

Fixed-order QCD corrections to top production near threshold

Yuichiro Kiyo

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Collaboration with: M.Beneke, K. Schuller, A. Penin

Talk based on Beneke-YK-Schuller, NPB714(2005)67

Snowmass 17.08.2005

Toward NNNLO $t\bar{t}$ threshold Xsection: 3rd order Coulomb Correction

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Part I

Introduction

ILC

Top quark is one of important physics at ILC

- Question to heavy top quark
- Theoretically clean: No hadronization due to $\Gamma_t = 1.5\text{GeV}$
Bigi-Dokshitzer-Khoze-Kuhn-Zerwas('86)

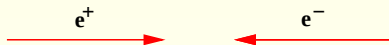
What will be measured in $e^+e^- \rightarrow t\bar{t}$ @ILC

- Top mass $\delta m_t \leq 50 \text{ MeV}$ Martinez-Miquel(02)
- Mom distribution Talk by Teubner
- SM parameters α_s , Γ_t , y_t + Anomalous coupling
Hioki-Gradkowski(03), talks by Uli, Jose, Kaustubh, Tim
- Spin/angular correlations
Park-Shadmi('96), Kodaira-Nasuno-Parke(99), Brandenburg-Si(05)

Precise **Theory** prediction is necessary to match **Exp** accuracy

→ We aim few % accuracy

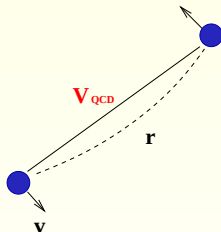
$t\bar{t}$ Production



- Beam energy : $\sqrt{s} \sim 2m_t$
- Threshold scan $\rightarrow$ shape of $\sigma_{t\bar{t}}(\sqrt{s})$

Talk by Gudi on 8/22

$t\bar{t}$ Production



- Nonrelativistic top: $v \sim \alpha_s \sim 0.16$
- $t\bar{t}$ in QCD potential: $V_{\text{QCD}}(r) = -\frac{C_F \alpha_s}{r}$
- Short distance bounding: $r \sim (m_t \alpha_s)^{-1} \ll \Lambda_{\text{QCD}}^{-1}$

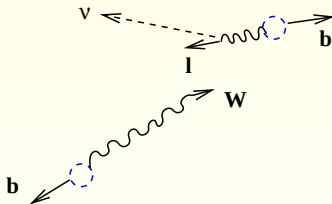
→ NR QCD is our tool

Fadin-Khoze('87), Strassler-Peskin('90), Sumino('93),

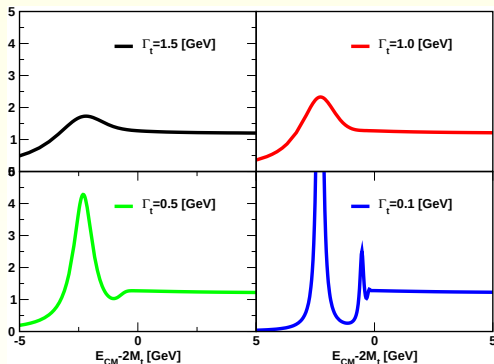
NNLO by Hoang-Teubner, Melnikov-Yelkhovsky,

Beneke-Signer-Smirnov, Yakovlev, Penin-Pipovarov, Nagano-Ota-Sumino

$t\bar{t}$ Production

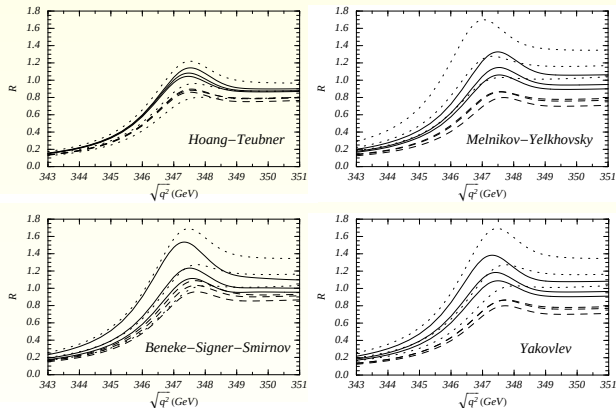


- Quick Decay: $t \rightarrow bW$
 → Hadronization effect does NOT enter
 Peter-Sumino('97), exp study; Ikematsu-Fujii-Hioki-Sumino-Takahashi(03)
- Effect of unstableness: Γ_t
 → interplay between QCD and EW corrections
 We treat the effect of Γ_t by replacement: $m_t \rightarrow m_t + i\Gamma_t$
 see for recent th attempt: Beneke-Chapovsky-Signer-Zanderighi(04),
 Hoang-Reisser(04)

$t\bar{t}$ threshold X section with Γ_t  $R(e^+e^- \rightarrow t\bar{t})$ varying Γ_t

- $e^+e^- \rightarrow \gamma^* \rightarrow t\bar{t}$ (No Z^* for simplicity)
- Here and hereafter only the QCD effect is discussed
- Finite width effect: $\sqrt{s} \rightarrow \sqrt{s} + i\Gamma_t$

