Cosmological Perturbations in Generalized Gravity Theories Including Tachyon

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The cosmological perturbation theory including tachyonic condensation was presented. Both of the scalar- and the tensor-type perturbations were considered. The power spectra based on the slow-roll inflations were derived.

1. INTRODUCTION

Cosmological perturbation theory [1] is important to investigate the large-scale structures in the universe and the cosmic microwave background radiation. Although Einstein's gravity theory has been succesful to study the perturbations using the various fluids and fields as the energy-momentum content, the relativistic gravity theories more general than Einstein's gravity are needed. These include variants of Einstein's gravity and more generalized forms with natural correction terms which appear in the quantum corrections or in the attempt of unified theories like string/M-theory program. Here we present the generalized forms of gravity theories expressed as actions in eqs. (20,31,45). The classical evolution and quantum generation processes are shown in unified forms.

We set $c \equiv 1 \equiv \hbar$.

2. EQUATION

We consider the Robertson-Walker spacetime with the metric

$$ds^2 = -\left(1+2\alpha\right) dt^2 - 2a\beta_{,\alpha} dt dx^\alpha + a^2 \left(g^{(3)}_{\alpha\beta} + 2\varphi g^{(3)}_{\alpha\beta} + 2\gamma_{,\alpha|\beta} + 2C_{\alpha\beta} \right) dx^\alpha dx^\beta, \quad (1)$$

where $a(t)$ is the cosmic scale factor and $dt \equiv a d\eta$. α, β, γ and φ are scalar-type perturbed order variables and $C_{\alpha\beta}$ is a (transverse-tracefree) tensor-type variable. $C_{\alpha\beta}$ is based on $g^{(3)}_{\alpha\beta}$ and a vertical bar indicates a covariant derivative based on $g^{(3)}_{\alpha\beta}$. We introduce $\chi \equiv a\left(\beta + a\dot{\gamma}\right)$ which is spatially gauge-invariant combination. The vector-type perturbation is neglected.

The energy-momentum tensors considering an imperfect fluid form are

$$T^0_0 = -\left(\bar{\mu} + \delta\mu\right), \quad T^0_\alpha = -\frac{1}{k}\left(\mu + p\right) v_{,\alpha},$$
$$T^\alpha_\beta = \left(\bar{p} + \delta p\right)\delta^\alpha_\beta + \Pi^\alpha_\beta, \quad (2)$$

where Π^α_β is a tracefree anisotropic stress; Π^α_β is based on $g^{(3)}_{\alpha\beta}$. An overbar indicates the background order quantities. The entropic perturbation e is defined as

$$e \equiv \delta p - c_s^2 \delta\mu, \quad c_s^2 \equiv \dot{p}/\dot{\mu}. \quad (3)$$

The anisotropic stress is decomposed as

$$\Pi_{\alpha\beta} \equiv \frac{1}{a^2}\left(\Pi_{,\alpha|\beta} - \frac{1}{3} g^{(3)}_{\alpha\beta}\Delta\Pi\right) + \Pi^{(t)}_{\alpha\beta}. \quad (4)$$

The background evolution equations are

$$H^2 = \frac{8\pi G}{3}\mu - \frac{K}{a^2}, \quad \dot{H} = -4\pi G\left(\mu + p\right) + \frac{K}{a^2} \quad (5)$$

Using Einstein's equations and the energy and momentum conservation equations we derive following scalar-type perturbation equations [2, 3]

$$\kappa \equiv 3\left(-\dot{\varphi} + H\alpha\right) + \frac{k^2}{a^2}\chi, \quad (6)$$
$$-\frac{k^2 - 3K}{a^2}\varphi + H\kappa = -4\pi G\delta\mu, \quad (7)$$
$$\kappa - \frac{k^2 - 3K}{a^2}\chi = 12\pi G\frac{a}{k}(\mu + p)v, \quad (8)$$
$$\dot{\chi} + H\chi - \alpha - \varphi = 8\pi G\Pi, \quad (9)$$
$$\dot{\kappa} + 2H\kappa + \left(3\dot{H} - \frac{k^2}{a^2}\right)\alpha,$$
$$= 4\pi G\left(\delta\mu + 3\delta p\right), \quad (10)$$
$$\delta\dot{\mu} + 3H\left(\delta\mu + \delta p\right),$$
$$= \left(\mu + p\right)\left(\kappa - 3H\alpha - \frac{k}{a}v\right), \quad (11)$$
$$\frac{\left[a^4(\mu + p)v\right]^\cdot}{a^4(\mu + p)} = \frac{k}{a}(\alpha + \frac{\delta p}{\mu + p}$$
$$-\frac{2}{3}\frac{k^2 - 3K}{a^2}\frac{\Pi}{\mu + p}). \quad (12)$$

