Status of NNLO Jet Calculations

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The perturbative description of jet observables is currently possible only to next-to-leading order (NLO) in perturbation theory. This level of accuracy turns out to be insufficient for several precision observables, thus limiting the physics interpretation of experimental jet production data. We discuss the current status of computations of next-to-next-to-leading order (NNLO) corrections to jet production.

1. INTRODUCTION

Jet production observables provide an important tool to study many aspects of the theory of quantum chromodynamics (QCD). They are used for a precise extraction of the strong coupling constant [1], as a constraint in global fits of parton distribution functions, and give many insights into the interplay of perturbative and non-perturbative physics.

At present, perturbative calculations of jet observables are available [2] to next-to-leading order (NLO) for up to $e^+e^- \to 4j$, $ep \to (3+1)j$ and $pp \to 3j$. All experimental data on these observables are in good agreement with the NLO QCD predictions. For many jet production observables, the experimental data have become so precise that they can be used as competitive determination of the strong coupling constant α_s. Using experimental data on $e^+e^- \to 3j$, $ep \to (2+1)j$ and $p\bar{p} \to 2j$ to extract α_s, one encounters several sources of uncertainty: while the statistical error on the data is negligible, systematical errors can be sizable. These arise from different sources: experimental jet energy scale (irrelevant in $e^+e^- \to 3j$), parton distribution functions (absent in $e^+e^- \to 3j$), hadronisation corrections and renormalisation/factorisation scale uncertainty (estimating the effect of perturbative terms beyond NLO). While the size of individual error estimates is clearly debatable, it is however clear that the scale uncertainty is a sizable, if not dominant, source of error in extractions of α_s from jet production at colliders. Therefore, competitive precision studies of QCD from jet production reactions in $2 \to 2$ and $1 \to 3$ kinematics clearly require next-to-next-to-leading order (NNLO) corrections.

Besides lowering this scale uncertainty in precision observables, one expects several further benefits from NNLO calculations. In leading order calculations of n-jet production, the final state is described by only n partons, and thus not sensitive on the jet clustering procedure (jet algorithm). NLO contains already $(n+1)$ partons, while NNLO goes up to $(n+2)$ partons. The higher perturbative orders do therefore start to resolve the precise details of the jet algorithm and allow for a better matching of theoretical and experimental implementation of it.

Using jet observables to constrain parton distribution functions (especially the gluon distribution), one is equally limited by the perturbative order of the calculation. While inclusive structure functions and splitting functions are known to NNLO [3], global fits including jet production data are currently restricted to NLO.

Confronting calculations of power corrections to jet observables with experimental data, one is also facing the issue of higher order perturbative corrections. Although these perturbative corrections depend on the collision energy (or the scale set by the final state jets) like some high power of the strong coupling constant, they are not easily distinguishable from genuine power corrections. Higher order perturbative corrections may therefore have a considerable influence on the extraction of genuine power correction effects from data.

In the following, we will outline the structure of NNLO calculations of jet production observables, describe their most important ingredients, review the current status of the NNLO calculation of $e^+e^- \to 3$ jets and provide an outlook on other observables.

Table I: The partonic channels contributing to $e^+e^- \to 3$ jets.

LO	$\gamma^* \to q\,\bar{q}g$	tree level
NLO	$\gamma^* \to q\,\bar{q}g$	one loop
	$\gamma^* \to q\,\bar{q}\,gg$	tree level
	$\gamma^* \to q\,\bar{q}\,q\bar{q}$	tree level
NNLO	$\gamma^* \to q\,\bar{q}g$	two loop
	$\gamma^* \to q\,\bar{q}\,gg$	one loop
	$\gamma^* \to q\,\bar{q}\,q\,\bar{q}$	one loop
	$\gamma^* \to q\,\bar{q}\,q\,\bar{q}\,g$	tree level
	$\gamma^* \to q\,\bar{q}\,g\,g\,g$	tree level

2. STRUCTURE OF NNLO CALCULATIONS

For an n-jet observable, several ingredients are required to obtain predictions accurate to NNLO: the two-loop n-parton matrix elements, the one-loop (n+1)-parton matrix elements and the tree level (n+2)-parton matrix elements. To illustrate the different processes contributing at a given order, we list them for $e^+e^- \to 3$ jets in Table I.

All these contributions contain infrared singularities, which are either explicit poles (in the two-loop n-parton matrix elements), or implicit poles (in the tree-level $(n+2)$-parton matrix elements), which become explicit only after integration over the phase space appropriate to n-jet final states, or a combination of both (in the one-loop $(n+1)$-parton matrix elements).

