

# The Chrono-geometrical Structure of Special and General Relativity: a Re-Visitation of Canonical Geometrodynamics.

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## Abstract

A modern re-visitation of the consequences of the lack of an intrinsic notion of instantaneous 3-space in relativistic theories leads to a reformulation of their kinematical basis emphasizing the role of non-inertial frames centered on an arbitrary accelerated observer. In special relativity the exigence of predictability implies the adoption of the 3+1 point of view, which leads to a well posed initial value problem for field equations in a framework where the change of the convention of synchronization of distant clocks is realized by means of a gauge transformation. This point of view is also at the heart of the canonical approach to metric and tetrad gravity in globally hyperbolic asymptotically flat space-times, where the use of Shanmugadhasan canonical transformations allows the separation of the physical degrees of freedom of the gravitational field (the tidal effects) from the arbitrary gauge variables. Since a global vision of the equivalence principle implies that only global non-inertial frames can exist in general relativity, the gauge variables are naturally interpreted as generalized relativistic inertial effects, which have to be fixed to get a deterministic evolution in a given non-inertial frame. As a consequence, in each Einstein's space-time in this class the whole chrono-geometrical structure, including also the clock synchronization convention, is dynamically determined and a new approach to the Hole Argument leads to the conclusion that "gravitational field" and "space-time" are two faces of the same entity. This view allows to get a classical scenario for the unification of the four interactions in a scheme suited to the description of the solar system or our galaxy with a deparametrization to special relativity and the subsequent possibility to take the non-relativistic limit.

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## I. INTRODUCTION

The main theoretical challenge in the next future will be to find a classical scenario allowing a unified description of gravity with the standard model of elementary particles (or its extensions), so to clarify the problems to be faced in trying to reconcile general relativity and quantum theory.

On one side particle physics is described by gauge theories in the inertial frames of Minkowski space-time. This implies the following conceptual problems:

A) The need of Einstein's convention for the synchronization of distant clocks (i.e. the definition of a notion of instantaneous 3-space, which is not intrinsically given in special relativity differently from the conformal structure identifying the allowed paths of light rays) to be made by the inertial observer describing the phenomena consistently with the relativity principle and the light postulates. However, all realistic observers are accelerated and there is no consensus on how to define relativistic non-inertial frames and on how non-inertial observers describe the dynamics. This fact, together with the need of a well-posed Cauchy problem for classical field equations like Maxwell's ones to have predictability, forces the adoption of the 3+1 point of view, of more general clock synchronization conventions and of a formulation in which the change of such conventions is realized by means of a gauge transformation not changing the physics but only its description (the appearances of phenomena).

B) The need of an implementation of the Poincare' group on either the configuration or the phase space of the relativistic system, with all the complications implied by the Lorentz signature of Minkowski space-time. As a consequence, relativistic kinematics is highly non trivial (think to the problem of the relativistic center of mass), so that relativistic particle mechanics, extended objects, fluids, relativistic statistical mechanics, classical field theory turn out to be extremely more complicated than their non-relativistic counterparts.

C) The gauge principle (the minimal coupling) together with manifest Lorentz covariance force us to introduce redundant non-physical gauge variables with the associated invariance under local Lie groups of gauge transformations acting on some internal space. As a consequence, all relativistic Lagrangians are singular, the action principles have to be studied with the second Noether theorem, the Hamiltonian formulation requires Dirac's theory of constraints and the measurable quantities are the gauge invariant Dirac observables (DO).

On the other hand, the relativistic description of gravity given by general relativity abandons the relativity principle and replaces it with the equivalence principle. Special relativity can be recovered only locally by a freely falling observer in a neighborhood where tidal effects are negligible. As a consequence, global inertial frames are not admitted and Einstein's geometrical view of the gravitational field implies the the 4-metric of space-time, the dynamical field mediating the gravitational interaction, also determines the line element (i.e. the chrono-geometrical structure) of the space-time. Therefore, the 4-metric teaches relativistic causality to all the other fields: now the conformal structure (the allowed paths of light rays) is point-dependent. The geometrical view is intimately connected with the principle of general covariance: the form-invariance of Einstein's equations in every 4-coordinate system (invariance under active diffeomorphisms) requires that all the fields have a tensorial character. But this implies the invariance of the action principle under passive diffeomorphisms (ordinary coordinate transformations). The shift from the Hilbert action to the ADM one,

needed for the Hamiltonian formulation, replaces the passive diffeomorphisms with local Noether transformations and again the second Noether theorem implies the need of Dirac's theory of constraints in phase space. The canonical formalism and the absence of global inertial frames require the 3+1 point of view as in special relativity: as a consequence the allowed space-times must be globally hyperbolic and the gauge equivalence of the allowed non-inertial frames is now implied by general covariance. The superiority of the manifestly covariant configuration approach (no choice of what is *time* in a relativistic framework) is illusory: to have predictability we are forced to face the Cauchy problem of Einstein's equations and only the Hamiltonian approach has the tools to do it. Moreover, the space-time must be asymptotically flat and without super-translations, so that the asymptotic ADM Poincare' generators are well defined: when the Newton constant is switched off they must tend to the Poincare' generators of special relativity so that ordinary particle physics can be recovered. In this class of (singularity-free) space-times there are asymptotic inertial observers to be identified with the fixed stars and there is a real temporal evolution (absence of the *frozen picture*) governed by the ADM energy. The ADM energy density is coordinate-dependent (the energy problem in general relativity), because it depends on the 8 gauge variables hidden in the 4-metric and its time-derivative. However, in the framework of Dirac's theory of constraints, the Shanmugadhasan canonical transformations allow to separate the two pairs of canonical variables describing the *tidal effects* from the gauge variables associated with the *relativistic generalized inertial effects* off-shell, i.e. *before* solving Hamilton equations. Only after a complete gauge fixing, namely after the choice of a non-inertial frame with its pattern of relativistic inertial forces, the Hamilton equations for the tidal degrees of freedom and matter (if present) become deterministic. Their solution with suitable Cauchy data identifies a non-inertial 4-coordinate system for the Einstein space-time, whose chrono-geometrical structure (including the clock synchronization convention) is dynamically determined.

These lectures are an introduction to a modern treatment of these topics starting from the Galilei space-time (Section II) and then discussing Minkowski space-time (Section III) and the previous class of Einstein space-times (Section IV).

In the Appendix there are some notions on Dirac's theory of constraints.

While in Ref.[1] there is a review of constraint theory (including the Shanmugadhasan canonical transformation and the notion of Dirac observable) and of its applications to gauge models in Minkowski space-time, in Refs. [2, 3] there is a detailed description of general relativity from this canonical point of view.

## II. THE CHRONO-GEOMETRICAL STRUCTURE OF NEWTON PHYSICS: GALILEI SPACE-TIME.

In Newton physics there are separated absolute notions of *time* and *space*, so that we can speak of absolute simultaneity and of instantaneous Euclidean 3-spaces with the associated Euclidean spatial distance notion. This non-dynamical chrono-geometrical structure is formalized in the so called Galilei space-time. The Galilei relativity principle assumes the existence of preferred inertial frames with inertial Cartesian coordinates, where free bodies move along straight lines (Newton's first law) and Newton's equations take the simplest form. In Galilei space-time inertial frames centered on inertial observers are connected by

the kinematical group of Galilei transformations. In Newton gravity the equivalence principle states the equality of inertial and gravitational mass. In non-inertial frames inertial (or fictitious) forces proportional to the mass of the body appear in Newton's equations.

For isolated systems the 10 generators of the Galilei group are Noether constants of motion. The Abelian nature of the Noether constants (the 3-momentum) associated to the invariance under translations allow to make a global separation of the center of mass from the relative variables (usually the Jacobi coordinates, identified by the centers of mass of subsystems, are preferred): in phase space this can be done with canonical transformations point both in the coordinates and in the momenta. Also the conserved Galilei boosts identify the center of mass. Instead the non-Abelian nature of the Noether constants (the angular momentum) associated with the invariance under rotations implies that there is no unique separation [4] of the relative variables in 6 orientational ones (the body frame in the case of rigid bodies) and in the remaining vibrational (or *shape*) ones. As a consequence, an isolated deformable body or a system of particles may rotate by changing the shape (the falling cat, the diver). In Ref.[5] there is a treatment of this part of the kinematics by means of canonical spin bases and of dynamical body frames, which can be extended to the relativistic case where the notions of Jacobi coordinates, reduced masses and tensors of inertia are absent and can be recovered only when extended bodies are simulated with multipolar expansions.

Other non-conventional aspects of non-relativistic physics are: A) The many-time formulation of classical particle dynamics [6] with as many first class constraints as particles. Like in the special relativistic case a distinction arises between physical positions and canonical configuration variables and a non-relativistic version of the no-interaction theorem emerges.

B) The possibility to gauge the kinematical and internal symmetry groups of an extended system (the simplest one being the Galilean one-time and two-time harmonic oscillator) at the position of the center of mass to define couplings to external gauge fields [7].

C) The possibility to define standard and generalized Newtonian gravity theories as gauge theories of the extended Galilei group [8] by studying the non-relativistic limit of the ADM action of metric gravity.

D) The quantum mechanics of particles in non-rigid non-inertial frames [9].

### **III. THE CHRONO-GEOMETRICAL STRUCTURE OF SPECIAL RELATIVITY: MINKOWSKI SPACE-TIME, CLOCK SYNCHRONIZATION, NON-INERTIAL FRAMES, PARAMETRIZED MINKOWSKI THEORIES.**

As a consequence of Einstein's Annus Mirabilis 1905, all special relativistic physical systems, defined in the inertial frames of Minkowski space-time, are manifestly covariant under the transformations of the kinematical Poincare' group connecting inertial frames (*relativity principle*). Minkowski space-time has an absolute (namely non-dynamical) chrono-geometrical structure. The *light postulates* say that the two-way (or round trip) velocity of light (only one clock is needed for its definition) is  $c$ , namely it is i) constant and ii)

isotropic. The Lorentz signature of its 4-metric tensor implies that every time-like observer can identify the light-cone (the conformal structure, i.e. the locus of the trajectories of light rays) in each point of the world-line. But there is *no notion of an instantaneous 3-space, of a spatial distance and of a one-way velocity of light between two observers* (the problem of the synchronization of distant clocks). Since the relativity principle privileges inertial observers and Cartesian coordinates  $x^\mu = (x^o = ct; \vec{x})$  with the time axis centered on them (inertial frames), the  $x^o = \text{const.}$  hyper-planes of inertial frames are usually taken as Euclidean instantaneous 3-spaces, on which all the clocks are synchronized. Indeed they can be selected with *Einstein's convention* for the synchronization of distant clocks to the clock of an inertial observer. This inertial observer  $A$  sends a ray of light at  $x_i^o$  to a second accelerated observer  $B$ , who reflects it towards  $A$ . The reflected ray is reabsorbed by the inertial observer at  $x_f^o$ . The convention states that the clock of  $B$  at the reflection point must be synchronized with the clock of  $A$  when it signs  $\frac{1}{2}(x_i^o + x_f^o)$ . This convention selects the  $x^o = \text{const.}$  hyper-planes of inertial frames as simultaneity 3-spaces and implies that with this synchronization the two-way (A-B-A) and one-way (A-B or B-A) velocities of light coincide and the spatial distance between two simultaneous point is the (3-geodesic) Euclidean distance.

However, real observers are never inertial and for them Einstein's convention for the synchronization of clocks is not able to identify globally defined simultaneity 3-surfaces, which could also be used as Cauchy surfaces for Maxwell equations. The 1+3 *point of view* tries to solve this problem starting from the local properties of an accelerated observer, whose world-line is assumed to be the time axis of some frame. Since only the observer 4-velocity is given, this only allows to identify the tangent plane of the vectors orthogonal to this 4-velocity in each point of the world-line. Then, both in special and general relativity, this tangent plane is identified with an instantaneous 3-space and 3-geodesic Fermi coordinates are defined on it and used to define a notion of spatial distance. However this construction leads to *coordinate singularities*, because the tangent planes in different points of the world-line will intersect each other at distances from the world-line of the order of the (linear and rotational) *acceleration radii* of the observer. Another type of coordinate singularity arises in all the proposed *uniformly rotating* coordinate systems: if  $\omega$  is the constant angular velocity, then at a distance  $r$  from the rotation axis such that  $\omega r = c$ , the  ${}^4g_{oo}$  component of the induced 4-metric vanishes. This is the so-called *horizon problem for the rotating disk*: the time-like 4-velocity of an observer sitting on a point of the disk becomes light-like in this coordinate system when  $\omega r = c$ .

While in particle mechanics one can formulate a theory of measurement for accelerated observers based on the *locality hypothesis*<sup>1</sup>, this methodology does not work with moving continuous media (for instance the constitutive equations of the electromagnetic field inside them in non-inertial frames are still unknown) and in presence of electromagnetic fields when their wavelength is comparable with the acceleration radii of the observer (the observer is not enough "static" to be able to measure the frequency of such a wave).

See Refs. [2, 10] for a review of these topics.

This state of affairs and the need of predictability (a well-posed Cauchy problem for field theory) lead to the necessity of abandoning the 1+3 point of view and to shift to the

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<sup>1</sup> Standard clocks and rods do not feel acceleration and at each instant the detectors of the instantaneously comoving inertial observer give the correct data.

3+1 one. In this point of view, *besides the world-line of an arbitrary time-like observer, it is given a global 3+1 splitting of Minkowski space-time*, namely a foliation of it whose leaves are space-like hyper-surfaces. Each leaf is *both* a Cauchy surface for the description of physical systems and an instantaneous (in general Riemannian) 3-space, namely a notion of simultaneity implied by a clock synchronization convention different from Einstein's one.

Even if it is unphysical to give initial data on a non-compact space-like hyper-surface, this is the only way to be able to use the existence and uniqueness theorem for the solutions of partial differential equations. In the more realistic mixed problem, in which we give initial data on the Earth and we add an arbitrary information on the null boundary of the future causal domain of the Earth (that is we prescribe the data arriving from the rest of the universe, the ones observed by astronomers), the theorem cannot be shown to hold!

