

# The Effect of Inhomogeneities on the Luminosity Distance–Redshift Relation: is Dark Energy Necessary in a Perturbed Universe?

A. Riotto

INFN, Sezione di Padova, Via Marzolo 8, I-35131 Italy

The luminosity distance–redshift relation is one of the fundamental tools of modern cosmology. Cosmological observations implementing the luminosity distance–redshift relation tell us that the Universe is presently undergoing a phase of accelerated expansion. This seems to call for a mysterious Dark Energy component with negative pressure. In this talk we argue that the need of a Dark Energy fluid may be challenged once a realistic inhomogeneous Universe is considered and that an accelerated expansion may be consistent with a matter-dominated Universe.

## 1. Introduction

One of the fundamental relations in cosmology is the one expressing the luminosity distance  $d_L$  of a cosmological source in terms of its redshift  $z$ . In recent years this relation has been exploited to measure the present value of the expansion rate, Hubble’s constant, with increasing accuracy. With the exploration of the Universe at redshifts of order unity, we now have information about the time evolution of the expansion rate [1]. A most surprising result is that the time evolution of the expansion rate does not seem to be described by a matter-dominated Friedmann-Robertson-Walker (FRW) homogeneous cosmological model of the Universe. The usual explanation for the discrepancy is that there is a new component of the energy density of the Universe, known as Dark Energy, that determines the recent evolution of the expansion rate. Of course all indications for Dark Energy are indirect; from cosmological data we only infer that the Universe is presently undergoing a phase of acceleration.

Since the luminosity distance–redshift relation is of such fundamental importance, we must understand any possible effects that would result in a relation different from the FRW prediction for a homogeneous Universe. The computation of the luminosity (or angular diameter) distance in a locally non homogeneous universe was first addressed by Zeldovich [2]). Dyer and Roeder [3] adopted the so-called empty beam approximation to derive an equation for the angular diameter distance. Our technique instead relies on the traditional perturbative approach, recently used in Refs. [4, 5, 6].

In this talk we study the change in the luminosity distance–redshift relation due to cosmological perturbations present in a background matter-dominated Universe with zero cosmological constant. In particular, we provide the generic expression for the luminosity distance–redshift relation at second order in perturbation theory, for any value of the physical redshift and any direction of observation. We only consider modifications to the luminosity distance–redshift relation of a matter-dominated Universe, although our

results can be easily extended to a Universe containing a mixture of matter and other fluids. Our findings may be used, for instance, to estimate the influence of lensing on the brightness of supernovae sources. What our second-order procedure does not account for is another class of terms, where a first-order small deflection in the light ray leads the geodesic to deflect to a region where the perturbations are sizeable relative to the neighbourhood of the geodesic in the unperturbed space-time.

It is well-known that the luminosity distance–redshift relation allows to extract the theoretical predictions for the local Hubble rate  $\mathcal{H}$  and deceleration parameter  $q$  upon expanding around the observer point at  $z = 0$ . In a perturbed Universe, however, both the Hubble constant and the deceleration parameter lose their deterministic nature; one has to consider the statistical nature of the vacuum fluctuations from which the present-day gravitational potential is originated. Therefore, the theoretical predictions for the expected values of the cosmological parameters are accompanied by a nonvanishing cosmic variance implying an intrinsic theoretical error. When calculating the variance of the deceleration parameter, we uncover an interesting infrared effect. We observe, at second order in perturbation theory, a large contribution to the variance from the cosmological perturbations with the largest wavelengths. If inflation is the origin of the cosmological perturbations [7], the beginning of inflation provides a cut-off to the infrared modes of the fluctuations. Our results suggest that if the super-Hubble modes have physical significance, they could yield a most important modification to the deceleration parameter. One might even speculate that a complete treatment of the effect could obviate the need for the Dark Energy assumption. Indeed, despite the fact that the homogeneous FRW model for the matter-dominated Universe and no cosmological constant predicts that the latter may not decelerate, *i.e.*  $q_0 = \frac{1}{2} > 0$ , because of the large variance, the true locally-determined value of the deceleration parameter has a non-zero probability of being less than zero. Put it differently, the theoretical prediction in

a perturbed matter-dominated model is not a single well-defined curve in the luminosity-distance redshift plane, but instead is represented by a finite confidence region whose size is determined by the cosmic variance.

The talk is organized as follows. In the next section we obtain the generic luminosity distance–redshift relation valid at any order in perturbation theory around the homogeneous FRW background. In Section III we express the luminosity distance–redshift relation in terms of the metric perturbations expanded up to second order. Section IV is devoted to the evaluation of the mean and the variance of the cosmological observables and to the discussion of their implications. Finally, Section IV present our conclusions and comments.

## 2. The generic luminosity distance–redshift relation

The goal of this section is to provide the reader with the generic formulation of the luminosity distance–redshift relation in a general setting. Our treatment closely follows the one given in Ref. [4]. In the next section, we will make use of this generic formulation to obtain the luminosity distance–redshift relation in a perturbed FRW Universe being the background matter-dominated Universe with zero cosmological constant.

In order to deal with the propagation of light from a given source to the observer, we make use of the conformal (Weyl) invariance of the electromagnetic field and define

$$d\hat{s}^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu = a^2 (\eta) g_{\mu\nu} dx^\mu dx^\nu, \quad (1)$$

where  $x^\mu = (\eta, x^i)$  are the space-time coordinates and the scale factor  $a$  is normalized to unity at the present conformal time  $\eta_0$  ( $a(\eta_0) = a_0 = 1$ ).

