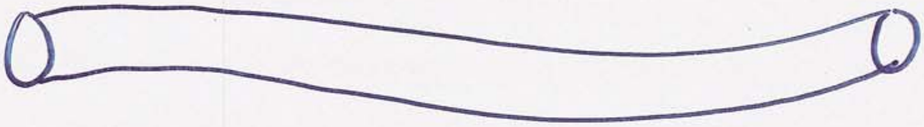
String Theory
for
Physicists



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Some good/bad news about string theory

Good: string theory is a consistent theory of quantum gravity

Bad: It's really a generator of an infinite number of mostly disconnected theories of quantum gravity, each around a different ground state.

No background independent truely off-shell formulation of string theory is known (yet).

Good: string theory is unique
i.e. there is only one distinct consistent
theory of "fundamental" strings

Bad: It has an infinite number of
continuously connected groundstates plus
a google of discrete ones.
There appears to be no vacuum
selection principle, other than
the stability of supersymmetric vacua,
which gives the wrong answer.

Good: string theory gives you chiral gauge theories, with big gauge groups, for free + complicated flavor structure at low energies is mapped into the geometry of extra dimensions

Bad: Doesn't like to give the Standard Model as the low energy theory.
A "typical" string compactification is either much simpler (with more SUSY and bigger gauge groups) or much more complicated (lot's of extra exotic matter extra U(1) gauge groups etc)

Good: string theory predicts supersymmetry and extra dimensions of space

Bad: It's happy to hide them both up at the Planck scale

Good: No length or energy scales are put in by hand; all scales should be determined dynamically

Bad: Appears to be too many $\overset{\text{hundreds!}}{\wedge}$ scalar fields (moduli) with too much SUSY to get determined dynamically; may be forced to appeal to cosmic initial conditions (the Landscape)

Good: string theory gives a microphysical description of (at least some) black holes, resolves their singularities

Bad: Doesn't seem to resolve the singularity of the Big Bang
(good for inflation, though)

Good: Lots of powerful dualities including weak ↔ strong coupling dualities and short ↔ long distance dualities

Bad: Can't tell what are the "fundamental" degrees of freedom.
String theory not necessarily a theory of strings

Good: unification of all the forces is almost for free, may need an (interesting) extra dimensional assist

Bad: In our most realistic string constructions so far, $SU(3)_c$ $SU(2)_W$ and $U(1)_Y$ have essentially nothing to do with each other: related to different features of complicated D-brane setups

Good: AdS/CFT duality shows that 10dim. string theory in a certain background is equivalent to a 4dim. gauge theory!! Use this e.g. to show that RHIC QCD physics maps onto quantum gravity/black holes

Bad: Adds more confusion: can't tell an extra dimension apart from technicolor

Good: We are starting to use string theory to learn tricks for perturbative QCD, understanding the QCD string, etc.

Bad: The QCD community was already doing fine, thank you.

Today's outline

→ from hadronic scattering to strings

→ classical vs quantum strings

→ why does string theory predict extra dimensions and supersymmetry

- string theory invented circa 1970 by Susskind, Nambu, Nielsen, ... as a theory of strong interactions

- today it is our leading theory of quantum gravity and unification

- how could one theory do two such different things?

strong interactions circa 1968:

2→2 hadronic processes $a+b \to c+d$

partial wave expansion

$$M(s,t) = \sum_{l=0}^{\infty} (2l+1) M_l(s) P_l(\cos\theta)$$

some hadronic coupling

$$M_l^{s\text{-channel}} \underset{\substack{\text{large } t\\ \text{fixed } s}}{\sim} \sum_J \frac{g_J^2\, t^J}{s-m_J^2}$$

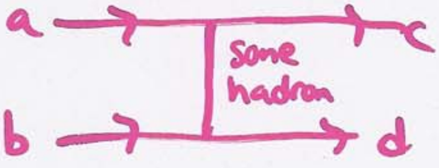
← need lots of derivatives to describe effective coupling

e.g. $\partial_\mu \partial_\nu \partial_\rho \partial_\lambda \phi^{\mu\nu\rho\lambda}$

$\sim t^J$ to leading order in large t

mass of some hadron of spin J

t-channel



$$M_l^{t\text{-channel}} \underset{\substack{\text{large } s\\ \text{fixed } t}}{\sim} \sum_J \frac{g_J^2\, s^J}{t-m_J^2}$$

i.e. high energy small angle scattering

problem: violates unitarity!

Let's take these "towers" of hadronic resonances seriously: note they obey $J \simeq \alpha' M^2 + \alpha_0$ for $\alpha' \simeq 0.85\ \text{GeV}^2$ and some α_0

This trajectory predicts a sequence of resonances with intrinsic spin 0, 1, 2, . . . and increasing mass. The resonances will have the total charge, baryon number, and strangeness of the initial and final states in Eq. 17-16. And if we resolve the reaction into separate total isotopic spin channels, as we may do, the resonances will also have the total isotopic spin of the channel. Thus the sequence of resonances will comprise a family of resonances with the same I, I_z, B, and S.

