

GENERAL RELATIVITY, GRAVITATIONAL ENERGY AND SPIN-TWO FIELD

Leszek M. SOKOŁOWSKI*

Astronomical Observatory and Centre for Astrophysics,
Jagellonian University,
Orla 171, Kraków 30-244, Poland

Abstract

In my lectures I will deal with three seemingly unrelated problems: i) to what extent is general relativity exceptional among metric gravity theories? ii) is it possible to define gravitational energy density applying field-theory approach to gravity? and iii) can a consistent theory of a gravitationally interacting spin-two field be developed at all? The connecting link to them is the concept of a fundamental classical spin-2 field. A linear spin-2 field introduced as a small perturbation of a Ricci-flat spacetime metric, is gauge invariant while its energy-momentum is gauge dependent. Furthermore, when coupled to gravity, the field reveals insurmountable inconsistencies in the resulting equations of motion. After discussing the inconsistencies of any coupling of the linear spin-2 field to gravity, I exhibit the origin of the fact that a gauge invariant field has the variational metric stress tensor which is gauge dependent. I give a general theorem explaining under what conditions a symmetry of a field Lagrangian becomes also the symmetry of the variational stress tensor. It is a conclusion of the theorem that any attempt to define gravitational energy density in the framework of a field theory of gravity must fail. Finally I make a very brief introduction to basic concepts of how a certain kind of a necessarily nonlinear spin-2 field arises in a natural way from vacuum nonlinear metric gravity theories (Lagrangian being any scalar function of Ricci tensor). This specific spin-2 field consistently interacts gravitationally and the theory of the field is promising.

*E-mail: UFLSOKOL@TH.IF.UJ.EDU.PL

1. Introduction

General relativity is just a point in the "space" of all existing and conceivable theories of gravitational interactions. Nevertheless all the theories other than Einstein's one, named "alternative theories of gravity", have rather bad reputation among most relativists: general relativity is enough complicated in itself and well confirmed by all known empirical data so that there seems no point in considering more intricate theories whose confirmation is either rather poor or presently non-existing at all. In fact, the alternative theories are some generalizations of general relativity, which is invariably used as a reference point for their construction. These modifications go in all possible directions making any attempt to confront them with experiment very difficult.

In spite of this criticism in the last three decades there has been considerable interest in some alternative theories. It comes from various directions but the generic attitude in it is to search for a deeper relationship between gravitational physics and elementary particle interactions. Quantum gravity and superstrings seem to indicate that definite modifications of general relativity are necessary. Though these fundamental theories are still rather fancy than a fact, they suggest that general relativity should be replaced in high energy approximation by an effective theory of gravity, which should be some metric nonlinear gravity (NLG) theory and it is here that a significant progress has been made since early 1980's. (Similar methods of investigation can be applied to purely affine and metric-affine gravity theories which are presented in other lectures). In NLG theories Einstein-Hilbert Lagrangian (the curvature scalar R) is replaced by any smooth scalar function of Riemann tensor, $L = f(R_{\alpha\beta\mu\nu}, g_{\mu\nu})$. All other axioms of general relativity hold for these theories. (General relativity and NLG theories may be studied in any spacetime dimension $d \geq 4$. In these lectures I will assume $d = 4$ unless otherwise is stated.) For f is arbitrary, the NLG theories form a densely populated neighbourhood of general relativity in the space of gravity theories. This raises a fundamental problem: is Einstein theory merely a point of this neighbourhood? In other terms: is general relativity distinguished merely by tradition and (relative) computational simplicity or does it occupy a preferred position with respect to the theories that surround it? It was shown by Magnano, Ferraris and Francaviglia [1] and independently by Jakubiec and Kijowski [2] that *NLG theories are mathematically, i.e. dynamically equivalent to general relativity and there are convincing arguments that at least some of them are also physically equivalent to it.* In a sense these theories represent Einstein

theory in disguise. General relativity is not surrounded by theories different from it and its closest neighbourhood consists of its own versions in distinct (usually unphysical) variables. General relativity is an isolated point in the space of gravity theories. Physical interpretation of these versions of Einstein theory is however quite subtle, see [3].

If general relativity preserves its distinguished and leading role, one should return to unsolved problems of this theory, e.g. to that of localizability of gravitational energy. A conventional wisdom claims that gravitational energy and momentum densities are nonmeasurable quantities since the gravitational field can always be locally transformed away according to Strong Equivalence Principle. Nevertheless since the very advent of general relativity there have been numerous attempts to construct a local concept of gravitational energy as it would be very useful in dealing with practical problems, e.g. a detailed description of cosmological perturbations in the early universe. Among various approaches to the problem a particularly promising one has been provided by the field theory formulation of gravity theory, according to which gravity is just a tensor field existing in Minkowski space and the latter may or may not be regarded as the spacetime of the physical world. In this formulation gravity is described by a Lagrangian field theory in flat spacetime for a spin-two field and its energy density is given by the metric (i.e. variational) energy-momentum tensor. In linearized around Minkowski space general relativity a linear spin-2 field arises and it is gauge invariant while its metric energy momentum tensor is not. Thus a naive field-theoretical approach to gravity fails. It is interesting on its own to explain why the energy-momentum tensor for the field does not inherit the symmetry property of the underlying Lagrangian. Another problem, closely related to the previous one, is whether can this defect be overcome at all: is there a linear spin-2 field which is dynamically equivalent to linearized gravity and possessing a gauge invariant energy-momentum tensor? The answer is no and no such field exists because the Lagrangian of the linearized general relativity is gauge invariant only in empty spacetimes ($R_{\mu\nu} = 0$). In general, the energy-momentum tensor does not inherit a symmetry of the Lagrangian if the symmetry does not hold in a generic curved spacetime.

This failure does not cancel interest in spin-2 fields since a certain kind of necessarily nonlinear spin-2 field arises in a natural way from vacuum NLG theories as a component of a multiplet of fields describing gravity. Any Lagrangian different from R and the Euler-Poincaré topological invariant density (Gauss-Bonnet term) gives rise to fourth order field equations and hence gravity has altogether eight (rather than two) degrees of freedom. The

”particle content” of these degrees of freedom is disclosed by decomposing the gravitational field into a multiplet of fields with definite spins. The decomposition is accomplished by using a specific Legendre transformation. One gets a triplet: spacetime metric (2 degrees of freedom), a scalar field (1 d.o.f.) and massive spin-2 field (5 d.o.f.). The Legendre transformation reduces the fourth-order theory to a second-order one. Actually there are two different Legendre transformations giving rise to two different (dynamically equivalent) versions of the resulting theory. Both versions take on the form of Einstein’s theory for the metric field with the other two fields of the triplet acting as a ”matter” source in Einstein field equations. For a special form of the original Lagrangian for NLG theory the scalar field disappears and the propagation of the metric field is governed by the nonlinear spin-2 field alone. Thus any NLG theory generates a consistent gravitational interaction for the nonlinear massive spin-2 field (in general also coupled to the scalar).

The latter outcome is quite unexpected and surprising since it has been known for more than three decades that a linear (and gauge invariant) spin-2 field cannot interact gravitationally in a consistent way, it can only propagate as a test field in an empty spacetime. For higher spin fields the situation is even worse: it was found that in order for these fields to exist on a Lorentzian manifold, restrictions on the curvature, being in fact consistency conditions, were required. These conditions are very severe: ”there is no easy way to have physical fields with spins > 2 on anything but a flat manifold” [4]. It is therefore a common belief among field theorists that Nature avoids the consistency problem by simply not creating fundamental spin-two (nor higher spin) fields except gravity itself. This is, however, not true and consistent dynamics for spin-2 fields in the framework of general relativity is provided by NLG theories; the fields must be nonlinear and their theory is built up in a way quite different from that for the linear (and inconsistent) field.

In these lectures I will follow the opposite order to that presented above. I will start from linear spin-2 field in flat spacetime and show that it is unphysical in the sense that it cannot exist (i.e. consistently interact gravitationally) in non-empty spacetimes. Then I will prove that no linear spin-2 field can have a gauge invariant energy-momentum tensor in flat spacetime. Finally I will discuss NLG theories and show how they generate the nonlinear massive spin-two field.

2. Linear free spin-2 field in Minkowski space

It is commonly accepted that a matter field of integer spin s is described by a symmetric tensor field with s indices, $\psi_{\mu_1 \dots \mu_s}$. Thus for $s = 2$ it is described by a tensor $\psi_{\mu\nu} = \psi_{\nu\mu}$. We shall focus our attention on spin-2 fields in this tensor representation¹. Whether the tensor ψ represents a gauge-dependent tensor potential or a measurable field strength depends on its dynamical properties. Assuming four-dimensionality of the spacetime we define spin (or helicity for massless matter fields) by the transformation properties of a tensor field under the rotation group in a three-dimensional space. Then a spin-2 field should have $2s + 1 = 5$ degrees of freedom. On the other hand the symmetric tensor $\psi_{\mu\nu}$ has 10 algebraically independent components. Thus *any* equations of motion should be compatible with a system of algebraic (and differential) constraints removing 5 spurious unphysical components ("unphysical modes"). In the case of massless fields of helicity two there are only two degrees of freedom and constraint equations should set to zero eight unphysical modes. In general, for spin $s \geq 1$ equations of motion must necessarily be supplemented by a number of constraints. This fact makes the dynamics of $s \geq 1$ fields rather complicated and for $s \geq 2$ excludes a consistent interaction of a linear field with gravity.

