Gravitational radiation: Lecture 1

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Lectures' content

Lecture 1:

- **Einstein equations for weak gravitational fields**
- **Propagation of GWs in vacuum: plane wave solution**
- **Interaction of GWs with free-falling particles**
- **Local Lorentz frame versus transverse traceless frame**
- **GWs carry off energy and angular-momentum**

Lectures' content

Lecture 2:

- **Quadrupolar wave generation in linearized Einstein theory**

- **Brief survey of GW sources:**

  – **Black hole and/or neutron star binaries**
  – **pulsars**
  – **supernovae**
  – **stochastic background**

References

Landau-Lifshitz: *Field Theory*, **Chap. 11, 13**

Schutz: *A first course in general relativity*, **Chap. 8, 9**

Weinberg: *Gravitation and Cosmology*, **Chap. 7, 10**

Misner-Thorne-Wheeler: *Gravitation*, **Chap. 8**

Course by Thorne available on the web: Lectures 4, 5 & 6

Relativistic units:

$G = 1 = c$ $\Rightarrow$ Mass, space and time have same units

$$1\,\mathrm{sec} \sim 3 \times 10^{10}\,\mathrm{cm}$$

$$1 M_\odot \sim 5 \times 10^{-6}\,\mathrm{sec}$$

A bit of gravitational-wave history

- **In 1916 Einstein realized the propagation effects at finite velocity in the gravitational equations and predicted the existence of wave-like solutions of the linearized vacuum field equations**

- **Works by Eddington, Einstein et al. in the 20-30s trying to understand whether the radiative degrees of freedom were physical**
  - **Complications and subtleties: Non-linearities and invariance under coordinate transformations**

- **The work by Bondi in the mid 50s, applied to self-gravitating systems like binaries made of neutron stars and/or black holes, proved that gravitational waves carry off energy and angular-momentum**

Brief summary of Einstein equations and notations

$$S = S_g + S_{\text{matter}}$$

$$S_g = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R \qquad -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} = T_{\mu\nu}$$

$\eta_{\mu\nu} = (-,+,+,+)$ **with** $\mu,\nu = 0,1,2,3$ **and** $i,j = 1,2,3$

by imposing the principle of minimal action

$$\int (G_{\mu\nu} - \frac{8\pi G}{c^4}T_{\mu\nu})\delta g^{\mu\nu}\sqrt{-g} = 0$$

$$\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}\,T_{\mu\nu}$$

Brief summary of Einstein equations [continued]

$$R^{\nu}_{\mu\rho\sigma} = \frac{\partial\Gamma^{\nu}_{\mu\sigma}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\nu}_{\mu\rho}}{\partial x^{\sigma}} + \Gamma^{\nu}_{\lambda\rho}\,\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\nu}_{\lambda\sigma}\,\Gamma^{\lambda}_{\mu\rho}$$

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda\nu}}{\partial x^{\rho}} + \frac{\partial g_{\lambda\rho}}{\partial x^{\nu}} - \frac{\partial g_{\rho\nu}}{\partial x^{\lambda}}\right)$$

more explicitly:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu}\partial x^{\rho}} + \frac{\partial^2 g_{\nu\rho}}{\partial x^{\mu}\partial x^{\sigma}} - \frac{\partial^2 g_{\mu\rho}}{\partial x^{\nu}\partial x^{\sigma}} - \frac{\partial^2 g_{\nu\sigma}}{\partial x^{\mu}\partial x^{\rho}}\right)$$

$$+\frac{1}{2}g_{\lambda\alpha}\left(\Gamma^{\lambda}_{\nu\rho}\,\Gamma^{\alpha}_{\mu\sigma} - \Gamma^{\lambda}_{\nu\sigma}\,\Gamma^{\alpha}_{\mu\rho}\right)$$

Bianchi identity: $R^{\lambda}_{\mu\nu\rho;\sigma} + R^{\lambda}_{\mu\sigma\nu;\rho} + R^{\lambda}_{\mu\rho\sigma;\nu} = 0$

Brief summary of Einstein equations [continued]

Ricci tensor: $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu}$

Scalar tensor: $R = g^{\mu\nu} R_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- **Non-linear equations with well-posed initial value structure**
- $4 \times 4 = 16$ **differential equations, but** $G_{\mu\nu}$ **and** $T_{\mu\nu}$ **are symmetric tensors** $\Rightarrow 10$ **differential equations, but because of Bianchi identity** ${G_{\mu\nu}}^{;\nu} = 0 \Rightarrow 6$ **differential equations to be solved when** $T_{\mu\nu}$ **is given**