The extra structure of the 3+1 splitting of Minkowski space-time allows to enlarge its atlas of 4-coordinate systems with the definition of *Lorentz-scalar observer-dependent radar 4-coordinates*  $\sigma^A = (\tau; \sigma^r)$ ,  $A = \tau, r$ . Here  $\tau$  is either the proper time of the accelerated observer or any monotonically increasing function of it, and is used to label the simultaneity leaves  $\Sigma_\tau$  of the foliation. On each leaf  $\Sigma_\tau$  the point of intersection with the world-line of the accelerated observer is taken as the origin of curvilinear 3-coordinates  $\sigma^r$ , which can be assumed to be globally defined since each  $\Sigma_\tau$  is diffeomorphic to  $R^3$ . To the coordinate transformation  $x^\mu \mapsto \sigma^A$  ( $x^\mu$  are the standard Cartesian coordinates) is associated an inverse transformation  $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ , where the functions  $z^\mu(\tau, \sigma^r)$  describe the embedding of the simultaneity surfaces  $\Sigma_\tau$  into Minkowski space-time. The 3+1 splitting leads to the following induced 4-metric (a functional of the embedding):  ${}^4g_{AB}(\tau, \sigma^r) = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A} {}^4\eta_{\mu\nu} \frac{\partial z^\nu(\sigma)}{\partial \sigma^B} = {}^4g_{AB}[z(\sigma)]$ , where  ${}^4\eta_{\mu\nu} = \epsilon(+ - - -)$  with  $\epsilon = \pm 1$  according to particle physics or general relativity convention respectively. The quantities  $z_A^\mu(\sigma) = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A}$  are cotetrad fields on Minkowski space-time.

To avoid the previously quoted coordinate singularities, an admissible 3+1 splitting of Minkowski space-time must have the embeddings  $z^\mu(\tau, \sigma^r)$  of the space-like leaves  $\Sigma_\tau$  of the associated foliation satisfying the Møller conditions on the coordinate transformation [11]

$$\begin{aligned} \epsilon^4 g_{\tau\tau}(\sigma) > 0, \quad \epsilon^4 g_{rr}(\sigma) < 0, \quad \begin{vmatrix} {}^4g_{rr}(\sigma) & {}^4g_{rs}(\sigma) \\ {}^4g_{sr}(\sigma) & {}^4g_{ss}(\sigma) \end{vmatrix} > 0, \quad \epsilon \det [{}^4g_{rs}(\sigma)] < 0, \\ \Rightarrow \det [{}^4g_{AB}(\sigma)] < 0. \end{aligned}$$

Moreover, the requirement that the foliation be well defined at spatial infinity may be satisfied by asking that each simultaneity surface  $\Sigma_\tau$  tends to a space-like hyper-plane there, namely we must have  $z^\mu(\tau, \sigma^r) \rightarrow x^\mu(0) + \epsilon_A^\mu \sigma^A$  for some set of orthonormal asymptotic tetrads  $\epsilon_A^\mu$ .

As a consequence, any admissible 3+1 splitting leads to the definition of a *non-inertial frame centered on the given time-like observer* and coordinatized with Lorentz-scalar observer-dependent radar 4-coordinates. While inertial frames centered on inertial observers are connected by the transformations of the Poincaré' group, the non-inertial ones are connected by *passive frame-preserving diffeomorphism*:  $\tau \mapsto \tau'(\tau, \sigma^r)$ ,  $\sigma^r \mapsto \sigma'^r(\sigma^s)$ . It turns

out that Møller conditions forbid uniformly rotating non-inertial frames: *only differentially rotating ones are allowed* (the ones used by astrophysicists in the modern description of rotating stars). In Refs.[10] there is a detailed discussion of this topic and there is the simplest example of 3+1 splittings whose leaves are space-like hyper-planes carrying admissible differentially rotating 3-coordinates ( $\sigma = |\vec{\sigma}|$ ;  $e_r^\mu$  are asymptotic space-like axes;  $\alpha_i$ ,  $i = 1, 2, 3$ , are Euler angles):

$$z^\mu(\tau, \vec{\sigma}) = x^\mu(\tau) + e_r^\mu R^r_s(\tau, \sigma) \sigma^s,$$

$$R^r_s(\tau, \sigma) \xrightarrow{\sigma \rightarrow \infty} \delta_s^r, \quad \partial_A R^r_s(\tau, \sigma) \xrightarrow{\sigma \rightarrow \infty} 0,$$

$$R^r_s(\tau, \sigma) = R^r_s(\alpha_i(\tau, \sigma)) = R^r_s(F(\sigma) \tilde{\alpha}_i(\tau)),$$

$$0 < F(\sigma) < \frac{1}{A\sigma}, \quad \frac{dF(\sigma)}{d\sigma} \neq 0.$$

Each function  $F(\sigma)$ , satisfying the Møller conditions given in the last line, defines an admissible differentially-rotating non-inertial frame centered on the world-line  $x^\mu(\tau)$  of an accelerated observer.

Moreover, it is shown in Ref.[10] that to each admissible 3+1 splitting are associated two congruences of time-like observers (the natural ones for the given notion of simultaneity):

- i) the Eulerian observers, whose unit 4-velocity field is the field of unit normals to the simultaneity surfaces  $\Sigma_\tau$ ;
- ii) the observers whose unit 4-velocity field is proportional to the evolution vector field of components  $\partial z^\mu(\tau, \sigma^r)/\partial \tau$ : in general this congruence is non-surface forming having a non-vanishing vorticity (like the congruence associated to a rotating disk).

The next problem is how to describe physical systems in non-inertial frames and how to connect different conventions for clock synchronization. The answer is given by *parametrized Minkowski theories* [12], [1]. Given any isolated system (particles, strings, fields, fluids) admitting a Lagrangian description, one makes the coupling of the system to an external gravitational field and then replaces the 4-metric  ${}^4g_{\mu\nu}(x)$  with the induced metric  ${}^4g_{AB}[z(\tau, \sigma^r)]$  associated to an arbitrary admissible 3+1 splitting. The Lagrangian now depends not only on the matter configurational variables but also on the embedding variables  $z^\mu(\tau, \sigma^r)$  (whose conjugate canonical momenta are denoted  $\rho_\mu(\tau, \sigma^r)$ ). Since the action principle turns out to be invariant under frame-preserving diffeomorphisms, at the Hamiltonian level there are four first-class constraints  $\mathcal{H}_\mu(\tau, \sigma^r) = \rho_\mu(\tau, \sigma^r) - l_\mu(\tau, \sigma^r) T^{\tau\tau}(\tau, \sigma^r) - z_s^\mu(\tau, \sigma^r) T^{\tau s}(\tau, \sigma^r) \approx 0$  in strong involution with respect to Poisson brackets,  $\{\mathcal{H}_\mu(\tau, \sigma^r), \mathcal{H}_\nu(\tau, \sigma_1^r)\} = 0$ . Here  $l_\mu(\tau, \sigma^r)$  are the covariant components of the unit normal to  $\Sigma_\tau$ , while  $z_s^\mu(\tau, \sigma^r)$  are the components of three independent vectors tangent to  $\Sigma_\tau$ . The quantities  $T^{\tau\tau}$  and  $T^{\tau s}$  are the components of the energy-momentum tensor of the matter inside  $\Sigma_\tau$  describing its energy- and momentum- densities. As a consequence, Dirac's theory of constraints implies that the configuration variables  $z^\mu(\tau, \sigma^r)$  are arbitrary *gauge variables*. Therefore, all the admissible 3+1 splittings, namely all the admissible conventions for clock synchronization, and all the admissible non-inertial frames centered on time-like observers are *gauge equivalent*.

By adding four gauge-fixing constraints  $\chi^\mu(\tau, \sigma^r) = z^\mu(\tau, \sigma^r) - z_M^\mu(\tau, \sigma^r) \approx 0$  ( $z_M^\mu(\tau, \sigma^r)$  being an admissible embedding), satisfying the orbit condition  $\det |\{\chi^\mu(\tau, \sigma^r), \mathcal{H}_\nu(\tau, \sigma_1^r)\}| \neq 0$ , we identify the description of the system in the associated non-inertial frame centered on a given time-like observer. The resulting effective Hamiltonian for the  $\tau$ -evolution turns out to contain the potentials of the *relativistic inertial forces* present in the given non-inertial frame. Since a non-inertial frame means the use of its radar coordinates, we see that already in special relativity *non-inertial Hamiltonians are coordinate-dependent quantities* like the notion of energy density in general relativity.

As a consequence, the gauge variables  $z^\mu(\tau, \sigma^r)$  describe the *spatio-temporal appearances* of the phenomena in non-inertial frames, which, in turn, are associated to extended physical laboratories using a metrology for their measurements compatible with the notion of simultaneity of the non-inertial frame (think to the description of the Earth given by GPS). Therefore, notwithstanding mathematics tends to use only coordinate-independent notions, physical metrology forces us to consider intrinsically coordinate-dependent quantities like the non-inertial Hamiltonians. For instance, the motion of satellites around the Earth is governed by a set of empirical coordinates contained in the software of NASA computers: this is a *metrological standard of space-time around the Earth* with a poorly understood connection with the purely theoretical coordinate systems. In a few years the European Space Agency will start the project ACES about the synchronization of a high-precision laser-cooled atomic clock on the space station with similar clocks on the Earth surface by means of microwave signals. If the accuracy of 5 picosec. will be achieved, it will be possible to make a coordinate-dependent test of effects at the order  $1/c^3$ , like the second order Sagnac effect (sensible to Earth rotational acceleration) and the general relativistic Shapiro time-delay created by the geoid [13]. The one-way velocity of light between an Earth station and the space station and the synchronization of the respective clocks are two faces of the same problem.

Inertial frames centered on inertial observers are a special case of gauge fixing in parametrized Minkowski theories. For each configuration of an isolated system there is an special 3+1 splitting associated to it: the foliation with space-like hyper-planes orthogonal to the conserved time-like 4-momentum of the isolated system. This identifies an intrinsic inertial frame, the *rest-frame*, centered on a suitable inertial observer (the Fokker-Pryce center of inertia of the isolated system) and allows to define the *Wigner-covariant rest-frame instant form of dynamics* for every isolated system (see Ref.[14] for the various forms of dynamics).

Let us remark that in parametrized Minkowski theories a relativistic particle with world-line  $x_i^\mu(\tau)$  is described only by the 3-coordinates  $\sigma^r = \eta_i^r(\tau)$  defined by  $x_i^\mu(\tau) = z^\mu(\tau, \eta_i^r(\tau))$  and by the conjugate canonical momenta  $\kappa_{ir}(\tau)$ . The usual 4-momentum  $p_{i\mu}(\tau)$  is a derived quantity satisfying the mass-shell constraint  $\epsilon p_i^2 = m_i^2$ . Therefore, we have a different description for positive- and negative- energy particles. All the particles on an admissible surface  $\Sigma_\tau$  are simultaneous by construction: *this eliminates the problem of relative times*, which for a long time has been an obstruction to the theory of relativistic bound states and to relativistic statistical mechanics (see Ref.[12] and its bibliography for these problems and the related no-interaction theorem).

Let us also remark that, differently from Fermi coordinates (a purely theoretical construction), radar 4-coordinates can be *operationally* defined. As shown in Ref.[10], given

four functions satisfying certain restrictions induced by the Møller conditions, the on-board computer of a spacecraft may establish a grid of radar 4-coordinates in its future.

The discovery of the rest-frame instant form made possible to develop a coherent formalism for all the aspects of relativistic kinematics both for N particle systems and continuous bodies and fields [15, 16] generalizing all known non-relativistic results [5]:

- i) the classification of the intrinsic notions of collective variables (canonical non-covariant center of mass; covariant non-canonical Fokker-Pryce center of inertia; non-covariant non-canonical Møller center of energy);
- ii) canonical bases of center-of-mass and relative variables;
- iii) canonical spin bases and dynamical body-frames for the rotational kinematics of deformable systems;
- iv) multipolar expansions for isolated and open systems;
- v) the relativistic theory of orbits (while the potentials appearing in the energy generator of the Poincare' group determine the relative motion, the determination of the actual orbits in the given inertial frame is influenced by the potentials appearing in the Lorentz boosts: the vanishing of the boosts is the natural gauge fixing to the rest-frame conditions and selects the covariant Fokker-Pryce center of inertia);
- vi) the Møller radius (a classical unit of length identifying the region of non-covariance of the canonical center of mass of a spinning system around the covariant Fokker-Pryce center of inertia; it is an effect induced by the Lorentz signature of the 4-metric; it could be used as a physical ultraviolet cutoff in quantization).

See Ref. [17] for a comprehensive review.

All these developments relied on an accurate study of the structure of the constraint manifold  $\bar{\gamma}$  (see the Appendix) from the point of view of the orbits of the Poincare' group. If  $p^\mu$  is the total momentum of the system, the constraint manifold has to be divided in four strata (some of them may be absent for certain systems) according to whether  $\epsilon p^2 > 0$ ,  $p^2 = 0$ ,  $\epsilon p^2 < 0$  or  $p^\mu = 0$ . Due to the different little groups of the various Poincare' orbits, the gauge orbits of different sectors will not be diffeomorphic. Therefore the manifold  $\bar{\gamma}$  is a *stratified* manifold and the gauge foliations of relativistic systems are nearly never nice, but rather one has to do with *singular* foliations.

For an acceptable relativistic system the stratum  $\epsilon p^2 < 0$  has to be absent to avoid tachyons. To study the strata  $p^2 = 0$  and  $p^\mu = 0$  one has to add these relations as extra constraints. For all the strata the next step is to do a canonical transformation from the original variables to a new set consisting of center-of-mass variables  $x^\mu$ ,  $p^\mu$  and of variables relative to the center of mass, both for particle and field systems. Let us consider the stratum  $\epsilon p^2 > 0$ . By using the standard Wigner boost  $L_\nu^\mu(p, \hat{p})$  ( $p^\mu = L_\nu^\mu(p, \hat{p})\hat{p}^\nu$ ,  $\hat{p}^\mu = \eta\sqrt{\epsilon p^2}(1; \vec{0})$ ,  $\eta = \text{sign } p^0$ ), one boosts the relative variables at rest. The new variables are still canonical and the base is completed by  $p^\mu$  and by a new canonical non-covariant <sup>2</sup> 4-center-of-mass coordinate  $\tilde{x}^\mu$ , differing from  $x^\mu$  for spin terms. The new relative variables are either Poincare' scalars or Wigner spin-1 vectors, transforming under the group O(3)(p)

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<sup>2</sup> This is a *universal breaking of manifest Lorentz covariance*, which appears for every isolated relativistic system, when one wants to identify a canonical 4-center of mass. Since there is no definition of relativistic center of mass enjoying all the properties of the non-relativistic one, it turns out [17] that the canonical 4-center of mass has only O(3) covariance like the quantum Newton-Wigner position operator.

of the Wigner rotations induced by the Lorentz transformations. A final canonical transformation [18], leaving fixed the relative variables, sends the center-of-mass coordinates  $\tilde{x}^\mu$ ,  $p^\mu$  in the new set  $p \cdot \tilde{x}/\eta\sqrt{\epsilon p^2} = p \cdot x/\eta\sqrt{\epsilon p^2}$  (the time in the rest frame),  $\eta\sqrt{\epsilon p^2}$  (the total mass),  $\vec{k} = \vec{p}/\eta\sqrt{\epsilon p^2}$  (the spatial components of the unit 4-velocity  $k^\mu = p^\mu/\eta\sqrt{\epsilon p^2}$ ,  $k^2 = 1$ ),  $\vec{z} = \eta\sqrt{\epsilon p^2}(\vec{x} - \tilde{x}^o\vec{p}/p^o)$ .  $\vec{z}$  is a non-covariant 3-center-of-mass canonical coordinate multiplied by the total mass: it is the classical analog of the Newton-Wigner position operator (like it,  $\vec{z}$  is covariant only under the little group  $O(3)(p)$  of the time-like Poincaré orbits). Analogous considerations could be done for the other sectors. In Refs. [19] there is the definition of other canonical bases, the spin bases, adapted to the spin Casimir of the Poincaré group, which made possible the quoted developments [15, 16, 17], [5].

