The Effects of Inhomogeneities on the Universe Today

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Plan of the talk

- **Short introduction to Inflation**
- **Short introduction to cosmological perturbations**
- **The influence of perturbations**
- **Dark Energy ?**
- **Conclusions**

Rocky Kolb (Fermilab/Chicago)
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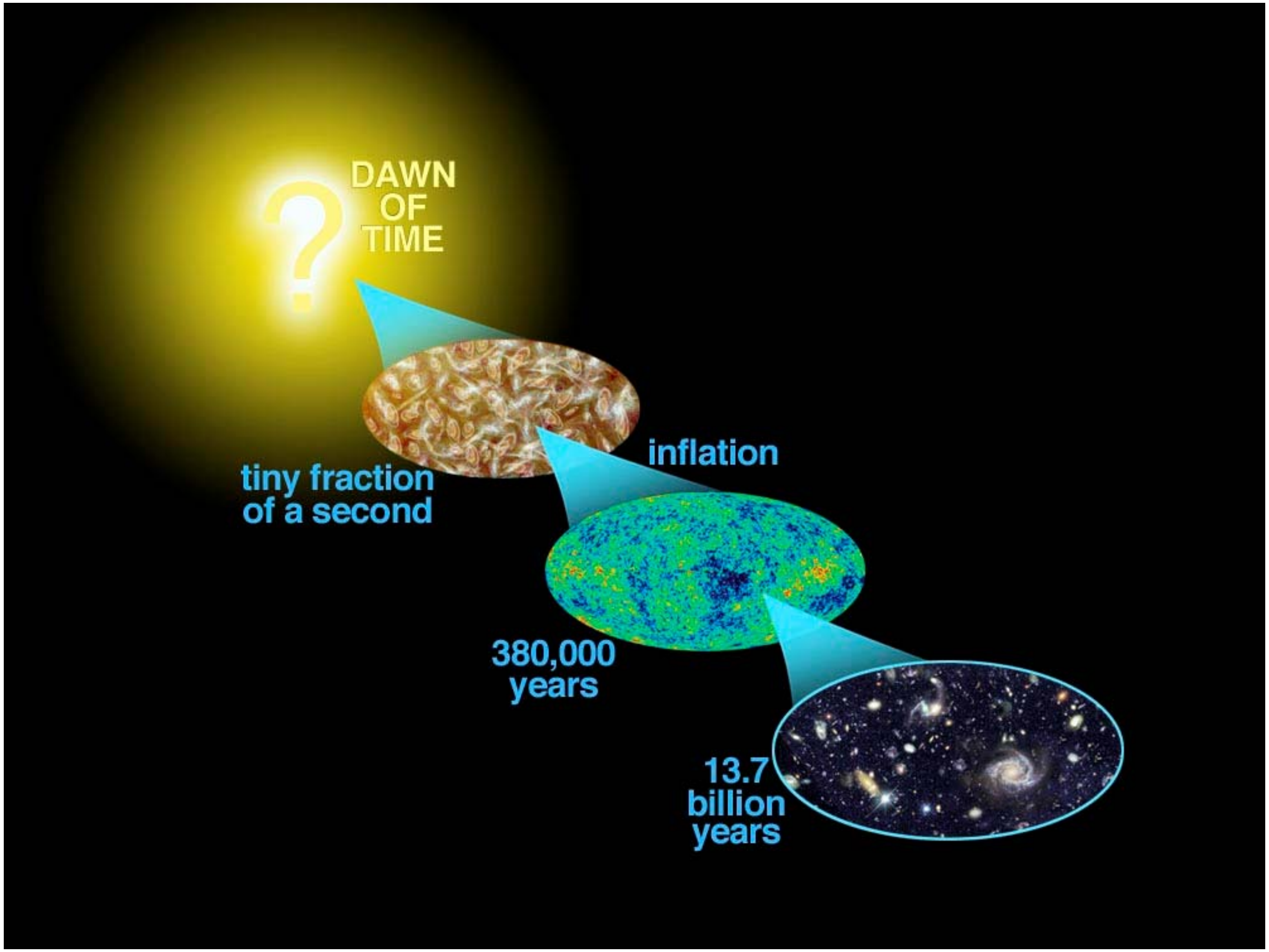
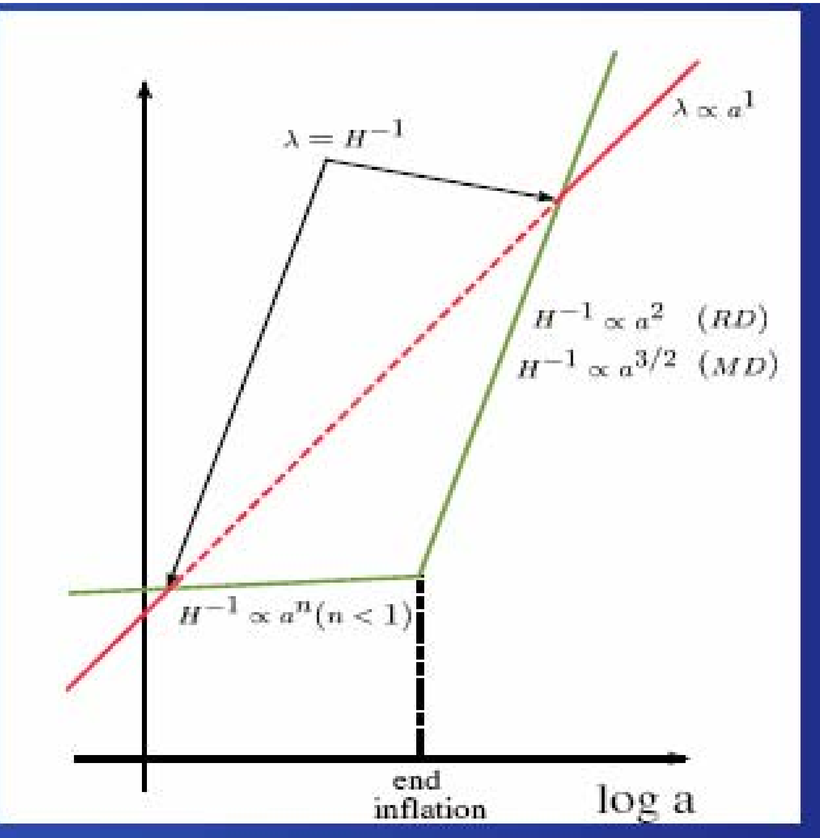
Enrico Barausse (Sissa,Trieste)
hep-ph/0409038, astroph/0410541, in preparation

From Inflation to the observed Universe



Inflation



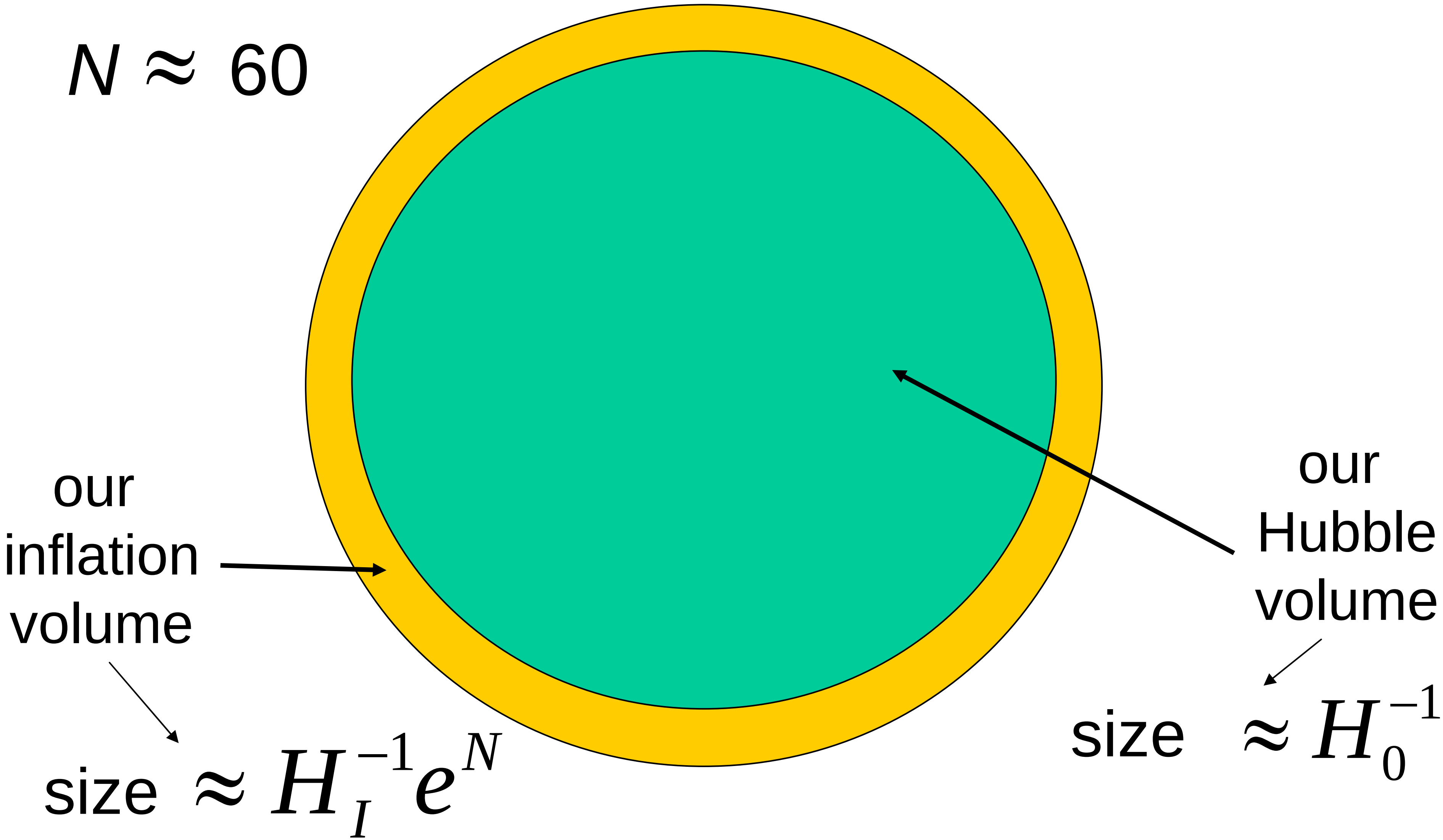
Require

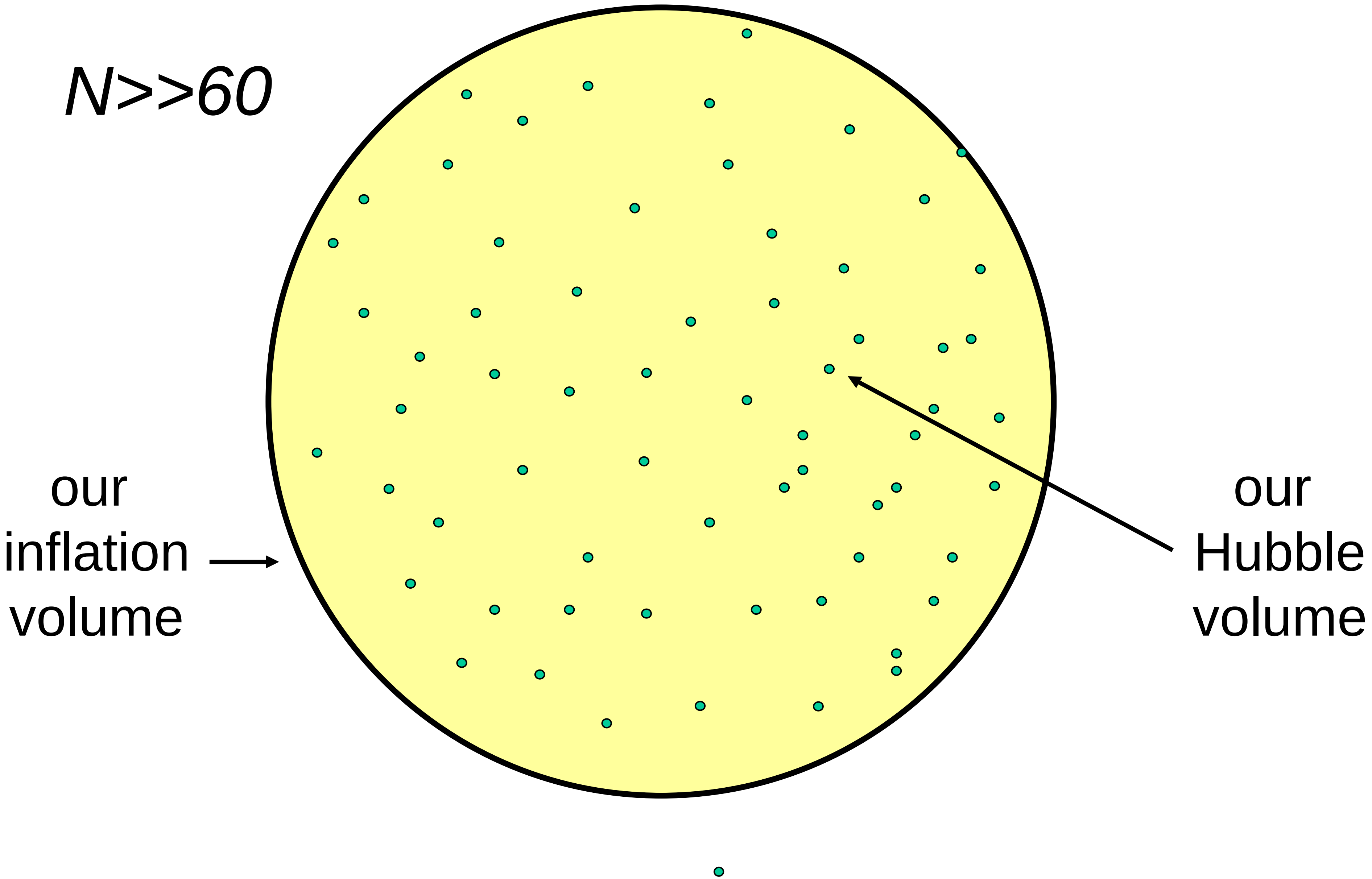
$$\frac{d}{dt}\left(\frac{\lambda}{R_H}\right) > 0 \Rightarrow \ddot{a} > 0$$