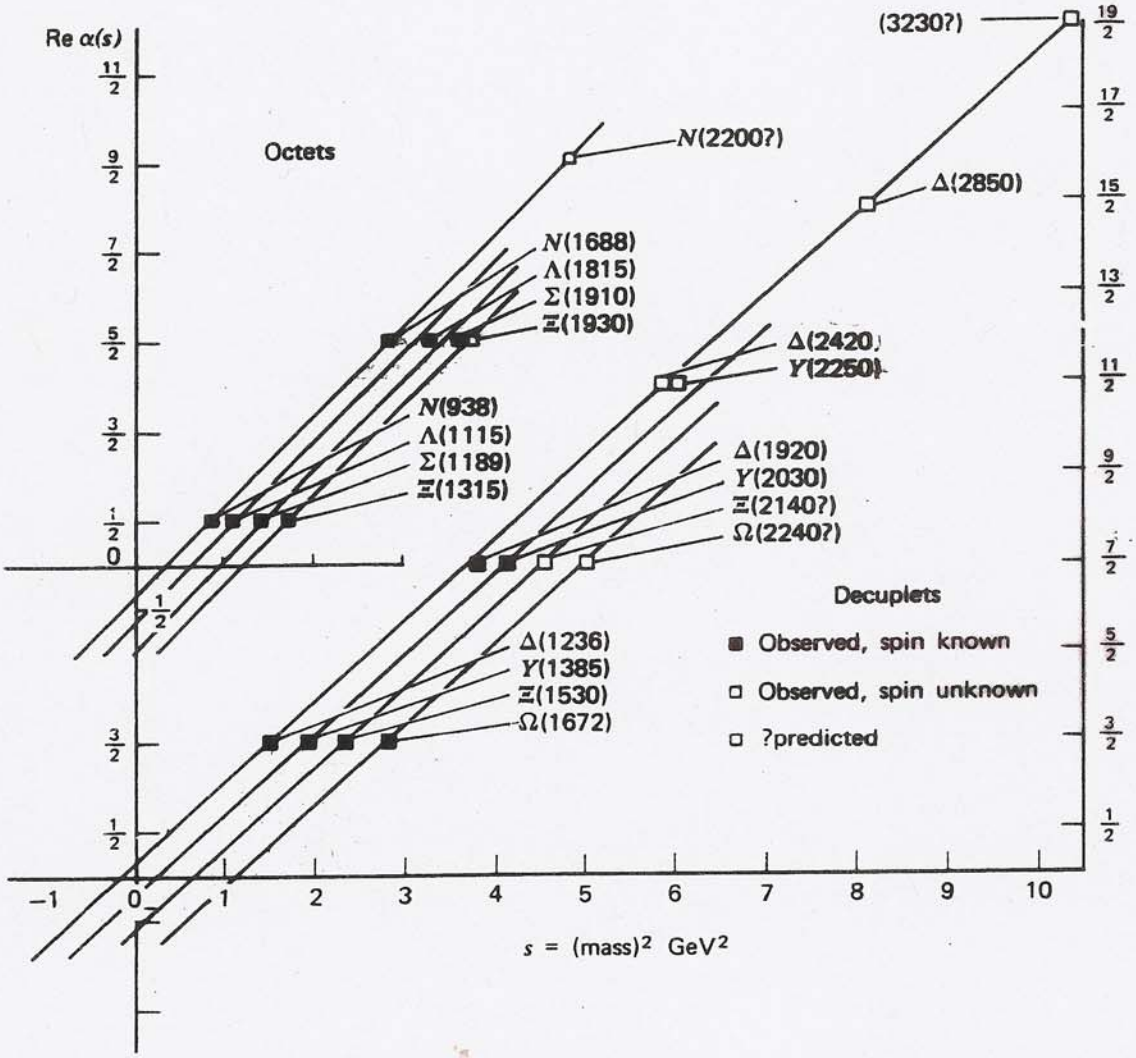


Fig. 17-8 A Chew-Frautschi plot for baryons. The lines, which are assumed to be Regge trajectories, connect baryon families with the same I, B, and S (17CHI). Reproduced with permission from "Evidence for Regge Poles and Hadron Collision Phenomena at High Energies," *Annual Review of Nuclear Science*, 22, 263 (1972). Copyright © by Annual Reviews, Inc., 1972. All rights reserved.