The definition of any n-jet-type observable acts differently on the n-parton, $(n+1)$-parton and $(n+2)$-parton channels. Therefore, each channel has to be computed separately, and the cancellation of infrared poles takes place only when adding the contributions from all channels.

In the following we explain the major challenges encountered in the computation of the purely virtual two-loop corrections to n-parton processes and in the implementation of the $n+1$ and $n+2$ parton processes.

2.1. Virtual Two-Loop Corrections

Enormous progress has been made in the calculation of two-loop $2 \to 2$ and $1 \to 3$ QCD matrix elements, which are now known for all massless parton–parton scattering processes [4] relevant to hadron colliders as well as for $\gamma^* \to q\bar{q}g$ [5] and its crossings [6].

These calculations became tractable owing to various technical developments. In particular, the systematic application of integration-by-parts [7–9] and Lorentz invariance [10] identities allowed the large number of Feynman integrals appearing in two-loop four-point matrix elements to be reduced to a small number of so-called master integrals. The master integrals relevant in the context of $2 \to 2$ and $1 \to 3$ processes are massless four-point functions with either all legs on-shell or three legs on-shell and one leg off-shell, computed earlier in [11].

2.2. Real Radiation at Tree-Level and One-Loop

The one-loop corrections to $2 \to 3$ and $1 \to 4$ matrix elements have been known for longer [12, 13] and form part of NLO calculations of the respective multi-jet observables [2, 14]. These NLO matrix elements naturally contribute to NNLO jet observables of lower multiplicity if one of the partons involved becomes unresolved (soft or collinear) [15]. In these cases, the infrared singular parts of the matrix elements need to be extracted and integrated over the phase space appropriate to the unresolved configuration to make the infrared pole structure explicit. Methods for the extraction of soft and collinear limits of one-loop matrix elements are worked out in detail in the literature [15–20].

As a final ingredient, the tree level $2 \to 4$ and $1 \to 5$ processes also contribute to $(2 \to 2)$- and $(1 \to 3)$-type jet observables at NNLO. These contain double real radiation singularities corresponding to two partons becoming simultaneously soft and/or collinear [21–24]. To determine the contribution to NNLO jet observables from these configurations, one has to find two-parton subtraction terms which coincide with the full matrix element and are still sufficiently simple to be integrated analytically in order to cancel their infrared pole structure with the two-loop virtual and the one-loop single-unresolved contributions. In the past, such configurations were only dealt with on a case-by-case basis in the context of specific calculations [21, 25–28], while no general method was available. Several methods have been proposed recently to accomplish this task [29–33]. Up to now, only one method has been fully worked through for observables of physical interest: the sector decomposition algorithm [34, 35]. In this method, both phase space and loop integrals are analytically decomposed into their Laurent expansion in dimensional regularisation, and the coefficients of the expansion are numerically integrated. Results have been obtained for $e^+e^- \to 2j$ [36], $pp \to H + X$ [37], muon decay [38] and $pp \to W + X$ [39] at NNLO. In contrast to all other approaches, in the sector decomposition method one does not have to integrate the subtraction term analytically.

In the following section, we present a new systematic method, named antenna subtraction, to construct NNLO subtraction terms. This method has been worked out in detail in [40].

3. ANTENNA SUBTRACTION METHOD

In electron-positron annihilation, an m-jet cross section at NLO is obtained by summing contributions from $(m+1)$-parton tree level and m-parton one-loop processes:

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^R_{NLO} - \mathrm{d}\sigma^S_{NLO}\right) + \left[\int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^S_{NLO} + \int_{\mathrm{d}\Phi_m} \mathrm{d}\sigma^V_{NLO}\right].$$

The cross section $\mathrm{d}\sigma^R_{NLO}$ is the $(m+1)$-parton tree-level cross section, while $\mathrm{d}\sigma^V_{NLO}$ is the one-loop virtual correction to the m-parton Born cross section $\mathrm{d}\sigma^B$. Both contain infrared singularities, which are explicit poles in $1/\epsilon$ in $\mathrm{d}\sigma^V_{NLO}$, while becoming explicit in $\mathrm{d}\sigma^R_{NLO}$ only after integration over the phase space. In general, this integration involves the (often iterative) definition of the jet observable, such that an analytic integration is not feasible (and also not appropriate). Instead, one would like to have a flexible method that can be easily adapted to different jet observables or jet definitions. Therefore, the infrared singularities of the real radiation contributions should be extracted using infrared subtraction terms. One introduces $\mathrm{d}\sigma^S_{NLO}$, which is a counter-term for $\mathrm{d}\sigma^R_{NLO}$, having the same unintegrated singular behaviour as $\mathrm{d}\sigma^R_{NLO}$ in all appropriate limits. Their difference is free of divergences and can be integrated over the $(m+1)$-parton phase space numerically. The subtraction term $\mathrm{d}\sigma^S_{NLO}$ has to be integrated analytically over all singular regions of the $(m+1)$-parton phase space. The resulting cross section added to the virtual contribution yields an infrared finite result.

