Computation of Transfer Maps from Surface Data

Using Elliptical Cylinders

Objective

- To obtain an accurate representation of the wiggler field that is analytic and satisfies Maxwell equations exactly. We want a vector potential that is analytic and $\nabla\times\nabla\times\boldsymbol{A}=0$.

- Using a Hamiltonian expressed as a series of homogeneous polynomials

$$K=-\sqrt{\frac{(p_t+q\phi)^2}{c^2}-(\vec{p}_\perp-q\vec{A}_\perp)^2}-qA_z=\sum_{\alpha=1}^{N}h_\alpha(z)K_\alpha(x,p_x y,p_y,\tau,p_\tau)$$

- ...we compute the design orbit and the transfer map about the design orbit to some order. We obtain a factorized symplectic map for single-particle orbits through the wiggler

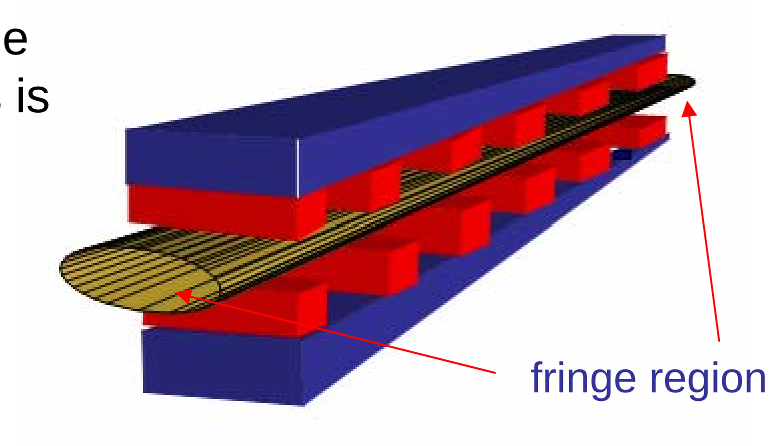
$$M\ =\ R_2 e^{:f_3:}e^{:f_4:}e^{:f_5:}e^{:f_6:}\ldots$$

- Use B-V data to find an accurate series representation of interior vector potential through order N in (x,y) deviation from design orbit.

$$A_x(x,y,z)=\sum_{k=1}^{N}a_k^x(z)P_k(x,y)$$

Fitting Wiggler Data

Place an imaginary elliptic cylinder between pole faces, extending beyond the ends of the magnet far enough that the field at the ends is effectively zero.



- Data on regular Cartesian grid

4.8cm in x, dx=0.4cm

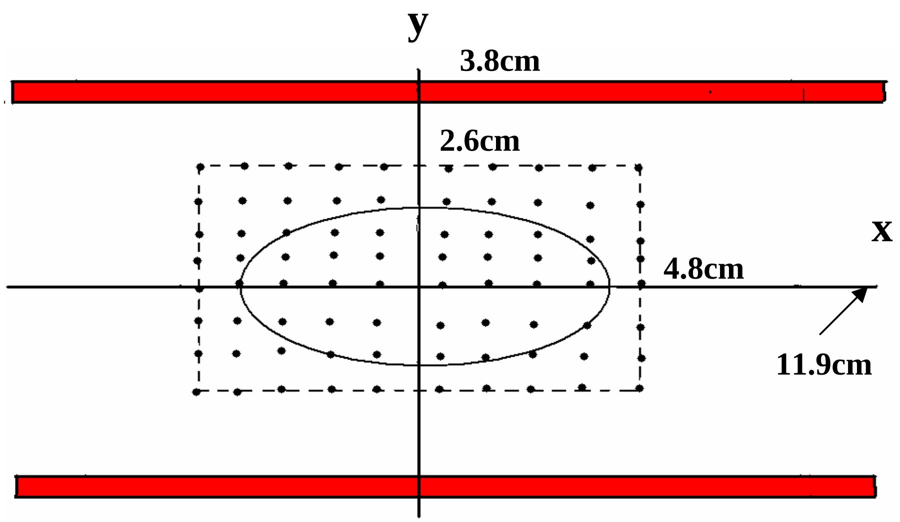
2.6cm in y, dy=0.2cm

480cm in z, dz=0.2cm

- Field components Bx, By, Bz in one quadrant given to a precision of 0.05G.

- Fitted onto elliptic cylindrical surface using bicubic interpolation to obtain the normal component.

- Compute the interior vector potential and all its desired derivatives.



Elliptic Coordinates

Defined by relations:

$$x = f\cosh u\cos v$$
$$y = f\sinh u\sin v$$

where f is the focal distance.

