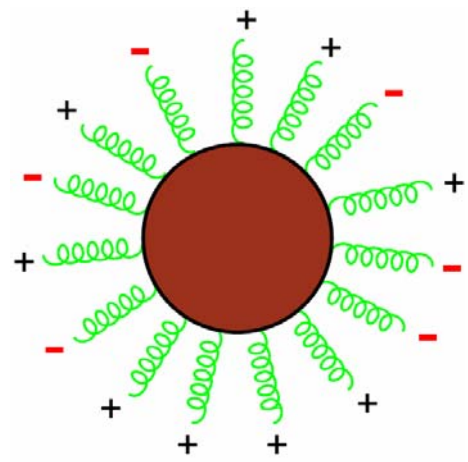
Practical Spinoffs from Twistor Space



Lance Dixon, SLAC

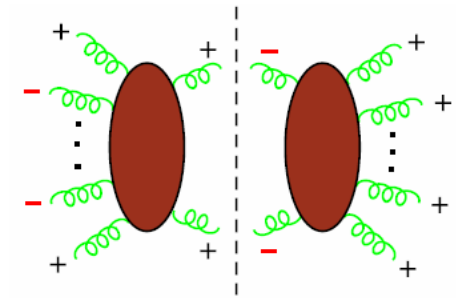
Loop Calculations WG, LCWS05

Stanford, March 21, 2005

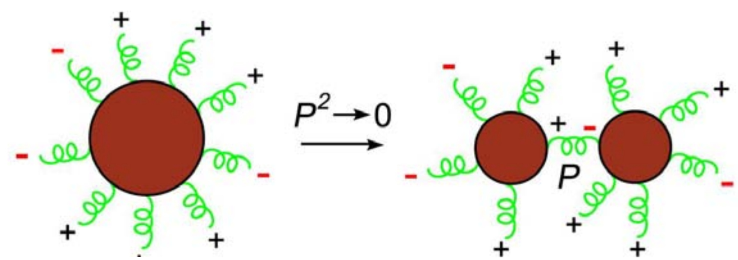
Motivation

- What are the basic analytic properties of scattering amplitudes?

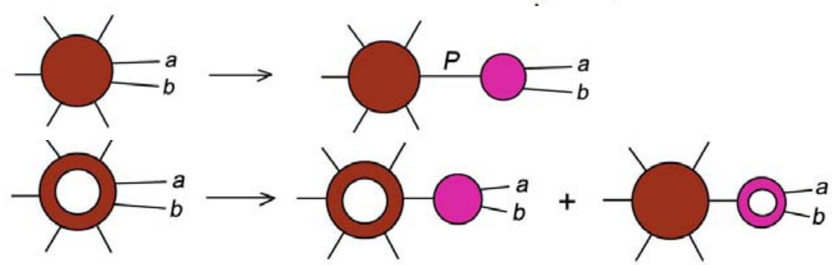
- Branch cuts



- Poles
  - multi-particle



  - collinear



Can we reconstruct scattering amplitudes directly from this information?

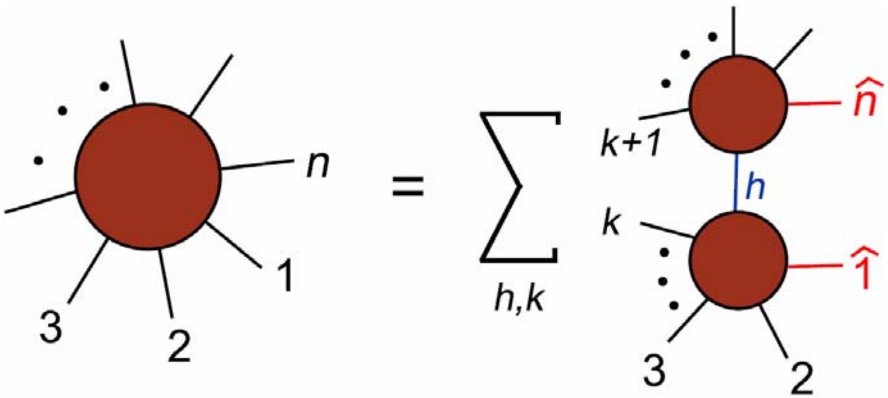
Outline

- BCF recursion relation for tree amplitudes: a simple example
- Where did it come from?
- What is twistor space (target space for Witten's twistor string theory)?
- CSW rules for trees
- BCFW proof of BCF recursion relation
- Recursion relations at loop level
- Conclusions

BCF recursion relation

Britto, Cachazo, Feng, hep-th/0412308

$$A_n(1,2,\ldots,n) = \sum_{h=\pm}\sum_{k=2}^{n-2} A_{k+1}(\hat{1},2,\ldots,k,-\hat{K}_{1,k}^{-h}) \times \frac{i}{K_{1,k}^2} A_{n-k+1}(\hat{K}_{1,k}^{h},k+1,\ldots,n-1,\hat{n})$$



A_{k+1} and A_{n-k+1} are on-shell tree amplitudes with fewer legs, evaluated with momenta shifted by a complex amount

Momentum shift

Describe using spinors, not Lorentz vectors k_i^μ

$$(\lambda_i)_\alpha = u_+(k_i) \qquad (\tilde\lambda_i)_{\dot\alpha} = u_-(k_i)$$

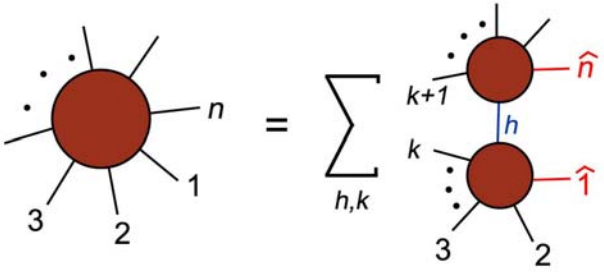
$$k_i^\mu(\gamma_\mu)_{\alpha\dot\alpha} = (\not{k}_i)_{\alpha\dot\alpha} = u_+(k_i)\bar{u}_-(k_i) = (\lambda_i)_\alpha(\tilde\lambda_i)_{\dot\alpha}$$