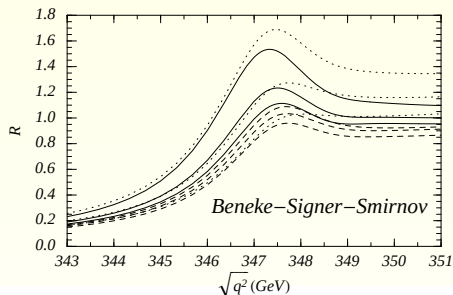
R_{NNLO} from Top Quark Working Group Report



Top Quark WG Report by A.H. Hoang et al.(2000)

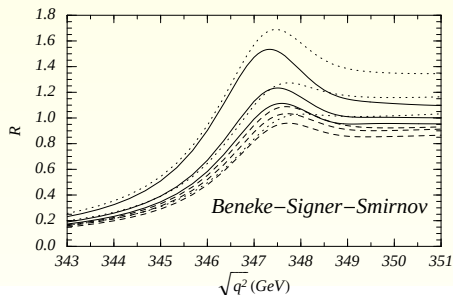
- Figs by 4 Groups: **Scheme difference**
- **Solid**/Dashed/Dotted Lines : **NNLO**/NLO/LO
- $\mu = 15, 30, 60$ GeV

What do we know? What should we do?



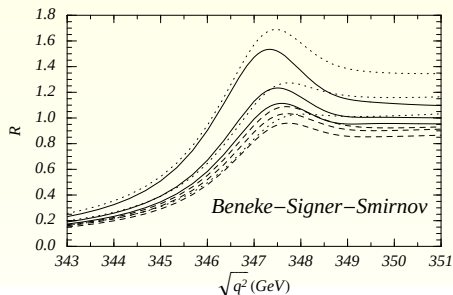
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- Nonrelativistic $1/c$ -expansion works? Necessary to sum up? ;
- Do we know the size of third order? $\Leftarrow$ Ultra Soft Corr

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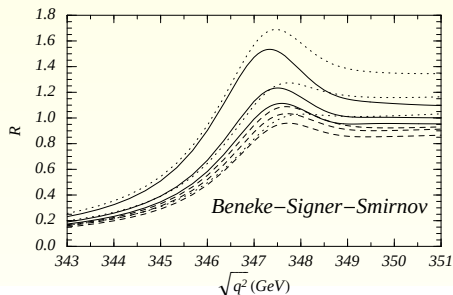
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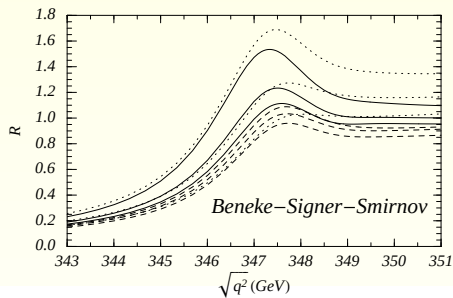


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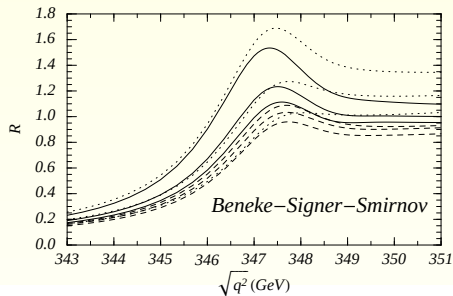


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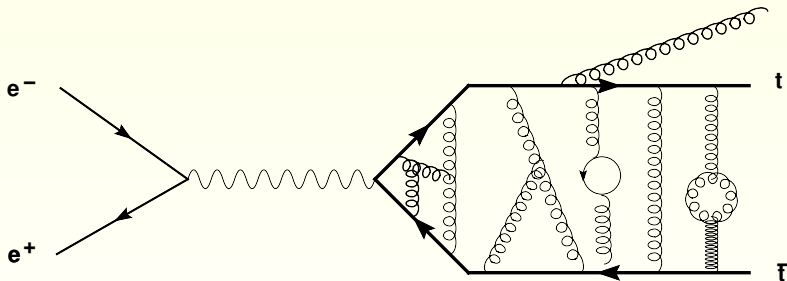
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work in progress with Beneke-Kiyo-Penin(Kurlsruhe)