There are four stages of constructing a theory for interacting fields.

- i) Choice of a mathematical object (field) describing classical matter carrying spin $s \geq 1$. As mentioned above we shall describe any continuous spin-2 matter by a symmetric tensor field $\psi_{\mu\nu}$.
- ii) Construction of free field theory in flat spacetime. The theory is first built for a massless field and the guiding principle is the postulate of a gauge invariance. Then for a massive field an appropriate mass term is added to the Lagrangian. The free field theory is consistent (after taking into account all the existing constraints).
- iii) Construction of self-interaction for the field. It is here that some troubles may arise. We shall not deal with this case.
- iv) Coupling of the field to other kinds of matter or to gravity. A genuine inconsistency arises when one attempts to introduce gravitational interactions for spin two.

¹In a 4-dimensional spacetime there exists another representation of a *linear* spin two: it is mathematically described by a 4-index tensor field $H_{\alpha\beta\mu\nu}$ having all algebraic symmetries of Riemann tensor [5]. Then the tensor $\psi_{\mu\nu}$ is just a superpotential for the field $H_{\alpha\beta\mu\nu}$. Since this alternative mathematical description cannot be used for the nonlinear spin-2 fields arising in NLG theories, we infer that the description in terms of symmetric tensors $\psi_{\mu\nu}$ is more fundamental and we shall not deal with the "Riemann tensor-like" representation of spin two.

To obtain linear equations of motion one assumes a Lagrangian quadratic in the field potential or field strength. Taking the variational derivative with respect to the 10 algebraically independent components of $\psi_{\mu\nu}$ one arrives at a system of 10 Lagrange equations, while, there are at most 5 physical degrees of freedom (in the massive case) and some components are redundant (represent a "pure gauge"). Then Lagrange field equations form a *degenerate system*: not all of them are second order hyperbolic (i.e. propagation) equations and a number of them are first order constraints on the initial data—they do not determine $\frac{\partial^2}{\partial t^2}\psi_{\mu\nu}$ for some components of ψ in terms of arbitrary initial data and actually represent restrictions on the data. These constraints will be referred to as *primary constraints*. Then applying various linear operations to Lagrange equations one can generate from them a number of *secondary constraints*². The secondary constraints allow one to transform the original system of Lagrange equations into a nondegenerate system of hyperbolic propagation equations for which the Cauchy problem is well posed.

It turns out that finding out the appropriate Lagrangian is not straightforward and Fierz and Pauli [6] who first did it in 1939 had to use a rather indirect procedure including introducing at an intermediate stage an auxiliary unphysical field which was set to zero at the end. The Lagrangian is quite complicated in comparison to that for a vector field and may be given in a number of equivalent (up to a total divergence term) forms.

As we are interested in gravitational interactions of spin-2 field, we omit the free field theory in flat spacetime and construct a Lagrangian for $\psi_{\mu\nu}$ in a curved spacetime.

3. Linear spin-2 field in a curved spacetime

The best way of constructing a theory for gravitationally interacting spin-2 field, massive or massless, is just to employ a gravitational perturbation analogy [7]. One takes any spacetime metric and perturbs it, $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$. The second variation of Einstein-Hilbert action $S[g] = \int d^4x \sqrt{-g} R$ evaluated at the "background" metric $g_{\mu\nu}$ is a functional quadratic in the metric fluctuations and if one identifies $\delta g_{\mu\nu}$ with $\psi_{\mu\nu}$ the functional provides an action giving rise to linear Lagrange equations for the field. Actually there is

²We do not define here the difference between primary and secondary constraints in a precise way. By primary constraints we mean the ones recognized in the system of field equations just by inspection (or possibly by taking a linear combination of the equations); secondary constraints are those generated by applying various differential operators to the system.

a considerable freedom in constructing a Lagrangian for the field giving rise to various inequivalent models. Different models arise if the variable to be varied is not the metric but a function of it. As a result the functional dependence of $\delta^2 S$ on the variations of the chosen variable is different in each case, while it is always chosen that $\psi_{\mu\nu}$ is equal to the metric perturbation. The Lagrangians and corresponding field equations differ by a number of terms involving Ricci tensor but not Weyl tensor. As a consequence all these models become equivalent in empty spacetime, $R_{\mu\nu} = 0$. Most of the models are nonminimal in the sense that they assume direct coupling of the field to Ricci tensor in their Lagrangians. And all models suffer the same deficiencies whenever $R_{\mu\nu} \neq 0$, i.e. when there is a source of gravity and these defects preclude a consistent theory of a linear purely spin-2 field in a generic spacetime. Here we give examples of two models.

The simplest choice of the variable to be varied is the metric $g^{\mu\nu}$ itself, then $\psi_{\mu\nu} = \delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}$. The action for $\psi_{\mu\nu}$ is defined as $S[\psi] \equiv \frac{1}{2}\delta(\delta S[g])$. Next one assumes that $\psi_{\mu\nu}$ is a non-geometric tensor field which interacts with gravity. In other words from now on $g_{\mu\nu}$ is not regarded as a fixed background metric but rather as a dynamical field coupled to $\psi_{\mu\nu}$. (This means that the equations of motion for the fields are *not* perturbation equations of a given solution for pure gravity.) To assign a mass to $\psi_{\mu\nu}$ one puts in the Lagrangian by hand a mass term which is appropriately chosen to avoid any additional scalar field. After making these operations a second order action for $\psi_{\mu\nu}$ reads (the action is linear in second derivatives)

$$S[\psi] = -\frac{1}{2} \int d^4x \sqrt{-g} [\psi^{\mu\nu} G_{\mu\nu}^L + \frac{1}{2} \psi \psi^{\mu\nu} G_{\mu\nu} + \frac{m^2}{2} (\psi_{\mu\nu} \psi^{\mu\nu} - \psi^2)] \quad (1)$$

where $\psi = g^{\mu\nu} \psi_{\mu\nu}$ and $G_{\mu\nu}^L$ is the linear in $\psi_{\mu\nu}$ part of Einstein tensor,

$$G_{\mu\nu}(g + \psi) = G_{\mu\nu}(g) + G_{\mu\nu}^L(\psi, g) + \dots,$$

and

$$G_{\mu\nu}^L(\psi, g) \equiv \frac{1}{2} (-\square \psi_{\mu\nu} + \psi_{\mu;\nu\alpha}^\alpha + \psi_{\nu;\mu\alpha}^\alpha - \psi_{;\mu\nu} - g_{\mu\nu} \psi^{\alpha\beta}_{;\alpha\beta} + g_{\mu\nu} \square \psi + g_{\mu\nu} \psi^{\alpha\beta} R_{\alpha\beta} - \psi_{\mu\nu} R); \quad (2)$$

here $\square T \equiv T_{;\alpha}^\alpha$. Adding Einstein-Hilbert action $S[g]$ to $S[\psi]$ one derives the full system of equations of motion consisting of Einstein's field equations $G_{\mu\nu} = T_{\mu\nu}(\psi, g)$ (we set $8\pi G = c = \hbar = 1$) where $T_{\mu\nu}$ is the variational energy-momentum tensor following from (1) and Lagrange equations

$$G_{\mu\nu}^L(\psi, g) + \frac{1}{2} G_{\mu\nu}(g) \psi - \frac{m^2}{2} (g_{\mu\nu} \psi - \psi_{\mu\nu}) = 0. \quad (3)$$

One sees that the simplest choice of the varied variable leads to a nonminimal model. We shall not investigate further this model and consider a minimal one.