We will use a “ $\hat{\phantom{x}}$ ” to mark quantities calculated in a space-time with the metric  $\hat{g}_{\mu\nu}$ ; quantities without “ $\hat{\phantom{x}}$ ” will be calculated with the metric  $g_{\mu\nu}$  instead. Furthermore, quantities with a “ $\sim$ ” stand for quantities in a perturbed Universe.

Using the geometric optics approximation, the energy-momentum tensor of a photon emitted by a given source is

$$\hat{T}^{\mu\nu} = A^2 \hat{k}^\mu \hat{k}^\nu, \quad (2)$$

where  $A$  is the scalar amplitude of the wave and  $\hat{k}^\mu$  is the photon four-momentum:

$$\begin{aligned} \hat{k}^\mu &= \frac{dx^\mu}{dv} = \hat{g}^{\mu\nu} \partial_\nu S, & \hat{k}^\mu \hat{k}_\mu &= 0, \\ \hat{k}^\nu \hat{\nabla}_\nu \hat{k}^\mu &= \frac{d^2 x^\mu}{dv^2} + \hat{\Gamma}_{\alpha\beta}^\mu \frac{dx^\alpha}{dv} \frac{dx^\beta}{dv} = 0. \end{aligned} \quad (3)$$

Here  $S$  is the phase (eikonal) of the wave and  $v$  is an affine parameter along the ray (*i.e.* along the photon trajectory). The evolution of the parameter  $A$  is provided by the continuity equation  $\hat{\nabla}_\nu \hat{T}^{\mu\nu} = 0$ : we have

$$\frac{dA}{dv} = -\frac{1}{2} A \hat{\theta}, \quad \hat{\theta} = \hat{\nabla}_\mu \hat{k}^\mu. \quad (4)$$

All these equations can be replaced by the corresponding ones in the metric  $g_{\mu\nu}$ . If we define

$$d\lambda = a^{-2} dv, \quad k^\mu = \frac{dx^\mu}{d\lambda} = a^2 \hat{k}^\mu, \quad (5)$$

we easily get

$$\begin{aligned} k_\mu &= \partial_\mu S, & k^\mu k_\mu &= 0, \\ k^\nu \nabla_\nu k^\mu &= \frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0, \\ \frac{d(Aa)}{d\lambda} &= -\frac{1}{2} A a \theta, & \theta &= \nabla_\mu k^\mu. \end{aligned} \quad (6)$$

To obtain the equation describing the transport of the optical scalar  $\theta$  (*expansion* of the null congruence  $k^\mu$ ) along the ray, it is useful to recall the concept of null (Newman-Penrose) tetrad: it is made of two real ( $k^\mu$  and  $m^\mu$ ) and two complex conjugate ( $t^\mu$  and  $\bar{t}^\mu$ ) 4-vectors, satisfying

$$\begin{aligned} k^\mu m_\mu &= \bar{t}^\mu t_\mu = 1, \\ k^\mu k_\mu &= m^\mu m_\mu = t^\mu t_\mu = k^\mu t_\mu = m^\mu t_\mu = 0. \end{aligned} \quad (7)$$

Defining the projector

$$\mathcal{P}_\nu^\mu \equiv t^\mu \bar{t}_\nu + \bar{t}^\mu t_\nu = \delta_\nu^\mu - k_\nu m^\mu - m_\nu k^\mu, \quad (8)$$

one can easily obtain the decomposition (the semi-colon indicates covariant differentiation in the conformal space-time with metric  $g_{\mu\nu}$ )

$$k_{\mu;\nu} = a_\nu k_\mu + b_\mu k_\nu + A_{\mu\nu}, \quad (9)$$

with

$$\begin{aligned} A_{\mu\nu} &= k_{\alpha;\beta} \mathcal{P}_\mu^\alpha \mathcal{P}_\nu^\beta = 2\Re \left( \frac{\theta}{2} t_\mu \bar{t}_\nu + \sigma t_\mu t_\nu \right), \\ a_\mu k^\mu &= b_\mu k^\mu = 0, \end{aligned} \quad (10)$$

where  $\theta$  is real and  $\sigma$  is complex. With such a decomposition we obtain

$$\theta = \nabla_\mu k^\mu, \quad |\sigma|^2 = \frac{1}{2} \left[ k_{\mu;\nu} k^{\mu;\nu} - \frac{\theta^2}{2} \right]. \quad (11)$$

The quantity  $\sigma$  is called *shear* of the null congruence. From the Ricci identity

$$k_{\mu;\nu\delta} - (\nu \leftrightarrow \delta) = R_{\sigma\mu\nu\delta} k^\sigma, \quad (12)$$

contracting the indices  $\mu$  and  $\nu$ , we obtain (see for instance [9])

$$\frac{d\theta}{d\lambda} = -R_{\mu\nu}k^\mu k^\nu - \frac{\theta^2}{2} - 2|\sigma|^2. \quad (13)$$

In a similar fashion, the transport equation for the shear  $\sigma$  can be derived by contracting (12) with  $\bar{t}^\mu \bar{t}^\nu k^\delta$  and assuming that  $\bar{t}^\mu$  is parallel-transported along the ray; we get

$$\frac{d\sigma}{d\lambda} = -\sigma\theta + C_{\alpha\beta\nu\mu}k^\alpha k^\mu \bar{t}^\nu \bar{t}^\beta, \quad k^\nu \nabla_\nu \bar{t}^\mu = 0, \quad (14)$$

where  $C_{\alpha\beta\nu\mu}$  is the Weyl tensor.