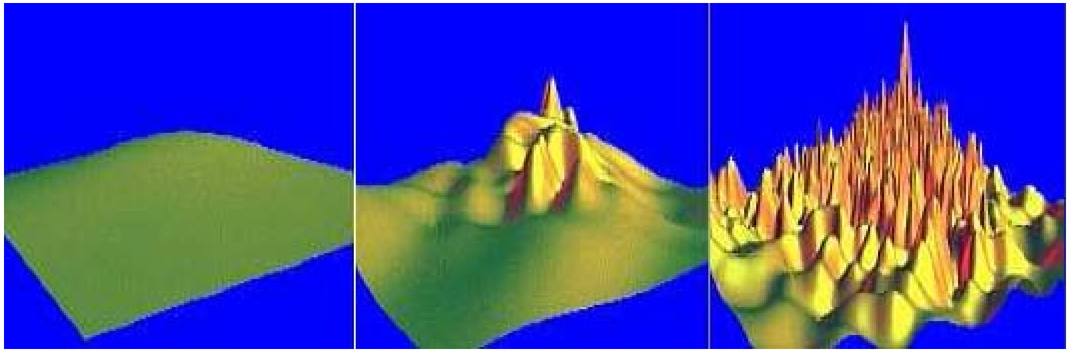
Inflation

- Inflation is attained when $\frac{\ddot{a}}{a} = H^2 \left(\frac{\dot{H}}{H^2} + 1 \right) > 0$
- If during inflation the Universe suffers a quasi-de Sitter phase $\dot{H} \approx 0$ and $H^2 \approx$ constant
- The scale factor grows exponentially, $a(t) = a\left(t_\star\right) \, e^{\int_{t_\star}^{t} \, H(t') \, dt'} \approx a\left(t_\star\right) \, e^N$
- $N = \int_{t_{\rm BI}}^{t_{\rm EI}} \, H(t') \, dt' \approx H \left(t_{\rm EI} - t_{\rm BI}\right) =$ number of e-foldings

$N \approx 60$

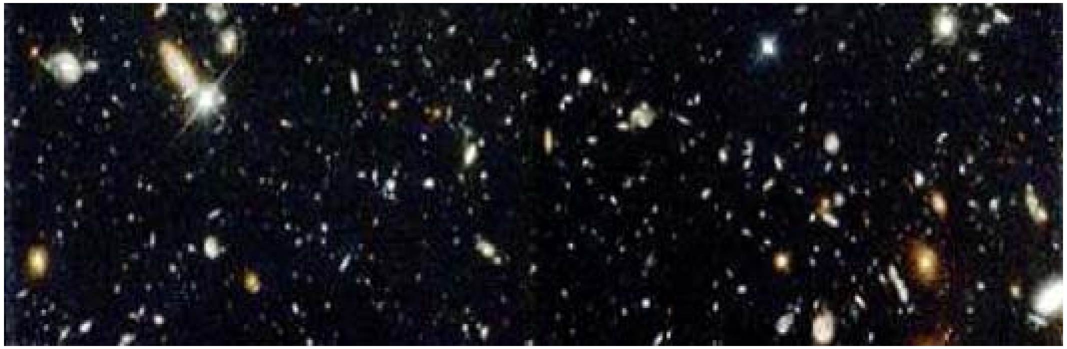

our
inflation
volume
size $\approx H_I^{-1} e^N$
our
Hubble
volume
size $\approx H_0^{-1}$

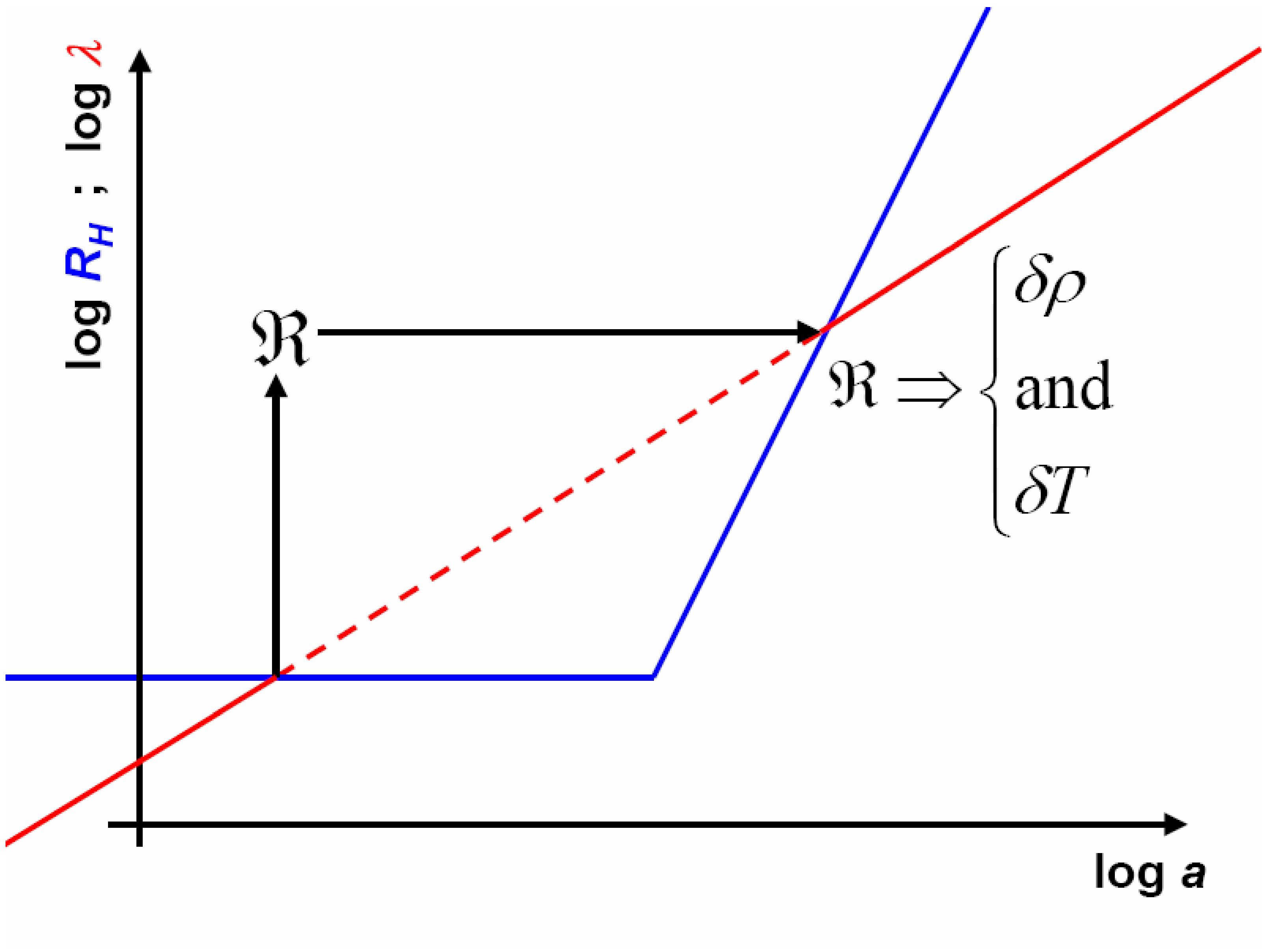

$N{>>}60$
our inflation volume
our Hubble volume



From quantum fluctuations to

the Large Scale Structure





log R_H ; log λ
$\Re$
$\Re \Rightarrow \begin{cases} \delta\rho \\ \text{and} \\ \delta T \end{cases}$
log a

Comoving curvature perturbation

If $\delta g_{00} = 2\psi$ (the gravitational potential)

$t \to t + \delta t$ implies $\psi \to \psi + H\,\delta t$ and $\delta\phi \to \delta\phi - \dot{\phi}\delta t$

$$\boxed{\mathcal{R} = \psi + H\frac{\delta\phi}{\dot{\phi}}}$$

Related to the gauge-dependent curvature perturbation ψ on a generic slicing and to inflaton perturbation $\delta\phi$ in that gauge

Power spectrum on superhorizon scales

On super-horizon scales

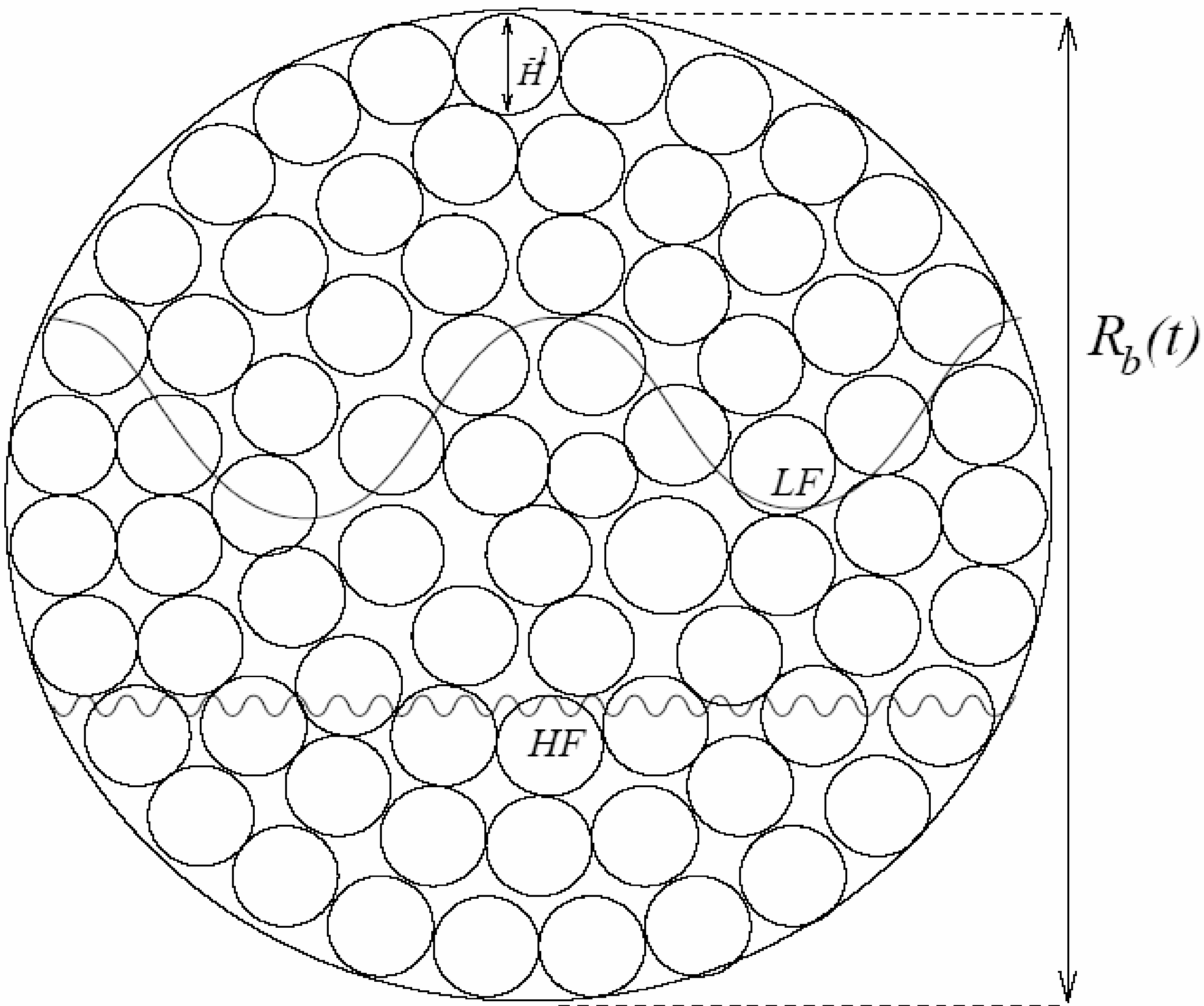
$$\mathcal{P}_{\mathcal{R}}(k) = \frac{1}{2M_p^2\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{n_{\mathcal{R}}-1}$$

where

$$n_{\mathcal{R}} - 1 = \frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k} = 2\eta - 6\epsilon$$

is the spectral index of curvature perturbations

In our local Universe both short- and long-wavelength modes are present



Evolution of $H(z)$ is a key quantity

- $H(z)$ is the most fundamental dynamical cosmological variable [e.g., all evidence for dark energy comes from evolution of $H(z)$].
- $H(z)$ in the FLRW (homogeneous/isotropic) model:

$$\boxed{\begin{cases} G_{00} = \kappa^2 T_{00} \qquad \left(\kappa^2 = 8\pi G_N\right) \\ 3\left(\dot{a}/a\right)^2 \equiv 3H^2 = \kappa^2 \rho \end{cases}}$$