Letting z=x+iy, w=u+iv we have

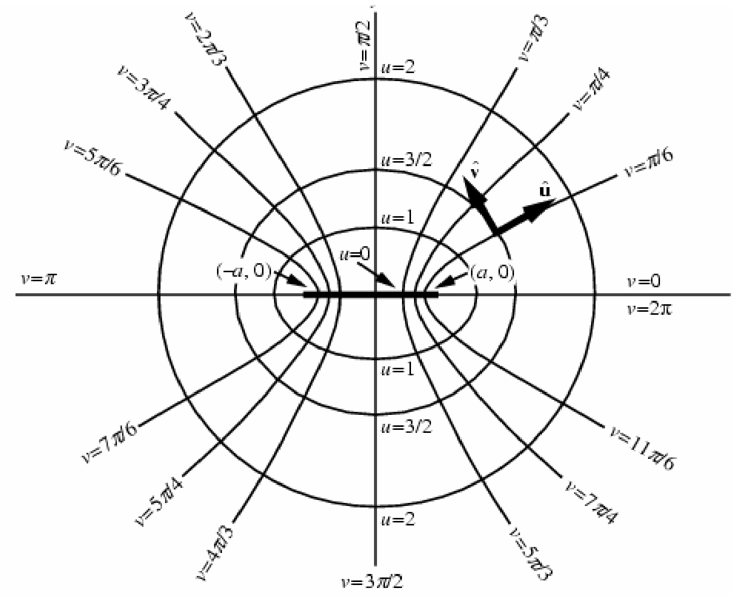
$$z = \Im(w) = f\cosh w$$

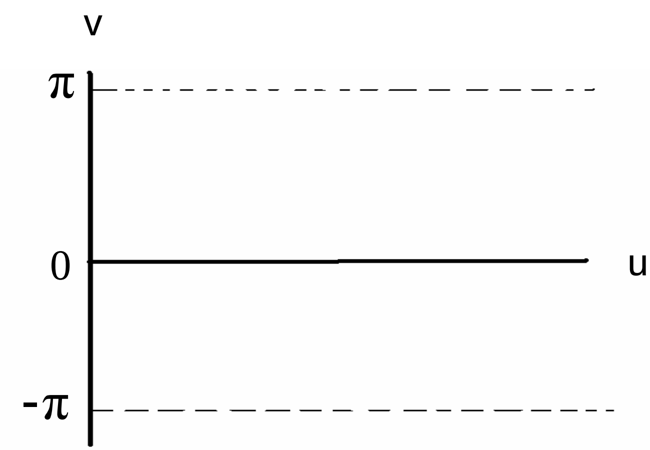
Jacobian:

$$J(u,v) = \left|\Im'(z)\right| = \left|f\sinh z\right| = f(\sinh^2 u + \sin^2 v)$$

Laplacian:

$$\nabla^2 = \frac{1}{J(u,v)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial}{\partial v^2}\right) + \frac{\partial^2}{\partial z^2}$$





- Interested in fitting air-core magnets, so we can use scalar potential satisfying $(\nabla_\perp^2 - k^2)\tilde{\psi}(u,v,k) = 0$

- Search for product solutions in elliptic coordinates

The solutions for V are

Mathieu functions

$ce_m(v,q)$ with λ = $a_m(q)$, ← even in v

$se_m(v,q)$ with λ = $b_m(q)$. ← odd in v

The associated solutions for U are

Modified Mathieu functions

$$Ce_m(u,q) = ce_m(iu,q),$$
$$Se_m(u,q) = -ise_m(iu,q).$$

For $\tilde{\Psi}(x,y,k)$ we make the Ansatz

$$\tilde{\psi}(x,y,k) = \sum_{n=0}^{\infty}\left[\left(\frac{F_n(k)}{Se_n'(u_b,k)}\right)Se_n(u,k)se_n(v,k) + \left(\frac{G_n(k)}{Ce_n'(u_b,k)}\right)Ce_n(u,k)ce_n(v,k)\right].$$

Boundary-Value Solution

- Normal component of field on bounding surface defines a Neumann problem with interior field determined by moment expansion on the boundary:

$$B_u(v,k)=\partial_u\tilde{\psi}=\sum_{n=0}^{\infty}F_n(k)se_n(v,k)+G_n(k)ce_n(v,k)$$

- Moments $F_n(k),G_n(k)$ on boundary are integrated against a kernel that falls off rapidly with large k, minimizing the contribution of high-frequency noise.

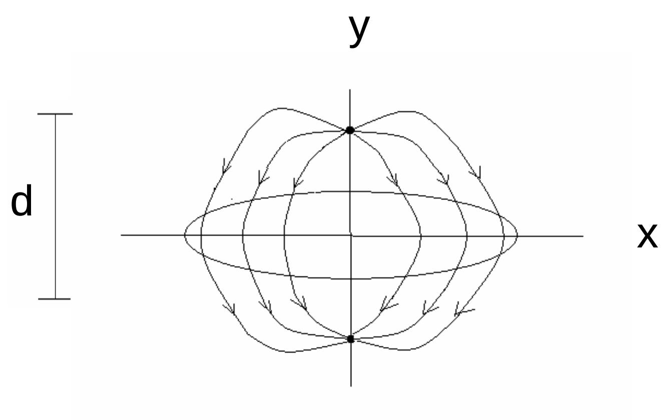
- On-axis gradients are found that specify the field and its derivatives.