The basic idea of the antenna subtraction approach at NLO is to construct the subtraction term [41] $\mathrm{d}\sigma^S_{NLO}$ from antenna functions. Each antenna function encapsulates all singular limits due to the emission of one unresolved parton between two colour-connected hard partons (tree-level three-parton antenna function). This construction exploits the universal factorisation of phase space and squared matrix elements in all unresolved limits, depicted in Figure 1. The individual antenna functions are obtained by normalising three-parton tree-level matrix elements to the corresponding two-parton tree-level matrix elements.

At NNLO, the m-jet production is induced by final states containing up to $(m+2)$ partons, including the one-loop virtual corrections to $(m+1)$-parton final states. As at NLO, one has to introduce subtraction terms for the $(m+1)$- and $(m+2)$-parton contributions. Schematically the NNLO m-jet cross section reads,

$$\begin{aligned}\mathrm{d}\sigma_{NNLO} &= \int_{\mathrm{d}\Phi_{m+2}} \left(\mathrm{d}\sigma^R_{NNLO} - \mathrm{d}\sigma^S_{NNLO}\right) + \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\sigma^S_{NNLO} \\ &\quad + \int_{\mathrm{d}\Phi_{m+1}} \left(\mathrm{d}\sigma^{V,1}_{NNLO} - \mathrm{d}\sigma^{VS,1}_{NNLO}\right) + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{VS,1}_{NNLO}\end{aligned}$$

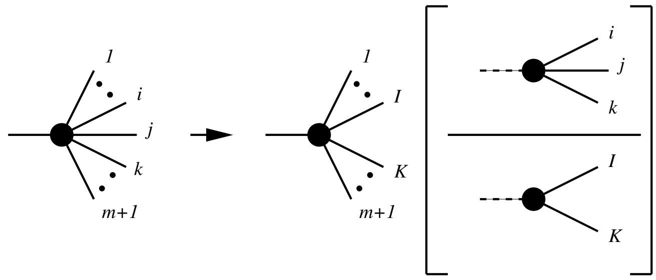

Figure 1: Illustration of NLO antenna factorisation representing the factorisation of both the squared matrix elements and the $(m+1)$-particle phase space. The term in square brackets represents both the antenna function and the antenna phase space.

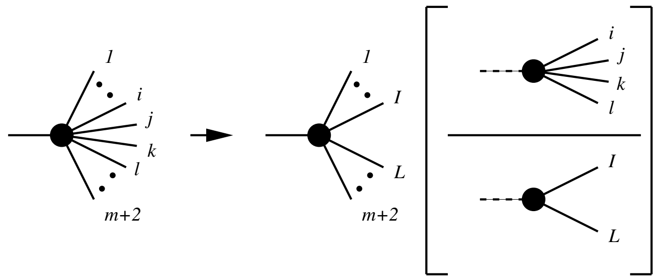

Figure 2: Illustration of NNLO antenna factorisation representing the factorisation of both the squared matrix elements and the $(m+2)$-particle phase space when the unresolved particles are colour connected. The term in square brackets represents both the antenna function and the antenna phase space.

$$+\int_{\mathrm{d}\Phi_m} \mathrm{d}\sigma^{V,2}_{NNLO} \; ,$$

where $\mathrm{d}\sigma^{S}_{NNLO}$ denotes the real radiation subtraction term coinciding with the $(m+2)$-parton tree level cross section $\mathrm{d}\sigma^{R}_{NNLO}$ in all singular limits [21–24]. Likewise, $\mathrm{d}\sigma^{VS,1}_{NNLO}$ is the one-loop virtual subtraction term coinciding with the one-loop $(m+1)$-parton cross section $\mathrm{d}\sigma^{V,1}_{NNLO}$ in all singular limits [15]. Finally, the two-loop correction to the m-parton cross section is denoted by $\mathrm{d}\sigma^{V,2}_{NNLO}$.

