

# WHAT IS CATEGORICAL RELATIVITY?

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The categorical relativity is a groupoid category of massive bodies in mutual motions. The relative velocity is defined to be the basis-free and coordinate-free binary morphism. In categorical relativity there is no need to distinguish separately the constant relative velocities (special relativity) from the variable accelerated velocities, hence the categorical relativity goes beyond boundary of the special relativity, and unify the kinematics of the special and the general relativity theories for curved spacetime.

The definition of relative velocity as the categorical morphism is contrasted with the reciprocal-velocity defined by means of the Lorentz boost in isometric special relativity.

Observer-dependence and the Lorentz-covariance are different concepts. During XX century the Lorentz covariance was the cornerstone of physical theory. We predict that Lorentz covariance as well as Lorentz invariance will dwindle in importance. Physics must be observer-free, rather than Lorentz-invariant.

In this paper instead of the individual time-plus-space split, we consider the algebra of many relative time-plus-space splits, as an operator algebra generated by idempotents. The kinematics of categorical relativity is ruled by associative Frobenius operator algebra, whereas the dynamics of categorical relativity needs the non-associative Frölicher-Richardson operator algebra.

In categorical relativity the *inverse* relative-velocity-morphism  $\mathbf{v}^{-1}$  is interior-observer-dependent, and not absolute as in the isometric exterior formulation where  $\mathbf{v}^{-1} \equiv -\mathbf{v}$ .

Electro-magnetics of moving bodies in categorical relativity is different from the isometric special relativity with Lorentz transformations, however this subject goes beyond the restricted scope of the present short introductory note.

*Keywords:* groupoid category; velocity-morphism; associative addition of relative velocities.

## 1. Galileo versus Descartes

Aether was introduced by René Descartes around 1650. Descartes conceive the space as an *absolute substance*, substratum, medium, that exists independently of any matter in it. Like a hall with sitting chairs-locations waiting for spectators. Galileo

claim instead, in 1632, that space is *relative*. The relativity theory is about the relativity of proper-time-plus-space splits, and not about the change of numerical coordinates.

Why does a space, a set of locations, not need be neither absolute nor primitive concept, as it was postulated by Descartes, and also by Newton in 1686? How could it be *relative*, *i.e.* dependent on another primitive concept? What is this primitive concept that space and time are derived? Galileo conceived space to be on massive bodies-dependent. Galilean space has no reality without the bodies that ‘it contains’. Galilean primitive concept from which space is derived is the relative *motion*, the relative *velocity* among massive bodies. Because of relative velocities, there are the multitude of relative spaces. If all massive bodies would be at rest relative to each other, zero relative velocities only, just one massive body, then space would be absolute and proper-time (a set of instants) would also be absolute. Why is space relative? Because there are relative velocities, and hence the systems of more than just one massive body in mutual motions [Galileo 1632].

In this way we come to a vicious circle: in order to define a velocity it is said, according to Newton, that we need firstly a primitive space and a primitive time, and a velocity is a derivative of a space with respect to a time. However, the very concept of a space (and a time), according to Galileo, is very much relative because of multitude of a priori relative velocities. Therefore one can ask what was first: chicken or egg? space & time or relative velocities among massive bodies? Émil Picard conceived motion and time to be dual: *Time is measuring a motion, and a motion is measuring a time.*

Every velocity is relative, it is a velocity of one massive body relative to another massive body. This is a binary function of ordered pair of massive bodies, like a set-valued Hom functor in category theory.

The set of all relative velocities among massive bodies is *not* a vector space of linear algebra because not every pair of such relative velocities can be composed (velocity addition is the partial operation), nor is the commutativity of composition (an additive Abelian group structure) applicable when it is defined. This holds equally well in Newtonian physics with absolute simultaneity (identified with the physical time), and as well as for relative simultaneity and finite light velocity. Moreover, every massive body possess its own identity velocity-morphism, it is the zero velocity of this object relative to himself. There is no universal unique zero velocity that would be massive-body-independent. The zero velocity of the Earth relative to Earth must *not* be identified with the zero velocity of the Sun relative to the Sun.