To calculate the luminosity distance, we need the energy flux per unit surface,  $\ell$ , measured by an observer with 4-velocity  $\hat{u}^\mu$

$$\ell = \sqrt{\hat{h}_{\mu\nu} (\hat{u}_\sigma \hat{T}^{\sigma\mu}) (\hat{u}_\delta \hat{T}^{\delta\nu})} = A^2 \omega^2, \quad (15)$$

where

$$\hat{h}_{\mu\nu} = \hat{g}_{\mu\nu} + \hat{u}_\mu \hat{u}_\nu, \quad \omega = -\hat{u}_\mu \hat{k}^\mu. \quad (16)$$

If the source has physical radius  $R$  (which we will eventually set to zero) and if we choose the affine parameter  $\lambda$  on the ray connecting the observer and the source such that  $\lambda = 0$ ,  $\lambda = \lambda_s$ ,  $\lambda = \lambda_s + \Delta\lambda_s$  correspond to the observer today, to the surface of the source and to its center, the power  $\mathcal{L}$  emitted by the source and measured by a comoving observer, is

$$\mathcal{L} = 4\pi R^2 \ell(\lambda_s). \quad (17)$$

The luminosity distance thus reads

$$d_L = R \sqrt{\frac{\ell(\lambda_s)}{\ell(0)}} = R \frac{A(\lambda_s)}{A(0)} (1 + \tilde{z}(\lambda_s)), \quad (18)$$

where the redshift  $\tilde{z}$  is defined by

$$1 + \tilde{z}(\lambda_s) = \frac{\omega(\lambda_s)}{\omega(0)}. \quad (19)$$

To summarize, the evaluation of the luminosity distance goes through the following steps: one has to solve for the photon trajectory from which one may deduce the frequency  $\omega$ ; then one is able to solve for the expansion parameter  $\theta$  and the shear  $\sigma$ ; finally one solves for the amplitude  $A$ .

### 3. The luminosity distance–redshift relation in a perturbed FRW Universe

The formalism we have summarized in the previous Section can now be applied to obtain the luminosity

distance–redshift relation in a perturbed FRW Universe. In particular, we are interested in a perturbative treatment up to second order. This Section contains only the main steps of the calculation, a plethora of more details can be found in Ref. [8].

First of all, we need to expand the wave four-vector of the photon reaching the observer at second order,

$$k^\mu = k_{(0)}^\mu + k_{(1)}^\mu + k_{(2)}^\mu. \quad (20)$$

Similarly, we expand the photon trajectory as

$$x^\mu(\lambda) = x_{(0)}^\mu(\lambda) + x_{(1)}^\mu(\lambda) + x_{(2)}^\mu(\lambda). \quad (21)$$

Both quantities  $k_{(r)}^\mu$  and the  $x_{(r)}^\mu$  ( $r = 0, 1, 2$ ) are fully determined by the geodesic equation

$$\begin{aligned} \frac{d}{d\lambda} (k_{(0)}^\mu + k_{(1)}^\mu + k_{(2)}^\mu) &= -\Gamma_{\alpha\beta}^\mu k_{(0)}^\alpha k_{(0)}^\beta \\ &= -\Gamma_{(1)\alpha\beta}^\mu k_{(0)}^\alpha k_{(0)}^\beta - \left( \Gamma_{(2)\alpha\beta}^\mu k_{(0)}^\alpha k_{(0)}^\beta \right. \\ &\quad \left. + 2\Gamma_{(1)\alpha\beta}^\mu k_{(0)}^\alpha k_{(1)}^\beta + \Gamma_{(1)\alpha\beta,\gamma}^\mu k_{(0)}^\alpha k_{(0)}^\beta x_{(1)}^\gamma \right), \end{aligned} \quad (22)$$

and by the initial conditions. We then need to perturb the expansion  $\theta$  of our null congruence

$$\theta = \theta_{(0)} + \theta_{(1)} + \theta_{(2)} : \quad (23)$$

these functions are determined by Eq. (13) and by the boundary conditions. Furthermore, since the right-hand side of Eq. (13) contains the square of the shear, we need also to solve Eq. (14) up to first order, again with suitable boundary conditions. After the quantities  $\theta$ ,  $\sigma$  and  $A$  are known, one is ready to calculate the luminosity distance–redshift relation. Sending the physical size  $R$  of the source (and therefore  $\Delta\lambda_s$ ) to zero, we get the following expression

$$\begin{aligned} d_L &= (1 + \tilde{z}(\lambda_s)) a_0 \lambda_s \left[ 1 - \frac{1}{2} \int_0^{\lambda_s} \theta_{(1)}(\lambda, \lambda_s) d\lambda \right. \\ &\quad \left. + \frac{1}{8} \left( \int_0^{\lambda_s} \theta_{(1)}(\lambda, \lambda_s) d\lambda \right)^2 \right. \\ &\quad \left. - \frac{1}{2} \int_0^{\lambda_s} \theta_{(2)}(\lambda, \lambda_s) d\lambda - k_{(1)}^0(\lambda_s) \right. \\ &\quad \left. - k_{(2)}^0(\lambda_s) + \frac{1}{2} k_{(1)}^0(\lambda_s) \int_0^{\lambda_s} \theta_{(1)}(\lambda, \lambda_s) d\lambda \right] \end{aligned} \quad (24)$$

This is the generic expression for the luminosity distance–redshift relation at second order in perturbation theory and for any value of the physical redshift  $\tilde{z}(\lambda_s)$  and any given direction of observation.