Part II

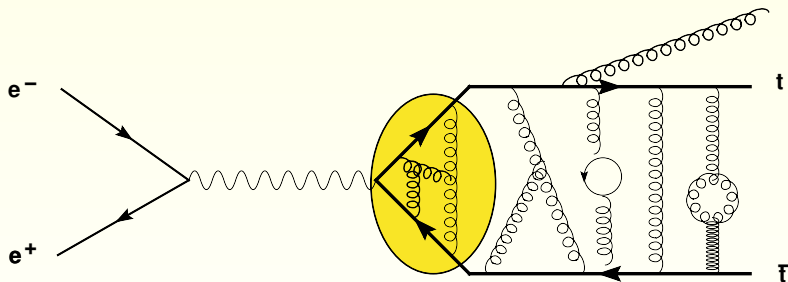
NNNLO Threshold Cross Section

Review of EFT Approach

QCD Corrections

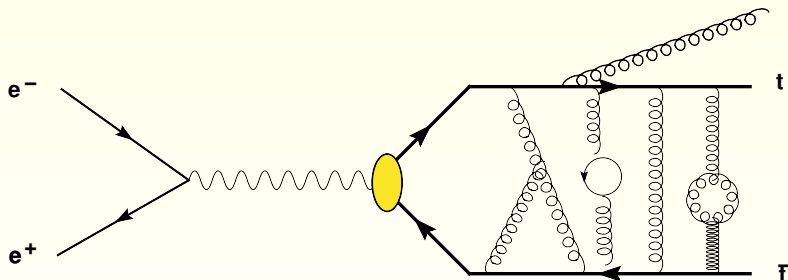


QCD Corrections



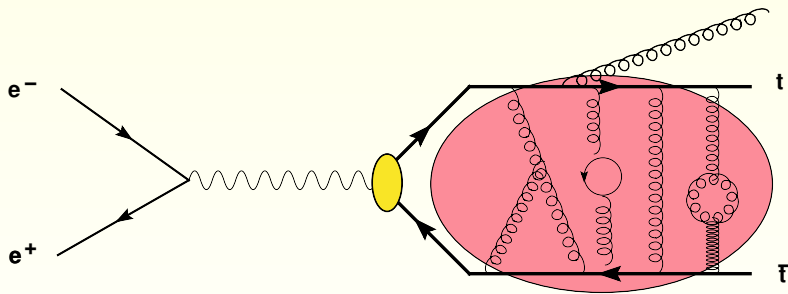
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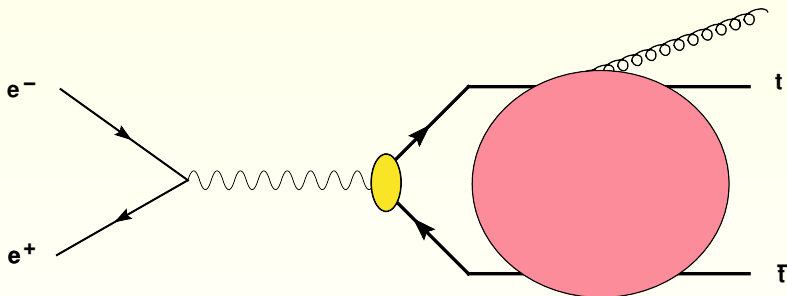
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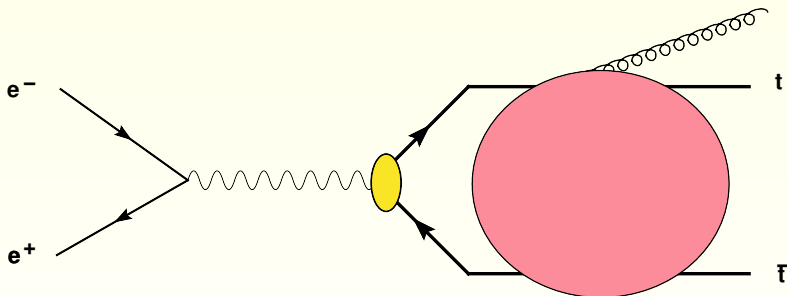
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We discuss $t\bar{t}$ boundstate dynamics

Potential NRQCD

- Integrate out **Hard**

$$\mathcal{L}_{\text{NRQCD}} = \psi^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m_t} \right) \psi + [\psi \rightarrow \chi] + \dots$$

$\Downarrow$

- Integrate out **Soft/Potential** gluons
(Pineda-Soto('98); Luke-Manohar-Rothstein('99))

$$\delta\mathcal{L}_{\text{pNRQCD}} = \int d\vec{r} [\psi^\dagger \chi] V_{\text{pot}}(r) [\chi^\dagger \psi]$$

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* EFT = $t\bar{t}$ -Potential and low energy gluon(Ultra soft gluon)

Potential NRQCD

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* EFT = $t\bar{t}$ -Potential and low energy gluon(Ultra soft gluon)

* $V_{\text{pot}} = -\frac{C_F \alpha_s(\mu)}{r} + \text{Higher Order Corr}$

Potentials are Wilson Coeff: $V_{pot}(r) [\psi^\dagger \chi](r) [\chi^\dagger \psi](0)$

- $V_{pot} = -\frac{C_F \alpha_s}{r} + \frac{C_2}{r^2} + C_3 \delta(\mathbf{r}) + \dots$
- Corr to the Coulomb potential
- Every potential (except a_3) is known to 3rd order
 - a_2 Schröder('99)
 - $a_{3,pade}$ Chishtie-Elias (01)
 - **ADM** IR Div Brambilla-Pineda-Soto-Vairo('99)
 - Complete 3rd Pot; Kniehl-Penin-Smirnov-Steinhauser(02)

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$$\tilde{V}_C = -\frac{C_F \alpha_s(\mathbf{q})}{r} \times \left[1 + \frac{\alpha_s(\mathbf{q})}{4\pi} a_1 \right]$$

$\rightarrow \ln \frac{\mu^2}{\mathbf{q}}$ is absorbed in to $\alpha_s(\mathbf{q})$
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→ $1/\epsilon$ ADM IR Divergence