Complex null momenta are also products of (different) spinors (degenerate 2 x 2 matrix): $0 = p^2 = \det(\not{p}) \quad \Rightarrow \quad (\not{p})_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda'_{\dot\alpha}$

The shift:

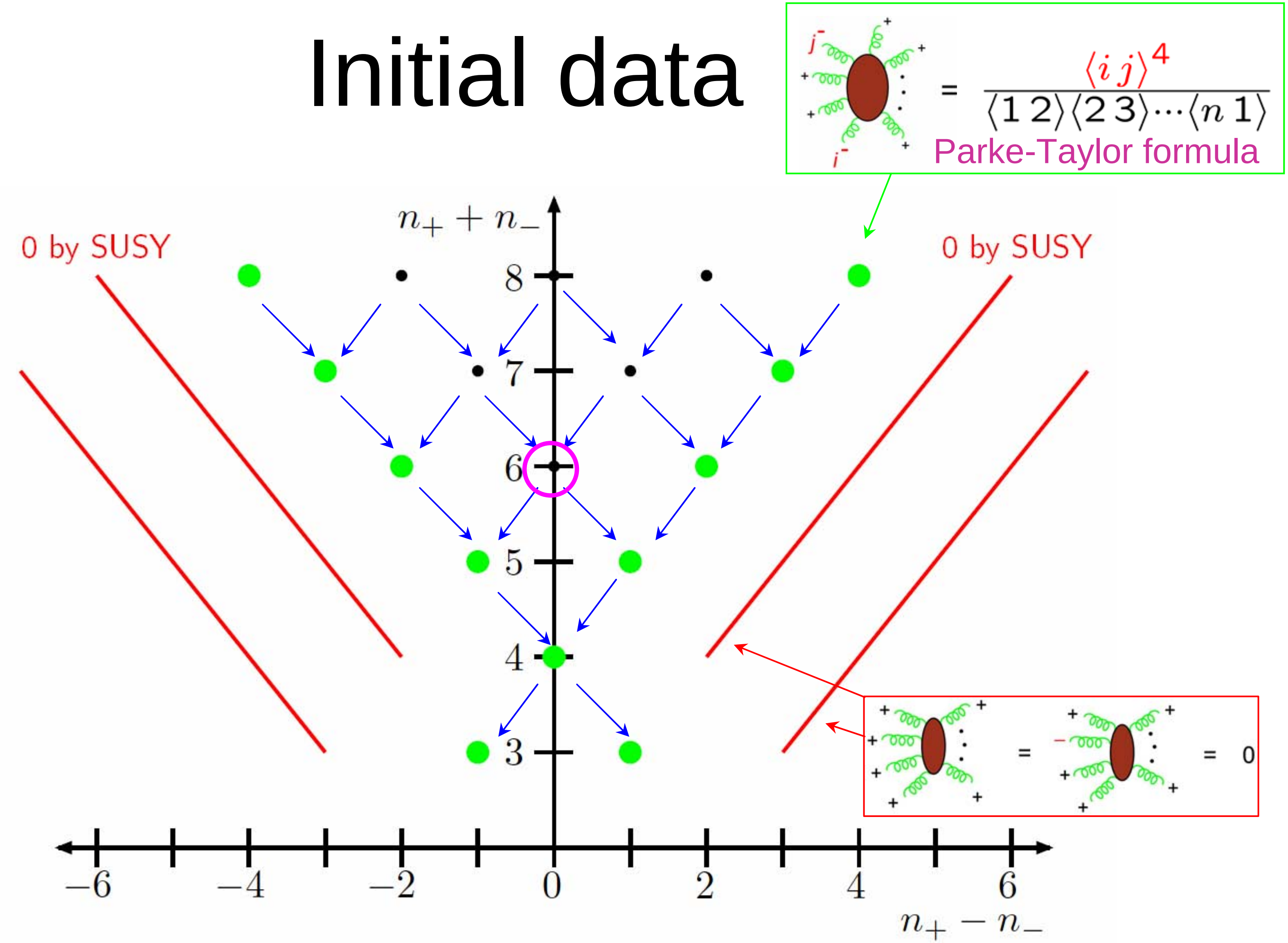
$$\hat\lambda_1 = \lambda_1 - \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle}\lambda_n \qquad \hat{\tilde\lambda}_1 = \tilde\lambda_1$$

$$\hat\lambda_n = \lambda_n \qquad \hat{\tilde\lambda}_n = \tilde\lambda_n + \frac{K_{1,k}^2}{\langle n^- | \not{K}_{1,k} | 1^- \rangle}\tilde\lambda_1$$



$\not{K}_{1,k}$ also shifted by $\propto \lambda_n \tilde\lambda_1$ to preserve momentum conservation