- Power series representation

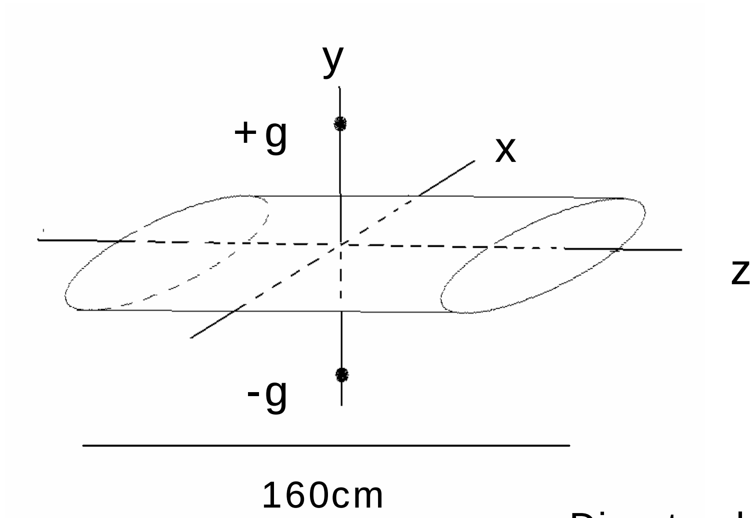
$$A_{\left\{\begin{smallmatrix}x\\y\end{smallmatrix}\right\}}=\sum_{m=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^l(m-1)!}{2^{2l}\,l!(l+m)!}\left\{\begin{matrix}x\\y\end{matrix}\right\}(x^2+y^2)\left[\mathrm{Re}(x+iy)^m C_{m,s}^{[2l+1]}(z)-\mathrm{Im}(x+iy)^m C_{m,c}^{[2l+1]}(z)\right]$$

$$A_z=\sum_{m=1}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^l(2l+m)(m-1)!}{2^{2l}\,l!(l+m)!}(x^2+y^2)\left[-\mathrm{Re}(x+iy)^m C_{m,s}^{[2l]}(z)+\mathrm{Im}(x+iy)^m C_{m,c}^{[2l]}(z)\right]$$

Dipole Field Test



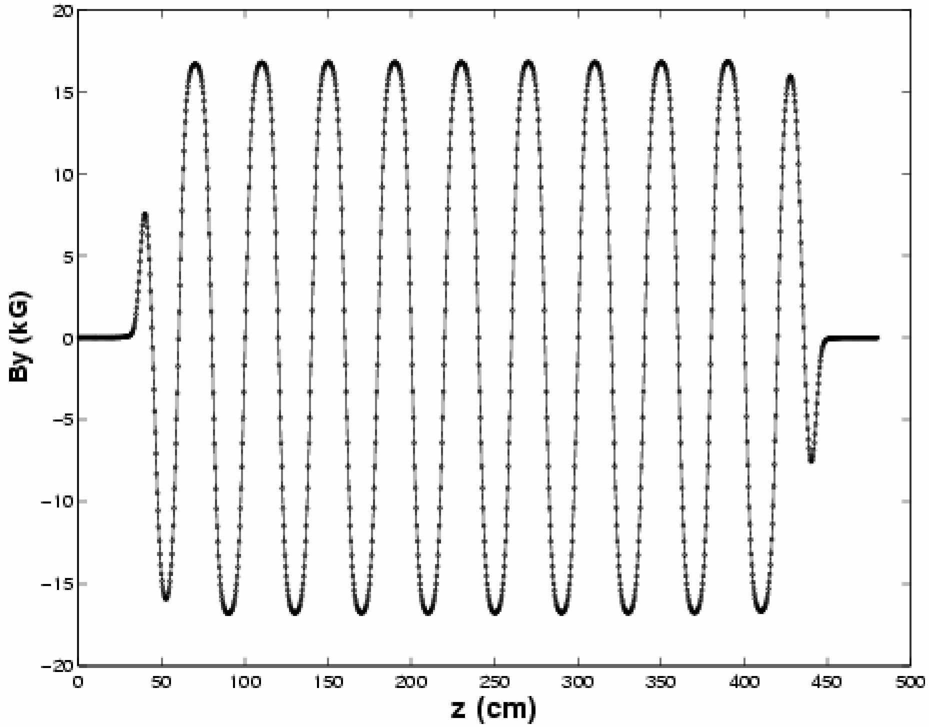
- Simple field configuration in which scalar potential, field, elliptical moments, and on-axis gradients can be determined analytically.
- Tested for two different aspect ratios: 4:3 and 5:1.



