MULTIFRACTAL STRUCTURE OF MATTER ON LARGE SCALES

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Abstract: On the one hand, the large scale structure of matter is arguably scale invariant, and, on the other hand, halos are recognized as prominent features of that structure. Therefore, we propose to model the dark matter distribution as sets of fractal distributions of halos. This model relies on the concept of multifractal as the most general scaling distribution, and on a plausible notion of halo as a mass concentration in a multifractal. We show that an analysis, according to this picture, of distributions produced in cosmological N-body simulations gives convincing evidence of scale invariance and actually yields a definite dimension for halos of similar mass, which grows as the mass diminishes.

1 Introduction

The large scale structure of matter can be studied by observing the distribution of the luminous matter, namely, galaxies, and by theoretical studies of the evolution of the dark matter. Galaxies seem to form a scale-invariant fractal distribution (on a range of scales) [1]. On the other hand, the dark-matter seems to concentrate in *halos*, which has given rise to the *halo model*. This model considers both the dark matter distribution within halos and the distribution of halos themselves. The matter distribution within halos is considered in *virial equilibrium* and smoother than the distribution of halos, which are very clustered.

Although the fractal model is well developed, it has not led to agreement on some important issues [2], suggesting that it needs to be improved. A natural generalization is the multifractal model [3]. As a new formulation of the multifractal model, we have recently proposed a model consisting of fractal distributions of halos and we have shown that it is supported by cosmological simulations [4]. This model assumes that halos of similar mass have a fractal distribution with a given dimension, which grows as the mass diminishes. The multifractal model is also supported by models of structure formation, in particular, by the *adhesion model*.

Thus, we begin with a brief review of the adhesion model and, then, we proceed to the mathematical definition of fractals and multifractals, focusing on the concept of singularities or mass concentrations, and on the realization of scale invariance. Then we study what implications a multifractal model has for the distribution of halos (defined as mass concentrations) and we test them against cosmological N-body simulations.

2 Structure of matter in the adhesion model

Let us begin by writing the Newtonian equations of motion of matter is cosmology in co-moving coordinates. If $\boldsymbol{v}$ is the velocity, we consider the peculiar velocity $\boldsymbol{u} = \boldsymbol{v} - H\boldsymbol{r}$) and deduce the equation:

$$\frac{d\boldsymbol{u}}{dt} + H\boldsymbol{u} = \boldsymbol{g}_T - \boldsymbol{g}_b \equiv \boldsymbol{g}\,, \tag{1}$$

where we have also defined a peculiar gravity acceleration $\boldsymbol{g}$ by subtracting the background acceleration $\boldsymbol{g}_b$, associated to the Hubble flow, from the total acceleration $\boldsymbol{g}_T$.

Given that $\nabla \times \boldsymbol{g} = 0$, there is no source of vorticity in Equation (1). In addition, the dissipative nature of Equation (1), due to the term $H\boldsymbol{u}$, implies that velocity components decay unless there is a source in the right-hand side. As there is no source of vorticity, the initial vorticity decays, and we can actually consider initial conditions with no vorticity. The linear evolution with these initial conditions is very simple: particles move along the gravitational field, namely, $\boldsymbol{x}(t, \boldsymbol{x}_0) = \boldsymbol{x}_0 + b(t)\,\boldsymbol{g}(\boldsymbol{x}_0)$, where $b(t)$ is the linear growth factor. The Zeldovich approximation consists of prolonging this linear solution into the nonlinear regime. If we define a new "time" $\tau = b(t)$, the Zeldovich approximation consists of free motion with "velocity" $\boldsymbol{g}(\boldsymbol{x}_0)$. In other words, after the redefinition of time, the "velocity" is the initial acceleration, and the "acceleration" is $d^2\boldsymbol{x}/d\tau^2 = 0$, despite the presence of actual acceleration in the time t. Singularities arise when the lines of motion of different particles cross. This situation is analogous to the formation of *caustics* in geometric optics and, in fact, the singularities produced in the Zeldovich approximation are also called caustics.