Suppose we sum over an <u>infinite</u> number of hadronic resonances

Two good things happen:

1) can fix unitarity problem

e.g. e^{-x} behaves better as $x \to \infty$ than does $\sum_{n=1}^{5} \frac{(-x)^n}{n!}$

2) it's possible for $M^{\text{s-channel}} = M^{\text{t-channel}}$

i.e. get s-t-u crossing symmetry from a <u>single</u> (infinite) sum

Veneziano dual amplitude (1968)

$$M(s,t) \approx B(-\alpha(s),-\alpha(t)) + B(-\alpha(s),-\alpha(u)) + B(-\alpha(t),-\alpha(u))$$

where
$$B(-\alpha(s),-\alpha(t)) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

with $\alpha(s) = \alpha' s + \alpha(0)$
$\alpha(t) = \alpha' t + \alpha(0)$ Regge trajectories

A single amplitude contains an infinite number of s, t, u channel poles

$$\boxed{\text{this is not a field theory!}}$$

note $B(-\alpha(s),-\alpha(t)) \underset{\substack{\text{large } s\\ \text{fixed } t}}{\sim} (-\alpha(s))^{\alpha(t)} \sim s^{\alpha(t)} \sim s^{\text{constant}}$

instead of s^J $J\to\infty$ as before

$\Rightarrow$ fixes unitarity

but Veneziano amplitude fails badly in the hard scattering limit: large s, large t

$$B(-\alpha(s),-\alpha(t)) \sim e^{-\alpha' s f(\theta)}$$

correct behavior (QCD!) is $\sim \frac{1}{s}$

what kind of physics gives towers of excitations that obey $J = \alpha' M^2 + \alpha(0)$?

Look at classical strings:

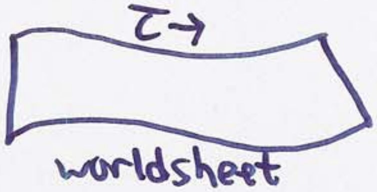
describe by $X^\mu(\sigma, \tau)$ $\mu = 0, 1, \ldots D-1$ spacetime index

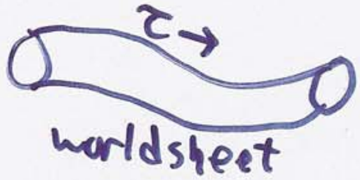
σ ↑ parametrizes where you are on the string

τ ↑ proper time

"open" strings have ends

"closed" strings are loops





Assume both spacetime and worldsheet version of Lorentz invariance, ask:

what is the EOM of a free classical (featureless) string?

Well, a free particle would obey

$$\ddot{X}^\mu \equiv \frac{d^2 X^\mu}{d\tau^2} = 0$$

solution: $X^\mu = X_0^\mu + p^\mu \tau$ $(c=1)$

so $\tau = \frac{t - t_0}{E}$

$$X^i = X_0^i + p^i \tau = X_0^i + \frac{p^i}{E}(t - t_0)$$

this would follow from an action

$$S_{particle} = \frac{1}{2}\int_{-\infty}^{\infty} d\tau \, \dot{X}^\mu \dot{X}_\mu$$

The classical string action must be similar + respect 2D Lorentz invariance, so it should be

$$S_{\substack{open \\ string}} = \frac{1}{2} T \int_{-\infty}^{\infty} d\tau \int_0^\pi d\sigma \, \partial^a X^\mu \partial_a X_\mu$$

where ∂_a stands for $(\partial_\tau, \partial_\sigma)$

and T = string tension, units of $\frac{\text{energy}}{\text{length}}$

EOM for classical string:

$$\ddot{X}^\mu - X''^\mu = 0$$

$\cdot = \partial_\tau$

$' = \partial_\sigma$

but also there is a surface term

$$[\delta X_\mu X'^\mu]_0^\pi \text{ must } = 0$$

$[\delta X_\mu X'^\mu]_0^\pi = 0$ means open string must obey boundary conditions

either

Neumann: $X^{\mu\prime} = 0$ at $\sigma = 0$ and $\sigma = \pi$

or

Dirichlet: $\delta X_\mu = 0$ at $\sigma = 0, \pi$

ie. the ends of the strings are fixed in spacetime

⇒ violation of translational invariance unless strings are tied to a dynamical object (a D-brane!)

what are the solutions to the EOM $\ddot{X}^\mu - X''^\mu = 0$?

general solution: $X^\mu(\sigma,\tau) = \underset{\uparrow}{X_R^\mu(\tau-\sigma)} + \underset{\uparrow}{X_L^\mu(\tau+\sigma)}$

arbitrary functions

\+ boundary conditions

open: $X'^\mu = 0$ $\sigma = 0, \pi$
(ties X_R^μ to X_L^μ)

closed: X_R^μ, X_L^μ periodic under $\sigma \to \sigma + \pi$

let's look at some simple classical solutions:

closed string: any two independent right + left moving profiles

a really simple case:

$$X^\mu = R\begin{pmatrix}\tau\\ \cos 2(\tau-\sigma)\\ \sin 2(\tau-\sigma)\\ 0\\ 0\\ \vdots\end{pmatrix} \quad \text{i.e.} \quad \begin{aligned} t &= R\tau\\ x &= R\cos 2(\tau-\sigma)\\ y &= R\sin 2(\tau-\sigma)\end{aligned}$$

just a spinning circle

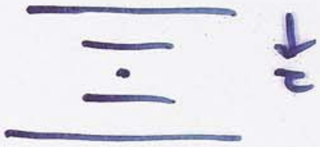
a simple open string solution:

$$X^\mu = \frac{L}{2}\begin{pmatrix}\tau\\ \cos\tau\cos\sigma\\ 0\\ 0\\ \vdots\end{pmatrix}$$

note $t \propto \tau$ with no loss of generality

satisfies $X'^\mu = 0$ at $\sigma = 0, \pi$

string contracts from length L to length 0 then snaps back again



another simple open string solution:

$$X^\mu = \begin{pmatrix} L_0\tau \\ L_1\cos\tau\cos\sigma \\ L_2\sin\tau\cos\sigma \\ 0 \\ \vdots \end{pmatrix}$$

this is a string spinning and stretching in the x-y plane

How fast is each element of the string moving?

$$|\vec{v}|^2 = \left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 = \frac{L_1^2}{L_0^2}\sin^2\tau\cos^2\sigma + \frac{L_2^2}{L_0^2}\cos^2\tau\cos^2\sigma$$

so the ends ($\sigma=0,\pi$) move the fastest:

$$|\vec{v}_{end}|^2 = \frac{L_1^2}{L_0^2}\sin^2\tau + \frac{L_2^2}{L_0^2}\cos^2\tau$$

Problem: they can move faster than c!

Special relativity demands we put an extra constraint on the system

For a particle: $\dot{X}^\mu \dot{X}_\mu = 0 \Rightarrow 1-\left(\frac{dx}{dt}\right)^2=0$ for $t=\tau$

so
For open string we want $\dot{X}^\mu \dot{X}_\mu \geq 0$ saturating the bound at $\sigma=0,$

so the correct relativistic constraint is

$$\dot{X}^\mu \dot{X}_\mu = \underbrace{-X^{\mu\prime} X'_\mu}_{\uparrow \text{ positive, vanishes at } \sigma=0,\pi}$$

so a classical relativistic string obeys both

$$\ddot{X}^{\mu} - X''^{\mu} = 0$$

and

$$\dot{X}^{\mu}\dot{X}_{\mu} + X'^{\mu}X'_{\mu} = 0$$

so

$$X^{\mu} = L\begin{pmatrix}\tau \\ \cos\tau\cos\sigma \\ \sin\tau\cos\sigma \\ 0 \\ \vdots\end{pmatrix} \quad \text{but} \quad X^{\mu} = L\begin{pmatrix}\tau \\ \cos\tau\cos\sigma \\ 0 \\ 0 \\ \vdots\end{pmatrix}$$

is OK — is not OK

transverse motion only
ends of string move at speed of light

what is the (spacetime) energy and angular momentum for this relativistic string?

For particle: $p^\mu = m\frac{dx^\mu}{d\tau}$ $\qquad J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$

For string: $p^\mu = T\int_0^\pi d\sigma\, \dot{X}^\mu$ $\qquad J^{\mu\nu} = T\int_0^\pi d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right)$

so for our simple open string solution

$$E = P^0 = T\int_0^\pi d\sigma\, \dot{X}^0 = \pi L T$$

$$J_z = J^{xy} = T\int_0^\pi d\sigma \left(X^x \dot{X}^y - X^y \dot{X}^x\right)$$

$$= TL^2 \int_0^\pi d\sigma \cos^2\sigma$$

$$= \frac{\pi}{2} TL^2 = \frac{1}{2\pi T}(P^0)^2$$

define $\alpha' \equiv \frac{1}{2\pi T}$

then $J_z = \alpha' M^2$ $\qquad$ Regge trajectory with slope α', intercept at zero

so strings mimic high energy small angle hadronic scattering

1st quantization of the string

Let's quantize $X^\mu(\sigma,\tau)$
(2nd quantization of $\Phi(X^\mu(\sigma,\tau))$ is too hard)

Need a canonical worldsheet hamiltonian H
→ dynamics will be generated by H as evolution in τ, not in physical time! Good idea for a theory (eventually) of quantum gravity

- need an action
- need to define its symmetries

what 2d worldsheet symmetries should we require?

- 2d Lorentz invariance (not necessarily)
- 2d general coordinate invariance yes!!

define "induced" 2d metric $h_{ab}(\sigma,\tau) \equiv \partial_a X^\mu \partial_b X_\mu$

spacetime scalar
worldsheet symmetric tensor

looks like a good starting point to make an action

Note, as in 4d general relativity

$d\sigma d\tau \sqrt{|\det h_{ab}|}$ is an invariant 2d "volume" element so a possible action is:

$$S = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{|\det h_{ab}|}$$

$$= \frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{|\dot{X}^\mu \dot{X}_\mu X'^\nu X'_\nu - \dot{X}^\mu X'_\mu \dot{X}^\nu X'_\nu|}$$

This Nambu-Goto action is just the spacetime area of the worldsheet

Problem: this is a nasty action

An equivalent starting point is the Polyakov action:

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{-\det \gamma_{ab}}\; \gamma^{ab} \partial_a X^\mu \partial_b X_\mu$$

where now $\gamma_{ab}(\sigma,\tau)$ is a general 2d metric fiel
<u>not</u> the induced metric

γ_{ab} is not really dynamical, can eliminate it algebraically from its EOM:

$$0 = T_{ab} = \frac{1}{2\pi\alpha'\sqrt{-\det\gamma}} \frac{\delta S}{\delta \gamma^{ab}} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\gamma_{ab} \partial_c X^\mu \partial^c X_\mu$$

solution: $\gamma_{ab} = \dfrac{2\partial_a X^\mu \partial_b X_\mu}{\partial_c X^\nu \partial^c X_\nu}$ $\quad \gamma^{ab} = \dfrac{\frac{1}{2}\partial^a X^\mu \partial^b X_\mu \partial_c X^\nu \partial^c X_\nu}{\partial^d X^\rho \partial_d X^\sigma \partial^e X_\rho \partial_e X_\sigma}$

and

$$\sqrt{-\det\gamma_{ab}}\; \gamma^{ab} \partial_a X^\mu \partial_b X_\mu \longrightarrow \sqrt{\det h_{ab}}$$

the Nambu action back again

The Polyakov action has more symmetry:

2d coordinate invariance: $\sigma, \tau \to \tilde{\sigma}(\sigma,\tau), \tilde{\tau}(\sigma,\tau)$

+ Weyl (scale) invariance:

$$\gamma_{ab}(\sigma,\tau) \to e^{\Lambda(\sigma,\tau)}\gamma_{ab}(\sigma,\tau)$$

Instead of eliminating γ_{ab} by its EOM, let's gauge-fix it with respect to the 2d coord. invariance:

$$\gamma_{ab} = e^{\Lambda(\sigma,\tau)}\eta_{ab} \qquad \eta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the EOM for $X^\mu(\sigma,\tau)$ is just what we had before:

$$\ddot{X}^\mu - X''^\mu = 0$$

but now we have derived the extra constraint:

$$0 = T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\eta_{ab}\partial_c X^\mu \partial^c X_\mu$$

ie.

$$0 = T_{00} = T_{11} = \frac{1}{4\pi\alpha'}\left[\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right]$$

and also

$$0 = T_{01} = T_{10} = \frac{1}{2\pi\alpha'}\dot{X}^\mu X'_\mu$$

Now canonically quantize $X^\mu(\sigma,\tau)$ in a Fock basis
= creation/annihilation operators for each string normal mode

e.g. closed string: $X^\mu(\sigma,\tau) = X_R^\mu(\tau-\sigma)+X_L^\mu(\tau+\sigma)$

with

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau-\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma}$$

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \alpha' p^\mu(\tau+\sigma) + \frac{i}{2}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}$$

where $\alpha_{-n}^\mu = \alpha_n^{\mu\dagger}$
for $n>0$ $\tilde{\alpha}_{-n}^\mu = \tilde{\alpha}_n^{\mu\dagger}$ are creation operators

while $\alpha_n, \tilde{\alpha}_n$ $n>0$ all annihilate the quantum vacuum state

and $[\alpha_n^{\mu\dagger}, \alpha_m^\nu] = n\,\eta^{\mu\nu}\delta_{nm}$

note: center of mass position is $\frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu = x^\mu + 2\alpha' p^\mu\tau$

and $P^\mu = T\int_0^\pi d\sigma\, \dot{X}^\mu = 2\pi T\alpha' p^\mu = p^\mu$

also, a more compact notation is to write

$\alpha_0^\mu = \tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}}\,p^\mu$
$\sigma^\pm \equiv \tau \pm \sigma$

so $$\frac{dX_R^\mu}{d\sigma^-} = \sqrt{2\alpha'}\sum_n \alpha_n^\mu e^{-2in(\tau-\sigma)}$$

$$\frac{dX_L^\mu}{d\sigma^+} = \sqrt{2\alpha'}\sum_n \tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)}$$

So the string has some ground state $|0,p\rangle$ and an infinite # of excitations

$$\alpha_1^{\dagger\mu}|0,p\rangle$$

$$\tilde{\alpha}_1^{\dagger\mu}|0,p\rangle \qquad \alpha_1^\dagger\alpha_1^\dagger\alpha_1^\dagger\alpha_2^\dagger\alpha_2^\dagger\alpha_3^\dagger\alpha_4^\dagger\tilde{\alpha}_1^\dagger\tilde{\alpha}_2^\dagger\tilde{\alpha}_2^\dagger\tilde{\alpha}_{137}^\dagger|0$$

$$\alpha_2^{\dagger\mu}|0,p\rangle$$

etc.

$$\alpha_1^\dagger\alpha_1^\dagger|0,p\rangle$$

But to find the physical states we must impose the constraints (at some fixed τ) either as operator conditions or on all the matrix elements

