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Analytical Estimation of Dynamic Aperture Limited by Wigglers in a Storage Ring

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Contents

- *Dynamic Apertures of Limited by Multipoles in a Storage Ring*
- *Dynamic Apertures Limited by Wigglers in a Storage Ring*
- *Application to TESLA damping ring*
- *Conclusions*

Dynamic Aperturs of Multipoles

Hamiltonian of a single multipole

$$H=\frac{p^{2}}{2}+\frac{K(s)}{2}x^{2}+\frac{1}{m!B\rho}\frac{\partial^{m-1}B_{z}}{\partial x^{m-1}}x^{m}L\sum_{k=-\infty}^{\infty}\delta(s^{*}-kL) \qquad \text{Eq. 1}$$

Where L is the circumference of the storage ring, and s is the place where the multipole locates (m=3 corresponds to a sextupole, for example).*

Important Steps to Treat the Perturbed Hamiltonian

- *Using action-angle variables*
- *Hamiltonian differential equations should be replaced by difference equations*

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

Since under some conditions the Hamiltonian don't have even numerical solutions

Standard Map

Near the nonlinear resonance, simplify the difference equations to the form of **STANDARD MAP**

$$\overline{I} = I + K_0 \sin\theta$$
$$\overline{\theta} = \theta + \overline{I}$$

Some explanations

Definition of TWIST MAP

$$\bar{x} = x + Kf(\theta)$$

$$\bar{\theta} = \theta + g(\bar{x}) \pmod{1}$$

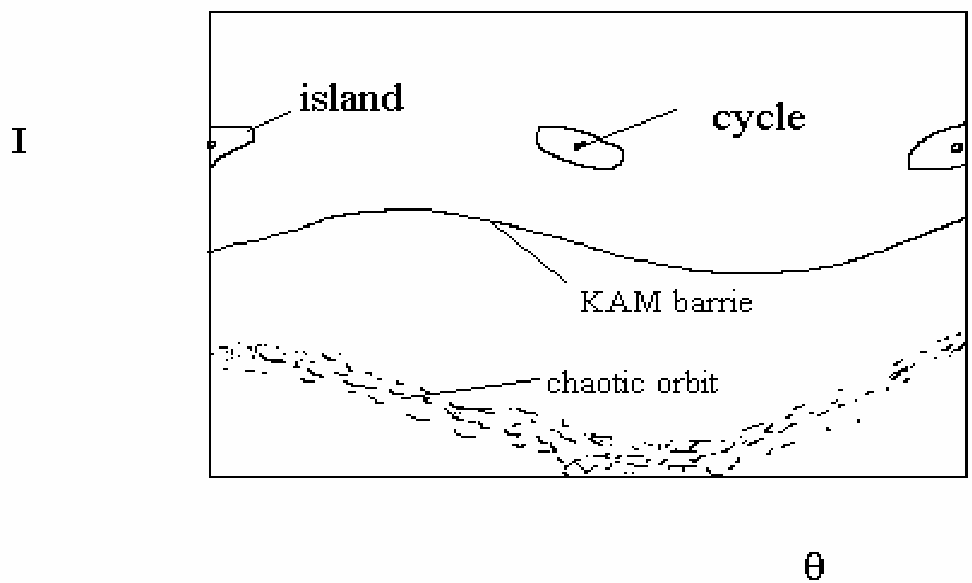
where

$$f(\theta + 1) = f(\theta)$$

$$\frac{dg(x)}{dx} \neq 0, \forall x$$

Some explanations

Classification of various orbits in a Twist Map, Standard Map is a special case of a Twist Map.



Stochastic motions

For Standard Map,

when $K_0 \geq 0.97164$ **global stochastic motion** *starts. Statistical descriptions of the nonlinear chaotic motions of particles are subjects of research nowadays. As a preliminary method, one can resort to* *Fokker-Planck* *equation .*

m=4 Octupole as an example

Step 1) Let m=4 in Eq. 1 *, and use canonical variables obtained from the unperturbed problem.*

Step 2) Integrate the Hamiltonian differential equation over a natural periodicity of L, the circumference of the ring

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m=4 Octupole as an example

Step 3)

$$\overline{J_1} = J_1 + A\sin 4\Phi_1$$

$$\overline{\Phi_1} = \Phi_1 + B\,\overline{J_1}$$

$$A = \left(\frac{J_1\beta_x^2(s_{m=4})}{2}\right)\left(\frac{b_3 L}{\rho}\right)$$

$$B = 2\beta_x^2(s_{m=4})\left(\frac{b_3 L}{\rho}\right)$$

$$K_0 = 4AB$$

m=4 Octupole as an example

Step 4) $K_0 = 4AB < 1(0.97164\quad)$

$$J_1 < \left(\frac{1}{2\beta_x^{\,2}(s_{m=4})}\right)\left(\frac{\rho}{|b_3|L}\right)$$

One gets finally

$$A_{dyna,oct,x} = (2J_1\beta_x(s))^{1/2} = \frac{\beta_x^{\,1/2}(s)}{\beta_x(s_{m=4})}2\beta_x^{\,2}(s_{m=4})\left(\frac{\rho}{|b_3|L}\right)^{1/2}$$

General Formulae for the Dynamic Apertures of Multipoles

$$A_{dyna,2m} = \sqrt{2\beta_x(s)}\left(\frac{1}{m\beta_x{}^m(s(2m))}\right)^{\frac{1}{2(m-2)}}\left(\frac{\rho}{|b_{m-1}|L}\right)^{\frac{1}{m-2}}$$