- Usual assumption for perturbed FLRW model:
  An inhomogeneous Universe follows the evolution of a FLRW model of the same average density.
- Issues:
  – gravity is nonlinear
  – gravity is a long-range force

Evolution of $H(z)$ is a key quantity

- $H(z)$ is the most fundamental dynamical cosmological variable [e.g., all evidence for dark energy comes from evolution of $H(z)$].
- $H(z)$ in the FLRW (homogeneous/isotropic) model:

(FLRW)

$$\begin{cases} G_{00} = \kappa^2 T_{00} \qquad \left(\kappa^2 = 8\pi G_N\right) \\ 3\left(\dot{a}/a\right)^2 \equiv 3H^2 = \kappa^2 \rho \end{cases}$$

(perturbed FLRW)

$$\begin{cases} G_{\mu\nu}\left(\vec{x},t\right) = G_{\mu\nu}^{\mathrm{FLRW}}\left(t\right) + \delta G_{\mu\nu}\left(\vec{x},t\right) \\ G_{00}^{\mathrm{FLRW}}\left(t\right) + \delta G_{00}\left(\vec{x},t\right) = \kappa^2 T_{00}\left(\vec{x},t\right) \\ 3\left(\dot{a}/a\right)^2 = \kappa^2\left(\langle\rho\rangle - \kappa^{-2}\langle\delta G_{00}\rangle\right) \end{cases}$$

- Can think of $\kappa^{-2}\langle\delta G_{00}\rangle$ as kinetic energy of the gravitational field.
- $(\dot{a}/a)^2$ <u>*is not*</u> $\kappa^2\langle\rho\rangle/3 \equiv H^2$.
- Usual soundbite: $\delta\rho/\rho$ small $\Rightarrow \langle\delta G_{00}\rangle$ small $\Rightarrow \delta H$ small.

Second-order perturbation theory

- Expand energy-momentum tensor and metric tensor to 2nd-order:

$$T_{\mu\nu} = T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)}$$

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} + g_{\mu\nu}^{(2)}$$

$T_{\mu\nu}^{(0)}$ & $g_{\mu\nu}^{(0)}$ are homogeneous & isotropic

- Einstein tensor to 2nd-order:

$$G_{\mu\nu} = G_{\mu\nu}^{(0)} + G_{\mu\nu}^{(1)} + G_{\mu\nu}^{(11)} + G_{\mu\nu}^{(2)}$$

- In synchronous gauge:

$$\delta g_{ij} = a^2(\tau)\left[-2\sum_{r=1}^{\infty}\frac{1}{r!}\psi^{(r)}\delta_{ij} + \sum_{r=1}^{\infty}\frac{1}{r!}\left(\partial_i\partial_j\chi^{(r)} + \partial_i\chi_j^{(r)} + \partial_j\chi_i^{(r)} + \chi_{ij}^{(r)}\right)\right]$$

$$(\partial^i\chi_i^{(r)} = 0; \quad \chi^i{}_i^{(r)} = 0, \quad \partial^i\chi_{ij}^{(r)} = 0)$$

- Metric perturbations in terms of peculiar gravitational potential φ:

$$\nabla^2\varphi(\vec{x}) = \frac{\kappa^2}{2}a^2\rho^{(0)}\left(\delta\rho^{(1)}(\vec{x})/\rho^{(0)}\right)$$

Metric perturbations in a

- φ is the peculiar gravitational potential
- related to $\delta\rho/\rho$ by Poisson equation $\nabla^2\varphi(\vec{x}) = \dfrac{\kappa^2}{2} a^2 \rho^{(0)} \dfrac{\delta\rho^{(1)}(\vec{x})}{\rho^{(0)}}$

□ $\psi^{(1)}(\vec{x},\tau) = \dfrac{5}{3}\varphi(\vec{x}) + \dfrac{\tau^2}{18}\nabla^2\varphi(\vec{x})$

□ $$\psi^{(2)}(\vec{x},\tau) = -\frac{50}{9}\varphi^2(\vec{x}) + \frac{5\tau^2}{18}\left[\varphi^{,k}(\vec{x})\varphi_{,k}(\vec{x}) + \frac{4}{3}\varphi(\vec{x})\nabla^2\varphi(\vec{x})\right] + \frac{\tau^4}{252}\left[\left(\nabla^2\varphi(\vec{x})\right)^2 - \frac{10}{3}\varphi^{,ik}(\vec{x})\varphi_{,ik}(\vec{x})\right]$$

□ …

Matter-dominated Universe

Einstein Equations – 2nd order

$$G_{\mu\nu}^{(0)} + G_{\mu\nu}^{(1)} + G_{\mu\nu}^{(11)} + G_{\mu\nu}^{(2)} = \kappa^2 \left(T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} \right)$$

$$\langle 00 \rangle : \quad G_{00}^{(0)} = \kappa^2 \left\langle T_{00} \right\rangle - \left\langle G_{00}^{(1)} + G_{00}^{(11)} + G_{00}^{(2)} \right\rangle$$

$$3\left(\frac{a'}{a^2} \right)^2 = \kappa^2 \left\langle \rho \right\rangle - \left\langle G_{00}^{(1)} + G_{00}^{(11)} + G_{00}^{(2)} \right\rangle$$

- In a perturbed universe $(a'/a^2)^2 = (\dot{a}/a)^2$ <u>*is not*</u> $\kappa^2\langle\rho\rangle/3$.
- a'/a^2 does not even describe the physical Hubble flow– description of Hubble flow comes from evolution of $\langle\rho\rangle$.
- Augment 00 equation with continuity equation:

$$D_\mu T^{\mu 0} = 0 \Rightarrow \dot{\rho} = \rho'/a = -\theta\rho$$

$\theta \equiv D_\mu u^\mu \quad u^\mu$ is the fluid 4-velocity

- Hubble flow described by θ. For FLRW, $\theta = 3H$.

The Hubble flow – 2nd order

- Hubble flow described by θ:

$$\langle\theta\rangle = \theta^{(0)} + \left\langle\theta^{(1)}\right\rangle + \left\langle\theta^{(11)}\right\rangle + \left\langle\theta^{(2)}\right\rangle = 3\frac{a'}{a^2} + \left\langle\theta^{(1)}\right\rangle + \left\langle\theta^{(11)}\right\rangle + \left\langle\theta^{(2)}\right\rangle$$

- Einstein equations relate a'/a^2 in terms of ρ & $G_{00}^{(1)},\ G_{00}^{(11)},\ G_{00}^{(2)}$

- Define a $\delta\theta$:

$$\frac{\langle\delta\theta\rangle}{3H} = \frac{\left\langle\theta^{(1)}\right\rangle + \left\langle\theta^{(11)}\right\rangle + \left\langle\theta^{(2)}\right\rangle}{3H} - \frac{\left\langle G_{00}^{(1)} + G_{00}^{(11)} + G_{00}^{(2)}\right\rangle}{6a^2H^2} - \frac{\left\langle G_{00}^{(1)}\right\rangle^2}{72a^4H^4}$$

- Notational point: could define $H = (\kappa^2\langle\rho\rangle/3)^{1/2}$ or $\cancel{H = (a'/a^2)^{1/2}}$

$$\boxed{H \equiv (\kappa^2\langle\rho\rangle/3)^{1/2}}$$

It is not a backreaction

$$\boxed{\frac{\langle \delta\theta \rangle}{3H} = \frac{\langle \theta^{(1)} \rangle + \langle \theta^{(11)} \rangle + \langle \theta^{(2)} \rangle}{3H} - \frac{\langle G_{00}^{(1)} + G_{00}^{(11)} + G_{00}^{(2)} \rangle}{6a^2 H^2} - \frac{\langle G_{00}^{(1)} \rangle^2}{72 a^4 H^4}}$$

- New terms present due to inhomogeneities
- New terms modify expansion
- But expansion does not *produce* inhomogeneities
  (although expansion does modify the growth of inhomogeneities)

$\langle\cdots\rangle$: Averaging

volume element: $\int d^3x\sqrt{\gamma(\tau,\vec{x})}$

$$\langle\mathcal{O}\rangle(\tau)\equiv\frac{\int d^3x\sqrt{\gamma(\tau,\vec{x})}\;\mathcal{O}(\tau,\vec{x})}{\int d^3x\sqrt{\gamma(\tau,\vec{x})}}$$

$$\sqrt{\gamma}=1+\frac{1}{2}\gamma^{(1)}(\tau,\vec{x})+\cdots=1-3\psi^{(1)}(\tau,\vec{x})+\cdots=1-5\varphi(\vec{x})-\frac{\tau^2}{6}\nabla^2\varphi(\vec{x})+\cdots$$

$$\langle\mathcal{O}^{(0)}\rangle=\mathcal{O}^{(0)}$$

$$\langle\mathcal{O}^{(1)}\rangle=\langle\mathcal{O}^{(1)}\rangle_1-3\langle\psi^{(1)}\mathcal{O}^{(1)}\rangle+3\langle\psi^{(1)}\rangle_1\langle\mathcal{O}^{(1)}\rangle_1 \qquad \langle\mathcal{O}^{(1)}\rangle_1\equiv\frac{\int d^3x\;\mathcal{O}^{(1)}(\tau,\vec{x})}{\int d^3x}$$

$$\langle\mathcal{O}^{(2)}\rangle(\tau)=\frac{\int d^3x\;\mathcal{O}^{(2)}(\tau,\vec{x})}{\int d^3x}$$

The Hubble flow – 2nd order

Averages involve integrals of the form

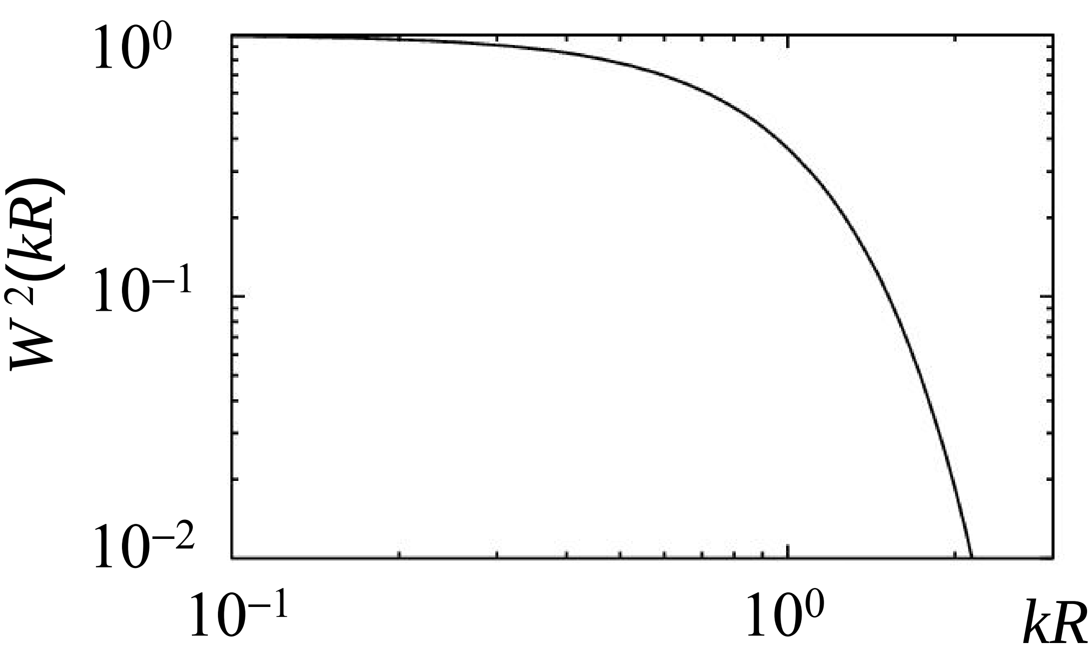
$$\langle \cdots \rangle \equiv V^{-1}(R) \int_{V(R)} (\cdots)\, dV$$