If relative velocity is not a vector of linear algebra, then one can ask with what mathematical concept the physical relative velocity could be identified?

## 2. Groupoid category

A category consists of a family of objects and a family of arrows/morphisms. Every arrow has a source object and a target object. Two arrows are said to be composable if the target object of one arrow is the same as the source object of the second arrow. The composition of composable arrows is assumed to be associative.

**Definition 2.1 (Groupoid category).** A category is said to be a *groupoid category*, if and only if every morphism has a two-sided inverse. In particular a *group* is a groupoid one-object-category, with just one object, hence with universal unique neutral element-morphism. A groupoid category is said to be *connected* if there is an arrow joining any two of its objects.

**Definition 2.2 (Terminal and initial object).** An object  $q$  is terminal if to each object  $p$  there is exactly one arrow  $p \rightarrow q$ . An object  $p$  is initial if to each object  $q$  there is exactly one arrow  $p \rightarrow q$ . A *null* object is an object which is both initial and terminal.

If  $p$  is terminal, the only arrow  $p \rightarrow p$  is the identity. Any two terminal objects are isomorphic [Mac Lane 1998, p. 20, p. 194].

## 3. Axiom 1: Categorical relativity is groupoid category

The categorical relativity is a connected groupoid category where *every* object is null. Each object is interpreted as some massive body, not necessarily inertial,

$$\varpi = \{\text{obj } \varpi, \text{arrows } \varpi\} = \{\text{massive bodies, relative velocities}\}. \quad (1)$$

Unique arrow from an object  $p$  to an object  $q$  is denoted by  $\varpi(p, q)$ , with analogy to Hom-set bifunctor. For each object  $p \in \text{obj } \varpi$ , a map

$$\text{obj } \varpi \ni q \xrightarrow{\varpi_p} \varpi(p, q) \in \text{arrows } \varpi, \quad (2)$$

extends to covariant representable functor among connected groupoid categories. The object  $p$  is representing-object for a functor  $\varpi_p$ . We call this structure the connected groupoid (1,1)-category, or enriched groupoid category. Categorical relativity is (arrows  $\varpi$ )-enriched groupoid category, rather than more restrictive concept of 2-category. This category is neither abelian, nor additive.

Each relative velocity is a categorical morphism (arrow that need not be a map because objects need not to be sets having elements). An arrow  $\varpi(p, q)$  is interpreted as a velocity-morphism of a body  $q \in \text{obj } \varpi$  relative to a body  $p \in \text{obj } \varpi$ , *i.e.* a velocity as measured by  $p$ . We say that the source (or domain) of velocity  $\varpi(p, q)$  is  $p$ , and the target (or codomain) of  $\varpi(p, q)$  is a body  $q$ . A category symbol  $\varpi$ , is interpreted as an actual velocity-measuring device. We display  $\varpi(p, q)$  as a categorical arrow (morphism, directed-path) which originates (is outgoing) at

observer  $p$ ,  $p$  is a node of the directed graph, and terminates (is incoming) at an observed body  $q$ ,

$$\dots \longrightarrow p \begin{array}{c} \xrightarrow{\varpi(p,q)} \\ \xleftarrow{\varpi(q,p)} \end{array} q \longrightarrow \dots; \quad (3)$$

Observer of  $\varpi(p, q)$  is  $p$ .      Observed body with  $\varpi(p, q)$  is  $q$ .  
Observed body with  $\varpi(q, p)$  is  $p$ .      Observer of  $\varpi(q, p)$  is  $q$ .

In categorical relativity there is no need to distinguish separately the constant velocities (special relativity) from the variable accelerated velocities, hence the categorical relativity goes beyond boundary of the special relativity, and unify the kinematics for curved spacetime..

There is not yet, neither the concept of a spacetime, nor the concept of a time, nor space. The relative velocity-morphism, and massive indivisible objects, are primary, postulated primitive concepts. Any massive body (observer, observed), is a null object in a category  $\varpi$ . The categorical null object is indivisible, like the Leibniz monad, it is not a set.