The gravitational wave evolution is given as

$$\ddot{C}^\alpha_\beta + 3H\dot{C}^\alpha_\beta + \frac{k^2 + 2K}{a^2}C^\alpha_\beta = 8\pi G\Pi^{(t)\alpha}{}_\beta. \quad (13)$$

2.1. Fluid

Using the Field-Shepley combination [5]

$$\Phi \equiv \varphi_v - \frac{K/a^2}{4\pi G(\mu + p)}\varphi_\chi, \quad (14)$$

we derive [6]

$$\dot\Phi = -\frac{H}{4\pi G(\mu+p)}\frac{c_s^2 k^2}{a^2}\varphi_\chi - \frac{H}{\mu+p}\left(e - \frac{2}{3}\frac{k^2}{a^2}\Pi\right), \quad (15)$$

$$\frac{H}{a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} = \frac{4\pi G(\mu+p)}{H}\Phi - 8\pi G H\Pi. \quad (16)$$

Combining eqs. (15,16) we derive the closed form second-order differential equations for both Φ and φ_χ

$$\frac{H^2 c_s^2}{a^3(\mu+p)}\left\{\frac{a^3(\mu+p)}{H^2 c_s^2}\left[\dot\Phi + \frac{H}{\mu+p}\left(e - \frac{2}{3}\frac{k^2}{a^2}\Pi\right)\right]\right\}^{\cdot} = -c_s^2\frac{k^2}{a^2}\left(\Phi - 2\frac{H^2}{\mu+p}\Pi\right), \quad (17)$$

$$\frac{\mu+p}{H}\left[\frac{H^2}{a(\mu+p)}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} + 8\pi G\frac{H^2}{\mu+p}\Pi\right]^{\cdot} = -c_s^2\frac{k^2}{a^2}\varphi_\chi - 4\pi G\left(e - \frac{2}{3}\frac{k^2}{a^2}\Pi\right). \quad (18)$$

For the tensor mode we have

$$\frac{1}{a^3}\left(a^3\dot C_{\alpha\beta}\right)^{\cdot} + \frac{k^2+2K}{a^2}C_{\alpha\beta} = 8\pi G\Pi^{(t)}_{\alpha\beta}. \quad (19)$$

2.2. Field

Considering an action for a minimally coupled scalar field [7–9]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2}\phi^{;c}\phi_{,c} - V(\phi)\right], \quad (20)$$

the gravitational field equation and the equation of motion are given as

$$G_{ab} = 8\pi G\left(\phi_{,a}\phi_{,b} - \frac{1}{2}\phi^{;c}\phi_{,c}g_{ab} - Vg_{ab}\right), \quad (21)$$

$$\phi^{;a}{}_{a} - V_{,\phi} = 0. \quad (22)$$

We have the same equations (5-12) with the following background and perturbed order fluid quantities

$$\mu \equiv \frac{1}{2}\dot\phi^2 + V, \quad p \equiv \frac{1}{2}\dot\phi^2 - V, \quad (23)$$

$$\delta\mu \equiv \dot\phi\delta\dot\phi - \dot\phi^2\alpha + V_{,\phi}\delta\phi,$$

$$\delta p \equiv \dot\phi\delta\dot\phi - \dot\phi^2\alpha - V_{,\phi}\delta\phi,$$

$$v \equiv \frac{k}{a}\frac{\delta\phi}{\dot\phi}, \quad \Pi = 0 = \Pi^{(t)\alpha}{}_{\beta}. \quad (24)$$

Eq. (14) becomes

$$\Phi \equiv \varphi_{\delta\phi} - \frac{K/a^2}{4\pi G\dot\phi^2}\varphi_\chi. \quad (25)$$

We derive

$$\dot\Phi = -\frac{H}{4\pi G\dot\phi^2}\frac{c_A^2 k^2}{a^2}\varphi_\chi, \quad (26)$$

$$\frac{H}{a}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot} = \frac{4\pi G\dot\phi^2}{H}\Phi, \quad (27)$$

where

$$c_A^2 \equiv 1 - (1 - c_s^2)\frac{3K}{k^2},$$

$$c_s^2 \equiv \frac{\dot p}{\dot\mu} = -1 - \frac{2\ddot\phi}{3H\dot\phi}. \quad (28)$$

