Approach to the Cosmological Constant Problem with a Diluting Mechanism by Extra-Dimensions

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The value of the cosmological constant has a gap of order 10^{-120} between observations and the standard theory. It is called the cosmological constant problem (CCP), which has been considered hard to overcome while few ten's of years. Recently a mechanism of diluting the cosmological constant by an extra dimensional effect has suggested by Dvali et al. [4]. That mechanism is that the crude cosmological constant is a Planck scale and exists in the 4 plus extra dimensional entire space, but we observe this diluted in the 4-dimensional brane embedded. Cho and Vilenkin have studied the 3-dimensional spherically symmetric extra space to examine the Dvali et al.'s diluting mechanism [5]. But out approach is observing the dependence of Hubble expansion rate H on the number of extra-dimensions n. So we have extended their model to an arbitrary $(4+n)$-dimensional one, and we have found new solutions numerically to get the relation between n and H, We have concluded that $n \geq 3$ is needed to obtain $H = 0$ if we assume the crude cosmological constant is the Planck mass.

1. INTRODUCTION

The energy density of the Universe is composed of the matter $\Omega_{M0} = 0.3$, and the cosmological constant $\Omega_{\Lambda 0} = 0.7$ [1]. The observed present value of the Hubble parameter is $H_0 \approx 10^{-33}$eV. These estimation implies the value of the cosmological constant is $\Lambda_4 \approx (10^{-3}\text{eV})^4$. On the other hand, a natural value of a constant contained in a gravitational theory is thought the Planck mass $M_p \approx 10^{18}$GeV. Both have a gap of order

$$\frac{(10^{-3}\text{eV})^4}{M_p^4} = 10^{-120}. \tag{1}$$

This inconsistency is called the cosmological constant problem (CCP) [2, 3] and still remains to be solved.

Dvali *et al.*[4] has suggested the mechanism of diluting the cosmological constant by using the extra-dimensional effect to overcome the CCP. This mechanism is that the observed effective cosmological constant becomes so small because whose energy is consumed to bend the bulk space even if the crude one is such large as the Planck scale. And they have introduced the conjecture such that

$$H = M_* \left(\frac{M_*^4}{\Lambda_4} \right)^{1/(n-2)}. \tag{2}$$

Where M_* is the $(4+n)$-dimensional Planck mass. If the number of extra dimensions n is greater than 3, Hubble parameter H is monotonically decreasing function of the cosmological constant Λ_4.

Cho and Vilenkin [5] have constructed the concrete model that has a 3-dimensional spherically symmetric extra space. In theirs model, physically acceptable solutions have features that bulk metric doesn't have a singularity at finite distance from the brane. In that paper, solutions whose asymptotic forms have the cigar ansatz and have the infinite bulk spaces were found. The relation between the scalar field's energy interpreted as the crude cosmological constant and the brane's expansion rate is calculated by using these solutions. And it is concluded that both have a positive correlation and can be linear fitted against the Dvali et al.'s conjecture (2).

Our approach is to observe the relation between the number of extra-dimensions and the brane's expansion rate with the energy scale fixed near the Planck mass, and obtain the condition of vanishing the expansion rate. So we have extended Cho and Vilenkin's model to the arbitrary $(4+n)$-dimensional one. We have found new solutions under the condition that the position of the singularity becomes as far as possible and the metric doesn't have the divergence.

We have concluded that Dvali et al.'s conjecture (2) cannot be reproduced by using our new bulk solutions in any dimension n. But we have found that the brane's expansion rate is a monotonically decreasing function of the number of the extra dimensions n. And the expansion rate can vanish at the specific dimension.

2. MODEL

In this section, the extended $(4+n)$-dimensional model is constructed.

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2.1. Space-time structure

In this model, the brane is assumed a 4-dimensional de-Sitter apace $\mathbf{dS}^4$, and the extra space is a spherically symmetric n-dimensional space $\mathbf{R} \times_{Cr} \mathbf{S}^{n-1}$. Where the number of extra dimensions n is greater than 2. The entire manifold is wrapped product of both spaces $\mathbf{R} \times_{Cr} \mathbf{S}^{n-1} \times_B \mathbf{dS}^4$, whose metric is

$$\begin{aligned} ds^2 &= dr^2 + C(r)^2 r^2 d\mathbf{\Omega}_{n-1}^2 \\ &+ B(r)^2 \left(-dt^2 + e^{2Ht} \sum_{i=1}^{3} dx^{i^2} \right). \end{aligned} \quad (3)$$

