VARIATIONAL APPROACH FOR COUPLED BACKWARD AND FORWARD WAVE EXCITATION IN FREE-ELECTRON LASERS*

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Abstract

In a recent paper [1] we have described a novel variational formulation for the propagation and generation of radiation in wave-guides The work underlines calculation of modal amplitude evolution rather than the calculation of the modal shapes, which is common in previous art. Modal amplitude evolution is important in electron devices such as free electron lasers and gyrotrons. The present paper deals with a variational derivation of a numerical scheme that can be used to study the build up of radiation in free electron lasers in the linear approximation.

INTRODUCTION

Interaction of radiation and plasma waves in many electron devices takes place inside an open or closed cylinder (wave guide) of some arbitrary cross-section (see figure 1 for a schematic illustration).

A well-known example is the free-electron laser, in which the electromagnetic field interacts with an electron beam in the presence of an undulator, generating high power coherent radiation. In order to achieve lasing, the radiation is being excited inside a resonator, dictating boundary conditions for both forward and backward waves (see figure 1). Solution of the electromagnetic radiation field inside

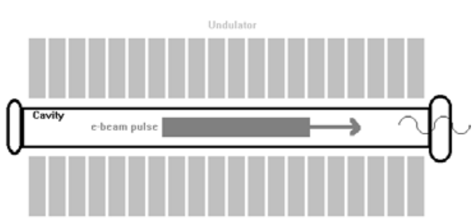

Figure 1: The FEL scheme

the resonator, requires simultaneous integration of the coupled excitation equations of forward and backward waves [10]. However, it becomes difficult to accommodate the different boundary conditions for both forward and backward modes in the same numerical integration scheme. Although the radiation power is built gradually in the direction of the electron beam propagation, the natural boundary conditions for the backward waves are given at the end of the interaction region. Thus it is desirable to develop a numerical procedure that allows non-local boundary conditions.

We suggest employing variational methods for calculating the total electromagnetic field, including excitation of forward and backward waves. Our developed variational principle is based on a modal representation of the total electromagnetic field in terms of the eigenmodes of the geometry in which the radiation is excited and formulation of the electromagnetic field action in the space-frequency domain.

Variational principles for electromagnetic field dynamics, including their interaction with matter are abundant in the literature [2]- [9]. Moreover, the behavior of the electromagnetic field inside a wave guide in terms of a variational principle was studied in many texts [4]- [9], most of the times in order to provide a basis for a numerical scheme. These works are concerned mainly with the derivations of eigenmodes for the case of non trivial geometries or an inhomogeneous refraction index. In this work we are not concerned with the modal form rather we assume that it is known. Our main concern is the development of the modal amplitude inside the wave guide due to its interaction with propagating charge. Three different variational principle describing the modal propagation inside a wave guide are introduced.

The structure of this paper is as follows: first we discuss the fundamentals of electromagnetic field presentation in the frequency domain, followed by a short review of the modal representation in a wave guide. Next the action is represented in terms of the mode amplitude. Then we introduce the quasi Hamiltonian which allows us to obtain a variational principle which generate first order equations in terms of the field amplitudes. Finally the introduction of backward-forward waves puts the variational principle in a particular simple form which concludes our report.

MODAL & SPECTRAL PRESENTATION OF ELECTROMAGNETIC FIELDS

The electromagnetic field in the time domain is described by the space-time electric $\mathbf{E}(\mathbf{r},t)$ and magnetic $\mathbf{H}(\mathbf{r},t)$ signal vectors. $\mathbf{r}$ stands for the (x,y,z) coordinates, where (x,y) are the transverse coordinates and z is the axis of propagation. The Fourier transform of the electric field is defined by:

$$\mathbf{E}(\mathbf{r},\omega) = \int_{-\infty}^{+\infty} \mathbf{E}(\mathbf{r},t)e^{+j\omega t}dt \quad (1)$$

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where ω is the angular frequency and $j = \sqrt{-1}$. Similar expression is defined for the Fourier transform $\mathbf{H}(\mathbf{r}, \omega)$ of the magnetic field. Since the electromagnetic signal is real, its Fourier transform satisfies $\mathbf{E}^*(\mathbf{r}, \omega) = \mathbf{E}(\mathbf{r}, -\omega)$.