A minimal model, which is unique up to field redefinitions, arises if the independent variable to be varied is the tensor density $\tilde{g}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$. The first variation of S is the same as in the nonminimal case presented above, $\delta_{\tilde{g}}S = \delta S$. A difference appears in evaluating the second variation, for, by definition, $\delta_{\tilde{g}}\delta\tilde{g}^{\mu\nu} \equiv 0$ while $\delta\delta\tilde{g}^{\mu\nu} \neq 0$ due to the relationship

$$\delta\tilde{g}^{\mu\nu} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta} + \delta g^{\mu\nu}\right).$$

The final outcome is a second order action for the field $\psi_{\mu\nu}$ which is related to $\delta\tilde{g}^{\mu\nu}$ according to the preceding equation by

$$\delta\tilde{g}^{\mu\nu} = -\sqrt{-g}(\psi^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\psi);$$

the action reads

$$\begin{aligned} S_{II}[\psi] &= -\frac{1}{4} \int d^4x \sqrt{-g} [\psi^{\mu\nu} (-\square\psi_{\mu\nu} + \psi^\alpha_{\mu;\nu\alpha} + \psi^\alpha_{\nu;\mu\alpha} - \psi_{;\mu\nu} - \\ &\quad g_{\mu\nu}\psi^{\alpha\beta}_{;\alpha\beta} + g_{\mu\nu}\square\psi) + m^2(\psi_{\mu\nu}\psi^{\mu\nu} - \psi^2)] \\ &= -\frac{1}{2} \int d^4x \sqrt{-g} [\psi^{\mu\nu} G_{\mu\nu}^{AD}(\psi, g) + \frac{m^2}{2}(\psi_{\mu\nu}\psi^{\mu\nu} - \psi^2)]. \end{aligned} \quad (4)$$

Here $G_{\mu\nu}^{AD}(\psi, g)$ is the curvature free part of $G_{\mu\nu}^L(\psi, g)$, i.e.

$$G_{\mu\nu}^{AD}(\psi, g) \equiv G_{\mu\nu}^L(\psi, g) - \frac{1}{2}(g_{\mu\nu}R^{\alpha\beta}\psi_{\alpha\beta} - R\psi_{\mu\nu}). \quad (5)$$

Hence the action (4) is the curvature free part of the action (1) for the non-minimal model. We stress that in both models the tensor field $\psi_{\mu\nu}$ (which should be a purely spin-2 field) is formally defined as a metric perturbation. One can also replace S_{II} by a first order action, to this end one expresses terms with second derivatives by appropriate terms made up of first derivatives of the field plus divergences which are discarded. The resulting action has an unambiguous form in any curved spacetime,

$$S_W[\psi] = \int d^4x \sqrt{-g} [L_W(\psi, g) - \frac{m^2}{4}(\psi_{\mu\nu}\psi^{\mu\nu} - \psi^2)] \quad (6)$$

with

$$L_W(\psi, g) = \frac{1}{4} (-\psi^{\mu\nu;\alpha}\psi_{\mu\nu;\alpha} + 2\psi^{\mu\nu;\alpha}\psi_{\alpha\mu;\nu} - 2\psi^{\mu\nu}_{;\nu}\psi_{;\mu} + \psi^{;\mu}\psi_{;\mu}). \quad (7)$$

This Lagrangian appeared first in the textbook [8] and will be referred hereafter to as *Wentzel Lagrangian*. Actually in Minkowski space the choice of a Lagrangian for $\psi_{\mu\nu}$ is not unique and a number of equivalent forms exist. For example one can replace the second term in (7) by a more symmetric term $\psi^{\mu\nu}{}_{;\nu}\psi_{\mu\alpha}{}^{;\alpha}$ and the resulting Lagrangian L_S differs from L_W by a full divergence. However in a curved spacetime the two Lagrangians become inequivalent as they differ by a curvature term, $L_S(\psi, g) = L_W + \text{div} + H$, where $H \equiv \psi^\nu{}_\mu\psi^{\mu\alpha}{}_{;[\nu\alpha]} = \frac{1}{2}\psi^{\alpha\beta}(\psi_\beta{}^\mu R_{\mu\alpha} + \psi^{\mu\nu} R_{\mu\alpha\beta\nu})$. We shall use L_W or its second order version (4).

For simplicity and possible physical relevance we shall investigate only the massless field; in the massive case the final conclusions are similar.

4. Massless field, gauge invariance and consistency

The action functionals (4) and (6) generate the variational energy-momentum tensor for the field (we shall deal with it below) and Lagrange equations (for $m = 0$)

$$E_{\mu\nu} = -G_{\mu\nu}^{AD}(\psi, g) = 0. \quad (8)$$

These form a degenerate system: only 6 out of 10 equations $E_{\mu\nu} = 0$ are hyperbolic propagation ones for $\psi_{\mu\nu}$, four equations $E_{0\mu}(g, \psi) = 0$ do not contain second time derivatives of the field and constitute primary constraints on the initial Cauchy data. There are no other primary constraints. To get a consistent dynamics one should replace the primary constraints by secondary constraints which allow to transform the primary ones in four missing propagation equations. In flat spacetime the procedure of deriving secondary constraints is different in the massive and massless case, while it turns out that in presence of curvature the situation is somewhat mixed. In Minkowski space the linear tensor $G_{\mu\nu}^L(\psi)$ coincides with $G_{\mu\nu}^{AD}(\psi)$ and it is well known from the theory of linear perturbations that the former satisfies the linearized version of Bianchi identity (in Cartesian coordinates)

$$\partial^\nu G_{\mu\nu}^L(\eta, \psi) \equiv 0. \quad (9)$$

Yet in a generic spacetime divergence of the field equations does not identically vanish,

$$\nabla^\nu E_{\mu\nu}(\psi, g) \equiv Q_\mu(\psi, g) = (-R_{\mu\alpha;\beta} + \frac{1}{2}R_{\alpha\beta;\mu} - R_{\mu\alpha}\nabla_\beta + \frac{1}{2}g_{\alpha\beta}R_\mu{}^\nu\nabla_\nu)\psi^{\alpha\beta} = 0, \quad (10)$$

to derive this expression one has applied the linearized Bianchi identity

$$\nabla^\nu G_{\mu\nu}^L(\psi, g) \equiv \nabla_\nu(\psi^{\nu\alpha}G_{\mu\alpha}) + \frac{1}{2}\psi^{\alpha\beta}{}_{;\mu}G_{\alpha\beta} - \frac{1}{2}\psi_{;\alpha}G^\alpha{}_\mu. \quad (11)$$

These are four secondary constraints for $\psi_{\mu\nu}$ in presence of gravitation. This outcome resembles the massive case in flat spacetime. However in flat spacetime the scalar equation arising as the trace of the field equations, $\eta^{\mu\nu} E_{\mu\nu}(\psi) = 0$, contains a term $m^2\psi$, which when combined with another scalar equation, $\nabla^\mu Q_\mu(\psi) = 0$, provides the necessary fifth constraint, $\psi = 0$. In the present case the trace of (8) is

$$g^{\mu\nu} E_{\mu\nu}(\psi, g) = -\psi_{;\mu}{}^{;\mu} + \psi^{\mu\nu}{}_{;\mu\nu}$$

and no linear combination of this equation with the other one, $\nabla^\mu Q_\mu(\psi, g) = 0$, can provide an additional constraint. There is no rigorous proof that the fifth constraint does not exist. According to general theory of constrained systems of partial differential equations [9] some constraints may appear only after applying derivatives of a very high order to the equations. However it seems unlikely that it may exist.

It is well known that in Minkowski space Wentzel Lagrangian, the second order action (4) and $G_{\mu\nu}^L(\eta, \psi)$ are gauge-invariant under the field transformations $\psi_{\mu\nu} \rightarrow \psi_{\mu\nu} + \delta\psi_{\mu\nu} = \psi_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$, where ξ_μ is an arbitrary vector field. This invariance allows one, as in the case of Maxwell field, to introduce five gauge conditions which work then as required secondary constraints. This fundamental feature of the field is lost in a generic spacetime. Under an infinitesimal transformation $\delta\psi_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$ the action (4) varies by

$$\delta_\xi S_{II}[\psi] = -2 \int d^4x \sqrt{-g} \xi^\mu Q_\mu \quad (12)$$

plus a surface term which vanishes if $\xi^\mu = 0$ on the boundary. Here $Q_\mu = \nabla^\nu E_{\mu\nu}(\psi, g) = -\nabla^\nu G_{\mu\nu}^{AD}(\psi, g)$. One sees that a condition for gauge invariance of the action and the secondary constraints are directly connected. On the other hand the gauge variation of the field equations is different,

$$\delta_\xi E_{\mu\nu} = -\delta_\xi G_{\mu\nu}^{AD} = -(R_{\mu\nu;\alpha} \xi^\alpha + R_{\mu\alpha} \xi^\alpha{}_{;\nu} + R_{\nu\alpha} \xi^\alpha{}_{;\mu} - g_{\mu\nu} R^{\alpha\beta} \xi_{\alpha;\beta} - \frac{1}{2} g_{\mu\nu} R_{;\alpha} \xi^\alpha). \quad (13)$$

This variation is independent of $\psi_{\mu\nu}$ (the field equations are linear) and for arbitrary ξ^μ is determined by Ricci tensor. Thus the massless theory is gauge invariant if the two independent conditions hold: $Q_\mu = 0$ and $\delta_\xi E_{\mu\nu} = 0$. The latter condition is satisfied only if the spacetime is empty, $R_{\mu\nu} = 0$, then also Q_μ vanishes.