Since the physical observable is the redshift and not the affine parameter  $\lambda_s$ , we have to trade  $\lambda_s$  in terms of  $\tilde{z}$  in Eq. (24). In order to achieve this we may make

use of the definition of the redshift in the unperturbed Universe

$$1 + z = \frac{a_0}{a(\eta_0 - \lambda_s)}, \quad \frac{\lambda_s}{\eta_0} = 1 - \frac{1}{\sqrt{1+z}}, \quad (25)$$

from which we deduce the relation between the physical redshift  $\tilde{z}$  and  $z$

$$\begin{aligned} 1 + \tilde{z}(\lambda_s) &= \frac{a_0}{a(\eta(\lambda_s))} \frac{(u_\mu k^\mu)(\lambda_s)}{(u_\mu k^\mu)(0)} \\ &= a_0 \left( 1 - \left( k_{(1)}^0(\lambda_s) + k_{(2)}^0(\lambda_s) \right) \right) \\ &\times \left[ a(\eta_{(0)}(\lambda_s)) + a'(\eta_{(0)}(\lambda_s)) (\eta_{(1)}(\lambda_s) + \eta_{(2)}(\lambda_s)) \right. \\ &\left. + \frac{1}{2} a''(\eta_{(0)}(\lambda_s)) \eta_{(1)}(\lambda_s)^2 \right]^{-1} \\ &= (1+z)(1+T_1(\lambda_s)+T_2(\lambda_s)), \end{aligned} \quad (26)$$

where  $\eta_{(1)}$  and  $\eta_{(2)}$  are respectively the first- and second-order expansion of the photon conformal time and

$$\begin{aligned} T_1(\lambda) &= \left( -\frac{a'(\eta_{(0)})}{a(\eta_{(0)})} \eta_{(1)} - k_{(1)}^0 \right) (\lambda), \\ T_2(\lambda) &= \left( -\frac{a'(\eta_{(0)})}{a(\eta_{(0)})} \eta_{(2)} - k_{(2)}^0 - \frac{a''(\eta_{(0)})}{2a(\eta_{(0)})} \eta_{(1)}^2 \right. \\ &\quad \left. + \left( \frac{a'(\eta_{(0)})}{a(\eta_{(0)})} \eta_{(1)} \right)^2 + \frac{a'(\eta_{(0)})}{a(\eta_{(0)})} \eta_{(1)} k_{(1)}^0 \right) (\lambda) \end{aligned} \quad (27)$$

We may therefore compute first the luminosity distance–redshift relation in terms of the parameter  $z$  and then as a function of the physical redshift  $\tilde{z}$  using Eqs. (26), (27) and (28) to express  $z$  as a function of  $\tilde{z}$ .

To compute all these quantities we have to expand the metric to second order. We adopt the comoving and synchronous gauge (the luminosity distance–redshift relation is a gauge-invariant concept as explicitly shown in Ref. [4] and therefore the choice of the gauge is arbitrary)

$$\begin{aligned} ds^2 &= -d\eta^2 + \gamma_{ij} dx^i dx^j, \\ \gamma_{ij} &= (1 - 2\psi_{(1)} - \psi_{(2)}) \delta_{ij} + \chi_{(1)ij} + \frac{1}{2} \chi_{(2)ij} \end{aligned} \quad (29)$$

where both  $\chi_{(1)ij}$  and  $\chi_{(2)ij}$  are traceless.

Assuming that the source and the observer are both comoving with the fluid, one has (at any order in perturbation theory)  $u^\mu = \delta_0^\mu$ . By solving Einstein's equations, the metric perturbations  $\psi_{(r)}$   $\chi_{(r)}$ ,  $r = 1, 2$ , can be expressed in terms of the peculiar gravitational

potential  $\varphi$  [10]:

$$\chi_{(1)ij} = -\frac{\eta^2}{3} \left( \varphi_{,ij} - \frac{1}{3} \delta_{ij} \nabla^2 \varphi \right), \quad (30)$$

$$\psi_{(1)} = \frac{5}{3} \varphi + \frac{\eta^2}{18} \nabla^2 \varphi, \quad (31)$$

at first order and

$$\psi_{(2)} = -\frac{50}{9} \varphi^2 - \frac{5\eta^2}{54} \varphi^{,k} \varphi_{,k} + \frac{\eta^4}{252} \left( -\frac{10}{3} \varphi^{,ik} \varphi_{,ik} + (\nabla^2 \varphi)^2 \right), \quad (32)$$

$$\begin{aligned} \chi_{ij}^{(2)} &= \frac{\eta^4}{126} \left( 19 \varphi^{,k}{}_{,i} \varphi_{,kj} - 12 \varphi_{,ij} \nabla^2 \varphi + 4 (\nabla^2 \varphi)^2 \delta_{ij} \right. \\ &\quad \left. - \frac{19}{3} \varphi^{,kl} \varphi_{,kl} \delta_{ij} \right) - \frac{10\eta^2}{9} \left( \varphi_{,i} \varphi_{,j} - \frac{1}{3} \varphi^{,k}{}_{,i} \varphi_{,k} \delta_{ij} \right) + \pi_{ij}, \end{aligned} \quad (33)$$

at second order (the well-known residual gauge ambiguity of the synchronous and comoving gauge has been fixed as in Ref. [10]). We have disregarded both linear vector modes (since they are not generated during inflation) and linear tensor modes (because their dynamical role is negligible). At second order tensor modes described by  $\pi_{ij}$  do not enter in the computation of the Hubble rate and the deceleration parameter. Furthermore, the values of the second-order potentials have been computed with a proper match to the initial conditions set by single-field models of inflation [11] (notice, however, that the initial conditions do not play a significant role, since physical observables like the Hubble constant and the deceleration parameter depend only upon the rate of change of cosmological inhomogeneities).

