New Numerical Methods to Evaluate Homogeneous Solutions of the Teukolsky Equation

R. Fujita, H. Tagoshi
Osaka University, Toyonaka 560-0043, Japan

We discuss a numerical method to compute the homogeneous solutions of the Teukolsky equation. We use the formalism developed by Mano, Suzuki and Takasugi, in which the homogeneous solutions of the radial Teukolsky equation are expressed in terms of two kinds of series of special functions, and the formulas for the asymptotic amplitudes are derived explicitly. In this formalism, we have to solve the continued fraction equation and determine the "renormalized angular momentum parameter". Although this parameter has been assumed to be real number, we find that it becomes complex number as the angular frequency becomes large. We check that these solutions of the continued fraction equation can be used to determine the homogeneous solution of the Teukolsky equation. We also compute the energy flux of the gravitational waves from a particle in circular orbits on the equatorial plane around a Kerr black hole and compare our results with a direct numerical integration method. We find that both methods produce consistent results. These facts prove the validity to use the complex solutions of the continued fraction equation to describe the homogeneous solutions of the Teukolsky equation.

1. INTRODUCTION

Laser Interferometer Space Antenna (LISA) is expected to be launch in 2012. LISA has the best sensitivity around mHz band and it can detect gravitational waves from inspirals of white dwarfs, neutron stars or solar mass black holes into a supermassive black hole and so on. Gair et al.[1] estimated the number of such event rates are expected to be about 600 by 3 years observation for the inspirals of $10M_\odot$ black holes into $10^6M_\odot$ supermassive black hole. If we can detect and observe such gravitational waves, we may extract information of these systems such as distance to the source, masses of binary, spin of the star, geometry of black hole space time and so on. In the matched filtering technique, which requires theoretical waveforms to be correlated with the data, we will need very accurate theoretical waveforms to extract astrophysical information of the source although we may not need it in detection.

To predict the waveforms of the extreme mass ratio inspirals, we adopt the black hole perturbation approach. In this approach, a supermassive black hole is taken to be a background (Kerr black hole), and we deal with a spiraling compact object as a source of the perturbation. In this context, the Teukolsky equation describes the evolution of a perturbation of the Kerr black hole spacetime. The standard approach to solve the Teukolsky equation is based on the Green function method. The Green function is expressed by two kinds of homogeneous solutions of the Teukolsky equation. The solution of the Teukolsky equation is obtained by the integration of Green's function multiplied by the source term, which is given by specifying the orbit of the point particle. The orbit of the particle is specified by the three constants of motion, the energy, the z-component of the angular momentum, and the Carter constant. When the point particle moves eccentric orbit on the equatorial plane, the orbits show "zoom-whirl" property as the eccentricity becomes large[2]. If the orbit of the point particle become more complicated, as in the standard case of the sources for LISA, we have to trace the orbit for much longer time than the dynamical time of the system in order to integrate the source term multiplied by the Green function with good accuracy. Although the accuracy of 10^{-5}, established in many previous works, may be sufficient to detect gravitational waves from the extreme mass ratio inspirals, it would be very helpful for the future data analysis of LISA if we had more efficient and accurate methods to compute the homogeneous solutions of the Teukolsky equation to calculate gravitational waves.

In Ref.[3], we adopt a formalism, originally developed by Leaver[4] and elegantly reformulated by Mano, Suzuki and Takasugi (MST formalism)[5, 6], in which the homogeneous solutions of the Teukolsky equation are expressed in series of special functions. We calculated the gravitational wave flux induced by a particle in a circular orbit on the equatorial plane around a Kerr black hole within accuracy at least 15 significant figures. And we showed the validity to use MST formalism to solve the Teukolsky equation. However when gravitational wave frequency became large, we could not determine "renormalized angular momentum parameter", ν, which is one of the most important variable in MST's formalism.

In this paper, we investigate the solution of the continued fraction equation by which we determine ν. Although this parameter has been believed to be real number, we find that we can find it in the complex region. Using these complex ν, we compute the energy flux of the gravitational waves from a particle in circular orbits on the equatorial plane around a Kerr black hole and compare these results with a direct numerical integration method. We find that both results are consistent. These facts prove the validity to use the complex solutions to describe the homogeneous solutions of the Teukolsky equation. Throughout this paper we use units with $G = c = 1$.

2. HOMOGENEOUS SOLUTIONS OF THE TEUKOLSKY EQUATION

The homogeneous Teukolsky equation is given by

$$\Delta^2 \frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{\ell m\omega}}{dr}\right) - V(r)R_{\ell m\omega} = 0. \qquad (1)$$

The potential $V(r)$ is given by

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda. \qquad (2)$$

Here $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$ with $r_\pm = M \pm \sqrt{M^2 - a^2}$, $K = (r^2 + a^2)\omega - ma$ and λ is the eigenvalue of the angular Teukolsky equation.