Eq. 2

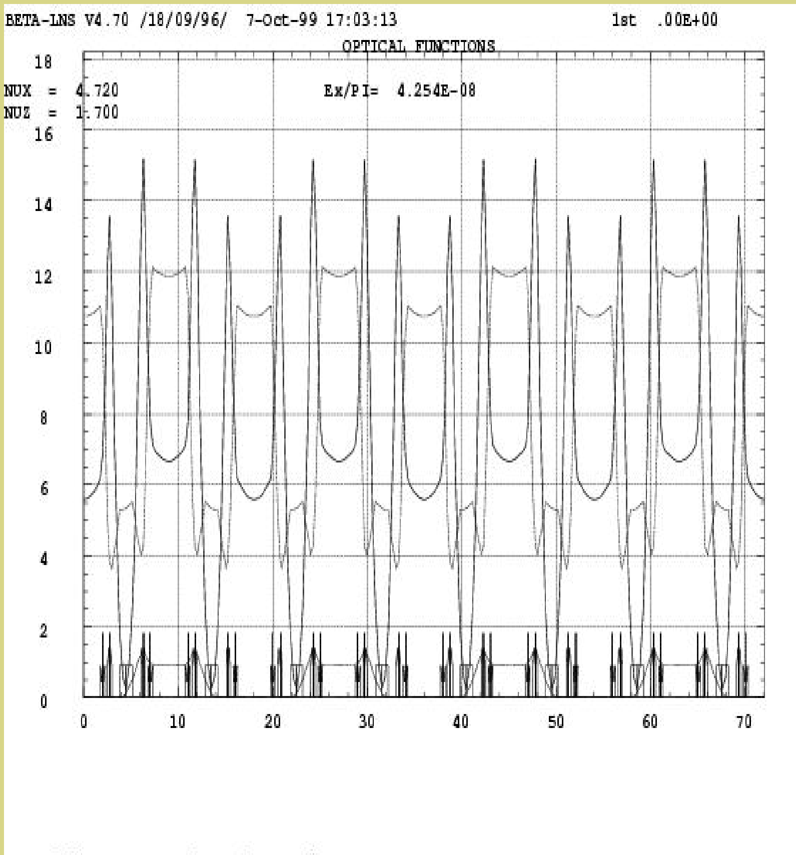
$$A_{dyna,total} = \frac{1}{\sqrt{\sum_i \frac{1}{A^2_{dyna,sext,i}} + \sum_j \frac{1}{A^2_{dyna,oct,j}} \sum_k \frac{1}{A^2_{dyna,deca,k}} + \ldots}}$$

Eq. 3

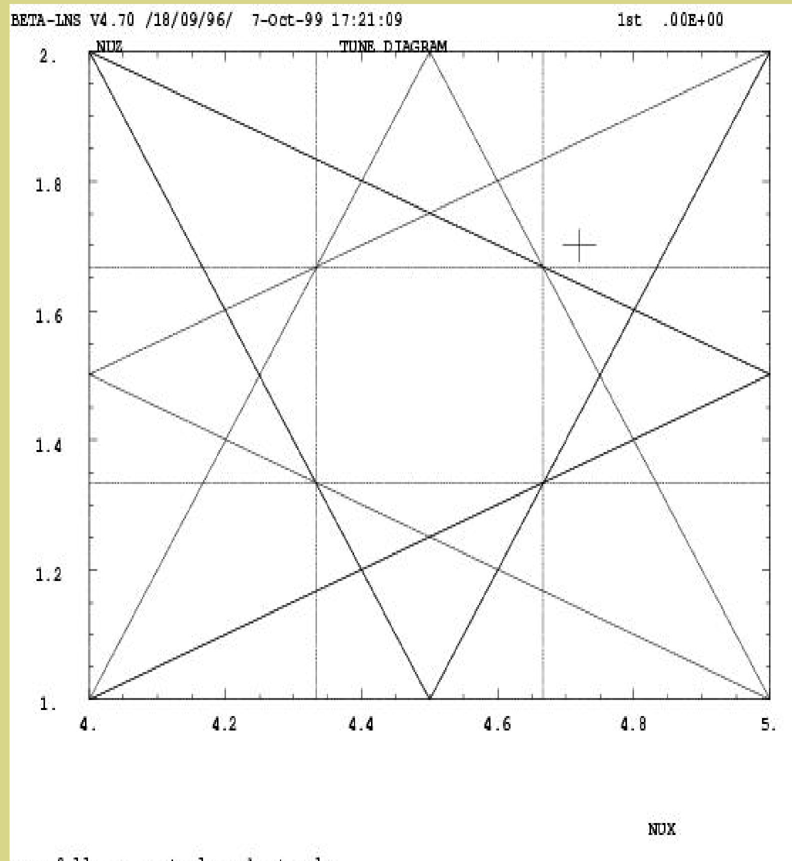
$$A_{dyna,sext,y} = \sqrt{\frac{\beta_x(s_1)}{\beta_y(s_1)}\left(A^2{}_{dyna,sext,x} - x^2\right)}$$