Einstein equations for weak gravitational fields in flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\left(\partial_\rho h^{\mu}_{\nu} + \partial_\nu h^{\mu}_{\rho} - \partial^\mu h_{\nu\rho}\right) + \mathcal{O}(|h|^2)$$

$$R_{\mu\nu\rho\sigma} = \eta_{\mu\alpha}\,\partial_\rho\Gamma^{\alpha}_{\nu\sigma} - \eta_{\mu\alpha}\,\partial_\sigma\Gamma^{\alpha}_{\nu\rho} + \mathcal{O}(|h|^2)$$

$$= \frac{1}{2}\left(h_{\mu\sigma,\nu\rho} + h_{\nu\rho,\mu\sigma} - h_{\mu\rho,\nu\sigma} - h_{\nu\sigma,\mu\rho}\right) + \mathcal{O}(|h|^2)$$

$$G_{\nu\sigma} = R_{\nu\sigma} - \frac{1}{2}\eta_{\nu\sigma}\,R =$$

$$\frac{1}{2}\left[h_{\mu\sigma,\nu}{}^{\mu} + h_{\mu\nu,\sigma}{}^{\mu} - h_{,\nu\sigma} - h_{\nu\sigma,\mu}^{\ \ \ \ \mu} - \eta_{\nu\sigma}h_{\mu\alpha,}{}^{\alpha\mu} + \eta_{\nu\sigma}h_{,\alpha}{}^{\alpha} + \mathcal{O}(|h|^2)\right]$$

Trace reverse tensor $\bar{h}_{\mu\nu}$

At linear order we can write: $h^{\mu}_{\beta} = \eta^{\mu\alpha}\, h_{\alpha\beta}$, $\qquad h = \eta^{\alpha\beta}\, h_{\alpha\beta}$

Introducing $\bar{h}^{\alpha\beta} = h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta}\, h$ $\qquad$ (note that $\bar{h} = -h$)

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\, R \simeq -\tfrac{1}{2}\left[\bar{h}_{\mu\nu,\lambda}{}^{\lambda} + \eta_{\mu\nu}\bar{h}_{\lambda\rho,}{}^{\lambda\rho} - 2\bar{h}_{\mu\lambda,\nu}{}^{\lambda} + \mathcal{O}(|\bar{h}|^2)\right]$$

Imposing the Lorenz gauge (or harmonic or De Donder gauge) $\bar{h}^{\mu\nu}{}_{,\nu} = 0$

Einstein field equations in linearized theory

$$G_{\mu\nu} \simeq -\tfrac{1}{2}\bar{h}_{\mu\nu,\lambda}{}^{\lambda} = \frac{8\pi G}{c^4}\, T^{\mu\nu} \Rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4}\, T^{\mu\nu}$$

Lorenz gauge can always be imposed

$x^\mu_{\rm new} = x^\mu + \xi^\mu(x)$ with ξ^μ an arbitrary and infinitesimal vector field

$$g^{\rm new}_{\mu\nu} = g_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

$$\bar{h}^{\mu\nu}_{\rm new} = \bar{h}^{\mu\nu} - \eta^{\mu\rho}\,\xi^\nu{}_{,\rho} - \eta^{\lambda\nu}\,\xi^\mu{}_{,\lambda} + \eta^{\mu\nu}\xi^\rho{}_{,\rho}$$

$$\bar{h}^{\mu\nu}_{{\rm new},\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \eta^{\lambda\nu}\,\xi^\mu{}_{,\lambda\nu} = 0 \Rightarrow \Box\,\xi^\mu = \bar{h}^{\mu\nu}{}_{,\nu}$$

- ξ^μ exists for any well behaved $\bar{h}^{\mu\nu}$

- ξ^μ is not unique, we can always add to it q^μ such that $\Box q^\mu = 0$

Propagation of GWs in vacuum

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{h}^{\mu\nu} = 0 \quad \text{with} \quad \partial_\nu \bar{h}^{\mu\nu} = 0$$

- Plane wave solution:

$$\bar{h}^{\mu\nu} = \mathcal{A}\,\epsilon^{\mu\nu}(\mathbf{k})\,e^{ik_\alpha\,x^\alpha} \quad \text{with} \quad k_\nu\,\epsilon^{\mu\nu} = 0 \quad \epsilon^{\mu\nu} \to \text{polarization tensor}$$

- General solution:

$$\bar{h}^{\mu\nu} = \mathrm{Re}\left[\int d^3k\,\mathcal{A}^{\mu\nu}(\mathbf{k})\,e^{ik_\alpha\,x^\alpha}\right] \text{ with } k^\mu = (\omega, \vec{k}) \text{ and } k^\mu\,\mathcal{A}_{\mu\nu} = 0$$

Using the freedom within Lorenz gauge $\Rightarrow$ we can determine the *only* physical radiative components in $\bar{h}^{\mu\nu} \Rightarrow \bar{h}^{\mu\nu}_{\rm TT}$

Imposing transverse-traceless gauge

We choose $q_\mu = B_\mu\, e^{ik_\alpha x^\alpha}$ with $k_\alpha k^\alpha = 0$ $\qquad (\Box q_\mu = 0)$

$$\mathcal{A}_{\mu\nu} = \mathcal{A}^{\rm old}_{\mu\nu} - i B_\mu k_\nu - i B_\nu k_\mu + i \eta_{\mu\nu} B^\rho k_\rho$$

We impose:

1. $\mathcal{A}_{\mu\nu} k^\nu = 0$
2. $\mathcal{A}_{\mu\nu} \eta^{\mu\nu} = 0$
3. If U^ν is a constant timelike unit vector $(U_\nu U^\nu = -1)$ we impose $\mathcal{A}_{\mu\nu} U^\mu = 0$

This set of equations determine B_μ

Imposing transverse-traceless gauge [continued]