When caustics form, the density diverges, so the Zeldovich approximation must necessarily fail, because it relies on the linear evolution. At any rate, the particle motion, whether linear or not, leads to caustics. When particles collide, they may not interact and continue their motion unaffected, as in optics, but this is not likely. In fact, our dark matter particles are best assumed to stand for blobs of real dark matter particles (in a *coarse-grained* picture), so we can ignore pressure or other small scale effects. With this assumption, it is natural to further assume the existence of *viscosity*, so the particles interchange momentum. We are thus led to replace free motion with the *Burgers equation* (originally an equation for compressible turbulence) [5, 6, 7]:

$$\frac{d\tilde{\boldsymbol{u}}}{d\tau} = \frac{\partial \tilde{\boldsymbol{u}}}{\partial \tau} + \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}} = \nu \nabla^2 \tilde{\boldsymbol{u}}, \quad \nabla \times \tilde{\boldsymbol{u}} = 0,$$

where $\tilde{\boldsymbol{u}}$ is the velocity in terms of the "time" τ. We can further make $\nu \to 0$ to preserve the validity of the Zeldovich approximation. This limit indeed recovers the Zeldovich approximation, except in the caustics, where it amounts to an *inelastic collision* prescription: the particles adhere to each other; hence, the *adhesion* model [5]. In consequence, this model produces the stabilization of caustics, which become the walls ("pancakes"), filaments and nodes that are typical of large scale structure.

The walls ("pancakes"), filaments and nodes are definite lower dimensional objects (in three dimensions). They are the ones expected to arise in a generic situation. Therefore, a *naive* picture of the large scale structure according to the adhesion model consists of a distribution of one, two and three-dimensional objects. This structure is a trivial example of multifractal distribution, in which there are only objects of integer dimension. However, as we say, this picture is naive: it only holds if the initial velocity field is smooth; but the natural initial velocity field is a non-smooth Gaussian random field [6]. The structure produced with these initial conditions resembles a self-similar distribution of walls, filaments and nodes that has been dubbed the *cosmic web*. In this "web" the mass concentrates, in addition to walls, filaments and nodes, in some regions rather than in others (because those objects concentrate there). A two dimensional version of this structure appears in Figure 1, where it is compared with the structure produced in a real cosmological N-body simulation. To describe and understand this type of structure, we need to introduce concepts of *fractal geometry*.

3 Random fractals and multifractals

Fractal geometry studies sets (or functions) that are irregular (non-smooth) and have *fine structure*, namely, detail at all scales. Consequently, fractal objects cannot be considered in the framework of traditional geometry. Fractal sets are usually non-denumerable sets of zero measure (in the mathematical sense). A typical example is the middle third Cantor set. Usually, the fine structure of

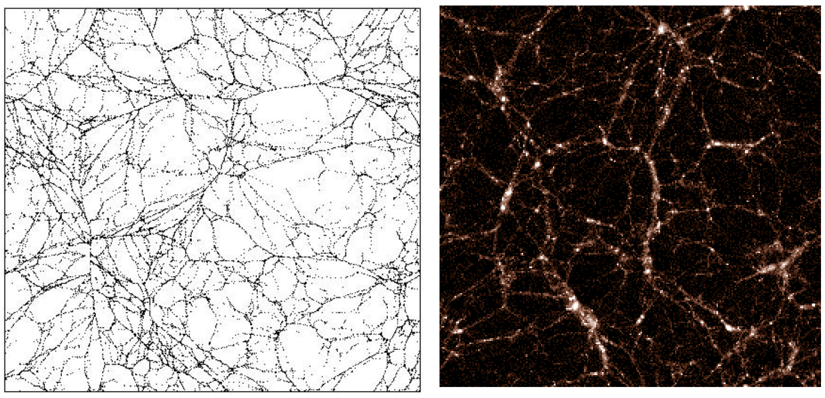

Figure 1: "Cosmic web" produced by the adhesion model (in two dimensions) with random initial conditions (left-hand side, reproduced with permission from [7]), and produced by the Virgo GIF2 N-body simulation (right-hand side, 1/256-thick slice).