The nature of the relative variables depends on the system. The first class constraints, once rewritten in terms of the new variables, can be manipulated to find suitable global and Lorentz scalar Abelianizations by means of Shanmugadhasan canonical transformations (see the Appendix). Usually there is a combination of the constraints which determines  $\eta\sqrt{\epsilon p^2}$ , i.e. the mass spectrum, so that the time in the rest frame  $p \cdot x/\eta\sqrt{\epsilon p^2}$  is the conjugated Lorentz scalar gauge variable. The other first class constraints eliminate some of the relative variables<sup>3</sup>: their conjugated coordinates are the other gauge variables. The DO (apart from the center-of-mass ones  $\vec{k}$  and  $\vec{z}$  describing a decoupled non-covariant observer) have to be extracted from the remaining relative variables and the construction shows that they will be either Poincaré scalars or Wigner covariant objects. In this way in each stratum preferred global Shanmugadhasan canonical transformations are identified, when no other kind of obstruction to globality is present inside the various strata.

See Ref. [1] for a list of the finite-dimensional relativistic particle systems, which have been described with Dirac theory of constraints and with parametrized Minkowski theories, with some comments on their quantization.

In particular in Ref.[20] there is the quantization of relativistic scalar and spinning particles in a class of non-inertial frames, whose simultaneity surfaces  $\Sigma_\tau$  are space-like hyperplanes with arbitrary admissible linear acceleration and carrying arbitrary admissible differentially rotating 3-coordinates. It is based on a multi-temporal quantization scheme for systems with first-class constraints, in which only the particle degrees of freedom  $\eta_i^r(\tau)$ ,  $\kappa_{ir}(\tau)$  are quantized. The gauge variables, describing the appearances (inertial effects) of the motion in non-inertial frames, are treated as c-numbers (like the time in the Schroedinger equation with a time-dependent Hamiltonian) and the physical scalar product does not depend on them. The previously quoted relativistic kinematics has made possible to separate the center of mass<sup>4</sup> and to verify that the spectra of relativistic bound states in non-inertial frames are only modified by inertial effects, being obtained from the inertial ones by means of a time-dependent unitary transformation. The non-relativistic limit [9] allows to recover the few existing attempts of quantization in non-inertial frames as particular cases.

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<sup>3</sup> In particular they eliminate the *relative energies* for systems of interacting relativistic particles (and for the string), so that *the conjugate relative times are gauge variables*, describing the freedom of the observer of looking at the particle at the same time in every allowed non-inertial frame [12].

<sup>4</sup> At the relativistic level this is done with a canonical transformation which is *point only* in the momenta [17].

Let us now look at what is known for other physically relevant systems. In non-inertial frames there is the reformulation of the Klein-Gordon equation [21], of the Dirac equation [22], of the electro-magnetic field [12] and of relativistic fluids [23] by means of parametrized Minkowski theories.

Inspired by Dirac <sup>5</sup>, the canonical reduction to Wigner-covariant generalized radiation gauges, with the determination of the physical Hamiltonian as a function of a canonical basis of DO, has been achieved for the following isolated systems <sup>6</sup>:

1) In the semi-classical approximation we get a consistent description of  $N$  charged particles plus the electromagnetic field if we use Grassmann-valued electric charges to regularize the Coulomb self-energies. In Ref.[25] the electromagnetic degrees of freedom are expressed in terms of the particle variables by means the Lienard-Wiechert solution and this allows to find the relativistic Darwin potential (or the Salpeter potential for spinning particles) starting from classical electrodynamics and not as a reduction from QFT.

2) Both the open and closed Nambu string, after an initial study with light-cone coordinates, have been treated [26] along these lines in the stratum  $\epsilon p^2 > 0$ . Both Abelian Lorentz scalar constraints and gauge variables have been found and globally decoupled, and a redundant set of DO  $[\vec{z}, \vec{k}, \vec{a}_n]$  has been found. It remains an open problem whether one can extract a global canonical basis of DO from the Wigner spin 1 vectors  $\vec{a}_n$ , which satisfy sigma-model-like constraints; if this basis exists, it would define the Liouville integrability of the Nambu string and would clarify whether there is any way to quantize it in four dimensions.

3) Yang-Mills theory with Grassmann-valued fermion fields [27] in the case of a trivial principal bundle over a fixed- $x^o$   $R^3$  slice of Minkowski space-time with suitable Hamiltonian-oriented boundary conditions; this excludes monopole solutions and, since  $R^3$  is not compactified, one has only winding number and no instanton number. After a discussion of the Hamiltonian formulation of Yang-Mills theory, of its group of gauge transformations and of the Gribov ambiguity (see the Appendix), the theory has been studied in suitable weighted Sobolev spaces where the Gribov ambiguity is absent [28, 29] and the global color charges are well defined. The global DO are the transverse quantities  $\vec{A}_{a\perp}(\vec{x}, x^o)$ ,  $\vec{E}_{a\perp}(\vec{x}, x^o)$  and fermion fields dressed with Yang-Mills (gluonic) clouds. The nonlocal and non-polynomial (due to the presence of classical Wilson lines along flat geodesics) physical Hamiltonian has been obtained: it is nonlocal but without any kind of singularities, it has the correct Abelian limit if the structure constants are turned off, and it contains the explicit realization of the abstract Mitter-Viallet metric.

4) SU(3) Yang-Mills theory with scalar particles with Grassmann-valued color charges [30] for the regularization of self-energies. It is possible to show that in this relativistic scalar quark model the Dirac Hamiltonian expressed as a function of DO has the property of asymptotic freedom.

5) The Abelian and non-Abelian SU(2) Higgs models with fermion fields [31], where the

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<sup>5</sup> Dirac [24] showed that the DO of the electromagnetic field are the transverse vector potential  $\vec{A}_\perp$  and the transverse electric field  $\vec{E}_\perp$  of the *radiation gauge*. When a fermion field is interacting with the electromagnetic field, the fermionic DO is a *fermion field dressed with a Coulomb cloud*.

<sup>6</sup> For them one only asks that the 10 conserved generators of the Poincaré algebra are finite so to be able to use group theory; theories with external fields can only be recovered as limits in some parameter of a subsystem of the isolated system.

symplectic decoupling is a refinement of the concept of unitary gauge. There is an ambiguity in the solutions of the Gauss law constraints, which reflects the existence of disjoint sectors of solutions of the Euler-Lagrange equations of Higgs models. The physical Hamiltonian and Lagrangian of the Higgs phase have been found; the self-energy turns out to be local and contains a local four-fermion interaction.

6) The standard  $SU(3)\times SU(2)\times U(1)$  model of elementary particles [32] with Grassmann-valued fermion fields. The final reduced Hamiltonian contains nonlocal self-energies for the electromagnetic and color interactions, but “local ones” for the weak interactions implying the non-perturbative emergence of 4-fermions interactions.

In inertial frames the quantization of DO can be faced with the standard methods. The main open problem is the quantization of the scalar Klein-Gordon field in non-inertial frames, due to the Torre and Varadarajan [33] no-go theorem, according to which in general the evolution from an initial space-like hyper-surface to a final one is *not unitary* in the Tomonaga-Schwinger formulation of quantum field theory. From the 3+1 point of view there is evolution only among the leaves of an admissible foliation and the possible way out from the theorem lies in the determination of all the admissible 3+1 splittings of Minkowski space-time satisfying the following requirements: i) existence of an instantaneous Fock space on each simultaneity surface  $\Sigma_\tau$  (i.e. the  $\Sigma_\tau$ 's must admit a generalized Fourier transform); ii) unitary equivalence of the Fock spaces on  $\Sigma_{\tau_1}$  and  $\Sigma_{\tau_2}$  belonging to the same foliation (the associated Bogoljubov transformation must be Hilbert-Schmidt), so that the non-inertial Hamiltonian is a Hermitean operator; iii) unitary gauge equivalence of the 3+1 splittings with the Hilbert-Schmidt property. The overcoming of the no-go theorem would help also in quantum field theory in curved space-times and in condensed matter (here the non-unitarity implies non-Hermitean Hamiltonians and negative energies).

#### IV. THE DYNAMICAL CHRONO-GEOMETRICAL STRUCTURE OF GENERAL RELATIVITY: THE REST-FRAME INSTANT FORM OF METRIC AND TETRAD GRAVITY AND THE ROLE OF NON-INERTIAL FRAMES.

In the years 1913-16 Einstein developed general relativity relying on the *equivalence principle* (equality of inertial and gravitational masses of bodies in free fall). It suggested him the impossibility to distinguish a uniform gravitational field from the effects of a constant acceleration by means of local experiments in sufficiently small regions where the effects of tidal forces are negligible. This led to the *geometrization* of the gravitational interaction and to the replacement of Minkowski space-time with a pseudo-Riemannian 4-manifold  $M^4$  with non vanishing curvature Riemann tensor. The principle of *general covariance* (see Ref.[34] for a review), at the basis of the tensorial nature of Einstein's equations, has the two following consequences:

- i) the invariance of the Hilbert action under *passive* diffeomorphisms (the coordinate transformations in  $M^4$ ), so that the second Noether theorem implies the existence of first-class constraints at the Hamiltonian level;
- ii) the mapping of solutions of Einstein's equations among themselves under the action of *active* diffeomorphisms of  $M^4$  extended to the tensors over  $M^4$  (dynamical symmetries of Einstein's equations).

The basic field of metric gravity is the 4-metric tensor with components  ${}^4g_{\mu\nu}(x)$  in an arbitrary coordinate system of  $M^4$ . The peculiarity of gravity is that the 4-metric field,

differently from the fields of electromagnetic, weak and strong interactions and from the matter fields, has a *double role*:

i) it is the mediator of the gravitational interaction (in analogy to all the other gauge fields);

ii) it determines the chrono-geometric structure of the space-time  $M^4$  in a dynamical way through the line element  $ds^2 = {}^4g_{\mu\nu}(x) dx^\mu dx^\nu$ .

As a consequence, the gravitational field *teaches relativistic causality* to all the other fields: for instance it tells to classical rays of light and to quantum photons and gluons which are the allowed trajectories for massless particles in each point of  $M^4$ .

Let us make a comment about the two main existing approaches to the quantization of gravity.

1) *Effective quantum field theory and string theory*. This approach contains the standard model of elementary particles and its extensions. However, since the quantization, namely the definition of the Fock space, requires a background space-time where it is possible to define creation and annihilation operators, one must use the splitting  ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu}^{(B)} + {}^4h_{\mu\nu}$  and quantize only the perturbation  ${}^4h_{\mu\nu}$  of the background 4-metric  $\eta_{\mu\nu}^{(B)}$  (usually  $B$  is either Minkowski or DeSitter space-time). In this way property ii) is lost (one uses the fixed non-dynamical chrono-geometrical structure of the background space-time), gravity is replaced by a field of spin two over the background (and passive diffeomorphisms are replaced by gauge transformations acting in an inner space) and the only difference among gravitons, photons and gluons lies in their quantum numbers.

2) *Loop quantum gravity*. This approach never introduces a background space-time, but being inequivalent to a Fock space, has problems to incorporate particle physics. It uses a fixed 3+1 splitting of the space-time  $M^4$  and it is a quantization of the associated instantaneous 3-spaces  $\Sigma_\tau$  (quantum geometry). However, there is no known way to implement a consistent unitary evolution (the problem of the super-hamiltonian constraint) and, since it is usually formulated in spatially compact space-times without boundary, there is no notion of a Poincare' group (and therefore no extra dimensions) and a problem of time (frozen picture without evolution).

For outside points of view on loop quantum gravity and string theory see Refs. [35, 36], respectively.

Let us remark that in all known formulations particle and nuclear physics are a chapter of the theory of representations of the Poincare' group in inertial frames in the spatially non-compact Minkowski space-time. As a consequence, if one looks at general relativity from the point of view of particle physics, the main problem to get a unified theory is how to reconcile the Poincare' group (the kinematical group of the transformations connecting inertial frames) with the diffeomorphism group implying the non-existence of global inertial frames in general relativity (special relativity holds only in a small neighborhood of a body in free fall).

Let us consider the ADM formulation of metric gravity [37] and its extension to tetrad gravity <sup>7</sup> obtained by replacing the ten configurational 4-metric variables  ${}^4g_{\mu\nu}(x)$

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<sup>7</sup> It is needed to describe the coupling of gravity to fermions; it is a theory of time-like observers endowed

with the sixteen cotetrad fields  ${}^4E_\mu^{(\alpha)}(x)$  by means of the decomposition  ${}^4g_{\mu\nu}(x) = {}^4E_\mu^{(\alpha)}(x) {}^4\eta_{(\alpha)(\beta)} {}^4E_\nu^{(\beta)}(x)$  [ $(\alpha)$  are flat indices].