In the $\sigma^\pm = \tau \pm \sigma$ notation, the 2 constraints are

$$0 = \frac{dX_L^\mu}{d\sigma^+}\frac{dX_{L\mu}}{d\sigma^+} \qquad 0 = \frac{dX_R^\mu}{d\sigma^-}\frac{dX_{R\mu}}{d\sigma^-}$$

we can Fourier transform to get the Fourier modes of the constraint operators:
(at $\tau=0$)

$$L_m \equiv -\frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\, e^{-2im\sigma}\frac{dX_R^\mu}{d\sigma^-}\frac{dX_{R\mu}}{d\sigma^-}$$

$$\tilde{L}_m \equiv -\frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\, e^{2im\sigma}\frac{dX_L^\mu}{d\sigma^+}\frac{dX_{L\mu}}{d\sigma^+}$$

"Virasoro operators"

since $\int_0^\pi d\sigma\, e^{2i(n_1+n_2-m)\sigma} = \pi\,\delta_{n_1+n_2-m,0}$

we get
$$L_m = -\frac{1}{2}\sum_n \alpha^\mu_{m-n}\,\alpha_{n\mu}$$
$$\tilde{L}_n = -\frac{1}{2}\sum_n \tilde{\alpha}^\mu_{m-n}\,\tilde{\alpha}_{n\mu}$$

the constraints are:

$\langle \text{phys } 1 \,|\, L_m \,|\, \text{phys } 2\rangle = 0$ for all m and all states $|\text{phys } 1\rangle, |\text{phys } 2\rangle$

or equivalently:

$L_m|\text{phys}\rangle = 0$ for all $m>0$

$\tilde{L}_m|\text{phys}\rangle = 0$

note:
$$L_0 = -\frac{1}{2}\alpha_0\alpha_0 - \sum_{n=1}^{\infty}\alpha_n^{\dagger\mu}\alpha_{n\mu}$$
$$= -\frac{\alpha'}{4}p_\mu p^\mu - \sum_{n=1}^{\infty}\alpha_n^{\dagger\mu}\alpha_{n\mu}$$
$$\tilde{L}_0 = -\frac{\alpha'}{4}p_\mu p^\mu - \sum_{n=1}^{\infty}\tilde{\alpha}_n^{\dagger\mu}\tilde{\alpha}_{n\mu}$$

since $\alpha_n^\dagger$ and α_n don't commute, these operators should be normal-ordered

$L_0 \rightarrow \,:L_0: - a$ $\qquad a =$ constant

$\tilde{L}_0 \rightarrow \,:\tilde{L}_0: - a$

use constraints to solve for the masses of the physical string states:

$$(L_0 - a)|\text{phys}\rangle = 0$$
$$(\tilde{L}_0 - a)|\text{phys}\rangle = 0$$

means
$$m^2 = p_\mu p^\mu = -8\pi T\Big[a + \underbrace{\sum_{n=1}^{\infty} \alpha_n^{\dagger\mu}\alpha_{n\mu}}_{-N}\Big] = -8\pi T\Big[a + \underbrace{\sum_{n=1}^{\infty} \tilde{\alpha}_n^{\dagger\mu}\alpha}_{-\tilde{N}}$$

so physical states of the closed string must have

$N = \tilde{N}$ "level matching"

and $m^2 = 8\pi T(N - a)$

note
$\alpha_1^{\dagger}\tilde{\alpha}_1^{\dagger}|0,p\rangle$ phys
but $\alpha_1^{\dagger}|0,p\rangle$ is <u>no</u>

how to compute "a"?

Think of $L_m, \tilde{L}_m$ as generators of an infinite-dimensional nonabelian gauge symmetry [= 2d coord inv]

Virasoro algebra
$$[L_m, L_n] = f_{mn}^{p} L_p$$
$$[\tilde{L}_m, \tilde{L}_n] = f_{mn}^{p} \tilde{L}_p$$

where
$$f_{mn}^{p} = (m-n)\delta_{m+n-p,0}$$

as in nonabelian gauge theory, covariant quantization means you need to add ghost fields bilinearly in the action:

then

$$L_m = L_m^0 - f_{mn}^{p} c^n b_p$$

$$\tilde{L}_m = \tilde{L}_m^0 - f_{mn}^{p} \tilde{c}^n \tilde{b}_p$$

with ghost modes c^n, b_n, $\tilde{c}^n$, $\tilde{b}_n$ obeying

$$\{c_m, b_n\} = \delta_{m+n,0}$$

$$\{\tilde{c}_m, \tilde{b}_n\} = \delta_{m+n,0}$$

$$c^m = c_{-m}$$

$$\tilde{c}^m = \tilde{c}_{-m}$$

so $L_m = L_m^D + L_m^g$

$$L_m^D = -\frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n}^{\mu} \alpha_{n\mu}$$

$$L_m^g = -(m-n) c_{-n} b_{m+n}$$

Now let's recompute the symmetry algebra but being careful about normal ordering

this will allow us to discover extra dimensions + tachyons!