Both types of NNLO subtraction terms can be constructed from antenna functions. In $\mathrm{d}\sigma^{S}_{NNLO}$, we have to distinguish four different types of unresolved configurations: (a) One unresolved parton but the experimental observable selects only m jets; (b) Two colour-connected unresolved partons (colour-connected); (c) Two unresolved partons that are not colour connected but share a common radiator (almost colour-unconnected); (d) Two unresolved partons that are well separated from each other in the colour chain (colour-unconnected). Among those, configuration (a) is properly accounted for by a single tree-level three-parton antenna function like used already at NLO. Configuration (b) requires a tree-level four-parton antenna function (two unresolved partons emitted between a pair of hard partons) as shown in Figure 2, while (c) and (d) are accounted for by products of two tree-level three-parton antenna functions.

In single unresolved limits, the one-loop cross section $\mathrm{d}\sigma^{V,1}_{NNLO}$ is described by the sum of two terms [15]: a tree-level splitting function times a one-loop cross section and a one-loop splitting function times a tree-level cross section. Consequently, the one-loop single unresolved subtraction term $\mathrm{d}\sigma^{VS,1}_{NNLO}$ is constructed from tree-level and one-loop three-parton antenna functions, as sketched in Figure 3. Several other terms in $\mathrm{d}\sigma^{VS,1}_{NNLO}$ cancel with the results from the integration of terms in the double real radiation subtraction term $\mathrm{d}\sigma^{S}_{NNLO}$ over the phase space appropriate to one of the unresolved partons, thus ensuring the cancellation of all explicit infrared poles in the difference $\mathrm{d}\sigma^{V,1}_{NNLO} - \mathrm{d}\sigma^{VS,1}_{NNLO}$.

Finally, all remaining terms in $\mathrm{d}\sigma^{S}_{NNLO}$ and $\mathrm{d}\sigma^{VS,1}_{NNLO}$ have to be integrated over the four-parton and three-parton antenna phase spaces. After integration, the infrared poles are rendered explicit and cancel with the infrared pole

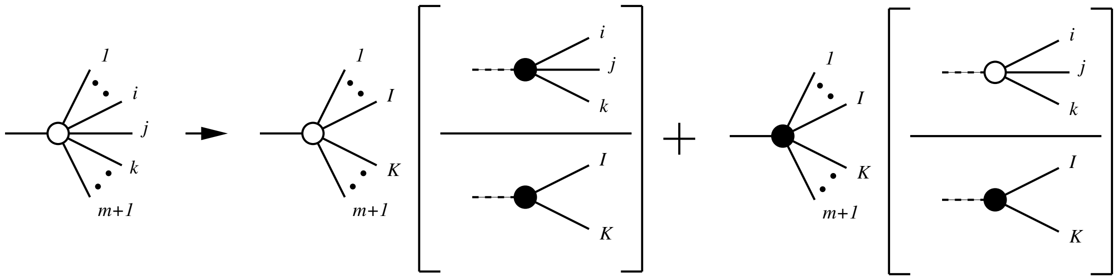

Figure 3: Illustration of NNLO antenna factorisation representing the factorisation of both the one-loop "squared" matrix elements (represented by the white blob) and the $(m+1)$-particle phase space when the unresolved particles are colour connected.

terms in the two-loop squared matrix element $\mathrm{d}\sigma^{V,2}_{NNLO}$.

The subtraction terms $\mathrm{d}\sigma^{S}_{NLO}$, $\mathrm{d}\sigma^{S}_{NNLO}$ and $\mathrm{d}\sigma^{VS,1}_{NNLO}$ require three different types of antenna functions corresponding to the different pairs of hard partons forming the antenna: quark-antiquark, quark-gluon and gluon-gluon antenna functions. In the past [14, 42], NLO antenna functions were constructed by imposing definite properties in all single unresolved limits (two collinear limits and one soft limit for each antenna). This procedure turns out to be impractical at NNLO, where each antenna function must have definite behaviours in a large number of single and double unresolved limits. Instead, we derive these antenna functions in a systematic manner from physical matrix elements known to possess the correct limits. The quark-antiquark antenna functions can be obtained directly from the $e^+e^- \to 2j$ real radiation corrections at NLO and NNLO [43]. For quark-gluon and gluon-gluon antenna functions, effective Lagrangians are used to obtain tree-level processes yielding a quark-gluon or gluon-gluon final state. The antenna functions are then obtained from the real radiation corrections to these processes. Quark-gluon antenna functions were derived [44] from the purely QCD (i.e. non-supersymmetric) NLO and NNLO corrections to the decay of a heavy neutralino into a massless gluino plus partons [45], while gluon-gluon antenna functions [46] result from the QCD corrections to Higgs boson decay into partons [47].

All tree-level three-parton and four-parton antenna functions and three-parton one-loop antenna functions are listed in [40], where we also provide their integrated forms, obtained using the phase space integration techniques described in [35].