Super-ACO

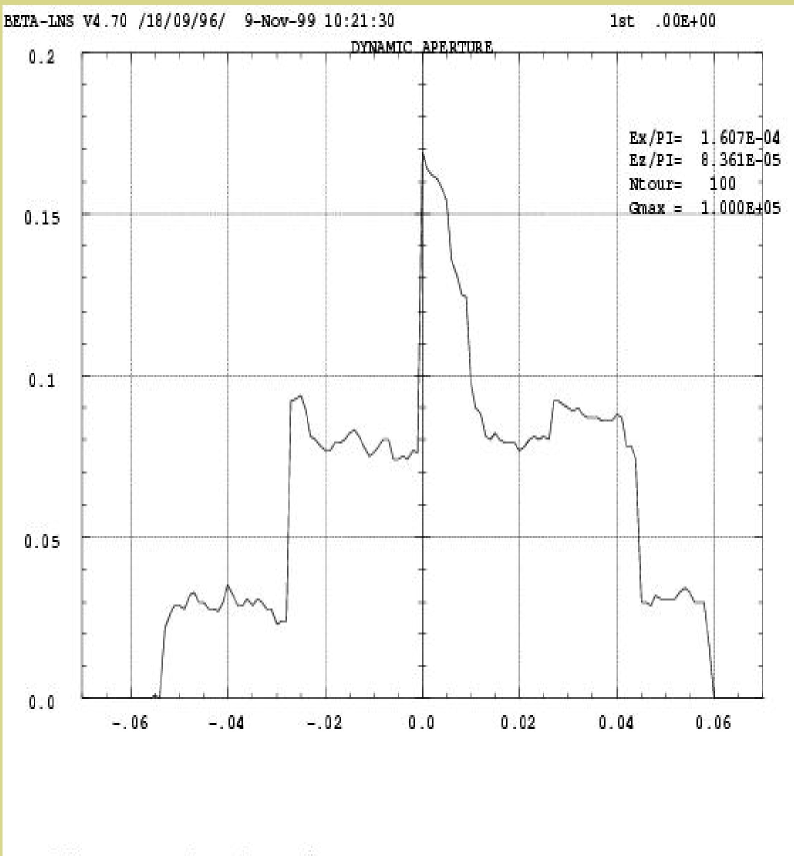
Lattice



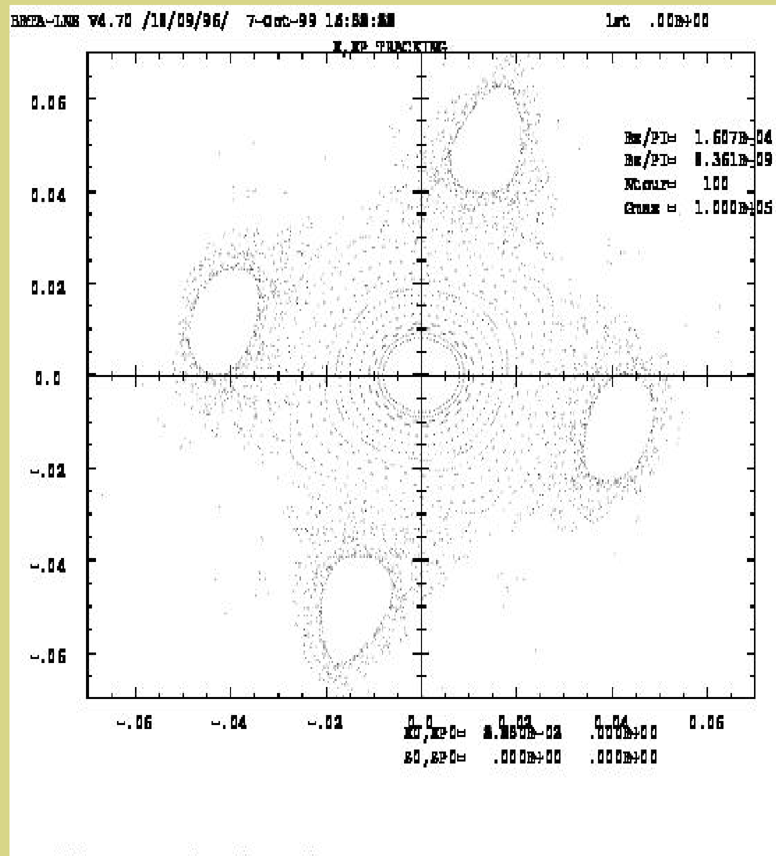
Working point



Single octupole limited dynamic aperture simulated by using BETA

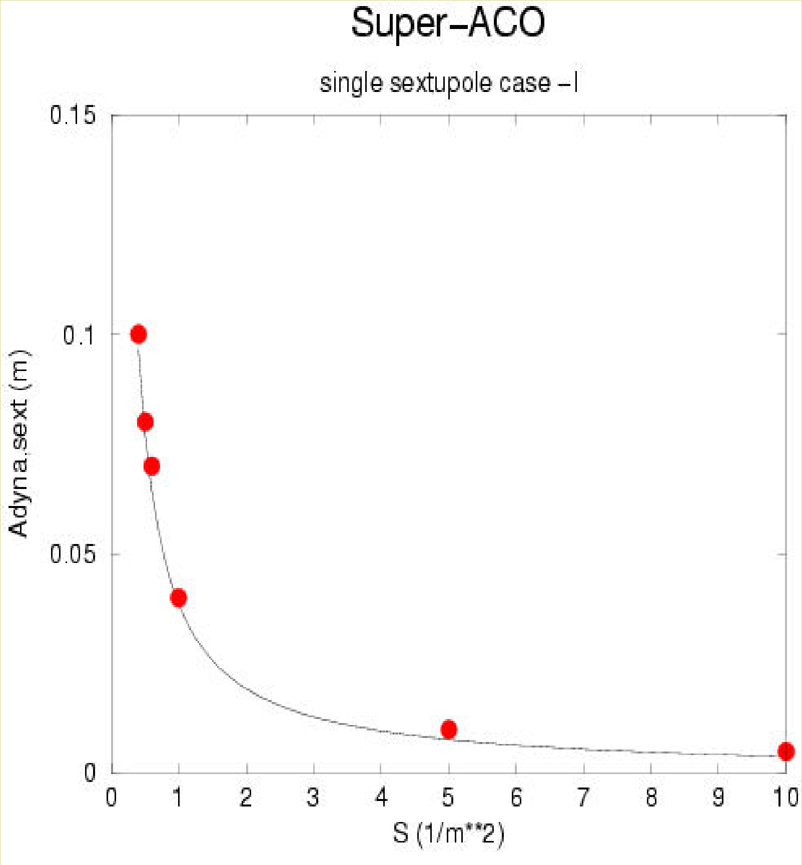


x-y plane

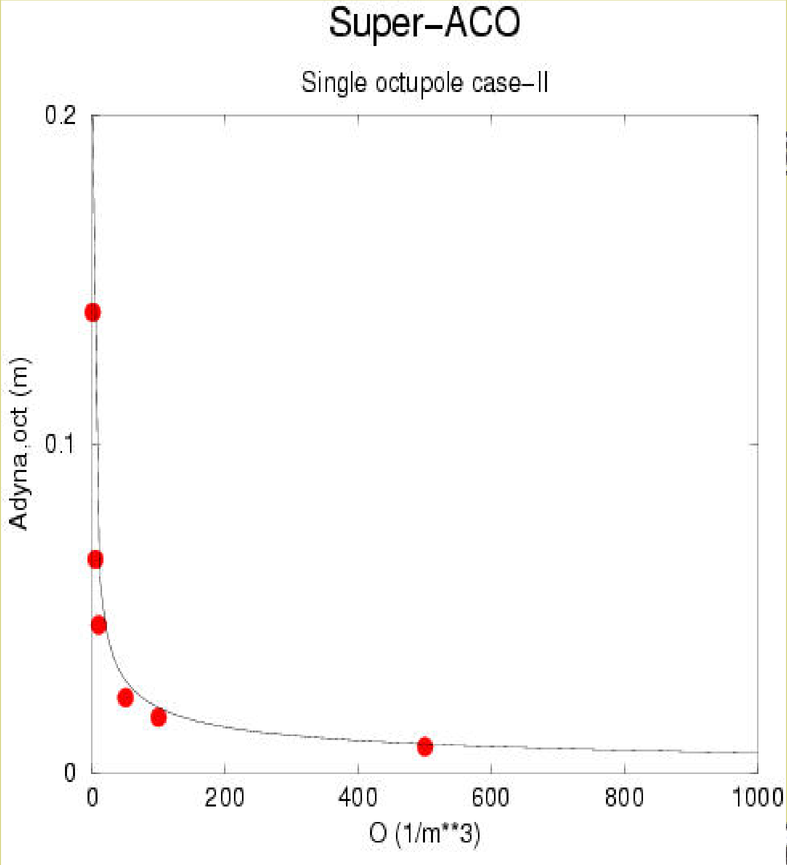


x-xp phase plane

Comparisions between analytical and numerical results

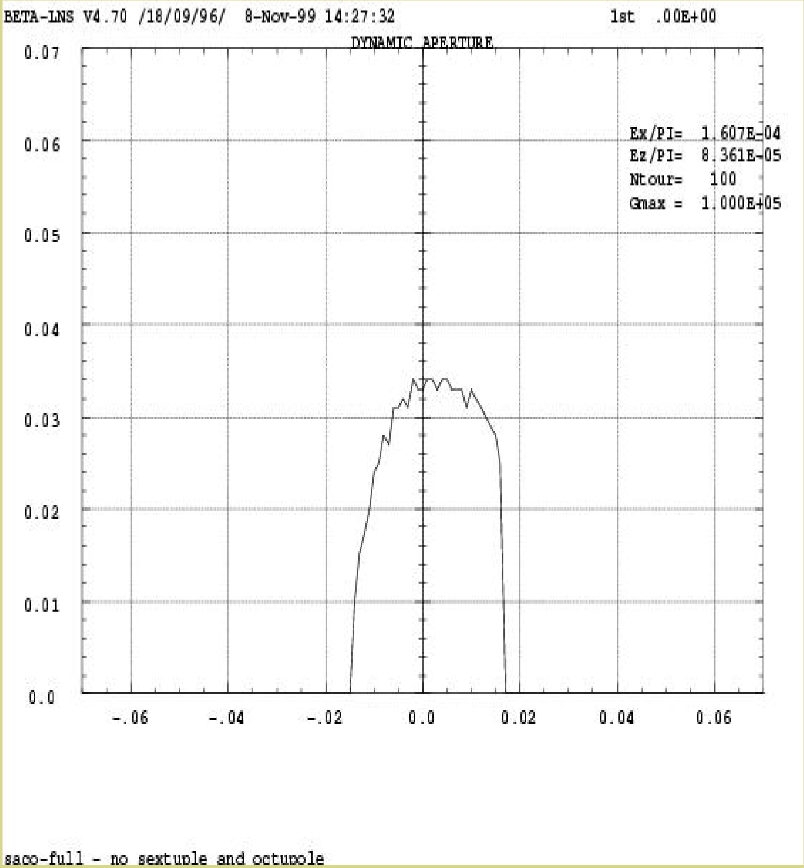


Sextupole

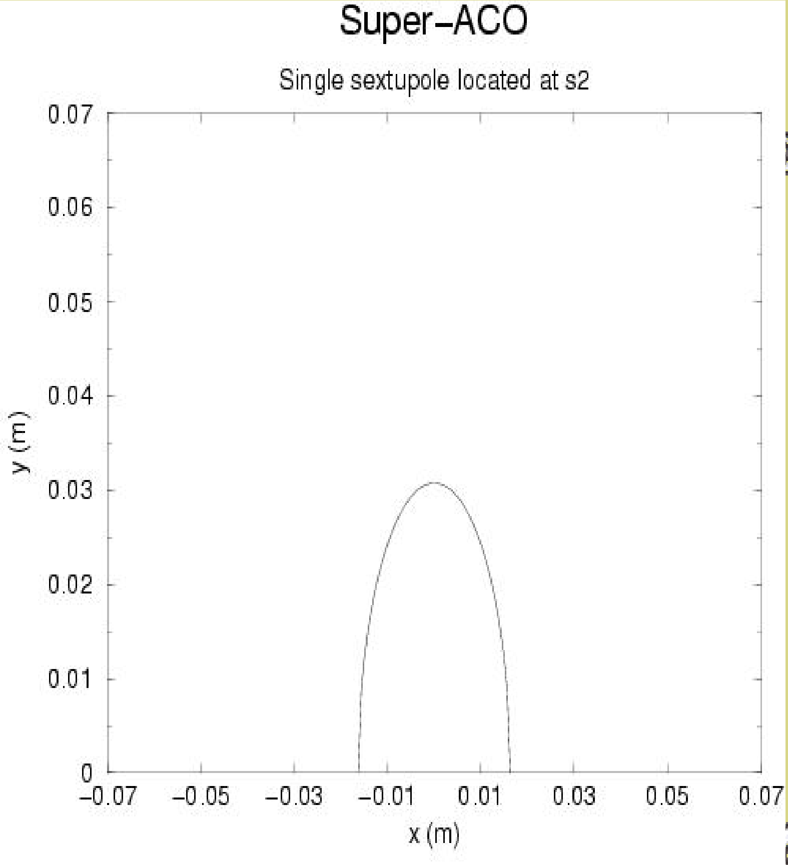


Octupole

2D dynamic apertures of a sextupole



Simulation result



Analytical result

Wiggler

Ideal wiggler magnetic fields

$$B_x = \frac{k_x}{k_y} B_0 \sinh(k_x x)\sinh(k_y y)\cos(ks)$$

$$B_y = B_0 \cosh(k_x x)\cosh(k_y y)\cos(ks)$$

$$B_z = -\frac{k}{k_y} B_0 \cosh(k_x x)\sinh(k_y y)\sin(ks)$$

$$k_x^2 + k_y^2 = k^2 = \left(\frac{2\pi}{\lambda_w}\right)^2$$

