Topos Theory and Spacetime Structure

JERZY KRÓL

Institute of Physics, University of Silesia, ul. Uniwersytecka 4
Katowice, 40-007/Poland
Jerzy Król_iriking@wp.pl

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According to the recently proposed model of spacetime various difficulties of quantum field theories and semiclassical quantum gravity on curved 4-Minkowski spacetimes gain new formulations leading to new solutions. The quantum mechanical effects appear naturally when diffeomorphisms are lifted to 2-morphisms between topoi. The functional measures can be well defined. Diffeomorphisms invariance and background independence are approached from the perspective of topoi. In the spacetimes modified at short distances by the internal structure of some topoi, the higher dimensional regions appear and field/strings duality emerges. We show that the model has natural extensions over extremly strong gravity sources in spacetime and shed light on the strong string coupling definition of B-type D-branes.

Keywords: Topos theory; quantum gravity; field theory.

1. Introduction

Considering spacetime as being modeled by smooth differentiable manifolds is widly accepted point of view in theoretical physics. Classical and quantum field theories (QFT) heavily rely on that concept. There are also known difficulties of QFT regarding the necessity to renormalize the theories, the trouble with the correct definition of the functional measure when the theory is formulated in terms of functional integrals. In the case of strings theory there is still missing either the background independent formulation or the connection with loop quantum gravity (QG) (see, e.g., [19]). Besides, it is known that semiclassical QG is not renormalized theory. The lack of successful formulation of QG is also the example of possible difficulties with the smooth manifold's model of spacetime.

In Refs. [13,14] we proposed and developed the model of spacetime where we assumed the modification of the normal manifold's structure of spacetime at short distances. The modification is given by the interpratation of suitably small regions (open domains diffeomorphic to $\mathbb{R}^4$) internally in some topoi. By the careful choice of topoi we were able to show that all above difficulties gained new perspactives and consequently the new way of solving them emerged.

In the next section we will discuss briefly these issues. In further sections we will present the model of spacetime which emerges naturally as extending the spcetime in the regime of extremly strong gravity and we will show the importance of the ideas when searching for the definition of the D-branes in the strong string coupling limit.

2. Topoi and the Model of Spacetime

The pioneering works about the role of topoi in physics are those by Lawvere [15], Benioff [5], Takeuti [20,21,22], Bell [4], Isham [7] and Isham and Butterfield [8,9].

Topoi are special categories which are models of the higher order intuitionistic logic [16]. The category of all sets and functions between them as morphisms (SET) is the example of a topos. However, SET is based on classical logic i.e. on 2-valued Boolean algebra. General topoi are based rather on Heyting algebras where the law of excluded midle does not hold. Our main concern here are *Grothendieck topoi* with the object of natural numbers [16].

Let us, following Ref. [13], consider the following model of spacetime:

At sufficiently small distances, those of order of the Planck length or bigger, the logic assigned to the description of spacetime regions, is weakened from classical to intuitionistic logic of some topoi.

The important thing is that the objects R^n in topoi are considered along with the different logic and set theory now being intuitionistic [13].

Boolean-valued models of Zermelo-Fraenkel set theory with the axiom of choice (ZFC) are also topoi [21]. Given a Hilbert space H of states and the lattice of all projections on closed subsets we can always choose the maximal Boolean algebras B_H of projections. Next, given any complete Boolean algebra we have Boolean-valued model of ZFC, V^{B_H} which is known to be the topos of sheaves of sets on B_H i.e. $Sh(B_H)$. Takeuti showed that [21]:

Theorem 1. *All real numbers from the object of real numbers R in $Sh(B_H)$, are exactly in one to one correspondence with the self-adjoint linear operators in H which are in B_H.*

Written down a classical expression in the topos $Sh(B_H)$ corresponds to the quantized expression provided the observables commute. Given the different maximal Boolean algebra of projections it can be spanned over non-commuting operators with those from the first B_H. Thus, quantization can be performed via interpretations in Boolean-valued topoi $Sh(B_H)$ [22,13].

Given a smooth topos, say Basel topos $\mathcal{B}$, it is known that its object of real numbers contains nonstandard infinitely big reals and infinitesimals (invertible and non-invertible) [17]. Moreover, any differential equation can be written down in this topos. That is why it was proposed in Refs. [13,12] to formulate divergent expressions from QFT's internally in the Basel topos and consider them as internally s-finite. Next, performing the renormalization of finitely many of the parameters

gives rise to the renormalization of infinitely many expressions. That can be important in semi-classical QG which is known to be nonrenormalized classically.