Where the brane's coordinate is (t, x^1, x^2, x^3) and H is the positive constant expansion rate. The extra space's coordinate is $(r, \theta_1, \ldots, \theta_{n-1})$ and $d\mathbf{\Omega}_{n-1}$ is the metric of an $n-1$ dimensional sphere $\mathbf{S}^{n-1}$. $C(r)r$, $B(r)$ are the radius of the extra space and the warp factor depending on r only. We adopt the Einstein-Hilbert action for the space-time dynamics such that

$$\mathcal{S}_{\mathrm{E-H}} = \frac{1}{2\kappa^2} \int d^{4+n}x \sqrt{-g} \mathcal{R}. \quad (4)$$

Where κ is the $(4+n)$-dimensional gravitational constant. In this paper, κ is independent of the number of extra dimensions to make clear the effect of a dilution. g and $\mathcal{R}$ are the determinant and the Ricci scalar of the metric (3).

2.2. Energy Momentum Tensor

The global defect in the n-dimensional spherically symmetric space is introduced to construct the brane, which is described by a multiplet of the scalar fields ϕ^i with a Lagrangian density,

$$\mathcal{S}_\phi = \int d^{4+n}x \sqrt{-g} \left[-\frac{1}{2} \partial^A \phi^i \partial_A \phi_i - V(\phi) \right]. \quad (5)$$

Where capital letters $(A, \ldots)$ and small letters $(i, \ldots)$ run from 1 to $4+n$ and from 1 to n respectively. Because we are thinking spherically symmetric solutions only, the scalar multiplet has been assumed to have a hedgehog configuration, $\phi^i = \phi(r) \cdot \xi^i / r$. Where $\phi(r)$ depends only on the radius coordinate r and ξ^i represent for the Cartesian coordinates of the extra space. The potential of the scalar field $V(\phi)$ has minimum at $|\phi^i| = \phi = \eta$ such that

$$V(\phi) = \frac{\lambda}{4} (\phi^2 - \eta^2)^2. \quad (6)$$

The energy created by the scalar field is interpreted as the crude cosmological constant existed in the entire space, so ϕ's vev holds about the Planck scale $\eta \approx 1/\kappa$.

2.3. Basic Equations

The Einstein equations and the EOM of the scalar field are obtained by the action,

$$\mathcal{S} = \mathcal{S}_{\mathrm{E-H}} + \mathcal{S}_\phi. \quad (7)$$

Independent components of the Einstein equations and EOM of the scalar field have the forms

$$\begin{aligned} G^\mu{}_\mu &= -\frac{1}{4} \frac{{}^{(4)}R}{B^2} + 3\frac{B''}{B} + 3\left(\frac{B'}{B}\right)^2 \\ &+ 3(n-1)\left(\frac{B'}{Br} + \frac{B'C'}{BC}\right) \\ &+ (n-1)\frac{C''}{C} + \frac{(n-2)(n-1)}{2}\left(\frac{C'}{C}\right)^2 \\ &+ n(n-1)\frac{C'}{Cr} + \frac{(n-2)(n-1)}{2}\left(\frac{1}{r^2} - \frac{1}{C^2 r^2}\right) \\ &= \kappa^2 \left[-\frac{\phi'^2}{2} - \frac{(n-1)\phi^2}{2C^2 r^2} - \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2 \right], \end{aligned} \quad (8)$$

$$\begin{aligned} G^r{}_r &= 6\left(\frac{B'}{B}\right)^2 + 4(n-1)\left(\frac{B'}{Br} + \frac{B'C'}{BC}\right) \\ &+ \frac{(n-2)(n-1)}{2}\left(\frac{C'}{C}\right)^2 + (n-2)(n-1)\frac{C'}{Cr} \\ &+ \frac{(n-2)(n-1)}{2}\left(\frac{1}{r^2} - \frac{1}{C^2 r^2}\right) - \frac{1}{2}\frac{{}^{(4)}R}{B^2} \\ &= \kappa^2 \left[\frac{\phi'^2}{2} - \frac{(n-1)\phi^2}{2C^2 r^2} - \frac{\lambda}{4}(\phi^2 - \eta^2)^2 \right], \end{aligned} \quad (9)$$

$$\begin{aligned} G^{\theta_i}{}_{\theta_i} &= -\frac{1}{2} \frac{{}^{(4)}R}{B^2} + 4\frac{B''}{B} + 6\left(\frac{B'}{B}\right)^2 \\ &+ 4(n-2)\left(\frac{B'}{Br} + \frac{B'C'}{BC}\right) + (n-2)\frac{C''}{C} \\ &+ \frac{(n-3)(n-2)}{2}\left(\frac{C'}{C}\right)^2 + (n-1)(n-2)\frac{C'}{Cr} \\ &+ \frac{(n-3)(n-2)}{2}\left(\frac{1}{r^2} - \frac{1}{C^2 r^2}\right) \\ &= \kappa^2 \left[-\frac{\phi'^2}{2} - \frac{(n-3)\phi^2}{2C^2 r^2} - \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2 \right] \end{aligned} \quad (10)$$