If $R_{\mu\nu} = 0$, i.e. the field $\psi_{\mu\nu}$ does not backreact gravitationally and there is no other matter, the field is gauge invariant and the invariance may be

employed to simplify the field equations by imposing a gauge condition. As in flat spacetime one assumes the harmonic de Donder gauge

$$(\psi^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\psi)_{;\nu} = 0$$

and it reduces eqs. (8) to

$$-2G_{\mu\nu}^{AD} = \psi_{\mu\nu;\alpha}{}^{;\alpha} + 2R_{\mu\alpha\nu\beta}\psi^{\alpha\beta} + \frac{1}{2}g_{\mu\nu}\psi_{;\alpha}{}^{;\alpha} = 0. \quad (14)$$

The trace of these equations forms a propagation equation for the scalar ψ , $\square\psi \equiv \psi_{;\alpha}{}^{;\alpha} = 0$. A residual gauge freedom still remains since the gauge condition is preserved if ξ^μ is a solution a wave equation $\square\xi^\mu = 0$ (the spacetime is empty). Computing divergence of this equation and using several times Bianchi identity and $R_{\mu\nu} = 0$ give rise to a scalar equation $\square\xi^\mu{}_{;\mu} = 0$. The residual gauge transformation alters the trace of $\psi_{\mu\nu}$ to $\psi' = \psi + 2\xi^\mu{}_{;\mu}$. For ψ and $\xi^\mu{}_{;\mu}$ satisfy the same wave equation one can choose such a solution ξ^μ that $\psi + 2\xi^\mu{}_{;\mu} = 0$ and in this gauge $\psi_{\mu\nu}$ is traceless. Finally one has five gauge conditions or secondary constraints, $\psi^{\mu\nu}{}_{;\nu} = 0 = \psi$ ensuring the purely spin-2 nature of the field and the field equations take on the form

$$\square\psi^{\mu\nu} + 2R_{\mu\alpha\nu\beta}\psi^{\alpha\beta} = 0. \quad (15)$$

These form a nondegenerate system of 10 hyperbolic propagation equations. As a final step in constructing a consistent dynamics one proves the proposition [10]:

if the equations of motion (15) hold throughout an empty spacetime and the following constraints restrict the initial data at $t = 0$: $E_{0\mu} = 0$ and $\psi = \psi_{;0} = \psi^{\mu\nu}{}_{;\nu} = 0$, then all the constraints, $\psi^{\mu\nu}{}_{;\nu} = 0 = \psi$ and $E_{0\mu} = 0$ are preserved in time. Here one must assume an additional initial data constraint $\psi_{;0} = 0$ at $t = 0$ to ensure vanishing of ψ in whole spacetime.

This gauge invariant theory describes a massless spin-2 field and is dynamically consistent. However the field cannot self-gravitate nor any other self-gravitating matter may be present. From the viewpoint of general relativity such a field is unphysical. It is therefore of crucial importance whether the theory in a general spacetime, described by equations (4) to (8), is dynamically consistent.

One conjectures that the theory (4)–(8) provides a "unified" description of a mixture of a purely spin-two field and a scalar one and the lack of the fifth constraint means that the scalar cannot be removed from the unifying quantity $\psi^{\mu\nu}$. It might be so if the four equations $Q_\mu = 0$ were genuine

constraints on $\psi^{\mu\nu}$. However eqs. (10) are defective in the following sense. It is natural to view them as first order differential constraints on the initial data for $\psi_{\mu\nu}$ at $t = 0$. But then replacing $R_{\mu\nu}$ by the energy–momentum tensor $T_{\mu\nu}$ for $\psi^{\mu\nu}$ one sees that they take on the following form:

$$Q'_\mu \equiv -\nabla_\beta(\psi^{\alpha\beta}T_{\mu\alpha}) + \frac{1}{2}\nabla_\alpha(\psi^\alpha_\mu T) - \frac{1}{4}\nabla_\mu(T\psi) + \frac{1}{2}T_{\alpha\beta;\mu}\psi^{\alpha\beta} + \frac{1}{2}\psi_{;\alpha}T^\alpha_\mu = 0. \quad (16)$$

The expression for $T_{\mu\nu}$ is extremely complicated (see below) and, what is more important, it contains $\psi_{\mu\nu;\alpha\beta}$, among them there are second time derivatives. Hence (16) actually are four nonlinear third order propagation equations. As a consequence there are no secondary constraints imposed on $\psi_{\mu\nu}$ which are preserved in time and which decouple the unphysical modes ensuring the existence of the correct number of degrees of freedom.

The opposite possibility is to consider $\psi_{\mu\nu}$ as a test field on a fixed background determined by Einstein equations $G_{\mu\nu} = t_{\mu\nu}(\phi)$ with $t_{\mu\nu}$ being the stress tensor for some matter ϕ . Then $E_{0\mu} = 0$ and $Q_\mu = 0$ are constraints on initial values of $\psi_{\mu\nu}$. The necessary condition for having a consistent dynamics for $\psi_{\mu\nu}$ is that both primary and secondary constraints are preserved in time. This can be shown only for very special cases [10]. In general there is no consistent dynamical description of this mixture of spin-2 and spin-0 fields on a given curved spacetime. Furthermore this approach is flawed by the assumption that the field does not self-gravitate.

The third possibility is to regard $Q_\mu = 0$ as restrictions on the spacetime metric. From (10) one sees that they contain third time derivatives of the metric and thus restrict it in the whole spacetime. It seems (though there is no rigorous proof) that there is only one admissible solution, $R_{\mu\nu} = 0$ and then one comes back to the gauge invariant case.

The final conclusion is [5, 7]: *massive linear spin-2 field is consistent only if it is a test field in an empty spacetime; then in the limit of vanishing mass it coincides with small gravitational fluctuations*. The same argument applies to the massless field. Inclusion of any non-minimal coupling to gravity cannot help [5]. A linear spin-2 field cannot be a source of gravity and in this sense is unphysical.

5. Symmetries and the metric stress tensor

It was mentioned in the introduction that the gauge invariant linear spin-2 field, which may be interpreted as a linearized gravitational perturbation

around an $R_{\mu\nu} = 0$ solution, has a gauge dependent energy–momentum tensor. This is a serious drawback of the theory for the lesson we have learnt from general relativity is that the adequate description of energy and momentum of any kind of matter, except for the gravitational field itself, is in terms of the variational (with respect to the spacetime metric) energy–momentum tensor, hereafter denoted as the metric stress tensor. In the gauge theories of particle physics the metric stress tensors for the gauge fields are all gauge invariant. This may arouse a conviction that this is a generic feature of any gauge invariant theory. However this is not the case. In general the metric stress tensor does not inherit the gauge–independence property of the underlying Lagrangian in Minkowski space.

An answer why it is so follows from a "folk theorem", rigorously stated and proven by Deser and McCarthy in [11] to the effect that the Poincaré generators, being spatial integrals of the metric stress tensor, are gauge invariant and thus unique. The theorem shows that in quantum field theory, where only global quantities, such as total energy and momentum of a quantum system, are physical (measurable) ones, the inevitable gauge dependence of the metric stress tensor (for fields carrying spin larger than one) is quite harmless. It follows from the proof that the gauge dependence of this tensor is due to the fact that the gauge transformations involve the spacetime metric. It is also stated in that work that this gauge dependence is unavoidable, there is no linear spin–2 field in flat spacetime with a gauge invariant stress tensor.

In a classical gauge invariant field theory any gauge dependence of the metric stress tensor is truly harmful since this tensor cannot act as the source in Einstein field equations and this defect makes the theory unphysical. Even if such a field is viewed as a test one in a fixed spacetime, its theory remains defective since the local conserved currents (which exist if there are Killing vectors) do not determine physical flows of energy or momentum through a boundary of a spatially bounded region.

It turns out that the gauge fields are not exceptional in the gauge symmetry breaking by the metric stress tensor for fields with spins $s > 1$. Actually it is a generic effect: for a symmetry transformation of the Lagrangian the metric stress tensor inherits the symmetry property provided that either the field equations hold or the transformation is metric independent [12]. Thus the spacetime metric plays a key role for all symmetries in a field theory and not only for gauge invariance. Here we follow the generic approach to the problem given in [12].