Then, after having restricted the model to globally hyperbolic, topologically trivial, spatially non-compact space-times (admitting a global notion of time), let us introduce a global 3+1 splitting of the space-time  $M^4$  and let choose the world-line of a time-like observer. As in special relativity, let us make a coordinate transformation to observer-dependent radar 4-coordinates,  $x^\mu \mapsto \sigma^A = (\tau, \sigma^r)$ , adapted to the 3+1 splitting and using the observer world-line as origin of the 3-coordinates. Again the inverse transformation,  $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ , defines the embedding of the leaves  $\Sigma_\tau$  into  $M^4$ . These leaves  $\Sigma_\tau$  (assumed to be Riemannian 3-manifolds diffeomorphic to  $R^3$ , so that they admit global 3-coordinates  $\sigma^r$  and a unique 3-geodesic joining any pair of points in  $\Sigma_\tau$ ) are both Cauchy surfaces and simultaneity surfaces corresponding to a convention for clock synchronization. For the induced 4-metric we get

$$\begin{aligned} {}^4g_{AB}(\sigma) &= \frac{\partial z^\mu(\sigma)}{\partial \sigma^A} {}^4g_{\mu\nu}(x) \frac{\partial z^\nu(\sigma)}{\partial \sigma^B} = {}^4E_A^{(\alpha)} {}^4\eta_{(\alpha)(\beta)} {}^4E_B^{(\beta)} = \\ &= \epsilon \begin{pmatrix} (N^2 - {}^3g_{rs} N^r N^s) & -{}^3g_{su} N^u \\ -{}^3g_{ru} N^u & -{}^3g_{rs} \end{pmatrix} (\sigma). \end{aligned}$$

Here  ${}^4E_A^{(\alpha)}(\tau, \sigma^r)$  are adapted cotetrad fields,  $N(\tau, \sigma^r)$  and  $N^r(\tau, \sigma^r)$  the lapse and shift functions and  ${}^3g_{rs}(\tau, \sigma^r)$  the 3-metric on  $\Sigma_\tau$  with signature  $(+++)$ . We see that in general relativity the quantities  $z_A^\mu = \partial z^\mu / \partial \sigma^A$  are no more cotetrad fields on  $M^4$  differently from what happens in special relativity: now they are only transition functions between coordinate charts, so that the dynamical fields are now the real cotetrad fields  ${}^4E_A^{(\alpha)}(\tau, \sigma^r)$  and not the embeddings  $z^\mu(\tau, \sigma^r)$ .

Let us try to identify a class of space-times and an associated suitable family of admissible 3+1 splittings able to incorporate particle physics and giving a model for the solar system or our galaxy (and hopefully allowing an extension to the cosmological context) with the following further requirements [38]:

1)  $M^4$  must be asymptotically flat at spatial infinity and the 4-metric must tend asymptotically at spatial infinity to the Minkowski 4-metric in every coordinate system (this implies that the 4-diffeomorphisms must tend to the identity at spatial infinity). Therefore, in these space-times there is an *asymptotic background 4-metric* and this will allow to avoid the decomposition  ${}^4g_{\mu\nu} = {}^4\eta_{\mu\nu} + {}^4h_{\mu\nu}$  in the bulk.

2) The boundary conditions on the fields on each leaf  $\Sigma_\tau$  of the admissible 3+1 splittings must be such to reduce the Spi group of asymptotic symmetries (see Ref.[39]) to the ADM Poincaré' group. This means that *super-translations* (direction-dependent quasi Killing vectors, obstruction to the definition of angular momentum in general relativity) must be absent, namely that all the fields must tend to their asymptotic limits in a direction-independent way (see Refs. [40]). This is possible only if the admissible 3+1 splittings have all the leaves  $\Sigma_\tau$  tending to Minkowski space-like hyper-planes orthogonal to the

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with a tetrad field, whose time-like axis is the unit 4-velocity of the observer and whose spatial axes are associated to a choice of three gyroscopes.

ADM 4-momentum at spatial infinity [38]. In turn this implies that every  $\Sigma_\tau$  is *the rest frame of the instantaneous 3-universe* and that there are asymptotic inertial observers to be identified with the *fixed stars*<sup>8</sup>. This requirement implies that the shift functions vanish at spatial infinity [ $N^r(\tau, \sigma^r) \rightarrow O(1/|\sigma|^\epsilon)$ ,  $\epsilon > 0$ ,  $\sigma^r = |\sigma| \hat{u}^r$ ], where the lapse function tends to 1 [ $N(\tau, \sigma^r) \rightarrow 1 + O(1/|\sigma|^\epsilon)$ ] and the 3-metric tends to the Euclidean one [ ${}^3g_{rs}(\tau, \sigma^u) \rightarrow \delta_{rs} + O(1/|\sigma|)$ ].

3) The admissible 3+1 splittings should have the leaves  $\Sigma_\tau$  admitting a generalized Fourier transform (namely they should be Lichnerowicz [41] 3-manifolds with involution, so to have the possibility to define instantaneous Fock spaces in a future attempt of quantization).

4) All the fields on  $\Sigma_\tau$  should belong to suitable weighted Sobolev spaces, so that  $M^4$  has no Killing vectors and Yang-Mills fields on  $\Sigma_\tau$  do not present Gribov ambiguities (due to the presence of gauge symmetries and gauge copies) [27].

In absence of matter the Christodoulou and Klainermann [42] space-times are good candidates: they are near Minkowski space-time in a norm sense, avoid singularity theorems by relaxing the requirement of conformal completeness (so that it is possible to follow solutions of Einstein's equations on long times) and admit gravitational radiation at null infinity.

Since the simultaneity leaves  $\Sigma_\tau$  are the rest frame of the instantaneous 3-universe, at the Hamiltonian level it is possible to define *the rest-frame instant form of metric and tetrad gravity* [38, 43]. If matter is present, the limit of this description for vanishing Newton constant will produce the rest-frame instant form description of the same matter in the framework of parametrized Minkowski theories and the ADM Poincaré' generators will tend to the kinematical Poincaré' generators of special relativity. Therefore we have obtained a model admitting *a deparametrization of general relativity to special relativity*. It is not known whether the rest-frame condition can be relaxed in general relativity without having super-translations reappearing, since the answer to this question is connected with the non-trivial problem of boosts in general relativity.

Let us now come back to ADM tetrad gravity. The time-like vector  ${}^4E_{(o)}^A(\tau, \sigma^r)$  of the tetrad field  ${}^4E_{(a)}^A(\tau, \sigma^r)$ , dual to the cotetrad field  ${}^4E_A^{(a)}(\tau, \sigma^r)$ , may be rotated to become the unit normal to  $\Sigma_\tau$  in each point by means of a standard Wigner boost for time-like Poincaré' orbits depending on three parameters  $\varphi_{(a)}(\tau, \sigma^r)$ ,  $a = 1, 2, 3$ :  ${}^4E_{(o)}^A(\tau, \sigma^r) = L^A{}_B(\varphi_{(a)}(\tau, \sigma^r)) {}^4\check{E}_{(o)}^B(\tau, \sigma^r)$ . This allows to define the following cotetrads adapted to the 3+1 splitting (the so-called *Schwinger time gauge*)  ${}^4\check{E}_A^{(o)}(\tau, \sigma^r) = (N(\tau, \sigma^r); 0)$ ,  ${}^4\check{E}_A^{(a)}(\tau, \sigma^r) = (N_{(a)}(\tau, \sigma^r); {}^3e_{(a)r}(\tau, \sigma^r))$ , where  ${}^3e_{(a)r}(\tau, \sigma^r)$  are cotriads fields on  $\Sigma_\tau$  (tending to  $\delta_{(a)r} + O(1/|\sigma|)$  at spatial infinity) and  $N_{(a)} = N {}^3e_{(a)r}$ . As a consequence, the sixteen cotetrad fields may be replaced by the fields  $\varphi_{(a)}(\tau, \sigma^r)$ ,  $N(\tau, \sigma^r)$ ,  $N_{(a)}(\tau, \sigma^r)$ ,  ${}^3e_{(a)r}(\tau, \sigma^r)$ , whose conjugate canonical momenta will be denoted as  $\pi_N(\tau, \sigma^r)$ ,  $\pi_{\check{N}_{(a)}}(\tau, \sigma^r)$ ,  $\pi_{\check{\varphi}_{(a)}}(\tau, \sigma^r)$ ,  ${}^3\pi_{(a)}^r(\tau, \sigma^r)$ .

The local invariance of the ADM action imply the existence of 14 first-class constraints (10 primary and 4 secondary):

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<sup>8</sup> In a future extension to the cosmological context they could be identified with the privileged observers at rest with respect to the background cosmic radiation.

- i)  $\pi_N(\tau, \sigma^r) \approx 0$  implying the secondary super-hamiltonian constraint  $\mathcal{H}(\tau, \sigma^r) \approx 0$ ;
- ii)  $\pi_{\bar{N}(a)}(\tau, \sigma^r) \approx 0$  implying the secondary super-momentum constraints  $\mathcal{H}_{(a)}(\tau, \sigma^r) \approx 0$ ;
- iii)  $\pi_{\bar{\varphi}(a)}(\tau, \sigma^r) \approx 0$ ;
- iv) three constraints  $M_{(a)}(\tau, \sigma^r) \approx 0$  generating rotations of the cotriads.

As a consequence there are 14 gauge variables describing the *generalized inertial effects* in the non-inertial frame defined by the chosen admissible 3+1 splitting of  $M^4$  centered on an arbitrary time-like observer. The remaining independent "two + two" degrees of freedom are the gauge invariant DO of the gravitational field describing *generalized tidal effects*. The same degrees of freedom emerge in ADM metric gravity, where the configuration variables  $N, N^r, {}^4g_{rs}$  with conjugate momenta  $\pi_N, \pi_{\bar{N}^r}, {}^3\Pi^{rs}$ , are restricted by 8 first-class constraints ( $\pi_N(\tau, \sigma^r) \approx 0 \rightarrow \mathcal{H}(\tau, \sigma^r) \approx 0, \pi_{\bar{N}^r}(\tau, \sigma^r) \approx 0 \rightarrow \mathcal{H}^r(\tau, \sigma^r) \approx 0$ ).

Again it is possible to make a separation of the gauge variables from the DO by means of a Shanmugadhasan canonical transformation. Since no-one knows how to solve the super-hamiltonian constraint (except that in the post-Newtonian approximation), the best we can do is to look for a quasi-Shanmugadhasan canonical transformation adapted to the other 13 first-class constraints (the only constraints to be Abelianized are  $M_{(a)}(\tau, \sigma^r) \approx 0$  and  $\mathcal{H}_{(a)}(\tau, \sigma^r) \approx 0$ ) [43]:

$$\begin{array}{|c|c|c|c|} \hline \varphi^{(a)} & N & N_r & {}^3e_{(a)r} \\ \hline \approx 0 & \approx 0 & \approx 0 & {}^3\tilde{\pi}_{(a)}^r \\ \hline \end{array} \longrightarrow \begin{array}{|c|c|c|c|c|c|c|c|} \hline \varphi^{(a)} & N & N_{(a)} & \alpha_{(a)} & \xi^r & \phi & r_{\bar{a}} \\ \hline \approx 0 & \pi_\phi & \pi_{\bar{a}} \\ \hline \end{array}$$

Here,  $\alpha_{(a)}(\tau, \sigma^r)$  are three Euler angles and  $\xi^r(\tau, \sigma^r)$  are three parameters giving a coordinatization of the action of 3-diffeomorphisms on the cotriads  ${}^3e_{(a)r}(\tau, \sigma^r)$ . The configuration variable  $\phi(\tau, \sigma^r) = \left(\det {}^3g(\tau, \sigma^r)\right)^{1/12}$  is the conformal factor of the 3-metric: it can be shown that it is the unknown in the super-hamiltonian constraint (also named the Lichnerowicz equation). The gauge variables are  $N, N_{(a)}, \varphi_{(a)}, \alpha_{(a)}, \xi^r$  and  $\pi_\phi$ , while  $r_{\bar{a}}, \pi_{\bar{a}}, \bar{a} = 1, 2$ , are the DO of the gravitational field (in general they are not tensorial quantities).

Even if we do not know the expression of the final variables in terms of the original ones, we note that this is a *point* canonical transformation with known inverse

$${}^3e_{(a)r}(\tau, \sigma^u) = {}^3R_{(a)(b)}(\alpha_{(e)}(\tau, \sigma^u)) \frac{\partial \xi^s(\tau, \sigma^u)}{\partial \sigma^r} \phi^2(\tau, \vec{\xi}(\tau, \sigma^u)) {}^3\hat{e}_{(b)s}(r_{\bar{a}}(\tau, \xi^u(\tau, \sigma^v))),$$

as implied by the study of the gauge transformations generated by the first-class constraints ( ${}^3\hat{e}_{(a)r}$  are reduced cotriads, which depend only on the two configurational DO  $r_{\bar{a}}$ ).

The point nature of the canonical transformation implies that the old cotriad momenta are linear functionals of the new momenta. The kernel connecting the old and new momenta satisfy elliptic partial differential equations implied by i) the canonicity conditions; ii) the super-momentum constraints  $\mathcal{H}_{(a)}(\tau, \sigma^r) \approx 0$ ; iii) the rotation constraints  $M_{(a)}(\tau, \sigma^r) \approx 0$ .

As already said, the first-class constraints are the generators of the Hamiltonian gauge transformations, under which the ADM action is quasi-invariant (second Noether theorem):

- i) The gauge transformations generated by the four primary constraints  $\pi_N(\tau, \sigma^r) \approx 0, \pi_{\bar{N}(a)}(\tau, \sigma^r) \approx 0$ , modify the lapse and shift functions, namely how densely the simultaneity surfaces are packed in  $M^4$  and which points have the same 3-coordinates on each  $\Sigma_\tau$ .

ii) Those generated by the three super-momentum constraints  $\mathcal{H}_{(a)}(\tau, \sigma^r) \approx 0$  change the 3-coordinates on  $\Sigma_\tau$ .

iii) Those generated by the super-hamiltonian constraint  $\mathcal{H}(\tau, \sigma^r) \approx 0$  transform an admissible 3+1 splitting into another admissible one by realizing a normal deformation of the simultaneity surfaces  $\Sigma_\tau$  [44]. As a consequence, *all the conventions about clock synchronization are gauge equivalent as in special relativity*.

iv) Those generated by  $\pi_{\bar{\varphi}(a)}(\tau, \sigma^r) \approx 0$ ,  $M_{(a)}(\tau, \sigma^r) \approx 0$ , change the cotetrad fields with local Lorentz transformations.

In the rest-frame instant form of tetrad gravity there are the three extra first-class constraints  $P_{ADM}^r \approx 0$  (vanishing of the ADM 3-momentum as *rest-frame conditions*). They generate gauge transformations which change the time-like observer whose world-line is used as origin of the 3-coordinates.

Finally let us see which is the Dirac Hamiltonian  $H_D$  generating the  $\tau$ -evolution in ADM canonical gravity. In *spatially compact space-times without boundary*  $H_D$  is a linear combination of the primary constraints plus the secondary super-hamiltonian and super-momentum constraints multiplied by the lapse and shift functions respectively (consequence of the Legendre transform). As a consequence,  $H_D \approx 0$  and in the reduced phase space we get a vanishing Hamiltonian. This implies the so-called *frozen picture* and the problem of how to reintroduce a temporal evolution<sup>9</sup>. Usually one considers the normal (time-like) deformation of  $\Sigma_\tau$  induced by the super-hamiltonian constraint as an evolution in a local time variable to be identified (the multi-fingered time point of view with a local either extrinsic or intrinsic time): this is the so-called *Wheeler-DeWitt interpretation*<sup>10</sup>.

