# Large $\beta_0$ corrections to the energy levels and wave function at N$^3$LO

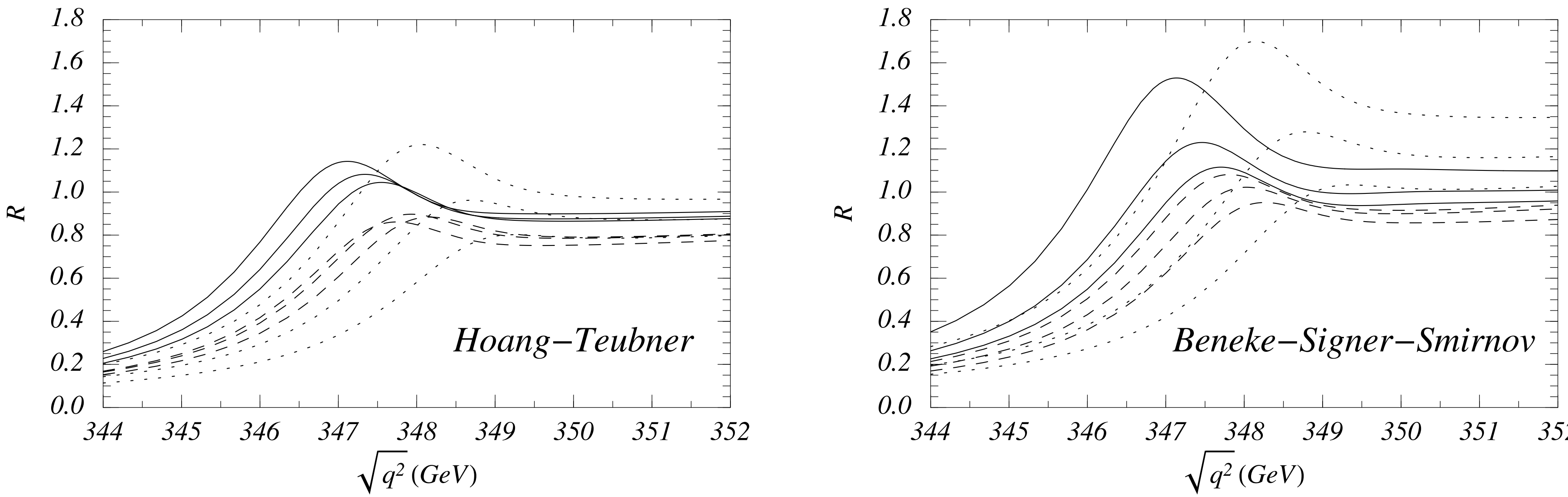
Matthias Steinhauser, University of Karlsruhe

LCWS05, March 2005

[In collaboration with A. Penin and V. Smirnov]

# Motivation I — top

$\sigma(e^+e^- \to t\bar{t})$ @ threshold




[A.H. Hoang, M. Beneke, K. Melnikov, T. Nagano, A. Ota, A.A. Penin, A.A. Pivovarov, A. Signer, V.A. Smirnov, Y. Sumino, T. Teubner, O. Yakovlev, A. Yelkhovsky'00]

pole mass, NNLO:

- 4 different groups
- no stability in position of peak
- no stability in normalization of peak

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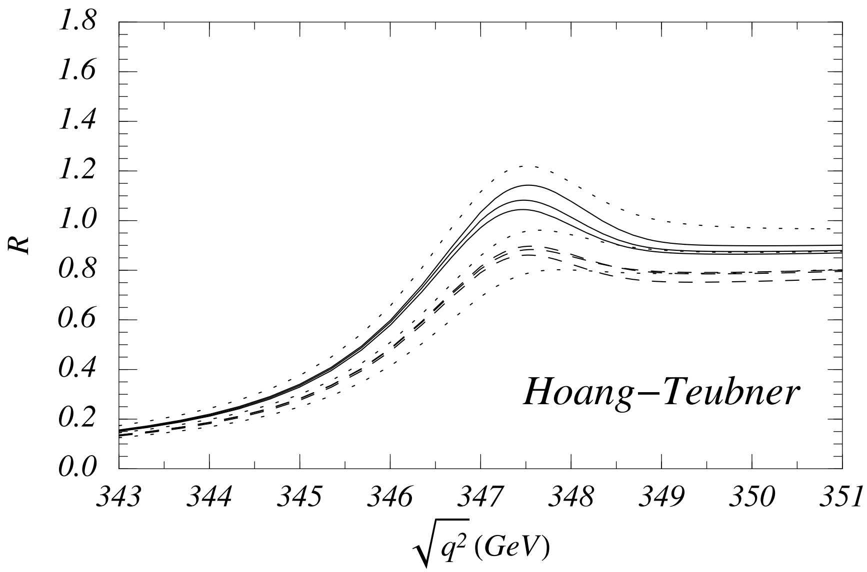


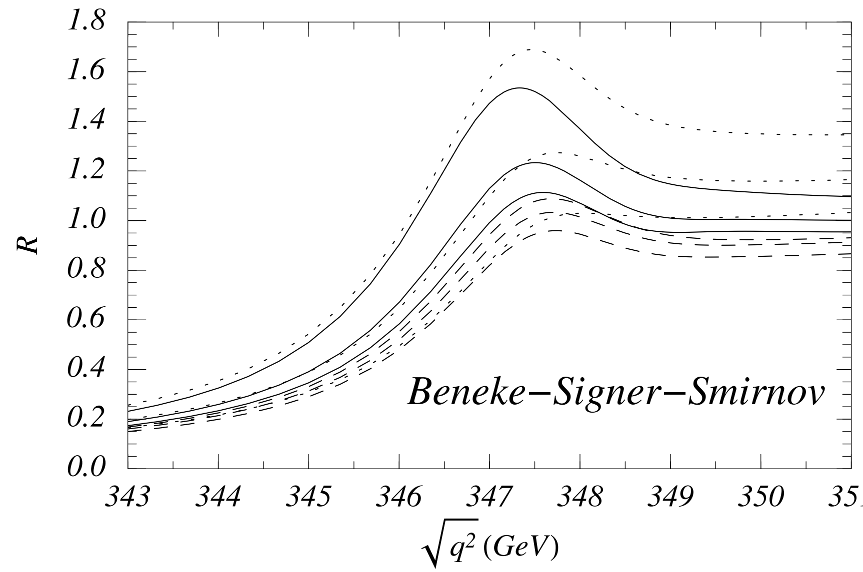


"Threshold" mass, NNLO:

- stability in position of peak
- no stability in normalization of peak

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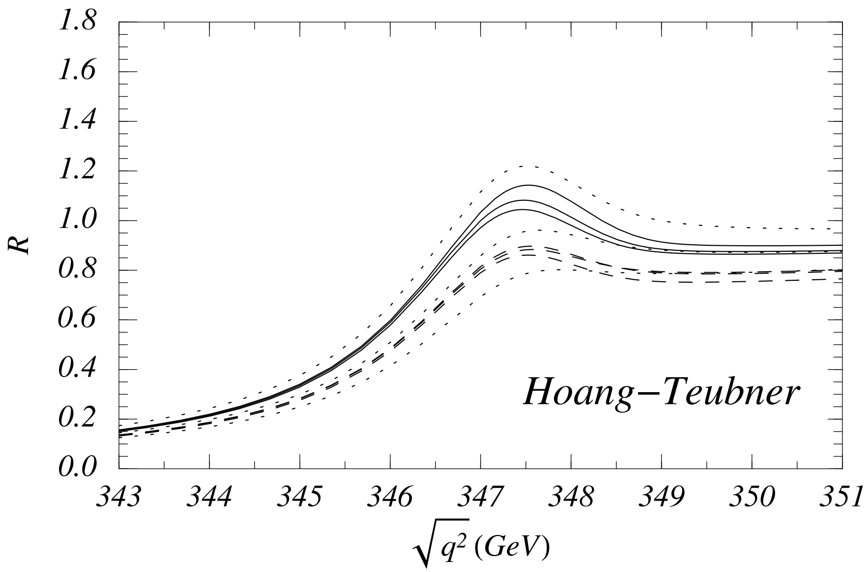


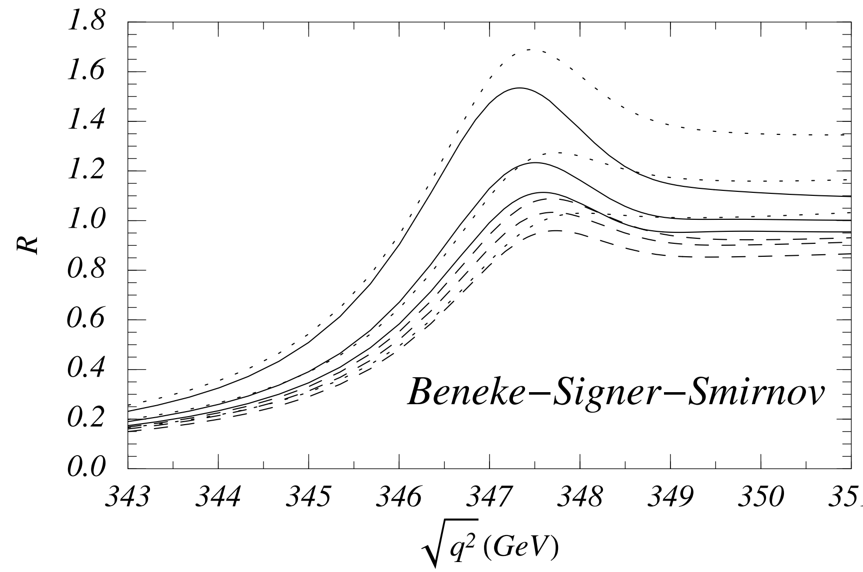


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Resummation of logarithms (to NNLL): [Hoang,Manohar,Stewart,Teubner'01,Hoang'04]

promising; however: not complete [Penin,Pineda,Smirnov,MS'04]

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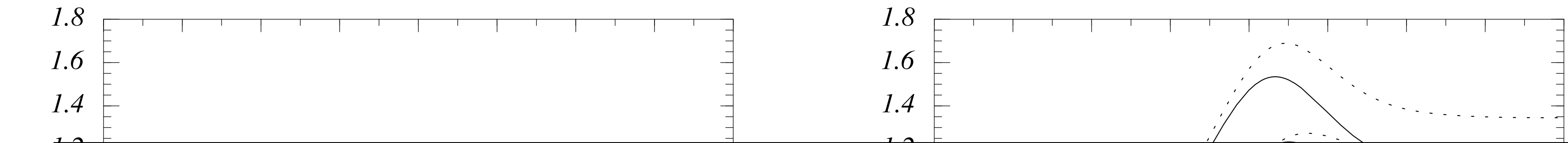
$\sigma(e^+e^- \to t\bar{t})$ @ threshold




Needed: N$^3$LO prediction for $\sigma(e^+e^- \to t\bar{t})$

i.e.: N$^3$LO corrections to $\delta E_n$, $\delta\psi_n$, $\delta G$

V.A. Smirnov, Y. Sumino, T. Teubner, O. Yakovlev,

A. Yelkhovsky'00]

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# Motivation II — bottom

Determination of $m_b$ from bottom system:

- low-moment SR:
  NNLO: $\delta m_b = 50$ MeV [Kühn,Steinhauser'01; Corcella,Hoang'03]
  N$^3$LO: still missing
- $\Upsilon(1S)$-system: $M_{\Upsilon(1S)} = 2m_b + E_1^{\text{p.t.}} + \delta^{\text{n.p.}} E_1$
  needed: $E_1^{\text{p.t.}}$ to N$^3$LO [Penin,MS'02]
  ➩ $\delta m_b = 70$ MeV
- $\Upsilon$ SR:
  NNLO: $\delta m_b = 50 - 100$ MeV
  [Penin,Pivovarov'98; Beneke,Signer'99; Melnikov,Yelkhovsky'00; Hoang'00]
  N$^3$LO: ???
  needed: $\delta E_n$ and $\delta \psi_n$ to N$^3$LO

# Framework

$$G(0,0,E) = \sum_{n=0}^{\infty} \frac{|\psi_n(0)|^2}{E-E_n}$$

$$|\psi_n(0)|^2 = |\psi_n^C(0)|^2 \left(1 + \delta\psi_n^{(1)} + \delta\psi_n^{(2)} + \delta\psi_n^{(3)} + \ldots\right)$$

$$E_n = E_n^C \left(1 + \delta E_n^{(1)} + \delta E_n^{(2)} + \delta E_n^{(3)} + \ldots\right)$$

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1. insert
2. expand for $E \to E_n^C$

single pole: $\Delta^{(n)}\psi$
double pole: $\Delta^{(n)}E$

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$$\delta G(0,0,E) \sim \langle G_C|\mathcal{H}|G_C\rangle + \langle G_C|\mathcal{H}|G_C|\mathcal{H}|G_C\rangle + \ldots$$

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$$\delta G(0,0,E) \sim \langle G_C|\mathcal{H}|G_C\rangle + \langle G_C|\mathcal{H}|G_C|\mathcal{H}|G_C\rangle + \dots$$

$$G_C(\mathbf{x},\mathbf{y},k) \sim \sum_{l=0}^{\infty}(2l+1)\sum_{m=0}^{\infty} \frac{L_m^{2l+1}(2kx)L_m^{2l+1}(2ky)m!}{(m+l+1-\alpha_s C_F m_q/(2k))(m+2l+1)!} P_l((\mathbf{xy})/xy)$$

➪ single, double and triple sums
➪ analytical solution in terms of $\zeta$ function possible

# Framework II

- potential NRQCD [Pineda,Soto'98,Brambilla,Pineda,Soto,Vairo'00]
- 2 expansion parameters: $\alpha_s$, $1/m_q$
- dynamical degrees of freedom:
  potential quarks, ultra-soft gluons
- effective Hamiltonian known to N$^3$LO [Kniehl,Penin,Smirnov,MS'02]

