The ariational formalism in the Tetrad gravitational theory with non-linear lagrangians and field equations in the Weyl-Cartan spacetime

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Dilaton matter, considered as a mathematical model for dark matter, generates in spacetime the Weyl-Cartan geometry. We investigate into variational procedure in Weyl-Cartan spacetime with nonmetricity of the Weyl type for non-linear lagrangians of the most general type presented within tetrad gravity theory and obtain gravitational field equations. We use method of independent variation with respect to tetrad and the general nonholonomic connection obeyed Weyl condition with the help undetermined lagrangians multipliers. In order to validate obtained equations we use strong differential identities. Also we showed the equivalence of our variational procedure and over variational methods. The obtained equations are used to investigate into the singularity problem in the early Universe.

1. THE WEYL-CARTAN SPACETIME

The basic concept of modern physics coming from Einstein establishes/claims that spacetime geometrical structure compatible with the properties of matter filling spacetime. That means that the matter dynamics determines the metric and the connection of the spacetime manifold and in turn is determined by the spacetime geometric properties. Within Einstein Gravitational theory it was possible to develop relative astrophysics and cosmology in Riemann spacetime successfully describing the basic structures of observable Universe. However the difficulties of classical theoretical cosmology and up-to date state of art in observation cosmology result in new problems of fundamental physics. One way to solve these problems suggested by many authors is to generalize Gravitation theory to spaces with more complicated geometrical structure, i.e. Riemann-Cartan space with curvature and torsion and general metric-affine spaces with curvature, torsion and non-metricity, in particular, Weyl-Cartan space. Weyl-Cartan space stands for a connected differentiable manifold with a connection $\Gamma^a_{b\mu}$, Lorenz metric g_{ab} and obeying of the Weyl non-metricity condition:

$$\nabla_\mu g^{ab} = Q^{ab}{}_\mu = \frac{1}{4} g^{ab} Q_\mu, \quad Q_\mu = g_{ab} Q^{ab}{}_\mu \tag{1}$$

Vector Q_μ is called Weyl vector. CW_4 geometry is characterized by Weyl non-metricity (1), curvature $R^a{}_{b\mu\nu}$ and torsion $T^a{}_{b\mu}$.

2. FIELD EQUATIONS

We represent the total Lagrangian density 4-form of the theory as follows [1]

$$L = L_G + L_m + \frac{1}{2}\sqrt{-g}\Lambda^\mu{}_{ab}\left(Q^{ab}{}_\mu - \frac{1}{4} g^{ab} Q_\mu\right), \quad \Lambda^\mu{}_{ab} g^{ab} = 0 \tag{2}$$

In (2) the Lagrangian density $L_G = L_R + L_G^{(2)}$ is used as the sum of linear Gilbert-Einstein Lagrangian density $L_R = f_0\sqrt{-g}R\,(f_0 = 1/(2k),\ \kappa = 8\pi G,\ c = 1)$ generalized to CW_4 space, and the term $L_G^{(2)}$ is a sum of arbitrary invariants that are squares of curvature tensors $R^a{}_{b\mu\nu}$, torsion $T^a{}_{b\mu}$ and nonmetricity $Q^{ab}{}_\mu$. In (2) L_μ is a Lagrangian density of matter that includes all possible sources of gravitational field. In the considered theory Weyl-Cartan space has the original fundamental status without any connection to the metric-affine gravitation theory that's why Weyl's condition (1) is allowed for by including the theory (2) in Lagrangian density (2) with the help of the Lagrange multipliers $\Lambda^\mu{}_{ab}$ before variation procedure.

In CW_4 space in tetrad variational formalism of the first order (generalized Palatiny formalism) independent variables are coupling components in anholonomical orthogonal basis $\Gamma^a{}_{b\mu}$, tetrad potentials $h^a{}_\mu$ and the components of metric tensor of tangent space g^{ab} and also indeterminate Lagrange multipliers $\Lambda^\mu{}_{ab}$. The appropriate variation field equations are obtained by varying action integral with the total Lagrangian density (2) with respect to this system of dynamic variations [3]:

$$\frac{\delta L}{\delta \Gamma^a{}_{b\mu}} = 0, \quad \frac{\delta L}{\delta h^a{}_\mu} = 0, \quad \frac{\delta L}{\delta g^{ab}} = 0, \quad \frac{\delta L}{\delta \Lambda^\mu{}_{ab}} = 0. \tag{3}$$