fractals is due to their being self-similar, in a strict or approximate sense. For example, random fractals cannot be strictly self-similar, but they actually have *statistical* self-similarity (see Figure 2). This similarity implies that parts of the set are statistically equivalent to the whole set, that is to say, in a technical language, they have the same distribution. In particular, the correlation functions must involve power laws.

The precise definition of fractal sets (whether deterministic or random) is not trivial. For example, [8] defined a fractal to be a set with *Hausdorff dimension* strictly greater than its *topological dimension* (the latter is always an integer, and equals zero for sets that we intuitively call sets of points, equals one for sets that we intuitively call sets of curves, etc.). There are other definitions of dimension: packing dimension, box-counting dimension, etc. Here, we do not need to consider mathematical subtleties of this kind: we shall always consider self-similar fractals (whether deterministic or random) such that all the dimensions are equal, in particular, equal to the *similarity dimension*. In general, a set made of m copies of itself scaled by a factor λ is such that $m = 1/\lambda^D$, where $D = -\log m/\log\lambda$ is the similarity dimension. For example, for the middle third Cantor set, $m = 2$, $\lambda = 1/3$, and $D = \log 2/\log 3$. We simply rename the similarity dimension fractal dimension.

Of course, in nature one cannot observe a mathematical fractal, that is, a set with an infinite number of points. In other words, natural fractals cannot have

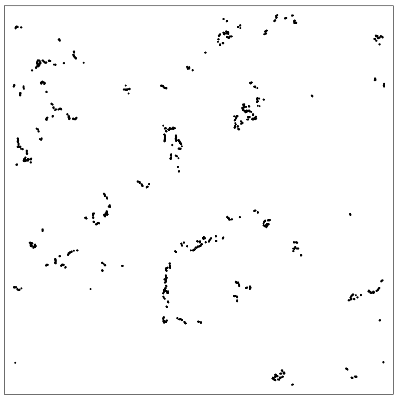

Figure 2: Random fractal with $D = 0.8$.

detail at all scales but only in a limited range of scales. Therefore, self-similarity must break down on small scales, where the discrete nature of the set becomes dominant. A useful description of random self-similar fractals is through the *number-radius* relation, which expresses the number of points in a ball of radius r centred on one point *averaged* over every point: it has to be the power law $N(r) = B\, r^D$, where D is the fractal dimension and B a constant. This relation replaces the relation $m = (1/\lambda)^D$ between the number of copies and the scaling factor for deterministic self-similar fractals. $N(r)$ is equivalent to the cumulative conditional probability, directly related with the two-point correlation function.

The fractal model in cosmology has its roots in old ideas about a hierarchical large scale structure of matter (the history of this subject is nicely told in [9]). It gained observational support from the measure of the galaxy-galaxy correlation, which is a power law within some range of scales [1]. Nowadays, the extent of this range and the actual fractal dimension of the galaxy distribution are controversial issues [2]. It has been proposed to amend the fractal model, for example, by introducing a scale-dependent fractal dimension. This breaks scale invariance and actually deprives the fractal model of its simplicity and appeal.

However, it is possible to reconcile scale-invariance (self-similarity) with observations of the galaxy distribution and cosmological N-body simulations. Indeed, the most general scale-invariant distribution is not a simple fractal, characterized by *one* fractal dimension, but a multifractal, characterized by a set of fractal dimensions.