1. Choosing $U^\mu = \delta^\mu_0 \Rightarrow \mathcal{A}_{\mu 0} = 0 \Rightarrow \mathcal{A}_{00} = \mathcal{A}_{x0} = \mathcal{A}_{y0} = \mathcal{A}_{z0} = 0$

assuming the wave travels along z

2. $\mathcal{A}_{\mu\nu} k^\nu = 0 \Rightarrow \mathcal{A}_{\mu z} = 0 \Rightarrow \mathcal{A}_{0z} = \mathcal{A}_{xz} = \mathcal{A}_{yz} = \mathcal{A}_{zz} = 0$

3. $\mathcal{A}_{\mu\nu} \eta^{\mu\nu} = 0 \Rightarrow \mathcal{A}_{xx} = -\mathcal{A}_{yy}$

$$\mathcal{A}^{\rm TT}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{A}_{xx} & \mathcal{A}_{xy} & 0 \\ 0 & \mathcal{A}_{xy} & -\mathcal{A}_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Only *two* **independent components:** $\mathcal{A}_{xx}$ **and** $\mathcal{A}_{xy}$

Linearly polarized waves in EM and GW theory

- **In EM theory linearly polarized <u>vectors</u> are:**

$\mathbf{e}_x$ and $\mathbf{e}_y$

- **In GW theory linearly polarized <u>tensors</u> are:**

$\mathbf{e}_+ = \mathbf{e}_x \times \mathbf{e}_x - \mathbf{e}_y \times \mathbf{e}_y$ and $\mathbf{e}_\times = \mathbf{e}_x \times \mathbf{e}_y + \mathbf{e}_y \times \mathbf{e}_x$

$(\mathbf{u} \times \mathbf{v})(\lambda, \mathbf{q}) = (\lambda \cdot \mathbf{u})(\mathbf{q} \cdot \mathbf{v})$

$$\mathbf{e}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{e}_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Circularly polarized waves in EM and GW theory

- **In EM theory circularly polarized vectors are:**

$$\mathbf{e}_R = \frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y) \qquad \text{and} \qquad \mathbf{e}_L = \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y)$$

- **In GW theory circularly polarized tensors are:**

$$\mathbf{e}_R = \frac{1}{\sqrt{2}}(\mathbf{e}_+ + i\mathbf{e}_\times) \qquad \text{and} \qquad \mathbf{e}_L = \frac{1}{\sqrt{2}}(\mathbf{e}_+ - i\mathbf{e}_\times)$$

$$\mathbf{e}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{e}_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

GWs have helicity 2

Any plane wave ψ which is transformed by a rotation of any angle θ around the direction of propagation into $\psi' = e^{i\,h\,\theta}\,\psi$ is said to have helicity h

Let us rotate the coordinate system around z by θ

$$x' = x\,\cos\theta + y\,\sin\theta \qquad y' = y\,\cos\theta - x\,\sin\theta$$

$$\mathbf{e}_{x'} = \mathbf{e}_x\,\cos\theta + \mathbf{e}_y\,\sin\theta \qquad \mathbf{e}_{y'} = -\mathbf{e}_x\,\sin\theta + \mathbf{e}_y\,\cos\theta$$

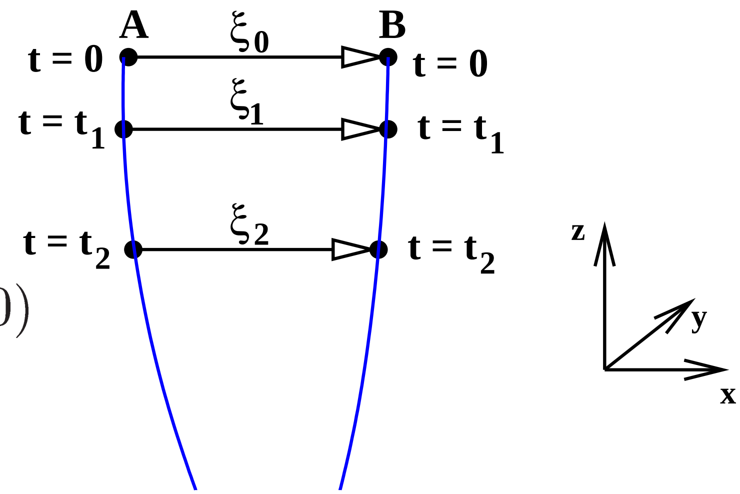
$$\mathbf{e}_{+'} = \mathbf{e}_+\,\cos 2\theta + \mathbf{e}_\times\,\sin 2\theta \qquad \mathbf{e}_{\times'} = -\mathbf{e}_+\,\sin 2\theta + \mathbf{e}_\times\,\cos 2\theta$$

Newtonian description of tidal gravity

- Two point particles A and B falling freely under the action of external Newtonian potential Φ

- A and B at time $t = 0$ are separated by small distance ξ and have equal velocity $\mathbf{v}_{\rm A}(0) = \mathbf{v}_{\rm B}(0)$

- For $t > 0$, A and B experiences slightly different gravitational potential and accelerations $\mathbf{g} = -\nabla\Phi$