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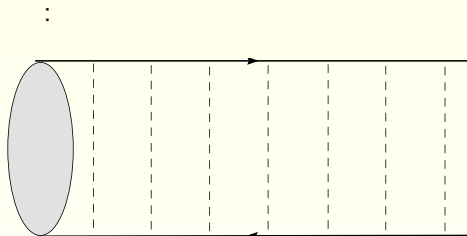
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→ $1/\epsilon$ ADM IR Divergence

⇒ cancels UltraSoft UV divergence

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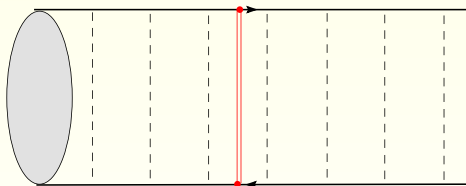
Diagrams for NNNLO



Coulomb gluons to be summed
up

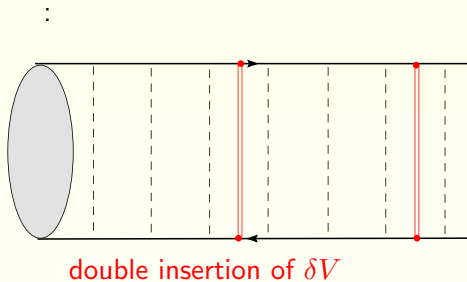
Diagrams for NNNLO

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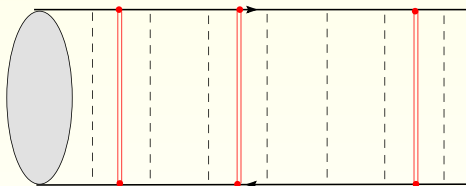
single insertion of δV

Diagrams for NNNLO



Diagrams for NNNLO

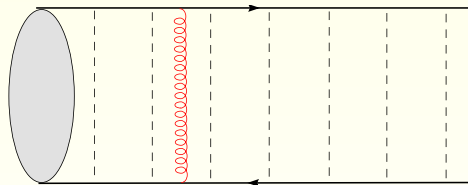
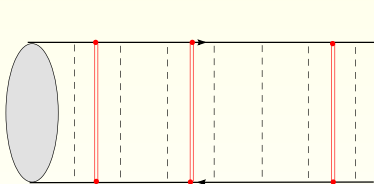
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triple insertion of δV

Diagrams for NNNLO

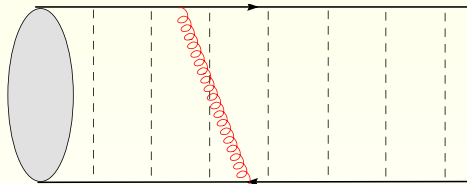
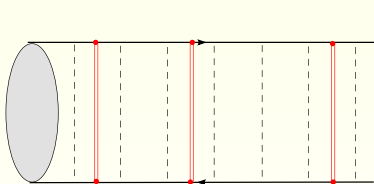
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Ultra Soft $k^2 \sim 0$