On the contrary, *in spatially non-compact space-times* the definition of functional derivatives and the existence of a well-posed Hamiltonian action principle (with the possibility of a good control of the surface terms coming from integration by parts) require the addition of the *DeWitt [47] surface term* (living on the surface at spatial infinity) to the Hamiltonian. It can be shown [38] that in the rest-frame instant form this term, together with a surface term coming from the Legendre transformation of the ADM action, leads to the Dirac Hamiltonian

$$H_D = \check{E}_{ADM} + (\text{constraints}) = E_{ADM} + (\text{constraints}) \approx E_{ADM}.$$

Here  $\check{E}_{ADM}$  is the *strong ADM energy*, a surface term analogous to the one defining the electric charge as the flux of the electric field through the surface at spatial infinity in electromagnetism. Since we have  $\check{E}_{ADM} = E_{ADM} + (\text{constraints})$ , we see that the non-vanishing part of the Dirac Hamiltonian is the *weak ADM energy*  $E_{ADM} = \int d^3\sigma \mathcal{E}_{ADM}(\tau, \sigma^r)$ , namely the integral over  $\Sigma_\tau$  of the ADM energy density (in electromagnetism this corresponds to the definition of the electric charge as the volume integral of matter charge density). Therefore there is no frozen picture but a consistent  $\tau$ -evolution.

However, the ADM energy density  $\mathcal{E}_{ADM}(\tau, \sigma^r)$  is a *coordinate-dependent quantity*, because it depends on the gauge variables (namely on the relativistic inertial effects present

<sup>9</sup> See Refs.[45] for the problem of time in general relativity.

<sup>10</sup> Kuchar [46] says that the super-hamiltonian constraint must not be interpreted as a generator of gauge transformations, but as an effective Hamiltonian.

in the non-inertial frame): this is the *problem of energy* in general relativity. Let us remark that in most coordinate systems  $\mathcal{E}_{ADM}(\tau, \sigma^r)$  does not agree with the pseudo-energy density defined in terms of the Landau-Lifschitz pseudo-tensor.

As a consequence, to get a deterministic evolution for the DO <sup>11</sup> we must fix the gauge completely, that is we have to add 14 gauge-fixing constraints satisfying an orbit condition and to pass to Dirac brackets. As already said, the correct way to do it is the following one:

i) Add a gauge-fixing constraint to the secondary super-hamiltonian constraint <sup>12</sup>. This gauge-fixing fixes the form of  $\Sigma_\tau$ , i.e. the convention for the synchronization of clocks. The  $\tau$ -constancy of this gauge-fixing constraint generates a gauge-fixing constraint to the primary constraint  $\pi_N(\tau, \sigma^r) \approx 0$  for the determination of the lapse function. The  $\tau$ -constancy of this new gauge fixing determines the Dirac multiplier in front of the primary constraint.

ii) Add three gauge-fixings to the secondary super-momentum constraints  $\mathcal{H}_{(a)}(\tau, \sigma^r) \approx 0$ . This fixes the 3-coordinates on each  $\Sigma_\tau$ . The  $\tau$ -constancy of these gauge fixings generates the three gauge fixings to the primary constraints  $\pi_{\bar{N}(a)}(\tau, \sigma^r) \approx 0$  and leads to the determination of the shift functions (i.e. of the appearances of gravito-magnetism). The  $\tau$ -constancy of these new gauge fixings determines the Dirac multipliers in front of the three primary constraints.

iii) Add six gauge-fixing constraints to the primary constraints  $\pi_{\bar{\varphi}(a)}(\tau, \sigma^r) \approx 0$ ,  $M_{(a)}(\tau, \sigma^r) \approx 0$ . This is a fixation of the cotetrad field which includes a convention on the choice and the transport of the three gyroscopes of every time-like observer of the two congruences associated to the chosen 3+1 splitting of  $M^4$ . Their  $\tau$ -constancy determines the six Dirac multipliers in front of these primary constraints.

iv) In the rest-frame instant form we must also add three gauge fixings to the rest-frame conditions  $P_{ADM}^r \approx 0$ . The natural ones are obtained with the requirement that the three ADM boosts vanish. In this way we select a special time-like observer as origin of the 3-coordinates (like the Fokker-Pryce center of inertia in special relativity [17]).

In this way all the gauge variables are fixed to be either numerical functions or well determined functions of the DO. As a consequence, in a completely fixed gauge (i.e. in a non-inertial frame centered on a time-like observer and with its pattern of inertial forces, corresponding to an extended physical laboratory with fixed metrological conventions) the ADM energy density  $\mathcal{E}_{ADM}(\tau, \sigma^r)$  becomes *a well defined function only of the DO* and the Hamilton equations for them with  $E_{ADM}$  as Hamiltonian are a hyperbolic system of partial differential equations for their determination. For each choice of Cauchy data for the DO on a  $\Sigma_\tau$ , we obtain a solution of Einstein's equations in the radar 4-coordinate system associated to the chosen 3+1 splitting of  $M^4$ .

A universe  $M^4$  (a *4-geometry*) is the equivalence class of all the completely fixed gauges with gauge equivalent Cauchy data for the DO on the associated Cauchy and simultaneity surfaces  $\Sigma_\tau$ . In each gauge we find the solution for the DO in that gauge (*the tidal effects*) and then the explicit form of the gauge variables (*the inertial effects*). Moreover, *also the extrinsic curvature of the simultaneity surfaces  $\Sigma_\tau$  is determined*. Since the simultaneity surfaces are

<sup>11</sup> See Refs.[48] for the modern formulation of the Cauchy problem for Einstein equations, which mimics the steps of the Hamiltonian formalism.

<sup>12</sup> The special choice  $\pi_\phi(\tau, \sigma^r) \approx 0$  implies that the DO  $r_{\bar{a}}$ ,  $\pi_{\bar{a}}$ , remain canonical even if we do not know how to solve this constraint.

asymptotically flat, it is possible to determine their embeddings  $z^\mu(\tau, \sigma^r)$  in  $M^4$ . As a consequence, differently from special relativity, the conventions for clock synchronization and the whole chrono-geometrical structure of  $M^4$  (gravito-magnetism, 3-geodesic spatial distance on  $\Sigma_\tau$ , trajectories of light rays in each point of  $M^4$ , one-way velocity of light) are *dynamically determined*.

Let us remark that, if we look at Minkowski space-time as a special solution of Einstein's equations with  $r_{\bar{a}}(\tau, \sigma^r) = \pi_{\bar{a}}(\tau, \sigma^r) = 0$  (zero Riemann tensor, no tidal effects, only inertial effects), we find [38] that the dynamically admissible 3+1 splittings (non-inertial frames) must have the simultaneity surfaces  $\Sigma_\tau$  *3-conformally flat*, because the conditions  $r_{\bar{a}}(\tau, \sigma^r) = \pi_{\bar{a}}(\tau, \sigma^r) = 0$  imply the vanishing of the Cotton-York tensor of  $\Sigma_\tau$ . Instead, in special relativity, considered as an autonomous theory, all the non-inertial frames compatible with the Møller conditions are admissible, namely there is much more freedom in the conventions for clock synchronization.

A first application of this formalism [49] has been the determination of post-Minkowskian background-independent gravitational waves in a completely fixed non-harmonic 3-orthogonal gauge with diagonal 3-metric. It can be shown that the requirements  $r_{\bar{a}}(\tau, \sigma^r) \ll 1$ ,  $\pi_{\bar{a}}(\tau, \sigma^r) \ll 1$  lead to a weak field approximation based on a Hamiltonian linearization scheme:

- i) linearize the Lichnerowicz equation, determine the conformal factor of the 3-metric and then the lapse and shift functions;
- ii) find  $E_{ADM}$  in this gauge and disregard all the terms more than quadratic in the DO;
- iii) solve the Hamilton equations for the DO.

In this way we get a solution of linearized Einstein's equations, in which the configurational DO  $r_{\bar{a}}(\tau, \sigma^r)$  play the role of the *two polarizations* of the gravitational wave and we can evaluate the embedding  $z^\mu(\tau, \sigma^r)$  of the simultaneity surfaces of this gauge explicitly.

Let us conclude with some remarks about the interpretation of the space-time 4-manifold in general relativity.

In 1914 Einstein, during his researches for developing general relativity, faced the problem arising from the fact that the requirement of general covariance would involve a threat to the physical objectivity of the points of space-time  $M^4$ , which in classical field theories are usually assumed to have a *well defined individuality*. This led him to formulate the Hole Argument. Assume that  $M^4$  contains a *hole*  $\mathcal{H}$ , that is an open region where all the non-gravitational fields vanish. It is implicitly assumed that the Cauchy surface for Einstein's equations lies outside  $\mathcal{H}$ . Let us consider an active diffeomorphism  $A$  which re-maps the points inside  $\mathcal{H}$ , but is the identity outside  $\mathcal{H}$ . For any point  $x \in \mathcal{H}$  we have  $x \mapsto D_A x \in \mathcal{H}$ . The induced active diffeomorphism on the 4-metric tensor  ${}^4g$ , solution of Einstein's equations, will map it into another solution  $D_A^* {}^4g$  ( $D_A^*$  is a dynamical symmetry of Einstein's equations) defined by  $D_A^* {}^4g(D_A x) = {}^4g(x) \neq D_A^* {}^4g(x)$ . As a consequence, we get two solutions of Einstein's equations with the same Cauchy data outside  $\mathcal{H}$  and it is not clear how to save the identification of the mathematical points of  $M^4$ .

Einstein avoided the problem with the pragmatic *point-coincidence argument*: the only real world-occurrences are the (coordinate-independent) space-time coincidences (like the intersection of two world-lines). However, the problem was reopened by Stachel [50] and

then by Earman and Norton [51] and this opened a rich philosophical debate that is still alive today.

If we insist on the reality of space-time mathematical points independently from the presence of any physical field (the *substantivalist* point of view in philosophy of science), we are in trouble with predictability.

If we say that  ${}^4g$  and  $D_A^* {}^4g$  describe the same universe (the so-called *Leibniz equivalence*), we lose any physical objectivity of the space-time points (the *relationist* point of view).

Stachel [50] suggested that a physical individuation of the point-events of  $M^4$  could be done only by using *four individuating fields depending on the 4-metric on  $M^4$* , namely that a tensor field on  $M^4$  is needed to identify the points of  $M^4$ .

On the other hand, *coordinatization* is the only way to individuate the points *mathematically* since, as stressed by Hermann Weyl [52]: "There is no distinguishing objective property by which one could tell apart one point from all others in a homogeneous space: at this level, fixation of a point is possible only by a *demonstrative act* as indicated by terms like *this* and *there*."

To clarify the situation let us remember that Bergmann and Komar [53] gave a *passive re-interpretation of active diffeomorphisms as metric-dependent coordinate transformations*  $x^\mu \mapsto y^\mu(x, {}^4g(x))$  restricted to the solutions of Einstein's equations (i.e. *on-shell*). It can be shown that on-shell ordinary passive diffeomorphisms and the on-shell Legendre pull-back of Hamiltonian gauge transformations are two (overlapping) dense subsets of this set of on-shell metric-dependent coordinate transformations. Since the Cauchy surface for the Hole Argument lies outside the hole (where the active diffeomorphism is the identity), it follows that the passive re-interpretation of the active diffeomorphism  $D_A^*$  must be an *on-shell Hamiltonian gauge transformation*, so that Leibniz equivalence is identified with gauge equivalence in the sense of Dirac constraint theory ( ${}^4g$  and  $D_A^* {}^4g$  belong to the same gauge orbit).

What remains to be done is to implement Stachel's suggestion according to which the *intrinsic pseudo-coordinates* of Bergmann and Komar [54] should be used as individuating fields. These pseudo-coordinates for  $M^4$  (at least when there are no Killing vectors) are four scalar functions  $F^A[w_\lambda]$ ,  $A, \lambda = 1, \dots, 4$ , of the four eigenvalues  $w_\lambda({}^4g, \partial^4g)$  of the spatial part of the Weyl tensor. Since these eigenvalues can be shown to be in general functions of the 3-metric, of its conjugate canonical momentum (namely of the extrinsic curvature of  $\Sigma_\tau$ ) and of the lapse and shift functions, the pseudo-coordinates are well defined in phase space and can be used as a label for the points of  $M^4$ .

The final step [55] is to implement the individuation of point-events by considering an arbitrary admissible 3+1 splitting of  $M^4$  with a given time-like observer and the associated radar 4-coordinates  $\sigma^A$  and by imposing the following gauge fixings to the secondary super-hamiltonian and super-momentum constraints (the only restriction on the functions  $F^A$  is the orbit condition)

$$\chi^A(\tau, \sigma^r) = \sigma^A - F^A[w_\lambda] \approx 0.$$

In this way we break completely general covariance and we determine the gauge variables  $\xi^r$  and  $\pi_\phi$ . Then the  $\tau$ -constancy of these gauge fixings will produce the gauge fixings determining the lapse and shift functions. After having fixed the Lorentz gauge freedom of

the cotetrads, we arrive at a completely fixed gauge in which, after the transition to Dirac brackets, we get  $\sigma^A \equiv \tilde{F}^A[r_{\bar{a}}(\sigma), \pi_{\bar{a}}(\sigma)]$ , namely that the radar 4-coordinates of a point in  $M_{3+1}^4$ , the copy of  $M^4$  coordinatized with the chosen non-inertial frame, are determined *off-shell* by the four DO of that gauge: in other words *the individuating fields are the genuine tidal effects of the gravitational field*. By varying the functions  $F^A$  we can make an analogous off-shell identification in every other admissible non-inertial frame. The procedure is consistent, because the DO know the whole 3+1 splitting  $M_{3+1}^4$  of  $M^4$ , being functionals not only of the 3-metric on  $\Sigma_\tau$ , but also of its extrinsic curvature.

Some consequences of this identification of the point-events of  $M^4$  are:

1) The space-time  $M^4$  and the gravitational field are essentially the same entity. The presence of matter modifies the solutions of Einstein equations, i.e.  $M^4$ , but does not play any role in this identification. Instead matter is fundamental for establishing a (still lacking) dynamical theory of measurement not using test objects. As a consequence, instead of the dichotomy substantialism/relationism, it seems that this analysis - as a case study limited to the class of space-times dealt with - may offer a new more articulated point of view, which can be named *point structuralism* (see Ref. [56]).