The relative *velocity* is the *primitive* notion, and this notion we are introducing as the *morphism* in the groupoid category of abstract observers. Each morphism of  $\varpi$  is a binary (interior) relative velocity. The velocity-morphism instead of the isometric Lorentz transformations in the Einstein special relativity. Why the name *categorical relativity*? Because the concept of the relative velocity we wish not associate neither with the concept of the vector space, nor with the Lorentz boost. We wish to see the relative velocity exclusively as the categorical morphism in groupoid category that is not abelian. This is why the relativity theory of space and time, in terms of the relative velocities-morphisms, is said to be the *categorical* relativity.

#### 4. Axiom 2: Categorical relativity is an algebra

Let  $\mathcal{F}$  be an associative, unital and commutative ring. We denote by  $\text{span}_{\mathcal{F}}\varpi$  the  $\mathcal{F}$ -module with a category  $\varpi$  as a set of basic vectors. This  $\mathcal{F}$ -module  $\text{span}_{\mathcal{F}}\varpi$  consists of all formal  $\mathcal{F}$ -linear combinations of the elements of  $\varpi$ , *i.e.* the formal combinations that mix objects with arrows.

It is postulated that an  $\mathcal{F}$ -module  $\text{span}_{\mathcal{F}}\varpi$  is an associative  $\mathcal{F}$ -algebra, called an algebra ‘of massive bodies/observers in the mutual relative motions’, and denoted by  $\text{Obs}(\varpi)$ . The algebra  $\text{Obs}(\varpi)$  is generated by (presented on) objects and arrows of  $\varpi$ , subject to the relations: It is postulated that every object  $p \in \text{obj}\varpi$  is an idempotent  $p^2 = p \in \text{Obs}(\varpi)$ , and every arrow of  $\varpi$  is nilpotent  $(\varpi(p, q))^2 = 0 \in \text{Obs}(\varpi)$ . Every object of  $\varpi$ , seen in an algebra  $\text{Obs}(\varpi)$ , and represented as an operator in  $\text{End}(\text{der}\mathcal{F})$ , looks like a pure state in quantum mechanics.

The above postulate have the following motivation. In order to have just one space, we need to fix one massive body as the reference system. Then the Minkowski proper-time is fixed by gravity potential tensor field  $g$  [Minkowski 1908]. The corre-

spondence, massive body  $\leftrightarrow$  idempotent, is motivated by a desire that every massive body  $p \in \text{End}(\text{der}\mathcal{F})$ , splits

$$\text{der}\mathcal{F} = (\ker p) \oplus (\text{imp}) = (\text{space}) \oplus (\text{time}). \quad (5)$$

The choice of one body, for example the Earth, as the reference system, *not* need coordinates. Such choice is coordinate-independent, and basis-independent. We call any massive body, the Earth, the Moon, an observer (no measuring devices, rods and clocks are involved).

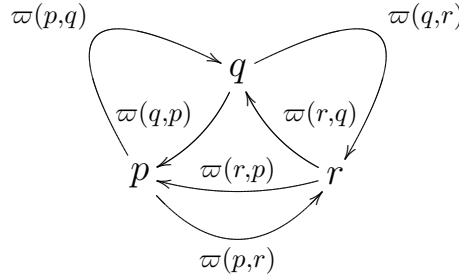


Fig. 1. A groupoid category  $\varpi$  of three massive bodies,  $p, q, r$ , with six binary velocities-morphisms. This category generate 9-dimensional algebra (Axiom 2). The identity arrows are suppressed.

An  $\mathcal{F}$ -dual  $\mathcal{F}$ -module is denoted by  $(\text{span}_{\mathcal{F}}\varpi)^*$ . There is the distinguished covector  $\text{tr} \in (\text{span}\varpi)^*$ ,  $\text{tr} : \text{span}\varpi \rightarrow \mathcal{F}$ ,

$$\text{If } p \text{ is an object of } \varpi, \text{ then: } \quad \text{tr } p = 1.$$

$$\text{If } \varpi(p, q) \text{ is an arrow of } \varpi, \text{ then: } \quad \text{tr } \varpi(p, q) = 0. \quad (6)$$

Therefore the covector  $\text{tr}$  is distinguishing an object  $p$ , from the identity arrow  $\varpi(p, p)$ .