Fourier transformation of the electric field results in a 'phasor-like' function $\widetilde{\mathbf{E}}(\mathbf{r}, \omega)$ defined in the positive frequency domain and related to the Fourier transform by:

$$\widetilde{\mathbf{E}}(\mathbf{r}, \omega) = 2\mathbf{E}(\mathbf{r}, \omega) u(\omega) \equiv \begin{cases} 2\mathbf{E}(\mathbf{r}, \omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases} \tag{2}$$

In the following we present the formalism employed throughout this paper for analyzing the excitation of electromagnetic fields by current sources distributed along a wave guide [1, 10, 12, 13]. The approach taken here utilizes representation of the total electromagnetic fields and their sources in terms of vector functions, which are the eigenmodes solutions of the medium, free of charge or current sources. The 'phasor like' quantities defined in (2) can be expanded in terms of transverse eigenmodes of the medium in which the field is excited and propagates. The perpendicular component of the electric and magnetic fields are given in any cross-section as a linear superposition of a complete set of transverse eigenmodes:

$$\begin{aligned} \widetilde{\mathbf{E}}_{\perp}(\mathbf{r}, \omega) &= \sum_q V_q(z, \omega) \widetilde{\mathcal{E}}_{q\perp}(x, y) \\ \widetilde{\mathbf{H}}_{\perp}(\mathbf{r}, \omega) &= \sum_q I_q(z, \omega) \widetilde{\mathcal{H}}_{q\perp}(x, y) \end{aligned} \tag{3}$$

The summations include propagating and cut-off TE and TM modes, for which $V_q(z, \omega)$ and $I_q(z, \omega)$ are the scalar amplitude of the electric and magnetic fields respectively and $\widetilde{\mathcal{E}}_{q\perp}(x, y)$ and $\widetilde{\mathcal{H}}_{q\perp}(x, y)$ are their respective profiles.

Expressions for the longitudinal component of the electric and magnetic fields are obtained after substituting the modal representation (3) of the fields into Maxwell's equations, where the Fourier transform of the current density $\mathbf{J}$, $\widetilde{\mathbf{J}}(\mathbf{r}, \omega)$ is introduced:

$$\begin{aligned} \widetilde{E}_z(\mathbf{r}, \omega) &= \sum_q I_q(z, \omega) \widetilde{\mathcal{E}}_{qz}(x, y) + \frac{1}{j\omega\epsilon} \widetilde{J}_z(\mathbf{r}, \omega) \\ \widetilde{H}_z(\mathbf{r}, \omega) &= \sum_q V_q(z, \omega) \widetilde{\mathcal{H}}_{qz}(x, y) \end{aligned} \tag{4}$$

By imposing the appropriate boundary conditions, the Maxwell vector equations are transformed into scalar differential ('transmission line') equations, which describe the evolution of the equivalent electric and magnetic amplitudes $V_q(z, \omega)$ and $I_q(z, \omega)$:

$$\begin{aligned} -\frac{dV_q(z, \omega)}{dz} &= -jk_{zq} I_q(z, \omega) + v_q(z, \omega) \\ -\frac{dI_q(z, \omega)}{dz} &= -jk_{zq} V_q(z, \omega) + i_q(z, \omega) \end{aligned} \tag{5}$$

where:

$$k_{zq} = \begin{cases} j\sqrt{k_{\perp q}^2 - k^2} = j|k_{zq}| & k < k_{\perp q} \\ \text{(cut-off modes)} & \\ \sqrt{k^2 - k_{\perp q}^2} = |k_{zq}| & k > k_{\perp q} \\ \text{(propagating modes)} & \end{cases} \tag{6}$$

is the axial wave number of mode q. For n electrons travelling under the influence of the wiggler magnetic flux density vector $\vec{B}$, v_q and i_q can be calculated for propagating modes as [13]:

$$\begin{aligned} v_q(z, \omega) &= -\frac{e}{\mathcal{N}_q} \sum_{i=1}^{n} \widetilde{\mathcal{E}}(x_i, y_i)_{qz} e^{+j\omega t_i(z)} \\ i_q(z, \omega) &= -\frac{e}{\mathcal{N}_q} \sum_{i=1}^{n} \frac{\vec{v}_{i\perp}}{v_{iz}} \cdot \widetilde{\mathcal{E}}(x_i, y_i)_{q\perp}^* e^{+j\omega t_i(z)} \end{aligned} \tag{7}$$