Pole location: d=4.7008cm
Pole strength: g=0.3Tcm^2
Semimajor axis: 1.543cm/4.0cm
Semiminor axis: 1.175cm/0.8cm
Boundary to pole: 3.526cm/3.9cm
Focal length: f=1.0cm/3.919cm
Bounding ellipse: u=1.0/0.2027

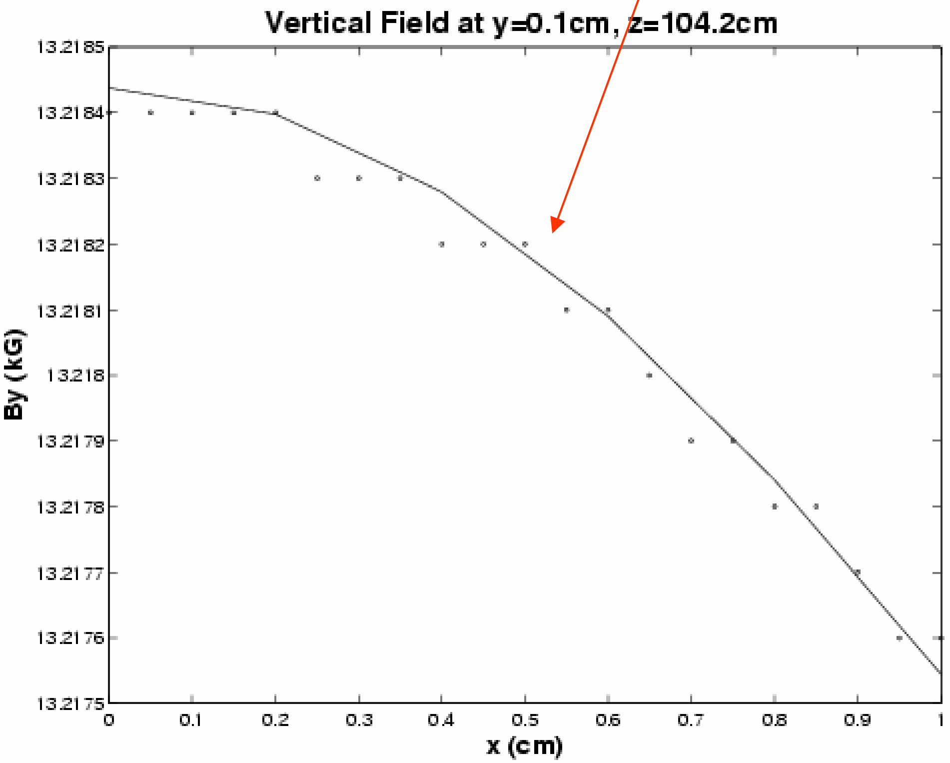
Direct solution for interior scalar potential
accurate to 3*10-10: set by convergence/roundoff
Computation of on-axis gradients C1, C3, C5
accurate to 2*10-10 before final Fourier transform
accurate to 2.6*10-6 after final Fourier transform

Fit to the Proposed ILC Wiggler Field Using Elliptical Cylinder



Fit to vertical field By
at x=0.4cm, y=0.2cm.