In the MST's formalism, the homogeneous solutions of the Teukolsky equation are expressed in terms of special functions [5, 6]. We consider a homogeneous solution, $R^{\rm in}_{lm\omega}$, which has purely ingoing wave property at the horizon, but has both ingoing wave property and outgoing property at infinity.

We transform the incoming solution $R^{\rm in}_{lm\omega}$ as

$$R^{\rm in}_{\ell m\omega} = e^{i\epsilon\kappa x}(-x)^{-s-i(\epsilon+\tau)/2}(1-x)^{i(\epsilon-\tau)/2}p_{\rm in}(x). \qquad (3)$$

We define $p_{\rm in}$ by

$$p_{\rm in}(x) \;=\; \sum_{n=-\infty}^{\infty} a_n F(a_n^\nu, b_n^\nu; c_n^\nu; x). \qquad (4)$$

where $x = \omega(r_+ - r)/\epsilon\kappa$, $\epsilon = 2M\omega, \kappa = \sqrt{1-q^2}, q = \frac{a}{M}, \tau = \frac{\epsilon - mq}{\kappa}$, $a_n^\nu = n + \nu + 1 - i\tau$, $b_n^\nu = -n - \nu - i\tau$, $c_n^\nu = 1 - s - i\epsilon - i\tau$ and $F(\alpha, \beta; \gamma; x)$ is hypergeometric function. We note that "renormalized angular momentum" ν does not appear in the original Teukolsky equation. This parameter is introduced in order to converge the series expansion Eq.(4) and determined later.

Substituting the Eq.(3) into Eq.(1), we find that the expansion coefficients of the series of hypergeometric functions $\{a_n\}$ satisfy three-term recurrence relation given by

$$\alpha_n^\nu a_{n+1} + \beta_n^\nu a_n + \gamma_n^\nu a_{n-1} = 0, \qquad (5)$$

where

$$\alpha_n^\nu \;=\; \frac{i\epsilon\kappa(n+\nu+1+s+i\epsilon)}{(n+\nu+1)} \times \frac{(n+\nu+1+s-i\epsilon)(n+\nu+1+i\tau)}{(2n+2\nu+3)}, \qquad (6)$$

$$\beta_n^\nu \;=\; -\lambda - s(s+1) + (n+\nu)(n+\nu+1) + \epsilon^2 + \epsilon(\epsilon - mq) + \frac{\epsilon(\epsilon - mq)(s^2+\epsilon^2)}{(n+\nu)(n+\nu+1)}, \qquad (7)$$

$$\gamma_n^\nu \;=\; -\frac{i\epsilon\kappa(n+\nu-s+i\epsilon)}{} \times \frac{(n+\nu-s-i\epsilon)(n+\nu-i\tau)}{(n+\nu)(2n+2\nu-1)}. \qquad (8)$$

In MST formalism, solving the Teukolsky equation is reduced to determine a parameter ν. Dividing Eq.(5) by a_n and setting $n = 0$, we obtain the following equation.

$$g_0(\nu) \equiv \beta_0^\nu + \alpha_0^\nu R_1 + \gamma_0^\nu L_{-1} = 0, \qquad (9)$$

where R_1 and L_{-1} are expressed by the continued fractions,

$$\begin{aligned} R_n(\nu) \;&\equiv\; \frac{a_n}{a_{n-1}} = \frac{-\gamma_n^\nu}{\beta_n^\nu + \alpha_n^\nu R_{n+1}} \\ &= \frac{-\gamma_n^\nu}{\beta_n^\nu - \dfrac{\alpha_n^\nu\gamma_{n+1}^\nu}{\beta_{n+1}^\nu - \dfrac{\alpha_{n+1}^\nu\gamma_{n+2}^\nu}{\beta_{n+2}^\nu - \cdots}}}, \end{aligned} \qquad (10)$$

$$\begin{aligned} L_n(\nu) \;&\equiv\; \frac{a_n}{a_{n+1}} = \frac{-\alpha_n^\nu}{\beta_n^\nu + \gamma_n^\nu L_{n-1}} \\ &= \frac{-\alpha_n^\nu}{\beta_n^\nu - \dfrac{\alpha_{n-1}^\nu\gamma_n^\nu}{\beta_{n-1}^\nu - \dfrac{\alpha_{n-2}^\nu\gamma_{n-1}^\nu}{\beta_{n-2}^\nu - \cdots}}}. \end{aligned} \qquad (11)$$

The minimal solution sequence of the expansion coefficients $\{a_n\}$ will be derived from Eqs.(10) and (11) if "renormalized angular momentum" ν is a root of the continued fraction equation Eq.(9). The problem is now to seek ν which satisfy Eq.(9).