In which $\vec{r}_i, \vec{v}_i$ are the six dimensional coordinate of each electron in phase space. And

$$t_i(z) = t_{0i} + \int_0^z \frac{dz'}{v_{iz}(z')} \tag{8}$$

The coordinates $\vec{r}_i, \vec{v}_i$ can be obtained by solving Newton's equation with the Lorentz force:

$$\frac{d}{dt}(\gamma_i \vec{v}_i) = -\frac{e}{m}\left[\vec{v}_i \times \vec{B}\right] \tag{9}$$

where $\gamma_i = \frac{1}{\sqrt{1 - (\frac{\vec{v}_i}{c})^2}}$. The normalization of the field amplitudes of each mode is made via each mode's complex Poynting vector power:

$$\mathcal{N}_q = \int\int_{c.s.} \left[\widetilde{\mathcal{E}}_{q\perp}(x, y) \times \widetilde{\mathcal{H}}_{q\perp}^*(x, y)\right] \cdot \hat{\mathbf{z}} dx dy \tag{10}$$

and the mode impedance is given by:

$$Z_q = \begin{cases} \sqrt{\frac{\mu}{\epsilon}} \frac{k}{k_{zq}} = \frac{\omega\mu}{k_{zq}} & \text{for TE modes} \\ \sqrt{\frac{\mu}{\epsilon}} \frac{k_{zq}}{k} = \frac{k_{zq}}{\omega\epsilon} & \text{for TM modes} \end{cases} \tag{11}$$

ε is the electric susceptibility and μ is the magnetic permeability.

The transmission-line equations (5) can also be written in the form:

$$\begin{aligned} V_q''(z, \omega) + k_{zq}^2 V_q(z, \omega) &= -v_q'(z, \omega) - jk_{zq} i_q(z, \omega) \\ I_q''(z, \omega) + k_{zq}^2 I_q(z, \omega) &= -jk_{zq} v_q(z, \omega) - i_q'(z, \omega) \end{aligned} \tag{12}$$

where $(')$ denotes a derivative in respect to z. Notice that only one of the equations in (12) needs to be solved, since solving for $V_q(z, \omega)$ we obtain immediately the solution for $I_q(z, \omega)$ through equation (5) .

THE ACTION IN A WAVE GUIDE

Following [1] the action of an electromagnetic field in a wave guide is given by:

$$\mathcal{A} = \frac{T^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sum_q \frac{\mathcal{N}_{q,n}^*}{k_{z_{q,n}}} L_{q,n}$$

$$
\begin{aligned}
L_{q,n} &\equiv \int \mathcal{L}_{q,n} dz \\
\mathcal{L}_{q,n} &\equiv \frac{1}{2} k_{z_{q,n}}^2 |V_{q,n}|^2 - \frac{1}{2} |\partial_z V_{q,n}|^2 - \frac{1}{2} \frac{\omega_n^2}{c^2 k_{\perp_q}^2} |v_{q,n}|^2 \\
&- \Re(v_{q,n}^* \partial_z V_{q,n}) - \Im(k_{z_{q,n}} i_{q,n} V_{q,n}^*) \qquad (13)
\end{aligned}
$$

All the quantities in the above expression are defined in the previous section. The quantity $\frac{\mathcal{N}_{q,n}^*}{k_{z_{q,n}}}$ is real and so are the Lagrangian $L_{q,n}$ and the Lagrangian density $\mathcal{L}_{q,n}$. Equating the variational derivative of the Lagrangian $L_{q,n}$ to zero will result in the second order equations given by (12).

In terms of the amplitude V and its complex conjugate V^*, the Lagrangian density $\mathcal{L}$ can be written as[1]:

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2} \{ k_z^2 V V^* - \partial_z V \partial_z V^* - \frac{\omega^2}{c^2 k_\perp^2} v v^* - \partial_z V v^* \\
&- \partial_z V^* v + j k_z i V^* - j k_z^* i^* V \} \qquad (14)
\end{aligned}
$$

SOME NUMERICAL ASPECTS

In order to understand the mathematical structure of equation (13) we shall write it in terms of real quantities. Representing all the quantities appearing in $\mathcal{L}$ (equation (13)) in terms of their real and imaginary parts we arrive at the result:

$$
\begin{aligned}
\mathcal{L} &\equiv \frac{1}{2} k_z^2 |V|^2 - \frac{1}{2} |\partial_z V|^2 - \frac{1}{2} \frac{\omega^2}{c^2 k_\perp^2} |v|^2 \\
&- \Re(v^* \partial_z V) - \Im(k_z i V^*) \\
&= \frac{1}{2} [k_z^2 (V_r^2 + V_i^2) - (\partial_z V_r^2 + \partial_z V_i^2) \\
&- \frac{\omega^2}{c^2 k_\perp^2} (v_r^2 + v_i^2) - \partial_z V_r v_r - \partial_z V_i v_i] \\
&- \frac{1}{2} \begin{cases} k_z (i_i V_r - i_r V_i) & \text{(propagating)} \\ |k_z| (i_r V_r - i_i V_i) & \text{(cut-off)} \end{cases} \qquad (15)
\end{aligned}
$$