Assume Gaussian window function

$$dV = 4\pi r^2 e^{-r^2/2R^2} dr \Rightarrow V(R) = (2\pi)^{3/2} R^3$$

Fourier transform of window function

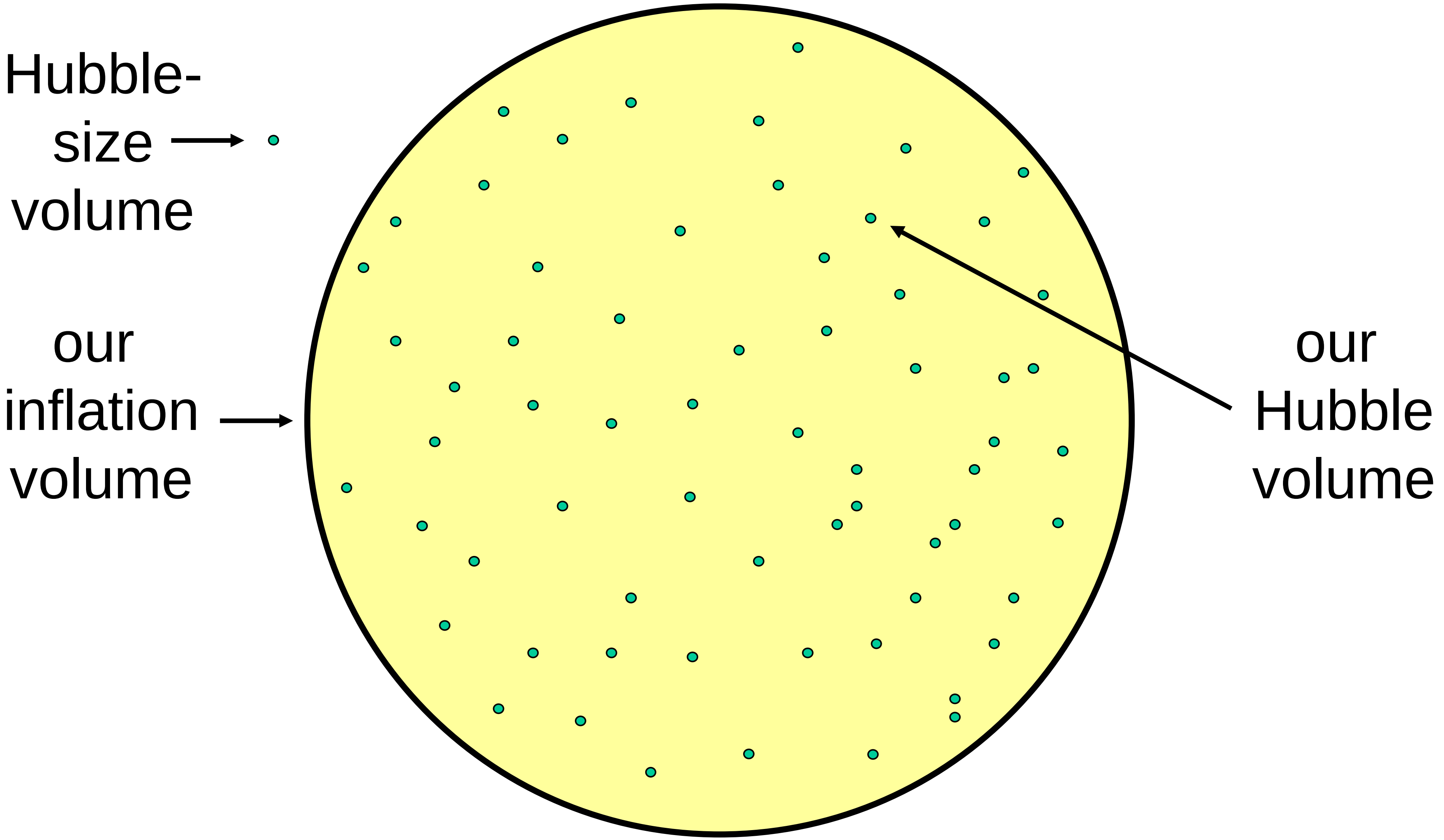
$$W(kR) = V^{-1}(R) \int_{V(R)} e^{-r^2/2R^2} \exp\left(i\vec{k}\cdot\vec{x}\right) dV$$



$$= e^{-k^2 r^2/2} \rightarrow \begin{cases} 1 & kr \rightarrow 0 \\ 0 & kr \rightarrow \infty \end{cases}$$

Mean of $\delta\theta$

• "Typical expected value" of $\delta\theta$ in $V(R) \Rightarrow$ ensemble average $\overline{\langle \ldots \rangle}$



Crucial point

Must remember the statistical nature of the vacuum fluctuations

Our Universe corresponds to a typical member of the ensemble of possible Universes

Mean of $\delta\theta$

- Express φ and its derivatives in terms of Fourier integrals

$$\varphi = \int \frac{d^3k}{(2\pi)^3} \varphi_{\vec{k}} \, e^{i\vec{k}\cdot\vec{x}}$$

- φ Gaussian random variable w/ zero mean $\Rightarrow$ N-point functions:

$$\overline{\varphi_{\vec{k}_1}\varphi_{\vec{k}_2}} = (2\pi)^3 \, \delta^3\left(\vec{k}_1 + \vec{k}_2\right) P_\varphi\left(k_1\right)$$

$$\begin{aligned}
\overline{\varphi_{\vec{k}_1}\varphi_{\vec{k}_2}\varphi_{\vec{k}_3}\varphi_{\vec{k}_4}} = (2\pi)^6 \Big[& \delta^3\left(\vec{k}_1 + \vec{k}_2\right)\delta^3\left(\vec{k}_3 + \vec{k}_4\right) P_\varphi\left(k_1\right) P_\varphi\left(k_3\right) \\
& + \delta^3\left(\vec{k}_1 + \vec{k}_3\right)\delta^3\left(\vec{k}_2 + \vec{k}_4\right) P_\varphi\left(k_1\right) P_\varphi\left(k_2\right) \\
& + \delta^3\left(\vec{k}_1 + \vec{k}_4\right)\delta^3\left(\vec{k}_2 + \vec{k}_3\right) P_\varphi\left(k_1\right) P_\varphi\left(k_2\right) \Big]
\end{aligned}$$

- $P_\varphi(k) = |\varphi_k|^2$ is the power spectrum of φ

- Cosmological Poisson equation relates $P_\varphi(k)$ and $\Delta^2(k,a)$

- $\Delta^2(k,a)$ is the power spectrum of $\delta\rho/\rho$

$\Delta^2(k,a)$

- Power spectrum of $\delta\rho/\rho$ in terms of $T^2(k)$, the transfer function, and $A \sim 10^{-5}$, a dimensionless amplitude:

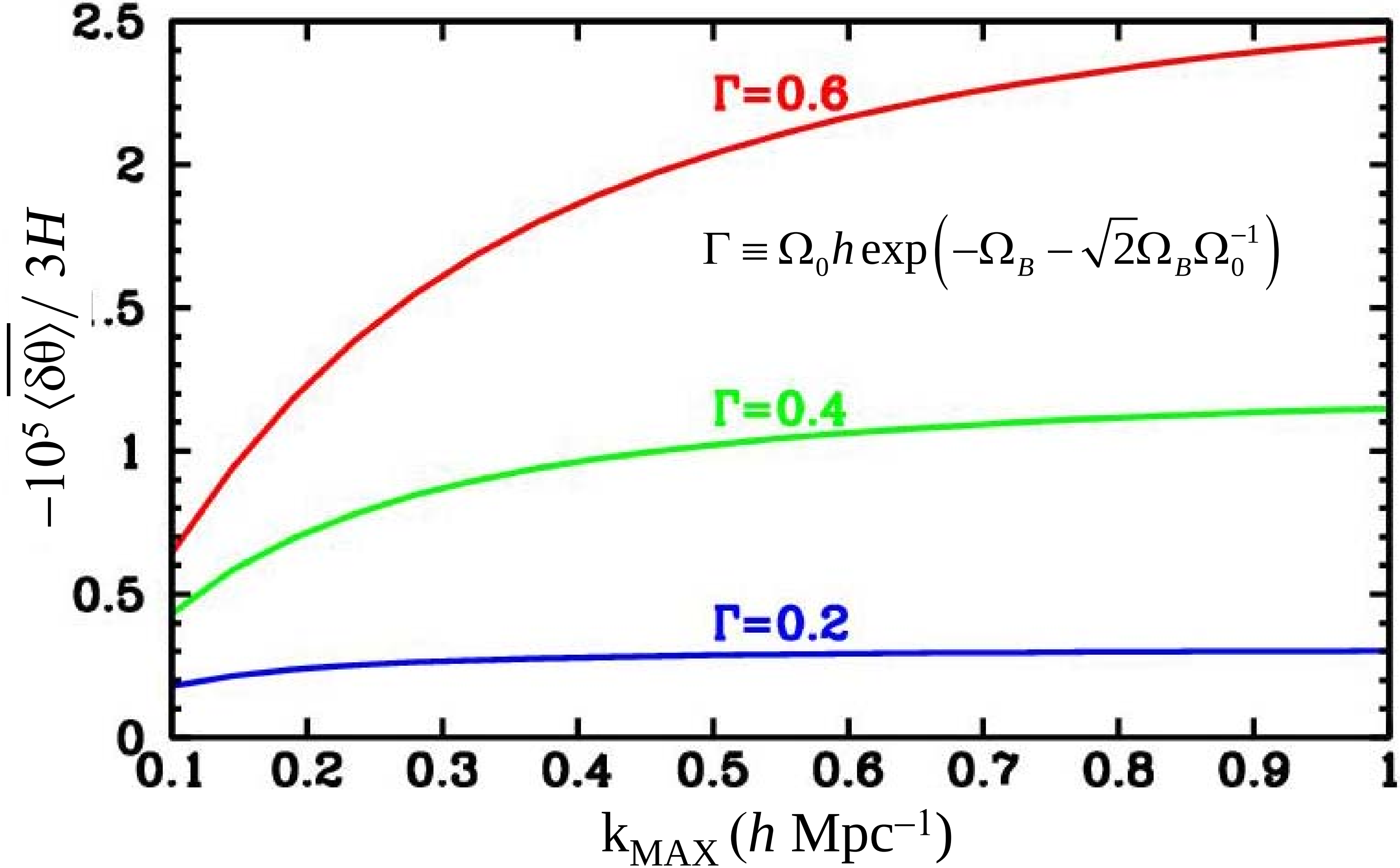
$$\Delta^2(k,a) = A^2\left(\frac{k}{aH}\right)^4 T^2(k)$$

Harrison–Zel'dovich spectrum

- For Harrison–Zel'dovich: $\Delta^2(k) \to \begin{cases} k^4 & k \to 0 \\ \ln^2(k) & k \to \infty \end{cases}$

- Other spectra, $\Delta^2(k) \to k^{3+n}$ as $k \to 0$ ($n = 1$ for H–Z)

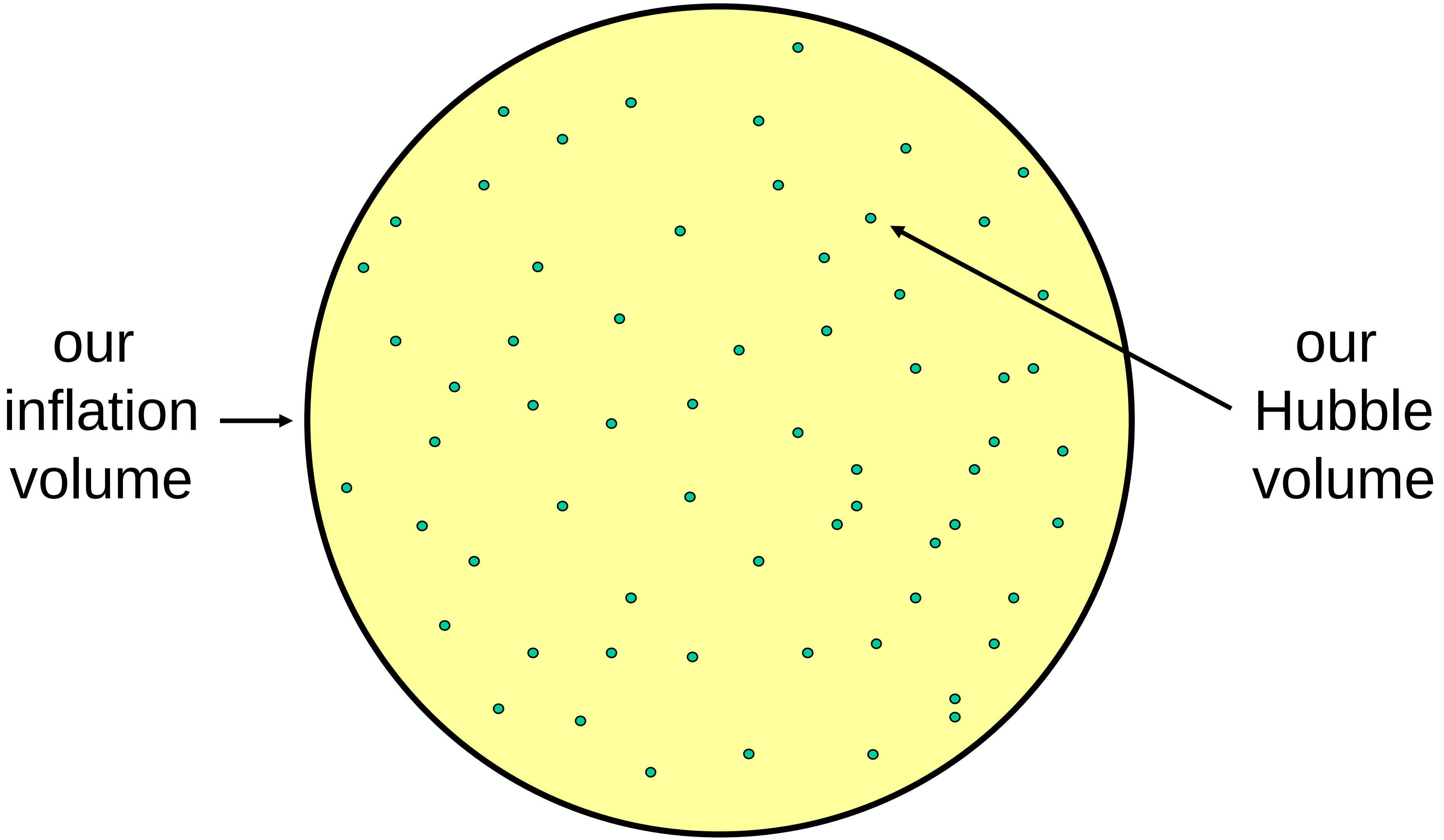
Mean of $\delta\theta$ (present value)