4. PRESENT STATUS OF $e^+e^- \to 3$ JETS

In [40, 48] we derived the $1/N^2$-contribution to the NNLO corrections to $e^+e^- \to 3$ jets. This colour factor receives contributions from $\gamma^* \to q\bar{q}ggg$ and $\gamma^* \to q\bar{q}q\bar{q}g$ at tree-level [49], $\gamma^* \to q\bar{q}gg$ and $\gamma^* \to q\bar{q}q\bar{q}$ at one-loop [13] and $\gamma^* \to q\bar{q}g$ at two-loops [5]. The four-parton and five-parton final states contain infrared singularities, which need to be extracted using the antenna subtraction formalism.

In this contribution, all gluons are effectively photon-like, and couple only to the quarks, but not to each other. Consequently, only quark-antiquark antenna functions appear in the construction of the subtraction terms.

Starting from the program `EERAD2` [14], which computes the four-jet production at NLO, we implemented the NNLO antenna subtraction method for the $1/N^2$ colour factor contribution to $e^+e^- \to 3j$. `EERAD2` already contains the five-parton and four-parton matrix elements relevant here, as well as the NLO-type subtraction terms.

The implementation contains three channels, classified by their partonic multiplicity: (a) in the five-parton channel, we integrate $\mathrm{d}\sigma^{R}_{NNLO} - \mathrm{d}\sigma^{S}_{NNLO}$; (b) in the four-parton channel, we integrate $\mathrm{d}\sigma^{V,1}_{NNLO} - \mathrm{d}\sigma^{VS,1}_{NNLO}$; (c) in the three-parton channel, we integrate $\mathrm{d}\sigma^{V,2}_{NNLO} + \mathrm{d}\sigma^{S}_{NNLO} + \mathrm{d}\sigma^{VS,1}_{NNLO}$. The numerical integration over these channels is carried out by Monte Carlo methods.

By construction, the integrands in the four-parton and three-parton channel are free of explicit infrared poles. In the five-parton and four-parton channel, we tested the proper implementation of the subtraction by generating trajectories of phase space points approaching a given single or double unresolved limit. Along these trajectories, we observe that the antenna subtraction terms converge locally towards the physical matrix elements, and that the cancellations among individual contributions to the subtraction terms take place as expected. Moreover, we checked the correctness of the subtraction by introducing a lower cut (slicing parameter) on the phase space variables, and observing that our results are independent of this cut (provided it is chosen small enough). This behaviour indicates that the subtraction terms ensure that the contribution of potentially singular regions of the final state phase space does not contribute to the numerical integrals, but is accounted for analytically.

As a final point, we noted in [40] that the infrared poles of the two-loop (including one-loop times one-loop) correction to $\gamma^* \to q\bar{q}g$ are cancelled in all colour factors by a combination of integrated three-parton and four-parton antenna functions. This highly non-trivial cancellation clearly illustrates that the antenna functions derived here correctly approximate QCD matrix elements in all infrared singular limits at NNLO. They also outline the structure of infrared cancellations in $e^+e^- \to 3j$ at NNLO, and indicate the structure of the subtraction terms in all colour factors.

5. OUTLOOK

In this talk, we discussed the theoretical prerequisites for performing precision QCD studies on jet production data at colliders, with focus on $e^+e^- \to 3j$. In particular, the precise extraction of the strong coupling constant α_s requires improved theoretical predictions to reduce the scale error inherent to calculations in perturbative QCD. At present, this extraction relies on the calculation of $e^+e^- \to 3$ jets at NLO accuracy, and we reported on progress towards the NNLO calculation.

This calculation requires a new method for the subtraction of infrared singularities which we call antenna subtraction. We introduced subtraction terms for double real radiation at tree level and single real radiation at one loop based on antenna functions. These antenna functions describe the colour-ordered radiation of unresolved partons between a pair of hard (radiator) partons. All antenna functions at NLO and NNLO can be derived systematically from physical matrix elements.

An immediate application of the method presented here is the calculation of the full NNLO corrections to $e^+e^- \to 3$ jets [50]. In total, this requires six colour factors: N^2, N^0, $1/N^2$, $N_F\,N$, N_F/N and N_F^2. Compared to the already computed $1/N^2$ colour factor, several new complications arise. While the coefficient of $1/N^2$ was free from angular correlations between the subtraction terms, most other colour factors contain these correlations, thus making the subtraction terms non-local. Also, at $1/N^2$, only the quark-antiquark system could act as an emitter, while radiation off quark-gluon and gluon-gluon systems is present in the other colour factors. At present, we have constructed the analytical form of the subtraction terms for all remaining colour factors, they are implemented and tested for N_F/N and N_F^2 now.