Notice that the cut-off modes Lagrangian density decouples into two separate Lagrangian densities:

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_r + \mathcal{L}_i \\
\mathcal{L}_r &= \frac{1}{2} [k_z^2 V_r^2 - (\partial_z V_r)^2 - \frac{\omega^2}{c^2 k_\perp^2} v_r^2 \\
&- \partial_z V_r v_r - |k_z| i_r V_r] \\
\mathcal{L}_i &= \frac{1}{2} [k_z^2 V_r i^2 - (\partial_z V_i)^2 - \frac{\omega^2}{c^2 k_\perp^2} v_i^2 \\
&- \partial_z V_i v_i + |k_z| i_i V_i] \qquad (16)
\end{aligned}
$$

while the propagating modes Lagrangian density cannot decouple. Using any type of discritization the Lagrangian density given in equation (15) will become a real bilinear form. For cut-off modes the form of $-\mathcal{L}$ appears to be positive since $k_z^2 = -|k_z^2|$ according to equation (6) . Thus the solution will correspond to the minimum of the bilinear form which can be found by standard numerical techniques such as the conjugate gradient method [15]. For propagating modes $k_z^2 = |k_z^2|$ the solution will correspond to a saddle point of the linear form and can be found using techniques such as the ones described in [16].

[1] From now on we will suppress the indices q, n

THE QUASI HAMILTONIAN

In certain cases it is desirable to obtain first order equations instead of the second order equation (12). In analytical mechanics [14] their is a well known technique to reach this goal using the Hamiltonian construction. Since L given in equation (13) is not strictly speaking a Lagrangian (time which appears in proper Lagrangians is replaced here by the longitudinal coordinate z) we will denote the analogue construction of the Hamiltonian a "quasi Hamiltonian". For convenience we introduce the Lagrangian density $\bar{\mathcal{L}}$:

$$
\begin{aligned}
\bar{\mathcal{L}} &= -2\mathcal{L} = \partial_z V \partial_z V^* - k_z^2 V V^* + \frac{\omega^2}{c^2 k_\perp^2} v v^* \\
&+ \partial_z V v^* + \partial_z V^* v - j k_z i V^* + j k_z^* i^* V \qquad (17)
\end{aligned}
$$

in which we utilized equation (14). Next we define the quasi canonical momentums of $V' \equiv \partial_z V$:

$$
\Pi \equiv \frac{\partial \bar{\mathcal{L}}}{\partial V'} = V'^* + v^* = -j k_z^* I^* \qquad (18)
$$

in which equation (5) is used. Notice that the quasi canonical momentums are proportional to I^*. Having done this we are in a position to define the quasi Hamiltonian density:

$$
\begin{aligned}
\mathcal{H} &\equiv V' \Pi + V'^* \Pi^* - \bar{\mathcal{L}} \\
&= |k_z|^2 |I|^2 + k_z^2 |V|^2 - j k_z I v^* + j k_z^* I^* v \\
&- \frac{k_z^2}{k_\perp^2} |v|^2 + j k_z i V^* - j k_z^* i^* V \\
&= k_z [k_z (|V|^2 \pm |I|^2) - j I v^* \pm j I^* v \\
&- \frac{k_z}{k_\perp^2} |v|^2 + j i V^* \mp j i^* V] \qquad (19)
\end{aligned}
$$

the upper sign should be attributed to propagating modes while the lower signs should be attributed to decaying modes. Thus $\bar{\mathcal{L}}$ can be written as:

$$
\begin{aligned}
\bar{\mathcal{L}} &= V' \Pi + V'^* \Pi^* - \mathcal{H} \\
&= k_z [\mp j I^* V' + j I V'^* - k_z (|V|^2 \pm |I|^2) + j I v^* \\
&\mp j I^* v - j i V^* \pm j i^* V + \frac{k_z}{k_\perp^2} |v|^2] \qquad (20)
\end{aligned}
$$

Our next step will be to take the variational derivative with respect to I and V and their complex conjugates of $\bar{L}$ which is defined as:

$$
\bar{L} = \int \bar{\mathcal{L}} dz = -2L \qquad (21)
$$

This will lead to equations (5) and their complex conjugates.