Combining eqs. (26,27) gives

$$\frac{H^2 c_A^2}{a^3\dot\phi^2}\left[\frac{a^3\dot\phi^2}{H^2 c_A^2}\dot\Phi\right]^{\cdot} = -c_A^2\frac{k^2}{a^2}\Phi, \quad (29)$$

$$\frac{\dot\phi^2}{H}\left[\frac{H^2}{a\dot\phi^2}\left(\frac{a}{H}\varphi_\chi\right)^{\cdot}\right]^{\cdot} = -c_A^2\frac{k^2}{a^2}\varphi_\chi. \quad (30)$$

Comparing with the ideal fluid equations in eqs. (17,18) we have c_s^2 replaced by c_A^2. The c_A has the role of wave speed of the perturbed field and the simultaneously excited metric; interpretation of c_A as the wave speed is properly valid only for $K = 0$.

For the tensor mode, eq. (19) remains valid in the field situation with $\Pi^{(t)}_{\alpha\beta} = 0$.

2.3. Generalized $f(\phi, R)$ gravity

An action is [3, 4, 10–12]

$$S = \int d^4x\sqrt{-g}[\frac{1}{2}f(\phi,R) - \frac{1}{2}\omega(\phi)\phi^{;c}\phi_{,c} - V(\phi) + L_{(m)}]. \quad (31)$$

This action includes $f(R)$ gravity, the scalar-tensor theory, the non-minimally coupled scalar field, the induced gravity, etc. The gravitational field equation and the equation of motion are given as

$$G_{ab} = \frac{1}{F}[T^{(m)}_{ab} + \omega\left(\phi_{,a}\phi_{,b} - \frac{1}{2}\phi^{;c}\phi_{,c}g_{ab}\right) + \frac{1}{2}\left(f - RF - 2V\right)g_{ab} + F_{,a;b} - F^{;a}{}_{a}g_{ab}], \quad (32)$$

$$\phi^{;a}{}_{a} + \frac{1}{2\omega}\left(\omega_{,\phi}\phi^{;c}\phi_{,c} + f_{,\phi} - 2V_{,\phi}\right) = 0, \quad (33)$$

where $F \equiv \frac{\partial f}{\partial R}$.

Equations (5-13) remain valid with the following effective fluid quantities

$$8\pi G\mu = \frac{1}{F}\left(\frac{1}{2}\omega\dot\phi^2 + \frac{RF - f + 2V}{2} - 3H\dot F\right),$$

$$\begin{aligned}
8\pi G p &= \frac{1}{F}(\frac{1}{2}\omega\dot{\phi}^2 - \frac{RF - f + 2V}{2} + \ddot{F} \\
&\quad +2H\dot{F}), \qquad (34)\\
8\pi G \delta\mu &= \frac{1}{F}\Bigg[\omega\dot{\phi}\delta\dot{\phi} + \frac{1}{2}\left(\omega_{,\phi}\dot{\phi}^2 - f_{,\phi} + 2V_{,\phi}\right)\delta\phi \\
&\quad -3H\delta\dot{F} + \left(3\dot{H} + 3H^2 - \frac{k^2}{a^2}\right)\delta F \\
&\quad + \left(3H\dot{F} - \omega\dot{\phi}^2\right)\alpha + \dot{F}\kappa\Bigg], \\
8\pi G \delta p &= \frac{1}{F}\Bigg[\omega\dot{\phi}\delta\dot{\phi} + \frac{1}{2}\left(\omega_{,\phi}\dot{\phi}^2 + f_{,\phi} - 2V_{,\phi}\right)\delta\phi \\
&\quad +\delta\ddot{F} + 2H\delta\dot{F} + (-\dot{H} - 3H^2 \\
&\quad +\frac{2}{3}\frac{k^2 - 3K}{a^2})\delta F \\
&\quad -\dot{F}\dot{\alpha} - \left(\omega\dot{\phi}^2 + 2\ddot{F} + 2H\dot{F}\right)\alpha - \frac{2}{3}\dot{F}\kappa\Bigg], \\
8\pi G T^0_\alpha &= \frac{1}{F}[\frac{1}{a}\left(-\omega\dot{\phi}\delta\phi - \delta\dot{F} + H\delta F + \dot{F}\alpha\right)_{,\alpha}], \\
8\pi G \Pi^\alpha_\beta &= \frac{1}{F}[\frac{1}{a^2}\left(\nabla^\alpha\nabla_\beta - \frac{1}{3}\delta^\alpha_\beta\Delta\right)\left(\delta F - \dot{F}\chi\right) \\
&\quad -\dot{F}\dot{C}^\alpha_\beta], \qquad (35)
\end{aligned}$$

where

$$\begin{aligned}
R &= 6\left(2H^2 + \dot{H} + \frac{K}{a^2}\right), \qquad (36)\\
\delta R &= 2[-\dot{\kappa} - 4H\kappa + \left(\frac{k^2}{a^2} - 3\dot{H}\right)\alpha \\
&\quad +2\frac{k^2 - 3K}{a^2}\varphi] = \delta\mu - 3\delta p. \qquad (37)
\end{aligned}$$

