A TOY MODEL OF THE COSMIC SINGULARITY

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We assume that evolution of the universe consists of the sequence of classical and quantum phases. Analyzes of the quantum phase is presented by examination of classical and quantum dynamics of a particle in the compactified Milne space. Our model offers some insight into the nature of the cosmic singularity.

Keywords: Cosmic singularity; Milne's space; particle's dynamics.

1. Introduction

Presently available cosmological data suggest that our universe emerged from a state with high density of physical fields [1] called the cosmic singularity. For modelling the very early universe it is necessary to understand the nature of the singularity.

It is attractive to assume that the singularity consists of contraction and expansion phases. This way one opens door for the cyclic universe models [2], which are free of the problem of beginning/end of the universe , i.e. creation/annihilation of space-time-matter-field from/into 'nothing'.

The cosmic singularity (CS) plays the key role because it joins each two consequtive classical phases. One of the simplest models of the CS offered by string/M theory is the compactified Milne space [3].

Any reasonable model of the CS should be able to describe quantum propagation of a fundamental object (e.g. particle, string, membrane,...) from the pre-singularity to post-singularity epoch. It is the most elementary criterion that should be satisfied. Some insight into the problem may be already achieved by studying classical and quantum dynamics of a test particle in two-dimensional spacetime. For more details concerning this lecture we recommend Ref. [4].

2. Compactified Milne Space

The two-dimensional compactified Milne, CM, space may be defined by the following mapping of $R^1 \times S^1$ into the three-dimensional Minkowski space

$$y^0(t,\theta) = t\sqrt{1+r^2}, \qquad y^1(t,\theta) = rt\sin(\theta/r), \qquad y^2(t,\theta) = rt\cos(\theta/r), \tag{1}$$

where $(t,\theta) \in R^1 \times S^1$. One has

$$\frac{r^2}{1+r^2}(y^0)^2 - (y^1)^2 - (y^2)^2 = 0, \tag{2}$$

where $r \in R^1_+$ is a constant labelling the compactification (in what follows we put $r = 1$ for simplicity). Equation (2) presents two cones with a common vertex at $(y^0, y^1, y^2) = (0,0,0)$. One may verify that the induced metric on (2) reads

$$ds^2 = -dt^2 + t^2 d\theta^2. \tag{3}$$

The coefficient of $d\theta^2$ in (3) disappears as $t \to 0$. Therefore, one may use the CM space to model a two-dimensional universe with the big-crunch/big-bang type singularity.

The space CM is locally isometric with the Minkowski space at each point, except the vertex $t = 0$. At the vertex there is a singularity. It is of removable type because any time-like geodesic from the lower cone ($t < 0$) linked with the vertex ($t = 0$) can be extended to the time-like geodesic of the upper-cone ($t > 0$). It is clear that such an extension cannot be unique, because at $t = 0$ the Cauchy problem for the geodesic equation is not well defined, due to the disappearance of space dimension.

The local symmetry of the CM space is described by the Lie algebra of the Killing vector fields

$$[\eta_1, \eta_2] = 0, \qquad [\eta_3, \eta_2] = \eta_1, \qquad [\eta_3, \eta_1] = \eta_2, \tag{4}$$

where

$$\eta_1 = \cosh\theta\,\frac{\partial}{\partial t} - \frac{\sinh\theta}{t}\,\frac{\partial}{\partial\theta}, \qquad \eta_2 = \sinh\theta\,\frac{\partial}{\partial t} - \frac{\cosh\theta}{t}\,\frac{\partial}{\partial\theta}, \qquad \eta_3 = \frac{\partial}{\partial\theta}. \tag{5}$$

The algebra is well defined for $t \neq 0$.

3. Dynamics of Particle

An action integral, $\mathcal{A}$, describing a relativistic test particle of mass m in a gravitational field g_{kl}, $(k,l = 0,1)$ may be defined by

$$\mathcal{A} = \int d\tau\, L(\tau), \qquad L(\tau) := \frac{m}{2}\,(\frac{\dot{x}^k\dot{x}^l}{e}g_{kl} - e), \qquad \dot{x}^k := dx^k/d\tau, \tag{1}$$

where τ is an evolution parameter, $e(\tau)$ denotes the 'einbein' on the world-line, x^0 and x^1 are time and space coordinates, respectively.