3.1 Random multifractals

Multifractal *measures* appear frequently in physics, for example, as *attractors* of dynamical systems [10]. In dynamical systems, the natural measure of some region represents the proportion of time that the system spends in it. For many systems, at long times, the trajectory is confined to a small region of the available space (an attractor), but the time that it spends in any subregion of it is very variable, so the natural measure is very irregular. In particular, large scale structure forms according to the Newton equations and can be considered an attractor of this many-dimensional dynamical system. Moreover, since gravity has no intrinsic scale, we expect a scale-invariant structure, namely, a multifractal.

Multifractal measures represent mass distributions spread according to highly irregular patterns, that is, with mass concentrations of very different magnitude [10]. This magnitude is referred to as "strength" and defined by the local dimension α (also called Lipschitz-Hölder exponent in the mathematical literature); it is such that

$$\alpha(\boldsymbol{x}) = \lim_{r \to 0} \frac{\log m[B(\boldsymbol{x}, r)]}{\log r}, \tag{2}$$

where $m[B(\boldsymbol{x}, r)]$ is the mass in the ball of radius r centred on $\boldsymbol{x}$. Informally, we write, $m[B(\boldsymbol{x}, r)] \sim r^{\alpha(\boldsymbol{x})}$. In a regular mass distribution, $\alpha = 3$ (constant), so mass concentrations $\alpha(\boldsymbol{x}) < 3$ are singularities. On the other hand, an ordinary fractal can be considered endowed with a uniform mass distribution over it, such that $\alpha < 3$ is the *constant* fractal dimension. Thus, in the context of multifractals, ordinary fractals are called *monofractals* (or *unifractals*). Full-fledged multifractals possess a range of dimensions α (or strenghts), namely, $0 < \alpha_{\min} \leq \alpha \leq \alpha_{\max}$. Every set of points in which α takes a definite value is a fractal set. Therefore, a multifractal can be considered as a set of fractals of various α. The *multifractal spectrum* $f(\alpha)$ is the function that gives the dimension of the fractal with exponent α.

Correlation moments

To analyse the structure of random multifractals, it is useful to introduce the scale-dependent statistical moments

$$M_q(r) \;\; = \;\; \int dm(\boldsymbol{x}) \; m[B(\boldsymbol{x}, r)]^{q-1}. \tag{3}$$

M_1 is the total mass, which we normalize to one. With this normalization, a multifractal measure can be interpreted as a probability measure. $M_2(r)$ is the integral of the two-point correlation function, that is, $N(r)$. Larger integer values

$q = 3, 4, \ldots$ are also employed in the analysis of the large scale matter distribution, but they are increasingly difficult to estimate. In addition, since we are dealing with singular non-uniform distributions, integer values of q are not sufficient and we have to consider the full set of moments $M_q(r)$ for $-\infty < q < \infty$. We can then define the function that gives the scaling behaviour of this full set of moments, namely,

$$\tau(q) = \lim_{r \to 0} \frac{\log M_q(r)}{\log r}. \tag{4}$$

For well-behaved multifractals, this function determines the multifractal spectrum through a Legendre transform [10]:

$$f(\alpha) = \inf_q [q\,\alpha - \tau(q)]\,, \tag{5}$$

where $\inf_q$ means the infimum with respect to q. If $\tau(q)$ is differentiable and *convex*,

$$\alpha(q) = \tau'(q).$$

Then, after inverting $\alpha(q)$, one obtains

$$f(\alpha) = q(\alpha)\,\alpha - \tau[q(\alpha)], \tag{6}$$

which is also convex, namely, $f''(\alpha) < 0$.