Initial data



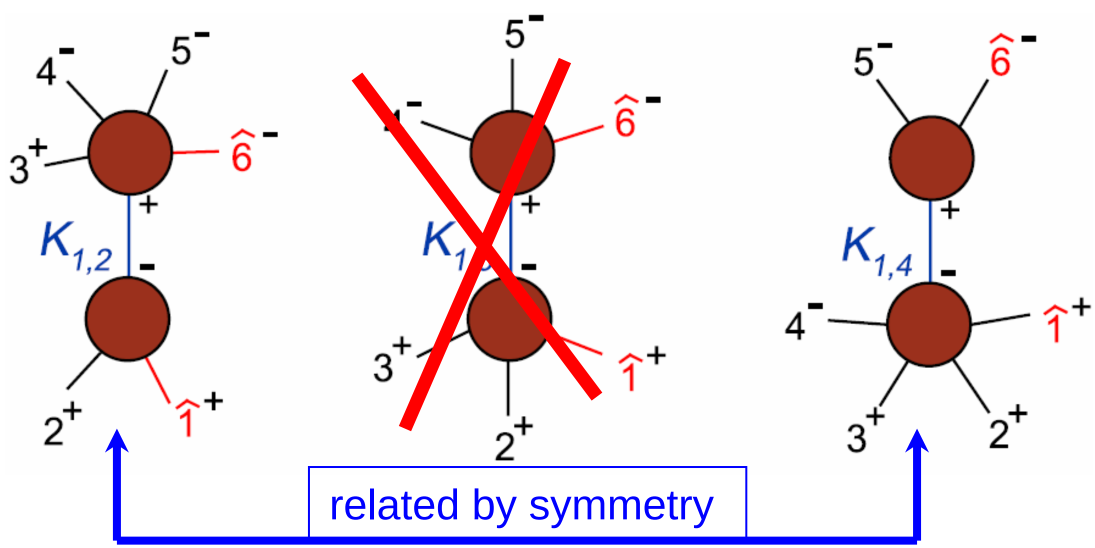
A 6-gluon example

220 Feynman diagrams for *gggggg*

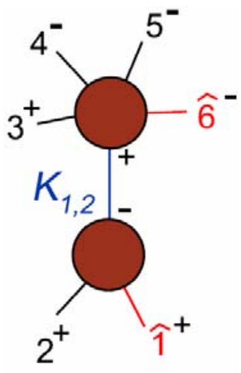
Helicity + color + MHV (--++++) results + symmetries

$\Rightarrow$ only $\boxed{A_6(1^+,2^+,3^+,4^-,5^-,6^-)}$, $A_6(1^+,2^+,3^-,4^+,5^-,6^-)$

3 BCF diagrams



The one $A_6(1^+,2^+,3^+,4^-,5^-,6^-)$ diagram



$$= -\frac{i}{s_{12}} \frac{[\hat{1}\,2]^3}{[2\,\hat{K}][\hat{K}\,\hat{1}]} \frac{[\hat{K}\,3]^3}{[3\,4][4\,5][5\,\hat{6}][\hat{6}\,\hat{K}]}$$

$$= -\frac{i}{s_{12}} \frac{[1\,2]^3}{([2\,\hat{K}]\langle\hat{K}\,6\rangle)(\langle 6\,\hat{K}\rangle[\hat{K}\,1])} \frac{(\langle 6\,\hat{K}\rangle[\hat{K}\,3])^3}{[3\,4][4\,5][5\,\hat{6}]([\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle)}$$

$$= i\,\frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]s_{612}\langle 2^-|(6+1)|5^-\rangle}$$

$$\begin{aligned} \langle 6\,\hat{K}\rangle[\hat{K}\,a] &= \langle 6\,1\rangle\,[1\,a] + \langle 6\,2\rangle\,[2\,a] \\ &= \langle 6^-|(1+2)|a^-\rangle \end{aligned}$$

$$[5\,\hat{6}] = [5\,6] + \frac{s_{12}[5\,1]}{\langle 6\,2\rangle[2\,1]} = \frac{\langle 5^+|(6+1)|2^+\rangle}{\langle 6\,2\rangle}$$

$$[\hat{6}\,\hat{K}]\langle\hat{K}\,6\rangle = \langle 6^+|(1+2)|6^+\rangle + s_{12} = s_{612}$$

Simple final form

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{\langle 6^-|(1+2)|3^-\rangle^3}{\langle 6\,1\rangle\langle 1\,2\rangle[3\,4][4\,5]\,s_{612}\langle 2^-|(6+1)|5^-\rangle} + \frac{\langle 4^-|(5+6)|1^-\rangle^3}{\langle 2\,3\rangle\langle 3\,4\rangle[5\,6][6\,1]\,s_{561}\langle 2^-|(6+1)|5^-\rangle}$$

Simpler than form found in 1980s
despite (because of?) spurious singularities $\langle 2^-|(6+1)|5^-\rangle$