The antenna subtraction method can be further generalised to NNLO corrections to jet production in lepton-hadron or hadron-hadron collisions. In these kinematical situations, the subtraction terms are constructed using the same antenna functions, but in different phase space configurations: instead of the $1 \to n$ decay kinematics considered here, $2 \to n$ scattering kinematics are required, which can also contain singular configurations due to single or double initial state radiation. These require new sets of integrated antenna functions.

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References

[1] O. Biebel, Phys. Rept. **340** (2001) 165;
S. Bethke, Phys. Rept. **403** (2004) 203 [hep-ex/0406058].

[2] L.J. Dixon and A. Signer, Phys. Rev. Lett. **78** (1997) 811 [hep-ph/9609460]; Phys. Rev. D **56** (1997) 4031 [hep-ph/9706285];
Z. Nagy and Z. Trocsanyi, Phys. Rev. Lett. **79** (1997) 3604 [hep-ph/9707309];
S. Weinzierl and D.A. Kosower, Phys. Rev. D **60** (1999) 054028 [hep-ph/9901277];
W.B. Kilgore and W.T. Giele, Phys. Rev. D **55** (1997) 7183 [hep-ph/9610433];
Z. Nagy and Z. Trocsanyi, Phys. Rev. Lett. **87** (2001) 082001 [hep-ph/0104315];
Z. Nagy, Phys. Rev. Lett. **88** (2002) 122003 [hep-ph/0110315]; Phys. Rev. D **68** (2003) 094002 [hep-ph/0307268];
J. Campbell and R.K. Ellis, Phys. Rev. D **65** (2002) 113007 [hep-ph/0202176].

[3] S. Moch, J.A.M. Vermaseren and A. Vogt, Nucl. Phys. B **688** (2004) 101 [hep-ph/0403192];
A. Vogt, S. Moch and J.A.M. Vermaseren, Nucl. Phys. B **691** (2004) 129 [hep-ph/0404111].

[4] Z. Bern, L.J. Dixon and A. Ghinculov, Phys. Rev. D **63** (2001) 053007 [hep-ph/0010075];
C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. B **601** (2001) 318 [hep-ph/0010212]; **601** (2001) 347 [hep-ph/0011094]; **605** (2001) 486 [hep-ph/0101304];
E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, Nucl. Phys. **605** (2001) 467 [hep-ph/0102201];
C. Anastasiou, E.W.N. Glover and M.E. Tejeda-Yeomans, Nucl. Phys. B **629** (2002) 255 [hep-ph/0201274];
E.W.N. Glover and M.E. Tejeda-Yeomans, JHEP **0306** (2003) 033 [hep-ph/0304169];
E.W.N. Glover, JHEP **0404** (2004) 021 [hep-ph/0401119];
Z. Bern, A. De Freitas and L.J. Dixon, JHEP **0109** (2001) 037 [hep-ph/0109078]; JHEP **0203** (2002) 018 [hep-ph/0201161]; JHEP **0306** (2003) 028 [hep-ph/0304168];
A. De Freitas and Z. Bern, JHEP **0409** (2004) 039 [hep-ph/0409007];
Z. Bern, A. De Freitas, L.J. Dixon, A. Ghinculov and H.L. Wong, JHEP **0111** (2001) 031 [hep-ph/0109079];
T. Binoth, E.W.N. Glover, P. Marquard and J.J. van der Bij, JHEP **0205** (2002) 060 [hep-ph/0202266].

[5] L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis and E. Remiddi, Nucl. Phys. B **627** (2002) 107 [hep-ph/0112081] and **642** (2002) 227 [hep-ph/0206067];
S. Moch, P. Uwer and S. Weinzierl, Phys. Rev. D **66** (2002) 114001 [hep-ph/0207043].

[6] T. Gehrmann and E. Remiddi, Nucl. Phys. B **640** (2002) 379 [hep-ph/0207020].

[7] G. 't Hooft and M. Veltman, Nucl. Phys. **B44** (1972) 189.

[8] F.V. Tkachov, Phys. Lett. **100B** (1981) 65.

[9] K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. **B192** (1981) 159.

[10] T. Gehrmann and E. Remiddi, Nucl. Phys. **B580** (2000) 485 [hep-ph/9912329].

