Non-Gaussian Galaxy Distributions

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The long-range force gravity produces all unique structures in the universe. Such structures, called self gravitating systems (SGS), are thought to represent the basic nature of gravity. Since the gravity is long-range and unshielded, we cannot expect a naïve Boltzmann statistical mechanics, which is fully based on the additivity of conserved quatities such as energy and angular momentum. Therefore in this paper, we return to the very basic stand point and try to find out what kind of statistical mechanics is applicable for SGS. We especially emphasize the two properties often appear in SGS systems: the non-extensive property and the long-tails in various distribution functions. In order to determine which property is essential in statistical description of SGS, we study four kinds of statistical mechanics, possessing all the combinations of these two properties: (1) Boltzmann, (2) Fractal matter, (3) Renyi, and (4) Tsallis statistical mechanics. We use Akaike Information Criteria for their fair comparison. We use the data of SDSS DR3 spectroscopic survey and apply the count-in-cell method. We conclude that only the Tsallis statistical mechanics properly describe the system. That is, both non-extensive and long-tail in distribution function are essential for SGS.

1. INTRODUCTION

Large-scale non-linear structures in the Universe such as clusters of galaxies and voids have some universal and coherent properties reflecting the force gravity. The detail of such structures, called self gravitating systems (SGS), are systematically observed especially recently. Such large structures are apparently made mainly by gravity. Since the gravity is long-range unshielded force, SGS generally have the following unique properties: they have non-extensive properties, and they have no absolute equilibrium state, and long-tails are often observed in their various distribution functions. Therefore, of course, the ordinary Boltzmann statistical mechanics, fully based on the extensive property, cannot be applied to SGS in a naïve form. Faced upon this situation, we are force to reconsider the basics of statistical mechanics. Actually there are some generalizations of the formalism in statistical mechanics in various fields in Physics. For the moment, we would like to examine which basic property is essential, among various phenomenological theories of statistical mechanics, for correctly describing the universal properties of SGS.

In our previous paper[1], we have tentatively explored the similar problem using the data set of CfAII South survey. Due to the smallness of the data set, we couldn't conclude the definite answer. Though we have obtained the conclusion that the long-tail in distribution function is at least essential, we couldn't sufficiently elucidate the non-extensive property. In this paper, we would like to obtain the definite answer applying the bland new data set of Sloan Digital Sky Survey DR3 spectroscopic catalog which includes enormous amount of galaxy data.

We start our discussion from describing the count-in-cell method, especially emphasizing on its generating functional. Then, using this formalism, we examine various statistical mechanics in comparison with observations.

2. THE COUNT-IN-CELL METHOD AND THE GENERATING FUNCTIONAL $f(0)$

We characterize the galaxy distributions by using the count-in-cell method, in which the probability to find exactly N galaxies within the volume V $f(N)$ becomes the central object. The most basic quantity would be the void probability $f(0)$. This quantity has a rich information and all the higher quantity $f(N)$ for $N \geq 1$ can be generated from $f(0)$ [2]. Actually, $f(0)$ is the assembly of the co-probability that each galaxy is located as some particular location in space. The expression for the probability $f(N)$ is given by

$$f(N) \equiv \frac{(-n)^N}{N!}\frac{d^N}{dn^N}f(0), \tag{1}$$

which we now derive in this section. In the above, n is the mean number density of galaxies.

Suppose a general statistical variable $f(x)$, which is a field on the three-dimensional space. Most generally, the partition function

$$\begin{aligned} Z[J] &= \left\langle \exp\left[\int_V J(x) f(x) d^3x\right]\right\rangle \\ &= \exp\left\langle -\int_V J(x) f(x) d^3x \right\rangle_c \\ &\equiv \exp W[J] \end{aligned} \tag{2}$$

has all the information of the statistical property of $f(x)$. In the above the brackets represent the functional integral of the field $f(x)$ (or the sum over all possible fluctuations=configurations $f(x)$), and $J(x)$ is a source field which is a small probe from outside. The whole part of the cumulants or the connected correlation functions, denoted as the brackets with a suffix c, are defined by the above equation by some appropriate expansions.