2) The reduced phase space of this model of general relativity is the space of abstract DO (pure tidal effects without inertial effects), which can be thought as four fields on an abstract space-time  $\tilde{M}^4 = \{\text{equivalence class of all the admissible non - inertial frames } M_{3+1}^4 \text{ containing the associated inertial effects}\}$ .

3) Each radar 4-coordinate system of an admissible non-inertial frame  $M_{3+1}^4$  has an associated *non-commutative structure*, determined by the Dirac brackets of the functions  $\tilde{F}^A[r_{\bar{a}}(\sigma), \pi_{\bar{a}}(\sigma)]$  determining the gauge.

4) Conjecture: there should exist privileged Shanmugadhasan canonical bases of phase space, in which the DO (the tidal effects) are also *Bergmann observables* [57], namely coordinate-independent (scalar) tidal effects.

As a final remark, let us note that these results on the identification of point-events are *model dependent*. In spatially compact space-times without boundary, the DO are *constants of the motion* due to the frozen picture. As a consequence, the gauge fixings  $\chi^A(\tau, \sigma^r) \approx 0$  (in particular  $\chi^\tau$ ) cannot be used to rebuild the temporal dimension: probably only the instantaneous 3-space of a 3+1 splitting can be individuated in this way.

## V. FUTURE DEVELOPMENTS.

I will finish with a list of the open problems in canonical metric and tetrad gravity for which there is a concrete hope to be clarified and solved in the near future.

i) A different Shanmugadhasan canonical transformation, adapted only to 10 constraints but allowing the addition of any kind of matter to the rest-frame instant form of tetrad gravity, has been recently found starting from a new parametrization of the 3-metric [58]. This transformation is the first explicit construction of a York map [59], in which the momentum conjugate to the conformal factor (the gauge variable controlling the convention for clock synchronization) is proportional to the trace  ${}^3K(\tau, \vec{\sigma})$  of the extrinsic curvature of the simultaneity surfaces  $\Sigma_\tau$ . Both the tidal variables and the gauge ones can be expressed in terms of

the original variables. The solution of the super-momentum constraints shows the existence of a generalized Gribov ambiguity connected with the gauge group of 3-diffeomorphisms. Also the Hamiltonian interpretation of harmonic gauges is given. Moreover, in a family of completely fixed gauges differing for the convention of clock synchronization, the deterministic Hamilton equations for the tidal variables and for the matter contain relativistic inertial forces determined by  ${}^3K(\tau, \vec{\sigma})$ , which change from attractive to repulsive where the trace change sign. These inertial forces do not have a non-relativistic counterpart (the Newton 3-space is absolute) and could support the proposal of Ref. [60]<sup>13</sup> that *dark matter* can be explained as an *inertial effect*. Also it would be interesting to have some understanding of how is distributed the gravitational energy density in different coordinate systems and how it depends on the convention for clock synchronization. Has this distribution any relevance for the dark energy problem?

ii) The York basis will allow to study the weak-field approximation to the two-body problem in a post-Minkowskian background-independent way by using a Grassmann regularization of the self-energies, following the track of Refs. [25]. The solution of the Lichnerowicz equation would allow to find the expression of *the relativistic Newton and gravito-magnetic action-at-a-distance potentials* between the two bodies (sources, among other effects, of the *Newtonian tidal effects*) and the coupling of the particles to the DO of the gravitational field (the genuine tidal effects) in various radar coordinate systems: it would amount to a re-summation of the  $1/c$  expansions of the Post-Newtonian approximation. Also the relativistic version of the quadrupole formula for the emission of gravitational waves from the binary system could be obtained. Moreover, we could understand better what replaces the spin-2 approximation of gravity on a fixed background space-time in a background-independent scheme and in non-harmonic gauges. Finally one could try to define a relativistic gravitational micro-canonical ensemble generalizing the Newtonian one developed in Ref.[62].

iii) With more general types of matter (relativistic fluids [23], scalar [21] and electromagnetic [12] fields) it should be possible to develop Hamiltonian numerical gravity based on the Shanmugadhasan canonical bases and to study post-Minkowskian approximations based on power expansions in Newton constant. Moreover one should look for strong-field approximations to be used in the gravitational collapse of a ball of fluid.

iv) Find the Hamiltonian formulation of the Newman-Penrose formalism (see Ref.[63]), in particular of the 10 Weyl scalars. Look for the Bergmann observables (the scalar tidal effects) and try to understand *which inertial effects may have a coordinate-independent form and which are intrinsically coordinate-dependent like the ADM energy density*. Look for the existence of a closed Poisson algebra of scalars and for Shanmugadhasan canonical bases incorporating the Bergmann observables, to be used to find new expressions for the super-hamiltonian and super-momentum constraints, hopefully easier to be solved.

v) Find the clock synchronization convention hidden in the Post-Newtonian metric used around the geoid for space navigation and for GPS: it deviates from Einstein's convention at the order  $1/c^2$ . Moreover try to understand which notions of instantaneous 3-space and of simultaneity are implied by the luminosity distance used in astrophysics and cosmology.

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<sup>13</sup> The model proposed in Ref.[60] is too naive as shown by the criticism in Refs.[61].

vi) Find all the admissible 3+1 splittings of Minkowski space-time which avoid the Torre-Varadarajan no-go theorem. Then adapt these 3+1 splittings to tetrad gravity and try to see whether it is possible to arrive at a multi-temporal background- and coordinate- independent quantization of the gravitational field, in which only the Bergmann observables (the scalar tidal effects) are quantized, maybe by using the non-commutative structure associated to each non-inertial system.

vii) Find the special relativistic version of Bell inequalities and the role of the non-local notions of clock synchronization convention and of separation of the relativistic center of mass from relative motions in the problem of entanglement relying on Ref.[20].

## APPENDIX A: DIRAC'S CONSTRAINT THEORY.

Dirac's theory of constraints [64, 65] is needed for the Hamiltonian formulation of special relativistic systems, gauge theories and general relativity, whose configuration description requires the use of singular Lagrangians with a degenerate Hessian matrix. The second Noether theorem is the basic tool to understand the properties of the associated Euler-Lagrange equations [66].

The local invariances (or quasi-invariances) of the singular Lagrangians under local Noether transformations, depending upon arbitrary functions of time or space-time, imply that there are certain arbitrary velocities. The Noether identities propagate this indetermination to other variables in the case of first-class constraints, while they lead to a final elimination of the arbitrariness in the case of the second-class ones. In the case of first class constraints the local Noether transformations are called gauge transformations. As a consequence, when first and second class constraints are present, some variables, the *gauge variables*, remain arbitrary; others are completely determined and can be eliminated; finally the true physical degrees of freedom, satisfying deterministic equations of motion, are the gauge invariant quantities, the *Dirac observables* (DO). One of the basic problems is the separation of these three types of quantities starting from the initial configuration variables. The only known algorithm for realizing this separation are the *Shanmugadhasan canonical transformations* [67, 68]. These transformations are implicitly assumed in the definition of the Faddeev-Popov measure [69] of the path integral and in the BRST method [70].

Let us review some of the main properties of constrained systems.

If a finite-dimensional system with configuration space  $Q$ <sup>14</sup> is described by a singular Lagrangian  $L$ , namely with a degenerate Hessian matrix ( $\det(\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j) = 0$ ), its Euler-Lagrange (EL) equations are in general a mixture of three types of equations

- i) some depending only on the  $q^i$  (holonomic constraints);
- ii) some depending only on  $q^i$  and  $\dot{q}^i$  (Lagrangian, in general non-holonomic, constraints and/or, when non projectable to phase space, intrinsic first order equations of motion violating the so called second order differential equation (SODE) conditions);
- iii) some depending on  $q^i$ ,  $\dot{q}^i$ ,  $\ddot{q}^i$  (genuine second order equations of motion, which however cannot be put in normal form, i.e. solved in the  $\ddot{q}^i$ ).

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<sup>14</sup>  $q^i$ ,  $i=1,\dots,N$ , are local coordinates in a global (assumed to exist for the sake of simplicity) chart of the atlas of  $Q$ ;  $(t, q^i(t))$  is a point in  $R \times Q$ , where  $R$  is the time axis;  $\dot{q}^i(t) = dq^i(t)/dt$

The solutions of the EL equations depend on arbitrary functions of time, namely they are not deterministic.

As shown in Refs.[66, 68], to each null eigenvalue of the Hessian matrix is associated a (quasi-) invariance of the action principle under a set of local Noether transformations depending upon an arbitrary function of time and a certain number of its time derivatives. As a consequence to each null eigenvalue is associated a chain of Noether identities, each one being the time derivative of the previous one, and a undetermined primary generalized velocity function. The Noether identities identify more equations of the types i) and ii), as combinations of the Euler-Lagrange equations and their time derivatives, and identify secondary undetermined velocity (or configuration) functions. In the simplest cases the chains are of two types:

- i) the first class ones, in which the primary and secondary velocity functions remain undetermined and the last identity is a combination of the EL equations and their time derivatives (contracted Bianchi identities, reducing the number of independent equations of motion);
- ii) the second-class ones, in which the last identity determines the originally arbitrary primary and secondary velocity functions.

Ordinary gauge theories are of the first type.

The canonical momenta  $p_i = \partial L / \partial \dot{q}^i$  are not independent: there are relations among them  $\phi_\alpha(q, p) \approx 0$ , called *primary Hamiltonian constraints*, which define a sub-manifold  $\gamma$  of the cotangent space  $T^*Q$ <sup>15</sup>. The canonical Hamiltonian  $H_c(q, p)$  has to be replaced by the Dirac Hamiltonian  $H_D = H_c + \sum_\alpha \lambda_\alpha(t) \phi_\alpha$ , which knows the restriction to the sub-manifold  $\gamma$ . The arbitrary Dirac multipliers  $\lambda_\alpha(t)$  are the phase space analogue of the arbitrary primary velocity functions<sup>16</sup>. The consistency requirement that the primary constraints are preserved in time,  $\partial_t \phi_\alpha = \{\phi_\alpha, H_D\} \approx 0$ , either produces *secondary Hamiltonian constraints* or determines some of the Dirac multipliers. This procedure is repeated for the secondary constraints<sup>17</sup> and so on. At the end there is a final set of constraints  $\chi_a \approx 0$  defining the final sub-manifold  $\bar{\gamma}$  of  $T^*Q$  on which the dynamics is consistently restricted, and a final Dirac Hamiltonian with a reduced set of arbitrary Dirac multipliers describing the remaining indetermination of the time evolution. The constraints are divided into two subgroups:

- i) the *first class* ones  $\chi_m^{(1)} \approx 0$ , having weakly zero Poisson bracket with all constraints and being the generators of the Hamiltonian version of the gauge transformations of the theory (the associated vector fields  $\{., \chi_m^{(1)}\}$  are tangent to  $\bar{\gamma}$ );
- ii) the *second class* ones  $\chi_n^{(2)} \approx 0$  (their number is even) with  $\det \left( \{\chi_{n_1}^{(2)}, \chi_{n_2}^{(2)}\} \right) \neq 0$ , corresponding to pairs of inessential eliminable variables (the associated vector fields are normal to  $\bar{\gamma}$ ).

The solutions of the Hamilton-Dirac equations with the final Dirac Hamiltonian depend on as many arbitrary functions of time as the final undetermined Dirac multipliers. The vec-

<sup>15</sup> The model is defined only on this sub-manifold; one uses the Poisson brackets of  $T^*Q$  in a neighborhood of  $\gamma$  and Dirac's weak equality  $\approx$  means that the equality sign cannot be used inside Poisson brackets

<sup>16</sup> Since there is no canonical form of these velocity functions, it is convenient to identify them with those functions which are mapped into the Dirac multipliers by the Legendre transform. This prescription eliminates every ambiguity in the Legendre transform [68]

<sup>17</sup> This is the *Dirac- Bergmann algorithm*, which can be shown to correspond to the projection to phase space of the Noether identities [66, 68].

tor fields associated with the first class constraints generate a foliation of the sub-manifold  $\bar{\gamma}$ : each leaf (*Hamiltonian gauge orbit*) contains all the configurations which are gauge equivalent and which have to be considered as the same physical configuration (*equivalence class of gauge equivalent configurations*); the canonical Hamiltonian  $H_c$  (if it is not  $H_c \approx 0$ ) generates an evolution which maps one gauge orbit into the others.

Therefore, the physical reduced phase space is obtained: i) by eliminating as many pairs of conjugate variables as second class constraints by means of the so called associated Dirac brackets; ii) by going to the quotient with respect to the foliation (a representative of the reduced phase space can be build by adding as many gauge-fixing constraints as first class ones, so to obtain a set of second class constraints if an orbit condition is satisfied). In general this procedure breaks the original manifest Lorentz covariance in the case of special relativistic systems.

Let us remark that only the primary first class constraints are associated with arbitrary Dirac multipliers.

The natural way to add gauge-fixing constraints when there are secondary first class constraints, is to start giving the gauge fixings to the secondary constraints. The consistency requirement that the gauge fixings be preserved in time will generate the gauge fixings for the primary first class constraints and the time constancy of these new gauge fixings will determine the Dirac multipliers eliminating every residual gauge freedom. The same method holds with chains of first class constraints of any length.

The DO are the *gauge invariant* functions on the reduced phase space, on which there is a deterministic evolution generated by the projection of the canonical Hamiltonian. Therefore, the main problem is to find a (possibly global) Darboux coordinate chart of the reduced phase space, namely a canonical basis of DO.

When there is reparametrization invariance of the original action, the canonical Hamiltonian vanishes and the reduced phase space is said frozen (like it happens in Hamilton-Jacobi theory). When this happens, both kinematics and dynamics are contained in the first class constraints describing the system: these can be interpreted as generalized Hamilton-Jacobi equations [71], so that the DO turn out to be the Jacobi data. When there is a kinematical symmetry group, like the Galileo or Poincaré groups, an evolution may be reintroduced by using the energy generator as Hamiltonian [66].

Now I will delineate the main steps for the determination of the DO in the case in which only *primary first class* constraints  $\phi_\alpha \approx 0$  are present at the Hamiltonian level.