Hamiltonian describing particle's motion

$$H_w = \frac{1}{2}(p_z^2 + (p_x - A_x \sin(ks))^2 + (p_y - A_y \sin(ks))^2)$$

where

$$A_x = \frac{1}{\rho_w k}\cosh(k_x x))\cosh(k_y y))$$

$$A_y = -\frac{1}{\rho_w k}\sinh(k_x x))\sinh(k_y y))\frac{k_x}{k_y}$$

Particle's transverse motion after averaging over one wiggler period

$$\frac{d^2x}{ds^2} = -\frac{k_x^2}{2\rho_w^2 k^2}\left(x + \frac{2}{3}k_x^2 x^3 + k^2 x y^2\right)$$

$$\frac{d^2y}{ds^2} = -\frac{k_y^2}{2\rho_w^2 k^2}\left(y + \frac{2}{3}k_y^2 y^3 + y x^2 \frac{k_x^2 k^2}{k_y^2}\right)$$

In the following we consider plane wiggler with K_x=0

One cell wiggler

- *One cell wiggler Hamiltonian*

$$H_{w,1} = H_0 + \frac{1}{4\rho^2} y^2 + \frac{k_y^2}{12\rho^2} y^4 \lambda_w \sum_{i=-\infty}^{\infty} \delta(s - iL)$$ Eq. 4

- *After comparing* Eq. 4 *with* Eq. 1 *one gets*

$$\frac{b_3}{\rho} L = \frac{k_y^2 \lambda_w}{3 \rho_w^2}$$

Using Eq. 2 *one gets*
one cell wiggler limited dynamic aperture

$$A_{1,y}(s) = \frac{\sqrt{\beta_y(s)}}{\beta_y(s_w)} \left(\frac{3\rho_w^2}{k_y^2 \lambda_w} \right)^{1/2}$$

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A full wiggler

Using Eq. 3 *one finds dynamic aperture for a full wiggler*

$$\frac{1}{A^2_{N_{w,y}}(s)} = \sum_{i=1}^{N_w} \frac{1}{A^2_{i,y}} = \sum_{i=1}^{N_w} \left(\frac{k_y^2}{3\rho_w^2 \beta_y(s)} \right) \beta_y^2(s_{i,w}) \frac{\lambda_w}{N_w}$$

or approximately $A_{N_{w,y}}(s) = \sqrt{\frac{3\beta_y(s)}{\beta^2_{y,m}}} \frac{\rho_w}{k_y \sqrt{L_w}}$

where $\beta_{y,m}$ *the beta function in the middle of the wiggler*

Multi-wigglers

Many wigglers (M)