For our purpose, we now specify the field $f(x)$ as discontinuous bare number density of galaxies:

$$\phi(x) \to \sum_{i=1}^{\infty} \delta^3(x_i - x). \tag{3}$$

Then the functional integration is reduced to the following multiple integration

$$\langle \cdots \rangle \to \int_V \cdots \frac{d^3x_1}{V}\frac{d^3x_2}{V}\frac{d^3x_3}{V}\cdots \tag{4}$$

since the distribution of galaxies at all the points $\{x_1, x_2, x_3, \ldots x_n, \ldots\}$ determines a field $\phi(x)$. It is apparent that the probability of finding one galaxy within a small volume d^3x_1 around the space position x_1 becomes $P(x_1)d^3x_1 = \langle \phi(x_1)\rangle d^3x_1$, and the joint probability of finding one galaxy within a small volume d^3x_1 around the space position x_1 *and* the other within d^3x_2 around x_2 would be $P(x_1,x_2)d^3x_1d^3x_2 = \langle \phi(x_1)\phi(x_2)\rangle d^3x_1d^3x_2$, and so on.

More useful, but complicated, quantity is the probability $\phi(0)$ of finding no galaxy within a fixed volume V. Suppose the volume V is divided into small pieces $\{v_1, \ldots, v_M\}$, each of which is of order $v = V/M$. Then this void probability is expressed as

$$\begin{aligned}\phi(0) &= \left\langle \prod_{m=1}^{M} (1 - \phi(x_m)v_m) \right\rangle \\ &= \left\langle \exp\left[-\sum_{m=1}^{M} \phi(x_m)v_m \right] + O(Mv^2) \right\rangle.\end{aligned} \tag{5}$$

In the above, the quantity $(1 - \phi(x_m)v_m)$ is the co-probability that a galaxy is located around x_m. Taking the continuous limit $M \to \infty$ with fixed V, and therefore $O(Mv^2) = O(V^2/M)$, the void probability simply reduces to

$$\begin{aligned}\phi(0) &= \left\langle \prod_{x \in V} (1 - \phi(x)d^3x) \right\rangle \\ &= \left\langle \exp\left[\int_V \phi(x)d^3x \right] \right\rangle \\ &= Z[1].\end{aligned} \tag{6}$$

Similarly, the probability of finding one galaxy within a small volume d^3x_1 around the space position x_1 *and* finding no other galaxies is given by

$$\begin{aligned}&P(x_1; 0)d^3x_1 \\ &= \lim_{M\to\infty} \left\langle \phi(x_1)v_1 \prod_{m=2}^{M} (1 - \phi(x_m)v_m) \right\rangle \\ &= \lim_{M\to\infty} \left\langle \phi(x_1)v_1 \exp\left[-\sum_{m=1}^{M} \phi(x_m)v_m \right] + O(Mv^2) \right\rangle \\ &= \left\langle \phi(x_1)\exp\left[\int_V \phi(x)d^3x \right] \right\rangle d^3x_1\end{aligned}$$

(7)

In the same way, the general probability $P(x_1, \ldots, x_N; 0)$ for finding N galaxies at around each N points and no other galaxies is given by

$$\begin{aligned}&P(x_1, \ldots, x_N; 0)d^3x_1..d^3x_N \\ &= \left\langle \begin{matrix} \phi(x_1)..\phi(x_N)d^3x_1..d^3x_N \times \\ \displaystyle\prod_{x \in V - \{x_1, x_2, \cdots, x_N\}} (1 - \phi(x)d^3x) \end{matrix} \right\rangle \\ &= \left\langle \phi(x_1)..\phi(x_N)\exp\left[\int_V \phi(x)d^3x \right] \right\rangle d^3x_1..d^3x_N\end{aligned}$$

(8)

These quantities should be clearly distinguished from $P(x_1, \ldots, x_N)$, which is the probability to find N galaxies without mentioning the existence of other galaxies. This probability is given by the expansion of the partition function $Z[J]$:

$$Z[J] = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int \begin{matrix} \langle \phi(x_1)\phi(x_2), \ldots \phi(x_l)\rangle \\ \times J(x_1)J(x_2), \ldots J(x_l) \\ \times dV_1dV_2..dV_l \end{matrix} \tag{9}$$

That is, in general,

$$\begin{aligned}P(x_1, \ldots, x_N) &\equiv \langle \phi(x_1)..\phi(x_N)\rangle \\ &= (-1)^N \left. \frac{\partial^N Z[J]}{\partial J(x_1)\ldots\partial J(x_N)} \right|_{J\to 0}\end{aligned} \tag{10}$$