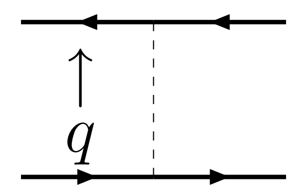
# Effective Hamiltonian

$$\mathcal{H} = (2\pi)^3\delta(\vec{q})\left(\frac{\vec{p}^{\,2}}{m_q} - \frac{\vec{p}^{\,4}}{4m_q^3}\right) + C_c(\alpha_s)V_C(|\vec{q}|) + C_{1/m}(\alpha_s)V_{1/m}(|\vec{q}|)$$
$$+\frac{\pi C_F\alpha_s(\mu)}{m_q^2}\left[C_\delta(\alpha_s) + C_p(\alpha_s)\frac{\vec{p}^{\,2}+\vec{p}^{\,\prime 2}}{2\vec{q}^{\,2}} + C_s(\alpha_s)\vec{S}^{\,2}\right]$$

Static potential: $\quad V_C(|\vec{q}|) = -\frac{4\pi C_F\alpha_s(|\vec{q}|)}{\vec{q}^2}$

$1/m_q$ potential: $\quad V_{1/m}(|\vec{q}|) = \frac{\pi^2 C_F\alpha_s^2(|\vec{q}|)}{m_q|\vec{q}|}$

"Breit" potential: $\quad \propto 1/m_q^2$




$\vec{q} = \vec{p}\,' - \vec{p}$

$$C_C(\alpha_s) = 1 + \frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}a_1 + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(|\boldsymbol{q}|)}{4\pi}\right)^3\left(a_3 + 8\pi^2 C_A^3\ln\frac{\mu^2}{\boldsymbol{q}^2}\right) + \ldots$$

$\vec{S}$: spin; $\quad L = 0$; $\quad$ no $1/m_q^3$ (tree-level) potential

# Perturbation theory for $E_n$

Energy level: $E_n = E_n^C + \delta E_n^{(1)} + \delta E_n^{(2)} + \delta E_n^{(3)} + \ldots \quad E_n^C = -\frac{C_F^2 \alpha_s^2 m_q}{4n^2}$

(a) $\delta E_n^{(1)}$ : $\langle G_C | \delta \mathcal{H}^{\rm NLO} | G_C \rangle$

(b) $\delta E_n^{(2)}$: $\langle G_C | \delta \mathcal{H}^{\rm NNLO} | G_C \rangle$, $2^{\rm nd}$ iteration of $\delta \mathcal{H}^{\rm NLO}$

(c) $\delta E_n^{(3)}$:

- $\langle G_C | \delta \mathcal{H}^{\rm N^3LO} | G_C \rangle$
- iteration of $\delta \mathcal{H}^{\rm NLO}$ and $\delta \mathcal{H}^{\rm NNLO}$; $3^{\rm rd}$ iteration of $\delta \mathcal{H}^{\rm NLO}$
- ultrasoft contribution:
  - $\langle G_C | \delta^{\rm us} \mathcal{H} | G_C \rangle$
  - retarded ultrasoft contribution

$$\delta E_1^{(3)} = \delta E_1^{(3)}\Big|_{\beta(\alpha_s)=0} + \delta E_1^{(3)}\Big|_{\beta(\alpha_s)}$$

$$\delta E_1^{(3)}\Big|_{\beta(\alpha_s)=0} = -E_1^C \frac{\alpha_s^3}{\pi} \Bigg\{ -\frac{a_1 a_2 + a_3}{32\pi^2} + \left[ -\frac{C_A C_F}{2} + \left( -\frac{19}{16} + \frac{S(S+1)}{2} \right) C_F^2 \right] a_1 + \left[ -\frac{1}{36} + \frac{\ln 2}{6} + \frac{L_{\alpha_s}}{6} \right] C_A^3$$
$$+ \left[ -\frac{49}{36} + \frac{4}{3}\left(\ln 2 + L_{\alpha_s}\right) \right] C_A^2 C_F + \left[ -\frac{5}{72} + \frac{10}{3}\ln 2 + \frac{37}{6} L_{\alpha_s} + \left( \frac{85}{54} - \frac{7}{6} L_{\alpha_s} \right) S(S+1) \right] C_A C_F^2$$
$$+ \left[ \frac{50}{9} + \frac{8}{3}\ln 2 + 3 L_{\alpha_s} - \frac{S(S+1)}{3} \right] C_F^3 + \left[ -\frac{32}{15} + 2\ln 2 + (1 - \ln 2) S(S+1) \right] C_F^2 T_F$$
$$+ \frac{49 C_A C_F T_F n_l}{36} + \left[ \frac{11}{18} - \frac{10}{27} S(S+1) \right] C_F^2 T_F n_l + \frac{2}{3} C_F^3 L_1^E \Bigg\}$$

[Kniehl,Penin,Smirnov,MS'01]

$$\delta E_1^{(3)}\Big|_{\beta(\alpha_s)} = E_n^C \left(\frac{\alpha_s}{\pi}\right)^3 \Bigg\{ 32\beta_0^3 L_\mu^3 + \left[ 40\beta_0^3 + 12 a_1 \beta_0^2 + 28\beta_1\beta_0 \right] L_\mu^2 + \Bigg[ \left( \frac{16\pi^2}{3} + 64\zeta(3) \right) \beta_0^3 + 10 a_1 \beta_0^2$$
$$+ \left( 40\beta_1 + \frac{a_1^2}{2} + a_2 + 8\pi^2 C_A C_F + \left( \frac{21\pi^2}{2} - \frac{16\pi^2}{3} S(S+1) \right) C_F^2 \right) \beta_0 + 3 a_1 \beta_1 + 4\beta_2 \Bigg] L_\mu$$
$$+ \left( -8 + 4\pi^2 + \frac{2\pi^4}{45} + 64\zeta(3) - 8\pi^2\zeta(3) + 96\zeta(5) \right) \beta_0^3 + \left( \frac{2\pi^2}{3} + 8\zeta(3) \right) a_1 \beta_0^2$$
$$+ \Bigg( \left( 8 + \frac{7\pi^2}{3} + 16\zeta(3) \right) \beta_1 - \frac{a_1^2}{8} + \frac{3}{4} a_2 + \left( 6\pi^2 - \frac{2\pi^4}{3} \right) C_A C_F$$
$$+ \left( 8\pi^2 - \frac{4\pi^4}{3} + \left( -\frac{4\pi^2}{3} + \frac{4\pi^4}{9} \right) S(S+1) \right) C_F^2 \Bigg) \beta_0 + 2 a_1 \beta_1 + 4\beta_2 \Bigg\}$$

[Penin,MS'02]

also known: $n = 2, 3$

$L_{\alpha_s} = -\ln(C_F \alpha_s)$, $L_\mu = \ln(\mu/(C_F \alpha_s m))$

[Penin,Smirnov,MS'05; Beneke,Kiyo,Schuller'05]