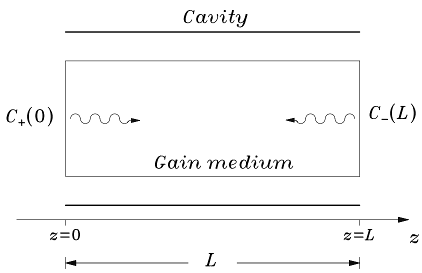

Figure 2: Interaction of the electromagnetic field in a gain medium

THE FORWARD-BACKWARD FORMULATION

In terms of V and I one can define the following new variables [13]:

$$C_+ \equiv \frac{1}{2}(V+I)e^{-jk_z z}, \quad C_- \equiv \frac{1}{2}(V-I)e^{jk_z z} \tag{22}$$

Or vice-versa as:

$$V = C_+ e^{jk_z z} + C_- e^{-jk_z z}, \quad I = C_+ e^{jk_z z} - C_- e^{-jk_z z} \tag{23}$$

Thus C_+ and C_- appear as the amplitudes of forward and backward waves respectfully (see figure 2) in the case of propagating modes. Inserting the above variables into $\bar{L}$ given in equation (21) we obtain for propagating modes:

$$\begin{aligned} \bar{L} &= jk_z\{2\int dz[C_-^* C_-' + C_+^{'*} C_+ - C_+^*\beta + C_+\beta^* \\ &+ C_-^*\alpha - C_-\alpha^* - j\frac{k_z}{2k_\perp^2}|v|^2] + (C_-^* C_+ e^{2jk_z z} \\ &- C_- C_+^* e^{-2jk_z z} - |C_-|^2 - |C_+|^2)|_0^{L_w}\} \end{aligned} \tag{24}$$

In which:

$$\alpha = \frac{1}{2}(v-i)e^{jk_z z} \qquad \beta = \frac{1}{2}(v+i)e^{-jk_z z} \tag{25}$$

At this stage one is tempted to discard the boundary term in the above equation since it appears to have no effect on the resulting equations, however, this will lead to unphysical boundary conditions and thus should be avoided. Taking the variational derivative we obtain the equations:

$$C_-' = -\alpha, \qquad C_+' = -\beta \tag{26}$$

and their complex conjugates which provides a truly elegant way to compute the field dynamics.

CONCLUSIONS

Three different action principles were obtained in this work: one in terms of the V modal amplitude leading to second order equations. Another principle was formulated in terms of the V, I amplitudes through the quasi Hamiltonian concept leading to first order equations. And finally an action principle in terms of the forward and backward modes were derived including the correct boundary conditions for those equations. It was outlined how the action can be used as a basis for a numerical scheme and that different numerical techniques should be utilized for propagating and cut-off modes.

REFERENCES

[1] Asher Yahalom, Yosef Pinhasi & Yuri Lurie "Spectral and Variational Principles of Electromagnetic Field Excitation in Wave Guides" accepted by Physics Letters A (2005) [online publication http://dx.doi.org/10.1016/j.physleta.2005.06.054].

[2] T. Fla, Physics of Plasmas **1** # 8 2409-2418 (1994).

[3] R. Elvsen & J. Larsson, Physica Scripta **47** # 4 571-575 (1993).

[4] I. Lindell, IEEE Trans. Microwave Theory and Techniques **40** 402 (1983).

[5] B. Little & H. Haus, IEEE Journal of Quantum Electronics **31** # 12 2258-2264 (1995).

[6] G. Berry, S. Burke, C. Smartt, T. Benson & P. Kendel, IEE Proceedings-Optoelectronics **142** # 1 66-75 (1995).

[7] S. Patrick & K. Webb, IEEE Transactions on Microwave Theory and Techniques **40** # 4 692-698 (1992).

[8] A. Bogolyubov, A. Delitsyn & I. Mogilevskii, Journal of Communications Technology and Electronics **45** supp2 s126-s130 (2000).

[9] I. Erteza & J. Goodman, Optics Communications, **115** # 1-2 52-56 (1995).

[10] N. Markuvitz & J. Schwinger, J. Appl. Phys. **22** 806 (1951)

[11] J. D. Jackson *Classical Electrodynamics* (John Wiley & Sons, New York, 2ND ED., 1975)

[12] R. E. Collin, *Foundations for Microwave Engineering*, (Mcgraw-Hill, Singapore, 1966)

[13] Y. Pinhasi, *Analysis of Super-Radiance Effects and of Transverse Mode Excitation and Coupling in Free-Electron Lasers* (PHD Thesis Submitted to the Senate of Tel-Aviv University, 1995)

[14] H. Goldstein *Classical Mechanics* (Addison-Wesley, Reading, M. 1950)

[15] W. H. Press, S. A. Teukolsky, W. T. Vetterling & B. P. Flannery *Numerical Recipes in C The Art of Scientific Computing - Second Edition* (Cambridge University Press 1992)

[16] M. R. Cui, Applied Numerical Mathematics, Volume 50, Issue 2 , 133-146 August (2004).