Introducing

$$\Phi \equiv \varphi_{\delta\phi} - 2\frac{K}{a^2}\frac{F}{\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}}\Psi,$$
$$\Psi \equiv \varphi_\chi + \frac{\delta F_\chi}{2F} \qquad (38)$$

we can derive

$$\dot{\Phi} = -\frac{2HF + \dot{F}}{\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}}\frac{c_A^2 k^2}{a^2}\Psi, \qquad (39)$$

$$\frac{H + \frac{\dot{F}}{2F}}{aF}\left(\frac{aF}{H + \frac{\dot{F}}{2F}}\Psi\right)^{\cdot} = \frac{\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}}{2HF + \dot{F}}\Phi, \qquad (40)$$

where

$$c_A^2 \equiv 1 - \left(6 + \frac{2\frac{\ddot{\phi}}{\dot{\phi}} - 3\frac{\dot{F}}{F} + \frac{\dot{E}}{E}}{H + \frac{\dot{F}}{2F}}\right)\frac{K}{k^2},$$
$$E \equiv F\left(\omega + \frac{3\dot{F}^2}{2F\dot{\phi}^2}\right). \qquad (41)$$

Combining eqs. (39,40) gives

$$\begin{aligned}
&\frac{\left(H + \frac{\dot{F}}{2F}\right)^2 c_A^2}{a^3\left(\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}\right)}\left[\frac{a^3\left(\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}\right)}{\left(H + \frac{\dot{F}}{2F}\right)^2 c_A^2}\dot{\Phi}\right]^{\cdot} \\
&= -c_A^2\frac{k^2}{a^2}\Phi, \qquad (42)\\
&\frac{\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}}{HF + \frac{1}{2}\dot{F}}\left[\frac{\left(H + \frac{\dot{F}}{2F}\right)^2}{a\left(\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}\right)}\left(\frac{aF}{H + \frac{\dot{F}}{2F}}\Psi\right)^{\cdot}\right]^{\cdot} \\
&= -c_A^2\frac{k^2}{a^2}\Psi. \qquad (43)
\end{aligned}$$

Here c_A can be interpreted as a wave speed of the perturbed field as well as the simultaneously excited metric.

For the tensor mode we have

$$\frac{1}{a^3 F}\left(a^3 F\dot{C}_{\alpha\beta}\right)^{\cdot} + \frac{k^2 + 2K}{a^2}C_{\alpha\beta} = 0. \qquad (44)$$

This equation is valid for general algebraic function of $f(\phi, R)$.

2.4. Tachyonic generalization

Considering an action [13]

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}f(R, \phi, X) + L_{(m)}\right], \qquad (45)$$

where $X \equiv \frac{1}{2}\phi^{,c}\phi_{,c}$, we have the gravitational field equation and the equation of motion

$$\begin{aligned}
G_{ab} &= \frac{1}{F}[T^{(m)}_{ab} + \frac{1}{2}(f - RF)g_{ab} + F_{,a;b} \\
&\quad -F^{;a}{}_{;a}g_{ab} - \frac{1}{2}f_{,X}\phi_{,a}\phi_{,b}], \qquad (46)\\
&\quad (f_{,X}\phi^{,c})_{;c} = f_{,\phi}. \qquad (47)
\end{aligned}$$

Equations (5-13) remain valid with the following effective fluid quantities

$$\begin{aligned}
8\pi G\mu &= \frac{1}{F}\left(f_{,X}X + \frac{FR - f}{2} - 3H\dot{F}\right), \\
8\pi G p &= \frac{1}{F}(-\frac{FR - f}{2} + \ddot{F} + 2H\dot{F}), \qquad (48)\\
8\pi G\delta\mu &= \frac{1}{F}\Bigg[-\frac{1}{2}(f_{,\phi}\delta\phi + f_{,X}\delta X) \\
&\quad -\frac{1}{2}\dot{\phi}^2(F_{,X}\delta R + f_{,X\phi}\delta\phi + f_{,XX}\delta X) \\
&\quad -f_{,X}\dot{\phi}\delta\dot{\phi} - 3H\delta\dot{F} \\
&\quad +\left(3\dot{H} + 3H^2 - \frac{k^2}{a^2}\right)\delta F + \dot{F}\kappa
\end{aligned}$$