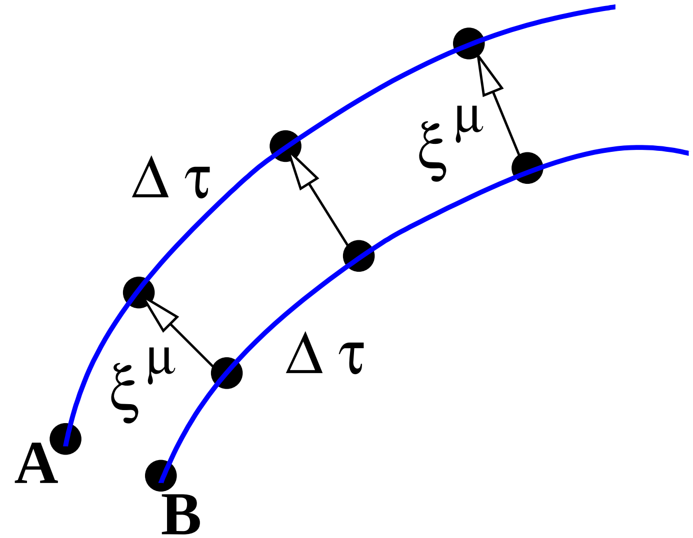


$$\xi^i = x^i_{\rm A} - x^i_{\rm B} \qquad \dot{\xi}^i = \dot{x}^i_{\rm A} - \dot{x}^i_{\rm B}$$

$$\ddot{\xi}^i = \ddot{x}^i_{\rm A} - \ddot{x}^i_{\rm B} = -\left(\frac{\partial \Phi}{\partial x^i}\right)_{\rm B} + \left(\frac{\partial \Phi}{\partial x^i}\right)_{\rm A} \simeq -\left(\frac{\partial^2 \Phi}{\partial x^i\,\partial x^j}\right)\,\xi^j$$

$\epsilon_{ij} = -\left(\frac{\partial \Phi}{\partial x^i\,\partial x^j}\right) \Rightarrow$ Newtonian tidal gravitational field

Equation of geodesic deviation



Pair of nearby freely-falling particles A and B traveling on trajectories $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu$

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau}$$

$$0 = \frac{d^2 (x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x + \xi) \frac{d(x^\nu + \xi^\nu)}{d\tau} \frac{d(x^\lambda + \xi^\lambda)}{d\tau}$$

Taking the difference and limiting to first order in ξ

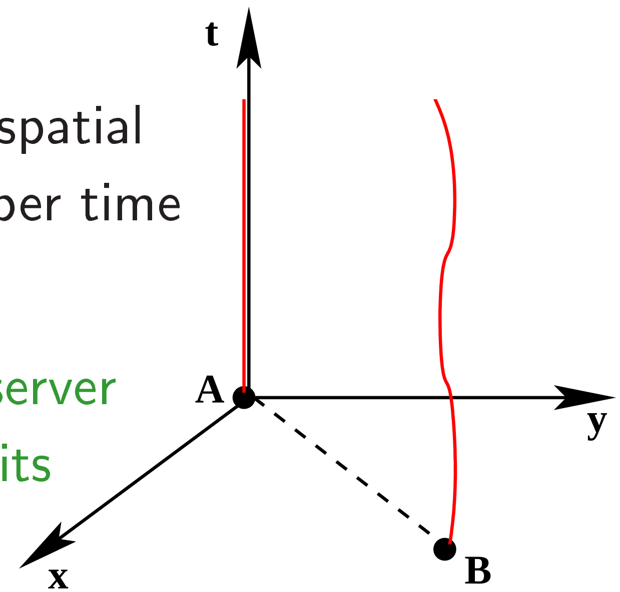
$$\nabla_U \nabla_U \xi^\lambda = R^\lambda_{\nu\mu\rho} \xi^\mu U^\nu U^\rho \qquad U^\alpha = \frac{dx^\alpha}{d\tau}$$

Interaction of GWs with free-falling particles using local Lorentz frame

- Two test particles A and B initially at rest one respect to the other in absence of GWs

- Local Lorentz frame attached to particle A, with spatial origin at $x^j = 0$ and coordinate time equal to proper time $x^0 = t$

- By definition of LLF, the metric $g_{\mu\nu}$ of a LLF observer reduces to Minkowski metric at the origin and all its first derivatives must vanish at the origin



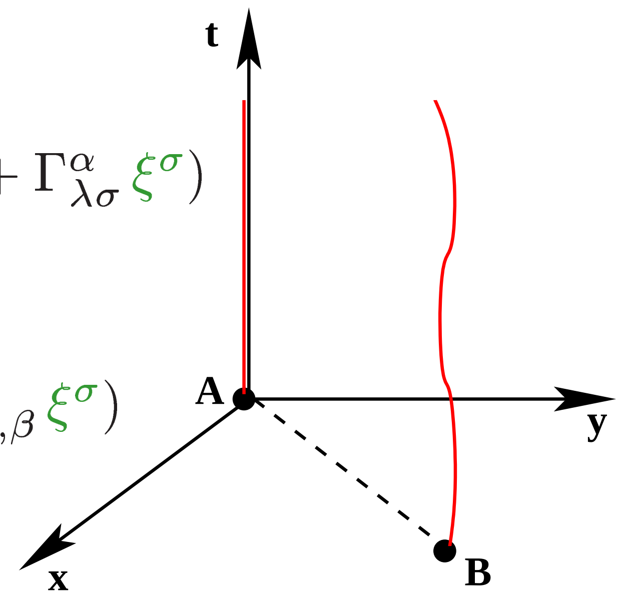
$$ds^2 = -dt^2 + d\mathbf{x}^2 + \mathcal{O}\left(\frac{|\mathbf{x}|^2}{\mathcal{R}}\right)$$