For every non empty word (string) of objects,  $p, q, r, \dots, s \in \text{obj}\varpi$ , the following properties and relation are postulated,

$$1 \leq \{\text{tr}(pqr \dots s)\}^2 = \text{tr}(pq)\text{tr}(qr)\text{tr}(r..) \dots \text{tr}(..s)\text{tr}(sp), \quad (7)$$

$$1 \leq \text{tr}(pq) = \text{tr}(qp), \quad \text{tr}(pqr) = \text{tr}(qpr), \quad (8)$$

$$\forall A \in \text{Obs}(\varpi), \quad qAp = \text{tr}(qAp) \left\{ p + \frac{1}{c} \varpi(p, q) \right\}, \quad c < \infty. \quad (9)$$

From relations (8-9), one can deduce the algebra multiplication table, the multiplication of arrows  $\{\varpi(p, q)\}$  with objects  $\{p, q, r, s, \dots\}$ , and arrows with arrows (we set  $c = 1$ ),

$$qp = \text{tr}(qp) \{ p + \varpi(p, q) \}, \quad (10)$$

$$q\varpi(p, r) = \left( \frac{\text{tr}(qrp)}{\text{tr}(rp)} - \text{tr}(pq) \right) \{ p + \varpi(p, q) \}, \quad (11)$$

$$\varpi(q, r)p = \frac{\text{tr}(rqp)}{\text{tr}(rq)} \{p + \varpi(p, r)\} - \text{tr}(qp) \{p + \varpi(p, q)\}, \quad (12)$$

$$\begin{aligned} \varpi(p, q)\varpi(r, s) &= \frac{1}{\text{tr}(qp)} \left( \frac{\text{tr}(qpsr)}{\text{tr}(sr)} - \text{tr}(qpr) \right) \{r + \varpi(r, q)\} \\ &\quad + \left( \text{tr}(pr) - \frac{\text{tr}(psr)}{\text{tr}(sr)} \right) \{r + \varpi(r, p)\}. \end{aligned} \quad (13)$$

Zero velocity of a body  $p$  relative to himself is  $\varpi(p, p) = \mathbf{0}_p$ . All these zero velocities are equal to unique zero of algebra  $\text{Obs}(\varpi)$  for all bodies, however they can *not* be identified with respect to associative composition of morphisms in a category  $\varpi$ , see below Section 7.

In [Cruz & Oziewicz 2006] we consider augmented unital algebra  $(\text{Obs } \varpi) \oplus \mathcal{F}$ , and pose a hypothesis that this trace-class algebra is a Frobenius algebra for each cardinality of family of objects.

### 5. Axiom 3: Scalar magnitude of arrow

The arrows of the category  $\varpi$  does not possess a linear structure (an arrow multiplied by a scalar is not an arrow), therefore the Euclidean structure of the linear algebra, the tensor product, is meaningless for them. The linear structure of an  $\mathcal{F}$ -algebra  $\text{Obs}\varpi$  is useless for the definition of the scalar magnitude of an arrow (and for defining an angle among arrows) because every arrow seen in an algebra  $\text{Obs}\varpi$  is nilpotent. An Euclidean angle between arrows with the same source only, and a scalar magnitude of each arrow, needs the following separate postulate

$$\frac{\varpi(q, p) \cdot \varpi(q, r)}{c^2} \equiv 1 - \frac{\text{tr}(pr)}{\text{tr}(pqr)} \implies \left( \frac{|\varpi(p, q)|}{c} \right)^2 = 1 - \frac{1}{\text{tr}(pq)}. \quad (14)$$

The above definition (14) is meaningful for  $c < \infty$  only.