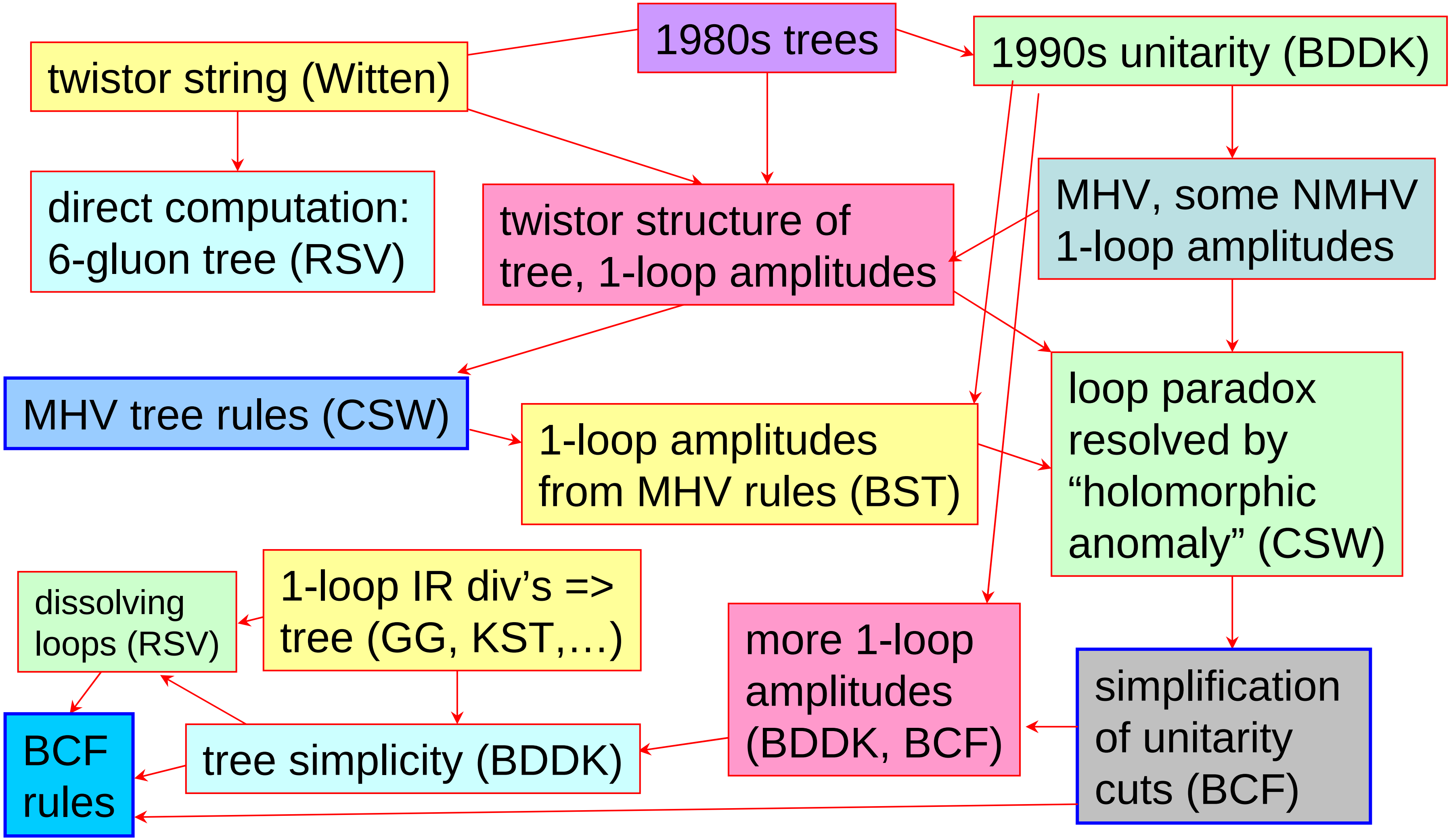
Mangano, Parke, Xu (1988)

$$-iA_6(1^+,2^+,3^+,4^-,5^-,6^-) = \frac{([1\,2]\langle 4\,5\rangle\langle 6^-|(1+2)|3^-\rangle)^2}{s_{61}s_{12}s_{34}s_{45}s_{612}} + \frac{([2\,3]\langle 5\,6\rangle\langle 4^-|(2+3)|1^-\rangle)^2}{s_{23}s_{34}s_{56}s_{61}s_{561}} + \frac{s_{123}[1\,2][2\,3]\langle 4\,5\rangle\langle 5\,6\rangle\langle 6^-|(1+2)|3^-\rangle\langle 4^-|(2+3)|1^-\rangle}{s_{12}s_{23}s_{34}s_{45}s_{56}s_{61}}$$

Relative simplicity even more striking for *n>6*; first noticed in *n=7* expressions found (before BCF relations) via IR divergences of one-loop amplitudes

Bern, Del Duca, LD, Kosower, hep-th/0410224

Where did rules come from?



What is twistor space?

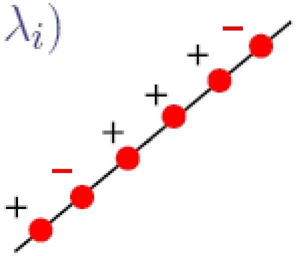
A "half-Fourier transform" of spinor space $(\lambda_a, \lambda_{\dot{a}})$ for each leg

$$\tilde{\lambda}_{\dot{a}} = i\frac{\partial}{\partial\mu^{\dot{a}}} \qquad \mu^{\dot{a}} = -i\frac{\partial}{\partial\tilde{\lambda}_{\dot{a}}}$$

- Twistor space is 4-dimensional: $(\lambda_1, \lambda_2, \mu^{\dot{1}}, \mu^{\dot{2}})$.
- Except for momentum conservation, $\delta(\sum_i k_i) = \int d^4x \exp(ix\sum_i \lambda_i\tilde{\lambda}_i)$, MHV tree amplitudes $\frac{\langle i\, j\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}$ only depend on RH spinors λ_i, not on LH spinors $\tilde{\lambda}_i$.

$$\int\prod_i d\tilde{\lambda}_i \exp(i\mu_i\tilde{\lambda}_i)\exp(ix\lambda_i\tilde{\lambda}_i)\times A(\lambda_i) \;\propto\; \prod_i \delta(\mu_i + x\lambda_i)$$