$\mathcal{R}$ being the curvature radius: $\mathcal{R}^{-2} = |R_{\mu\nu\rho\sigma}|$

Geodesic deviation equation in LLF

$$\nabla_U \nabla_U \xi^\alpha = R^\alpha_{\nu\lambda\rho}\, \xi^\lambda \frac{dx^\nu}{dt} \frac{dx^\rho}{dt}$$

$$\nabla_U \nabla_U \xi^\alpha = U^\beta \nabla_\beta (U^\lambda \nabla_\lambda \xi^\alpha) = U^\beta U^\lambda \nabla_\beta (\xi^\alpha{}_{,\lambda} + \Gamma^\alpha_{\lambda\sigma}\, \xi^\sigma)$$



In the LLF of particle A:

$$\Gamma^\sigma_{\alpha\beta} = 0 \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\alpha = U^\beta U^\lambda (\xi^\alpha{}_{,\lambda\beta} + \Gamma^\alpha_{\lambda\sigma,\beta}\, \xi^\sigma)$$

$$U^\alpha = \delta^\alpha_0 \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\alpha = \ddot{\xi}^\alpha + \Gamma^\alpha_{0\sigma,0}\, \xi^\sigma$$

$$\dot{g}_{\mu\nu} \sim \mathcal{O}\left(\frac{|\mathbf{x}|^2}{\mathcal{R}}\right) \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\alpha = \ddot{\xi}^\alpha$$

Assuming $\xi^0 = 0 \quad \Rightarrow \quad \dfrac{d^2 \xi^j}{dt^2} = R^j{}_{0i0}\, \xi^i$

Geodesic deviation equation in LLF [continued]

If $\bar{x}^\mu$ and $\bar{g}^{\mu\nu}$ refer to TT gauge $(h_\times^{\rm TT} = 0, h_+^{\rm TT} \neq 0)$:

$$\bar{g}^{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{\rm TT} & 0 & 0 \\ 0 & 0 & -h^{\rm TT} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The local Lorentz metric $g_{\mu\nu}$ should reduce to Minkowski metric at the origin and all its first derivatives must vanish

$$\bar{t} = t - \dot{h}^{\rm TT}\,(x^2 - y^2)/4$$
$$\bar{x} = x - h^{\rm TT}\,x/2$$
$$\bar{y} = y + h^{\rm TT}\,y/2$$
$$\bar{z} = z + \dot{h}^{\rm TT}\,(x^2 - y^2)/4$$

$$g^{\mu\nu} = \eta_{\mu\nu} - 2\begin{pmatrix} \Phi(t) & 0 & 0 & \Phi(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi(t) & 0 & 0 & \Phi(t) \end{pmatrix}$$

$$\Phi(t) = -\frac{1}{4}\ddot{h}^{\rm TT}\,(x^2 - y^2) \Rightarrow \frac{d^2\xi^j}{dt^2} = -\frac{\partial^2\Phi}{\partial x^j \partial x^k}\,\xi^k$$

Geodesic deviation equation in LLF [continued]

$$R^i_{0j0} = \frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial \bar{x}^\nu}{\partial x^0}\frac{\partial \bar{x}^\lambda}{\partial x^j}\frac{\partial \bar{x}^\rho}{\partial x^0} R^{\rm TT\,\mu}{}_{\nu\lambda\rho} \simeq R^{\rm TT\,i}{}_{0j0} \quad \Rightarrow \quad R^i_{0j0} = \frac{1}{2}\,\ddot{h}^{\rm TT}_{ij}$$

$$\frac{d^2\xi^j}{dt^2} = R^j{}_{0i0}\,\xi^i = \frac{1}{2}\,\ddot{h}^{\rm TT}_{ij}\,\xi^j(0) \quad \Rightarrow \quad \delta\xi^i = \frac{1}{2}\,h^{\rm TT}_{ij}\,\xi^j(0)$$

- The acceleration of particle B in the LLF of particle A is: $a^i = \frac{F^i}{m_B} = \frac{1}{2}\,\ddot{h}^{\rm TT}_{ij}\,\xi^j(0)$

- The observer in LLF of particle A concludes that particle B is subjected to the force $\mathbf{F}$, whereas for an observer in LLF of particle B, particle B is just free falling

In GW interferometers: A → mirror at beam splitter; B → mirror at end of arm cavity

$$\frac{\delta\xi}{\xi(0)} = \frac{\delta\xi}{L} = h \qquad \text{If } L \sim 3 \text{ km},\ h \sim 10^{-21} \quad \Rightarrow \quad \delta\xi \sim 10^{-16}\ \text{m!}$$

Geodesic deviation equation in LLF [continued]

The LLF is useful to do calculations as long as we can use the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}\left(\frac{\mathbf{x}^2}{\mathcal{R}^2}\right)$, i.e., as long as we can disregard $\mathbf{x}^2$ corrections

$$\mathcal{R}^{-2} = |R_{\mu\nu\rho\sigma}| \sim |R_{i0j0}| \sim \ddot{h} \sim \frac{h}{\lambda_{\rm GW}^2}$$