### 6. Module over an algebra of observers

An  $\mathcal{F}$ -algebra of observers,  $\text{Obs}(\varpi)$ , is abstractly isolated from the concepts of spacetime, time and space. The indivisible objects of  $\varpi$  are not yet located neither in a space, nor in a time. A ring  $\mathcal{F}$  is interpreted as an  $\mathbb{R}$ -algebra of the classical measurements, in case that  $\mathcal{F}$  is commutative.

The spacetime manifold of events is usually identified with  $\text{alg}(\mathcal{F}, \mathbb{R})$ . However we prefer to see the concept of the spacetime encoded in an  $\text{Obs}(\varpi)$ -module, *i.e.* in an algebra morphism, from an associative  $\mathcal{F}$ -algebra  $\text{Obs}\varpi$ , into endomorphism algebra of the Lie  $\mathcal{F}$ -module  $\text{der}\mathcal{F}$ ,

$$\text{Obs}(\varpi) \xrightarrow[\text{morphism}]{\text{associative algebra}} \text{End}(\text{der}\mathcal{F}) \simeq (\text{der}\mathcal{F}) \otimes (\text{der}\mathcal{F})^*. \quad (15)$$

The Lie  $\mathcal{F}$ -module  $\text{der}\mathcal{F}$  of derivations of the ring  $\mathcal{F}$ , and the dual  $\mathcal{F}$ -module,  $(\text{der}\mathcal{F})^*$ , of the differential forms, are considered as  $\text{Obs}(\varpi)$ -modules.

A massive body  $p \in \text{obj}\varpi$  is represented by  $(1, 1^*)$ -tensor field  $p \in \text{End}(\text{der}\mathcal{F})$ , where  $p^2 = p$  must be the minimal polynomial. Every object of  $\varpi$ , seen in an algebras,  $\text{Obs}(\varpi)$  and  $\text{End}(\text{der}\mathcal{F})$ , looks like a pure state in quantum mechanics.

A pull back (transpose) of  $p$  is denoted by  $p^*$ , it is a  $(1^*, 1)$ -tensor field. The gravity potential tensor field  $g = g^* \in (\text{der}\mathcal{F}) \otimes (\text{der}\mathcal{F})$ , and his inverse  $g^{-1} \in (\text{der}\mathcal{F})^* \otimes (\text{der}\mathcal{F})^*$ , are considered as an invertible Grassmann  $\mathcal{F}$ -algebra morphisms from differential multi-forms to multi-vector fields  $(\text{der}\mathcal{F})^\wedge$ ,

$$n^* \xrightarrow{p^*} n^* \begin{array}{c} \xrightarrow{g} \\ \xleftarrow{g^{-1}} \end{array} n \xrightarrow{p} n, \quad \forall n \in \mathbb{N}. \quad (16)$$

In this convention,  $1 \in \mathbb{N}$  denotes (a grade of) a Lie  $\mathcal{F}$ -module of vector fields  $\text{der}\mathcal{F}$ ,  $2 \in \mathbb{N}$  denotes an  $\mathcal{F}$ -module of bivector fields  $(\text{der}\mathcal{F}) \wedge (\text{der}\mathcal{F})$ ,  $1^*$  is a module of Pfaffian differential one-forms  $(\text{der}\mathcal{F})^*$ , etc. Therefore  $g \in \text{alg}(n^*, n)$ , however,  $\text{obj}(\varpi) \ni p \rightsquigarrow p \in \text{der}(n, n)$ . Here  $\rightsquigarrow$  means ‘extends’, and  $\text{der}(n, n)$  is short for  $\text{der}((\text{der}\mathcal{F})^\wedge n, (\text{der}\mathcal{F})^\wedge n)$ .