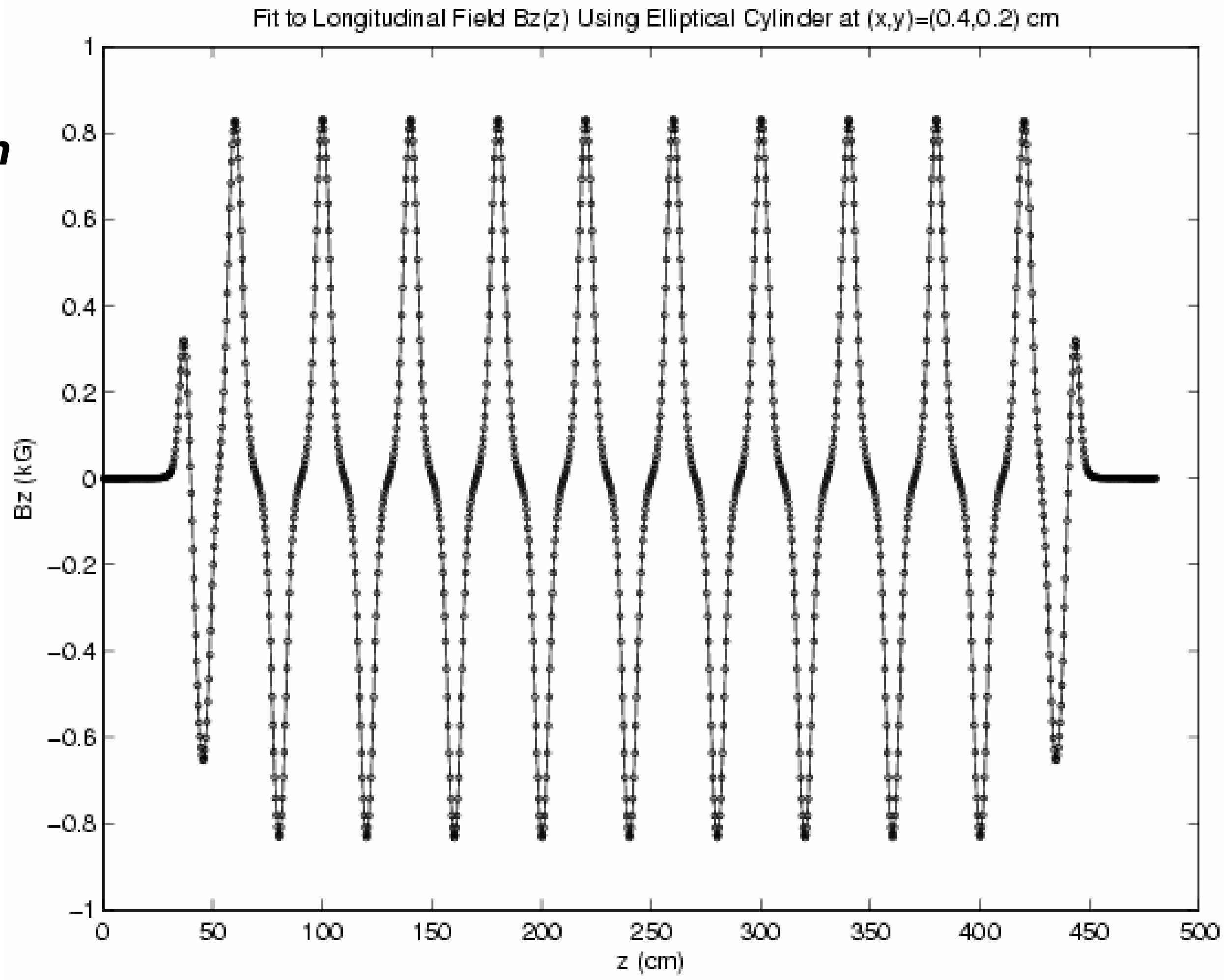
Note precision of data.