[11] V.A. Smirnov, Phys. Lett. **B460** (1999) 397 [hep-ph/9905323];
J.B. Tausk, Phys. Lett. **B469** (1999) 225 [hep-ph/9909506];
V.A. Smirnov and O.L. Veretin, Nucl. Phys. **B566** (2000) 469 [hep-ph/9907385];
C. Anastasiou, T. Gehrmann, C. Oleari, E. Remiddi and J.B. Tausk, Nucl. Phys. **B580** (2000) 577 [hep-ph/0003261];
T. Gehrmann and E. Remiddi, Nucl. Phys. **B** (Proc. Suppl.) **89** (2000) 251 [hep-ph/0005232];
C. Anastasiou, J.B. Tausk and M.E. Tejeda-Yeomans, Nucl. Phys. **B** (Proc. Suppl.) **89** (2000) 262 [hep-ph/0005328];
T. Gehrmann and E. Remiddi, Nucl. Phys. B **601** (2001) 248 [hep-ph/0008287]; **601** (2001) 287 [hep-ph/0101124].

[12] Z. Bern, L.J. Dixon and D.A. Kosower, Phys. Rev. Lett. **70** (1993) 2677 [hep-ph/9302280];
Z. Kunszt, A. Signer and Z. Trocsanyi, Phys. Lett. B **336** (1994) 529 [hep-ph/9405386];
Z. Bern, L.J. Dixon and D.A. Kosower, Nucl. Phys. B **437** (1995) 259 [hep-ph/9409393].

[13] E.W.N. Glover and D.J. Miller, Phys. Lett. B **396** (1997) 257 [hep-ph/9609474];
Z. Bern, L.J. Dixon, D.A. Kosower and S. Weinzierl, Nucl. Phys. B **489** (1997) 3 [hep-ph/9610370];

J.M. Campbell, E.W.N. Glover and D.J. Miller, Phys. Lett. B **409** (1997) 503 [hep-ph/9706297];
Z. Bern, L.J. Dixon and D.A. Kosower, Nucl. Phys. B **513** (1998) 3 [hep-ph/9708239].

[14] J. Campbell, M.A. Cullen and E.W.N. Glover, Eur. Phys. J. C **9** (1999) 245 [hep-ph/9809429].

[15] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, Nucl. Phys. B **425** (1994) 217 [hep-ph/9403226];
D.A. Kosower, Nucl. Phys. B **552** (1999) 319 [hep-ph/9901201];
D.A. Kosower and P. Uwer, Nucl. Phys. B **563** (1999) 477 [hep-ph/9903515];
Z. Bern, V. Del Duca and C.R. Schmidt, Phys. Lett. B **445** (1998) 168 [hep-ph/9810409];
Z. Bern, V. Del Duca, W.B. Kilgore and C.R. Schmidt, Phys. Rev. D **60** (1999) 116001 [hep-ph/9903516].

[16] S. Catani and M. Grazzini, Nucl. Phys. B **591** (2000) 435 [hep-ph/0007142].

[17] D.A. Kosower, Phys. Rev. Lett. **91** (2003) 061602 [hep-ph/0301069].

[18] S. Weinzierl, JHEP **0307** (2003) 052 [hep-ph/0306248].

[19] C. Anastasiou, Z. Bern, L.J. Dixon and D.A. Kosower, Phys. Rev. Lett. **91**, 251602 (2003) [hep-th/0309040];
Z. Bern, L.J. Dixon and D.A. Kosower, JHEP **0408** (2004) 012 [hep-ph/0404293]; S.D. Badger and E.W.N. Glover, JHEP **0407** (2004) 040 [hep-ph/0405236].

[20] C. Anastasiou, L.J. Dixon, K. Melnikov and F. Petriello, Phys. Rev. Lett. **91** (2003) 182002 [hep-ph/0306192]; Phys. Rev. D **69** (2004) 094008 [hep-ph/0312266].

[21] A. Gehrmann-De Ridder and E.W.N. Glover, Nucl. Phys. B **517** (1998) 269 [hep-ph/9707224].

[22] J. Campbell and E.W.N. Glover, Nucl. Phys. B **527** (1998) 264 [hep-ph/9710255].

[23] S. Catani and M. Grazzini, Phys. Lett. B **446** (1999) 143 [hep-ph/9810389]; Nucl. Phys. B **570** (2000) 287 [hep-ph/9908523].

[24] F.A. Berends and W.T. Giele, Nucl. Phys. B **313** (1989) 595;
V. Del Duca, A. Frizzo and F. Maltoni, Nucl. Phys. B **568** (2000) 211 [hep-ph/9909464];
T.G. Birthwright, E.W.N. Glover, V.V. Khoze and P. Marquard, JHEP **0505** (2005) 013 [hep-ph/0503063].