In MHV tree amplitudes, all points lie on a line



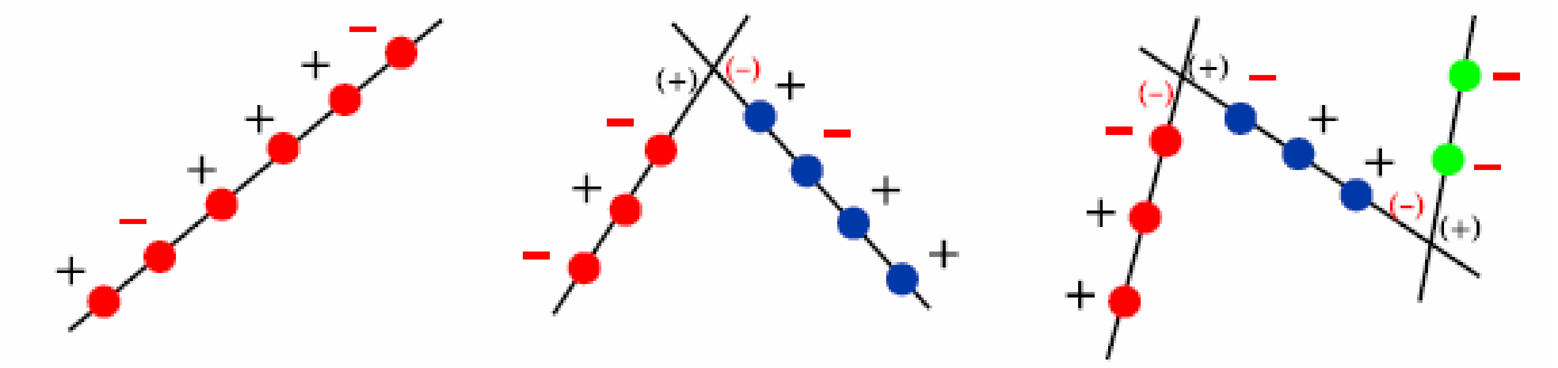
Twistor structure of trees

Can determine empirically using differential operators on helicity amplitudes

- i, j, k have collinear support if A annihilated by
  $F_{ijkL} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K \to \langle i\, j\rangle \frac{\partial}{\partial \tilde\lambda_k^{\dot a}} + \langle j\, k\rangle \frac{\partial}{\partial \tilde\lambda_i^{\dot a}} + \langle k\, i\rangle \frac{\partial}{\partial \tilde\lambda_j^{\dot a}}$
  for $L = \dot a$.

Find:

$n_- = 2$ (MHV) $\quad n_- = 3$ (NMHV) $\quad n_- = 4$ (NNMHV)



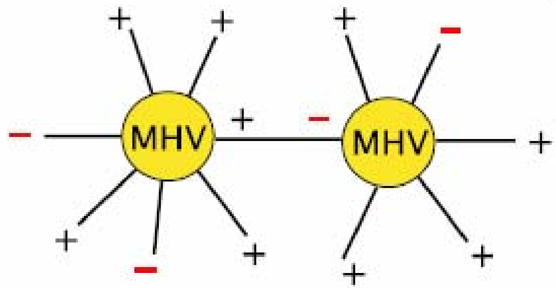
MHV rules

Cachazo, Svrček, Witten (2004)

- Continue MHV amplitudes off-shell:

$$A_n^{\text{tree},\text{MHV},ij}(1^*) = \frac{\langle i\, j\rangle^4}{\langle 1^*\, 2\rangle \ldots \langle n\, 1^*\rangle} = \frac{\langle i\, j\rangle^4}{\langle \eta^+|\, 1\, |2^+\rangle \ldots \langle n^-|\, 1\, |\eta^-\rangle}$$

η is null, $\eta^2 = 0$. Sew vertices with "scalar" propagators. $\dfrac{1}{p^2}$



- Results independent of η, agree (numerically) with Feynman.

MHV rules (cont.)

Rules are quite efficient, can be extended to:

massless fermions

Georgiou, Khoze, hep-th/0404072;
Wu, Zhu, hep-th/0406146;
Georgiou, Glover, Khoze, hep-th/0407027

Higgs bosons (via *Hgg* effective vertex)

LD, Glover, Khoze, hep-th/0411092;
Badger, Glover, Khoze, hep-th/0412275

vector bosons (W,Z,γ^*)

Bern, Forde, Kosower, Mastrolia, hep-th/0412167

Still, each extension requires a little thought, and initial "proofs" of correctness were partly based on empirical agreement.

Recent extension to massive quarks a bit different

Schwinn, Weinzierl, hep-th/0503015

BCFW proof of BCF

Britto, Cachazo, Feng, Witten, hep-th/0501052

Very simple, general – Cauchy's theorem + factorization

Let complex momentum shift depend on z

$$\hat{\lambda}_1 = \lambda_1 + z\lambda_n \qquad \hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 \qquad \Rightarrow \quad A(0) \;\to\; A(z)$$

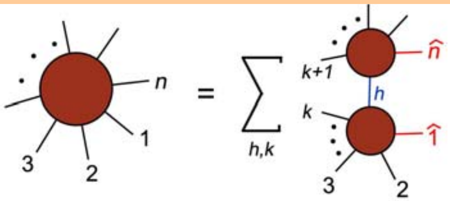
$$\hat{\lambda}_n = \lambda_n \qquad \hat{\tilde{\lambda}}_n = \tilde{\lambda}_n - z\tilde{\lambda}_1$$

Cauchy: $A(\infty) = 0 \quad \Rightarrow$

$$0 = \frac{1}{2\pi i}\oint dz\,\frac{A(z)}{z} = A(0) + \sum_k \mathrm{Res}[\frac{A(z)}{z}]|_{z=z_k}$$

simple pole in z for each possible factorization; residue = [BCF term]

$$z_k = -\frac{K_{1,k}^2}{\langle n^-|\not{K}_{1,k}|1^-\rangle} \qquad \Leftrightarrow \qquad \hat{K}_{1,k}^2(z) = 0$$



Recursion at one loop

Bern, LD, Kosower, hep-th/0501240

For simplicity, we considered special one-loop amplitudes with no cuts, only poles: $A_n^{1-\text{loop}}(1^\pm, 2^+, 3^+, \ldots, n^+)$

Still not quite a trivial extension of BCF because of:

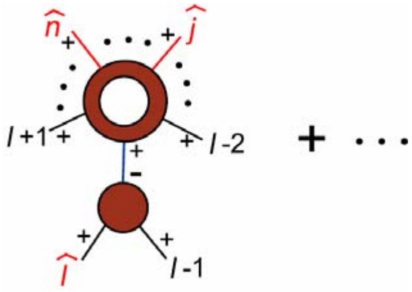
- difficulties with $z \to \infty$ for $A_n^{1-\text{loop}}(1^+, 2^+, 3^+, \ldots, n^+)$
  $\Rightarrow$ need triple-shift

$$\hat{\hat{\tilde\lambda}}_j = \tilde\lambda_j - z\tilde\lambda_l - z\frac{\langle n\, j\rangle}{\langle l\, j\rangle}\tilde\lambda_n$$

$$\hat\lambda_l = \lambda_l + z\lambda_j$$

$$\hat\lambda_n = \lambda_n + z\frac{\langle n\, j\rangle}{\langle l\, j\rangle}\lambda_j$$

$\Rightarrow$ 4-term recursion relation $\sim$



Recursion at one loop (cont.)

- double pole for one z_k in $A_n^{\text{1-loop}}(1^-, 2^+, 3^+, \ldots, n^+)$

  $\Rightarrow$ need ansatz for single pole underneath

  $\Rightarrow$ $(n-2)$-term recursion relation

Agrees with (off-shell) recursive result but expressions much simpler

Mahlon, hep-ph/9312276

$$
\begin{aligned}
&A_{6;1}(1^-,2^+,3^+,4^+,5^+,6^+) \\
&= i\,\frac{N_p}{96\pi^2}\Bigg[\frac{\langle 1^-|(2+3)|6^-\rangle^3}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 4\,5\rangle^2\,s_{123}\,\langle 3^-|(1+2)|6^-\rangle} + \frac{\langle 1^-|(3+4)|2^-\rangle^3}{\langle 3\,4\rangle^2\,\langle 5\,6\rangle\,\langle 6\,1\rangle\,s_{234}\,\langle 5^-|(3+4)|2^-\rangle} \\
&\qquad + \frac{[2\,6]^3}{[1\,2]\,[6\,1]\,s_{345}}\left(\frac{[2\,3]\,[3\,4]}{\langle 4\,5\rangle\,\langle 5^-|(3+4)|2^-\rangle} - \frac{[4\,5]\,[5\,6]}{\langle 3\,4\rangle\,\langle 3^-|(1+2)|6^-\rangle} + \frac{[3\,5]}{\langle 3\,4\rangle\,\langle 4\,5\rangle}\right) \\
&\qquad - \frac{\langle 1\,3\rangle^3\,[2\,3]\,\langle 2\,4\rangle}{\langle 2\,3\rangle^2\langle 3\,4\rangle^2\,\langle 4\,5\rangle\,\langle 5\,6\rangle\,\langle 6\,1\rangle} + \frac{\langle 1\,5\rangle^3\,\langle 4\,6\rangle\,[5\,6]}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle^2\langle 5\,6\rangle^2} \\
&\qquad - \frac{\langle 1\,4\rangle^3\,\langle 3\,5\rangle\,\langle 1^-|(2+3)|4^-\rangle}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle^2\langle 4\,5\rangle^2\,\langle 5\,6\rangle\,\langle 6\,1\rangle}\Bigg]
\end{aligned}
$$

Conclusions

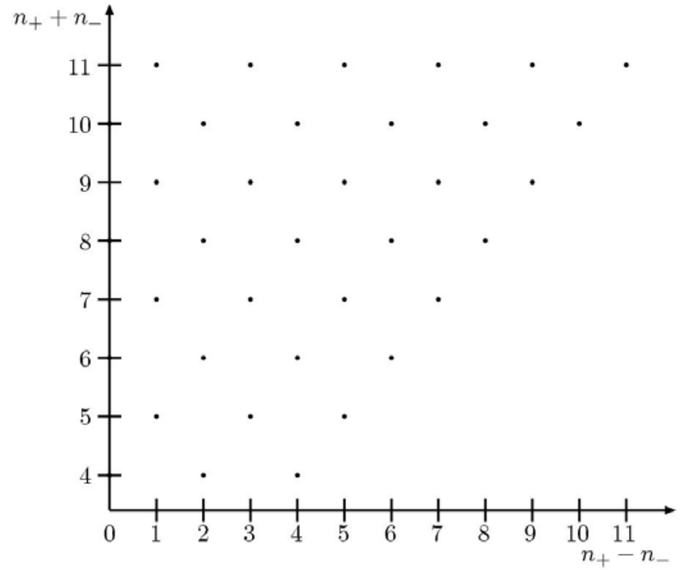
- Much progress in computational techniques gauge theories in last year or so is attributable (directly or indirectly) to development of twistor string theory
- So far, practical spinoffs mostly for trees, and for loops in supersymmetric theories
- However, loop-level versions of BCF recursion relations look promising
- Try to determine polynomial terms in non-SUSY (QCD) loop amplitudes this way (unitarity for branch-cut terms)
- Expect much more progress along these lines in future