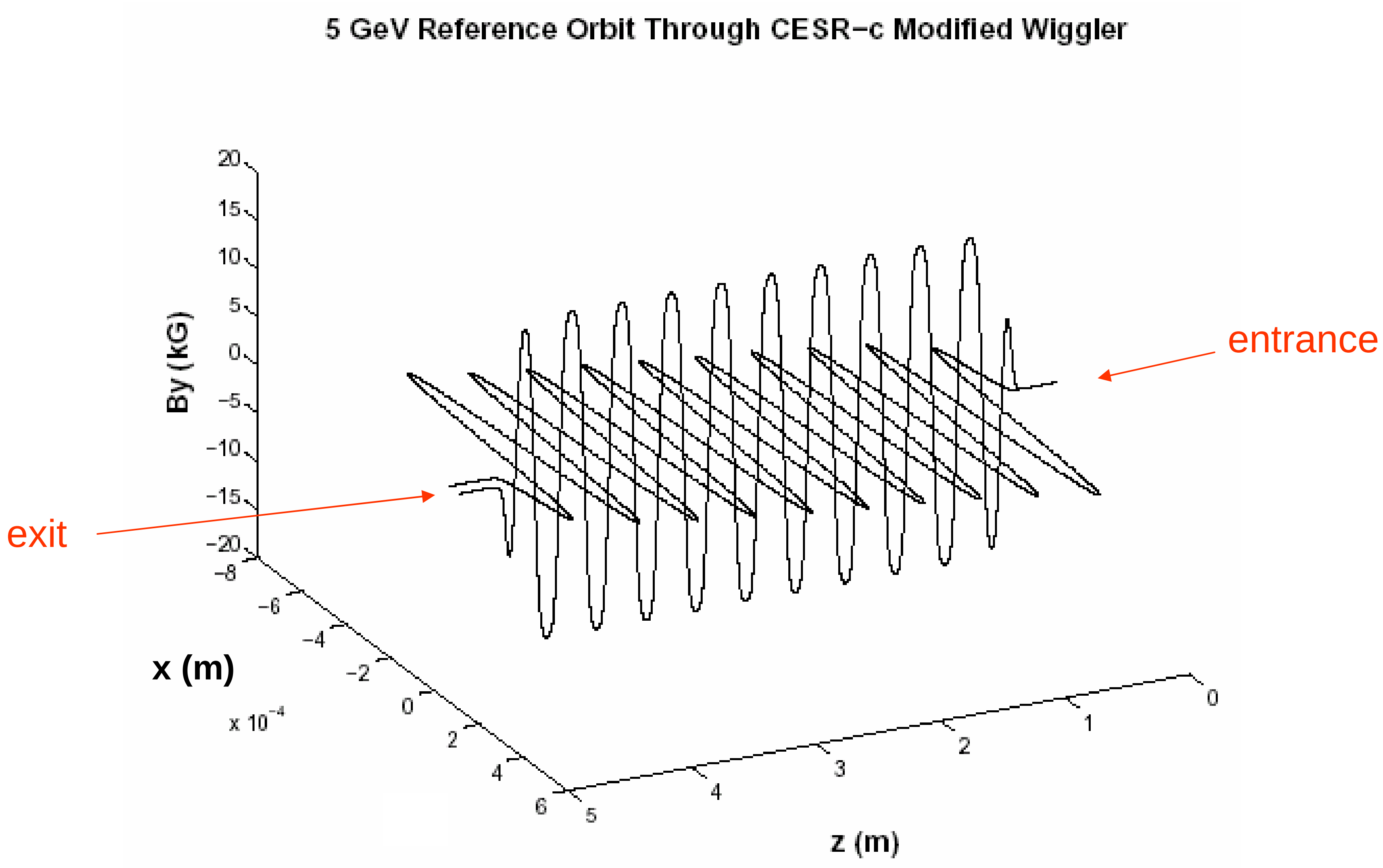


Fit to the Proposed ILC Wiggler Field Using Elliptical Cylinder

No information about Bz was used to create this plot.

Fit to longitudinal field Bz at x=0.4cm, y=0.2cm.

