\omega \gg m_{1,2}$, which corresponds to ultrarelativistic particles.

With help of Eqs. (15) and (18) as well as in the high frequency approximation we obtain the expressions for the energy densities of the fields $\varphi_{1,2}$ in the following form:

$$\begin{aligned} \mathfrak{H}_1(t) =& \omega \left\{ \sin^2(2\theta) \sin^2[\Delta(\omega)t] + \mathcal{O}\left(\frac{m_a^2}{\omega^2}\right) \right\}, \\ \mathfrak{H}_2(t) =& \omega \left\{ 1 - \sin^2(2\theta) \sin^2[\Delta(\omega)t] + \mathcal{O}\left(\frac{m_a^2}{\omega^2}\right) \right\}, \end{aligned} \tag{19}$$

where

$$\Delta(\omega) = \frac{\mathcal{E}_1(\omega) - \mathcal{E}_2(\omega)}{2} \to \frac{\Delta m^2}{4\omega} + \mathcal{O}\left(\frac{m_a^2}{\omega^2}\right).$$

Here we use the common notation $\Delta m^2 = m_1^2 - m_2^2$.

Now we can introduce the probability which corresponds to the transitions from the second eigenstate $\varphi_2$ to the eigenstate $\varphi_k$ with $k = 1, 2$: $P_{2\to k}^{(\mathrm{scalar})}(t) = \mathfrak{H}_k(t)/\omega$. Using Eqs. (19) we obtain for the probabilities,

$$P_{2\to 1}^{(\mathrm{scalar})}(t) = \sin^2(2\theta) \sin^2\left(\frac{\Delta m^2}{4\omega} t\right), \tag{20}$$

$$P_{2\to 2}^{(\mathrm{scalar})}(t) = 1 - \sin^2(2\theta) \sin^2\left(\frac{\Delta m^2}{4\omega} t\right). \tag{21}$$

The function $P_{2\to 1}^{(\mathrm{scalar})}(t)$ in Eq. (20) is usually called the transition probability whereas the function $P_{2\to 2}^{(\mathrm{scalar})}$ in Eq. (21) – the survival probability. It should be noted that Eqs. (20) and (21) reproduce the common quantum mechanical expressions for transition and survival probabilities of neutrino flavor oscillations in vacuum.

## 3. EVOLUTION OF SPINOR PARTICLES

Now let us discuss the evolution of $N$ coupled spinor fields. The Lagrangian for this system has the form

$$\mathcal{L}(\boldsymbol{\nu}) = \sum_{k=1}^{N} \mathcal{L}_0(\nu_k) - \sum_{\substack{i,k=1\\ i\neq k}}^{N} g_{ik}\bar{\nu}_i\nu_k,$$

where $\boldsymbol{\nu} = (\nu_1, \dots, \nu_N)$ and

$$\mathcal{L}_0(\nu_k) = \bar{\nu}_k(\mathrm{i}\gamma^\mu\partial_\mu - \mathfrak{m}_k)\nu_k,$$

is the free field Lagrangian for $\nu_k$ and $\mathfrak{m}_k$ are the masses of $\nu_k$.

Analogously to Sec. 2 (see also Refs. [13, 14]) we set the initial condition problem for the considered system,

$$\nu_i(\mathbf{r}, 0) = \xi_i(\mathbf{r}), \qquad (22)$$

where $\xi_i(\mathbf{r})$ are the given functions. It should be mentioned that we have to set only one initial condition in Eq. (22) since the Dirac equation is the first order differential equation. Now it is necessary to find the fields distributions for $\nu_i(\mathbf{r}, t)$ at $t > 0$ which would be consistent with Eq. (22).

With help of Refs. [13, 14] we obtain for $\nu_j(\mathbf{r}, t)$ the following expression:

$$\nu_j(\mathbf{r}, t) = \sum_{ia=1}^{N} M_{ja}M_{ai}^{-1} \times \int \mathrm{d}^3\mathbf{r}' S_a(\mathbf{r}' - \mathbf{r}, t)(-\mathrm{i}\gamma^0)\xi_i(\mathbf{r}'), \qquad (23)$$

where

$$S_a(\mathbf{r}, t) = (\mathrm{i}\gamma^\mu\partial_\mu + m_a)D_a(\mathbf{r}, t), \quad x^\mu = (t, \mathbf{r}),$$

is the Pauli-Jordan function for a spinor field and $m_a$ are the masses of mass eigenstates which are related to $\mathfrak{m}_i$ by the formula,

$$\mathfrak{m}_i = \sum_{a=1}^{N} |M_{ia}|^2 m_a.$$

It should be noted that Eq. (23) is in agreement with the initial conditions in Eq. (22).