On the other hand, our probability is more directly related with $P(x_1, \ldots, x_N; 0)$, which can be expressed as

$$\begin{aligned}P(x_1, \ldots x_N; 0) &= \left\langle \phi(x_1)..\phi(x_N)e^{-\int_V \phi} \right\rangle \\ &= (-1)^N \left. \frac{\partial^N Z[J]}{\partial J(x_1)\ldots\partial J(x_N)} \right|_{J\to 1}.\end{aligned} \tag{11}$$

Actually the probability of finding exactly N galaxies within the volume V is given by

$$\begin{aligned}\phi(N) &= \frac{1}{N!}\int_{V_1}\cdots\int_{V_N} P(x_1, x_2, \ldots x_N; 0) \\ &= \frac{(-1)^N}{N!}\int_{V_1}\cdots\int_{V_N} \left. \frac{\partial^N Z[J]}{\partial J(x_1)\ldots\partial J(x_N)} \right|_{J\to 1}.\end{aligned} \tag{12}$$

If we factor out the mean number density n from the density field $\phi(x)$ as $\phi(x) = n(1 + \delta(x))$, where $\delta(x)$ denotes deviation from the average, then we have

$$\int_V \left. \frac{\partial Z[J]}{\partial J(x)} \right|_{J\to 1} = n\frac{\partial Z[1]}{\partial n}, \tag{13}$$

and similarly

$$\int_{V_1}\cdots\int_{V_N} \left. \frac{\partial^N Z[J]}{\partial J(x_1)\cdots\partial J(x_N)} \right|_{J\to 1} = n^N \frac{\partial^N Z[1]}{\partial n^N}. \tag{14}$$

Using this form and Eqs.(6) (12), we finally have the compact expression

$$\phi(N) = \frac{(-n)^N}{N!}\frac{\partial^N}{\partial n^N}\phi(0), \tag{15}$$

which we now apply in the following arguments.

1411

3. VARIOUS THEORIES OF STATISTICAL MECHANICS

In this section, we apply the following four theories of statistical mechanics classified by (non)-extensive property and the long tails of the associated distribution function.

Table1. Four models of statistical mechanics are compared. All the combinations of the two basic properties are exhaustively included.

Theory	Extensive?	Long-tail?
Boltzmann	Yes	No
Fractal	No	No
Renyi	Yes	Yes
Tsallis	No	Yes

3.1. Boltzmann statistical mechanics

We first consider a simple model in which the galaxy distribution is supposed to obey the ordinary Boltzmann statistical mechanics with grand canonical ensemble. It is apparent that the genuine Boltzmann statistical mechanics is not suitable for the galaxy distributions. Therefore let us introduce a possible small deviation from it, and consider the virial parameter β which measures the deviation from the dynamical-equilibrium. Then we have the following expression for the pressure

$$pV = NT(1-\beta). \tag{16}$$

The distribution function of void is defined to be the probability of finding no galaxy in any part of the volume V [3],

$$f(0) = e^{-\bar{N}(1-\beta)}. \tag{17}$$

As is explained in the previous section, this is the generating functional of the general probability $f(N)$ the probability of finding N galaxies in the fixed volume V [2][1].

3.2. Fractal model

Motivated by the fact that various observational data suggest that the matter distribution in the Universe shows fractal nature, we investigate a simple mono-fractal distribution (α beeing the fractal dimension) model with the ordinary Boltzmann statistical mechanics. Reminding the number $\bar{N} = nV$ depends on the scale r in this fractal model, the void probability is given by

$$f(0) = e^{-\bar{N}} = \exp\left[-n\frac{\pi^{\alpha/2} r^{\alpha}}{\Gamma(\frac{\alpha}{2}+1)}\right]. \tag{18}$$

3.3. Rényi Statistical Mechanics

Rényi statistical mechanics is a generalization of the ordinary Boltzmann statistical mechanics by introducing a new form of entropy.

$$S[p] = \frac{1}{1-q}\ln\left(\sum_i p_i^q\right) \tag{19}$$

This Rényi entropy reduces, in the limit $q \to 1$, to the ordinary Boltzmann entropy. Because of the extensive property of this form, the total entropy of N galaxies is simply expressed as the sum of all the entropies of a single galaxy:

$$S_{N\,galaxy} = \sigma N. \tag{20}$$

The distribution function is given by

$$p_{N,E} = \left\{1 - \frac{1-q}{T}\left(E - \bar{E} - \mu\left(N - \bar{N}\right)\right)\right\}^{\frac{1}{1-q}}, \tag{21}$$

which maximizes the above Rényi entropy. This has a long-tail with power-law shape.