Diagrams for NNNLO

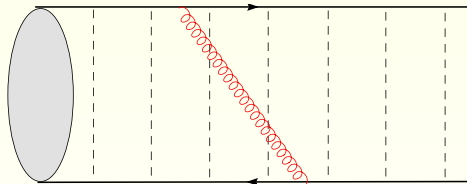
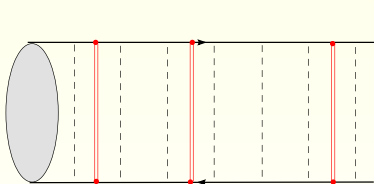
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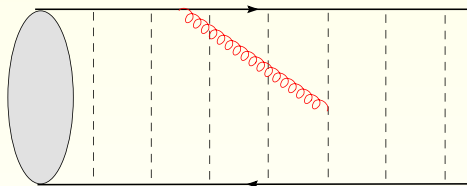
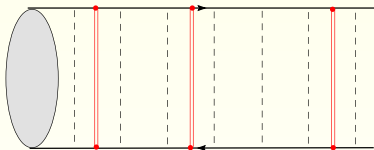
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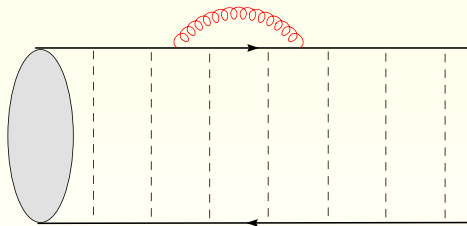
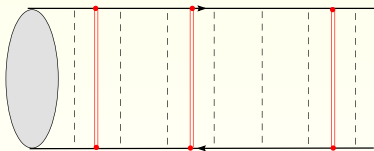
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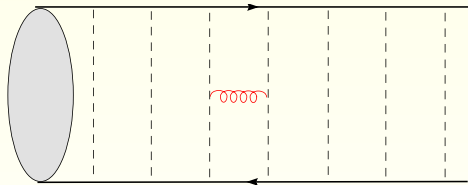
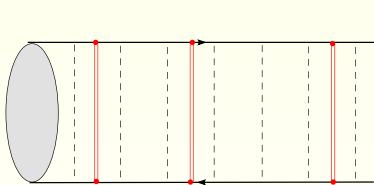
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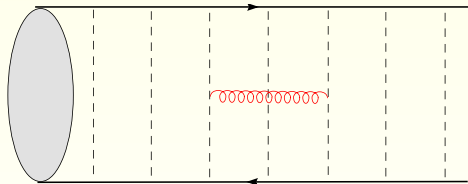
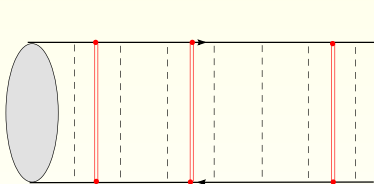
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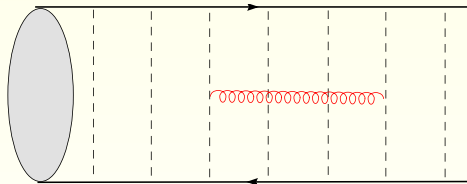
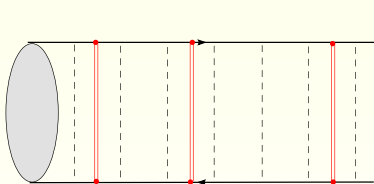
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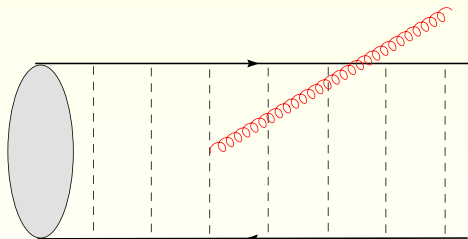
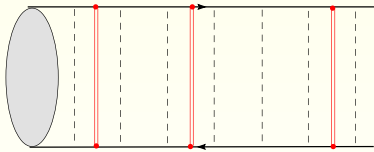
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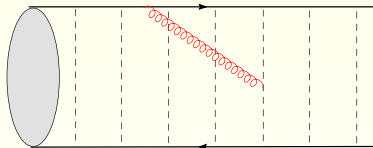
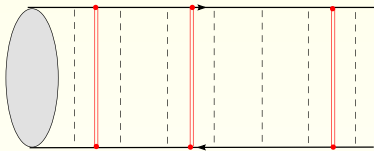
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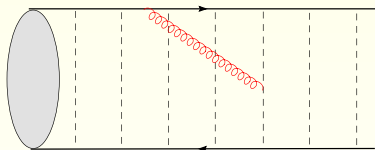
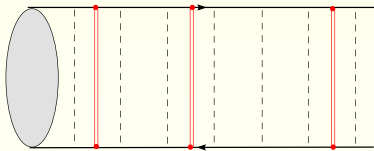
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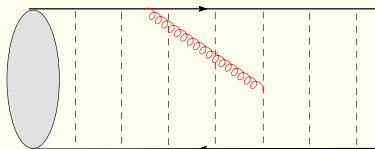
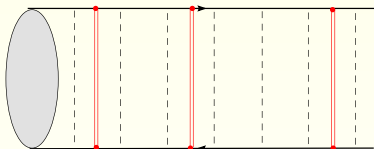


Ultra Soft $k^2 \sim 0$

- Coulombic pot insertions $\Rightarrow$ Complete (Beneke-YK-Schuller(05))
- Singular pot insertions $\Rightarrow$ UV renormalization is necessary
(work in progress with Beneke-Schuller)
- Ultrasoft Corr $\Rightarrow$ UV renormalization is necessary
(work in progress with Beneke-Penin)

Diagrams for NNNLO

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- Singular pot insertions $\Rightarrow$ UV renormalization is necessary
(work in progress with Beneke-Schuller)
- Ultrasoft Corr $\Rightarrow$ UV renormalization is necessary
(work in progress with Beneke-Penin)

Part III

Coulomb Corrections

Based on Beneke-Kiyo-Schuller, NPB714(2005)67.

(Coulomb) X Section near Threshold; $E = \sqrt{s} - 2m_t \sim 0$

$$\sigma_{t\bar{t}} \sim \text{Im} \langle 0 | \frac{1}{H - E - i\Gamma_t} | 0 \rangle + \text{non-potential}(\mu_{US})$$

$$H = \frac{p^2}{m_t} - \frac{C_F \alpha_s}{r} + \delta V_C(r) + \frac{\#}{r^2} + \# \delta(\vec{r}) + \dots,$$

- We do NOT discuss singular potentials

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$$H = \frac{p^2}{m_t} - \frac{C_F \alpha_s}{r} + \delta V_C(r),$$

- Coulomb pot: $\delta \tilde{V}_C = -\frac{C_F \alpha_s}{r}$
 $\times \left[\frac{\alpha_s}{4\pi} (\beta_0 \ln(\mu^2/q^2) + a_1) + \dots \right.$
 $\left. + \left(\frac{\alpha_s}{4\pi} \right)^3 (\clubsuit \ln^3 + \clubsuit \ln^2 + \clubsuit \ln + a_3 + 8\pi^2 C_A^3 \ln(\mu_{US}^2/q^2)) \right]$

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- δV_C is renormalization free $\Rightarrow$ Num evaluation is possible