The last equation (3) is Weyl's condition of nonmetricity (3). Allowing for this condition we obtane the following variational equations

$$f_0 P^{\mu b}{}_a - H^{\mu b}{}_a = J^{\mu b}{}_a + \Lambda^{\mu b}{}_a \tag{4}$$

$$f_0\left(R^\mu{}_a + \tilde{R}^\mu{}_a - h^\mu{}_a R\right) - H^\mu{}_a = \overset{(m)}{t}{}^\mu{}_a \tag{5}$$

$$f_0\left(R_{ab} - \frac{1}{2} g_{ab} R\right) + H_{ab} = \overset{(m)}{T}_{ab} + \frac{1}{2}\hat{\nabla}_\mu \Lambda^\mu{}_{ab} \tag{6}$$

where $\hat{\nabla}_\mu = \dot{\nabla}_\mu - \frac{1}{2}Q_\mu = \nabla_\mu + T_\mu - \frac{1}{2}Q_\mu$, $T_\mu = T^\nu{}_{\mu\nu}$.

In (4)-(6) we use the symbols according to [1]

$$\frac{\delta L_G^{(2)}}{\delta \Gamma^a{}_{b\mu}} = \sqrt{-g}H^{\mu b}{}_a, \quad \frac{\delta L_G^{(2)}}{\delta h^a{}_\mu} = \sqrt{-g}H^\mu{}_a, \quad \frac{\delta L_G^{(2)}}{\delta g^{ab}} = \sqrt{-g}H_{ab}.$$

In equation (4) tensor P is called Palatiny tensor that is given by

$$P^{\mu b}{}_a = M^{\mu b}{}_a - g^{bc} h^\mu{}_{[c} Q_{a]} + 2h^\mu_{[c} Q^{cb}{}_{a]}$$

where $M^\mu{}_{ba} = T^\mu{}_{ba} + 2h^\mu{}_{[a} T_{b]}$ is modified torsion tensor. The left parts of equations (5), (6) contain the following reduces of curvature tensor

$$R_{ab} = R^\sigma{}_{a\sigma b},\ R^\mu{}_a = h^\mu{}_b R^b{}_a,\ \tilde{R}^\mu{}_a = R^{\mu\sigma}{}_{a\sigma},\ R = g^{ab} R_{ab}.$$

The right parts of equations (4)-(6) contain tensors describing the sources of gravitational field $\sqrt{-g}\,\overset{(m)}{t}{}^\mu{}_a = \frac{\delta L_m}{\delta h^a{}_\mu}$ – canonical tensor energy-momentum, $\sqrt{-g}\,\overset{(m)}{T}{}^\mu{}_a = \frac{\delta L_m}{\delta g^{ab}}$ – metric tensor of energy-momentum, $\sqrt{-g}J^{\mu b}{}_a = \frac{\delta L_m}{\delta \Gamma^a{}_{b\mu}}$ tensor of dilaton-spin momentum [2].

By simmetrization of the first equation with respect to indexes a and b one can obtain the equation for indeterminate Lagrange multipliers

$$\Lambda^{\mu}{}_{ab} = f_0 P^{\mu}{}_{(ab)} - H^{\mu}{}_{(ab)} - J^{\mu}{}_{(ab)}$$

and eliminate indeterminate multipliers from (6). Then we antisimmetrizate the also take trace of this equation with respect of these indexes equation (4) with respect to indexes a and b and. As a result, taking into account the properties of Palatiny tensor [1]

$$\hat{\nabla}_{\lambda} P^{\lambda\mu}{}_{a} = R_{a}{}^{\mu} - \tilde{R}^{\mu}{}_{a},\ P^{\lambda a}{}_{a} = 0, \qquad (7)$$

we obtain the equation of gravitation field in tetrad gravitation theory in Weyl-Cartan space

$$f_0 P^{\mu}{}_{[ab]} - H^{\mu}{}_{[ab]} = J^{\mu}{}_{[ab]},\ H^{\mu a}{}_{a} = J^{\mu a}{}_{a} \qquad (8)$$

$$f_0\left(R^{\mu}{}_{a} + \tilde{R}^{\mu}{}_{a} - h^{\mu}{}_{a} R\right) - H^{\mu}{}_{a} = \overset{(m)}{t}{}^{\mu}{}_{a} \qquad (9)$$

$$f_0\left(R_{(ab)} - \frac{1}{2}\hat{\nabla}_{\mu} P^{\mu}{}_{(ab)} + \frac{1}{2} g_{ab} R\right) + H_{ab} + \frac{1}{2}\hat{\nabla}_{\mu} H^{\mu}{}_{(ab)} = \overset{(m)}{T}{}_{ab} - \frac{1}{2}\hat{\nabla}_{\mu}\Lambda^{\mu}{}_{(ab)} \qquad (10)$$

The direct calculation proves that equation (10) is the consequence of equations (8), (9) and identity (7).

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