$$
\begin{aligned}
& + \left(3H\dot{F} + f_{,X}\dot{\phi}^2 \right) \alpha \Bigg], \\
8\pi G \delta p \;=\; & \frac{1}{F} \Bigg[\frac{1}{2} \left(f_{,\phi}\delta\phi + f_{,X}\delta X \right) + \delta\ddot{F} + 2H\delta\dot{F} \\
& + \left(-\dot{H} - 3H^2 + \frac{2}{3}\frac{k^2 - 3K}{a^2} \right) \delta F \\
& - \frac{2}{3}\dot{F}\kappa - \dot{F}\dot{\alpha} - 2\left(\ddot{F} + H\dot{F} \right)\alpha \Bigg], \\
8\pi G T^0_\alpha \;=\; & \frac{1}{F} [\frac{1}{a} \left(\frac{1}{2} f_{,X}\dot{\phi}\delta\phi - \delta\dot{F} + H\delta F + \dot{F}\alpha \right)_{,\alpha}], \\
8\pi G \Pi^\alpha_\beta \;=\; & \frac{1}{F} [\frac{1}{a^2} \left(\nabla^\alpha\nabla_\beta - \frac{1}{3}\delta^\alpha_\beta\Delta \right) \left(\delta F - \dot{F}\chi \right) \\
& - \dot{F}\dot{C}^\alpha_\beta], \qquad (49)
\end{aligned}
$$

where

$$
X = -\frac{1}{2}\dot{\phi}^2, \quad \delta X = -\dot{\phi}\delta\dot{\phi} + \dot{\phi}^2\alpha. \qquad (50)
$$

The equation of motion gives

$$
\frac{1}{a^3}\left(a^3 \dot{\phi} f_{,X} \right)^{\cdot} + f_{,\phi} = 0, \qquad (51)
$$

$$
\begin{aligned}
& f_{,X}[\delta\ddot{\phi} + \left(3H + \frac{\dot{f}_{,X}}{f_{,X}} \right) \delta\dot{\phi} + \frac{k^2}{a^2}\delta\phi \\
& + \dot{\phi}\left(3\dot{\varphi} - \dot{\alpha} - \frac{k^2}{a^2}\chi \right)] \\
& + 2 f_{,\phi}\alpha + \frac{1}{a^3}\left(a^3\dot{\phi}\delta f_{,X} \right)^{\cdot} + \delta f_{,\phi} = \delta T^{(c)}, \qquad (52)
\end{aligned}
$$

where

$$
\delta f = f_{,\phi}\delta\phi + f_{,X}\delta X + f_{,R}\delta R. \qquad (53)
$$

We introduce a more generalized form

$$
\Phi \equiv \varphi_{\delta\phi} - \frac{K}{a^2}\frac{2F}{X f_{,X} + \frac{3\dot{F}^2}{2F}}\Psi, \qquad (54)
$$

where Ψ is the same as in eq. (38). We can derive

$$
\dot{\Phi} = -\frac{2HF + \dot{F}}{X f_{,X} + \frac{3\dot{F}^2}{2F}} \frac{c_A^2 k^2}{a^2}\Psi, \qquad (55)
$$

$$
\frac{H + \frac{\dot{F}}{2F}}{aF}\left(\frac{aF}{H + \frac{\dot{F}}{2F}}\Psi \right)^{\cdot} = \frac{X f_{,X} + \frac{3\dot{F}^2}{2F}}{2HF + \dot{F}}\Phi, \qquad (56)
$$

where

$$
\begin{aligned}
c_A^2 \;\equiv\; & \frac{X f_{,X} + \frac{3\dot{F}^2}{2F}}{X f_{,X} + 2X^2 f_{,XX} + \frac{3\dot{F}^2}{2F}} \\
& \{ 1 - \Bigg[3 + \frac{X(f_{,X} + 2X f_{,XX}) + \frac{3\dot{F}^2}{2F}}{X f_{,X} + \frac{3\dot{F}^2}{2F}} \\
& (3 + \frac{\frac{\dot{X}}{X} - 3\frac{\dot{F}}{F} + \frac{\dot{E}}{E}}{H + \frac{\dot{F}}{2F}}) \Bigg] \frac{K}{k^2} \}, \qquad (57)
\end{aligned}
$$

$$
E \equiv -\frac{F}{2X}\left(X f_{,X} + \frac{3\dot{F}^2}{2F} \right). \qquad (58)
$$