Let ϕ be a dynamical field or a multiplet of fields described by a generally covariant action functional with a Lagrangian density $\sqrt{-g}L(\phi, g_{\mu\nu})$, residing

in a curved spacetime with a (dynamical or background) metric $g_{\mu\nu}$; for simplicity we assume that L does not depend on second and higher derivatives of ϕ . Let $\phi = \varphi(\phi', \xi, g)$ be any invertible transformation of the dynamical variable, which in general involves the metric tensor and a non-dynamical vector or tensor field ξ and its first covariant derivative $\nabla\xi$. The transformation is arbitrary with the exception that we exclude the tensor (or spinor) transformations of the field under a mere coordinate transformation. As a consequence, as opposed to many authors we do not view the transformation group of the dynamical variables induced by spacetime diffeomorphisms as a gauge group. The transformation need not be infinitesimal. Under the change of the dynamical field one sets

$$L(\phi, g) = L(\varphi(\phi', \xi, g), g) \equiv L'(\phi', \xi, g). \quad (17)$$

It is convenient to define the variational stress tensor (signature is $-+++$) by the following expression, which is equivalent to the standard definition,

$$\delta_g(\sqrt{-g}L) \equiv -\frac{1}{2}\sqrt{-g}[T_{\mu\nu}(\phi, g)\delta g^{\mu\nu} + \text{div}], \quad (18)$$

where div means a full divergence which is usually dropped; we will mark its presence from time to time to display an exact equality. In evaluating the variation in eq. (18) one assumes that ϕ is a fundamental field, i.e. is *not* affected by metric variations, $\delta_g\phi = 0$. This is the case of the vector potential (one-form) A_μ in electrodynamics while A^μ is already metric dependent with $\delta_g A^\mu = A_\nu\delta g^{\mu\nu}$. Hence, in evaluating $\delta_g L$ one takes into account only the explicit dependence of L on $g_{\mu\nu}$ and $g_{\mu\nu,\alpha}$ (or covariantly, on $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^\alpha$).

In terms of the new field ϕ' and of the transformed Lagrangian L' , the stress tensor of the theory is re-expressed as follows:

$$\delta_g(\sqrt{-g}L'(\phi', \xi, g)) \equiv -\frac{1}{2}\sqrt{-g}[T'_{\mu\nu}(\phi', \xi, g)\delta g^{\mu\nu} + \text{div}]. \quad (19)$$

To evaluate $T'_{\mu\nu}(\phi', \xi, g)$ one assumes that the appropriate (covariant or contravariant) components of the field ξ are metric independent, i.e. $\delta_g\xi = 0$, while metric variations of the new dynamical field ϕ' are determined by the inverse transformation $\phi' = \varphi^{-1}(\phi, \xi, g)$, i.e.

$$\delta_\varphi\phi' = \frac{\partial\varphi^{-1}}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial\varphi^{-1}}{\partial g^{\mu\nu},\alpha}\delta g^{\mu\nu},\alpha. \quad (20)$$

We denote this variation by $\delta_\varphi\phi'$ to emphasize that ϕ' and $g_{\mu\nu}$ are *not* independent fields: the value $\phi'(p)$ at any point p depends both on $\phi(p)$ and

$g_{\mu\nu}(p)$. Any scalar or tensor function $f(\phi', \nabla\phi', g)$ depends on the metric both explicitly (including the connection Γ) and implicitly via ϕ' , therefore its metric variation is determined by the *substantial* (or *total*) variation $\bar{\delta}_g$, $\bar{\delta}_g f \equiv \delta_g f + \delta_\varphi f$. Here $\delta_g f$ is the variation taking into account only the explicit metric dependence of f , i.e.

$$\delta_g f \equiv \frac{\partial f}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial f}{\partial \nabla\phi'} \delta_g \nabla\phi'. \quad (21)$$

On the other hand the variation δ_φ takes into account the metric dependence of f via $\phi' = \varphi^{-1}(\phi, \xi, g)$, then

$$\delta_\varphi f \equiv \frac{\partial f}{\partial \phi'} \delta_\varphi \phi' + \frac{\partial f}{\partial \nabla\phi'} \delta_\varphi \nabla\phi' \quad (22)$$

with $\delta_\varphi \phi'$ given by (20), thus δ_φ commutes with the covariant derivative ∇ . For a function $f(\phi, \nabla\phi, g)$ the operators $\bar{\delta}_g$ and δ_g coincide, i.e.

$$\bar{\delta}_g f(\phi, \nabla\phi, g) = \delta_g f(\phi, \nabla\phi, g) \frac{\partial f}{\partial g^{\mu\nu}} \delta g^{\mu\nu} + \frac{\partial f}{\partial \nabla\phi} \delta_g \nabla\phi. \quad (23)$$

Accordingly, δ_g on the l.h.s. of eq. (19) should be replaced by $\bar{\delta}_g$.

The identity (17), which is valid for *all* ϕ , $g_{\mu\nu}$ and transformations φ , implies $\delta_g L(\phi, g) = \bar{\delta}_g L'(\phi', \xi, g)$, what in turn implies the crucial equality

$$T_{\mu\nu}(\phi, g) = T'_{\mu\nu}(\phi', \xi, g); \quad (24)$$

in general the two tensors depend differently on their arguments. It should be stressed that in this way one has not constructed the stress tensor for a different theory (also described by the Lagrangian L') in which ϕ' would represent a new, metric-independent field variable. Here, one is dealing with field redefinitions within the same theory, not with transformations relating theories which are dynamically equivalent but different in their physical interpretation.

We are interested in finding out a generic relationship between $T_{\mu\nu}(\phi, g)$ and $T_{\mu\nu}(\phi', g)$. To this end we explicitly evaluate $T'_{\mu\nu}(\phi', \xi, g)$ from the definition (19) and then apply the equality (24). It is convenient to write the transformed Lagrangian as a sum

$$L'(\phi', \xi, g) \equiv L(\phi', g) + \Delta L'(\phi', \xi, g), \quad (25)$$

this is a definition of $\Delta L'(\phi', \xi, g)$. This splitting allows one to obtain $T_{\mu\nu}(\phi', g)$ upon applying $\bar{\delta}_g$ to this formula. It is furthermore convenient to make the inverse transformation in the term $\Delta L'$, then

$$\Delta L'(\phi', \xi, g) = \Delta L'(\varphi^{-1}(\phi, \xi, g), \xi, g) \equiv \Delta L(\phi, \xi, g), \quad (26)$$

and this is a definition of $\Delta L(\phi, \xi, g)$. Then eq. (25) takes the form

$$L'(\phi', \xi, g) = L(\phi', g) + \Delta L(\phi, \xi, g). \quad (27)$$

The Euler operator of Lagrange equations for ϕ arising from $L(\phi, g)$ is as usual

$$E(\phi) \equiv \frac{\delta L}{\delta \phi} = \frac{\partial L}{\partial \phi} - \nabla \left(\frac{\partial L}{\partial \nabla \phi} \right); \quad (28)$$

the tensor $E(\phi)$ has the same rank and symmetry as the field ϕ . We can now evaluate $T'_{\mu\nu}(\phi', \xi, g)$: employing the definition of $\bar{\delta}_g$ and equations (23), (24) and (28), after some manipulations and dropping a full divergence one arrives at the fundamental relationship

$$\begin{aligned} \delta g^{\mu\nu} [T_{\mu\nu}(\phi', g) - T_{\mu\nu}(\phi, g) + g_{\mu\nu} \Delta L(\phi, \xi, g)] \\ - 2E(\phi') \delta_\varphi \phi' - 2\delta_g \Delta L(\phi, \xi, g) = 0. \end{aligned} \quad (29)$$

This is an identity (up to a total divergence) valid for any field, Lagrangian and any field transformation. We remark that if all full divergence terms were kept in the derivation of the identity, a divergence term would replace zero on the r.h.s. of eq. (29). However a total divergence cannot cancel the last three terms on the l.h.s. of the identity since in general these terms do not sum up into a divergence. Therefore the difference $T_{\mu\nu}(\phi', g) - T_{\mu\nu}(\phi, g)$ does not vanish in general.

The transformation $\phi \mapsto \phi'$ is a *symmetry transformation* of the theory (of the Lagrangian) iff $L'(\phi', \xi, g) = L(\phi', g) + \text{div}$, i.e. if $\Delta L(\phi, \xi, g) = \text{div}$ or is zero. Equivalently, symmetry implies that $L(\phi, g) = L(\phi', g) + \text{div}$. These equalities should hold identically for a symmetry independently of whether the field equations are satisfied or not. According to the proposition “the metric variation of a divergence is another divergence”, adding a covariant divergence to $L(\phi, g)$ does not affect the variational stress tensor; in a similar way one shows that the variation with respect to the dynamical field ϕ of a full divergence gives rise to another divergence, thus the Lagrange field equations remain unaffected too. It is worth stressing that we impose no restrictions on the transformation φ and on possible symmetries — they should only smoothly depend on the components of a vector or tensor field ξ ; discrete transformations, like reflections, are excluded. The identity (29) has deeper consequences usually when the transformation φ depends on the spacetime metric (possibly through covariant derivatives of the field ξ). For any symmetry (29) reduces to

$$\delta g^{\mu\nu} [T_{\mu\nu}(\phi', g) - T_{\mu\nu}(\phi, g)] - 2E(\phi') \delta_\varphi \phi' = 0 \quad (30)$$

since for $\Delta L = \text{div}$ the two terms containing ΔL can be combined into a divergence and it can be discarded. As a trivial example, consider Maxwell electrodynamics. Here $\phi = A_\mu$, $\phi' = A_\mu + \partial_\mu f$ with arbitrary f ; since the gauge transformation is metric independent, $\delta_\varphi \phi' = 0$. Then the term $E(\phi')\delta_\varphi \phi' = \nabla_\nu F^{\mu\nu}\delta_\varphi \phi'$ vanishes giving rise to the gauge invariance of $T_{\mu\nu}$ independently of Maxwell equations.