In case of CM space the Lagrangian, for $t \neq 0$, reads

$$L(\tau) = \frac{m}{2e}\,(t^2\dot{\theta}^2 - \dot{t}^2 - e^2). \tag{2}$$

The action (1) is invariant under reparametrization with respect to τ. This gauge symmetry leads to the constraint

$$\Phi := (p_\theta/t)^2 - (p_t)^2 + m^2 = 0, \tag{3}$$

where $p_t := \partial L/\partial \dot{t}$ and $p_\theta := \partial L/\partial \dot{\theta}$ are canonical momenta.

Variational principle applied to (1) gives the equations of motion of a particle

$$\frac{d}{d\tau}p_t - \frac{\partial L}{\partial t} = 0, \qquad \frac{d}{d\tau}p_\theta = 0, \qquad \frac{\partial L}{\partial e} = 0. \tag{4}$$

Thus, during evolution of the system p_θ is conserved. Owing to the constraint (11), p_t blows up as $t \rightarrow 0$ for $p_\theta \neq 0$. However, trajectories of a test particle, i.e. nonphysical particle, coincide (by definition) with time-like geodesics of an empty spacetime, and there is no obstacle for such geodesics to reach/leave the singularity.

The dynamics of our system can be parametrized by three dynamical integrals

$$I_1 = p_t \cosh\theta - p_\theta \frac{\sinh\theta}{t}, \qquad I_2 = p_t \sinh\theta - p_\theta \frac{\cosh\theta}{t}, \qquad I_3 = p_\theta. \tag{5}$$

Making use of (5) we may rewrite the constraint (11) in the form

$$\Phi = I_2^2 - I_1^2 + m^2 = 0. \tag{6}$$

Solution to (4), for $t < 0$ or $t > 0$, may be parametrized by the dynamical integrals (5) as follows [4]

$$\theta(t) = -\sinh^{-1}\left(\frac{I_3}{mt}\right) + \tanh^{-1}\left(\frac{I_2}{I_1}\right) =: -\sinh^{-1}\left(\frac{c_1}{mt}\right) + c_2. \tag{7}$$

Dynamical integrals in terms of the new variables c_1 and c_2 read

$$I_1 = m\cosh(c_2), \quad I_2 = m\sinh(c_2), \quad I_3 = c_1. \tag{8}$$

Algebra of observables

$$\{I_1, I_2\} = 0, \quad \{I_3, I_2\} = I_1, \quad \{I_3, I_1\} = I_2, \tag{9}$$

is isomorphic to the Killing vectors algebra if

$$\{\cdot,\cdot\} := \frac{\partial\,\cdot}{\partial c_1}\frac{\partial\,\cdot}{\partial c_2} - \frac{\partial\,\cdot}{\partial c_2}\frac{\partial\,\cdot}{\partial c_1}, \qquad \{c_1, c_2\} = 1. \tag{10}$$

Thus, $c_1 \in R^1$ ('momentum') and $c_2 \in S^1$ ('position') play the role of canonical coordinates. It is clear that the phase space of the system has the topology $R^1 \times S^1$.

Due to the Cauchy problem at the singularity $t = 0$, there exists an ambiguity in passage from the pre-singularity to post-singularity era. In what follows we present only two examples of the passage. More examples can be found in [4].

3.1. *'Simple' propagation across the singularity*

This propagation is defined by the condition

$$\lim_{t\to 0^-} I_k = \lim_{t\to 0^+} I_k, \qquad k = 1,2,3. \tag{11}$$

Topology of the phase space is $S^1 \times R^1$. Algebra of canonical variables reads $\{c_1, c_2\} = 1$. For the quantization purpose one has to redefine this algebra [4], since $c_2 \in S^1$ has non-trivial topology, as follows

$$U_2 := \exp ic_2, \qquad < \cdot, \cdot >:= \Big(\frac{\partial\, \cdot}{\partial c_1}\frac{\partial\, \cdot}{\partial U_2} - \frac{\partial\, \cdot}{\partial U_2}\frac{\partial\, \cdot}{\partial c_1}\Big)U_2. \tag{12}$$

New form of the algebra is $< c_1, U_2 >= U_2$.

In what follows, *by quantization we mean finding an (essentially) self-adjoint representation of the algebra of the canonical variables.*

Mapping of classical observables into operators is defined as

$$c_1 \to \hat{c_1}\psi(\beta) := -i\frac{d}{d\beta}\psi(\beta), \qquad U_2 \to \hat{U_2}\psi(\beta) := e^{i\beta}\psi(\beta), \qquad \psi \in \Omega_\lambda. \tag{13}$$

The domain of $\hat{c_1}$ and $\hat{U_2}$ is defined to be

$$\Omega_\lambda := \{\psi \in L^2(\mathbb{S}^1) \mid \psi \in \mathcal{C}^\infty[0, 2\pi],\ \psi^{(n)}(2\pi) = e^{i\lambda}\psi^{(n)}(0)\}, \tag{14}$$

where $n = 0, 1, 2, \ldots$ and $0 \le \lambda < 2\pi$. One may verify that the representation is essentially self-adjoint on Ω_λ.