The Euler-Lagrange equations associated with a singular Lagrangian do not determine the gauge part of the extremals. However it cannot be totally arbitrary, but must be compatible with the algebraic properties of the Noether gauge transformations induced by the first class constraints under which the action is either invariant or quasi-invariant as implied by the second Noether theorem. In the Hamiltonian formulation these properties are contained in the structure constants, or functions, of the Poisson brackets of the first-class constraints among themselves [ $\{\phi_\alpha, \phi_\beta\} = C_{\alpha\beta\gamma}\phi_\gamma$ ,  $\{\phi_\alpha, H_c\} = C_{\alpha\beta}\phi_\beta$ ] and the gauge arbitrariness of the trajectories is described by the Dirac multipliers appearing in the Dirac Hamiltonian. In both formulations one has to add extra equations, the either Lagrangian or Hamiltonian *multi-temporal equations* [72], to have a consistent determination of the gauge part of the trajectory. These equations are obtained by rewriting the variables  $q^i(t)$ ,  $p_i(t)$  in the form

$q^i(t, \tau_\alpha)$ ,  $p_i(t, \tau_\alpha)$ , and by assuming that the original t-evolution generated by the Dirac Hamiltonian  $H_D = H_c + \sum_\alpha \lambda_\alpha(t) \phi_\alpha$  is replaced by:

- i) a deterministic t-evolution generated by  $H_c$ ;
- ii) a  $\tau_\alpha$ -evolution (re-absorbing the arbitrary Dirac multipliers  $\lambda_\alpha(t)$ ), for each  $\alpha$ , generated in a suitable way by the first class constraints  $\phi_\alpha$ .

The  $\tau_\alpha$ -dependence of  $q^i$ ,  $p_i$  determined by these multi-temporal (or better multi-parametric) equations, which are integrable due to the first-class property of the constraints, describes their dependence on the gauge orbit, once Cauchy data for the Hamilton-Dirac equations for the DO are given.

When the Poisson brackets of the Hamiltonian first class constraints imply a canonical realization of a Lie algebra, the extra Hamiltonian multi-temporal equations have the first class constraints as Hamiltonians and the time parameters (replacing the Dirac multipliers) are the coordinates of a group manifold for a Lie group whose algebra is the given Lie algebra: they enter in the multi-temporal equations via a set of left invariant vector fields  $Y_\alpha$  on the group manifold [ $Y_\alpha A(q, p) = \{A(q, p), \phi_\alpha\}$ ]. In the ideal case in which the gauge foliation of  $\bar{\gamma}$  is nice, all the leaves (or gauge orbits) are diffeomorphic and, in the simplest case, all of them are diffeomorphic to the group manifold of a Lie group. In this ideal case to rebuild a gauge orbit from one of its points (and therefore to determine the gauge part of the trajectories passing through that point) one needs the Lie equations associated with the given Lie group: *the Hamiltonian multi-temporal equations are generalized Lie equations describing all the gauge orbits simultaneously*. In a generic case this description holds only locally for a set of diffeomorphic orbits, also in the case of systems invariant under diffeomorphisms.

Once one has solved the multi-temporal equations, the next step is the determination of a *Shanmugadhasan canonical transformation* [67, 68]. In the finite dimensional case general theorems [73] connected with the Lie theory of function groups [74] ensure the existence of local canonical transformations from the original canonical variables  $q^i$ ,  $p_i$ , in terms of which the first class constraints (assumed globally defined) have the form  $\phi_\alpha(q, p) \approx 0$ , to canonical bases  $P_\alpha$ ,  $Q_\alpha$ ,  $P_A$ ,  $Q_A$ , such that

- i) the equations  $P_\alpha \approx 0$  locally define the same original constraint manifold (the  $P_\alpha$  are an Abelianization of the first class constraints);
- ii) the  $Q_\alpha$  are the adapted Abelian gauge variables describing the gauge orbits (they are a realization of the times  $\tau_\alpha$  of the multi-temporal equations in terms of variables  $q^i$ ,  $p_i$ );
- iii) the  $Q_A$ ,  $P_A$  are an adapted canonical basis of DO.

Therefore the problem of the search of the DO becomes the problem of finding Shanmugadhasan canonical transformations. The strategy is to find Abelianizations  $P_\alpha$  of the original constraints, to solve the multi-temporal equations for  $q^i$ ,  $p_i$  associated with the  $P_\alpha$ , to determine the multi-times  $Q_\alpha = \tau_\alpha$  and to identify the DO  $P_A$ ,  $Q_A$  from the remaining original variables, i.e. from those their combinations independent from  $P_\alpha$  and  $Q_\alpha$ .

In gauge field theories the situation is more complicated, because the theorems ensuring the existence of the Shanmugadhasan canonical transformation have not been extended to the infinite-dimensional case and one must use heuristic extrapolations of them. One of the reasons is that some of the constraints can now be interpreted as elliptic equations and they can have zero modes. Let us consider the stratum  $\epsilon p^2 > 0$  of free Yang-Mills theory as a prototype and its first class constraints, given by the Gauss laws and by the vanishing of the

time components of the canonical momenta. The problem of the *zero modes* will appear as a *singularity structure of the gauge foliation* of the allowed strata, in particular of the stratum  $\epsilon p^2 > 0$ . This phenomenon was discovered in Ref.[75] by studying the space of solutions of Yang-Mills and Einstein equations, which can be mapped onto the constraint manifold of these theories in their Hamiltonian description. It turns out that the space of solutions has a *cone over cone* structure of singularities: if we have a line of solutions with a certain number of symmetries, in each point of this line there is a cone of solutions with one less symmetry. In the Yang-Mills case the *gauge symmetries* of a gauge potential are connected with the generators of its stability group, i.e. with the subgroup of those special gauge transformations which leave invariant that gauge potential (this is the Gribov ambiguity for gauge potentials; there is also a more general Gribov ambiguity for field strengths, the *gauge copies* problem; for all these problems see Ref. [27] and its bibliography).

Since the Gauss laws are the generators of the gauge transformations (and depend on the chosen gauge potential through the covariant derivative), this means that for a gauge potential with non trivial stability group those combinations of the Gauss laws corresponding to the generators of the stability group cannot be any more first class constraints, since they do not generate effective gauge transformations but special symmetry transformations. This problematic has still to be clarified, but it seems that in this case these components of the Gauss laws become *third class constraints*, which are not generators of true gauge transformations. This new kind of constraints was introduced in Refs.[66, 68] in the finite dimensional case as a result of the study of some examples, in which the Jacobi equations (the linearization of the Euler-Lagrange equations) are singular, i.e. some of their solutions are not infinitesimal deviations between two neighboring extremals of the Euler-Lagrange equations. This interpretation seems to be confirmed by the fact that the singularity structure discovered in Ref.[75] follows from the existence of singularities of the linearized Yang-Mills and Einstein equations. These problems are part of the Gribov ambiguity, which, as a consequence, induces an extremely complicated stratification and also singularities in each Poincaré stratum of  $\bar{\gamma}$ .

Other possible sources of singularities of the gauge foliation of Yang-Mills theory in the stratum  $\epsilon p^2 > 0$  may be:

- i) different classes of gauge potentials identified by different values of the field invariants;
- ii) the orbit structure of the rest frame (or Thomas) spin  $\vec{S}$ , identified by the Pauli-Lubanski Casimir  $W^2 = -\epsilon p^2 \vec{S}^2$  of the Poincaré' group.

The final outcome of this structure of singularities is that the reduced phase-space, i.e. the space of the gauge orbits, is in general a *stratified manifold with singularities* [75]. In the stratum  $\epsilon p^2 > 0$  of the Yang-Mills theory these singularities survive the Wick rotation to the Euclidean formulation and it is not clear how the ordinary path integral approach and the associated BRS method can take them into account (they are zero measure effects). The search of a global canonical basis of DO for each stratum of the space of the gauge orbits can give a definition of the measure of the phase space path integral, but at the price of a non polynomial Hamiltonian. Therefore, if it is not possible to eliminate the Gribov ambiguity (assuming that it is only a mathematical obstruction without any hidden physics), the existence of global DO for Yang-Mills theory is very problematic.

See Ref.[58] for the recently discovered generalized Gribov ambiguity in metric and tetrad gravity arising in the solution of the super-momentum constraints after having done the York map.

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- [1] L.Lusanna, *Towards a Unified Description of the Four Interactions in Terms of Dirac-Bergmann Observables*, invited contribution to the book *Quantum Field Theory: a 20th Century Profile* of the Indian National Science Academy, ed. A.N.Mitra, foreward F.J.Dyson (Hindustan Book Agency, New Delhi, 2000) (hep-th/9907081).
- [2] L.Lusanna, *General Covariance and its Implications for Einstein's Space-Times*, talk at ERE2005 *A Century of Relativistic Physics*, Oviedo 2005 and at QG05 *Constrained Dynamics and Quantum Gravity 05*, Cala Gonone (Sardinia) 2005 (gr-qc/0510024).
- [3] L.Lusanna, *The Chronogeometrical Structure of Special and General Relativity: towards a Background-Independent Description of the Gravitational Field and Elementary Particles* (2004), in *General Relativity Research Trends*, ed. A.Reiner, Horizon in World Physics vol. 249 (Nova Science, New York, 2005) (gr-qc/0404122).
- [4] R.G.Littlejohn and M.Reinsch, *Gauge Fields in the Separation of Rotations and Internal Motions in the N Body Problem*, Rev.Mod.Phys. **69**, 213 (1997).
- [5] D.Alba, L.Lusanna and M.Pauri, *Dynamical Body Frames, Orientation-Shape Variables and Canonical Spin Bases for the Nonrelativistic N-Body Problem* J.Math.Phys. **43**, 373 (2002) (hep-th/0011014).
- [6] G.Longhi, L.Lusanna and J.M.Pons, *On the Many-Time Formulation of Classical Particle Dynamics*, J.Math.Phys. **30**, 1893 (1989).
- [7] R.De Pietri, L.Lusanna and M.Pauri, *Gauging Kinematical and Internal Symmetry Groups for Extended Systems: the Galilean One-Time and Two-Times Harmonic Oscillator*, Class.Quantum Grav. **11**, 1417 (1996).
- [8] R.De Pietri, L.Lusanna and M.Pauri, *Standard and Generalized Newtonian Gravities as 'Gauge' Theories of the Extended Galilei Group., I. The Standard Theory*, Class.Quantum Grav. **12**, 219 (1995); *II. Dynamical 3-Space Theoreies*, Class.Quantum Grav. **12**, 255 (1995).
- [9] D.Alba, *Quantum Mechanics in Non-Inertial Frames with a Multi-Temporal Quantization Scheme: II) Non-Relativistic Particles*, to appear in Int.J.Mod.Phys. (hep-th/0504060).
- [10] D.Alba and L.Lusanna, *Simultaneity, Radar 4-Coordinates and the 3+1 Point of View about Accelerated Observers in Special Relativity* (2003) (gr-qc/0311058); *Generalized Radar 4-Coordinates and Equal-Time Cauchy Surfaces for Arbitrary Accelerated Observers* (2005), submitted to General Relativity and Gravitation (gr-qc/0501090).
- [11] C.Møller, *The Theory of Relativity* (Oxford, Oxford University Press, 1957).
- [12] L.Lusanna, *The N- and 1-Time Classical Description of N-Body Relativistic Kinematics and the Electromagnetic Interaction*, Int. J. Mod. Phys. **A12**, 645-722 (1997).
- [13] L.Blanchet, C.Salomon, P.Teyssandier and P.Wolf, *Relativistic Theory for Time and Frequency Transfer to Order  $1/c^3$* , Astron.Astrophys. **370**, 320 (2000).  
For ACES see C.Salomon et al., *Cold Atoms in Space and Atomic Clocks: ACES*, C.R.Acad.Sci.Paris, t. **2**, Se-rie 4, 1313 (2001) and [www.spaceflight.esa.int/projects/index.cfm?act=default.page](http://www.spaceflight.esa.int/projects/index.cfm?act=default.page).
- [14] P.A.M. Dirac, *Forms of Relativistic Dynamics*, Rev.Mod.Phys. **21**, 392-399 (1949).
- [15] D.Alba, L.Lusanna and M.Pauri, *Centers of Mass and Rotational Kinematics for the Relativistic N-Body Problem in the Rest-Frame Instant Form* J.Math.Phys. **43**, 1677-1727 (2002) (hep-th/0102087).
- [16] D.Alba, L.Lusanna and M.Pauri, *Multipolar Expansions for Closed and Open Systems of*

- Relativistic Particles*, J. Math.Phys. **46**, 062505, 1-36 (2004) (hep-th/0402181).
- [17] D.Alba, L.Lusanna and M.Pauri, *New Directions in Non-Relativistic and Relativistic Rotational and Multipole Kinematics for N-Body and Continuous Systems* (2005), in *Atomic and Molecular Clusters: New Research*, ed.Y.L.Ping (Nova Science, New York, 2006) (hep-th/0505005).
- [18] G.Longhi and L.Lusanna, *Bound-State Solutions, Invariant Scalar Products and Conserved Currents for a Class of Two-Body Relativistic Systems*, Phys.Rev. **D34**, 3707 (1986).
- [19] A.Lucenti, L.Lusanna and M.Pauri, *Dirac Observables and Spin Bases for N Relativistic Particles*, J.Phys. **A31**, 1633 (1998).
- [20] D.Alba and L.Lusanna, *Quantum Mechanics in Non-Inertial Frames with a Multi-Temporal Quantization Scheme: I) Relativistic Particles* (hep-th/0502060), to appear in Int.J.Mod.Phys.
- [21] L.Lusanna and M.Materassi, *A Canonical Decomposition in Collective and Relative Variables of a Klein-Gordon Field in the Rest-Frame Wigner-Covariant Instant Form*, Int.J.Mod.Phys. **A15**, 2821 (2000) (hep-th/9904202).
- [22] F.Bigazzi and L.Lusanna, *Dirac Fields on Spacelike Hypersurfaces, their Rest-Frame Description and Dirac Observables*, Int.J.Mod.Phys. **A14**, 1877 (1999) (hep-th/9807054).
- [23] L. Lusanna and D. Nowak-Szczepaniak, *The Rest-Frame Instant Form of Relativistic Perfect Fluids with Equation of State  $\rho = \rho(\eta, s)$  and of Nondissipative Elastic Materials.*, Int. J. Mod. Phys. **A15**, 4943 (2000).  
D.Alba and L.Lusanna, *Generalized Eulerian Coordinates for Relativistic Fluids: Hamiltonian Rest-Frame Instant Form, Relative Variables, Rotational Kinematics* Int.J.Mod.Phys. **A19**, 3025 (2004) (hep-th/020903).
- [24] P.A.M.Dirac, *Gauge Invariant Formulation of Quantum Electrodynamics*, Can.J.Phys. **33**, 650 (1955).
- [25] H.Crater and L.Lusanna, *The Rest-Frame Darwin Potential from the Lienard-Wiechert Solution in the Radiation Gauge*, Ann.Phys. (N.Y.) **289**, 87-177 (2001) (hep-th/0001046).  
D.Alba, H.Crater and L.Lusanna, *The Semiclassical Relativistic Darwin Potential for Spinning Particles in the Rest-Frame Instant Form: Two-Body Bound States with Spin 1/2 Constituents*, Int.J.Mod.Phys. **A16**, 3365-3477 (2001) (hep-th/0103109).
- [26] F.Colomo, G.Longhi and L.Lusanna, *Classical Solutions of the Many-Time Functional Equations of Motion of the Nambu String*, Int.J.Mod.Phys. **A5**, 3347 (1990); *Classical Canonical Observables of the Nambu String*, Mod.Phys.Letters **A5**, 17 (1990).  
F.Colomo and L.Lusanna, *Wigner Theory of the Nambu String: I. The Open String*, Int.J.Mod.Phys. **A7**, 1705 and *Wigner Theory of the Nambu String: II. The Closed String*, Int.J.Mod.Phys. **A7**, 4107 (1992).
- [27] L.Lusanna, *Classical Yang-Mills Theory with Fermions. I. General Properties of a System with Constraints*, Int.J.Mod.Phys. **A10**, 3531-3579 (1995); *II. Dirac's Observables*, Int.J.Mod.Phys. **A10**, 3675-3757 (1995).
- [28] V.Moncrief, *Gribov Degeneracies: Coulomb Gauge Conditions And Initial Value Constraints*, J.Math.Phys. **20**, 579 (1979).
- [29] M.Cantor, *Elliptic Operators and the Decomposition of Tensor Fields*, Bull.Am.Math.Soc. **5**, 235 (1981); *Some Problems of Global Analysis on Asymptotically Simple Manifolds*, Comp. Math. **38**, 3 (1979); *The Existence of Non-Trivial Asymptotically Flat Initial Data for Vacuum Spacetimes*, Commun.Math.Phys. **57**, 83 (1977).
- [30] D.Alba and L.Lusanna, *The Classical Relativistic Quark Model in the Rest Frame Wigner Covariant Coulomb Gauge*, Int.J.Mod.Phys. **A13**, 3275 (1998).