It is useful to define the quantity $D(q) = \tau(q)/(q-1)$, which is constant for a monofractal, and, in general, decreases with q. The most important values of q are $q = 0, 1$. At $q = 0$, we have $f(\alpha_0) = D(0) = -\tau(0)$. Since $f'(\alpha_0) = q = 0$, the set of points with exponent α_0 has the largest fractal dimension. For self-similar multifractals, this value equals the fractal dimension of the *measure's support*, which is the smallest closed set such that its complement has zero measure. The value $q = 1$ gives the *measure's dimension* $D(1)$, for a value of $\alpha_1 = \alpha(1)$ such that $\alpha_1 = f(\alpha_1) = D(1)$. The corresponding singularity set defines the *measure's concentrate*, that is, the set outside of which the measure is essentially negligible. Note that $f'(\alpha_1) = q(\alpha_1) = 1$ and convexity of $f(\alpha)$ imply $f(\alpha) \leq \alpha$ for any α (with equality at α_1).

The mathematical definition of q-moments provided by the integral of Equation (3) may not be convenient for calculations. We use instead *coarse multifractal analysis* [10]: we put an ℓ-mesh of cubes in the total volume (which is itself a large cube) and consider the mass inside each cube. Then we define

$$M_q(\ell) = \sum_i m_i^q\,, \tag{7}$$

where the sum is over non-empty cubes. (This method of calculating moments has been used in cosmology under the name of "counts-in-cells".) We let $\ell \to 0$ to define the function $\tau(q)$ according to Equation (4), after replacing r with ℓ.

4 Distribution of halos and scale invariance

In the halo model, the full dark matter distribution is composed of irregularly distributed halos which have a smoother inner distribution. The halo model relies, to some extent, on the results of cosmological N-body simulations. If the simulated cosmological Newton equations have a dynamical attractor, it must be scale invariant, except that homogeneity must be recovered on large scales and that there is a *lower cutoff* to scaling. This lower cutoff appears in two forms: as mass *discreteness*, corresponding to the finite number of particles, and as a *softening* of the gravitational force, which induces smoothness on small scales. The effective lower cutoff is the larger of the two, normally, the linear size of the *initial* volume per particle (initially, the particles are homogeneously distributed). As the existence of halos breaks scale invariance, halos cannot be much larger than this basic volume; on the other hand, there is no reason to consider them smaller, because we want them to have as many particles as possible. In conclusion, a natural definition of halos in cosmological simulations is to assign them the size of this basic volume. Of course, as long as the evolution is linear, there is about one particle per basic volume (by definition), and halos only arise in the nonlinear stage, as they concentrate particles from other regions that become *voids*.

On scales larger than the halo size, scale invariance holds, so the correlation functions of halos are power laws. We shall see that on these scales the mass distribution is multifractal. Therefore, we can identify halos with mass concentrations in a multifractal. Thus, every halo population formed by equal mass halos is a fractal, although different populations must have different dimensions. We can describe this difference between populations as a kind of *bias*, albeit of *non-linear* type.

4.1 The fractal distributions of halos in a multifractal

We have applied coarse multifractal analysis to the redshift $z = 0$ positions of the ΛCDM GIF2 simulation, with 400^3 particles in a volume of $(110\ h^{-1}\ \text{Mpc})^3$ and force-softening length $7\,h^{-1}$ Kpc, from the Virgo Consortium (`http://www.mpa-garching.mpg.de/Virgo/data_download.html#GIF2`). Since the halo size must be larger than the volume per particle and we operate with powers of two, we take it to be $\ell_H = 256^{-1}$ (in box-length units). Then, we classify the halos according to their masses (equivalent to their exponents). We can directly test if each halo population is a fractal by calculating its number radius relation $N(r)$.

The study of fractal distributions of equal mass halos serves two purposes: first, to give a proof of multifractality and, second, to provide the homogeneity scale.