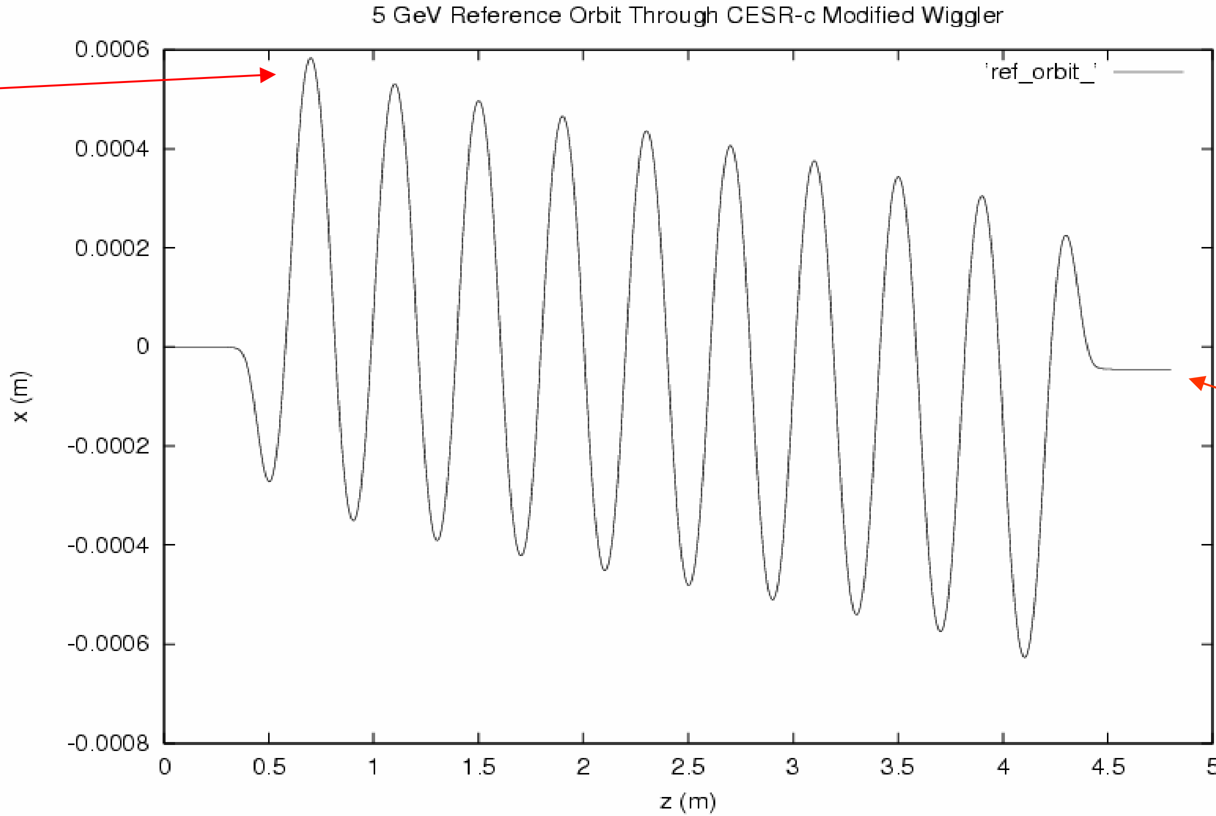




5 GeV Reference Orbit Through CESR−c Modified Wiggler
By (kG)
x (m)
x 10^-4
z (m)
entrance
exit

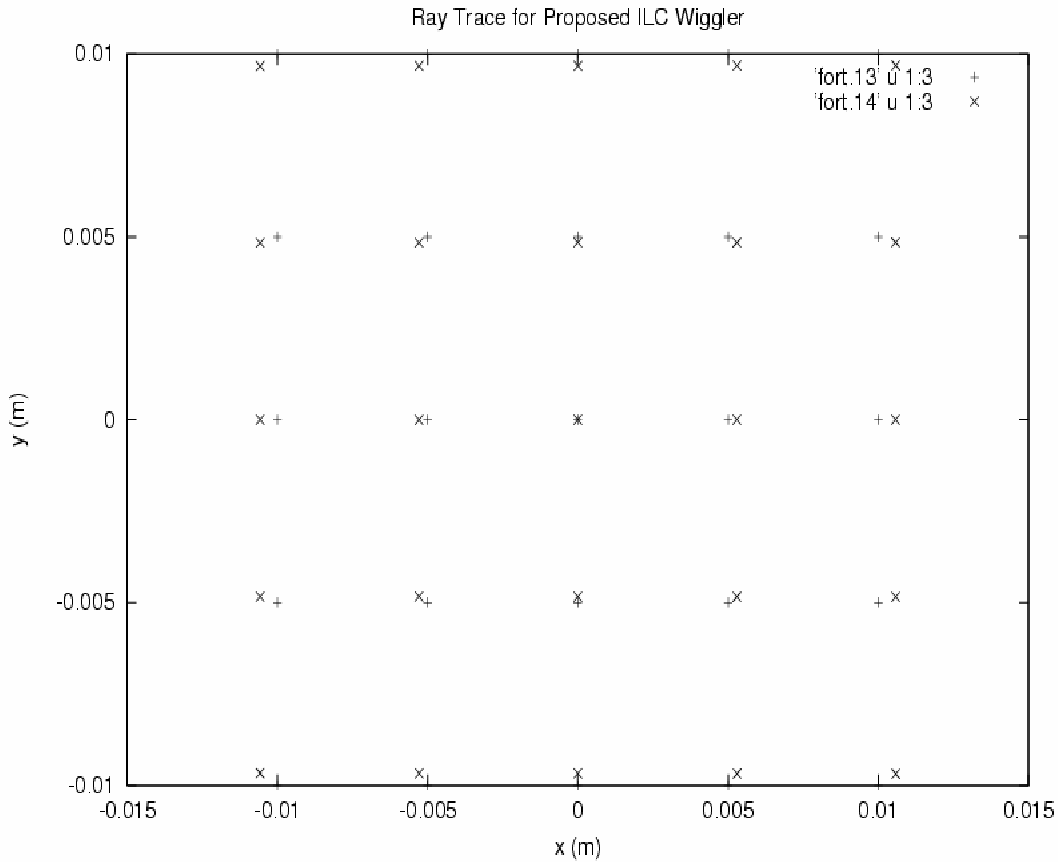
Reference orbit through proposed ILC wiggler at 5 GeV