The void probability is given by

$$f(0) = \{1 + (1-q)N\sigma\}^{-1}. \tag{22}$$

We note that, since the effect of the change in the parameter σ can be absorbed into the change in the parameter q, this probability is essentially independent of σ.

3.4. Tsallis Statistical Mechanics

Tsallis proposed a non-extensive entropy [4]

$$S[P] = \frac{1}{1-q}\left(\left(\sum_i P_i^{\frac{1}{q}}\right)^{-q} - 1\right). \tag{23}$$

When we compose two independent systems A and B, the total Tsallis entropy satisfies the non-extensive relation:

$$S_{A+B} = S_A + S_B + (1-q)S_A S_B, \tag{24}$$

as is easily shown from the above entropy form. The distribution function

$$p_{N,E} = \frac{1}{\Xi_q}\left\{1 - \frac{1-q}{T}\left(E - \bar{E} - \mu\left(N - \bar{N}\right)\right)\right\}^{\frac{1}{1-q}} \tag{25}$$

maximizes the above Tsallis entropy. In the limit $q \to 1$, this expression reduces to the ordinary Boltzmann distribution as previously.

Since the theory is non-extensive, the total entropy of N galaxies is given by

$$S_{N\,galaxy} = \frac{\{1 + (1-q)\sigma\}^N - 1}{1-q}. \tag{26}$$

Using this non-extensive property, we obtain the final void probability as

$$f(0) = \{1 + (1-q)\sigma\}^{\frac{-N}{1-q}} \times [1 + N\ln\{1 + (1-q)\sigma\}]^{\frac{q}{1-q}}. \tag{27}$$

4. COMPARISON WITH OBSERVATION

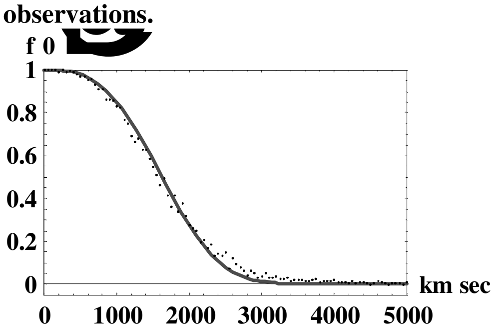
We use the observational data of SDSS DR3 spectroscopic catalog. (http://www.sdss.org/dr3) It is amazing that this data set includes 374,767 galaxies and the spectroscopic area is 4188 square degree. This is sufficient amount of data for us to select appropriate theory of statistical mechanics, as we will see soon.

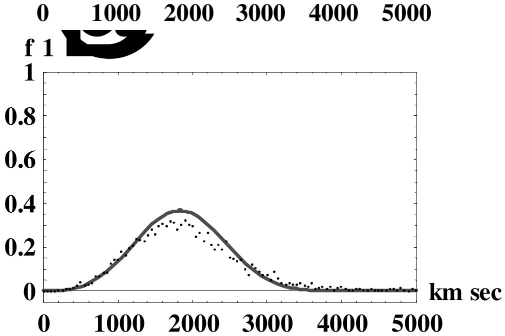
In order to cleanly apply the count-in-cell method, we need large extension of special volume for a single cell. Moreover, for the uniform data set excluding the boundary region of the total observation region, we use only the data region RA 150 - 210 degree and DEC 45 – 67 degree. From the magnitude-redshift diagram (Fig. 1), we choose the lower limit of the absolute magnitude -

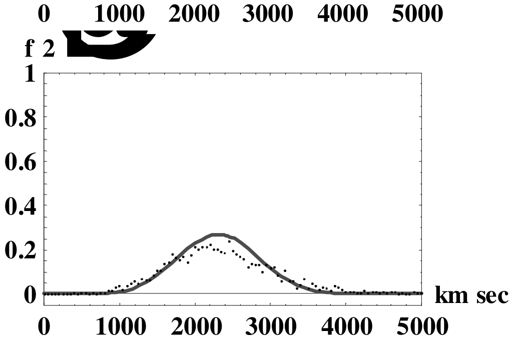
4.1. Boltzmann Statistical Mechanics

If we apply the Boltzmann statistical mechanics to the SDSS data set, we obtain the value b=0.98 as the best fit for $f(0)$ within this theory. This fact (b » 1) means the observation does not allow the generalization of genuine Boltzmann statistical mechanics toward the deviation from the dynamical equilibrium. We fix this b and plotted the fit for higher probabilities in Fig. 2.