As it was demonstrated in Sec. 2 we have to rewrite Eq. (23) in the momentum representation in order to avoid computational difficulties. Finally we get

$$\nu_j(\mathbf{r}, t) = \sum_{ia=1}^{N} M_{ja}M_{ai}^{-1} \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} e^{\mathrm{i}\mathbf{pr}} S_a(-\mathbf{p}, t)(-\mathrm{i}\gamma^0)\xi_i(\mathbf{p}), \qquad (24)$$

where

$$S_a(-\mathbf{p}, t) = \bigg[\cos\mathcal{E}_a t - \mathrm{i}\frac{\sin\mathcal{E}_a t}{\mathcal{E}_a}(\boldsymbol{\alpha}\mathbf{p} + \beta m_a)\bigg](i\beta), \qquad (25)$$

and

$$\xi_i(\mathbf{p}) = \int \mathrm{d}^3\mathbf{r} e^{-\mathrm{i}\mathbf{pr}}\xi_i(\mathbf{r}),$$

are the Fourier transforms of the Pauli-Jordan function and the initial conditions. Here we adopt the common notation for the Dirac matrices $\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}$ and $\beta = \gamma^0$.

It is interesting to verify that in the classical field theory approach the total intensity of all the fields $\nu_k$ is conserved. We assume that

$$\mathfrak{I}(0) = \sum_{i=1}^{N} \int \mathrm{d}^3\mathbf{r}|\xi_i(\mathbf{r})|^2 = \sum_{i=1}^{N} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}|\xi_i(\mathbf{p})|^2, \qquad (26)$$

has the given value. Now we are interested in the following quantity:

$$\mathfrak{I}(t) = \sum_{j=1}^{N} \int \mathrm{d}^3\mathbf{r}|\nu_j(\mathbf{r}, t)|^2.$$

With help of Eq. (24) this expression can be rewritten in the form

$$\mathfrak{I}(t) = \sum_{jikab=1}^{N} [M_{ja}M_{ai}^{-1}]^* M_{jb}M_{bk}^{-1} \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\mathrm{d}^3\mathbf{r}\, e^{\mathrm{i}(\mathbf{q}-\mathbf{p})\mathbf{r}} \times \xi_i^\dagger(\mathbf{p})[S_a(-\mathbf{p}, t)(-\mathrm{i}\gamma^0)]^\dagger \times S_b(-\mathbf{q}, t)(-\mathrm{i}\gamma^0)\xi_k(\mathbf{q}). \qquad (27)$$

Again using the fact that the matrix $M_{ja}$ is the unitary one, we obtain for Eq. (27),

$$\mathfrak{I}(t) = \sum_{ika=1}^{N} M_{ia}M_{ak}^{-1} \times \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\xi_i^\dagger(\mathbf{p})[S_a(-\mathbf{p}, t)(-\mathrm{i}\gamma^0)]^\dagger \times S_a(-\mathbf{p}, t)(-\mathrm{i}\gamma^0)\xi_k(\mathbf{p}). \qquad (28)$$

With help of Eq. (25) one can show that

$$[S_a(-\mathbf{p}, t)(-\mathrm{i}\gamma^0)]^\dagger S_a(-\mathbf{p}, t)(-\mathrm{i}\gamma^0) = 1.$$

Therefore, taking into account Eq. (26), we represent Eq. (28) in the form

$$\mathfrak{I}(t) = \sum_{i=1}^{N} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} |\xi_i(\mathbf{p})|^2 = \mathfrak{I}(0), \qquad (29)$$

which shows that the the total intensity is conserved.

In the following we consider the evolution of only two coupled spinor fields to make the results more illustrative. Thus we assume that the initial conditions are: $\xi_1(\mathbf{r}) = 0$ and $\xi_2(\mathbf{r}) = e^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}}\xi_0$. The discussion of such a choice of initial conditions is given in Sec. 2. The mixing matrix $M_{ja}$ can be also represented in the same form as in Eq. (17). Using general Eq. (24) one obtains for the fields distributions of $\nu_{1,2}$:

$$\begin{aligned}
\nu_1(\mathbf{r},t) =& e^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}} \sin\theta\cos\theta \bigg\{ \cos[\mathcal{E}_1(\omega)t] - \cos[\mathcal{E}_2(\omega)t] \\
& - \mathrm{i}\frac{\sin[\mathcal{E}_1(\omega)t]}{\mathcal{E}_1(\omega)}(\boldsymbol{\alpha}\boldsymbol{\omega} + \beta m_1) \\
& + \mathrm{i}\frac{\sin[\mathcal{E}_2(\omega)t]}{\mathcal{E}_2(\omega)}(\boldsymbol{\alpha}\boldsymbol{\omega} + \beta m_2) \bigg\} \xi_0, \\
\nu_2(\mathbf{r},t) =& e^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}} \bigg\{ \sin^2\theta\cos[\mathcal{E}_1(\omega)t] + \cos^2\theta\cos[\mathcal{E}_2(\omega)t] \\
& - \mathrm{i}\sin^2\theta\frac{\sin[\mathcal{E}_1(\omega)t]}{\mathcal{E}_1(\omega)}(\boldsymbol{\alpha}\boldsymbol{\omega} + \beta m_1) \\
& - \mathrm{i}\cos^2\theta\frac{\sin[\mathcal{E}_2(\omega)t]}{\mathcal{E}_2(\omega)}(\boldsymbol{\alpha}\boldsymbol{\omega} + \beta m_2) \bigg\} \xi_0. \qquad (30)
\end{aligned}$$