Since the term $E(\phi')\delta_\varphi \phi'$ is different from zero for fields not being solutions and for metric-dependent symmetry transformations, one arrives at the conclusion: *the metric energy-momentum tensor for a theory having a symmetry does not possess this symmetry*. It is only for solutions, $E(\phi') = E(\phi) = 0$, that the energy-momentum tensor does possess the same symmetry, $T_{\mu\nu}(\phi', g) = T_{\mu\nu}(\phi, g)$. In physics one is mainly interested in quantities built up of solutions of equations of motion, but from the mathematical viewpoint it is worth noticing that the symmetry property is *not* carried over from S to $\delta_g S$.

Now we can return to the gauge invariant spin-2 field and its gauge dependent stress tensor. In gauge theories of particle physics the field potentials are exterior forms since the fields carry spin one. Then the gauge transformations are independent of the spacetime metric and the identity (30) implies gauge invariance of the stress tensor for arbitrary fields, not only for solutions. Yet it is characteristic for gauge theories that for integer spins larger than one a gauge transformation necessarily involves covariant derivatives of vector or tensor fields [13], giving rise to gauge dependent stress tensors.

We know that Wentzel Lagrangian for spin-2 field is gauge invariant in empty spacetimes ³, i.e. $L_W(\psi, g) = L_W(\psi', g) + \nabla_\alpha A^\alpha(\psi, \xi, g)$ under $\psi_{\mu\nu} \mapsto \psi'_{\mu\nu} \equiv \psi_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}$ for some vector A^α . In a generic curved spacetime the "gauge" transformation is no more a symmetry since

$$L_W(\psi, g) = L_W(\psi', g) + \text{div} + \Delta L_W(\psi, \xi, g) \quad (31)$$

with

$$\Delta L_W = -\delta_\xi L_W = 2\xi^\mu Q_\mu. \quad (32)$$

This expression either follows immediately from eq. (12) (the second order Lagrangian in (4) and L_W are equivalent up to a divergence in any spacetime) or may be directly derived from (7). For $\delta_g \psi_{\mu\nu} = 0 = \delta_g \xi_\mu$ one gets for this gauge transformation

$$\delta_\varphi \psi'_{\mu\nu} = -2\xi_\alpha \delta \Gamma_{\mu\nu}^\alpha. \quad (33)$$

³In our previous work [12] there is an erroneous statement about this property in sect. 3 and particularly eq. (42) in that work is misleading.

In this case the fourth term in the identity (29) reads

$$-2E^{\mu\nu}(\psi')\delta_\varphi\psi'_{\mu\nu} = 4E^{\mu\nu}(\psi')\xi_\alpha\delta\Gamma_{\mu\nu}^\alpha$$

and discarding the full divergence arising from $\delta g^{\mu\nu}{}_{;\alpha}$ one arrives at the following explicit form of (29) for the linear massless spin-2 field and the transformation $\psi'_{\mu\nu} = \psi_{\mu\nu} + 2\xi_{(\mu;\nu)}$,

$$\begin{aligned} & \delta g^{\mu\nu} \{ T_{\mu\nu}(\psi', g) - T_{\mu\nu}(\psi, g) + g_{\mu\nu} \Delta L(\psi, \xi, g) \\ & + 2\nabla_\alpha [2\xi_{(\mu} E_{\nu)}^\alpha(\psi', g) - \xi^\alpha E_{\mu\nu}(\psi', g)] \} \\ & - 2\delta_g \Delta L(\psi, \xi, g) = 0. \end{aligned} \quad (34)$$

One is interested in evaluating this identity for $R_{\mu\nu}(g) = 0$, what will be symbolically denoted by $g = r$. One has $\Delta L_W|_{g=r} = \text{div}$ while $\delta_g \Delta L_W|_{g=r} \neq 0$. In fact, from (10) one can write $\Delta L_W = 2\xi_\mu P^{\mu\alpha\beta} \psi_{\alpha\beta}$, where $P^{\mu\alpha\beta}$ is made up of Ricci tensor and covariant derivative operators. Then $\delta_g \Delta L_W|_{g=r} = 2\xi_\mu \delta_g (P^{\mu\alpha\beta} \psi_{\alpha\beta})$ and this variation does not vanish in empty spacetimes and even in Minkowski space is different from zero.

Let us denote the expression in square brackets in (34) by $F_{\mu\nu}^\alpha(\psi', \xi, g)$. For $R_{\mu\nu} = 0$ the gauge invariance implies for any $\psi_{\mu\nu}$ that $E_{\mu\nu}(\psi', r) = E_{\mu\nu}(\psi, r)$ and then $\nabla_\alpha F_{\mu\nu}^\alpha(\psi', \xi, g)|_{g=r} = \nabla_\alpha F_{\mu\nu}^\alpha(\psi, \xi, r)$. Assuming that $\psi_{\mu\nu}$ is a solution in an empty spacetime, $E_{\mu\nu}(\psi, r) = 0$, one gets that $\nabla_\alpha F_{\mu\nu}^\alpha(\psi, \xi, r) = 0$. Thus, the identity (34) reduces for solutions and for $R_{\mu\nu} = 0$, to

$$\delta g^{\mu\nu} \{ T_{\mu\nu}(\psi', r) - T_{\mu\nu}(\psi, r) \} - 2 \delta_g \Delta L_W(\psi, \xi, g)|_{g=r} = 0. \quad (35)$$

This relationship (not an identity) shows that the stress tensor is *not* gauge invariant even in flat spacetime. In other terms, the symmetry properties of the Lagrangian in Minkowski (or empty) space are insufficient for determining symmetry properties of the metric stress tensor in this spacetime. In this sense classical field theory in flat spacetime is incomplete and a complete and logically closed theory should be formulated in a generic spacetime.

Finally we give for completeness the explicit covariant form in Minkowski space of the gauge dependent metric stress tensor generated by Wentzel Lagrangian,

$$\begin{aligned} T_{\mu\nu}^W(\psi, \eta) &= -2\psi_{\alpha\beta;(\mu} \psi_{\nu)}^{\alpha\beta} + \frac{1}{2}\psi_{\alpha\beta;\mu} \psi^{\alpha\beta}{}_{;\nu} + 2\psi_{(\mu}{}^{\alpha\beta} \psi_{\nu)(\alpha;\beta)} \\ &+ \frac{1}{4}g_{\mu\nu} \left(-\psi^{\alpha\beta;\sigma} \psi_{\alpha\beta;\sigma} + 2\psi^{\alpha\beta;\sigma} \psi_{\sigma\alpha;\beta} \right). \end{aligned} \quad (36)$$

In deriving it one assumes that the covariant derivatives commute.

6. Gauge symmetry and gravitational energy

The fact that the stress tensor $T_{\mu\nu}^W$ depends on the gauge was known long ago [5]. More interesting is the problem whether there exists a Lagrangian $L_K(\psi, g)$, which is equivalent to L_W at least in Minkowski space, but which generates a different, gauge-invariant stress tensor $T_{\mu\nu}^K$ in this spacetime. The no-go theorem stating that such gauge invariant stress tensor does not exist was given in [11]. The authors of that work did not publish a detailed proof and only referred to the underlying "folk" wisdom. According to S. Deser, for all gauge fields (with metric dependent gauge transformations) in flat spacetime the manifest covariance of energy-momentum density objects is incompatible with their gauge invariance, i.e. these objects are either covariant or gauge invariant but not both [14]. In fact, one can always remove (in a non-covariant way) the non-physical components of the fields, so that the result will have no residual gauge dependence; notice however that this is not the same as producing a gauge-independent definition in the usual sense. An alternative direct proof of the no-go theorem based on the relationship (35) was then given in [12].

The physical relevance of the linear massless spin-two field $\psi_{\mu\nu}$ in flat spacetime stems from the fact that it is dynamically equivalent to linearized General Relativity and it is an unquestionable requirement that all viable theories of gravity should dynamically coincide in the weak-field approximation with the linearized GR, i.e. gravitation should be described in this limit by the field. Hence this field is closely related to the problem of gravitational energy density: it is applied in the field theory approach to gravitation, according to which gravity is just a tensor field existing in Minkowski space, which may be (though not necessarily) the spacetime of the physical world. In these theories of gravity the metric energy-momentum tensor again serves as the most appropriate local description of energy for the field [15]. The best and most recent version of field theory of gravitation given in [15] satisfies this and other requirements imposed on any gravity theory. However, while the linearized Lagrange equations of their theory are gauge invariant (as being equivalent to those for $\psi_{\mu\nu}$), their metric stress tensor in this approximation shares the defect of all the metric stress tensors for $\psi_{\mu\nu}$, i.e. breaks the gauge symmetry. From the obvious condition that the stress tensor should have the same gauge invariance as the field equations, it follows that also this approach to gravity does not furnish a physically acceptable notion of gravitational energy density.