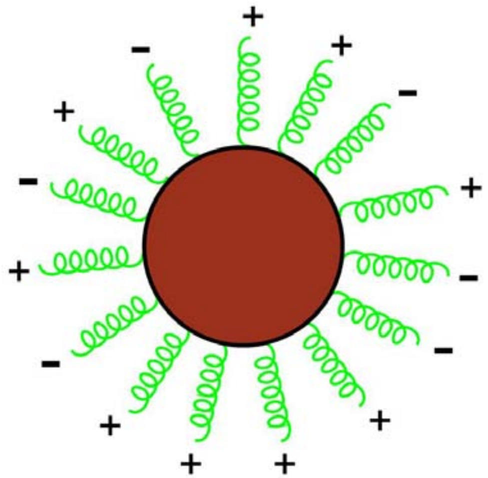
Extra slides

A few references

- Topological string in twistor space   Witten, hep-th/0312171
- MHV (CSW) rules   Cachazo, Svrcek, Witten, hep-th/0403047
- All 7-point N=4 1-loop amplitudes   Bern, LD, Kosower hep-th/0410224
- Generalized unitarity & N=4 1-loop amplitudes
  Britto, Cachazo, Feng, hep-th/0412103
- All n-point NMHV N=4 1-loop amplitudes   Bern, LD, Kosower hep-th/0412210
- Dissolving loops into trees   Roiban, Spradlin, Volovich, hep-th/0412265
- BCF rules   Britto, Cachazo, Feng, hep-th/0412308
- BCFW proof   Britto, Cachazo, Feng, Witten, hep-th/0501052
- Recursive rules at 1 loop   Bern, LD, Kosower hep-th/0501240