The composition,  $p \circ g$ , and their transpose,  $(p \circ g)^* = g \circ p^*$ , are both morphism with the same domain and codomain, and therefore one can ask that a massive body  $p \in \text{End}(\text{der}\mathcal{F})$  is metric-compatible ( $g$ -orthogonal),

$$p^2 = p \implies \text{tr}p = \dim \text{imp},$$

$$p^2 = p \quad \& \quad \text{tr}p = 1 \quad \& \quad g \circ p^* = p \circ g \iff p = \frac{P \otimes (g^{-1}P)}{g^{-1}(P \otimes P)}. \quad (17)$$

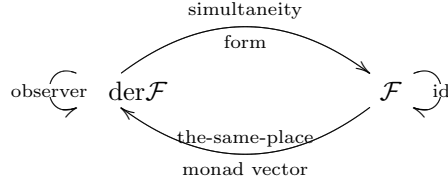
The above compatibility of observer  $p$  with the gravitational potential  $g$  (17), is postulated in the present paper, however seems that this compatibility must be tested experimentally rather than postulated a priori.

Let  $g = g^*$  be a Lorentzian metric tensor field of signature  $(- + + +)$ , not necessarily curvature-free. An idempotent is said to be time-like if his every non-zero eigenvector (a monad field) is time-like,  $pP = P \implies P^2 < 0$ . The space-like simultaneity of an observer  $p$  must be given by the Einstein-Minkowski proper time,  $-g^{-1}P$  [Minkowski 1908]. This time-like differential Pfaff form,  $-g^{-1}P$ , encode the unique empirical and metric-dependent simultaneity of an observer  $p$ . Metric-compatible observer is conformally invariant.

One can set all monad fields (the eigenvectors of observers) to be normalized,  $pP = P$  with  $P^2 = -1$ . For more motivations about the concept of observer as  $(1, 1)$ -tensor field, we refer to [Cruz and Oziewicz 2003].

## 7. Associative addition of relative velocities

When the time-like vector  $P$  is an eigenvector of an operator  $p$ ,  $pP = P$ , and the time-like vector  $Q$  is eigenvector for  $q$ ,  $qQ = Q$ , then  $\varpi(p, q)R = (-P \cdot R)\varpi(P, Q)$  is space-like vector  $\forall R$ . Definition of space-like relative binary velocity,  $\varpi(P, Q)$ , as a



$$p = (\text{monad vector}) \otimes (\text{simultaneity form})$$

Fig. 2. An observer as the split-idempotent.

coordinate-free and basis-free orthogonal projection, we proposed in [Świerk 1988], where we observed that such relative velocity can not parameterize the isometric Lorentz boost. The same definition was introduced independently by many Authors, Matolcsi [1993, p. 191], Bini, Carini and Jantzen [1995], Gottlieb [1996], Matolcsi and Goher [2001, p. 89, Definition (18)], ...

The addition law of the space-like relative velocities-projections were derived by many Authors, [Świerk 1988, Matolcsi 1993, §4.3; Bini, Carini & Jantzen 1995]. In [Oziewicz 2005] we stressed that this addition is associative, and compared with non-associative addition of velocities parameterizing the isometric Lorentz boost. Here we are showing how this addition follows from the associativity of the algebra  $\text{Obs}\varpi$ .

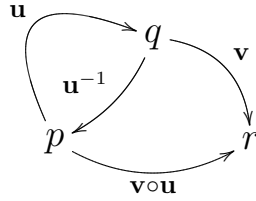


Fig. 3. Associative composition of velocities-morphisms.

**Theorem 7.1 (Associative addition of relative velocities).** See Figure 3. Set  $\mathbf{u} \equiv \varpi(p, q)$ ,  $\text{source}(\mathbf{u}) = p$ , and  $\mathbf{v} \equiv \varpi(q, r)$ . We adhere to the arabic convention of the composition of morphisms, read from the right to the left,

$$\varpi(q, r) \circ \varpi(p, q) = \varpi(p, r) = \mathbf{v} \circ \mathbf{u}. \tag{18}$$

The binary composition of composable arrows is associative and has the following  $p$ -dependent explicit form,

$$\left(1 - \frac{\mathbf{v} \cdot \mathbf{u}^{-1}}{c^2}\right)(\mathbf{v} \circ \mathbf{u}) = \mathbf{u} + \left(1 - \frac{\mathbf{u}^2}{c^2}\right)\mathbf{v}p + \frac{1}{c}(\mathbf{v} \cdot \mathbf{u}^{-1})p. \tag{19}$$