Here ${}^{(4)}R = 12H^2$ represents for the 4-dimensional Ricci scalar depending on the expansion rate of the brane. The prime denotes the differentiation with respect to r. The equation of motion of the scalar field is

$$\begin{aligned} \phi'' &+ (n-1)\left(\frac{4}{(n-1)}\frac{B'}{B} + \frac{C'}{C} + \frac{1}{r}\right)\phi' \\ &- (n-1)\frac{\phi}{C^2 r^2} - \lambda\phi(\phi^2 - \eta^2) = 0. \end{aligned} \quad (11)$$

Eq.(9) imposes the constraint when solving eq. (8), (10) and (11) as the second-order differential equations for B, C and ϕ.

3. ASYMPTOTIC SOLUTIONS

We have found asymptotic solutions which can be written as the exact analytic forms. They were gotten by solving eq.(8), (9), (10) and (11) analytically where r has a large value.

3.1. $H = 0$ case

In case of $H = 0$, asymptotic solutions can be found as follows. If $n \geq 3$,

$$\phi(\infty) = \eta, \tag{12}$$

$$B^2(\infty) = (\text{arbitrary const.}), \tag{13}$$

$$C^2(\infty) = 1 - \frac{(\kappa\eta)^2}{n-2}, \tag{14}$$

where $(\kappa\eta)^2 \leq n-2$. From eq. (14), the sphere $\mathbf{S}^{n-1}$ has a solid angle deficit such that

$$\Delta\Omega = \frac{2\pi^{n/2}}{\Gamma(n/2)} \cdot \frac{(\kappa\eta)^2}{n-2}. \tag{15}$$

Where Γ is a gamma function. As $\kappa\eta$ approaches to $\sqrt{n-2}$, the deficit angle will consume the entire area.

In the previous works [5], the cigar type solutions in $n = 3$ case are studied. These solutions have asymptotic forms $\sqrt{\lambda}\eta C r \to$ constant. Exact analytic solutions for arbitrary $n \geq 2$ have been found such that

$$(\kappa\phi)^2 = \frac{2(n^2-4) - (n-1)(\kappa\eta)^2}{(n+5)}, \tag{16}$$

$$B = \frac{H}{\sqrt{\lambda}\eta k} \sin(\sqrt{\lambda}\eta k r), \tag{17}$$

$$\lambda\eta^2 C^2 r^2 = \frac{(n-1)(n+5)(\kappa\eta)^2}{2(n+2)[(\kappa\eta)^2 - (n-2)]}. \tag{18}$$

Where,

$$k = \sqrt{\frac{n+2}{2(n+5)^2}} \frac{(\kappa\eta)^2 - (n-2)}{\kappa\eta}. \tag{19}$$

3.2. $H \neq 0$ case

In case of $H \neq 0$, $B(\infty)$ is not allowed to be a constant yet. Instead, the linear form can hold such that

$$\phi(\infty) = \eta, \tag{20}$$

$$B^2(\infty) = \frac{3}{n+2} H^2 r^2, \tag{21}$$

$$C^2(\infty) = -\frac{(\kappa\eta)^2 - (n-2)}{n+2}. \tag{22}$$

Where $(\kappa\eta)^2 \geq n-2$ and $C^2(\infty)$ takes a negative value, which is meaning the sphere sector $\mathbf{S}^{n-1}$ has a sign $(-1)^{n-1}$ for $n \geq 2$.

4. NUMERICAL RESULTS

Differential equations to solve have a set of three parameters $(n, \kappa\eta, (\kappa/\lambda^{1/2})H)$. It is found by the numerical integration that the proper relation among them is obtained under the condition that the point of a singularity becomes as far as possible. We call sets of parameters this relation holds eigen values and call solutions with eigen values physically proper solutions. Solutions obtained from parameters deviated from the eigen value have a divergence of B or C. Similar situations are considered in [5] for the cigar ansatz, but we adopted our new solutions found numerically.