Perturbation in QM: $H_0 = \frac{p^2}{m_t} - \frac{C_F \alpha_s}{r}$

Iteration of the Coulomb potentials

$$\left\langle \frac{1}{H_0 + \delta V_C - E} \right\rangle = \langle G_0 \rangle - \langle G_0 \delta V_1 G_0 \rangle + \langle G_0 \delta V_1 G_0 \delta V_1 G_0 \rangle + \dots$$

with $\langle G_0 \rangle = \langle 0 | \frac{1}{H_0 - E} | 0 \rangle$ and $\delta V_C = \delta V_1 + \delta V_2 + \delta V_3$,

- N³LO Corrections are given by:
 - * Single: $\langle G_0 \delta V_3 G_0 \rangle$
 - * Double: $\langle G_0 \delta V_1 G_0 \delta V_2 G_0 \rangle$
 - * Triple: $\langle G_0 \delta V_1 G_0 \delta V_1 G_0 \delta V_1 G_0 \rangle$

Results

- We have calculated the Green function $\langle \frac{1}{H_0 + \delta V_C - E - i\Gamma_t} \rangle$
treating δV_C as iterative interaction
→ Result is too lengthy to show in a slide

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 - * Double and triple sums are left, can be done **Numerically**
 - * Expressed by well known special functions
- Pole structure $\Leftrightarrow$ **bound-state**: $\sum_n \frac{|\Psi_n(0)|^2}{E_n - E - i\Gamma_t}$
 - * Pole and residue were obtained **Analytically**
 - * Expressed by multi- ξ and nested $S_{i,j,k}(n)$
 → This result fit in a page

Coulomb Energy

$$E_n^{(3)} = E_n^{(0)} \left(\frac{\alpha_s}{4\pi} \right)^3 \{ [32\beta_0^3 L^3 + L^2 (-56\beta_0^3 + 28\beta_0\beta_1 + 24\beta_0^2 c_{E,1}) + L (16\beta_0^3 - 16\beta_0\beta_1 + 4\beta_2 - 12\beta_0^2 c_{E,1} + 6\beta_1 c_{E,1} + 8\beta_0 c_{E,2})] + c_{E,3} + 32\pi^2 C_A^3 [\ln \left(\frac{n \mu_{\text{US}}}{m C_F \alpha_s} \right) + S_1(n)] \},$$

$$S_i = S_i(n) = \sum_k^n \frac{1}{k^i} \text{ is harmonic-sum}$$

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Every coefficient of $L = \ln \left(\frac{n\mu}{m C_F \alpha_s} \right)$ is determined by RG we used this to check the result

Coulomb Energy

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$$c_{E,1} = 2a_1 + 4S_1\beta_0,$$

1st Corr

Coulomb Energy

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$$c_{E,1} = 2a_1 + 4S_1\beta_0,$$

$$c_{E,2} = a_1^2 + 2a_2 + 4S_1\beta_1 + 4a_1\beta_0[3S_1 - 1] + \beta_0^2[S_1(12S_1 - 8 - \frac{8}{n}) + 16S_2 - 8nS_3 + \frac{2\pi^2}{3} + 8n\xi(3)],$$

2nd Corr

Coulomb Energy

$$\begin{aligned}
 E_n^{(3)} = E_n^{(0)} \left(\frac{\alpha_s}{4\pi} \right)^3 & \{ [32\beta_0^3 L^3 + L^2 (-56\beta_0^3 + 28\beta_0\beta_1 + 24\beta_0^2 c_{E,1}) + L (16\beta_0^3 - 16\beta_0\beta_1 + 4\beta_2 \\
 & - 12\beta_0^2 c_{E,1} + 6\beta_1 c_{E,1} + 8\beta_0 c_{E,2})] + c_{E,3} + 32\pi^2 C_A^3 [\ln \left(\frac{n\mu_{\text{US}}}{mC_F\alpha_s} \right) + S_1(n)] \}, \\
 c_{E,1} = 2a_1 + 4S_1\beta_0, \\
 c_{E,2} = a_1^2 + 2a_2 + 4S_1\beta_1 + 4a_1\beta_0[3S_1 - 1] + \beta_0^2[S_1(12S_1 - 8 - \frac{8}{n}) + 16S_2 - 8nS_3 + \frac{2\pi^2}{3} + 8n\xi(3)], \\
 c_{E,3} = 2a_1a_2 + 2a_3 + 2a_1^2\beta_0[4S_1 - 5] + 4a_2\beta_0[4S_1 - 1] + 4a_1\beta_1[3S_1 - 1] + 4S_1\beta_2 + \beta_0\beta_1[S_1(28S_1 \\
 - 16 - \frac{24}{n}) + 36S_2 - 16nS_3 + \frac{7\pi^2}{3} + 16n\xi(3)] + a_1\beta_0^2[S_1(48S_1 - 56 - \frac{32}{n}) + 64S_2 - 32nS_3 + 8 + \frac{8\pi^2}{3} \\
 + 32n\xi(3)] + \beta_0^3[S_1(S_1(32S_1 - 56 - \frac{32}{n}) + 96S_2 - 64nS_3 + 16 + \frac{16}{n} + \frac{32\pi^2}{3} + 64n\xi(3)) + S_2(8nS_2 \\
 + 16n^2S_3 - 32 - \frac{16}{n} - \frac{40n\pi^2}{3} - 16n^2\xi(3)) + S_3(96 + 16n + 8n^2\pi^2) - 104nS_4 + 48n^2S_5 - 144S_{2,1} \\
 + 224nS_{3,1} - 32n^2S_{3,2} - 96n^2S_{4,1} - \frac{4\pi^2}{3} + \frac{2n\pi^4}{45} + \xi(3)(32 - 16n - 8n^2\pi^2) + 96n^2\xi(5)]
 \end{aligned}$$