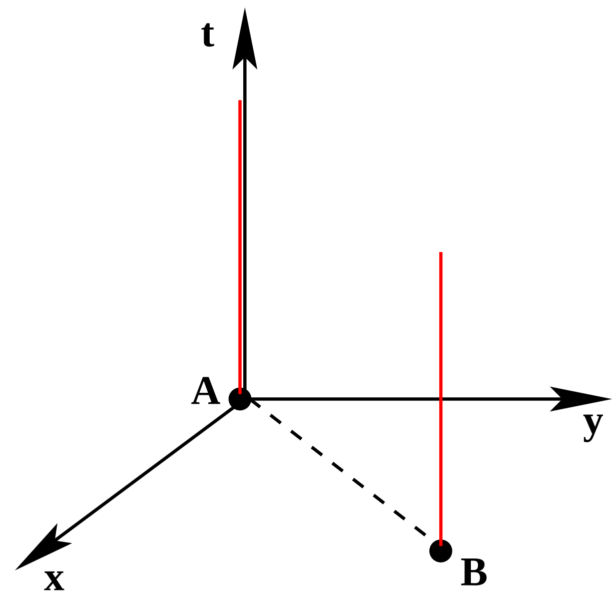
$$\frac{\mathbf{x}^2}{\mathcal{R}^2} \sim \frac{L^2\, h}{\lambda_{\rm GW}^2} \ll 1 \quad \text{if} \quad L \ll \lambda_{\rm GW}$$

- Ground-based detectors (LIGO, VIRGO, TAMA, GEO): $L \ll \lambda_{\rm GW} \sim 10^3$ km

- Space-based detectors (LISA): $L \sim \lambda_{\rm GW} \sim 5 \times 10^6$ km

Interaction of GWs with free-falling particles using TT gauge

- Two test particles A and B initially at rest one respect to the other in absence of GWs



- U^α being the 4-velocity of particle A

$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu}\, U^\mu\, U^\nu = 0$$

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta}\,(h^{\mathrm{TT}}_{\beta\mu,\nu} + h^{\mathrm{TT}}_{\nu\beta,\mu} - h^{\mathrm{TT}}_{\mu\nu,\beta})$$

- Initially $U^\alpha = \delta^\alpha_0 \quad \Rightarrow$

$$\frac{dU^\alpha}{d\tau} = -\Gamma^\alpha_{00} = -\frac{1}{2}\eta^{\alpha\beta}\,(h^{\mathrm{TT}}_{0\beta,0} + h^{\mathrm{TT}}_{\beta 0,0} - h^{\mathrm{TT}}_{00,\beta}) = 0!$$

The particles A and B do not move!

Equivalence between TT gauge and local Lorentz gauge

Proper distance in the two frames (assume that A and B are along x-axis and *only* $h_+ \neq 0$)

- LLF:

$$(\Delta s)^2 = g_{xx}\,(\Delta x)^2$$

$$g_{xx} = 1 \quad \text{but} \quad (\Delta x)^2 = (L + \tfrac{1}{2} h_+\, L)^2$$

$$\Rightarrow \Delta s = L\,\left(1 + \tfrac{1}{2} h_+\right)$$

- TTF:

$$(\Delta s)^2 = g_{xx}\,(\Delta x)^2$$

$$g_{xx} = 1 + h_+ \quad \text{but} \quad (\Delta x)^2 = L^2$$

$$\Rightarrow \Delta s = L\,\left(1 + \tfrac{1}{2} h_+\right)$$

Proper distances are the same!

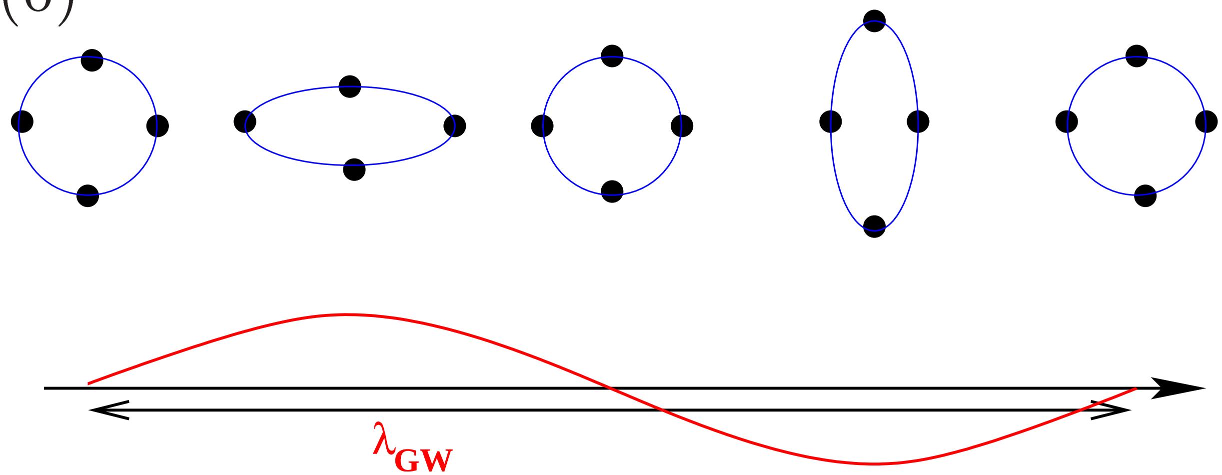
Interaction between GW and ring of free-falling particles: $h_{+}^{\rm TT}$