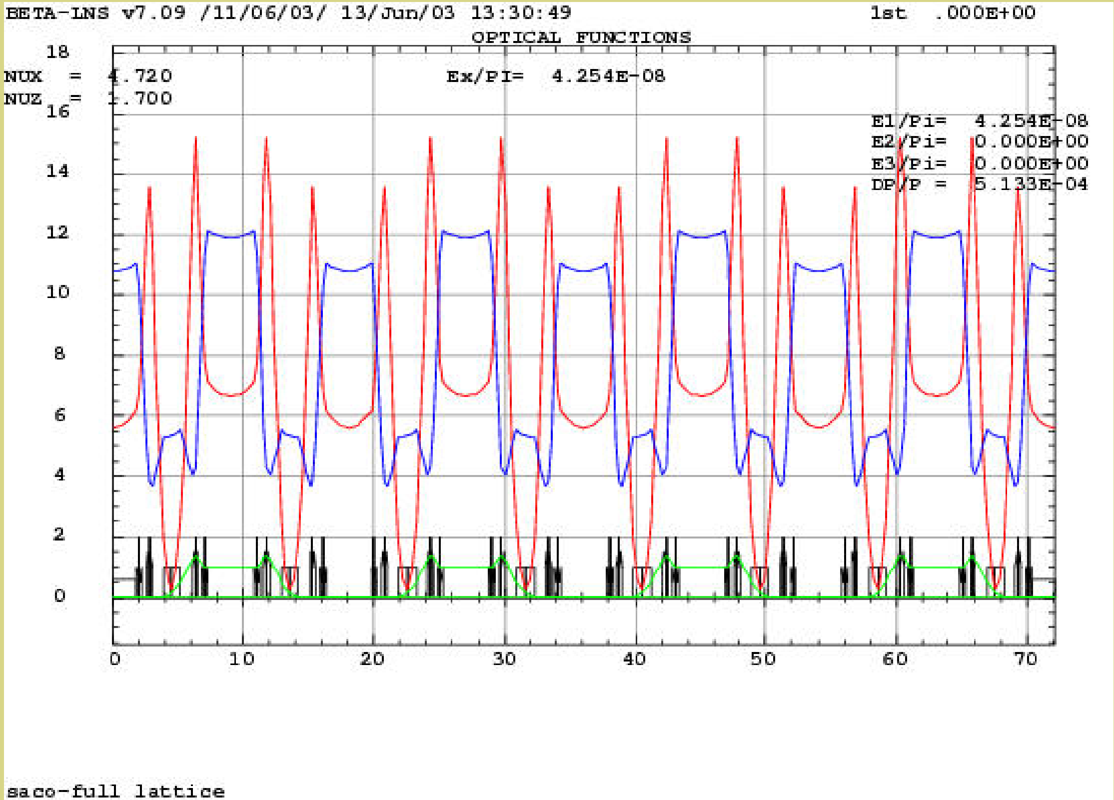
$$A_{total,y}(s)=\frac{1}{\sqrt{\frac{1}{A_y^2(s)}+\sum_{j=1}^{M}\frac{1}{A_{j,w,y}^2(s)}}}$$

Dynamic aperture in horizontal plane

$$A_{dyna,wigl,x}=\sqrt{\frac{\beta_{y,m}}{\beta_{x,m}}\left(A^2_{dyna,wigl,y}-y^2\right)}$$

Numerical example: Super-ACO

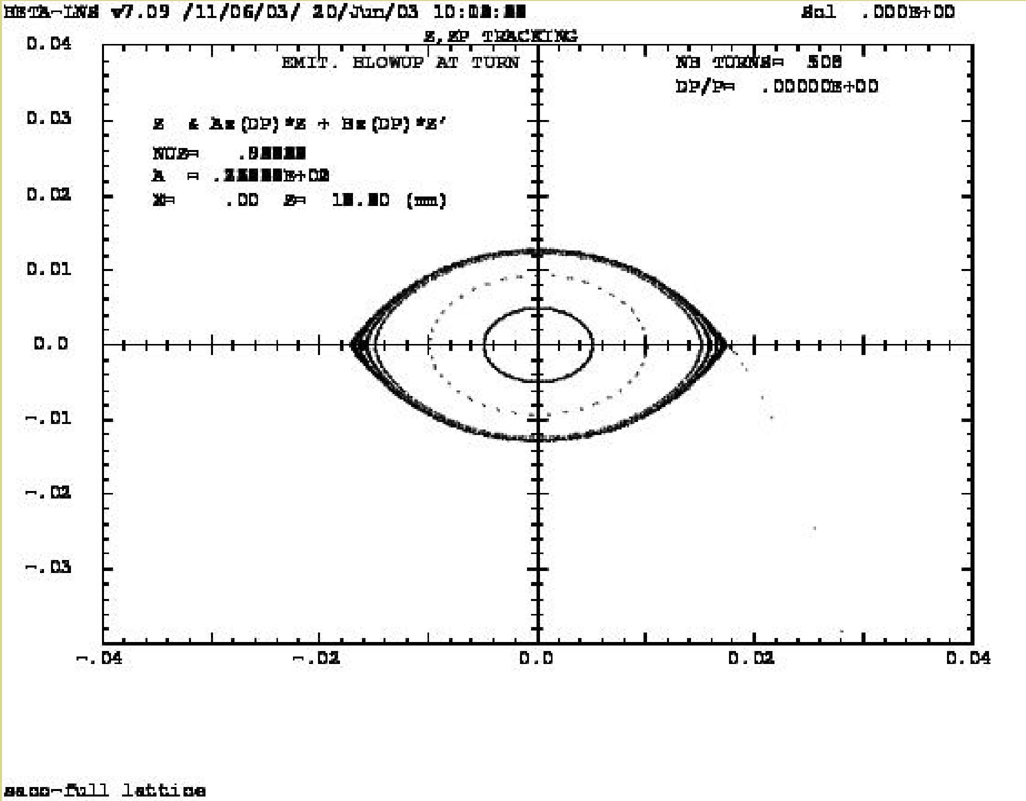
Super-ACO lattice with wiggler switched off



Super-ACO (one wiggler)