## 4. Results and implications

We are now ready to extract from the findings of the previous Section the relevant cosmological parameters in a realistic perturbed Universe. To this purpose, let us recall that in a generic unperturbed FRW Universe the following relation holds

$$d_L = \frac{c}{\mathcal{H}_0} \left[ z + \frac{z^2}{2} (1 - q_0) + \mathcal{O}(z^3) \right], \quad (34)$$

where  $\mathcal{H}_0$  is the unperturbed Hubble constant and  $q_0$  is the unperturbed deceleration parameter. Therefore, an observer measuring a relation of the type  $d_L = \mathcal{A} + \mathcal{B}\tilde{z} + \mathcal{C}\tilde{z}^2 + \dots$  would conclude that  $\mathcal{B}$  and  $\mathcal{C}$  provide a measure of the Hubble constant and deceleration parameter at the present epoch. To make contact with this procedure, we may expand the generic formula (24) for the luminosity distance (valid at any value of the redshift  $\tilde{z}$ ) around  $\tilde{z} = 0$ . In this way, we may determine the value of the Hubble constant and the

deceleration parameter that would be measured by an observer having at her/his disposal sources with redshifts  $\tilde{z} \lesssim 1$  in a *perturbed* Universe. Performing properly the angular average  $\langle \dots \rangle_\Omega$  and comparing

this expansion of Eq. (24) with Eq. (34), we infer the expression for  $\tilde{\mathcal{H}}_0$  and for  $\tilde{q}_0$ :

$$\begin{aligned} \langle \tilde{\mathcal{H}}_0 \rangle_\Omega &= \mathcal{H}_0 \left[ 1 - \frac{2}{\mathcal{H}_0} \left( \frac{1}{2} \psi'_{(1)} + \frac{1}{4} \psi'_{(2)} + \psi_{(1)} \psi'_{(1)} + \frac{1}{12} (\chi_{(1)}^{ij})' \chi_{(1)ij} \right) \right] \\ &= \mathcal{H}_0 \left[ 1 - \left( \frac{1}{18} \nabla^2 \varphi - \frac{5}{108} (\nabla \varphi)^2 + \frac{5}{27} \varphi \nabla^2 \varphi \right) \left( \frac{2}{\mathcal{H}_0} \right)^2 - \left( \frac{1}{189} \varphi^{,ij} \varphi_{,ij} + \frac{1}{252} (\nabla^2 \varphi)^2 \right) \left( \frac{2}{\mathcal{H}_0} \right)^4 \right] \quad (35) \\ \langle \tilde{q}_0 \rangle_\Omega &= \frac{1}{2} \left[ 1 + \frac{2}{\mathcal{H}_0} \left( 2\psi'_{(1)} + \psi'_{(2)} + 4\psi_{(1)} \psi'_{(1)} + \frac{1}{3} (\chi_{(1)}^{ij})' \chi_{(1)ij} \right) \right. \\ &\quad + \left( \frac{2}{\mathcal{H}_0} \right)^2 \left( \frac{1}{2} \psi''_{(1)} + \frac{1}{4} \psi''_{(2)} + \frac{9}{4} (\psi'_{(1)})^2 + \psi_{(1)} \psi''_{(1)} + \frac{7}{40} (\chi_{(1)ij})' (\chi_{(1)}^{ij})' + \frac{1}{12} (\chi_{(1)}^{ij})'' \chi_{(1)ij} \right) \\ &\quad \left. + \left( \frac{2}{\mathcal{H}_0} \right)^3 \left( \frac{1}{2} \psi'_{(1)} \psi''_{(1)} + \frac{1}{60} (\chi_{(1)ij})' (\chi_{(1)}^{ij})'' \right) \right] \\ &= \frac{1}{2} \left[ 1 + \left( \frac{5}{18} \nabla^2 \varphi + \frac{25}{27} \varphi \nabla^2 \varphi - \frac{25}{108} (\nabla \varphi)^2 \right) \left( \frac{2}{\mathcal{H}_0} \right)^2 + \left( \frac{1}{30} (\nabla^2 \varphi)^2 + \frac{23}{270} \varphi^{,ij} \varphi_{,ij} \right) \left( \frac{2}{\mathcal{H}_0} \right)^4 \right] \quad (36) \end{aligned}$$

where  $\psi_{(r)}$ ,  $\chi_{(r)}$  ( $r = 1, 2$ ) and  $\varphi$  are evaluated at the observer's position  $\mathbf{x}_{(0)} = 0$  and at the present time  $\eta_0$  (note that Eq. (35) agrees with Eq. (39) of [12]). Eqs. (35) and (36) are among the main results of our talk. The key point is that the values (35) and (36) of the Hubble constant and the deceleration parameter in a perturbed Universe are not deterministic. Indeed, we must consider the statistical nature of the vacuum fluctuations from which the present-day linear gravitational potential  $\varphi$  is originated. This implies that the gravitational potential does not have well-defined values, but one can only define the probability of finding a given value at a given point in space. In a given realization of the perturbed Universe, the values of the cosmological parameters may change in regions of the Universe which are causally disconnected today. Therefore, it is unavoidable that the theoretical predictions for the values of the cosmological parameters come with a nonvanishing cosmic variance which implies an intrinsic theoretical error.

It will turn out that the variance can be large and get its largest contribution from the infrared super-Hubble modes of the metric perturbations.