Fig.2. Various probabilities in Boltzmann statistical mechanics. Dots are calculated from the SDSS observations.

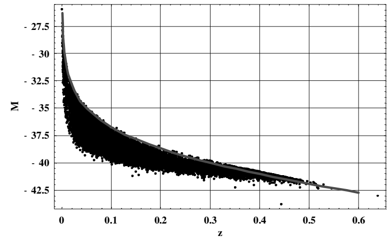


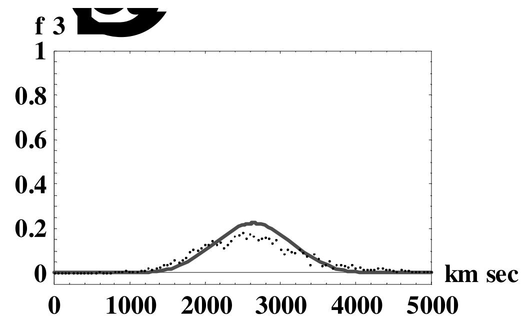


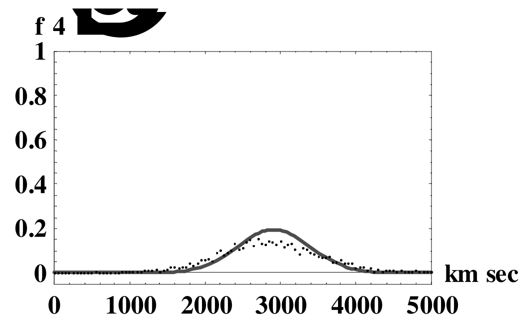


37.83 and the redshift distance z < 0.16. This constructs a volume limited sample. K-correction is also included.

Fig. 1: A diagram of absolute-magnitude v.s. redshift distance. The red line is the detection limit of the observation.



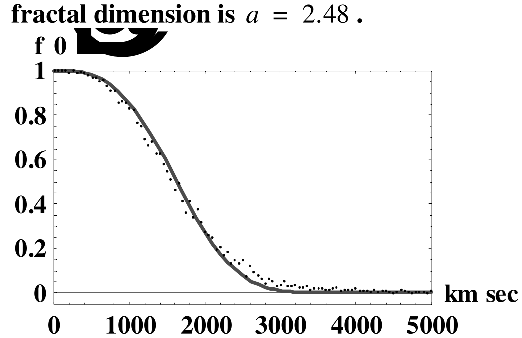


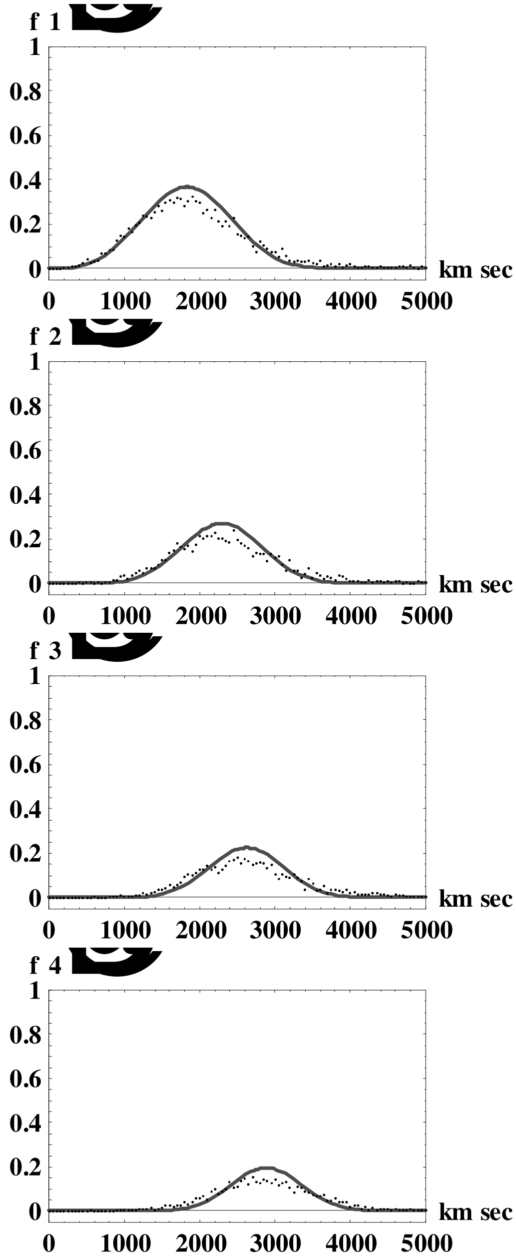


4.2. Fractal model

If we apply the fractal model to SDSS data set, the fractal dimension a turns out to be $a = 2.48$. Various probabilities are plotted in Fig.3.