$\rho_w(m)=2.7$ $A_{y,n}(m)=0.017$ $A_{y,a}(m)=0.019$

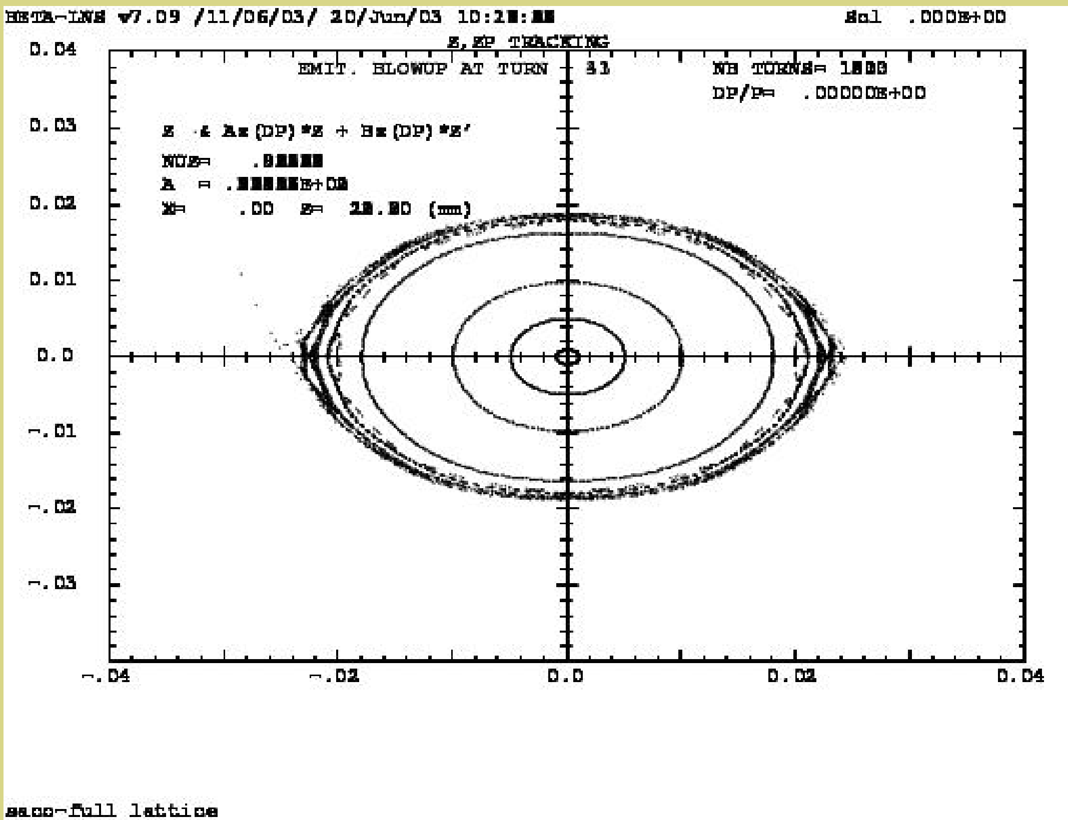
$\beta_{y,m}(m)=13$ $l_w(m)=0.17584$ $L_w(m)=3.5168$



Super-ACO (one wiggler)

$\rho_w(m)=3$ $A_{y,n}(m)=0.023$ $A_{y,a}(m)=0.024$

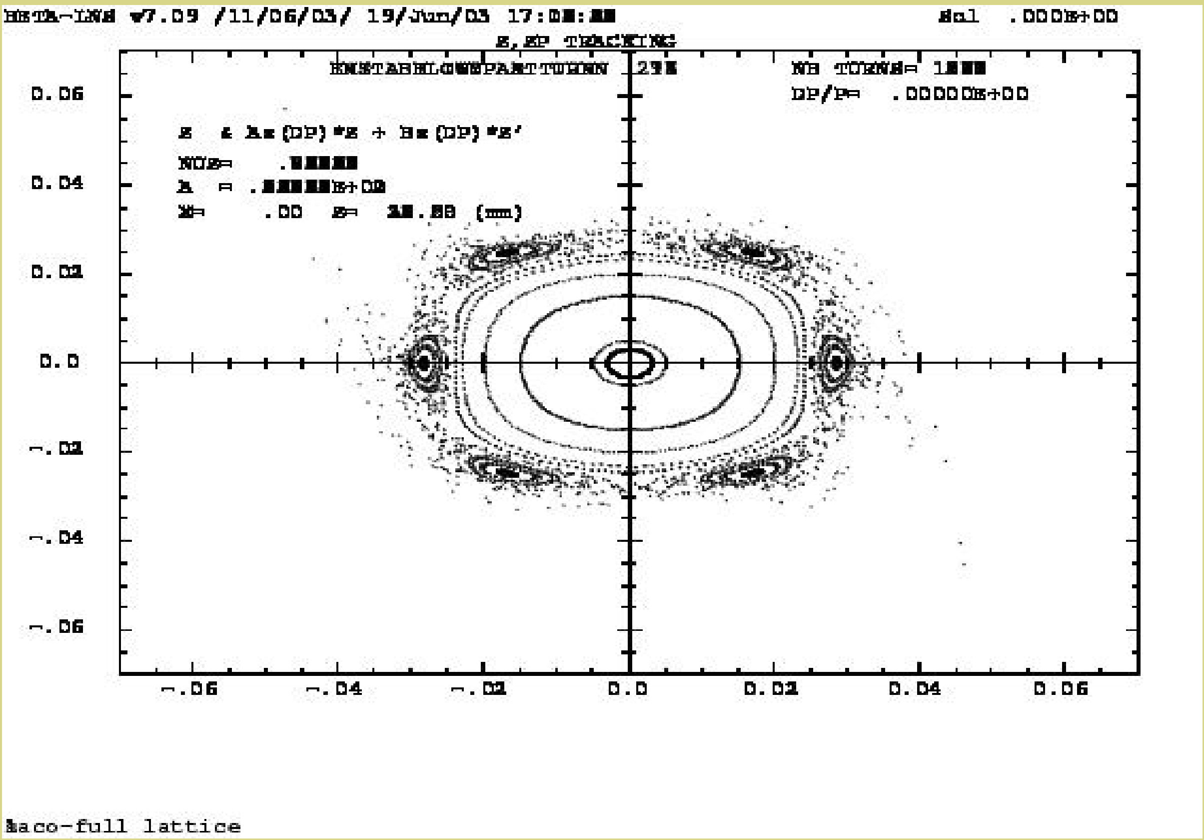
$\beta_{y,m}(m)=10.7$ $l_w(m)=0.17584$ $L_w(m)=3.5168$