Combining eqs. (55,56) we have

$$
\begin{aligned}
& \frac{\left(H + \frac{\dot{F}}{2F} \right)^2 c_A^2}{a^3\left(X f_{,X} + \frac{3\dot{F}^2}{2F} \right)} \left[\frac{a^3\left(X f_{,X} + \frac{3\dot{F}^2}{2F} \right)}{\left(H + \frac{\dot{F}}{2F} \right)^2 c_A^2}\dot{\Phi} \right]^{\cdot} \\
& = -c_A^2\frac{k^2}{a^2}\Phi, \qquad (59) \\
& \frac{X f_{,X} + \frac{3\dot{F}^2}{2F}}{HF + \frac{1}{2}\dot{F}} \left[\frac{\left(H + \frac{\dot{F}}{2F} \right)^2}{a\left(X f_{,X} + \frac{3\dot{F}^2}{2F} \right)} \left(\frac{aF}{H + \frac{\dot{F}}{2F}}\Psi \right)^{\cdot} \right]^{\cdot} \\
& = -c_A^2\frac{k^2}{a^2}\Psi. \qquad (60)
\end{aligned}
$$

c_A^2 differs clearly from $c_s^2 \equiv \dot{p}/\dot{\mu}$. Contrary to the minimally coupled scalar field and the generalized $f(\phi, R)$ gravity theory the wave speed is non-trivial even for $K = 0$.

For the tensor mode, eq. (44) remains the same.

3. CLASSICAL EVOLUTION

Our basic scalar-type perturbation equations can be written in the following forms

$$
\dot{\Phi} = -2x_1\frac{k^2}{a^2}\Psi, \qquad (61)
$$

$$
\frac{1}{x_2}\left(x_2\Psi \right)^{\cdot} = \frac{1}{2}x_3\Phi. \qquad (62)
$$

Introducing

$$
\begin{aligned}
& \bar{z} \equiv c_A z \equiv \sqrt{a x_2 x_3}, \quad c_A \equiv \sqrt{x_1 x_3}; \quad v \equiv z\Phi, \\
& u \equiv x_2\frac{1}{\bar{z}}\Psi, \qquad (63)
\end{aligned}
$$

we have

$$
v = \frac{2}{c_A\bar{z}}\left(\bar{z}u \right)', \quad u = -\frac{1}{2k^2}\frac{z}{c_A}\left(\frac{v}{z} \right)', \qquad (64)
$$

where a prime indicates a time derivative based on η with $dt \equiv a d\eta$. Then

$$
\begin{aligned}
& v'' + \left(c_A^2 k^2 - \frac{z''}{z} \right) v \\
& = a^2 z\left[\frac{1}{az^2}\left(az^2\dot{\Phi} \right)^{\cdot} + c_A^2\frac{k^2}{a^2}\Phi \right] = 0, \qquad (65) \\
& u'' + \left[c_A^2 k^2 - \frac{(1/\bar{z})''}{(1/\bar{z})} \right] u \\
& = \frac{a^2 x_2}{\bar{z}}\left\{ \frac{\bar{z}^2}{a x_2}\left[\frac{a}{\bar{z}^2}\left(x_2\Psi \right)^{\cdot} \right]^{\cdot} + c_A^2\frac{k^2}{a^2}\Psi \right\} = 0. (66)
\end{aligned}
$$

In these wave equations, c_A becomes the wave speed of the fluctuating fluid or field and the simultaneously excited metric.

For the tensor mode, we use

$$z_t \equiv a\sqrt{Q_t}, \quad v_t \equiv z_t \Phi, \tag{67}$$

with $\Phi = C_{\alpha\beta}$ or $h_{\ell\mathbf{k}}$, then

$$\begin{aligned} & v_t'' + \left(c_T^2 k^2 - \frac{z_t''}{z_t} \right) v_t \\ & = a^2 z_t \left[\frac{1}{a z_t^2} \left(a z_t^2 \dot{\Phi} \right)^{\cdot} + c_T^2 \frac{k^2}{a^2} \Phi \right] = 0. \end{aligned} \tag{68}$$