# Perturbation theory for $\psi_n$

## Wave function:

$$|\psi_n(0)|^2 = |\psi_n^C(0)|^2 \left(1 + \delta\psi_n^{(1)} + \delta\psi_n^{(2)} + \delta\psi_n^{(3)} + \ldots\right) + \ldots$$

$$|\psi_n^C(0)|^2 = \frac{C_F^3 \alpha_s^3 m_q^3}{8\pi n^3}$$

$$\delta\psi_n^{(3)} = \underbrace{K_2 \ln^2 \alpha_s + K_1 \ln \alpha_s + K_0\Big|_{\beta_0^3}}_{\text{known}} + \underbrace{K_0\Big|_{\text{rem}}}_{\text{unknown}}$$

$$\begin{aligned} V_C(r) = & -\frac{C_F \alpha_s}{r}\Bigg\{1 + \frac{\alpha_s}{4\pi}\left(8\beta_0 L_r + a_1\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \Bigg[64{\beta_0}^2 L_r^2 + \left(16 a_1 \beta_0 + 32\beta_1\right) L_r \\ & + a_2 + \frac{16\pi^2}{3}{\beta_0}^2\Bigg] + \left(\frac{\alpha_s}{4\pi}\right)^3 \Bigg[512{\beta_0}^3 L_r^3 + \left(192 a_1 {\beta_0}^2 + 640\beta_0\beta_1\right) L_r^2 \\ & + \left(128\pi^2{\beta_0}^3 + 24 a_2\beta_0 + 64 a_1\beta_1 + 128\beta_2 + 16\pi^2 C_A^3\right) L_r \\ & + a_3 + 16\pi^2 a_1 {\beta_0}^2 + 1024\zeta(3){\beta_0}^3 + \frac{160\pi^2}{3}\beta_0\beta_1\Bigg] + \mathcal{O}(\alpha_s^4)\Bigg\} \end{aligned}$$

$$L_r = \ln(e^{\gamma_E}\mu r)$$

# $\delta\psi_n^{(3)}\Big|_{\ln^2\alpha_s}$ and $\delta\psi_n^{(3)}\Big|_{\ln\alpha_s}$

$$
\begin{aligned}
\delta_{\ln^2\alpha_s}\psi_1^{(3)} &= \frac{\alpha_s^3}{\pi}\left\{\left(-2C_AC_F+\left(-4+\frac{4}{3}S(S+1)\right)C_F^2\right)\beta_0-\frac{2}{3}C_A^2C_F\right.\\
&\left.+\left(-\frac{41}{12}+\frac{7}{12}S(S+1)\right)C_AC_F^2-\frac{3}{2}C_F^3\right\}\ln^2\left(C_F\alpha_s\right)
\end{aligned}
$$

[Kniehl,Penin'00; Manohar,Stewart'01]

$$
\begin{aligned}
\delta_{\ln\alpha_s}\psi_1^{(3)} &= \frac{\alpha_s^3}{\pi}\left\{\left[\left(-3+\frac{2\pi^2}{3}\right)C_AC_F+\left[\frac{4\pi^2}{3}-\left(\frac{10}{9}+\frac{4\pi^2}{9}\right)S(S+1)\right]C_F^2\right]\beta_0\right.\\
&+\left[-\frac{3}{4}C_AC_F+\left(-\frac{9}{4}+\frac{2}{3}S(S+1)\right)C_F^2\right]a_1+\frac{1}{4}C_A^3+\left(\frac{59}{36}-4\ln 2\right)C_A^2C_F\\
&+\left[\frac{143}{36}-4\ln 2-\frac{19}{108}S(S+1)\right]C_AC_F^2+\left[-\frac{35}{18}+8\ln 2\right.\\
&\left.-\frac{1}{3}S(S+1)\right]C_F^3+\left[-\frac{32}{15}+2\ln 2+(1-\ln 2)\,S(S+1)\right]C_F^2T_F\\
&\left.+\frac{49}{36}C_AC_FT_Fn_l+\left[\frac{8}{9}-\frac{10}{27}S(S+1)\right]C_F^2T_Fn_l\right\}\ln\left(C_F\alpha_s\right)
\end{aligned}
$$

[Kniehl,Penin,Smirnov,MS'02;Hoang'04]

# new: $\delta\psi_n^{(3)}\Big|_{\beta_0^3}$

$$\delta\psi_1^{(3)}\Big|_{\beta_0^3} = \left(\frac{\beta_0\alpha_s}{\pi}\right)^3\Bigg[80L_1^3+\left(52-\frac{80\pi^2}{3}\right)L_1^2+\left(-40-6\pi^2+\frac{10\pi^4}{9}+200\zeta(3)\right)L_1$$
$$-20+\frac{22\pi^2}{3}-\frac{7\pi^4}{5}+\frac{4\pi^6}{105}+112\zeta(3)-12\pi^2\zeta(3)-16\zeta(3)^2-40\zeta(5)\Bigg]$$

$$\delta\psi_2^{(3)}\Big|_{\beta_0^3} = \left(\frac{\beta_0\alpha_s}{\pi}\right)^3\Bigg[80L_2^3+\left(332-\frac{160\pi^2}{3}\right)L_2^2+\left(308-\frac{266\pi^2}{3}+\frac{40\pi^4}{9}+400\zeta(3)\right)L_2$$
$$-361+\frac{73\pi^2}{3}-\frac{26\pi^4}{45}+\frac{32\pi^6}{105}+496\zeta(3)-48\pi^2\zeta(3)-128\zeta(3)^2-160\zeta(5)\Bigg]$$

$$\delta\psi_3^{(3)}\Big|_{\beta_0^3} = \left(\frac{\beta_0\alpha_s}{\pi}\right)^3\Bigg[80L_3^3+\left(612-80\pi^2\right)L_3^2+\left(\frac{2893}{3}-228\pi^2+10\pi^4+600\zeta(3)\right)L_3$$
$$-\frac{100679}{54}+\frac{183\pi^2}{2}+\frac{52\pi^4}{15}+\frac{36\pi^6}{35}+1374\zeta(3)-108\pi^2\zeta(3)-432\zeta(3)^2$$
$$-360\zeta(5)\Bigg]$$

[Penin,Smirnov,MS'05; Beneke,Kiyo,Schuller'05]

$$L_n = \ln\frac{n\mu}{C_F\alpha_s(\mu)m_q}$$

# Excited states of bottomonium

$$M_{\Upsilon(nS)} = 2m_b + E_n^{\rm p.t.} + \delta^{\rm n.p.} E_n$$

$\rho_n = \frac{E_n - E_1}{2m_b + E_1}$ ( does not suffer from renormalon ambiguities)

$$10^2 \times \rho_2^{\rm p.t.} = 1.49\,(1 + 0.79_{\rm NLO} + 1.18_{\rm NNLO} + 1.21_{\rm N^3LO} + \ldots) = 6.2^{+1.7}_{-1.2}$$

$$10^2 \times \rho_2^{\rm exp} = 5.95$$

$$10^2 \times \rho_3^{\rm p.t.} = 1.77\,(1 + 0.92_{\rm NLO} + 1.37_{\rm NNLO} + 1.55_{\rm N^3LO} + \ldots) = 8.6^{+2.4}_{-1.8}$$

$$10^2 \times \rho_3^{\rm exp} = 9.46$$

➪ convergence not good

➪ $\rm N^3LO$ terms needed to match experimental result; impressive agreement

➪ non-perturbative contribution small !?