Super-ACO (one wiggler)

$\rho_w(m)=4$ $A_{y,n}(m)=0.033$ $A_{y,a}(m)=0.034$

$\beta_{y,m}(m)=9.5$ $l_w(m)=0.17584$ $L_w(m)=3.5168$



Super-ACO (one wiggler)

$\rho_w(m)=4$ $\beta_{y,m}(m)=9.5$ $L_w(m)=3.5168$

$l_w(m)=0.08792$ $A_{y,n}(m)=0.016$ $A_{y,a}(m)=0.017$

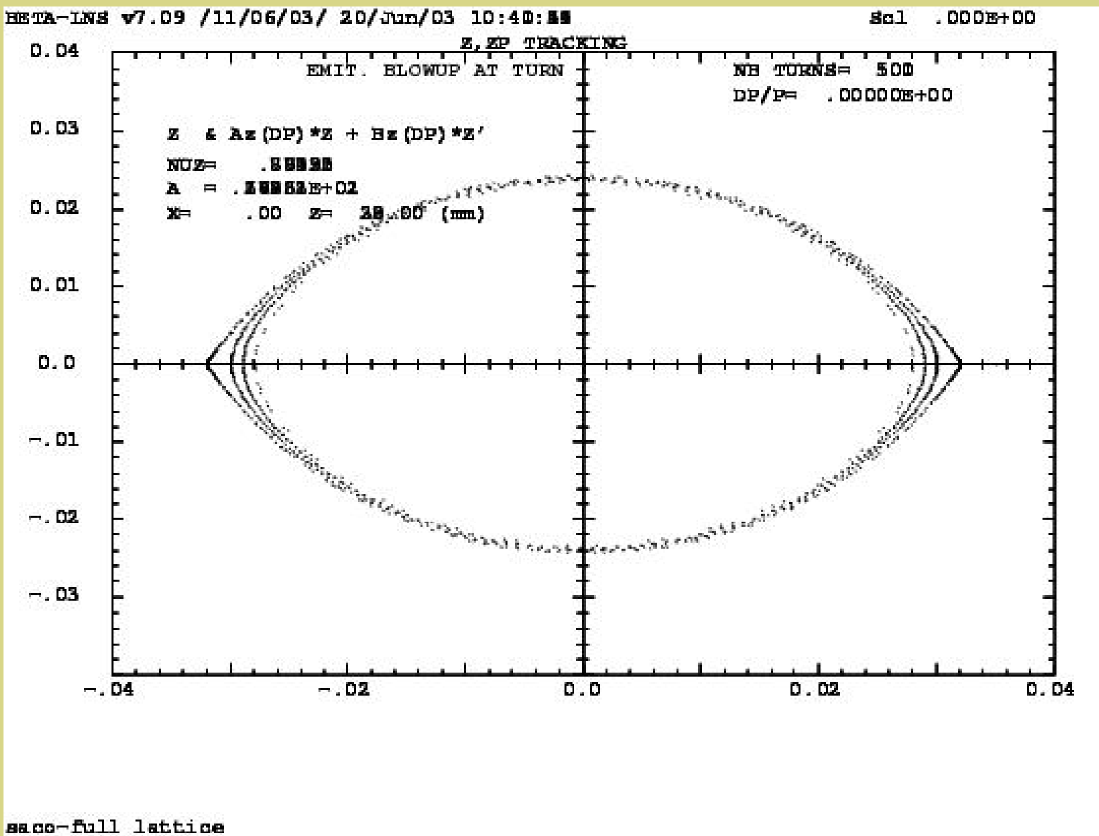
$l_w(m)=0.17584$ $A_{y,n}(m)=0.033$ $A_{y,a}(m)=0.034$

$l_w(m)=0.35168$ $A_{y,n}(m)=0.067$ $A_{y,a}(m)=0.067$

Super-ACO (two wigglers)

$\rho_w(m)=6$ $A_{y,n}(m)=0.032$ $A_{y,a}(m)=0.03$

$\beta_{y,m}(m)=13.75$ $l_w(m)=0.17584$ $L_w(m)=3.5168$



Application to TESLA Damping Ring

$E=5GeV$ $Bo=1.68T$ $\lambda_w = 0.4m$

$N_w = 12$ $\beta_{y,1} = 9m$ (at the entrance of the wiggler)

$\beta_{y,2} = 15m$ (at the exit of the wiggler)

The total number of wigglers in the damping ring is 45.

The vertical dynamic aperture due to 45 wiggler is $A_{total,y} = 2.1cm$

Conclusions

1) *Analytical formulae for the dynamic apertures limited by multipoles in general in a storage ring are derived.*

2) *Analytical formulae for the dynamic apertures limited by wigglers in a storage ring are derived.*

3) *Both sets of formulae are checked with numerical simulation results.*

4) *These analytical formulae are useful both for experimentalists and theorists in any sense.*

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