3.2. *'Complex' propagation across the singularity*

The propagation is defined by the condition

$$\lim_{t\to 0^-} |p_\theta| = \lim_{t\to 0^+} |p_\theta|. \tag{15}$$

The topology of phase space is found to be $S^1 \times R^1 \times S^1 \times Z_2$. The algebra of canonical variables reads

$$\langle c_1, U_2\rangle = U_2, \qquad \langle c_1, U_3\rangle = \varepsilon U_3, \quad \langle U_2, U_3\rangle = 0, \quad \varepsilon = \pm 1, \tag{16}$$

where

$$\langle \cdot, \cdot\rangle := \Big(\frac{\partial\, \cdot}{\partial c_1}\frac{\partial\, \cdot}{\partial U_2} - \frac{\partial\, \cdot}{\partial U_2}\frac{\partial\, \cdot}{\partial c_1}\Big)U_2 + \Big(\frac{\partial\, \cdot}{\partial c_1}\frac{\partial\, \cdot}{\partial U_3} - \frac{\partial\, \cdot}{\partial U_3}\frac{\partial\, \cdot}{\partial c_1}\Big)U_3. \tag{17}$$

Mapping of observables into operators is is defined to be

$$c_1 \to \hat{c_1}\psi(\beta)f_\varepsilon\varphi(\alpha) := -i\frac{d}{d\beta}\psi(\beta)f_\varepsilon\varphi(\alpha), \qquad \hat{\varepsilon}f_\varepsilon = \varepsilon f_\varepsilon \qquad \varepsilon = \pm 1, \tag{18}$$

$$U_2 \to \hat{U_2}\psi(\beta)f_\varepsilon\varphi(\alpha) := e^{i\beta}\psi(\beta)f_\varepsilon\varphi(\alpha), \tag{19}$$

$$U_3 \to \hat{U_3}\psi(\beta)f_\varepsilon\varphi(\alpha) := e^{i\beta\hat{\varepsilon}}e^{i\alpha}\psi(\beta)f_\varepsilon\varphi(\alpha) := e^{i\beta\varepsilon}\psi(\beta)f_\varepsilon e^{i\alpha}\varphi(\alpha). \tag{20}$$

The domain of the algebra is defined as $\Omega_\lambda \otimes E \otimes \Omega_\lambda$, where E denotes the Hilbert space spanned by the eigenvectors of $\hat{\varepsilon}$. The algebra of quantum observables reads

$$[\hat{c_1}, \hat{U_2}] = \hat{U_2}, \qquad [\hat{c_1}, \hat{U_3}] = \hat{\varepsilon}\hat{U_3}, \qquad [\hat{U_2}, \hat{U_3}] = 0. \tag{21}$$

The representation is essentially self-adjoint.

4. Ambiguity of Quantization

Since $0 \leq \lambda < 2\pi$ in Ω_λ, there are ∞-many unitarily non-equivalent representations, i.e. ∞-many quantum systems corresponding to one classical system. However this ambiguity can be removed by the imposition of time-reversal invariance upon the algebra of observables. In the case of simple propagation this procedure means

$$\hat{T}\hat{c_1}\hat{T}^{-1} = -\hat{c_1}, \qquad \hat{T}\hat{U_2}\hat{T}^{-1} = \hat{U_2}^{-1}, \tag{1}$$

where $\hat{T}$ is anti-unitary operator representing time-reversal transformation. The formal reasoning at the level of operators should be completed by the corresponding one at the level of the domain space Ω_λ of the algebra (1). Following step-by-step the method of the imposition of the time-reversal invariance upon the dynamics of a test particle in de Sitter's space, presented in [5], leads to the result that the range of the parameter λ must be restricted to the two values: $\lambda = 0$ and $\lambda = \pi$. Similar considerations may be carried out in the case of complex propagation.