Various difficulties with rigorous definition of functional measure on some spaces of functions, as nonexistence of the positive definite measure or measures which are not gaussian can be cured when taking the spatial topos $Sh(M)$ [13]. M is a smooth compact n-manifold on which the space of functions is defined. In the topos $Sh(M)$ reals from the object of real numbers are in one to one correspondence with the continuous functions from $C^0(M)$. Thus, integration functional measures correspond to the internal measures on R [13].

In the case of pointless topoi, one can relate the diffeomorhisms invariance and background independence of QFT with the formulation of the appropriate expressions in such a topos [10,13].

When some QFT's are formulated internally in the special pointless topos, constructed in Ref. [14], they appear to be dual to some higher dimensional set-theoretical strings theory. The origins of the duality field theory/strings emerges at the level of varying set theory. If the topos used is again Basel topos the higher dimensional regions are naturally presnt in 4-spacetimes modified locally by the topos [14].

3. Strong Gravity and Spacetime Structure

Given the model of spacetime we can try to find topoi which would be suitable for the correct description of spacetime at the regime of very strong gravity. Classically the spacetime is extremly curved at such regimes, although locally there always exists a flat Minkowski's frame. When passing from frame to frame one is confined by diffeomorphisms at least for the smooth category. The idea now is to broaden the class of acceptable morphisms between the coordinate frames such that some 2-morphisms are allowed. 2-morphisms are natural functors between topoi in the category of all topoi and geometric morphisms [16].

Suppose that in 4-Minkowski spacetime there is a quantum system which is characterized by a Hilbert space H of its states. The spacetime is locally R^4. We try to deform R^4 such that we are able to reach quantum Hilbert space H. The deformation, however, should preserve the generalized structure of spacetime according to the proposed model.

Thus, let us think about the topology of R^4 as about the lattice $L_{\mathcal{O}}$ of open subsets defining a locale [16]. The Hilbert space H determines the lattice of projections on the closed linear subspaces of H. The deformation is defined by substituting the structure of the locale lattice by the Hilbert space lattice of projections onto the closed subspaces [20].

The above procedure results in 2-morphism

$$f : Sh(L_{\mathcal{O}}) \to Sh(B_H) \tag{1}$$

where B_H is some maximal Boolean algebra of projections chosen from the lattice L_H.

$Sh(B_H)$ is the Takeuti topos which was discussed in Sec. (2). $Sh(L_{\mathcal{O}})$ is $Sh(R^4)$. Thus, we have the change of algebraical and topological structures driven by the above 2-morphism. Such a change, as correlated with the change of topoi, gives rise to the replacing SET by Boolean valued Takeuti topos. Note that one Hilbert space generates many different Boolean valued topoi $Sh(B_H)$ corresponding to different maximal Boolean algebras B_H.

The idea is to correlate the above 2-morphisms between the topoi with extremly strong gravity. The deformation of spacetime due to the presence of such gravity sources is described by 2-morphisms rather than diffeomorphisms between R^4's patches. Similarly as diffeomorphisms, representing the change of coordinate frames, generate the curvature of spacetime, the 2-morphisms generate a kind of 2-curvature. The 2-curvature, in turn, is connected with the possibility to describe consistently quantum matter in the regime of very strong gravity. This follows from the property of Takeuti's topoi which replace consistently real numbers by the self-adjoint operators on H. Thus classical expressions, depending on real arguments, as coordinates and momenta, become self-adjoint operators. The quantization of matter is performed due to the generalized change of local coordinates (see, e.g., [13]). Moreover, the model of spacetime emerges which is consistent with the very strong gravity and quantum matter.

We can formulate the following:

Theorem 2. *(i) Suppose H is a Hilbert space of states of some quantum system. If the density of matter and energy is sufficiently big, the quantum system modifies spacetime such that local coordinates patch $\mathbb{R}^4$, by coordinates of which the quantum system is defined, becomes the object R^4 in the Takeuti's topos $Sh(B_H)$.*

(ii) The generalized spacetime allows for the change of coordinates $\mathbb{R}^4 \to R^4$ which corresponds to some 2-morphism between the topoi as in Eq. (1).

(iii) The change of coordinates corresponds to the quantization of matter [20].

4. Generalized Spacetimes and GR

In this section we want to analyze the following problem: up to what extent the generalized spacetimes can be considered as solutions of, possibly generalized, Einstein equations (EE's)? What are the generalized EE's?