- [31] L.Lusanna and P.Valtancoli, *Dirac's Observables for for the Higgs Model. I. The Abelian Case*, Int.J.Mod.Phys. **A12**, 4769 (1997) (hep-th/9606078) and *Dirac's Observables for for the Higgs Model. II. The Non-Abelian SU(2) Case*, Int.J.Mod.Phys. **A12**, 4797 (1997). (hep-th/9606079).
- [32] L.Lusanna and P.Valtancoli, *Dirac's Observables for the SU(3) × SU(2) × U(1) Standard Model*, Int.J.Mod.Phys. **A13**, 4605-4690 (1998) (hep-th/9707072).
- [33] Torre, C.G. and Varadarajan, M. *Functional Evolution of Free Quantum Fields*, Clas. Quantum Grav. **16**, 2651-2668 (1999).  
A.D.Helfer, *The Hamiltonian of Linear Quantum Fields* (hep-th/990811).  
A.Arageorgis, J.Earman and L.Ruetsche, *Weyling the Time Away: the Non-Unitary Implementability of Quantum Field Dynamics on Curved Spacetimes*, Studies in History and Philosophy of Modern Physics **33**, 151 (2002).
- [34] J.Norton, *General Covariance and the Foundations of General Relativity: Eight Decades of Dispute*, Rep.Prog.Phys. **56**, 791-858 (1993).
- [35] H.Nicolai, K.Peeters and M.Zamaklar, *Loop Quantum Gravity: an Outside View* (hep-th/0501114).
- [36] L.Smolin *How Far are we from Quantum Gravity?* (hep-th/0303185).
- [37] R.Arnowitz, S.Deser and C.W.Misner, *The Dynamics of General Relativity*, in L. Witten (ed.), *Gravitation: an Introduction to Current Research*, (pp. 227-265) (Wiley, New York, 1962).
- [38] L.Lusanna, *The Rest-Frame Instant Form of Metric Gravity*, Gen.Rel.Grav. **33**, 1579-1696 (2001) (gr-qc/0101048).
- [39] R.M.Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [40] T.Regge and C.Teitelboim, *Role of Surface Integrals in the Hamiltonian Formulation of General Relativity*, Ann.Phys. (N.Y.) **88**, 286-318 (1974).  
R.Beig and N.O'Murchadha, *The Poincare' Group as the Symmetry Group of Canonical General Relativity*, Ann.Phys. (N.Y.) **174**, 463-498 (1987).
- [41] A.Lichnerowicz, *Propagateurs, Commutateurs et Anticommutateurs en Relativite Generale*, in Les Houches 1963, *Relativity, Groups and Topology*, eds. C.DeWitt and B.DeWitt (Gordon and Breach, New York, 1964).  
C.Moreno, *On the Spaces of Positive and Negative Frequency Solutions of the Klein-Gordon Equation in Curved Space-Times*, Rep.Math.Phys. **17**, 333 (1980).
- [42] D.Christodoulou and S.Klainerman, *The Global Nonlinear Stability of the Minkowski Space*. (Princeton University Press, Princeton, 1993).
- [43] L.Lusanna and S.Russo, *A New Parametrization for Tetrad Gravity*, Gen.Rel.Grav. **34**, 189-242 (2002) (gr-qc/0102074).  
R.De Pietri, L.Lusanna, L.Martucci and S.Russo, *Dirac's Observables for the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge*, Gen.Rel.Grav. **34**, 877-1033 (2002) (gr-qc/0105084).
- [44] C.Teitelboim, *The Hamiltonian Structure of Spacetime*, in *General Relativity and Gravitation*, ed. A.Held, vol.I (Plenum, New York, 1980).
- [45] C.J.Isham, *Canonical Quantum Gravity and the Problem of Time*, in *Integrable Systems, Quantum Groups and Quantum Field Theories*, eds.L.A.Ibort and M.A.Rodriguez, Salamanca 1993 (Kluwer, London, 1993).  
K.Kuchar, *Time and Interpretations of Quantum Gravity*, in Proc.4th Canadian Conf. on *General Relativity and Relativistic Astrophysics*, eds. G.Kunstatter, D.Vincent and J.Williams (World Scientific, Singapore, 1992).

- [46] K.Kuchar, *Canonical Quantum Gravity*, in G.Kunstatter, D.E.Vincent, and J.G.Williams (eds.), *Cordoba 1992, General Relativity and Gravitation*, (pp.119-150) (gr-qc/9304012).
- [47] B.DeWitt, *Quantum Theory of Gravity, I) The Canonical Theory*, Phys.Rev. **160**, 1113-1148 (1967); *II) The Manifestly Covariant Theory*, **162**, 1195-1239 (1967).
- [48] A. Rendall, *Local and Global Existence Theorems for the Einstein Equations*, Online Journal Living Reviews in Relativity **1**, n. 4; *ibid.* (2000) **3**, n. 1; (gr-qc/0001008).  
H. Friedrich and A. Rendall, *The Cauchy Problem for Einstein Equations*, in B.G.Schmidt (ed.), *Einstein's Field Equations and their Physical Interpretation* (Berlin, Springer, 2000) (gr-qc/0002074).  
M.Sanchez, *Cauchy Hypersurfaces and Global Lorentzian Geometry*, (math.DG/0604265).
- [49] J.Agresti, R.De Pietri, L.Lusanna and L.Martucci, *Hamiltonian Linearization of the Rest-Frame Instant Form of Tetrad Gravity in a Completely Fixed 3-Orthogonal Gauge: a Radiation Gauge for Background-Independent Gravitational Waves in a Post-Minkowskian Einstein Space-Time*, Gen.Rel.Grav. **36**, 1055-1134 (2004) (gr-qc/0302084).
- [50] J.Stachel, *Einstein's Search for General Covariance, 1912-1915*. Ninth International Conference on General Relativity and Gravitation, Jena (1980), ed.E.Schmutzner (Cambridge Univ. Press, Cambridge, 1983).
- [51] J.Earman and J.Norton, *What Price Spacetime Substantivalism? The Hole Story*, British Journal for the Philosophy of Science **38**, 515-525 (1987).
- [52] H.Weyl, *Groups, Klein's Erlangen Program*. Quantities, ch.I, sec.4 of *The Classical Groups, their Invariants and Representations*, 2nd ed., (pp.16) (Princeton University Press, Princeton, 1946).
- [53] P.G.Bergmann and A.Komar, *The Coordinate Group Symmetries of General Relativity*, Int.Theor.Phys., **5**, 15-28 (1972).
- [54] P.G.Bergmann and A.Komar, *Poisson Brackets between Locally Defined Observables in General Relativity*, Phys.Rev.Lett. **4**, 432-433 (1960).  
P.G.Bergmann, *The General Theory of Relativity*, in S.Flugge (ed.), *Handbuch der Physik*, Vol. IV, *Principles of Electrodynamics and Relativity*, (pp. 247-272) (Springer, Berlin, 1962).
- [55] L.Lusanna and M.Pauri, *Explaining Leibniz-Equivalence away: Dis-solution of the Hole Argument and Physical Individuation of Point-Events*, talk at the Oxford Conference on Space-time Theory (2004), to appear in *Studies in History and Philosophy of Modern Physics* (gr-qc/0604087); *The Physical Role of Gravitational and Gauge Degrees of Freedom in General Relativity. I: Dynamical Synchronization and Generalized Inertial Effects; II: Dirac versus Bergmann Observables and the Objectivity of Space-Time*, Gen.Rel.Grav. **38**, 187 and 229 (2006) (gr-qc/0403081 and 0407007).
- [56] M.Dorato and M.Pauri, *Holism and Structuralism in Classical and Quantum General Relativity*, Pittsburgh-Archive, ID code 1606, forthcoming in (2006). S.French and D.Rickles (eds.), *Structural Foundations of Quantum Gravity* (Oxford University Press, Oxford, 2004).
- [57] P.G.Bergmann, *Observables in General Relativity*, Rev.Mod.Phys. **33**, 510-514 (1961).
- [58] D.Alba and L.Lusanna, *The York Map as a Shanmugadhasan Canonical Transformation in Tetrad Gravity and the Role of Non-Inertial Frames in the Geometrical View of the Gravitational Field* (gr-qc/0604086).
- [59] J.Isemberg and J.E.Marsden, *The York map is a Canonical Transformation*, J.Gem.Phys. **1**,85 (1984).
- [60] F.I.Cooperstock and S.Tieu, *General Relativity resolves Galactic Rotation Curves without Exotic Dark Matter* (astro-ph/0507619) and *Perspectives on Galactic Dynamics via General*

- Relativity* (astro-ph/0512048).
- [61] M.Korzynski, *Singular Disk of Matter in the Cooperstock-Tieu Galaxy Model*, (astro-ph/0508377).  
D.Vogt, *Presence of Exotic Matter in the Cooperstock and Tieu Galaxy Model* (astro-ph/0510750).  
D.Garfinkle, *The Need for Dark Matter in Galaxies*, (astro-ph/0511082).  
D.J.Cross, *Comments on the Cooperstock-Tieu Galaxy Model*, (astro-ph/0601191).
- [62] E.V.Votyakov, H.I.Hidmi, A.De Martino and D.H.E.Gross, *Microcanonical Mean-Field Thermodynamics of Self-Gravitating and Rotating Systems*, Phys.Rev.Lett. **89**, 031101 (2002) (cond-mat/0202140).
- [63] J.Stewart, *Advanced General Relativity* (Cambridge University Press, Cambridge, 1993).
- [64] P.A.M.Dirac, *Generalized Hamiltonian Dynamics*, Can.J.Math. **2**, 129 (1950); *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Monographs Series (Yeshiva University, New York, N.Y., 1964).
- [65] M.Henneaux and C.Teitelboim, *Quantization of Gauge Systems* (Princeton Univ. Press, Princeton, 1992).
- [66] L.Lusanna, *The Second Noether Theorem as the Basis of the Theory of Singular Lagrangians and Hamiltonian Constraints*, Riv. Nuovo Cimento **14**, n.3, 1 (1991).  
L.Lusanna, *An Enlarged Phase Space for Finite-Dimensional Constrained Systems, Unifying their Lagrangian, Phase- and Velocity- Space Descriptions*, Phys.Rep. **185**, 1 (1990).  
M.Chaichian, D.Louis Martinez and L.Lusanna, *Dirac's Constrained Systems: the Classification of Second Class Constraints*, Ann.Phys.(N.Y.)**232**, 40 (1994).
- [67] S.Shanmugadhasan, *Canonical Formalism for Degenerate Lagrangians*, J.Math.Phys. **14**, 677 (1973).
- [68] L.Lusanna, *The Shanmugadhasan Canonical Transformation, Function Groups and the Extended Second Noether Theorem*, Int.J.Mod.Phys. **A8**, 4193 (1993).
- [69] L.D.Faddeev and V.N.Popov, *Feynman Diagrams for the Yang-Mills Field*, Phys.Lett. **B25**, 30 (1967).
- [70] M.Henneaux, *Hamiltonian Form of the Path Integral for Theories with a Gauge Freedom*, Phys.Rep. **126**, 1 (1985).
- [71] D.Dominici, J.Gomis, G.Longhi and J.M.Pons, *Hamilton-Jacobi Theory for Constrained Systems*, J.Math.Phys. **25**, 2439 (1984).
- [72] L.Lusanna, *Lagrangian and Hamiltonian Many-Times Equations*, J.Math.Phys. **31**, 2126 (1990).
- [73] J.A.Schouten and W.V.D.Kulk, *Pfaff's Problem and Its Generalizations* (Clarendon, Oxford, 1949).
- [74] S.Lie, *Theorie der Transformation Gruppe*, Vol. II (B.G.Teubner, Leipzig, 1890).  
A.R.Forsyth, *Theory of Differential Equations*, Vol. V, Ch. IX (Dover, New York, 1959).  
L.P.Eisenhart, *Continuous Groups of Transformations* (Dover, New York, 1961).  
R.O.Fulp and J.A.Marlin, *Integrals of Foliations on Manifolds with a Generalized Symplectic Structure.*, Pacific J. Math. **67**, 373 (1976); *Function Groups Associated with Constraint Submanifolds*, Rep.Math.Phys. **18**, 295 (1980).
- [75] J.M.Arms, J.E.Marsden and V.Moncrief, *Symmetry and Bifurcations of Momentum Mappings*, Commun.Math.Phys. **78**, 455 (1981).  
J.M.Arms, *Symmetry and Solution Set Singularities in Hamiltonian Field Theories*, Acta Phys.Pol. **B17**, 499 (1986).

L.Bos and M.J.Gotay, *Singular Angular Momentum Mappings*, J.Diff.Geom. **24**, 181 (1986).