**Proof.** The velocity-addition follows from the associativity of an algebra  $\text{Obs}\varpi$ ,  $r(qp) = (rq)p$ , and from Axiom 2, formula (10),

$$\begin{aligned} r(qp) &= \text{tr}(qp)\text{tr}(rp)\{\varpi(p, r) + p\} + \text{tr}(qp)r\varpi(p, q), \\ (rq)p &= \text{tr}(qp)\text{tr}(rq)\{\varpi(p, q) + p\} + \text{tr}(rq)\varpi(q, r)p. \end{aligned} \quad (20)$$

Therefore

$$\begin{aligned} \text{tr}(qp)\text{tr}(rp)\varpi(p, r) &= \text{tr}(qp)\text{tr}(rq)\varpi(p, q) + \text{tr}(r, q)\varpi(q, r)p \\ &\quad + \text{tr}(qp)\text{tr}(r(q-p))p - \text{tr}(qp)r\varpi(p, q). \end{aligned} \quad (21)$$

The multiplication of an object  $r$  with an arrow  $\varpi(p, q)$ , as given by formula (11), is expressed in terms of  $\{\varpi(p, r) + p\}$ . This gives

$$\begin{aligned} \varpi(p, r) &= \varpi(q, r) \circ \varpi(p, q) \\ &= \frac{\text{tr}(pqr)}{\text{tr}(pr)} \left( \varpi(p, q) + \frac{1}{\text{tr}(pq)} \varpi(q, r)p \right) + c \left( \frac{\text{tr}(pqr)}{\text{tr}(pr)} - 1 \right) p. \end{aligned} \quad (22)$$

Finally we need use Axiom 3, expression (14).

The associativity of addition follows from considering a system of four objects/massive bodies,  $p(qrs) = (pqr)s$ .  $\square$

Ungar observed that the addition of velocities in isometric special relativity is non-associative [Ungar 1988, 2001]. In contrast, the addition of velocities in categorical relativity, Theorem 7.1 and expression (19), is observer-dependent (depends on observer  $p$  that is the source of the velocity-arrow  $\mathbf{u}$ ), however this addition operation is associative.

If  $\varpi(p, q)$  is a velocity of  $q$  relative to  $p$ , then the  $\circ$ -inverse velocity-morphism,  $\varpi(q, p) = (\varpi(p, q))^{-1}$ , is a velocity of  $p$  relative to  $q$ .

$$\begin{aligned} \varpi(p, q) \circ \varpi(q, p) &= \mathbf{0}_q \neq \mathbf{0}_p = \varpi(p, p) = \varpi(q, p) \circ \varpi(p, q) \\ &= \text{tr}(pq)\varpi(p, q) + \varpi(q, p)p + c(\text{tr}(pq) - 1)p. \end{aligned} \quad (23)$$

## 8. Galilean addition

Two objects,  $p$  and  $q$ , possess the same simultaneity iff  $\text{tr}(pq) = 1$ . Therefore in the Galilean algebra of observers the trace of arbitrary string of objects must be  $\text{tr}(pq\dots) = 1$ . The Galilean algebra of observers is presented on idempotent-objects, and on the following relations, not independent,

$$\begin{aligned} qp = q = p + \varpi(p, q), & \quad \varpi(p, q)\varpi(r, s) = 0, \\ q\varpi(p, r) = 0, & \quad \varpi(q, r)p = \varpi(q, r). \end{aligned} \quad (24)$$

Hence Axiom 2 with (9) must be replaced by the reciprocal relative velocity  $\varpi(p, q) = q - p$ , *i.e.* relative velocity in Galilean relativity is exactly the *skew* symmetric function of his arguments,

$$\varpi(q, r) \circ \varpi(p, q) = \varpi(p, q) + \varpi(q, r) = (q - p) + (r - q) = \varpi(p, r). \quad (25)$$

Therefore,  $\lim_{c \rightarrow \infty} (\mathbf{v} \circ \mathbf{u}) = \mathbf{v} + \mathbf{u}$ , because  $\mathbf{v}p = \mathbf{v}$ .

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