scales as $(1+z)^{-1}$

Variance of $\delta\theta$

- What we really want is $\delta\theta$ in <u>our</u> Hubble volume



Variance of $\delta\theta$

$$\mathrm{Var}\left[\langle\cdots\rangle\right] \equiv \overline{\left(\langle\cdots\rangle^2\right)} - \left(\overline{\langle\cdots\rangle}\right)^2$$

$$\left[\frac{\langle\delta\theta\rangle}{3H}\right]^2 = \left[\frac{\langle\theta^{(1)}\rangle + \langle\theta^{(11)}\rangle + \langle\theta^{(2)}\rangle}{3H} - \frac{\langle G_{00}^{(1)} + G_{00}^{(11)} + G_{00}^{(2)}\rangle}{6a^2H^2} - \frac{\langle G_{00}^{(1)}\rangle^2}{72a^4H^4}\right]^2$$

$$= \left\{\frac{1}{a^2H^2}\left[-\frac{5}{9}\langle\nabla^2\varphi\rangle_1 - \frac{25}{9}\langle\varphi\rangle_1\langle\nabla^2\varphi\rangle_1 - \frac{23\tau^2}{216}\langle\nabla^2\varphi\rangle_1\langle\nabla^2\varphi\rangle_1\right.\right.$$

$$\left.\left. + \frac{25}{27}\langle\varphi\nabla^2\varphi\rangle + \frac{25}{54}\langle\varphi^{,i}\varphi_{,i}\rangle - \frac{\tau^2}{27}\langle\nabla^2\varphi\nabla^2\varphi\rangle + \frac{\tau^2}{27}\langle\varphi^{,ij}\varphi_{,ij}\rangle\right]\right\}^2$$

Variance of $\delta\theta$ – sample terms

$$\mathrm{Var}\left[\left\langle \nabla^2 \varphi \right\rangle_1\right] = \frac{9}{4} a^4 H^4 \int_0^\infty \frac{dk}{k} \Delta^2(k,a) W^2(kR)$$

$$\mathrm{Var}\left[\left\langle \varphi \nabla^2 \varphi \right\rangle\right] = \left(\frac{9}{8} a^4 H^4\right)^2 \int_0^\infty \frac{dk}{k_1^3} \Delta^2(k_1,a) \int_0^\infty \frac{dk}{k_2^3} \Delta^2(k_2,a) \left(\frac{k_1^2}{k_2^2} + \frac{k_2^2}{k_1^2} + 2\right) \times 2W^2(k_1 R) W^2(k_2 R)$$

- $W^2(kR)$ acts as a “filter function” regulating ultraviolet …
- … <u>*but*</u> Var[$\langle \varphi \nabla^2 \varphi \rangle$] singular in the infrared if $n \leq 1$!
- as is Var[$\langle \varphi \rangle_1 \langle \nabla^2 \varphi \rangle_1$]
- traces to the infrared behavior of Var[$\langle \varphi \rangle_1$]

Variance of $\delta\theta$ – infrared nature

$$\mathrm{Var}\left[\langle\varphi\rangle_1\right]=\frac{9}{4}a^4H^4\int_0^\infty\frac{dk}{k^5}\Delta^2(k,a)W^2(kR) \qquad \overline{\langle\varphi\rangle_1}=0$$

$$\mathrm{Var}\left[\langle\nabla^2\varphi\rangle_1\right]=\frac{9}{4}a^4H^4\int_0^\infty\frac{dk}{k}\Delta^2(k,a)W^2(kR) \qquad \overline{\langle\nabla^2\varphi\rangle_1}=0$$

- Interested in $R=R_H$
- For $k < k_H$: $W(kR_H) \to 1$ and $T^2(k) \to 1$, so $\Delta^2(k,a)W^2(kR) \to k^{3+n}$

$$\mathrm{Var}\left[\langle\varphi\rangle_1\right]_{IR}\to\int_0^{k_H}\frac{dk}{k^5}k^{3+n} \qquad \mathrm{Var}\left[\langle\nabla^2\varphi\rangle_1\right]_{IR}\to\int_0^{k_H}\frac{dk}{k}k^{3+n}$$

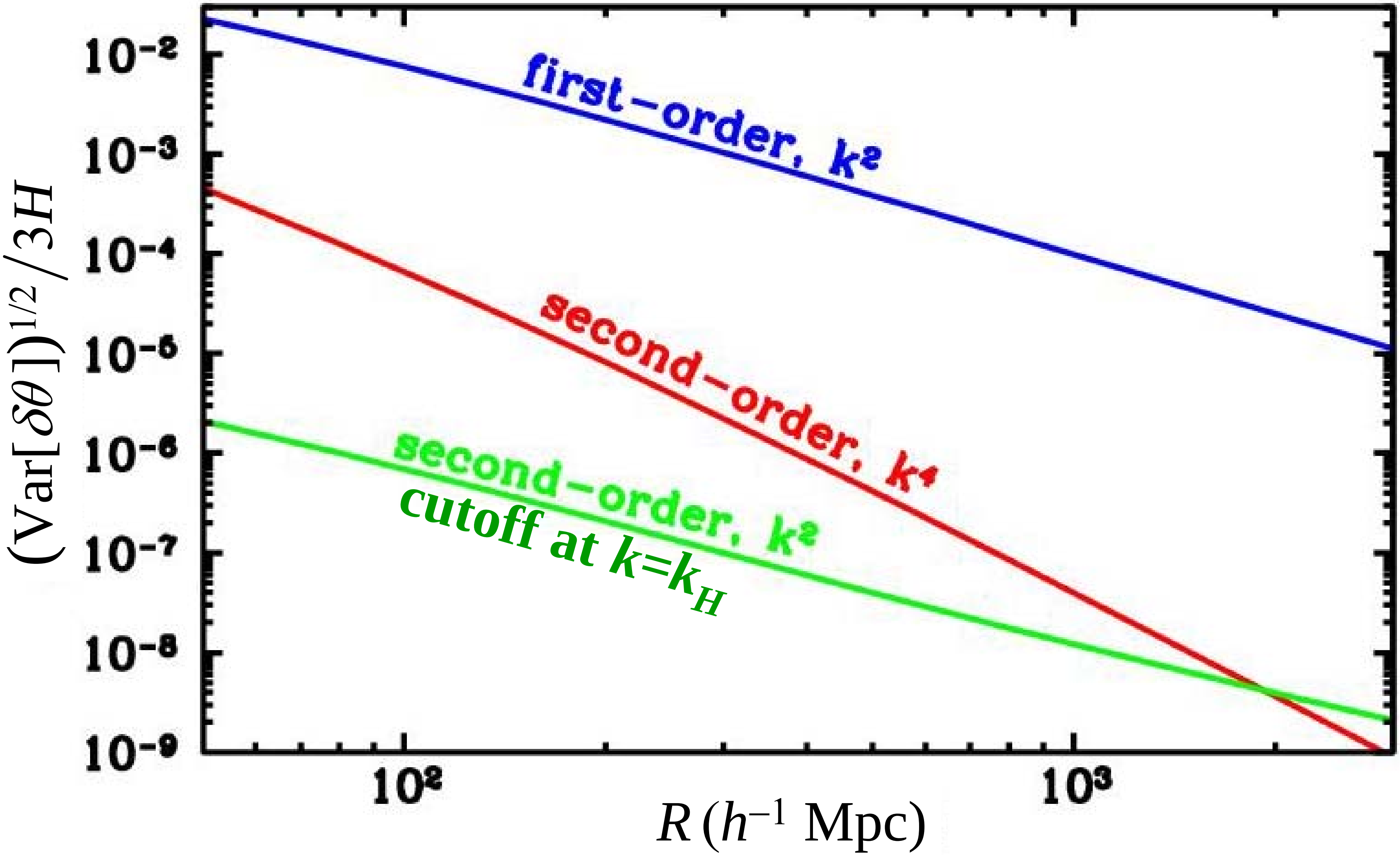
$$\mathrm{Var}\left[\langle\varphi\nabla^2\varphi\rangle\right]_{IR}\to\int_0^{k_H}\frac{dk_1}{k_1^5}k_1^{3+n}\int_0^{k_H}\frac{dk_2}{k_2}k_2^{3+n}=\mathrm{Var}\left[\langle\varphi\rangle_1\right]_{IR}\mathrm{Var}\left[\langle\nabla^2\varphi\rangle_1\right]_{IR}$$

Variance of $\delta\theta$

How to make sense of infrared singularity?

1. variance is infrared finite–bluer than Harrison-Zel'dovich
2. somehow infrared singular terms cancel
3. physical cutoff at Hubble radius: $k_{\mathrm{MIN}} = k_H$

Variance of $\delta\theta$



k^4 scales as $(1+z)^{-2}$; k^2 scales as $(1+z)^{-1}$

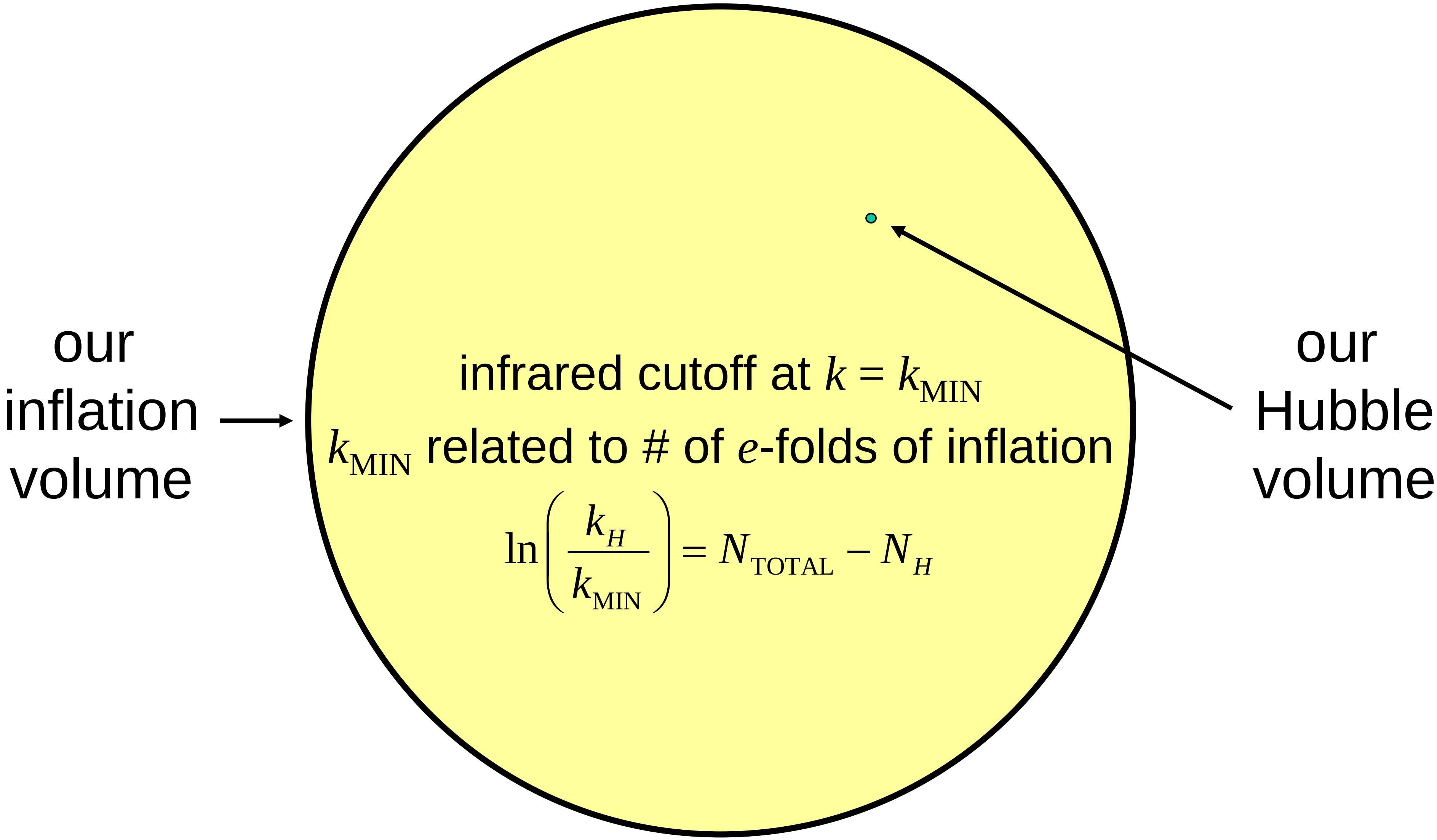
Variance of $\delta\theta$

How to make sense of infrared singularity?