[25] D.A. Kosower and P. Uwer, Nucl. Phys. B **674** (2003) 365 [hep-ph/0307031].

[26] D. de Florian and M. Grazzini, Nucl. Phys. B **616** (2001) 247 [hep-ph/0108273]; **704** (2005) 387 [hep-ph/0407241].

[27] A. Daleo, C.A. Garcia Canal and R. Sassot, Nucl. Phys. B **662** (2003) 334 [hep-ph/0303199];
A. Daleo, D. de Florian and R. Sassot, Phys. Rev. D **71** (2005) 034013 [hep-ph/0411212].

[28] K.L. Adamson, D. de Florian and A. Signer, Phys. Rev. D **65** (2002) 094041 [hep-ph/0202132]; Phys. Rev. D **67** (2003) 034016 [hep-ph/0211295].

[29] D.A. Kosower, Phys. Rev. D **67** (2003) 116003 [hep-ph/0212097].

[30] S. Weinzierl, JHEP **0303** (2003) 062 [hep-ph/0302180].

[31] W.B. Kilgore, Phys. Rev. D **70** (2004) 031501 [hep-ph/0403128].

[32] M. Grazzini and S. Frixione, JHEP **0506** (2005) 010 [hep-ph/0411399].

[33] G. Somogyi, Z. Trocsanyi and V. Del Duca, JHEP **0506** (2005) 024 [hep-ph/0502226].

[34] G. Heinrich, Nucl. Phys. Proc. Suppl. **116** (2003) 368 [hep-ph/0211144];
C. Anastasiou, K. Melnikov and F. Petriello, Phys. Rev. D **69** (2004) 076010 [hep-ph/0311311];
T. Binoth and G. Heinrich, Nucl. Phys. B **693** (2004) 134 [hep-ph/0402265];
G. Heinrich, Nucl. Phys. Proc. Suppl. **135** (2004) 290 [hep-ph/0406332]; **157** (2006) 43 [hep-ph/0601232]; hep-ph/0601062.

[35] A. Gehrmann-De Ridder, T. Gehrmann and G. Heinrich, Nucl. Phys. B **682** (2004) 265 [hep-ph/0311276].

[36] C. Anastasiou, K. Melnikov and F. Petriello, Phys. Rev. Lett. **93** (2004) 032002 [hep-ph/0402280].

[37] C. Anastasiou, K. Melnikov and F. Petriello, Phys. Rev. Lett. **93** (2004) 262002 [hep-ph/0409088]; Nucl. Phys. B **724** (2005) 197 [hep-ph/0501130].

[38] C. Anastasiou, K. Melnikov and F. Petriello, hep-ph/0505069.

[39] K. Melnikov and F. Petriello, hep-ph/0603182.

[40] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, JHEP **0509** (2005) 056 [hep-ph/0505111].

[41] W.T. Giele and E.W.N. Glover, Phys. Rev. D **46** (1992) 1980;
Z. Kunszt and D.E. Soper, Phys. Rev. D **46** (1992) 192;

S. Frixione, Z. Kunszt and A. Signer, Nucl. Phys. B **467** (1996) 399 [hep-ph/9512328];
S. Catani and M.H. Seymour, Nucl. Phys. B **485** (1997) 291; **510** (1997) 503(E) [hep-ph/9605323].

[42] D.A. Kosower, Phys. Rev. D **57** (1998) 5410 [hep-ph/9710213]; Phys. Rev. D **71** (2005) 045016 [hep-ph/0311272].

[43] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Nucl. Phys. B **691** (2004) 195 [hep-ph/0403057].

[44] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Phys. Lett. B **612** (2005) 36 [hep-ph/0501291].

[45] H.E. Haber and D. Wyler, Nucl. Phys. B **323** (1989) 267.

[46] A. Gehrmann-De Ridder, T. Gehrmann and E.W.N. Glover, Phys. Lett. B **612** (2005) 49 [hep-ph/0502110].

[47] F. Wilczek, Phys. Rev. Lett. **39** (1977) 1304;
M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Phys. Lett. B **78** (1978) 443.

[48] A. Gehrmann-De Ridder, T. Gehrmann and E. W. N. Glover, Nucl. Phys. Proc. Suppl. **135** (2004) 97 [hep-ph/0407023].

[49] K. Hagiwara and D. Zeppenfeld, Nucl. Phys. B **313** (1989) 560;
F.A. Berends, W.T. Giele and H. Kuijf, Nucl. Phys. B **321** (1989) 39;
N.K. Falck, D. Graudenz and G. Kramer, Nucl. Phys. B **328** (1989) 317.

[50] A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover and G. Heinrich, work in progress.