Here one touches a subtle problem of what is actually measurable in gravitational physics. If one views a field theory approach to gravity as a different theory of gravity then one may claim that the "gauge" transformation

actually maps one solution of field equations to another *physically distinct* solution. Then the energy density need not be gauge invariant and measurements of energy may be used to discriminate between two physically different solutions related by the “gauge” transformation (which should then be rather called a “symmetry transformation”). The gravitational field of [15] or the spin-2 field with Wentzel Lagrangian would then be measurable quantities rather than gauge potentials. If the transformation of these fields is not an internal gauge but corresponds to a change of physical state, this raises a difficult problem of finding out a physical interpretation of it. Clearly it is not a transformation between reference frames.

Here we adopt the opposite view that the field theory approach to gravity is merely an auxiliary procedure for constructing notions which are hard to define in the framework of GR. It is commonly accepted that in the weak field limit of GR the spacetime metric is measurable only in a very restricted sense: if two almost Cartesian coordinate systems are related by an infinitesimal translation $x'^{\mu} = x^{\mu} + \xi^{\mu}$, then no experiment can tell the difference of their metrics while the curvature tensor has the same components in both systems. This implies that all different coordinate systems connected by this transformation actually represent the same physical reference frame and from the physical viewpoint the transformation is an internal gauge symmetry [16]. Thus, showing a mathematical equivalence of the corresponding field equations is insufficient to achieve compatibility of a given approach to gravity with the linearized GR. The weak field gravity should be described by a gauge potential. In consequence, any gravitational energy density should be a gauge invariant quantity.

We conclude that the above no-go theorem closes one line of research of gravitational energy density. This makes the quest of this notion harder than previously.

7. Nonlinear massive spin-two field generated by NLG theories

As mentioned in Introduction, a radically different approach to the notion of consistently gravitationally interacting spin-2 field is provided by metric nonlinear gravity theories. Unfortunately in these lectures we have no time and space to do justice to this theory, we can only signal the basic concepts and results. For an almost comprehensive exposition of the subject we refer the reader to our paper [17] and references therein.

Dynamical evolution of a Lorentzian manifold $(M, \psi_{\mu\nu})$ is determined in

the framework of a generic NLG theory by the Lagrangian density

$$L = \sqrt{-\tilde{g}} f(\tilde{g}_{\mu\nu}, \tilde{R}_{\alpha\beta\mu\nu}(\tilde{g}_{\mu\nu}))$$

where f is any smooth scalar function. This evolution and particle content of the theory is studied using Legendre transformation method [1, 2, 18]. One need not view $\tilde{g}_{\mu\nu}$ as a physical spacetime metric, actually whether $\tilde{g}_{\mu\nu}$ or its "canonically conjugate" momentum is the measurable quantity determining all spacetime distances in physical world should be determined only after a careful examination of the physical content of the theory, rather than prescribed *a priori*. Formally $\tilde{g}_{\mu\nu}$ plays both the role of a metric tensor on M and is a kind of unifying field which will be decomposed in a multiplet of fields with definite spins; pure gravity is described in terms of the fields with the metric being a component of the multiplet. Except for Hilbert–Einstein and Euler–Poincaré topological invariant densities the resulting variational Lagrange equations are of fourth order. The Legendre transformation technique allows one to deal with fully generic Lagrangians; from the physical standpoint, however, there is no need to investigate a generic f . Firstly, in the bosonic sector of low energy field theory limit of string effective action one gets in the lowest approximation the Hilbert–Einstein Lagrangian plus terms quadratic in the curvature tensor. Secondly, to obtain an explicit form of field equations and to deal with them effectively one needs to invert the appropriate Legendre transformation and in a generic case this amounts to solving nonlinear matrix equations. Hindawi, Ovrut and Waldram [19] have given arguments that a generic NLG theory has eight degrees of freedom and the same particle spectrum as in the quadratic Lagrangian below, the only known physical difference lies in the fact that in the generic case one expects multiple nontrivial (i.e. different from flat spacetime) ground state solutions. This result can be also derived from the observation that after the Legendre transformation the kinetic terms in the resulting (Helmholtz) Lagrangian are universal, and only the potential terms keep the trace of the original nonlinear Lagrangian. If the latter is a polynomial of order higher than two in the curvature tensor, the Legendre map is only locally invertible and this leads to multivalued potentials, generating a ground state solution in each "branch"; yet the form of the potential could produce additional dynamical constraints, affecting the number of degrees of freedom, only in non-generic cases. The physically relevant Lagrangians in field theory depend quadratically on generalized velocities and then conjugate momenta are linear functions of the velocities. For both conceptual and practical purposes it is then sufficient to envisage a quadratic Lagrangian

$$L = \tilde{R} + a\tilde{R}^2 + b\tilde{R}_{\mu\nu}(\tilde{g})\tilde{R}^{\mu\nu}(\tilde{g}). \quad (37)$$

In four dimensions the term $\tilde{R}_{\alpha\beta\mu\nu}\tilde{R}^{\alpha\beta\mu\nu}$ can be eliminated via Gauss–Bonnet theorem). The Lagrangian cannot be purely quadratic: it is known from the case of restricted NLG theories (Lagrangian depends solely on the curvature scalar, $L = f(\tilde{R})$) that the linear term \tilde{R} is essential [3] and we will see that the same holds for Lagrangians explicitly depending on Ricci tensor $\tilde{R}_{\mu\nu}$. The coefficients a and b have dimension $[\text{length}]^2$; contrary to some claims in the literature there are no grounds to presume that they are of order $(\text{Planck length})^2$ unless the Lagrangian (37) arises from a more fundamental theory (e.g. string theory) where \hbar is explicitly present. Otherwise in a pure gravity theory the only fundamental constants are c and G ; then a and b need not be new fundamental constants, they are rather related to masses of the gravitational multiplet fields. We assume that the NLG theory with the Lagrangian (37) is an independent one, i.e. it inherits no features or relationships from a possible more fundamental theory.

Such a theory gives rise to a massive nonlinear spin-2 field (and a massive scalar field) in two ways. Firstly, one assumes that $\tilde{g}_{\mu\nu}$ is the spacetime metric and then one can both lower the order of the equations of motion and generate additional fields describing gravity in a way analogous to replacing Lagrange formalism by canonical one in classical mechanics. One introduces ”canonical momenta” conjugate to Christoffel connection for $\tilde{g}_{\mu\nu}$ using Legendre transformations with respect to the irreducible parts of Ricci tensor [18]:

$$\sqrt{-\tilde{g}}\pi^{\mu\nu} = \frac{\partial L}{\partial S_{\mu\nu}} \quad \text{and} \quad \sqrt{-\tilde{g}}\phi = \frac{\partial L}{\partial \tilde{R}} \quad (38)$$

where $S_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{4}\tilde{R}\tilde{g}_{\mu\nu}$. The fields $\pi^{\mu\nu}$ and ϕ turn out to be massive and carry spin two and zero respectively. The original Lagrangian L is then replaced by a Helmholtz Lagrangian L_H generating second order field equations for the triplet of the fields [18]. It is remarkable that for $\tilde{g}_{\mu\nu}$ one gets exactly Einstein’s field equations $\tilde{G}_{\mu\nu}(\tilde{g}) = T_{\mu\nu}(\tilde{g}, \pi, \phi)$ with a stress tensor for the nongeometric part of the triplet which is however indefinite [10]. In a weak-field limit one recovers the well known Stelle’s results for a quadratic L [20].