1. variance is infrared finite ($\delta\rho/\rho$ bluer than $n=1$ as $k \to 0$)
2. somehow infrared-singular terms cancel
3. physical cutoff at Hubble radius: $k_{\mathrm{MIN}} = k_H$
4. use cutoff determined by duration of inflation (sensitive to unknown nature of perturbations on scales greater than the Hubble radius)

Variance of $\delta\theta$

- What we really want is $\delta\theta$ in our Hubble volume



Variance of $\delta\theta$

How to make sense of infrared singularity?

1. variance is infrared finite–bluer than Harrison-Zel'dovich
2. somehow infrared singular terms cancel
3. introduce cutoff at Hubble radius: $k_{\mathrm{MIN}} = k_H$
4. use cutoff determined by duration of inflation (sensitive to unknown nature of perturbations on scales greater than the Hubble radius)

Are super–Hubble perturbations physical?

- constant φ can be scaled out equations …
- … but can't get rid of $\varphi\nabla^2\varphi$!
- could Var[$\delta\theta$] be large enough to give $\langle \delta\theta \rangle / 3H \sim \mathcal{O}(1)$?

Crucial technical points

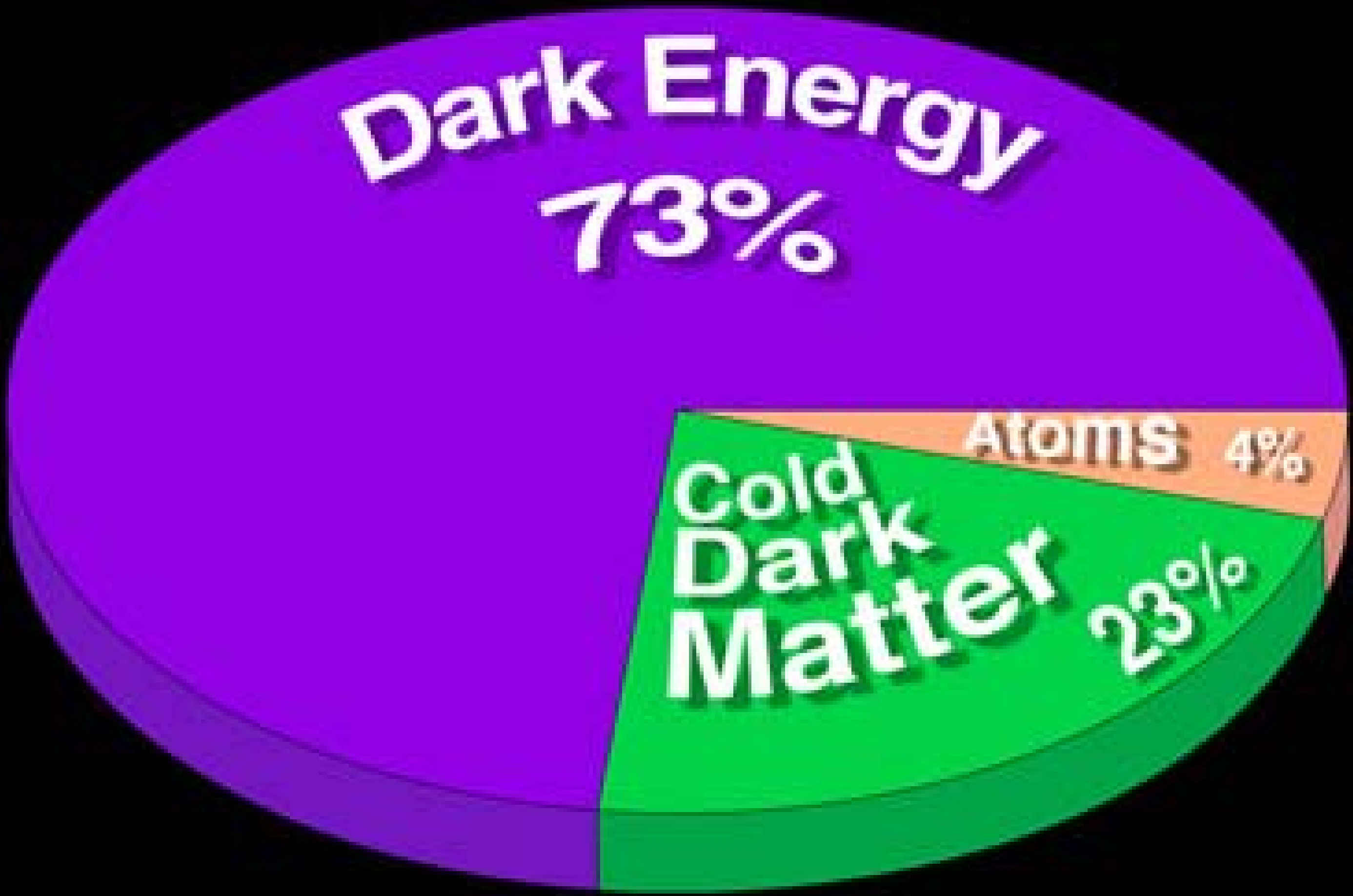
- $$\frac{\delta\rho}{\rho} \propto \nabla^2 \varphi$$ ⇒ ***on superhorizon scales density perturbs are perturbative***

- $$\varphi \nabla^2 \varphi$$ ***comes from*** $$e^{-\varphi} \nabla^2 \varphi$$

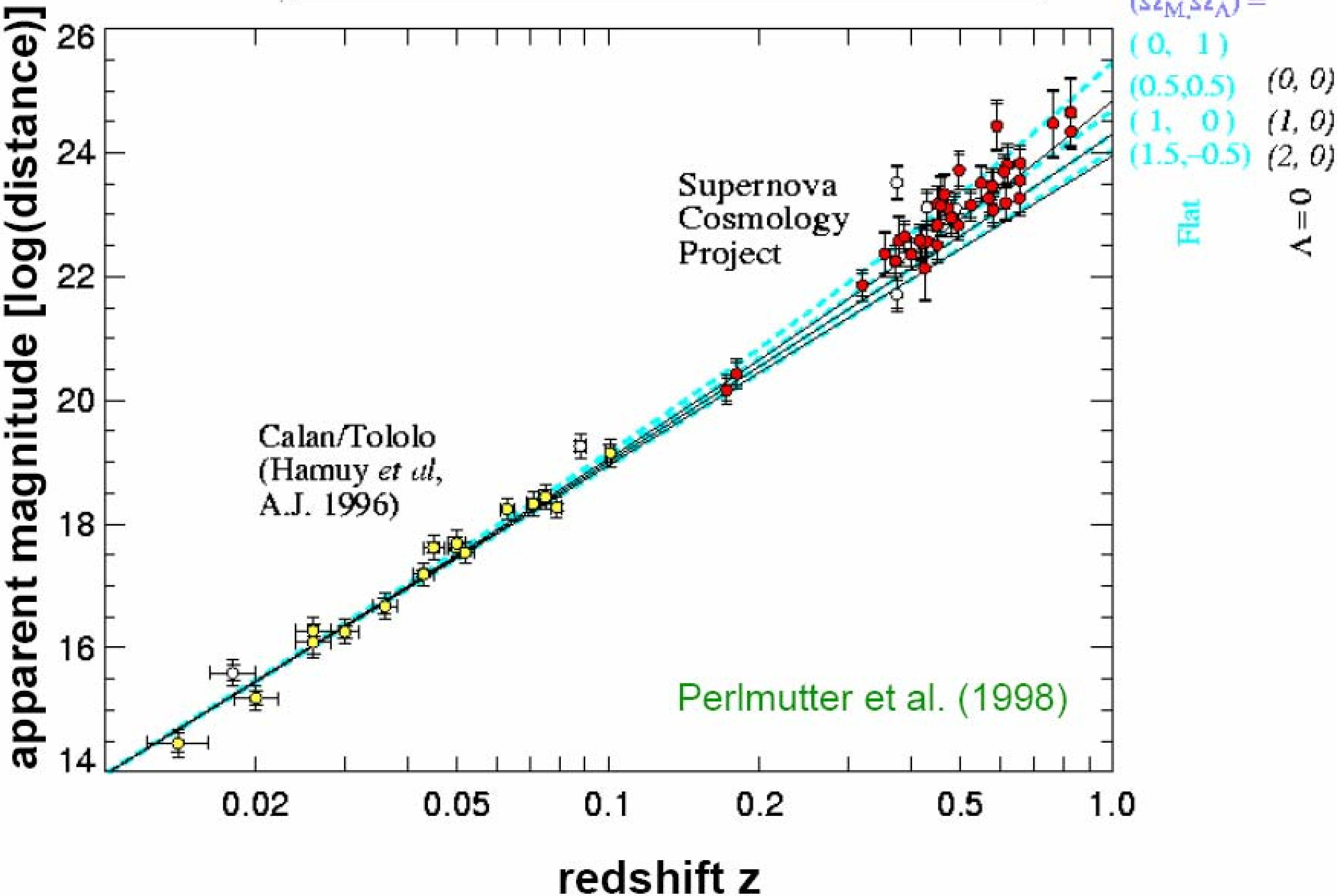
may go beyond second-order

E.W. Kolb, S. Matarrese, A. Notari and A.R., astroph/0410541

Entertaining conjecture



Type Ia supernova



Field equations

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \qquad H \equiv \frac{\dot{a}}{a} \qquad \text{expansion rate}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right) \qquad q \equiv -\frac{\ddot{a}}{a}\frac{1}{H^2} \quad \text{deceleration parameter}$$

$$\rho = \rho_M + \rho_R + \rho_\Lambda + \ldots\ldots$$

whatever:	$p_w = w\rho_w$	$w = p/\rho$	$\rho_w \propto (1+z)^{3(1+w)}$
matter:	$p_M = 0$	$w = 0$	$\rho_M \propto (1+z)^3$
radiation:	$p_R = \rho_R/3$	$w = 1/3$	$\rho_R \propto (1+z)^4$
vacuum:	$p_\Lambda = -\rho_\Lambda$	$w = -1$	$\rho_\Lambda \propto (1+z)^0$

Distance-luminosity relation

$F = \dfrac{L}{4\pi d_L^2}$ defines luminosity distance - "know" L, measure F

$4\pi d_L^2$ = area of ${}^2S$ centered on source at time of detection, t_0

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \Rightarrow \text{Area} = 4\pi a^2(t_0) r^2$$

Conservation of energy: flux redshifted: $(1+z)^2 = \left[a(t_0)/a(t_1)\right]^2$

redshift of energy × redshift of time interval: $(1+z)^2$

$$F = \frac{L}{4\pi a^2(t_0) r^2 (1+z)^2} \quad \Rightarrow \quad \boxed{d_L = a(t_0)\, r\, (1+z)}$$

light from comoving coordinate r reaches us now

redshifted by an amount $(1+z) = a(t_0)/a(t)$

Distance-luminosity relation

$$\boxed{d_L = a(t_0)\, r\,(1+z)}$$

$$\boxed{ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)}$$

light travels on geodesics

$$ds^2 = 0$$

$$\int \frac{dr}{\sqrt{1-kr^2}} = \int \frac{dt}{a(t)} = \int \frac{da}{H(a)a^2}$$

$$H^2 = H_0^2\left[(1-\Omega_{\mathrm{TOTAL}})(1+z)^2 + \Omega_M(1+z)^3 + \ldots\right]$$

$$\Omega_i = \rho_i \Big/ \left(3H_0^2 \big/ 8\pi G\right)$$

$$\Omega_{\mathrm{TOTAL}} = \Omega_M + \Omega_\Lambda + \Omega_R + \Omega_w + \ldots \qquad (1-\Omega_{\mathrm{TOTAL}}) \propto k$$

$$\int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_0^z \frac{a^{-1}(t_0)H_0^{-1} \quad dz'}{\sqrt{(1-\Omega_0)(1+z')^2 + \Omega_M(1+z')^3 + \Omega_w(1+z')^{3(1+w)} + \ldots}}$$