In the case of $(\kappa\eta)^2 \leq n-2$, non-singular solution exists when the brane is not expanding such that $H = 0$. We have solved the Einstein equations and the EOM by an numerical method with the initial condition, $B(0) = C(0) = 1$, $B'(0) = C'(0) = 0$ and $\phi(0) = 0$. The sixth condition can be determined by the constraint (9) automatically. An example of the solution is discussed in [5]

If the case of $(\kappa\eta)^2 \geq n-2$, arbitrary H including $H = 0$ leads to a divergence of B or C at finite distance from the origin and the singularity is formed. We call this point r_{sing}. At the specific H, the distance of singularity becomes as far as possible and has has a local maximum, where the divergence vanishes. We call this point r_{f}. For example, the solution with the eigen value $(n, \kappa\eta, (\kappa/\lambda^{1/2})H) = (3, 1.09, 0.003786056)$ is shown in Fig. 1. It is noticed that the $B(r)$ vanishes at finite r but $C(r)$ doesn't diverge. Fig. 2 shows the relations between η and H with n fixed at some values. The larger η corresponds to the larger H. This tendency can be naturally understood like the Friedmann equation. Besides, each lines approach to the point $(\kappa\eta, H) = (\sqrt{n-2}, 0)$. Fig. 3 shows the relations between n and H with η fixed at some values. At $n = 0$ in this figure, the values led from the normal

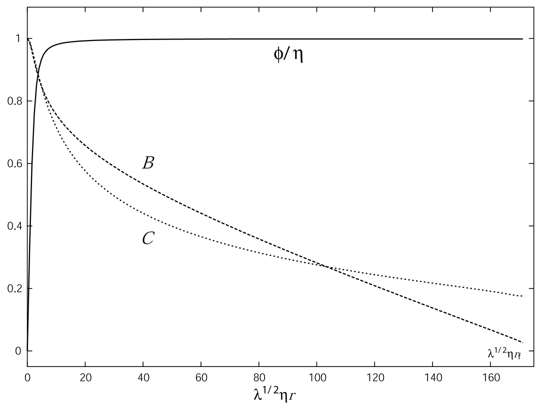

Figure 1: This graph shows the physically proper solution with the eigen value $(n, \kappa\eta, (\kappa/\lambda^{1/2})H) = (3, 1.09, 0.003786056)$. B is approaching to 0 at finite r_f.

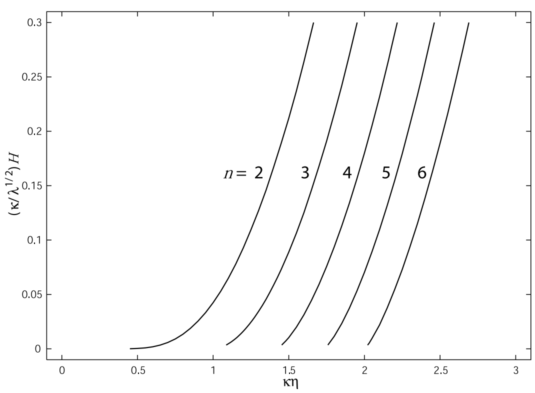

Figure 2: The relations between η and H with n fixed respectively. $H = 0$ seems to be established at $\kappa\eta = \sqrt{n-2}$.

Friedmann equation, $H^2 = \kappa^2\rho/3,\ \rho = \lambda\eta^4/4$ are also indicated. This figure shows that the expansion rate determined normally by the Friedmann equation is suppressed as a number of extra-dimensions increases, where H vanished at specific dimension. And these lines connect to the $(\kappa\eta) \leq n-2$ case. This effect can be considered the diluting cosmological constant.

Finally, it should be mentioned that the Dvali et al.'s conjecture (2) is not reproduced with our solutions as shown in Fig. 2.

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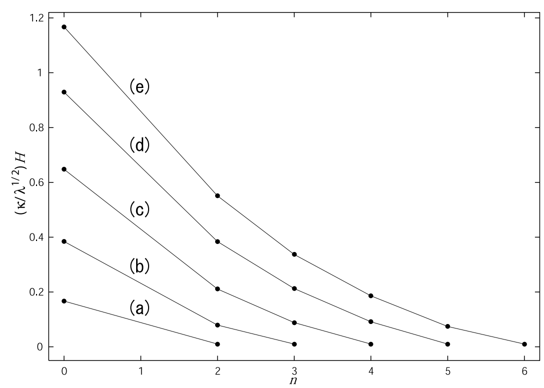

Figure 3: The relations between n and H. (a), (b), (c), (d) and (e) were given by fixing $\kappa\eta = 0.760, 1.15, 1.50, 1.80, 2.01$ respectively. Values the normal Friedmann equation holds are also indicated at $n = 0$. As a number of extra-dimensions increase, the expansion rate is suppressed. The end points of each lines are $(n, H) = (n, 0.01)$.

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