Maximum deviation 6mm



Exit displacement

Ray trace for proposed ILC wiggler



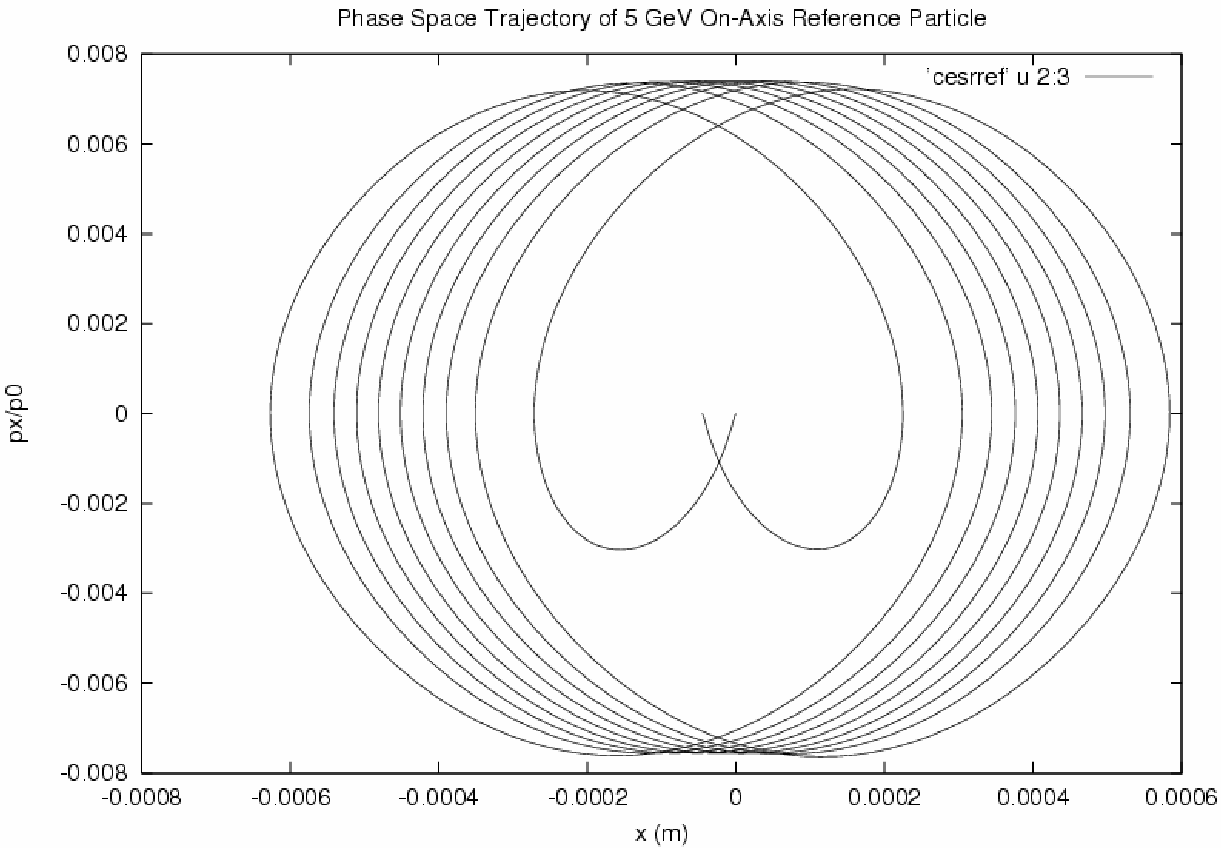
Initial grid of spacing 5mm in the xy plane.

+ initial values, x final values +.

Defocusing in x, focusing in y

Phase space trajectory of 5 GeV on-axis reference particle

Mechanical momentum



REFERENCE ORBIT DATA

At entrance:

x (m) = 0.000000000000000E+000
can. momentum p_x = 0.000000000000000E+000
mech. momentum p_x = 0.000000000000000E+000
y (m) = 0.000000000000000E+000
mech. momentum p_y = 0.000000000000000E+000
angle phi_x (rad) = 0.000000000000000E+000
time (s) = 0.000000000000000E+000
p_t/(p0c) = -1.0000000052213336

At exit:

x (m) = -4.534523825505101E-005
can. momentum p_x = 1.245592900543683E-007
mech. momentum p_x = 1.245592900543683E-007
y (m) = 0.000000000000000E+000
mech. momentum p_y = 0.000000000000000E+000
angle phi_x (rad) = 1.245592900543687E-007
time of flight (s) = 1.60112413288E-008
p_t/(p0c) = -1.0000000052213336

Bending angle (rad) = 1.245592900543687E-007

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matrix for map is :

1.05726E+00	4.92276E+00	0.00000E+00	0.00000E+00	0.00000E+00	-5.43908E-05
2.73599E-02	1.07323E+00	0.00000E+00	0.00000E+00	0.00000E+00	-4.82684E-06
0.00000E+00	0.00000E+00	9.68425E-01	4.74837E+00	0.00000E+00	0.00000E+00
0.00000E+00	0.00000E+00	-1.14609E-02	9.76409E-01	0.00000E+00	0.00000E+00
3.61510E-06	-3.46126E-05	0.00000E+00	0.00000E+00	1.00000E+00	9.87868E-05
0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00	1.00000E+00

nonzero elements in generating polynomial are :

f(28)=f(30 00 00)=-0.86042425633623D-03
f(29)=f(21 00 00)= 0.56419178301165D-01
f(33)=f(20 00 01)=-0.76045220664105D-03
f(34)=f(12 00 00)=-0.25635788141484D+00

focusing

defocusing

. . . . Currently through f(923) – degree 6.

Advantages of Surface Fitting

- Uses functions with known (orthonormal) completeness properties and known (optimal) convergence properties.
- Maxwell equations are exactly satisfied. (Other procedures.)
- Error is globally controlled. The error must take its extrema on the boundary, where we have done a controlled fit.
- Insensitivity to errors due to Laplace kernel smoothing. Improves accuracy in higher derivatives. Insensitivity to noise improves with increased distance from the surface: advantage over circular fitting.
- Careful benchmarking.