Distance-luminosity relation

$$\boxed{d_L = a(t_0)\, r\, (1+z)}$$

$$a(t_0)r \text{ from } \int_0^r \frac{dr'}{\sqrt{1-kr'^2}} = \int_0^z \frac{a^{-1}(t_0) H_0^{-1} \quad dz'}{\sqrt{(1-\Omega_{\text{TOTAL}})(1+z')^2 + \Omega_M (1+z')^3 + \ldots}}$$

Example: matter + lambda $\longrightarrow$ $\Omega_{\text{TOTAL}} = \Omega_M + \Omega_\Lambda$

Program:

- **measure** d_L **(via** $d_L^2 = L / 4\pi F$**) and** z
- **input** Ω_i **and calculate** $a(t_0)r$

$$\boxed{\begin{array}{c} H_0 d_L = z + O(z^2) \\ \Omega_i \hookleftarrow \end{array}}$$

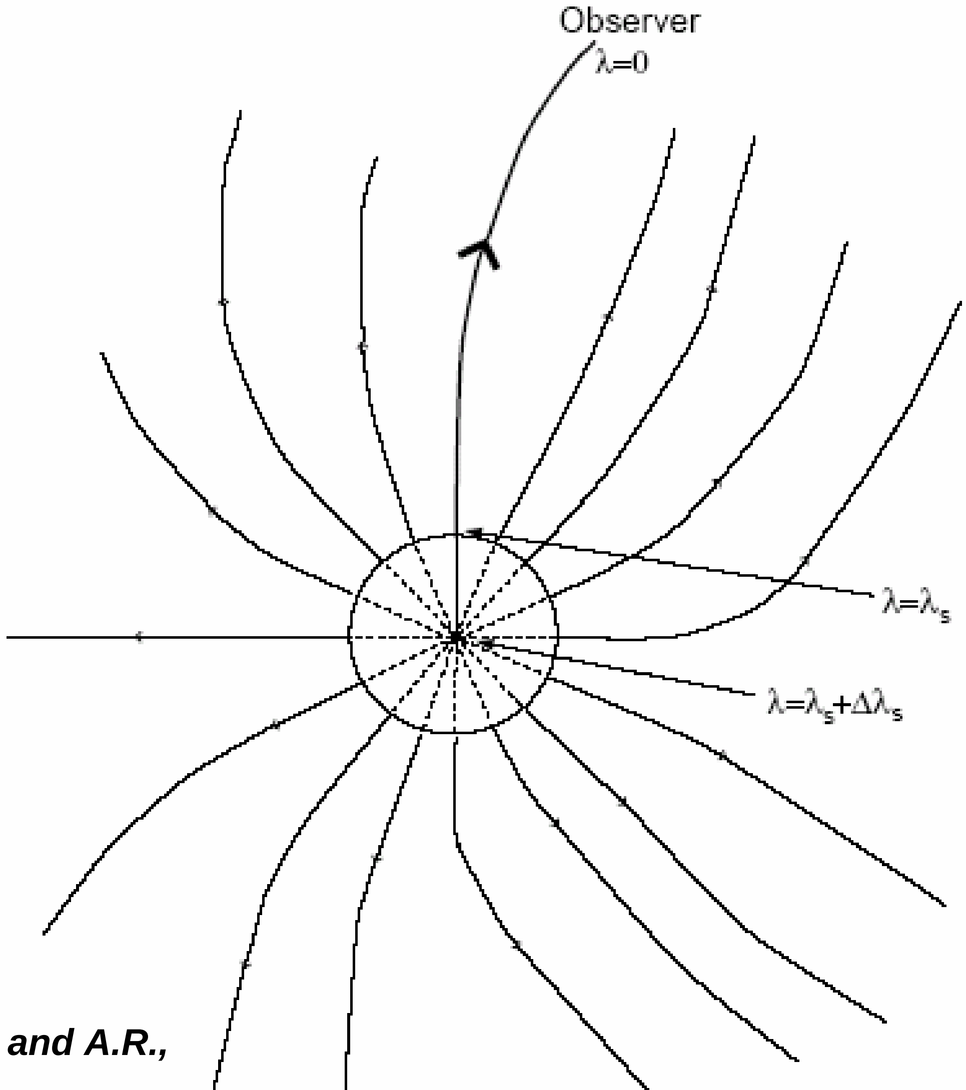
Unperturbed Universe

$$\boxed{q_0 = \frac{\Omega_0}{2} + \frac{3}{2}\sum_i w_i \Omega_i}$$

Acceleration implies a fluid with negative pressure

Dark Energy

Distance-redshift relation in a perturbed Universe



E. Barausse, S. Matarrese and A.R., In preparation

Distance-redshift relation in a perturbed Universe

observer $\lambda = 0$

$$k_\mu k^\mu = 0$$

$$k^\nu \nabla_\nu k^\mu = 0$$

photon momentum

source $\lambda = \lambda_s$

Distance-redshift relation in a perturbed Universe

$$d\hat{s}^2 = a(\eta)^2 \, ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = a(\eta)^2 \, g_{\mu\nu} dx^\mu dx^\nu \ , \qquad x^\mu = \left(\eta, x^i\right)$$

$$\hat{T}^{\mu\nu} = A^2 \hat{k}^\mu \hat{k}^\nu$$ **energy momentum tensor**

$$\hat{k}^\mu \hat{k}_\mu = 0, \qquad \hat{k}^\nu \hat{\nabla}_\nu \hat{k}^\mu = \frac{d^2 x^\mu}{dv^2} + \hat{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{dv} \frac{dx^\beta}{dv} = 0$$

Photon equations

$$\hat{\nabla}_\nu \hat{T}^{\mu\nu} = 0$$

$$\frac{dA}{dv} = -\frac{1}{2} A\hat{\theta} \,, \qquad \hat{\theta} := \hat{\nabla}_\mu \hat{k}^\mu$$

$\hat{\theta}$ ***expansion of the null congruence along the photon trajectory***

$$d\lambda := a^{-2} dv$$

$$\frac{d\theta}{d\lambda} = -R_{\mu\nu}k^{\mu}k^{\nu} - \frac{\theta^2}{2} - 2|\sigma|^2$$

$$|\sigma| = \sqrt{\frac{1}{2}\left[k_{(\mu;\nu)}k^{(\mu;\nu)} - \frac{\theta^2}{2}\right]}$$

Shear

Distance-redshift relation in a perturbed Universe

$$\ell = \sqrt{\hat{h}_{\mu\nu} \left(\hat{u}_\sigma \hat{T}^{\sigma\mu}\right) \left(\hat{u}_\delta \hat{T}^{\delta\nu}\right)} = A^2 \omega^2$$

Energy flux per unit surface measured by an observer with four-velocity $\hat{u}_\mu$

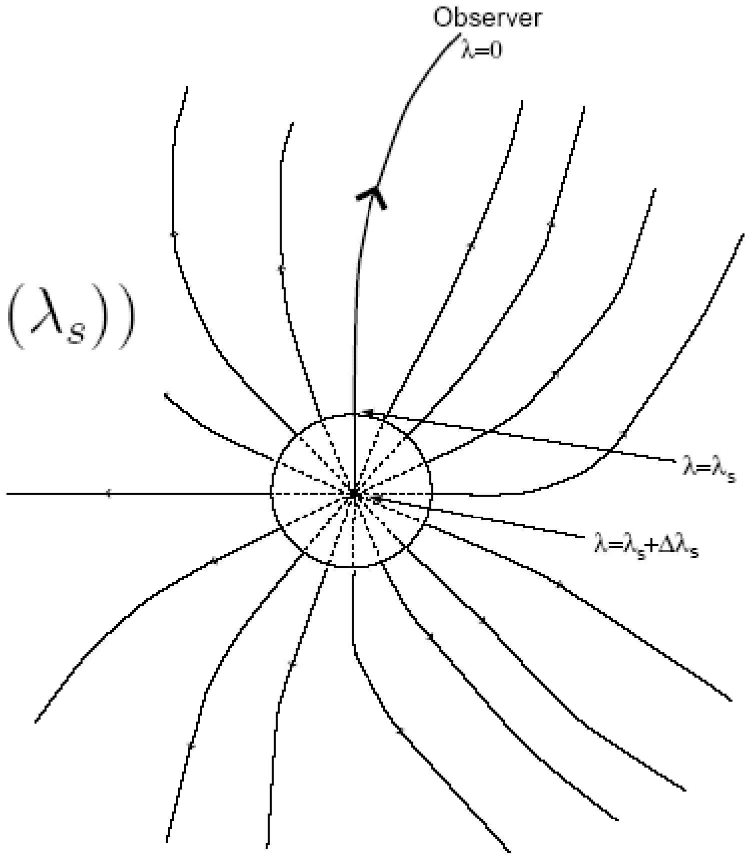
$$\hat{h}_{\mu\nu} = \hat{g}_{\mu\nu} + \hat{u}_\mu \hat{u}_\nu \ , \qquad \omega = -\hat{u}_\mu \hat{k}^\mu$$

Distance-redshift relation in a perturbed Universe

$$L = 4\pi R^2 \ell\left(\lambda_s\right)$$

$$d_L = R\sqrt{\frac{\ell\left(\lambda_s\right)}{\ell\left(0\right)}} = R\frac{A\left(\lambda_s\right)}{A\left(0\right)}\left(1+z\left(\lambda_s\right)\right)$$

$$1+z\left(\lambda_s\right) := \frac{\omega\left(\lambda_s\right)}{\omega\left(0\right)}$$



Distance-redshift relation in a perturbed Universe

$$d_L = \frac{c}{\widetilde{H}_0}\left[z + \frac{z^2}{2}\left(1 - \tilde{q}_0\right) + O\left(z^3\right)\right]$$

$$\langle \widetilde{H}_0 \rangle_\Omega = H_0\left[1 - \frac{2}{H_0}\left(\frac{1}{2}\phi'_{(1)} + \frac{1}{4}\phi'_{(2)} + \phi_{(1)}\phi'_{(1)} + \frac{1}{30}\left(\chi^{ij}_{(1)}\right)'\chi_{(1)ij}\right)\right]$$

$$\begin{aligned}\langle \tilde{q}_0 \rangle_\Omega = \frac{1}{2}\Bigg[1 &+ \frac{2}{H_0}\left(2\phi'_{(1)} + \phi'_{(2)} + 4\phi_{(1)}\phi'_{(1)} + \frac{2}{15}\left(\chi^{ij}_{(1)}\right)'\chi_{(1)ij}\right) \\ &+ \left(\frac{2}{H_0}\right)^2\left(\frac{1}{2}\phi''_{(1)} + \frac{1}{4}\phi''_{(2)} + \frac{9}{4}\left(\phi'_{(1)}\right)^2 + \phi_{(1)}\phi''_{(1)} + \frac{7}{40}\left(\chi_{(1)ij}\right)'\left(\chi^{ij}_{(1)}\right)' + \frac{1}{30}\left(\chi^{ij}_{(1)}\right)''\chi_{(1)ij}\right) \\ &+ \left(\frac{2}{H_0}\right)^3\left(\frac{1}{2}\phi'_{(1)}\phi''_{(1)} + \frac{1}{60}\left(\chi_{(1)ij}\right)'\left(\chi^{ij}_{(1)}\right)''\right)\Bigg]\end{aligned}$$

$$\langle \quad \rangle_\Omega := \frac{1}{4\pi}\int d\Omega$$

Distance-redshift relation in a perturbed Universe

$$\langle \widetilde{H}_0 \rangle_\Omega = H_0 \left[1 - \left(\frac{1}{18} \nabla^2 \varphi - \frac{5}{108} (\nabla \varphi)^2 + \frac{5}{27} \varphi \nabla^2 \varphi \right) \left(\frac{2}{H_0} \right)^2 + \frac{1}{3780} \left(22 \varphi^{,ij} \varphi_{,ij} - 29 \left(\nabla^2 \varphi \right)^2 \right) \left(\frac{2}{H_0} \right)^4 \right]$$

$$\langle \widetilde{q}_0 \rangle_\Omega = \frac{1}{2} \left[1 + \left(\frac{5}{18} \nabla^2 \varphi + \frac{25}{27} \varphi \nabla^2 \varphi - \frac{25}{108} (\nabla \varphi)^2 \right) \left(\frac{2}{H_0} \right)^2 + \frac{1}{135} \left(7 \left(\nabla^2 \varphi \right)^2 + 4 \varphi^{,ij} \varphi_{,ij} \right) \left(\frac{2}{H_0} \right)^4 \right]$$

Crucial aspects

- ***The Hubble parameter and the deceleration parameter are not deterministic***
- ***Because of the statistical nature of vacuum fluctuations, the gravitational potential does not have well-defined values***
- ***The theoretical predictions of the cosmological parameters come with a nonvanishing cosmological variance implying an intrinsic theoretical error***