3rd Corr: All $\xi(i, j, k)$ were reduced to $\xi(i)$, $S_{i,j,k} = S_{i,j,k}(n)$ is nested harmonic-sum

Coulomb Wave Func(constant part)

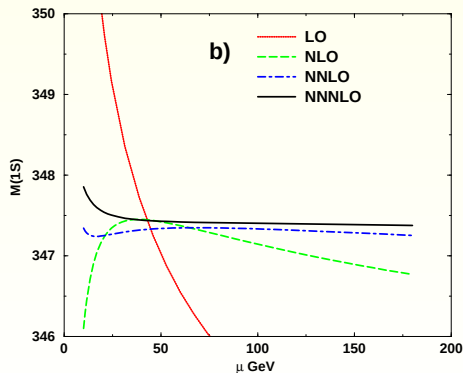
$$\begin{aligned}
 c_{\psi,3} = & a_1^3 + 6a_1a_2 + 3a_3 + 10a_1^2\beta_0[S_1 + 2nS_2 - \frac{31}{10} - \frac{n\pi^2}{3}] + 10a_2\beta_0[S_1 + 2nS_2 \\
 & - \frac{8}{5} - \frac{n\pi^2}{3}] + 8a_1\beta_1[S_1 + 2nS_2 - \frac{7}{4} - \frac{n\pi^2}{3}] + 2\beta_2[S_1 + 2nS_2 - 1 - \frac{n\pi^2}{3}] \\
 & + \beta_0\beta_1[S_1(22S_1 + 40nS_2 - 44 - \frac{36}{n} - \frac{20n\pi^2}{3}) + S_2(8n^2S_2 + 14 - 16n - \frac{8n^2\pi^2}{3}) \\
 & + 64nS_3 - 40n^2S_4 - 56nS_{2,1} + 32n^2S_{3,1} + 8 + \frac{(21 + 16n)\pi^2}{6} + \frac{2n^2\pi^4}{9} + 48n\xi(3)] \\
 & + a_1\beta_0^2[S_1(40S_1 + 80nS_2 - 116 - \frac{60}{n} - \frac{40n\pi^2}{3}) + S_2(20n^2S_2 + 40 - 72n - \frac{20n^2\pi^2}{3}) \\
 & + 140nS_3 - 100n^2S_4 - 120nS_{2,1} + 80n^2S_{3,1} + 48 + (5 + 12n)\pi^2 + \frac{5n^2\pi^4}{9} + 100n\xi(3)] \\
 & + \beta_0^3[S_1(4S_1(4S_1 + 16nS_2 - 19 - \frac{6}{n} - \frac{8n\pi^2}{3}) + 8S_2(3n^2S_2 + 2 - 14n - n^2\pi^2) \\
 & + 104nS_3 - 120n^2S_4 - 112nS_{2,1} + 96n^2S_{3,1} + 80 + \frac{64}{n} + \frac{(58 + 56n)\pi^2}{3} + \frac{2n^2\pi^4}{3} \\
 & + 120n\xi(3)) + S_2(-4n(17 + 2n)S_2 + 72n^2S_3 - 96n^2S_{2,1} + 64n^3S_{3,1} - 96 + 16n \\
 & - \frac{24}{n} - \frac{8(5 - n)n\pi^2}{3} - 8n^2\xi(3)) + S_3(-16n^3S_3 + 64 - 16n - 20n^2\pi^2 + 32n^3\xi(3))]
 \end{aligned}$$

More.....

$$\begin{aligned}
& +S_4(68n + 40n^2 + \frac{64n^3\pi^2}{3}) - 312n^2S_5 + 144n^3S_6 + S_{2,1}(48n - 120 + 16n^2\pi^2) \\
& - 32S_{3,1}(\frac{15n}{2} + n^2 + \frac{n^3\pi^2}{3}) + 384n^2S_{3,2} + 576n^2S_{4,1} - 224n^3S_{4,2} - 256n^3S_{5,1} \\
& + 256nS_{2,1,1} + 64n^2S_{2,2,1} - 64n^3S_{2,3,1} - 448n^2S_{3,1,1} + 192n^3S_{4,1,1} - 8 - \frac{8(2+n)\pi^2}{3} \\
& - \frac{(83+10n)n\pi^4}{45} + \frac{4n^3\pi^6}{105} + \xi(3)(48 - 80n - 12n^2\pi^2 - 16n^3\xi(3)) - 40n^2\xi(5) \].
\end{aligned}$$

Toponium mass

- Complete NNNLO 1S mass is known by Penin-Steinhauser (2002)
- Extraction of $m_{\overline{\text{MS}}}$ was discussed by Sumino-Y.K.(2002).