# $\Upsilon$ SR

Compare exp. and th. moments: $\mathcal{M}_k^{\mathrm{th}} \stackrel{!}{=} \mathcal{M}_k^{\mathrm{exp}}$

Experimental input: masses and leptonic width of
$\Upsilon$ resonances

Theory input: (corrections to) energy and wave function;
non-perturbative effects under control

Estimate N$^3$LO corrections:

$k \geq 20$: corrections to ground state energy dominate
$\delta\bar{m}_b(\bar{m}_b)^{\mathrm{N^3LO}} > 0$ small

$k = 4$: corrections to wave function become important
$\delta\bar{m}_b(\bar{m}_b)^{\mathrm{N^3LO}} \approx -100$ MeV

➪ Wait for complete N$^3$LO result

# Peak for $\sigma(e^+e^- \to t\bar{t})$ at N$^3$LO

Normalization of peak

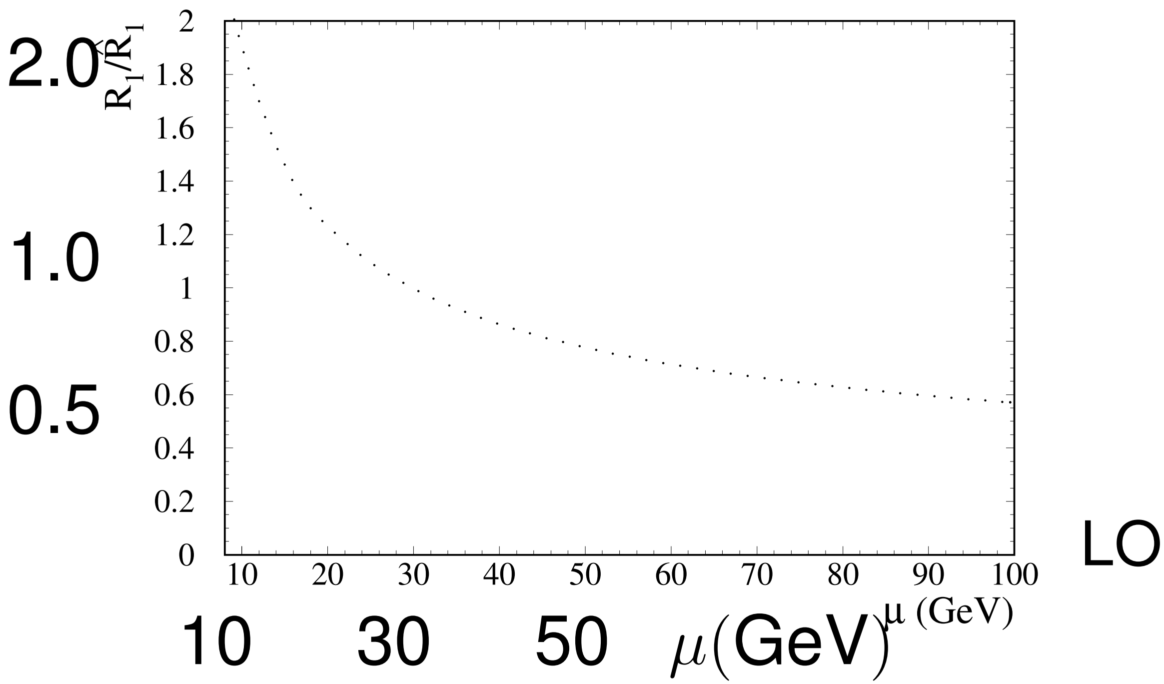
$$R_{\rm res}(e^+e^- \to t\bar{t}) = \frac{6\pi N_c Q_t^2}{m_t^2 \Gamma_t}\; c_v^2 |\psi_1(0)|^2$$

$c_v$: hard matching coefficient [Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97]

$|\psi_1(0)|^2$: $\ln^{1,2}\alpha_s$, $\beta_0^3$ terms known at N$^3$LO [Kniehl,Penin,Smirnov,MS'02;Hoang'04]

[Penin,Smirnov,MS'05]

$R_{\rm res}(\mu)/R_{\rm res}^{\rm LO}(\mu_s)$



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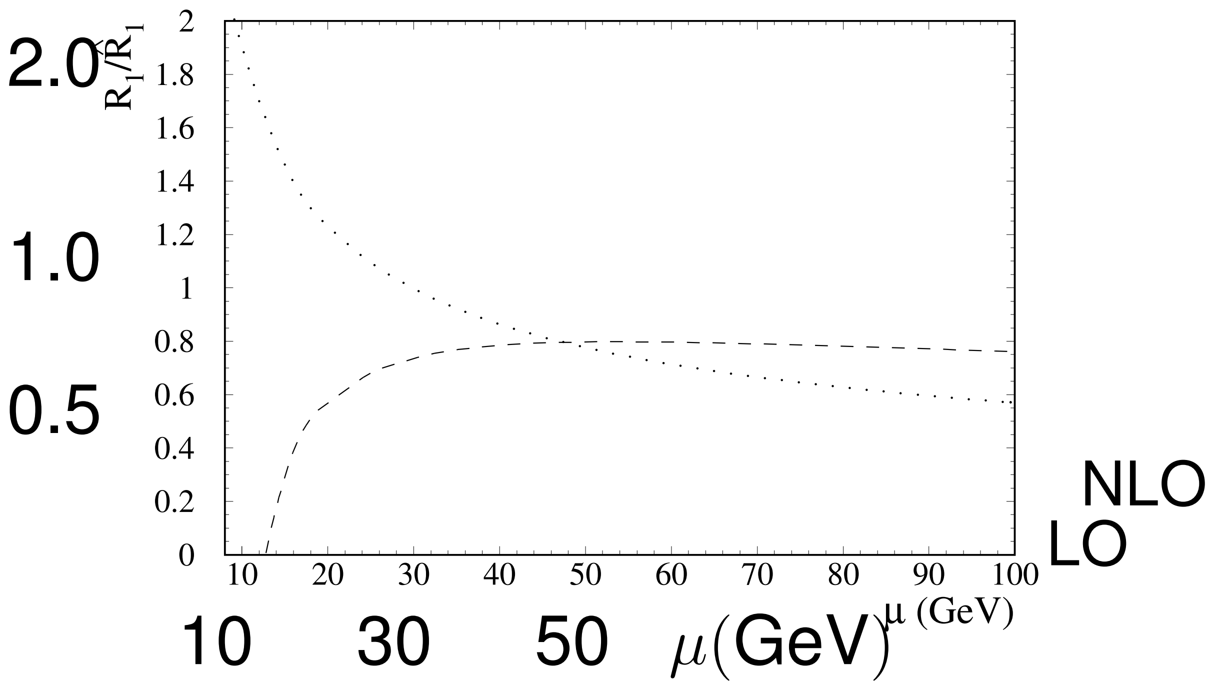