Variance of q

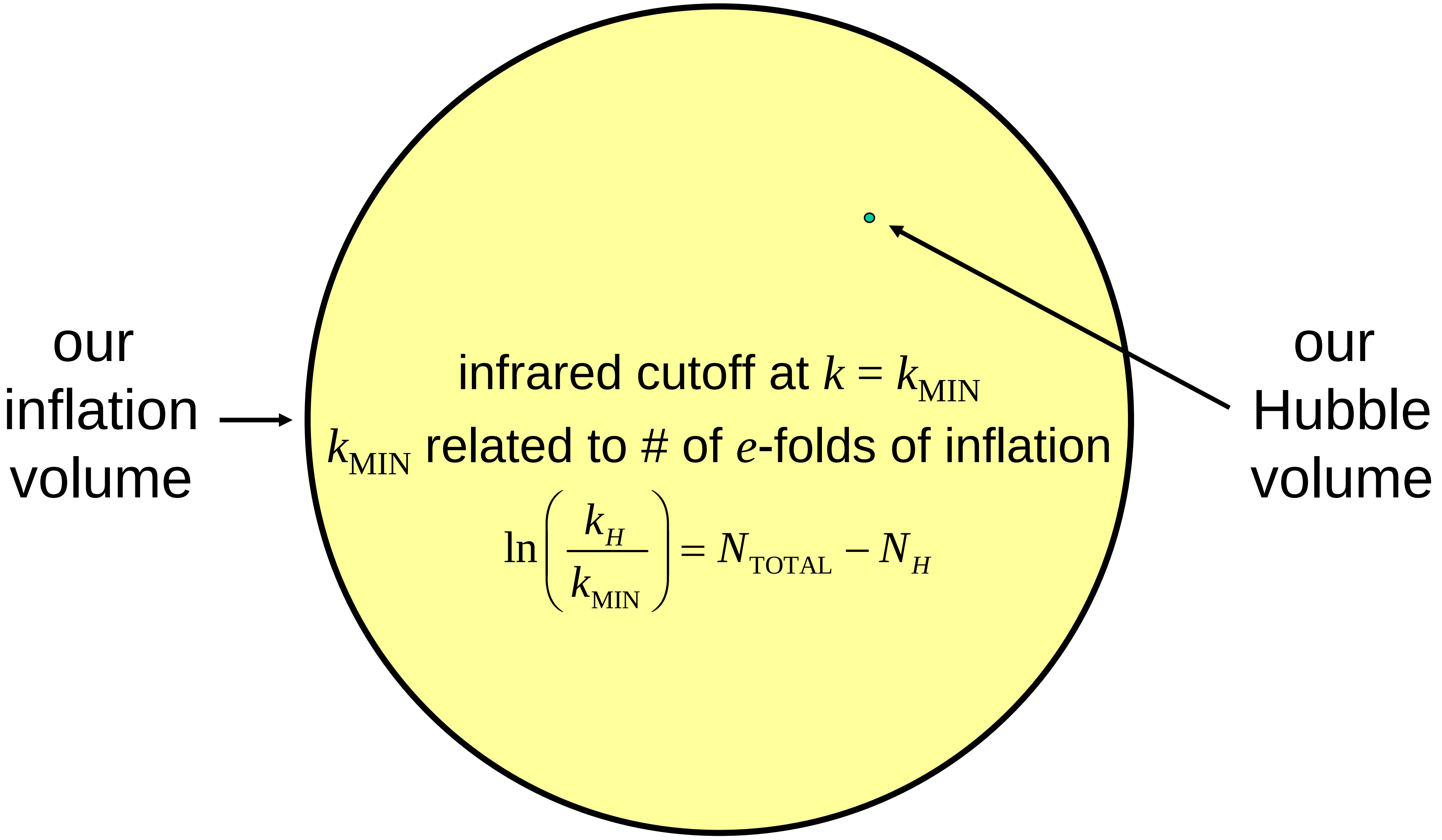
Are super–Hubble perturbations physical?

Constant φ can be scaled out of equations

...but can't get rid of $\varphi \nabla^2 \varphi$!

Variance of q

- What we really want is q in our Hubble volume



$$\mathrm{Var}\left[\varphi\nabla^2\varphi\right] \simeq \left(\frac{9}{4}a_0^4 H_0^4\right)^2 \int \frac{dk_1}{k_1}\Delta^2(k_1, a_0) \int \frac{dk_2}{k_2^5}\Delta^2(k_2, a_0)$$

for a flat spectrum, n=1

$$\frac{\sqrt{\mathrm{Var}\left[\langle\widetilde{q}\rangle_\Omega\right]}}{q_0} \simeq 10^{-10} \ln \frac{k_{\mathrm{MAX}}}{k_{\mathrm{MIN}}} \simeq 1 \qquad (10^{18.8} \quad e\text{-folds!})$$

$$\Delta^2(k) \propto k^{3+n} \text{ with } 0 < (1-n) \ll 1$$

$$n = 0.94 \qquad \sim 700 \;\; e\text{-folds}$$

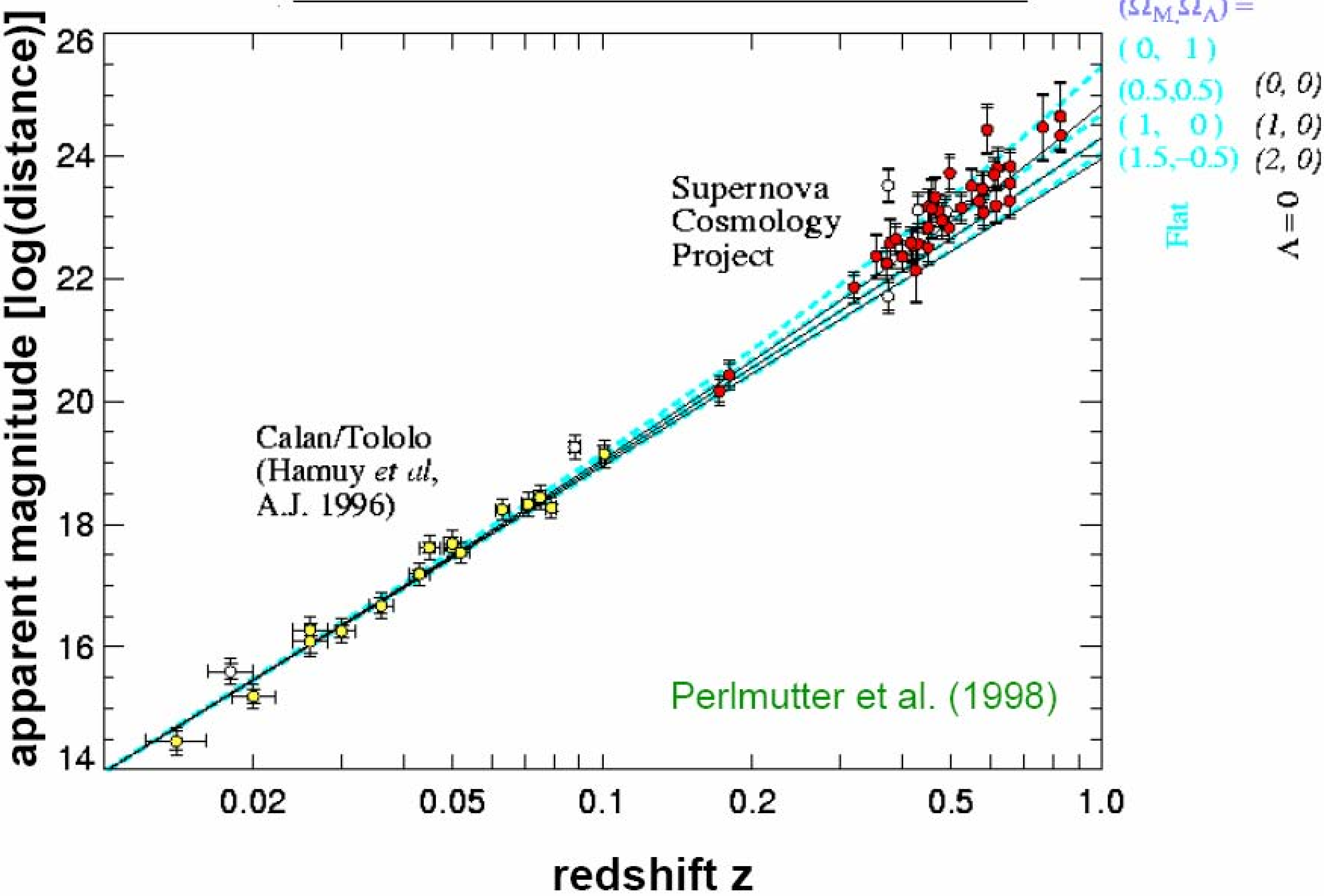
The Universe accelerates

Dark Energy is present

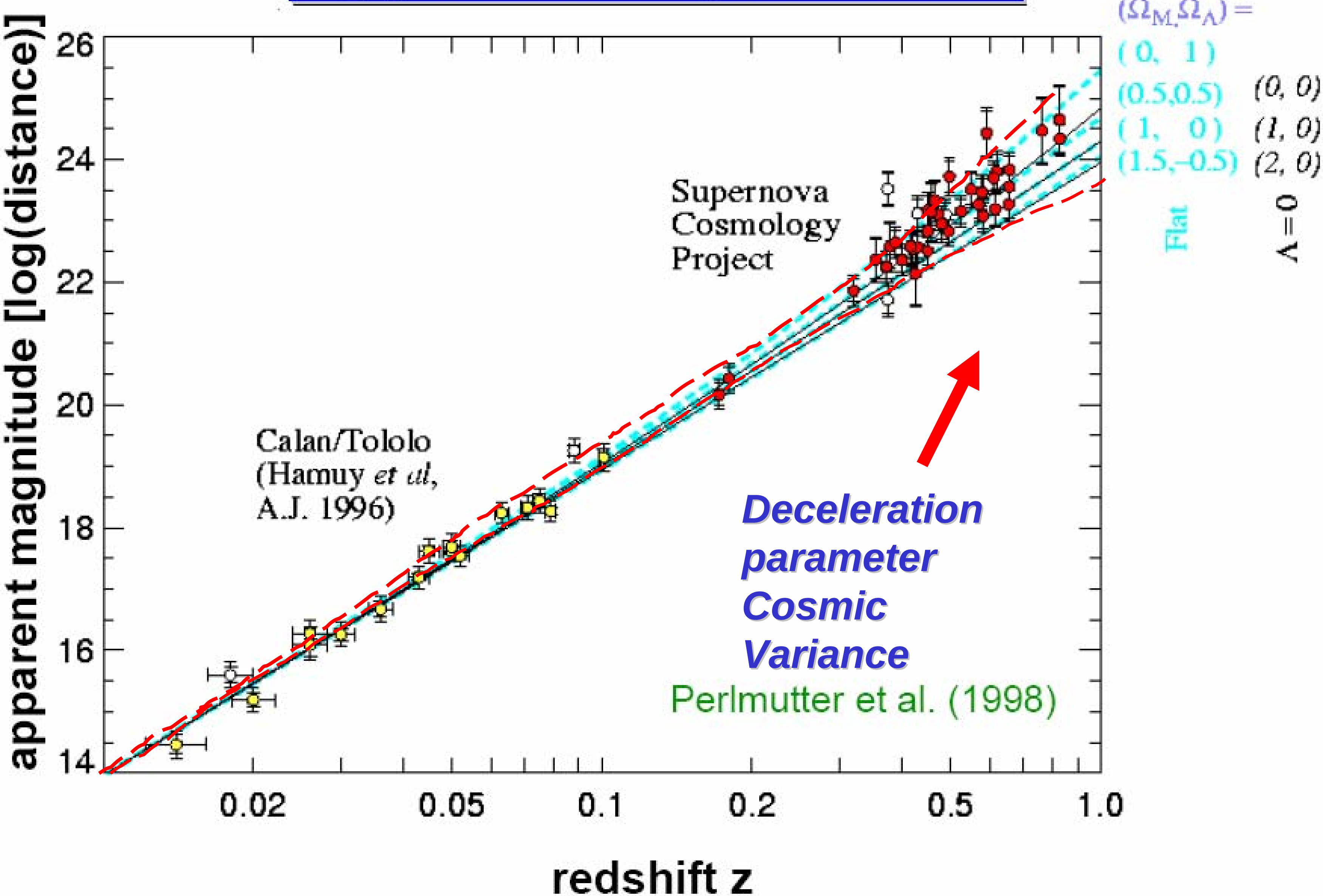
or

It is matter-dominated and the value of deceleration parameter predicted by the theory at the unperturbed level comes with a large cosmological variance: negative values of q are allowed

Type Ia supernova



Type Ia supernova



Conclusions

- ***Inflation generates long wavelength perturbations***
- ***Super-Hubble modes introduce intrinsic theoretical errors***
- ***New effects to study***
- ***No need of Dark Energy?***