The second approach is more sophisticated. One assumes that $\tilde{g}_{\mu\nu}$ appearing in L is a kind of a unifying field and does *not* coincide with the physical spacetime metric and $\tilde{g}_{\mu\nu}$ is not a geometric quantity. The genuine measurable metric field should be recovered from L via a Legendre transformation [1, 2]

$$g^{\mu\nu} \equiv \left| \det\left(\frac{\partial L}{\partial \tilde{R}_{\alpha\beta}}\right) \right|^{-1/2} \frac{\partial L}{\partial \tilde{R}_{\mu\nu}}. \quad (39)$$

If this transformation can be inverted one expresses the canonical "velocity" $\tilde{R}_{\mu\nu}$ in terms of the "positions and momenta", $\tilde{R}_{\mu\nu}(\tilde{g}) = r_{\mu\nu}(g^{\alpha\beta}, \tilde{g}_{\alpha\beta})$. To view $g^{\mu\nu}$ as a spacetime metric one assumes that it is nonsingular, i.e. $\det(\partial f / \partial \tilde{R}_{\mu\nu}) \neq 0$ and $g_{\mu\nu}$ is its inverse. In other terms one maps $(M, \tilde{g}_{\mu\nu})$ onto $(M, g_{\mu\nu})$ and from now on one treats $\tilde{g}_{\mu\nu}$ as some matter tensor field on the spacetime $(M, g_{\mu\nu})$. From now on all tensor indices will be raised and lowered with the aid of this metric. At this point, to make the following equations more readable, we alter our notation and denote the original tensor field $\tilde{g}_{\mu\nu}$ by $\psi_{\mu\nu}$ and its inverse $\tilde{g}^{\mu\nu}$ by $\gamma^{\mu\nu}$. Then

$$\psi^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} \psi_{\alpha\beta} \quad \text{and} \quad \gamma^{\mu\alpha} \psi_{\alpha\nu} = \delta_{\nu}^{\mu} = \psi^{\mu\alpha} \gamma_{\alpha\nu}.$$

The fields $g_{\mu\nu}$ and $\psi_{\mu\nu}$ will be referred to as Einstein frame, $\text{EF} = \{g_{\mu\nu}, \psi_{\mu\nu}\}$. For the generic Lagrangian $\psi_{\mu\nu}$ is actually a mixture of fields carrying spin two and zero. Notice that for f as in (37),

$$g^{\mu\nu} = \left| \frac{\psi}{g} \right|^{1/2} [(1 + 2a\tilde{R})\gamma^{\mu\nu} + 2b\tilde{R}^{\mu\nu}], \quad (40)$$

hence for $\psi_{\mu\nu}$ being Lorentzian and close to Minkowski metric, $g_{\mu\nu}$ is also Lorentzian and close to flat metric (and thus invertible). This shows the importance of the linear term in (37). Here one meets the subtle problem which frame is physical, the original one consisting solely of the unifying field $\psi_{\mu\nu}$ or EF. It was argued in [3] that energy density is very sensitive to field redefinitions and thus is a good indicator of which variables are physical. For a restricted NLG theory, $L = f(\tilde{R})$, Einstein frame is the physical one [3].

As in classical mechanics one replaces the Lagrangian by the Hamiltonian,

$$H(g, \psi) \equiv \sqrt{-g} g^{\mu\nu} r_{\mu\nu}(g, \psi) - (-\det \psi_{\alpha\beta})^{\frac{1}{2}} f(\psi_{\mu\nu}, r_{\mu\nu}) \quad (41)$$

and then the latter by a Helmholtz Lagrangian

$$L_H(g, \psi) \equiv \sqrt{-g} g^{\mu\nu} \tilde{R}_{\mu\nu}(\psi) - H(g, \psi). \quad (42)$$

The Helmholtz action $S_H = \int d^4x L_H$ generates Hamilton equations for $g_{\mu\nu}$ and $\psi_{\mu\nu}$ as variational Lagrange equations and these are of *second order*. Introducing a tensor being the difference of the Christoffel connections for the two tensors, $Q_{\mu\nu}^{\alpha} \equiv \tilde{\Gamma}_{\mu\nu}^{\alpha}(\psi) - \Gamma_{\mu\nu}^{\alpha}(g)$, and applying the following identity valid for *any* two nonsingular tensor fields [1],

$$K_{\mu\nu} \equiv \tilde{R}_{\mu\nu}(\psi) - R_{\mu\nu}(g) \equiv \nabla_{\alpha} Q_{\mu\nu}^{\alpha} - \nabla_{(\mu} Q_{\nu)\alpha}^{\alpha} + Q_{\mu\nu}^{\alpha} Q_{\alpha\beta}^{\beta} - Q_{\mu\beta}^{\alpha} Q_{\nu\alpha}^{\beta}, \quad (43)$$

(all covariant derivatives are with respect to $g_{\mu\nu}$) one can finally express L_H in the form (up to a divergence term)

$$L_H = \sqrt{-g}[R(g) + K(g, \psi) - 2V(g, \psi)] \equiv \sqrt{-g}[R(g) + g^{\mu\nu}(Q_{\mu\nu}^\alpha Q_{\alpha\beta}^\beta - Q_{\mu\beta}^\alpha Q_{\nu\alpha}^\beta)] - \sqrt{-g}[g^{\mu\nu}r_{\mu\nu}(g, \psi) - (\frac{\det \psi}{g})^{\frac{1}{2}}f(\psi_{\mu\nu}, r_{\mu\nu})]. \quad (44)$$

Here $K \equiv g^{\mu\nu}K_{\mu\nu}$ being the quadratic polynomial in $Q_{\mu\nu}^\alpha$ (one sees from (43) that K contains a full divergence term which may be discarded) is a kinetic Lagrangian for $\psi_{\mu\nu}$ and is *universal* (is independent of the form of f) while the potential V is determined by the original Lagrangian. It is a straightforward calculation to show that the theory based on L_H is dynamically equivalent to that based on $L = \sqrt{-\psi}f$ [1, 2]. It is far from being obvious that it is possible to define the genuine metric $g_{\mu\nu}$ in such a way that the gravitational part of the Helmholtz Lagrangian is exactly equal to the curvature scalar. In this sense *Einstein general relativity is a universal Hamiltonian image (under a Legendre transformation) of any NLG theory.*

The field equations $\delta S_H/\delta g^{\mu\nu} = 0$ are just Einstein ones,

$$G_{\mu\nu}(g) = T_{\mu\nu}(g, \psi) \equiv Q_{\alpha(\mu;\nu)}^\alpha - Q_{\mu\nu;\alpha}^\alpha - Q_{\mu\nu}^\alpha Q_{\alpha\beta}^\beta + Q_{\mu\beta}^\alpha Q_{\alpha\nu}^\beta + r_{\mu\nu} + \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}(Q_{\alpha\beta;\lambda}^\lambda - Q_{\lambda\alpha;\beta}^\lambda + Q_{\alpha\beta}^\sigma Q_{\sigma\lambda}^\lambda - Q_{\alpha\lambda}^\sigma Q_{\sigma\beta}^\lambda - r_{\alpha\beta}). \quad (45)$$

This $T_{\mu\nu}$ is the metric stress tensor following from (44); the quickest way of deriving it is to apply the identity (43). In general the stress tensor is indefinite and the energy density is not determined by initial data since it depends on $\psi_{\mu\nu;\alpha\beta}$. The kinetic part of it (made up of $Q_{\mu\nu}^\alpha$) is universal while the potential part is determined by $r_{\mu\nu}(g, \psi)$ and does not depend explicitly on f . This stress tensor is rather complicated, nevertheless it is considerably simpler than that for the linear inconsistent field previously discussed.

Lagrange equations $\delta S_H/\delta \psi_{\mu\nu} = 0$ are too complicated to be presented here [17]. These are quasi-linear second order equations whose "kinetic" part is universal (due to (44)). This universality shows that also from a mathematical viewpoint there is no need to study a generic NLG theory—to find out the physical content of all these theories it is sufficient to investigate the simplest case: the original L being a quadratic function of the curvature. The particle content is the same for all cases: in a weak-field limit of (44) we find that $g_{\mu\nu}$ describes the massless graviton (helicity 2) while $\psi_{\mu\nu}$ is a mixture of massive spin-2 and spin-0 fields. Our results are in agreement with Stelle [20, 21] who studied the quadratic Lagrangian (37).

We choose the Lagrangian $L = \sqrt{-\psi}(\hat{R} + a\tilde{R}^2 - 3a\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu})$ with $a = \text{const} > 0$ of dimension (length)². The fourth-order field equations then imply $\tilde{R} = 0$. This corresponds to absence of the scalar canonical momentum ϕ defined

in (38), $\phi = \text{const.}$ The same holds in EF for $\psi_{\mu\nu}$ which is subject to one algebraic and four differential constraints and these together ensure that the field describes purely spin-2 particles with five degrees of freedom. The field cannot be massless, $m_\psi^2 = \frac{1}{3a}$.

The theory of nonlinear spin-2 field is quite promising. Lagrange equations in Einstein frame look rather formidable at first sight, nevertheless it was possible to find out a nontrivial exact solution [17] in the spacetime of a plane-fronted gravitational wave with parallel rays (a *pp* wave). The field has a ghost-like nature (found long ago in the linear approximation), however recent arguments by Hawking and Hertog [22] undermine the common conviction that any viable theory of quantum gravity should be unitary and causal, i.e. should exclude negative energy solutions and ghosts. Their arguments, based on a scalar field model, are not directly related to the spin-two field theory studied here. Nevertheless they are in accord with conclusions of [17] that appearance of ghostlike features in the linear approximation to the spin-two field theory does not dismiss it.

8. Final remarks

The content of my lectures delivered at the School was to some extent different from the written version. It turned out that the problem of physical interpretation and viability of restricted nonlinear gravity theories (Lagrangian $L = f(R)$ without an explicit dependence on Ricci tensor) was still a matter of hot debate and it was why I decided to devote to it a part of my talks. As I was unable to present new arguments on the subject besides those already existing in the literature, I have not written down this part of my lectures. I advice the interested reader to consult my joint paper with Guido Magnano [3] and my conference talk [23].

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