In the large-scale limits, with $c_A^2 k^2 \ll z''/z$ and $(1/\bar{z})''/(1/\bar{z})$

$$\begin{aligned} \Phi(k,\eta) &= \frac{1}{z} v = C(k)\{1 + k^2 [\int^\eta \bar{z}^2 \left(\int^\eta \frac{d\eta}{z^2} \right) d\eta \\ & \quad - \int^\eta \bar{z}^2 d\eta \int^\eta \frac{d\eta}{z^2}]\} - 2\tilde{d}(k) k^2 \int^\eta \frac{d\eta}{z^2}, \\ \Psi(k,\eta) &= \frac{\bar{z}}{x_2} u = C(k) \frac{1}{2x_2} \int^\eta \bar{z}^2 d\eta \\ & \quad + \tilde{d}(k) \frac{1}{x_2} \{1 + k^2 [\int^\eta \frac{1}{z^2} \left(\int^\eta \bar{z}^2 d\eta \right) d\eta \\ & \quad - \int^\eta \bar{z}^2 d\eta \int^\eta \frac{d\eta}{z^2}]\}. \end{aligned} \tag{69}$$

The C (d)-mode is relatively growing (decaying) in the expanding phase of the background world. So, to the leading order in the large-scale expansion the C-mode of Φ remains constant whereas the one of Ψ changes its behavior according to the background evolution. Ignoring the transient mode

$$\Phi(k,\eta) = C(k). \tag{70}$$

In the small-scale limits, with $c_A^2 k^2 \gg z''/z$ and $(1/\bar{z})''/(1/\bar{z})$, we have

$$v(k,\eta) = z\Phi = c_{v_1} e^{i c_A k \eta} + c_{v_2} e^{-i c_A k \eta}, \tag{71}$$

$$\begin{aligned} u(k,\eta) &= \frac{x_2}{\bar{z}} \Psi \\ &= \frac{i}{2k} (-c_{v_1} e^{i c_A k \eta} + c_{v_2} e^{-i c_A k \eta}), \end{aligned} \tag{72}$$

where we assumed $c_A =$ constant.

In case

$$z \propto |\eta|^q, \quad c_A^2 = \text{constant}, \tag{73}$$

we have

$$\frac{z''}{z} = \frac{q(q-1)}{\eta^2} \equiv \frac{n}{\eta^2}, \tag{74}$$

and exact solutions

$$\begin{aligned} \Phi(k,\eta) &= \frac{\sqrt{\pi|\eta|}}{2z} [c_1(k) H_\nu^{(1)}(c_A k|\eta|) \\ & \quad + c_2(k) H_\nu^{(2)}(c_A k|\eta|)], \end{aligned} \tag{75}$$

$$\begin{aligned} \Psi(k,\eta) &= -\frac{\sqrt{\pi|\eta|}}{2z} \frac{a c_A}{2k x_1} [c_1(k) H_{\nu-1}^{(1)}(c_A k|\eta|) \\ & \quad + c_2(k) H_{\nu-1}^{(2)}(c_A k|\eta|)], \end{aligned} \tag{76}$$

where

$$\nu \equiv \frac{1}{2} - q = \sqrt{n + \frac{1}{4}}. \tag{77}$$

4. QUANTUM GENERATION

The perturbed action is [8, 16]

$$\begin{aligned} \delta^2 S &= \frac{1}{2} \int a z^2 \left(\dot{\Phi}^2 - c_A^2 \frac{1}{a^2} \Phi^{|\alpha} \Phi_{,\alpha} \right) dt d^3x, \\ &= \frac{1}{2} \int \left(v'^2 - c_A^2 v^{|\alpha} v_{,\alpha} + \frac{z''}{z} v^2 \right) d\eta d^3x. \end{aligned} \tag{78}$$

In terms of the mode function we have [9, 14]

$$\begin{aligned} \Phi_{\mathbf{k}}(\eta) &= \frac{\sqrt{\pi|\eta|}}{2z} [c_1(k) H_\nu^{(1)}(c_A k|\eta|) \\ & \quad + c_2(k) H_\nu^{(2)}(c_A k|\eta|)], \end{aligned} \tag{79}$$

$$\begin{aligned} \Psi_{\mathbf{k}}(\eta) &= -\frac{\sqrt{\pi|\eta|}}{2z} \frac{a c_A}{2k x_1} [c_1(k) H_{\nu-1}^{(1)}(c_A k|\eta|) \\ & \quad + c_2(k) H_{\nu-1}^{(2)}(c_A k|\eta|)], \end{aligned} \tag{80}$$

where

$$|c_2(k)|^2 - |c_1(k)|^2 = 1. \tag{81}$$

4.1. Power spectra

We evaluate the power-spectrum based on a vacuum expectation value given by

$$\begin{aligned} \mathcal{P}_{\hat{\Phi}}(k,t) &\equiv \frac{k^3}{2\pi^2} \int \langle \hat{\Phi}(\mathbf{x}+\mathbf{r},t) \hat{\Phi}(\mathbf{x},t) \rangle_{\rm vac} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r \\ &= \frac{k^3}{2\pi^2} |\Phi_k(t)|^2. \end{aligned} \tag{82}$$