3. SOLUTIONS OF THE CONTINUED FRACTION EQUATION

We can determine a parameter ν by solving the continued fraction equation Eq.(9). When $\epsilon = 2M\omega$ is small, there is an analytic expression of a solution ν in the form of a series of ϵ given by

$$\begin{aligned} \nu \;=\; l + \frac{1}{2l+1}\Bigg[&-2 - \frac{s^2}{l(l+1)} - \frac{(l^2-s^2)^2}{(2l-1)2l(2l+1)} \\ &+ \frac{[(l+1)^2 - s^2]^2}{(2l+1)(2l+2)(2l+3)}\Bigg]\epsilon^2 + O(\epsilon^3). \end{aligned} \qquad (12)$$

Thus, when $\epsilon \to 0$, ν takes integer value, $\nu = l$. As an example case, we consider $l = m = 2$, $q = 0$. As ω increases from 0, the solution ν decreases from l and approaches $l - 1/2$ at around $M\omega = 0.36$. If $M\omega$ is larger than 0.36, the real solution ν disappear. The function $g_0(x)$ is shown in Fig.1 in the cases $M\omega \leq 0.36$ and in Fig.2 in the cases $M\omega > 0.36$. At first glance, in Fig.2, there seems to be a solution at $x = l - 1/2 = 3/2$, $x = 1$ or $x = 2$. These are solutions of $g_0(x) = 0$ mathematically, but they don't produce the homogeneous solution of the Teukolsky equation.

Next we consider the solution of $g_0(z) = 0$ in the complex plane of z in the case $M\omega$ is above the critical value and we can not find the real solution. In Fig.3, we plot a contour of $|g_0(z)|$ in the case $M\omega = 0.5$. We find that there is a minimal around $\mathrm{Re}(z) = 3/2$ and $\mathrm{Im}(z) = 0.36$. In Fig.4, we plot $\mathrm{Re}(g_0(z))$ and $\mathrm{Im}(g_0(z))$ as a function of $\mathrm{Im}(z)$ at $\mathrm{Re}(z) = 3/2$. It is evident that there is a solution at around $z = 3/2 + 0.36i$. The precise value of the solution is $z = 3/2 + 0.36188061539416i$. It is also suggested from Fig.3 that there are no solutions other than this value. We find that when there are no real solutions, complex solution always exists. Further, the real part of the complex solutions are always half-integer or integer. These property is the same in the case of other parameters, s, l, m and q.

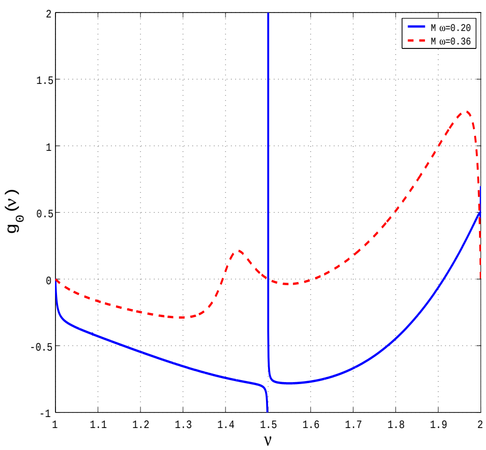

Figure 1: The function $g_0(x)$ for $M\omega = 0.20$ and 0.36. $s = -2$ and $q = 0$. There are real solutions in these cases.

4. NUMERICAL RESULTS

In this section, we confirm the results of Sec.3 by computing the energy flux of the gravitational waves induced by a test particle orbiting in circular and equatorial plane around a Kerr black hole. The complete formulas are given in Appendix A of [3]. We compare our results, in the case $r_0 = 1.55M, q = 0.99$, and ν is complex, with that of Kennefick[1], which are

[1]This data was calculated by D. Kennefick based on his previous work[2], and was kindly provided for us.

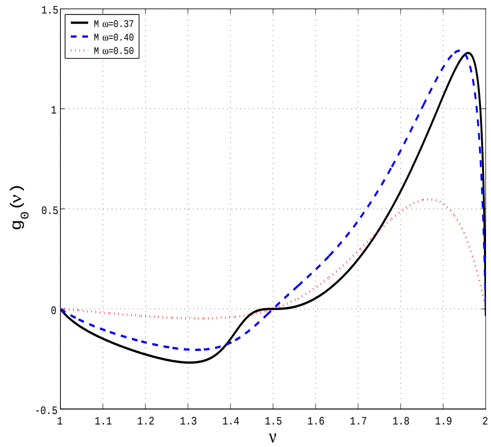

Figure 2: The function $g_0(x)$ for $M\omega = 0.37$, 0.40 and 0.50. $s = -2$ and $q = 0$. There are no real solution in these cases.