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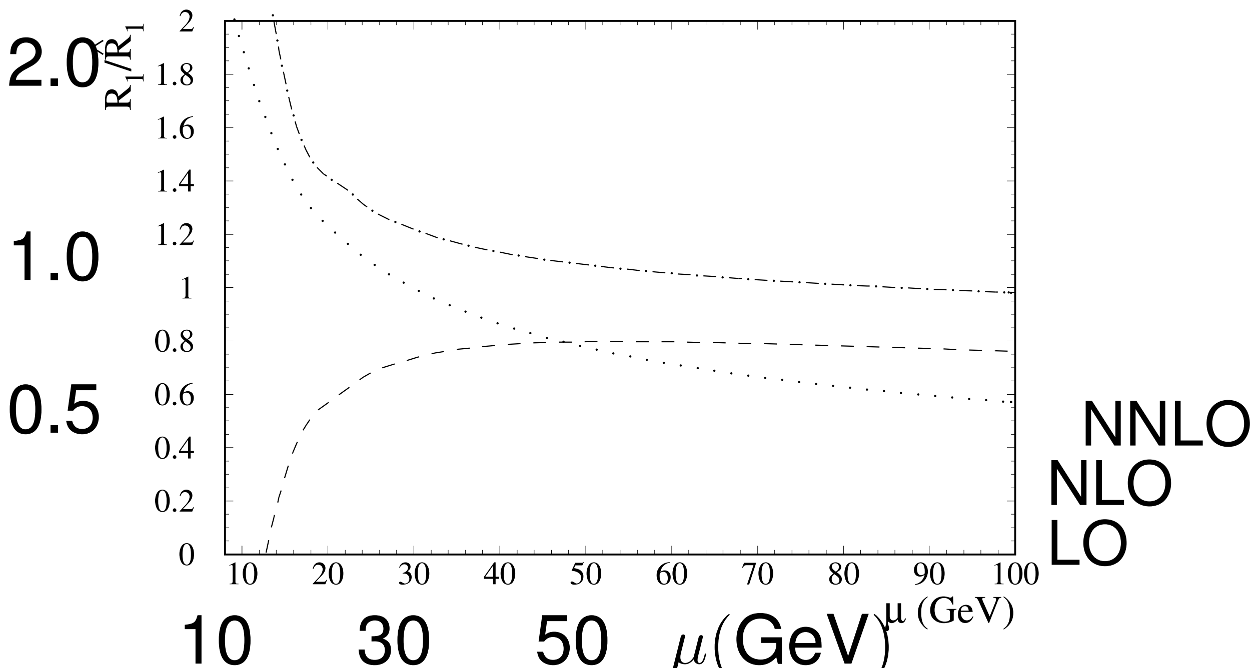


# Peak for $\sigma(e^+e^- \to t\bar{t})$ at N$^3$LO

Normalization of peak

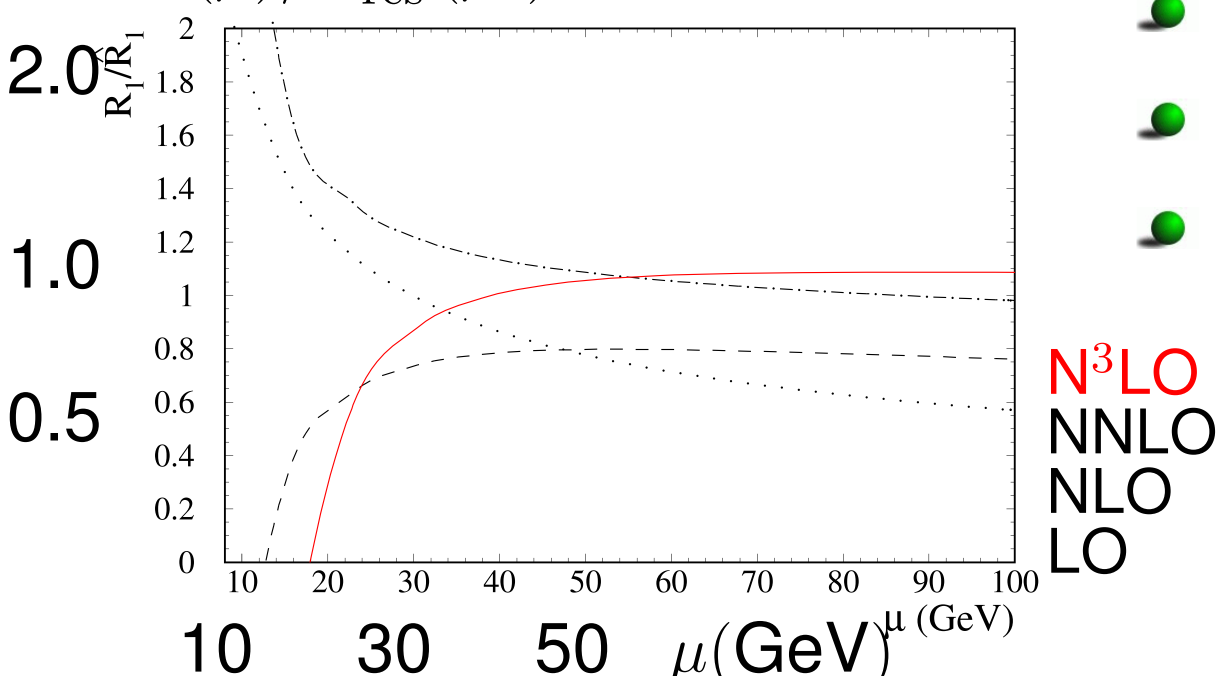
$$R_{\rm res}(e^+e^- \to t\bar{t}) = \frac{6\pi N_c Q_t^2}{m_t^2 \Gamma_t} \; c_v^2 |\psi_1(0)|^2$$

$c_v$: hard matching coefficient [Czarnecki,Melnikov'97; Beneke,Signer,Smirnov'97]

$|\psi_1(0)|^2$: $\ln^{1,2}\alpha_s$, $\beta_0^3$ terms known at N$^3$LO [Kniehl,Penin,Smirnov,MS'02;Hoang'04]

[Penin,Smirnov,MS'05]