5. Regularization

The ambiguity connected with the Cauchy problem at the singularity $t = 0$ may be removed by regularization of the compactified Milne [6]. Suppose that physical (not test) particle modifies spacetime at $t = 0$ in such a way that the compact space dimension does not contract to a point, but to some 'small' value ϵ. The following mapping defines the regularization

$$y^0_\epsilon(t,\theta) := t\sqrt{1+r^2}, \tag{1}$$

$$y^1_\epsilon(t,\theta) := r\,\sqrt{t^2+\epsilon^2}\,\sin(\theta/r), \qquad y^2_\epsilon(t,\theta) := r\,\sqrt{t^2+\epsilon^2}\,\cos(\theta/r). \tag{2}$$

One has

$$\frac{r^2}{1+r^2}(y^0_\epsilon)^2 - (y^1_\epsilon)^2 - (y^2_\epsilon)^2 = -\epsilon^2 r^2. \tag{3}$$

The regularized space has big-bounce type singularity, instead of big-crunch/big-bang type singularity of the compactified Milne space. The induced metric on the regularized space has the property

$$ds^2_\epsilon = -(1+\frac{r^2\epsilon^2}{t^2+\epsilon^2})dt^2 + (t^2+\epsilon^2)d\theta^2 \;\longrightarrow\; -dt^2 + t^2 d\theta^2 = ds^2, \tag{4}$$

as $\epsilon \longrightarrow 0$. Solution to the equations for geodesics reads

$$\theta_\epsilon(t) = \beta + p\int_{-\infty}^{t} d\tau\sqrt{\frac{1+\frac{r^2\epsilon^2}{\tau^2+\epsilon^2}}{(\tau^2+\epsilon^2)^2+p^2(\tau^2+\epsilon^2)}} \tag{5}$$

and one has

$$\lim_{\epsilon\longrightarrow 0}\theta_\epsilon(t) = \beta - \sinh^{-1}(p/t) = \theta(t), \qquad (p,\beta)\in R^1\times S^1. \tag{6}$$

It is clear that the compactified Milne space leads to the phase space (of the simple propagation)

$$\Gamma := \{(\beta, p) \mid \beta \in R^1 \; mod\; 2\pi,\; p \in R^1\} = S^1 \times R^1. \tag{7}$$

Making use of the group theoretical quantization method (group of motions of the Euclidean plane $E(2)$ is the canonical group of Γ) leads to the following irreducible unitary representation of $E(2)$ (see, e.g. [7])

$$[U(\alpha)\psi](\beta) := \psi[(\beta - \alpha) \; mod\; 2\pi], \qquad \text{for rotations} \quad z \rightarrow e^{i\alpha} z, \tag{8}$$

and

$$[U(t)\psi](\beta) := e^{-i(a\cos\beta + b\sin\beta)}\psi(\beta), \qquad \text{for translations} \quad z \rightarrow z + t, \tag{9}$$

where $\psi \in L^2(S^1),\; z = |z|e^{i\beta},\; t = a + bi$. Applying the Stone theorem to (8) and (9) leads to the essentially self-adjoint representation of the algebra $e(2)$

$$[\hat{A}, \hat{B}] = 0, \quad [\hat{p}, \hat{B}] = -i\hat{A}, \quad [\hat{p}, \hat{A}] = i\hat{B}, \tag{10}$$

where

$$\hat{p}\varphi(\beta) := -i\frac{\partial}{\partial\beta}\varphi(\beta), \quad \hat{A}\varphi(\beta) := \sin\beta\; \varphi(\beta), \quad \hat{B}\varphi(\beta) := \cos\beta\; \varphi(\beta) \tag{11}$$

and the domain space is the same as in the case of real propagation. The algebras (10) and the algebra of the latter case are identical owing to the relations

$$e^{i\beta} = \cos\beta + i\sin\beta, \qquad [\cos\beta, \sin\beta] = 0. \tag{12}$$

It seems that *any regularization* of the singularity of the compactified Milne space leads to the phase space with the topology $S^1 \times R^1$.

6. Conclusions

The compactified Milne space represents simple, but non-trivial model of singular universe. The propagation of a *quantum* test particle is well defined mathematically (up to the Cauchy problem at $t = 0$). Some hypothesis concerning the nature of the singularity can be put forward by specific choice of particle's transitions across the singularity. In the case that there is no clear reason to choose a specific transition across the singularity, it acts as 'generator' of uncertainty in the propagation of a particle from the pre-singularity to the post-singularity era.

The most urgent improvement to make is an extension of our results to higher dimensional Milne space required for compactification of higher dimensional quantum phase to four dimensional classical phase. It can be done by making use of the method used in the quantization of particle dynamics in higher dimensional de Sitter's space [8]. Other improvement should take into account the effect of the particle's own gravitational field on its motion [9].

Our considerations concern point-like objects. Next natural step would be examination of dynamics of extended objects like strings or membranes. According to string/M theory (see, e.g. [10]), they are more elementary than point particles.

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