GW propagating along z-axis

Case: $h_{xx}^{\rm TT} = -h_{yy}^{\rm TT} \equiv h_{+}^{\rm TT} \neq 0 \qquad h_{xy}^{\rm TT} = h_{yx}^{\rm TT} \equiv h_{\times}^{\rm TT} = 0$

$$\delta\xi_x = +\frac{1}{2} h_{xx}^{\rm TT}\, \xi_x(0)$$

$$\delta\xi_y = -\frac{1}{2} h_{xx}^{\rm TT}\, \xi_y(0)$$



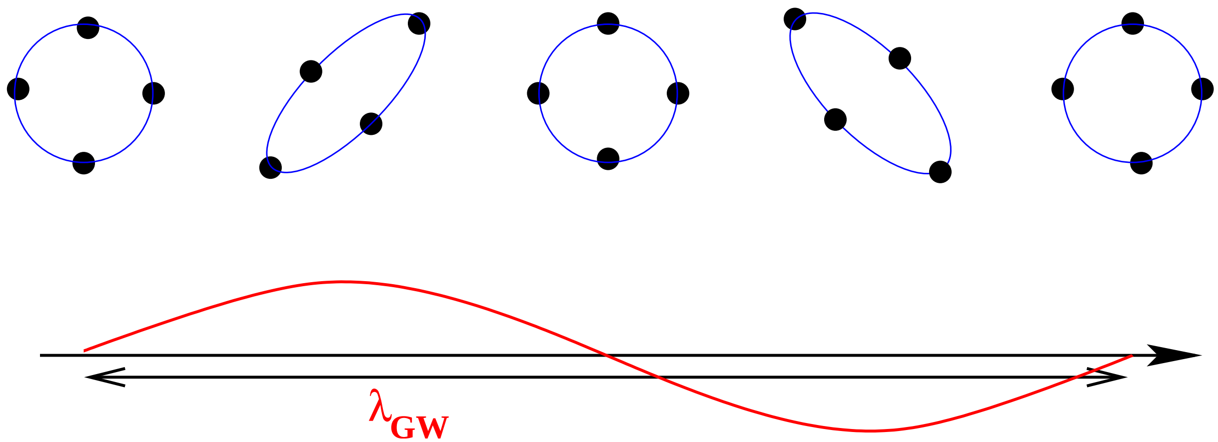
Interaction between GW and ring of free-falling particles: $h_\times^{\rm TT}$

GW propagating along z-axis

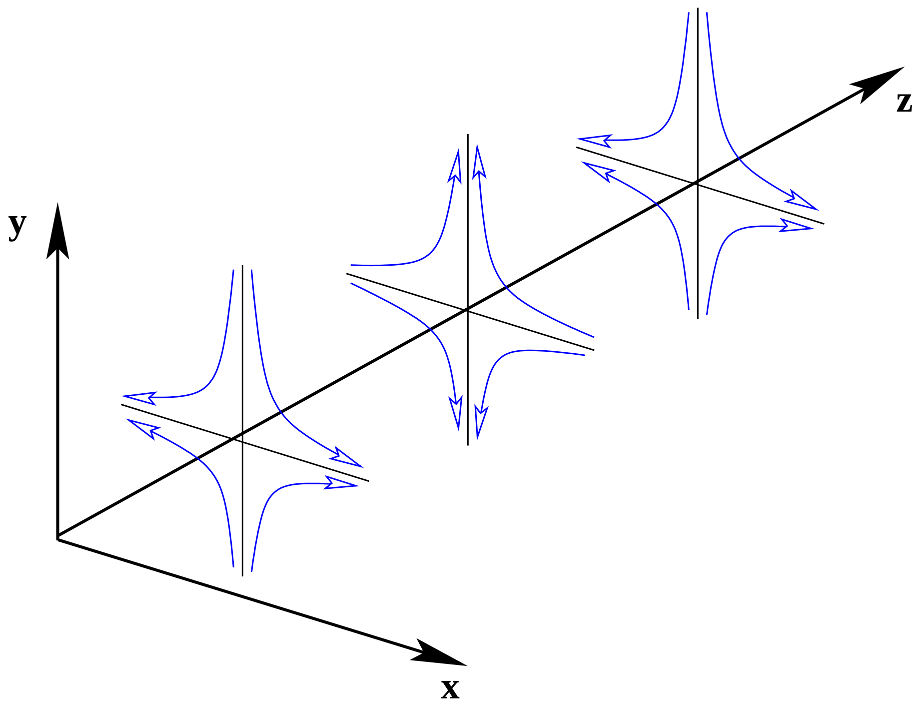
Case: $h_{xx}^{\rm TT} = -h_{yy}^{\rm TT} \equiv h_+^{\rm TT} = 0 \qquad h_{xy}^{\rm TT} = h_{yx}^{\rm TT} \equiv h_\times^{\rm TT} \neq 0$