Let us evaluate the expected uncertainty in the determination of the deceleration parameter, we treat the metric fluctuation  $\varphi$  as a Gaussian random variable with zero statistical mean over a volume which, in the inflationary picture, has a size much larger than the present-day Hubble radius. The variable  $\varphi$  takes random values over different "realizations". We may

express the variance in terms of the matter power spectrum. The procedure would be to fix a spherical domain of large volume  $\mathcal{V}$  surrounding the observer and containing the most distant sources of interest (we are interested in comparing the theoretical predictions of a perturbed Universe to the ones obtained in a unperturbed FRW homogeneous classical background. Since the quantities relative to the unperturbed model are recovered in the zero-momentum limit, the volume average has to be performed over a volume of size at least as large as the present-day Hubble radius). However, since the main contribution to the variance comes from the longest wavelengths, this volume averaging is irrelevant (as a technical remark, we stress that the low-pass filtering procedure over the volume  $\mathcal{V}$  should have been performed before Taylor expanding the luminosity distance-redshift relation (24) around the observer point at  $\tilde{z} = 0$ . This amounts to cutting off the spurious contribution of ultraviolet modes to the means and variances of physical observables which would appear performing the average after expanding in powers of the physical redshift). We will express  $\varphi$  and its derivatives in terms of a Fourier integral, so

$$\begin{aligned} \varphi &= \int \frac{d^3 k}{(2\pi)^3} \varphi_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}, \quad \varphi_{,i} = \int \frac{d^3 k}{(2\pi)^3} i k_i \varphi_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}, \\ \nabla^2 \varphi &= - \int \frac{d^3 k}{(2\pi)^3} k^2 \varphi_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}, \quad \text{etc.} \end{aligned} \quad (37)$$

The Fourier components  $\varphi_{\vec{k}}$  satisfy

$$\overline{\varphi_{\vec{k}}} = 0, \quad \overline{\varphi_{\vec{k}_1} \varphi_{\vec{k}_2}} = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) P_\varphi(k_1) \quad (38)$$

$$\begin{aligned} \overline{\varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \varphi_{\vec{k}_4}} &= (2\pi)^6 \left\{ \delta^{(3)}(\vec{k}_1 + \vec{k}_2) \delta^{(3)}(\vec{k}_3 + \vec{k}_4) \right. \\ &\quad \times P_\varphi(k_1) P_\varphi(k_3) + (\text{cyclic terms}) \left. \right\}, \end{aligned}$$

where  $\overline{(\dots)}$  denotes the statistical average and  $P_\varphi(k)$  is the power spectrum of the gravitational potential. We can express  $P_\varphi(k)$  in terms of the matter power spectrum as

$$P_\varphi(k) \equiv \frac{9\pi^2}{2} a_0^4 \mathcal{H}_0^4 \frac{\Delta^2(k, a_0)}{k^7}, \quad (39)$$

where  $\Delta^2(k, a_0)$  is the (dimensionless) power spectrum of the matter density fluctuations linearly extrapolated to the present time. We express the power spectrum  $\Delta^2(k, a_0)$  in terms of the transfer function  $T^2(k)$ . For a Harrison–Zel’dovich spectrum, the power spectrum is

$$\Delta^2(k, a_0) = \mathcal{A}^2 \left( \frac{k}{a_0 \mathcal{H}_0} \right)^4 T^2(k), \quad (40)$$

where  $\mathcal{A}$  is the dimensionless amplitude,  $\mathcal{A} = 1.9 \times 10^{-5}$ .

An analysis similar to the one performed in Ref. [12] shows that the biggest contribution to the variance of the deceleration parameter comes from the terms of the type  $\varphi \nabla^2 \varphi$  whose variance is

$$\begin{aligned} \text{Var} [\varphi \nabla^2 \varphi] &\simeq \left( \frac{9}{4} a_0^4 \mathcal{H}_0^4 \right)^2 \int \frac{dk_1}{k_1} \Delta^2(k_1, a_0) \\ &\quad \times \int \frac{dk_2}{k_2^5} \Delta^2(k_2, a_0) \\ &\simeq \text{Var} [\varphi] \text{Var} [\nabla^2 \varphi]. \end{aligned} \quad (41)$$

Taking into account that  $T(k) \rightarrow 1$  when  $k \rightarrow 0$ , we conclude that the variance of the deceleration parameter is sensitive also to the infrared modes of the Harrison–Zel’dovich power spectrum and is

$$\frac{\sqrt{\text{Var} [\langle \tilde{q} \rangle_\Omega]}}{q_0} \simeq \mathcal{A}^2 \ln \frac{k_{\text{MAX}}}{k_{\text{MIN}}} \sim 10^{-10} \ln \frac{k_{\text{MAX}}}{k_{\text{MIN}}}, \quad (42)$$

where  $k_{\text{MIN}}$  is the infrared cut-off and  $k_{\text{MAX}}$  is the ultraviolet cut-off set by the averaging volume. We can take it to coincide with the horizon volume, that is  $k_{\text{MAX}} = \mathcal{H}_0^{-1}$ .

Rather than a Harrison–Zel’dovich spectrum, if we assume a slightly red spectrum so that  $\Delta^2(k) \propto k^{3+n_s}$  with  $0 < (1 - n_s) \ll 1$ , then the logarithmic term in Eq. (42) is replaced by  $(1 - n_s)^{-1} (k_{\text{MAX}}/k_{\text{MIN}})^{(1-n_s)}$ . Now this will give unit variance if

$$\ln k_{\text{MAX}}/k_{\text{MIN}} \simeq (45 + \ln(1 - n_s))/(1 - n_s). \quad (43)$$

The variance of the deceleration parameter is infrared sensitive. There are various ways to avoid such a sensitivity:

- If the power spectrum is blue, *i.e.*  $n_s > 1$ , the variance does not pick up a large contribution from the long wavelengths;
- One may imagine to fix the physical infrared cut-off at a momentum scale corresponding to the present-day Hubble radius, but in the inflationary picture there is no strong motivation to do so since inflation is likely to have lasted for a period much longer than the minimum required number of  $e$ -folds  $\sim 60$ . After inflation, the size of the Universe which is relatively homogeneous is of the order of  $\ell_* \sim H_1 e^N$ , where  $N$  is the total number of  $e$ -folds and  $H_1$  is the Hubble rate during inflation. As the length scale  $\ell_*$  is much larger than the present-day Hubble size, it seems reasonable to sum up all the super-Hubble modes up to a length scale corresponding to  $\ell_*$  at the end of inflation (one should also keep in mind that on scales larger than  $\ell_*$  the Universe may become extremely inhomogeneous due to quantum fluctuations produced during inflation and a considerable part of the physical volume of the entire Universe may remain forever in the inflationary phase [17]).
- The presence of an infrared sensitivity might be only an artifact of the perturbative expansion and it might disappear when dealing with a complete nonperturbative approach. To check this possibility, a computation beyond second order is needed, even though we do not see any a priori reason why such cancellation should manifest itself at higher orders.