Fig.3. Various probabilities in fractal model. The fractal dimension is $a = 2.48$.

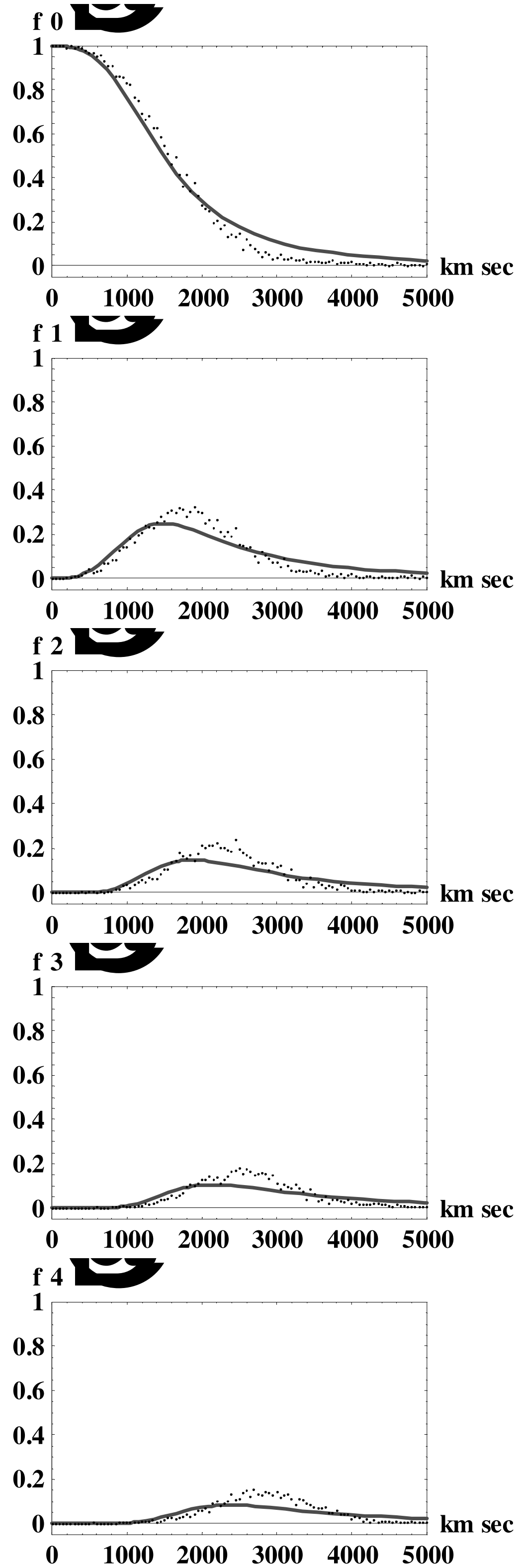




4.3. Rényi Statistical Mechanics

If we apply the Rényi statistical mechanics to the SDSS data set, we obtain the parameter value $q = 0.97$. as the best fit for $f(0)$ within this theory. This fact (q » 1) means the observation does not allow the generalization of genuine Boltzmann statistical mechanics holding the additivity. We fix this q and plotted the higher probabilities in Fig. 4.