Sumino-YK(02)

Summary of toponium mass

- N^3LO Corr to E_n is complete;
Beneke-YK-Schuller&Penin-Smirnov-Steinhauser&Kniehl-Penin
- QCD corr to the peak position is under control

$$M_{t\bar{t}(1S)} = (350 + 0.85_{LO} + 0.05_{NLO} - 0.13_{N^2LO} + 0.01_{N^3LO})\text{GeV}$$

$$\text{using } m_{t,PS} = 175, \mu = 32.6\text{GeV}, \Lambda_{QCD}^{(5)} = 208\text{MeV}$$

- 4-loop relation between $\overline{MS}$ pole masses is necessary to keep the NNNLO accuracy completely.

Wave function and σ_C

$$\sigma_C \sim \text{Im } G(E + i\Gamma_t)$$

We performed two analyses for threshold cross section taking NNNLO Coulomb corr into account,

- Analytical cal at NNNLO taking $\delta V_{1,2,3}$ as **ITERATIVE** interaction, **dropping followings**;

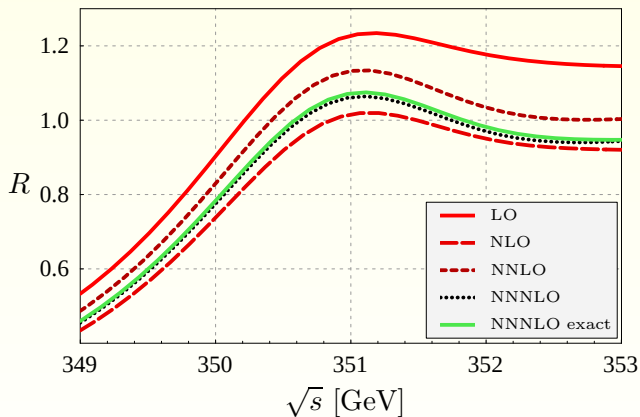
$$\langle \delta V_1 \times \delta V_1 \times \delta V_1 \times \delta V_1 \rangle \sim \text{4th order}$$

$$\langle \delta V_1 \times \delta V_1 \times \delta V_2 \rangle \sim \text{4th order}$$

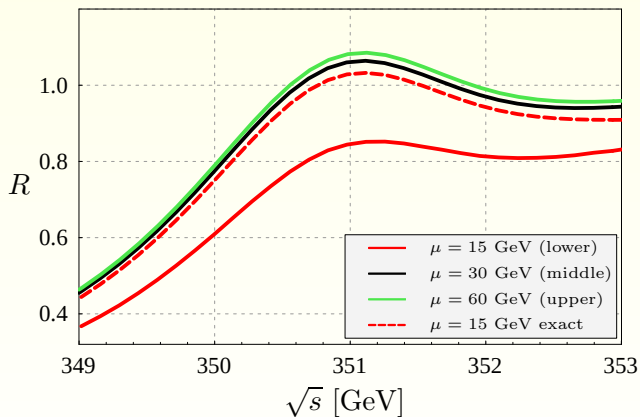
$$\langle \delta V_1 \times \delta V_3 \rangle \sim \text{4th order}$$

$$\vdots$$

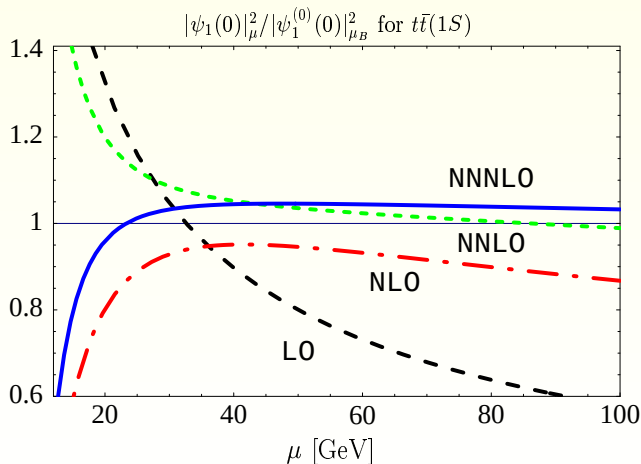
- **NUMERICALLY** solved schrödinger Eq. with
 $V = V_C + \delta V_1 + \delta V_2 + \delta V_3 \quad \equiv \text{“ N}^3\text{LO Exact”}$

X section; $\text{LO} \rightarrow \text{NLO} \rightarrow \text{NNLO} \rightarrow \text{NNNLO}$ 

X section; X section; scale dependence



Wave function and μ dependence



Summary of Wave func and σ

- N³LO Coulomb Corr to σ , E_n , $|\Psi_n(0)|^2$
- Comparison between analytic and numerical method for σ_C
 - * Naively taking $\mu \sim mC_F\alpha_s/n$ is not working.
Large correction and large μ -dependence
- The strong μ -dependence, $\mu \leq mC_F\alpha_s/n$ is an artifact of iterative method
 - $\Leftrightarrow$ "Num Exact" result does NOT have this behavior
- We estimated error; $\Delta\sigma_C \leq 5\%$