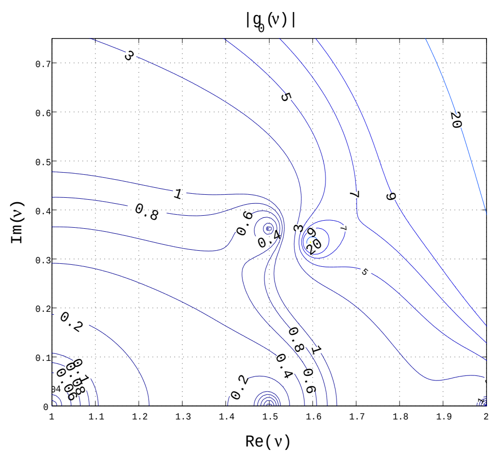

Figure 3: The $|g_0(z)|$ in the case $M\omega = 0.50$, $s = -2$, $l = m = 2$ and $q = 0$. There is a complex solution at $\nu = 1.5 + 0.36188061539416i$.

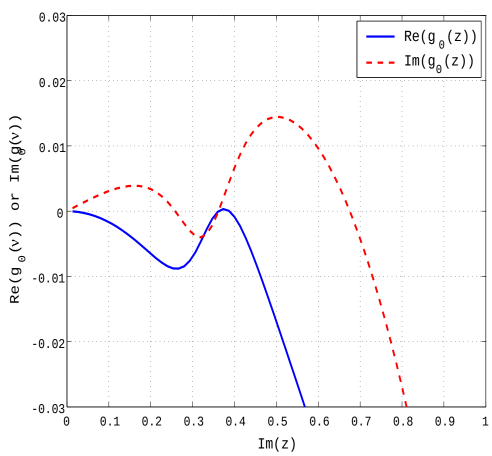

Figure 4: The real and imaginary part of the function $g_0(z)$ at $\mathrm{Re}(z) = 1.5$ in the case $M\omega = 0.50$, $s = -2$, $l = m = 2$ and $q = 0$. The real and imaginary part become 0 at $\mathrm{Im}(z) = 0.36188061539416$.

Table I Relative error of the energy flux, up to $\ell = 7$, between Ref.[2] and our results when $r_0 = 1.55M, q = 0.99$, and ν is complex.

l	$\mid m \mid$	Re(ν)	Im(ν)	Numerical integration	This paper	Relative error
2	2	1.5	1.1374192131794	$3.568050135 \times 10^{-2}$	$3.568033154338851 \times 10^{-2}$	4.76×10^{-6}
3	3	2.5	1.4249576682707	$2.152962111 \times 10^{-2}$	$2.152959342790158 \times 10^{-2}$	1.29×10^{-6}
4	4	3.5	1.7677955367662	$1.230541176 \times 10^{-2}$	$1.230541952573211 \times 10^{-2}$	6.31×10^{-7}
5	4	4.5	0.3735768429955	$1.933923538 \times 10^{-5}$	$1.933924400940079 \times 10^{-5}$	4.46×10^{-7}
5	5	4.5	2.1437000387424	$7.259841388 \times 10^{-3}$	$7.259849874157195 \times 10^{-3}$	1.17×10^{-6}
6	5	5.5	0.8032470628900	$1.536518877 \times 10^{-5}$	$1.536520289551414 \times 10^{-5}$	9.19×10^{-7}
6	6	5.5	2.5395166388946	$4.404590776 \times 10^{-3}$	$4.404599359654937 \times 10^{-3}$	1.95×10^{-6}
7	6	6.5	1.2521347909128	$1.148793520 \times 10^{-5}$	$1.148795883734041 \times 10^{-5}$	2.06×10^{-6}
7	7	6.5	2.9481395894691	$2.726943363 \times 10^{-3}$	$2.726949666682515 \times 10^{-3}$	2.31×10^{-6}

computed by numerical integration, in Table I. We find that our numerical results agree with that of [2] for 5-7 significant figures, which are consistent with the accuracy of the data by the numerical integration method. This fact shows the validity to use the complex ν to calculate the homogeneous solutions of the Teukolsky equation.

5. SUMMARY AND CONCLUDING REMARKS

We investigated solution of the continued fraction equation, which is derived by Leaver and Mano et al. in the formalism deal with the homogeneous solutions of the Teukolsky equation, and determine the "renormalized angular momentum" ν. Although we could not find ν in real region when gravitational wave frequency becomes large, we can find it in the complex region. In order to verify the existence of complex ν, we compute the power radiated by gravitational waves from a particle in circular orbit in equatorial plane around a Kerr black hole using complex ν and compare our results with that of a direct numerical integration method. We find that relative errors of both methods are about 10^{-5}. Thus, the range of the gravitational frequency in which the MST formalism can be used for numerical analysis becomes much wider than that where ν is assumed to be real. We will apply MST's formalism using the complex solutions of ν to compute the gravitational waves from a compact star in generic orbits around a supermassive black hole in the future.

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