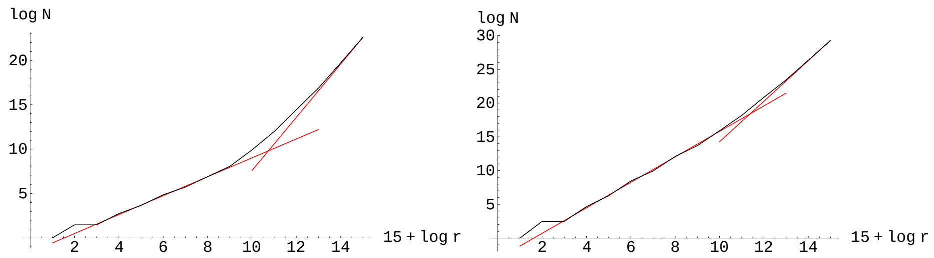

Figure 3: Number-radius relations for two halo populations from the GIF2 N-body simulation: heavy haloes with 750 to 1000 particles and light haloes with 100 to 150 particles. Their fractal dimensions are $D = 1.1$ and $D = 1.9$, respectively, and both have a transition to homogeneity at about the same scale (logarithms are to base $\sqrt{2}$ and the total size is normalized to unity).

Actually, we need a range of masses in each population to have a sufficient number of halos. Hence, we choose two separated ranges, with 750 to 1000 particles and with 100 to 150 particles, defining two populations consisting of 2508 and 25 556 members, respectively. Results of the analysis appear in Figure 3, namely, the log-log plots of the number-radius functions for the respective spatial distributions. They have scaling ranges corresponding to quite different fractal dimensions, namely, $D_1 = 1.1$ and $D_2 = 1.9$. The scale of transition to homogeneity, is $\ell_0 \simeq 1/4$ ($27\,h^{-1}$ Mpc).

4.2 Multifractal spectrum

To actually prove multifractality, we need to show that the multifractal spectrum is *stable* within a scale range. Therefore, we compute the coarse multifractal spectrum by the method of moments, according to Equation 6, for various $\ell_n = 2^{-n}$, and plot the results in Figure 4. We can see that the multifractal spectrum is indeed stable in the parts where the curves overlap. For the decreasing part of $f(\alpha)$, which corresponds to negative q, we have only two curves. Their agreement is sufficient for smaller α but they separate for larger α ($\alpha > 4$). This was to be expected, for these values correspond to the most depleted areas, which suffer from undersampling.

Here are some consequences that can be drawn from Figure 4:

- The measure's dimension is about 2.5 (the measure is concentrated in a set that is not far from three dimensions).

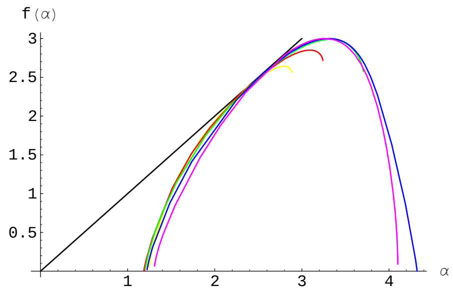

Figure 4: Full multifractal spectrum $f(\alpha)$ for $\log_2 \ell = -9, -8, -7, -6, -5$ (yellow, red, green, blue, magenta). We have also plot the diagonal line to show how the multifractal spectrum is placed under it and touches it at the measure's dimension $\alpha \simeq 2.5$.

- The largest fractal dimension is 3 (the dimension of the measure's support).
- The measure's support corresponds to $\alpha(q = 0) \simeq 3.3 > 3$. We see that it is formed by points with vanishing density, that is, belonging to voids.

5 Conclusions

Our fundamental conclusion is that scale invariance can be reconciled with the halo model, if we define halos as having the size of the scale at which scale invariance is broken. This conclusion agrees with the picture of large scale structure as a *multifractal attractor* of the cosmological Newton equations.

Regarding the fractal debate, we draw two conclusions. First, the extent of the fractal range, namely, the scale of transition to homogeneity, in our multifractal analysis is $\ell_0 \simeq 27\, h^{-1}$ Mpc. This value corresponds to a dark-matter simulation, but it roughly agrees with values found for the galaxy distribution. Second, it seems misleading to attribute a *unique* fractal dimension to the galaxy distribution: halo populations of different mass have different dimension, so galaxy populations of different luminosity may have different dimension.

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