In the large-scale limit we have

$$\begin{aligned} & \mathcal{P}_{\hat{\Phi}}^{1/2}(\mathbf{k},\eta) \\ & = \frac{H}{2\pi} \frac{1}{aH|\eta|} \frac{\Gamma(\nu)}{\Gamma(3/2)} \left(\frac{k|\eta|}{2} \right)^{3/2-\nu} \frac{1}{c_A^\nu z/a}. \end{aligned} \tag{83}$$

For the tensor-type perturbation we have $\hat{\Phi} = \hat{C}^\alpha_\beta$ and we need additional $\sqrt{2}$ factor [15], with c_T replacing c_A, thus

$$\begin{aligned} \mathcal{P}_{\hat{C}_{\alpha\beta}}^{1/2}(\mathbf{k},\eta) &= \sqrt{16\pi G} \frac{H}{2\pi} \frac{1}{aH|\eta|} \frac{\Gamma(\nu_t)}{\Gamma(3/2)} \\ & \quad \left(\frac{k|\eta|}{2} \right)^{3/2-\nu_t} \frac{1/\sqrt{8\pi G}}{c_T^{\nu_t} z_t/a}. \end{aligned} \tag{84}$$

For $\nu = 0$ we we have additional $2\ln(c_A k|\eta|)$ factor.

In the super-horizon scale we identify [17]

$$\mathcal{P}_{\hat{\Phi}} \equiv \mathcal{P}_{\Phi}, \tag{85}$$

where

$$\begin{aligned} \mathcal{P}_{\Phi}(k,t) &\equiv \frac{k^3}{2\pi^2} \int \langle \Phi(\mathbf{x}+\mathbf{r},t)\Phi(\mathbf{x},t)\rangle_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r \\ &= \frac{k^3}{2\pi^2} |\Phi(k,t)|^2 \end{aligned} \tag{86}$$

is a power-spectrum based on the spatial averaging.

Since the growing modes of Φ are conserved in the large scale limit, the final classical power spectra of the large-scale structure and the gravitational wave $\mathcal{P}_{\Phi}$ is the same as the $\mathcal{P}_{\hat{\Phi}}$ generated from the quantum fluctuations in the early universe.

Spectral indices are defined as

$$n_S - 1,\ n_T \equiv \frac{\partial \ln \mathcal{P}_{\Phi}}{\partial \ln k}, \tag{87}$$

thus

$$\mathcal{P}_{\Phi} \propto k^{n_S - 1},\ k^{n_T}. \tag{88}$$

Assuming the simplest vacuum state, i.e., $c_2 = 1$ and $c_1 = 0$, etc., we have

$$n_S - 1,\ n_T = 3 - 2\nu = 2 + 2q. \tag{89}$$

For near Harrison-Zel'dovich spectra ($n_S - 1 \simeq 0 \simeq n_T$) the quadrupole anisotropy of the CMB is

$$\begin{aligned} \langle a_2^2 \rangle &= \langle a_2^2 \rangle_S + \langle a_2^2 \rangle_T \\ &= \frac{\pi}{75} \mathcal{P}_{\varphi_{\delta\phi}} + 7.74 \frac{1}{5} \frac{3}{32} \mathcal{P}_{C_{\alpha\beta}}, \end{aligned} \tag{90}$$

which is valid for $K = 0 = \Lambda$. The ratio between two types of perturbations is

$$r_2 \equiv \frac{\langle a_2^2 \rangle_T}{\langle a_2^2 \rangle_S} \simeq 3.46 \frac{\mathcal{P}_{C_{\alpha\beta}}}{\mathcal{P}_{\varphi_{\delta\phi}}}. \tag{91}$$

From eqs. (83,84) we have

$$r_1 \equiv \frac{\mathcal{P}_{C_{\alpha\beta}}}{\mathcal{P}_{\varphi_{\delta\phi}}} = 2 \left[\left(\frac{k|\eta|}{2} \right)^{\nu - \nu_t} \frac{\Gamma(\nu_t)}{\Gamma(\nu)} \frac{c_A^{\nu-1}}{c_T^{\nu_t}} \frac{\bar{z}}{z_t} \right]^2 \tag{92}$$

Therefore, if the background evolution during the quantum generation stage satisfies eq. (73) we can read the power spectra using eqs. (82,79). In the large-scale limit we have the power spectra in eqs. (83,84). It is noticeable that our results in §3 and 4 are generally valid in our generalized gravity theories in unified forms.

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