Let us write down the EE's as follows:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

The equations are invariant with respect to the diifeomorphic choice of any coordinate frame. When we allow for the extension of the possible choices of the coordinate frames by 2-morphisms, the question arises about the meaning of the generalized 2-covariance of the equations. The 2-covariance of the left hand side can be different than the covariance of the stress energy tensor $T_{\mu\nu}$. This is possible since in the case of a general topos the internal and external seeing of the objects is different (in the

case of SET they coincide). Thus, the $G_{\mu\nu}$ under 2-morphisms becomes internally interpreted in the topos $Sh(B_H)$. This gives $\hat{G}_{\mu\nu}$ as the modification of $G_{\mu\nu}$.

$T_{\mu\nu}$, in turn, is understood externally, i.e. written with respect to the real numbers seen as self-adjoint operators on H. Whereas, $G_{\mu\nu}$ is written down with respect to the real numbers R, as the internal real numbers object in the topos $Sh(B_H)$. Thus, left hand side of EE's is flexible enough and becomes interpreted in the intuitionistic logic of the topoi; the right hand side of EE's is governed by classical logic. Such an extension of diffeomorphisms into 2-morphisms makes that the structure of spacetime unifies gravity with the quantum matter. The price to pay is the intuitionistic logic of the gravity side.

5. Sheaf Structure of B-branes and Topoi

It is known [1] that the correct definition of quantum D-branes in general spacetime backgrounds is currently out of reach of mathematicions and physicists. However, in the simplified situations one can manage this problem via the sheaf-like description of B-branes in terms of derived categories [18]. The simplifications rely mainly on the following points:

1. The target of string theory is the space $\mathbb{R}^{1,3} \times K$ for some compact space K.
2. The string coupling is sending to zero.
3. Dealing with only finite dimensional Hilbert spaces of open strings.

The result by Douglas [6] expresses deep relationship between BPS B-branes and derived categories:

Theorem 3. *A B-type BPS Brane on the Calabi-Yau manifold M is a Π stable object in the derived category of coherent sheaves on M.*

Holomorphic coherent sheaves are especially well suited for the description of B-type branes and the spectra of open strings between such branes [18].

We propose that when trying to extend these ideas over strong string couplings one should deal with more fundamental than holomorhic level of sheaves constructions, namely sheaves of sets on suitable submanifolds in Calabi-Yau manifolds. The holomorphic structure of branes gives very good calculational results in the weak string coupling limit. However, in the non-zero couplings the machinery is stopped. That is why, keeping the sheaves constructions valid but considering more general sheaves of sets, one has the possibility that branes in the varying set-theoretical environment can extend the usual picture. The point is that also logic and set theory should be deformed rather than kept as classical, with the usual understanding of the axioms of choice and the excluded midle. Knowing that the constructions of Grothendieck topoi heavily rely on sheaves of sets and that topoi generate their own (intuitionistic) logic and set theory [16], the proposition as above looks natural.

In the case of B-type branes corresponding to the objects in the derived category of coherent sheaves, one is confined with the morphisms of the derived category

when translating one brane into another. What we want to do is to consider further "categorification", in the sense of Baez and Dolan (see e.g. [2,3]), and to consider some strict relations, in the sense of the morphisms in the derived category, as holding up to morphisms between sheaves of sets in suitable topos.

The equations governing the branes are deformed and should be considered as written down in the topos of sheaves of sets on some submanifold $X \subset K$. The submanifold X is the same around which branes, understood classically, wrapped. In the limit of small string coupling one consideres holomorphic sheaves on X and this suffices to generate various open string spectra between branes [18]. In the case of strong coupling limit the deformation due to topoi as above, occurs. Similarly, as spacetime is deformed at short distances by some topoi, branes in strongly interacting strings are also deformed due to some topoi.

6. Discussion

In the presented model of spacetime logic can be weakened at small distances. Logic is rather a feature of a physical system or region of spacetime than the universal ambient environment unchanged whatsever the scales of energies are considered. A kind of relativization of logic is thus proposed (see also Bell[4]).

In the case of exotic smooth structures on open 4-manifolds similar thinking was proved to be useful [10,11]. The structures are considered as invariant classical objects with respect to the changes of the metalevel generated by the shift SET$\rightarrow \mathcal{B}$ where $\mathcal{B}$ is smooth Basel topos. The shift can be understood as 2-morphism between topoi. Thus, the model of spacetime, discussed in this paper, indicates that exotic smooth R^4's, rather than standard smooth $\mathbb{R}^4$, are possibly more appropriate for the correct applications of the model especially when the covariance in the model is approached.

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