$R_{\rm res}(\mu)/R_{\rm res}^{\rm LO}(\mu_s)$

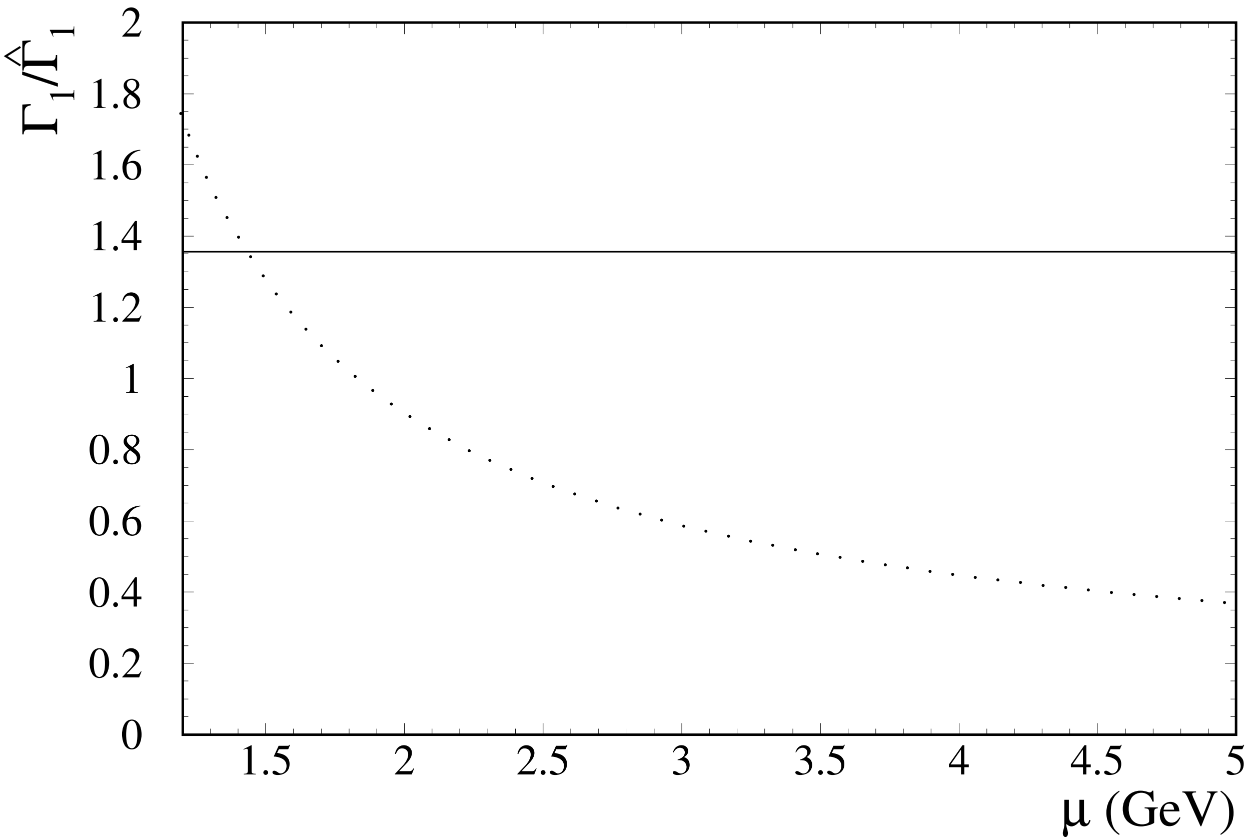



- nice $\mu$ dependence
- first sign of convergence
- complete N$^3$LO-constant needed!

# $\Gamma(\Upsilon(1S) \to e^+e^-)$

$\Gamma^{\text{exp}} = 1.31$ **keV**

$\Gamma(\mu)/\Gamma^{\text{LO}}(\mu_s)$




experiment

LO

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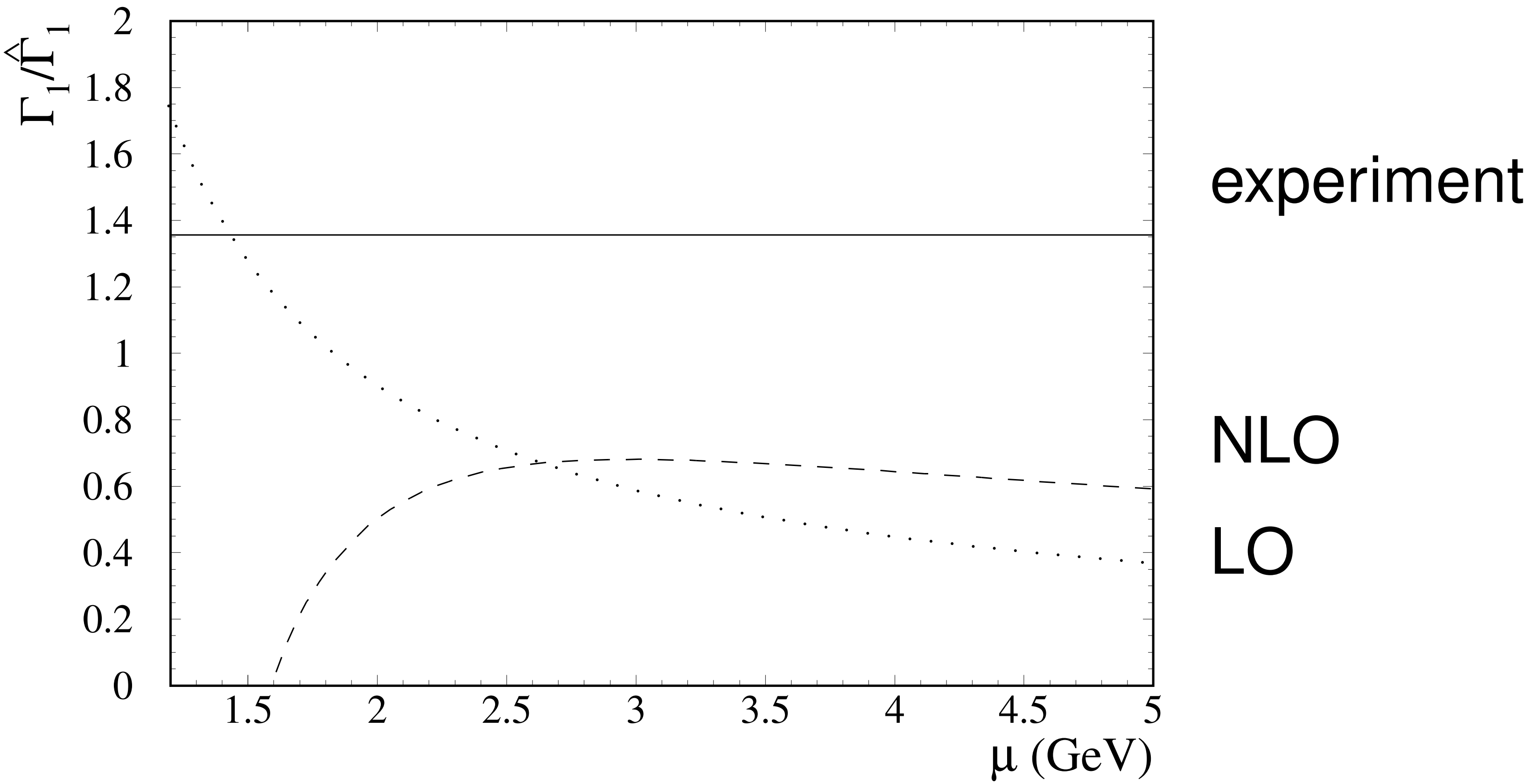