$$L^D_m = -\frac{1}{2}\Big[\underset{\substack{\downarrow\\ \alpha^\dagger\alpha}}{\sum_{-\infty}^{0}\alpha_{n\mu}\alpha^\mu_{m-n}} + \underset{\substack{\downarrow\\ \alpha\alpha}}{\sum_{1}^{m-1}\alpha^\mu_{m-n}\alpha_{n\mu}} + \underset{\substack{\downarrow\\ \alpha^\dagger\alpha}}{\sum_{m}^{\infty}\alpha^\mu_{m-n}\alpha_{n\mu}}$$

$$L^D_{-m} = -\frac{1}{2}\Big[\underset{\substack{\downarrow\\ \alpha^\dagger\alpha}}{\sum_{-\infty}^{-m}\alpha_{n\mu}\alpha^\mu_{m-n}} + \underset{\substack{\downarrow\\ \alpha^\dagger\alpha^\dagger}}{\sum_{-1}^{-(m-1)}\alpha^\mu_{-m-n}\alpha_{n\mu}} + \underset{\substack{\downarrow\\ \alpha^\dagger\alpha}}{\sum_{0}^{\infty}\alpha^\mu_{-m-n}\alpha_{n\mu}}$$

we want to compute
$[L_m, L_{-m}]$ using

$$[AB, CD] = A[B,C]D + [A,C]BD + CA[B,D] + C[A,D]B$$

so $[\alpha^\dagger\alpha, \alpha^\dagger\alpha] \sim \alpha^\dagger\alpha$ doesn't need normal ordering

so only the finite sums contribute to a normal ordering constant!

$$\frac{1}{4}\sum_{n=1}^{m-1}\sum_{p=1}^{m-1}[\alpha^\mu_{m-n}\alpha_{n\mu}, \alpha^\nu_{p-m}\alpha_{-p\nu}]$$

$$\begin{aligned} = \frac{1}{4}\sum_{n=1}^{m-1}\sum_{p=1}^{m-1}\Big\{&(-n\delta_{n+p-m}\delta^\nu_\mu)\alpha^\mu_{m-n}\alpha_{-p\nu} \\ &+((n-m)\delta_{p-n}\eta^{\mu\nu})\alpha_{n\mu}\alpha_{-p\nu} \\ &+(-n\delta_{n-p}\eta_{\mu\nu})\alpha^\nu_{p-m}\alpha^\mu_{m-n} \\ &+((n-m)\delta_{m-n-p}\delta^\mu_\nu)\alpha^\nu_{p-m}\alpha_{n\mu}\Big\}\end{aligned}$$

this gives $\frac{1}{4}\sum_{n=1}^{m-1}\{-n\alpha^\mu_{m-n}\alpha_{n-m\,\mu} + (n-m)\alpha^\mu_n\alpha_{-n\,\mu}$

$-n\,\alpha^\mu_{n-m}\alpha_{m-n\,\mu} + (n-m)\alpha^\mu_{-n}\alpha_{n\,\mu}\}$

simplify by using $\sum_{n=1}^{m-1}(-n)\alpha^\mu_{m-n}\alpha_{n-m\,\mu} = \sum_{n=1-m}^{-1} -(n+m)\alpha^\mu_{-n}\alpha_{n\,\mu}$

$= \sum_{n=1}^{m-1}(n-m)\alpha^\mu_n\alpha_{-n\,\mu}$

so we just get

$$\frac{1}{2}\sum_{n=1}^{m-1}(n-m)(\alpha^\mu_n\alpha_{-n\,\mu} + \underbrace{\alpha^\mu_{-n}\alpha_{n\,\mu}}_{\text{already } \alpha^\dagger\alpha})$$

gives $\frac{1}{2}\sum_{n=1}^{m-1}(n-m)[\alpha^\mu_n,\alpha_{-n\,\mu}] = \frac{1}{2}\sum_{n=1}^{m-1}(n-m)(-n)\delta^\mu_\mu$

$$= \frac{D}{2}\sum_{n=1}^{m-1} n(m-n)$$

use $\sum_{n=1}^{m} n = \frac{1}{2}m(m+1)$ (Gauss)

$\sum_{n=1}^{m} n^2 = \frac{1}{6}m(m+1)(2m+1)$

So $\frac{D}{2}\sum_{n=1}^{m}(-n^2+mn) = \frac{D}{2}\left[\frac{1}{6}m(m+1)(2m+1) + \frac{1}{2}m^2(m+1)\right]$

$$= \frac{D}{12}(m^3-m)$$

Uh-oh: a nonzero normal ordering contribution means the symmetry is <u>anomalous</u>

i.e. not respected by the quantum theory

__But__ let's check the ghost piece: maybe there is a cancellation

$$L^g_m = \Big[\underset{\overset{\downarrow}{b^\dagger c}}{\sum_{-\infty}^{-(m+1)} (m-n) b_{m+n} c_{-n}} + \underset{\overset{\downarrow}{bc}}{\sum_{-m}^{-1} (m-n) b_{m+n} c_{-n}} - \underset{\overset{\downarrow}{c^\dagger b}}{\sum_{0}^{\infty} (m-n) c_{-n} b_{m+n}} \Big]$$