March of the *n*-gluon helicity amplitudes





n_+ positive-helicity gluons

n_- negative-helicity gluons

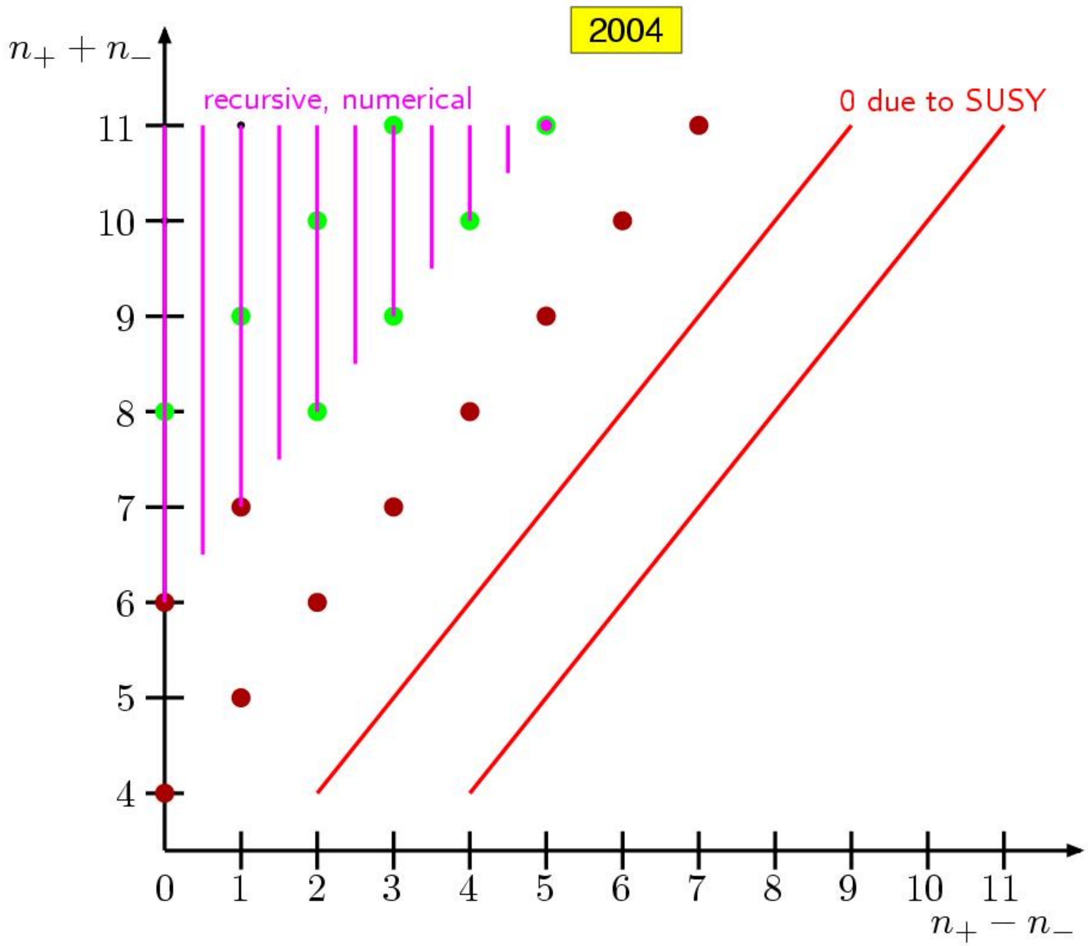
$n = n_+ + n_- \ \geq\ 4$

$n_+ \geq n_-$ by parity

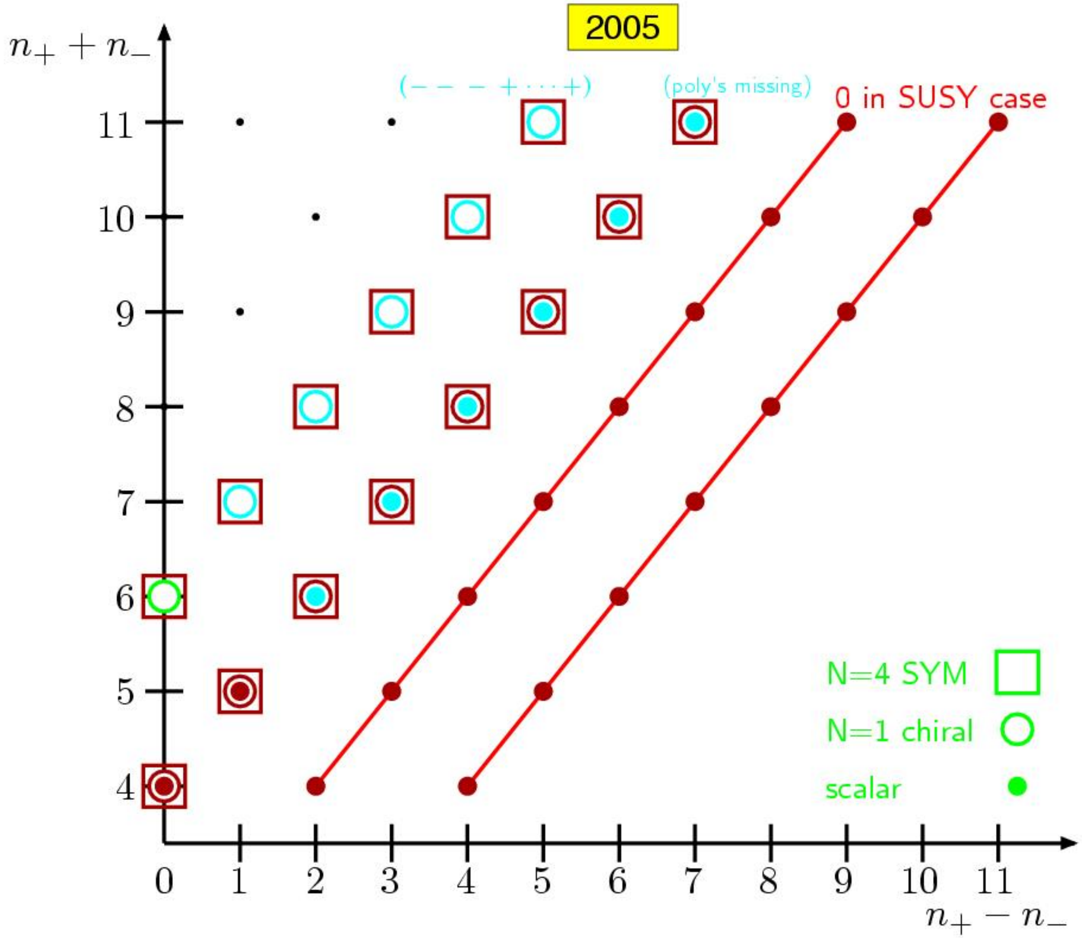
At 1-loop, QCD decomposable into

N=4 SYM, N=1 chiral, scalar contributions

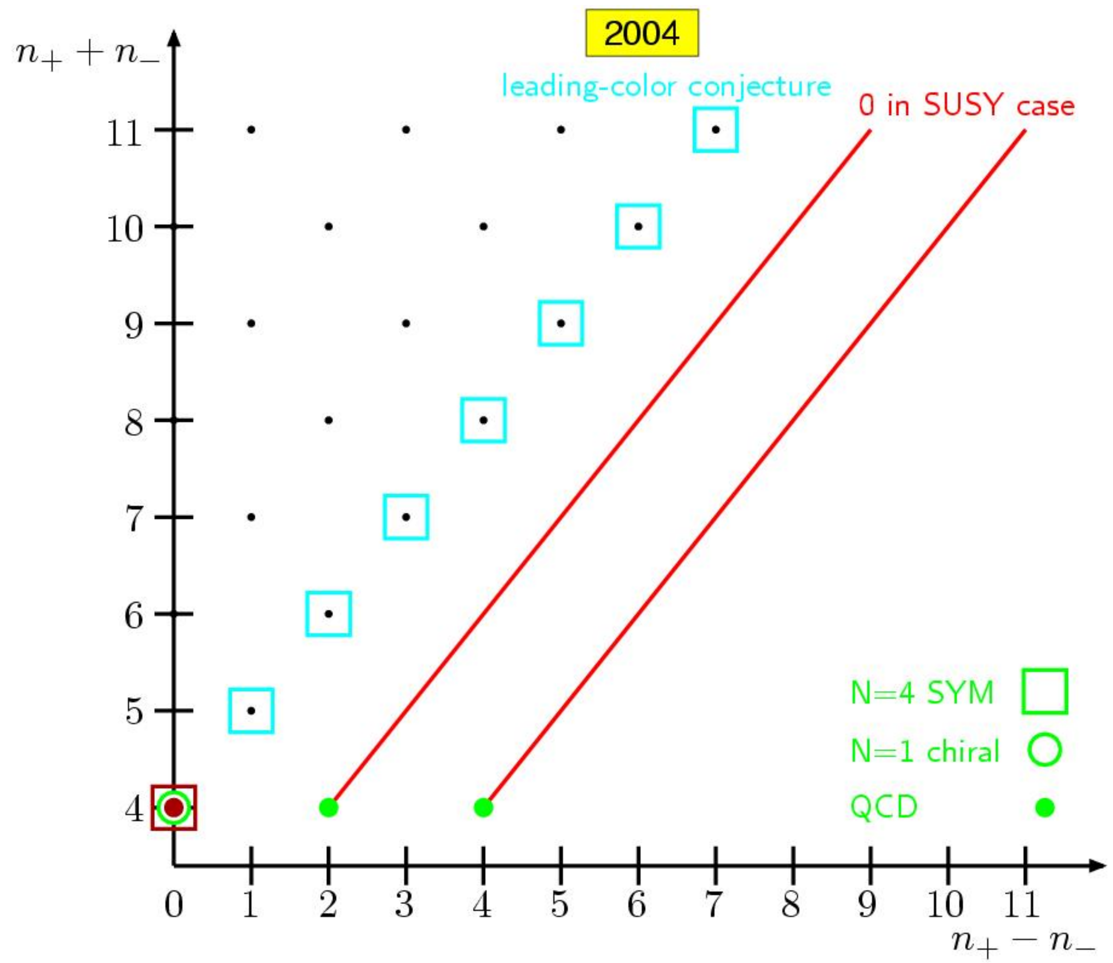
March of the tree amplitudes



March of the 1-loop amplitudes



March of the 2-loop amplitudes



March of the 3-loop amplitudes