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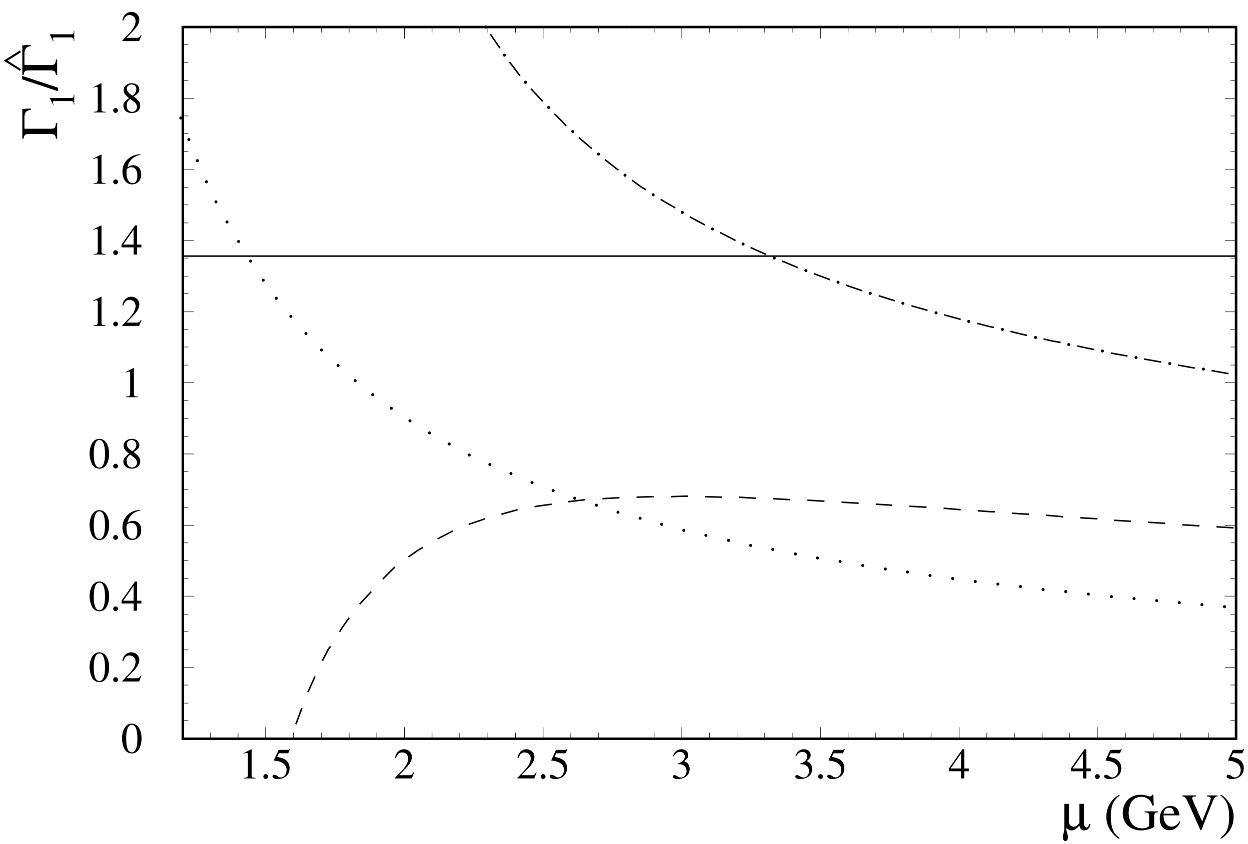



experiment

NNLO

NLO

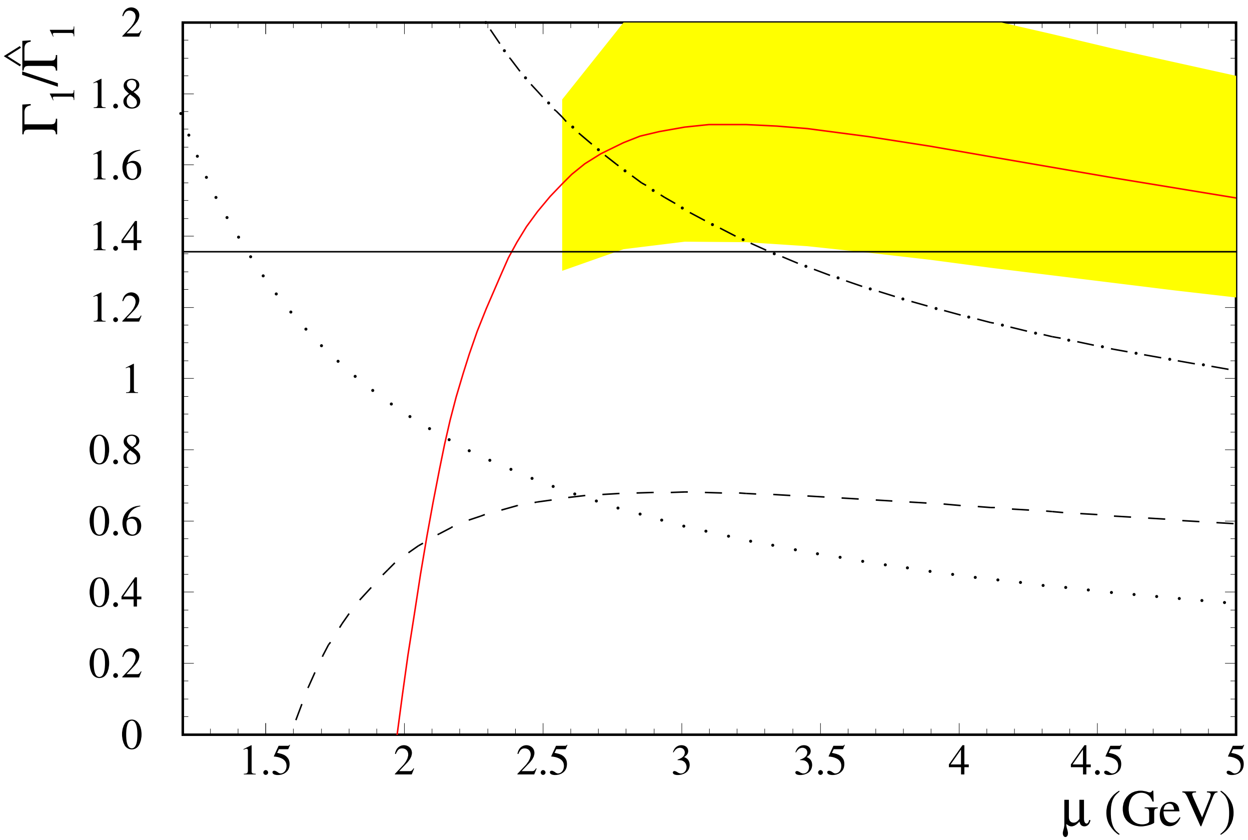
LO

# $\Gamma(\Upsilon(1S) \to e^+e^-)$

$\Gamma^{\text{exp}} = 1.31$ keV

$\boxed{\alpha_s(M_Z) = 0.118 \pm 0.003}$

$\Gamma(\mu)/\Gamma^{\text{LO}}(\mu_s)$




N$^3$LO
experiment
NNLO
NLO
LO

# Summary

- N$^3$LO corrections to energy levels, $\delta E_n^{(3)}$: complete
- N$^3$LO corrections to wave function, $\delta \psi_n^{(3)}$: $\ln^2$, $\ln$, $\beta_0^3$
- stabilization of pertubative series observed
  - excited states of bottomonium
  - $\Upsilon(1S)$ leptonic width
  - $\Upsilon$ sum rules
  - $t\bar{t}$ threshold production
- Third-order corrections to $\delta\psi$ to be completed