Since our findings reveal the crucial role played by the infrared modes, a deeper understanding of their physical significance is certainly desirable (for references dealing with the physical significance of the super-Hubble modes *neglecting* the gradients see Refs. [18] and [13]. Recently, it has been claimed [19] that “the relativistic zero-pressure fluid perturbed to second order in a flat Friedmann background coincides with the Newtonian result” and that “there are no relativistic correction terms even near and beyond the horizon to the second-order perturbation”. However, this claim relies on disregarding second-order tensor modes and is not justified, as implied by the discussion of Ref. ([20]), where it was stressed the importance of second-order tensor modes for the correct recovery of the Newtonian limit from the full relativistic theory). A large variance of the deceleration parameter  $\frac{\sqrt{\text{Var}[\langle \tilde{q} \rangle_\Omega]}}{q_0}$  is caused by the fact that the cosmological perturbations on super-Hubble scales are time-dependent when gradients are consistently taken into account. Indeed, it is easy to convince oneself that, were the infrared part of the perturbations  $\psi_{(r)}$ ,  $\chi_{(r)}$  ( $r = 1, 2$ ) constant in time, they might be eliminated from the metric (29) by a simple rescaling of

the spatial coordinates which allows to remain in the synchronous gauge. Such a freedom is lost if the inhomogeneities have a non-trivial time-dependence on super-Hubble scales. Consider, for instance, the first-order perturbation  $\psi_{(1)} = \frac{5}{3}\varphi + \frac{7}{18}\nabla^2\varphi$ . It contributes to the metric (29) by a piece  $(1 - \frac{10}{3}\varphi)\delta_{ij}$  which can be rescaled to  $\delta_{ij}$  by a transformation of the spatial coordinates. However, this change is not for free, it gives rise to the crucial piece proportional to  $\varphi\nabla^2\varphi$  at second order as it is easily realized by inspecting the transformations of the perturbations listed in Ref. [10]. The fact that the variance gets its largest contribution from a piece proportional to  $\varphi\nabla^2\varphi$  does not then come as a surprise, it manifests the impossibility of rescaling out the super-Hubble modes.

In the following, we set the infrared cut-off to the value fixed by the beginning of inflation which, in turn, depends upon the total number of  $e$ -folds  $N$  of the inflationary period.

The ratio  $(k_{\text{MAX}}/k_{\text{MIN}})$  is predicted by the inflationary theory to be (we are adopting natural units)

$$\frac{k_{\text{MAX}}}{k_{\text{MIN}}} \simeq 10^{-30} \left( \frac{T_{\text{RH}}}{H_{\text{I}}} \right) e^N, \quad (44)$$

where  $T_{\text{RH}}$  is the reheating temperature at the beginning of the radiation era after the end of inflation.

For the variance to be of the order of the background value  $q_0 = \frac{1}{2}$ , the perturbation spectrum would have to extend to a factor of  $\exp(6 \times 10^{18})$  ( $10^{18.8}$   $e$ -folds!) times the present Hubble radius (for  $T_{\text{RH}} \simeq H_{\text{I}}$ ) in the case of the Harrison-Zel'dovich spectrum. However, if, for instance,  $n_s = 0.94$  on super-Hubble-radius scales, then a variance of order unity is obtained if the perturbation spectrum extends  $N \sim 700$   $e$ -folds beyond the Hubble radius. Since the present Hubble radius corresponds to a scale that crossed the Hubble radius about 60  $e$ -folds before the end of inflation, if inflation lasted more than  $\sim 700$   $e$ -folds with a super-Hubble-radius spectral index of  $n_s = 0.94$ , then the effect of super-Hubble-radius perturbations on the locally observed value of the deceleration parameter would be sizeable.

What are the practical implications of our findings? What observations tell us is that the Universe is presently undergoing a phase of accelerated expansion, *i.e.* that the deceleration parameter is negative. Indeed, the unexpected faintness of high-redshift Type Ia supernovae (SNe Ia), as measured by two independent teams [1], has been interpreted as evidence that the expansion of the Universe is accelerating. In an *unperturbed* FRW Universe, the deceleration parameter is uniquely determined by the relative densities of the various fluids with their own equation of state

$$q_0 = \frac{1}{2}\Omega_0 + \frac{3}{2}\sum_i w_i \Omega_i, \quad (45)$$

where  $\Omega_0$  is the present-day total energy density parameter and  $\Omega_i$  are the relative contributions of the various components with equation of state  $w_i = P_i/\rho_i$  ( $P$  and  $\rho$  are the pressure and energy density, respectively). Therefore, the observation of a negative value of the deceleration parameter seems to call for the presence of a “Dark Energy” component with negative equation of state [14] and abundance  $\Omega_\Lambda \sim 0.7$ .