$$L^g_{-m} = \Big[\underset{\overset{\downarrow}{b^\dagger c}}{\sum_{-\infty}^{-1} (-m-n) b_{-m+n} c_{-n}} + \underset{\overset{\downarrow}{b^\dagger c^\dagger}}{\sum_{0}^{m-1} (-m-n) b_{-m+n} c_{-n}} - \underset{\overset{\downarrow}{c^\dagger b}}{\sum_{m}^{\infty} (-m-n) c_{-n} b_{-m+n}} \Big]$$

use $\{c_m, b_n\} = \delta_{m+n,0}$

so $[b_{m-n} c_n, b_{p-m} c_{-p}] = b_{m-n} \{c_n, b_{p-m}\} c_{-p}$
$\qquad\qquad - b_{p-m} \{b_{m-n}, c_{-p}\} c_n$

so $[L^g_m, L^g_{-m}]$ gives a normal ordering piece $\displaystyle\sum_{n=1}^{m} (m+n)(n-2m) \{b_{m-n} c_{n-m} - \underset{\overset{\downarrow}{b^\dagger c}}{b_{-n} c_n}\}$

$\hookrightarrow$ gives $\displaystyle\sum_{n=1-m}^{0} (2m+n)(n-m) \{b_{-n}, c_n\}$

$$= \sum_{n=0}^{m-1} (2m-n)(-n-m) \{b_n, c_{-n}\}$$

$$= \sum_{n=0}^{m-1} (-2m^2 - mn + n^2) = -2m^2(m) - m\tfrac{1}{2}m(m+1) + \tfrac{1}{6}m(m+1)(2m+1)$$

$$= -2m^3 - \tfrac{1}{2}m^3 - \tfrac{1}{2}m^2 + \tfrac{1}{3}m^3 + \tfrac{1}{2}m^2 + \tfrac{1}{6}m$$

$$= \tfrac{1}{6}(m - 13m^3)$$

so all together

$$[L_m, L_n] = (m-n)L_{m+n} + \left\{ \frac{D}{12}(m^3-m) + \frac{1}{6}(m-13m^3) \right\} \delta_{m+n,0}$$

last step: redefine $L_0 \rightarrow L_0 + a$ on the r.h.s.

so

$$[L_m, L_n] = (m-n)L_{m+n} + \left\{ \frac{D-26}{12} m^3 - \frac{D-2}{12} m + 2ma \right\} \delta_{m+n,0}$$

Conclusion: the 2d coordinate invariance has a quantum anomaly

<u>unless</u>, $D = 26$

<u>and</u> $a = 1$

→ The closed (bosonic) string theory must live in a 26 dimensional spacetime!

→ bad news: since $a=1$ the ground state $|0,p\rangle$ has mass-squared

$$m^2 = -8\pi T a = -8\pi T$$

a tachyon!

good news: the 1st excited state with $N=\tilde{N}=1$ is

$$\alpha_1^{\dagger\mu}\tilde{\alpha}_1^{\dagger\nu}|0,p\rangle$$

with $m^2 = 8\pi T(N-a) = 0$ <u>massless</u>

describes a D-dimensional massless

→ symmetric tensor (a graviton!!!)

→ antisymmetric tensor (Kalb-Ramond field) (would be just a pseudoscalar in $D=4$)

→ massless scalar "dilaton"

so closed bosonic string is a
26 dimensional theory of quantum gravity
+ matter + other long range forces + a tachyonic
instability of its ground state

Supersymmetry

How to solve the tachyon problem?

→ want to remove it by new constraints
or a consistent truncation of the spectrum

→ need a new local worldsheet symmetry

Ramond: add 2dim. fermions and a 2d bose-fermi symmetry

$$S_{RNS} = \frac{1}{2} T \int d\sigma d\tau \left(\partial_a X^\mu \partial^a X_\mu + i \bar{\psi}^\mu \gamma^a \partial_a \psi_\mu \right)$$

worldsheet SUSY : $\delta X^\mu = \bar{\epsilon} \psi^\mu$, $\delta \psi^\mu = -i \gamma^a \epsilon \partial_a X^\mu$

L_m picks up a new piece L_m^f

result : $D_{critical} \to 10$

$a \to 1/2$

note
$\psi^\mu(\sigma,\tau)$ is either
periodic (R) or
antiperiodic (NS)
in σ

The spectrum now has spacetime fermions and enough mass-degenerate states to make spacetime SUSY multiplets

$\Rightarrow$ so we suspect we can truncate the physical states consistently to just the part which has spacetime SUSY

This removes the tachyon, since the Dirac eqn has no tachyonic solutions (so SUSY never has tachyons)

Conclusion: 10-dimensional superstrings are consistent theories of quantum gravity + $D=10$ spacetime SUSY

Friday lecture: some black holes
some string phenomenology