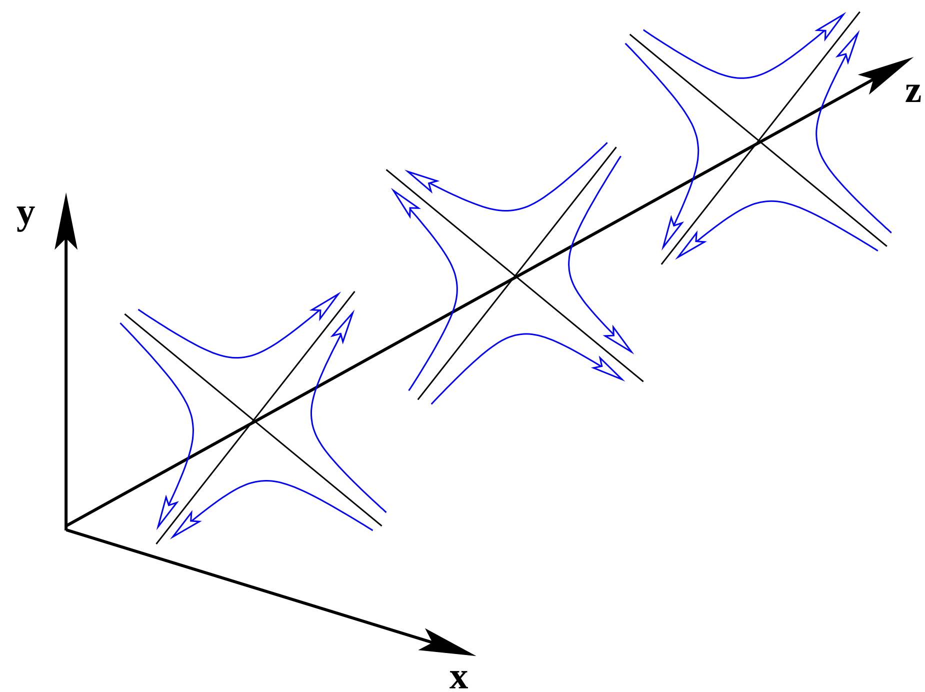
$$\delta\xi_x = +\frac{1}{2} h_{xy}^{\rm TT}\, \xi_y(0)$$

$$\delta\xi_y = +\frac{1}{2} h_{xy}^{\rm TT}\, \xi_x(0)$$



Lines of force for $h_{+}^{\rm TT}$ and $h_{\times}^{\rm TT}$





Energy-momentum pseudo-tensor

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with} \quad |h_{\mu\nu}| \ll 1$$

$$R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, R^{(1)} = \frac{8\pi G}{c^4}\,(T_{\mu\nu} + \tau_{\mu\nu}) \qquad R^{(1)}_{\mu\nu} \sim [\partial\partial h]_{\mu\nu}$$

$$\tau_{\mu\nu} = \frac{c^4}{8\pi G}\left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, R - R^{(1)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\, R^{(1)}\right) \sim [\partial h\, \partial h]_{\mu\nu} + \cdots$$

$$t^{\nu\lambda} = \eta^{\nu\mu}\,\eta^{\lambda\alpha}\,(\underbrace{T_{\mu\alpha}}_{\rm matter's\, EMT} + \underbrace{\tau_{\mu\alpha}}_{\rm grav\, field\, EMPT})$$

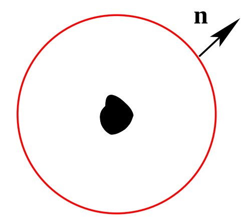
$t^{\mu\nu}$ is locally conserved $\Rightarrow t^{\mu\nu}{}_{,\nu} = 0$

[matter EMT satisfies $T^{\mu\nu}{}_{;\nu} = 0$]

GWs carry off energy and angular momentum

${t^{\mu\nu}}_{,\nu} = 0 \quad \Rightarrow$ for any finite system of volume V bounded by a surface S

$$\Rightarrow P^\lambda = \frac{d}{dt}\int_V t^{0\lambda}\, d^3x = -\int_S t^{i\lambda}\,\mathbf{n}_i\, dS$$



$P^\lambda \quad \Rightarrow$ EM vector of matter and gravitational field; $\quad t^{i\lambda} \quad \Rightarrow$ flux

$$\lambda = 0 \quad \Rightarrow \quad \frac{dE}{dt} = -\int_S t^{i0}\,\mathbf{n}_i\, dS = -\int \tau^{i0}\,\mathbf{n}_i\, dS$$

For a plane GW propagating along z-axis, oscillating at frequency $f_{\rm GW}$:

$$c < \tau^{0z} > = \frac{c^5}{16\pi G} < \frac{1}{2}\dot{h}_{xx}^2 + \frac{1}{2}\dot{h}_{yy}^2 + \dot{h}_{xy}^2 > \sim \frac{c^3}{G} f_{\rm GW}^2\, h^2$$

Supernova at 20 kpc: $c < \tau^{0z} > \sim 400 \frac{\rm erg}{\rm cm^2\,sec}\left(\frac{f_{\rm GW}}{\rm 1kHz}\right)^2 \left(\frac{h}{10^{-21}}\right)^2$

Comparison with other kind of radiation from supernovae

- From neutrino $\sim 10^5 \frac{\rm erg}{\rm cm^2\, sec}$ during 10 secs
- From optical radiation $\sim 10^{-4} \frac{\rm erg}{\rm cm^2\, sec}$ during one week