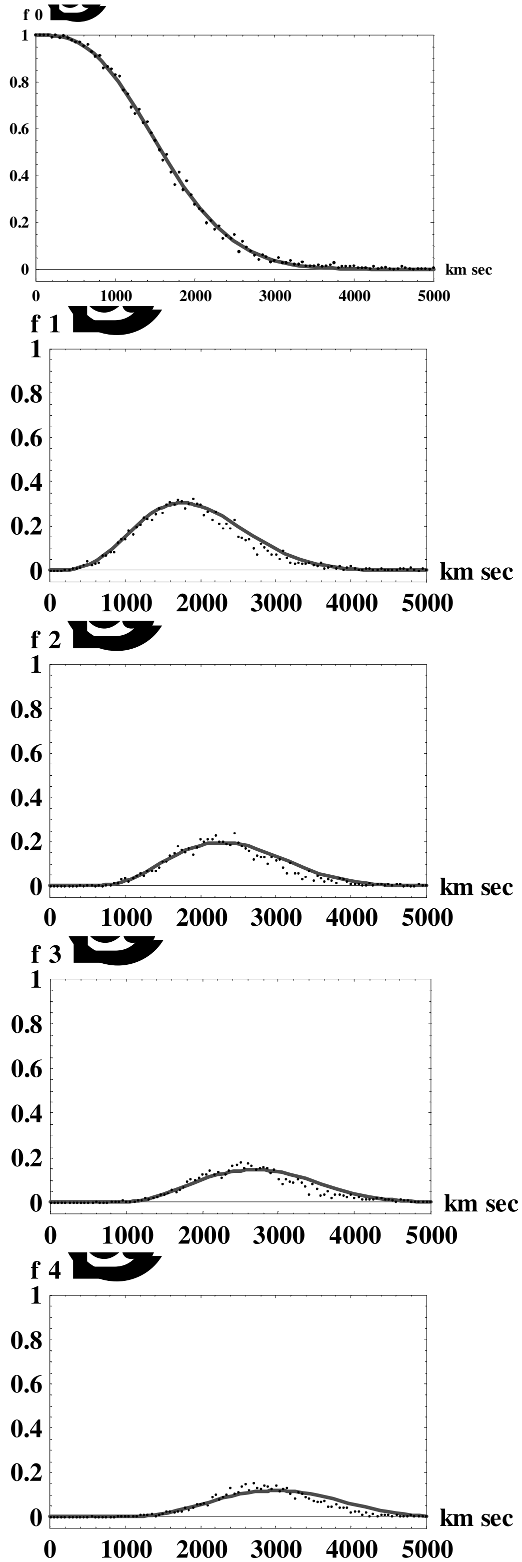
Fig.4. Various probabilities in Rényi statistical mechanics. The best parameter is $q = 0.97$.



4.4. Tsallis Statistical Mechanics

If we apply Tsallis statistical mechanics to the SDSS data set, we obtain the values $q = -1.84$ and $s = 0.0081$ as the best fit for $f(0)$ within this theory. We fix these parameters and plot the higher probabilities in Fig. 5.

Fig.5. Various probabilities in Tsallis statistical mechanics. The best fit parameters are $q = -1.84$ **and** $s = 0.0081$.



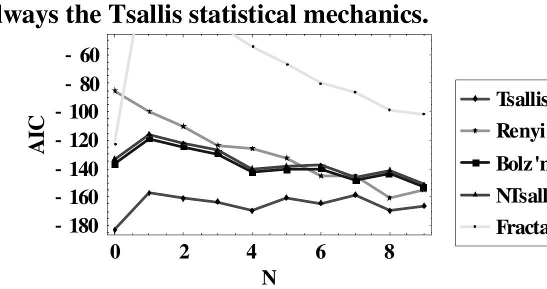
5. COMPARISON OF MODELS USING AIC

In the above, we have studied four kinds of statistical mechanics, in which the number of free parameters are different from each other. Despite this fact, we need a fair measure to choose the most appropriate theory of statistical mechanics to describe galaxy distributions. For this purpose, we introduce the Akaike Information Criteria (AIC)[5], in which the theory with much more number of parameters is imposed penalty. AIC measure is defined as

$$\begin{aligned} AIC &= (\#\,\text{datapoints}) \times (1 + \log_{10} 2\pi(\text{variance})) \\ &\quad + 2 \times (1 + \text{number of free parameters}) \end{aligned} \tag{28}$$

and the better model has the smaller AIC measure.

Fig.6. The AIC measure for various theories. We have used all the data $f(0), f(1), \ldots, f(9)$. **The bet fit is always the Tsallis statistical mechanics.**



From the result in Fig.6., it is apparent that the Tsallis statistical mechanics always has the smallest AIC measure, i.e. the most appropriate theory. Therefore we conclude that both the non-extensive property and the existence of long-tail in distribution function are essential for describing the distribution of galaxies. All the other theories of statistical mechanics, with only a single property at most, have been rejected by SDSS analysis.

The next step toward our problem would be to clarify the physical nature of these two properties and the theoretical justification of Tsallis statistical mechanics as well as the derived values of parameters. We would like to report these analysis soon in separate publications.

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