The most interesting case is the high frequency approximation of the initial conditions given in Eq. (22), i.e. $\omega \gg m_{1,2}$. Therefore, Eqs. (30) can be rewritten in the form

$$\begin{aligned}
\nu_1(\mathbf{r},t) =& -e^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}} \sin 2\theta \sin[\Delta(\omega)t] \\
& \times \{\sin[\sigma(\omega)t] + \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\cos[\sigma(\omega)t]\}\,\xi_0 \\
& + \mathcal{O}(m_a/\omega), \\
\nu_2(\mathbf{r},t) =& e^{\mathrm{i}\boldsymbol{\omega}\mathbf{r}} \{\sin[\Delta(\omega)t]\cos 2\theta - \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\cos[\Delta(\omega)t]\} \\
& \times \{\sin[\sigma(\omega)t] + \mathrm{i}(\boldsymbol{\alpha}\mathbf{n})\cos[\sigma(\omega)t]\}\,\xi_0 \\
& + \mathcal{O}(m_a/\omega), \qquad (31)
\end{aligned}$$

where

$$\sigma(\omega) = \frac{\mathcal{E}_1(\omega) + \mathcal{E}_2(\omega)}{2} \to \omega + \frac{m_1^2 + m_2^2}{4\omega} + \mathcal{O}\left(\frac{m_a^2}{\omega^2}\right).$$

We showed above that the total intensity of all the fields $\nu_k$, which is proportional to a squared fields distributions, is conserved in the considered system. Therefore we suppose that the measurable quantity of a classical spinor particle is the intensity, $I_k(t) = |\nu_k(\mathbf{r},t)|^2$. Analogously to Sec. 2 we introduce the probabilities for transitions from the second eigenstate to the eigenstate $k = 1, 2$ as $P_{2\to k}^{(\mathrm{spinor})}(t) = I_k(t)$. Using Eqs. (31) we express the probabilities in the following way:

$$P_{2\to 1}^{(\mathrm{spinor})}(t) = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4\omega}t\right), \qquad (32)$$

$$P_{2\to 2}^{(\mathrm{spinor})}(t) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4\omega}t\right). \qquad (33)$$

One can observe that these expressions for transition (32) and survival (33) probabilities coincide with the analogous Eqs. (20) and (21) for the coupled scalar fields. Note that Eqs. (32) and (33) are consistent with Eq. (29). The obtained probabilities formulas are the same as in the quantum mechanical treatment of neutrino flavor oscillations in vacuum.

## 4. CONCLUSION

In conclusion we note that the evolution of coupled scalar as well as spinor fields has been described in frames of classical field theory. First we have considered the system of $N$ arbitrary coupled complex scalar fields and solved the initial conditions problem for this system. The general fields distributions consistent with the initial conditions have been found. Note that these expressions exactly took into account the Lorentz invariance and were valid in (3+1)-dimensional space-time. The energy conservation law has been analyzed for the considered system. Then we have discussed the simple case of two coupled fields and have constructed the energy densities of each scalar field. It has been demonstrated that in ultrarelativistic limit these expressions were analogous to the transition and the survival probabilities of neutrino flavor oscillations in vacuum. The calculations in the present paper have refined the results of our previous work [13] where analogous problem was analyzed in (1+1)-dimensional space-time.

The case of $N$ coupled classical spinor fields has been also considered in the present work. We have formulated the initial condition problem and constructed the general expressions for the fields distributions which were valid in (3+1)-dimensional space-time and were Lorentz invariant. The total intensity of all coupled spinor fields has been computed. It has been demonstrated that this quantity was conserved independently of the initial conditions. Then we have discussed the case of rapidly oscillating initial conditions which corresponded to ultrarelativistic particles. In this case as well as for only two coupled fermions we have obtained the intensities of the fields in question. The expressions for the intensities were also revealed to coincide with transition and survival probabilities of neutrino flavor oscillations in vacuum. It should be mentioned that transition and survival probabilities for coupled classical spinor fields turned out to be the same as the analogous formulae for scalar fields.

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