The need of a mysterious Dark Energy fluid seems to be challenged once a realistic *perturbed* Universe is considered. Our results show that the theoretical predictions for the local Hubble rate and the deceleration parameter are affected by a cosmic variance whose size may be large, depending upon the value of the spectral index and the overall duration of inflation. Suppose, as we have done so far, that the Universe is globally flat and matter-dominated with  $\Omega_M \sim 0.3$  and the *background* value  $q_0 = \frac{1}{2} > 0$ . Because of the large variance, however, the true locally-determined value of the deceleration parameter has non-zero probability of being less than zero! In other words, in a perturbed Universe, acceleration might not imply the existence of Dark Energy. Fig. 1 gives a qualitative sketch of the effect of the cosmic variance predicted in our talk as far as the magnitude-redshift relation is concerned: the theoretical prediction in a matter-dominated model is not a single well-defined curve, but instead is represented by a finite region whose size is determined by the cosmic variance.

## 5. Conclusions and comments

Let us close with some comments. First of all, let us notice that in a perturbed Universe the theory is unable to provide the expected value of the Hubble constant (35) and deceleration parameter (36) since they depend upon the “bare” unobservable Hubble constant  $\mathcal{H}_0$ . Therefore, in order to compute the probability that a typical observer measures values of the Hubble constant and deceleration parameter in agreement with the observations, a complete procedure would require marginalizing over the bare Hubble constant  $\mathcal{H}_0$  with some physical prior. However, predicting a large variance for the deceleration parameter can be already regarded as an indication that the tree-level value  $q_0 = \frac{1}{2}$  may not be in conflict with observing a locally accelerating Universe. A large variance does not imply the breakdown of perturbation theory either. Indeed, the density contrast  $\delta\rho/\rho \sim (\varphi\nabla^2\varphi/a^2\mathcal{H}^2)$  (valid in the synchronous gauge for *all* scales) is at most of order unity. We have checked that a nonperturbative approach – along the lines of Ref. [13] – leads to similar conclusions and that  $\varphi\nabla^2\varphi$  is in fact the first term of the expansion of  $e^{-\varphi\nabla^2\varphi}$  [16]. To get a physical intuition of the reason why there is a large contribution to the variance of the deceleration parameter one might think in terms

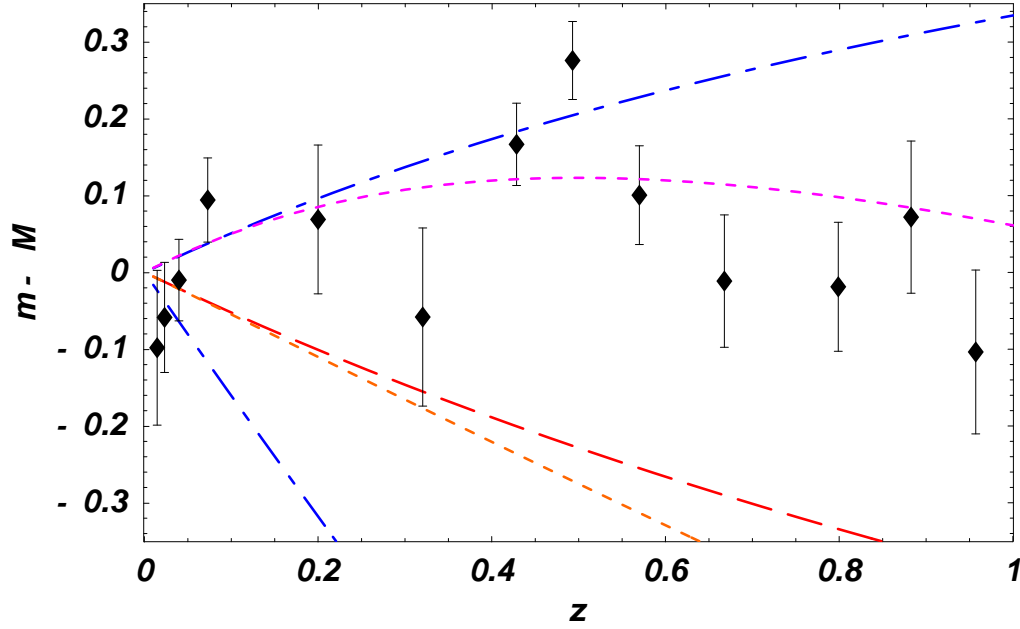


Figure 1: The residual Hubble diagram with respect to an empty Universe for a collection of type Ia Supernovae taken from Ref. [21]. The apparent magnitude  $m$  and the absolute magnitude  $M$  are related to the luminosity distance by the relation  $(m - M) = -5 + 5 \log_{10} d_L(\text{pc})$ . The red long-dashed line corresponds to  $q_0 = 1/2$  for a flat matter-dominated Universe (up to  $\mathcal{O}(z^3)$  terms); the blue dot-dashed lines correspond to the values  $q = q_0 \pm 1$ , *i.e.* to the possible size of the cosmic variance band expected in a perturbed Universe; the orange short-dashed line corresponds to the prediction of the flat matter-dominated unperturbed Universe and the short-dashed purple line to a flat unperturbed Universe with  $\Omega_\Lambda = 0.7$  and  $\Omega_M = 0.3$ .

of the energy density contrast. Indeed, a large variance of the  $\varphi \nabla^2 \varphi$  term implies a large variance of  $\delta\rho/\rho$  when summing up every Fourier mode of the perturbations. Such a variance is dominated by the infrared part of the spectrum and consequently is seen by an observer restricted to a region of size comparable to the present-day Hubble radius as a classical energy density background. Such interpretation is however not entirely correct because we are dealing with variances and not with averages of the physical observables. When inspecting the left-hand side of Einstein equations, one may regard the contribution from the first- and second-order gravitational potentials as kinetic energy of the gravitational field. Bringing them to the right-hand side and upon averaging, one may think of such terms as contributing to the classical energy density. However, this reasoning does not directly apply to variances.

Instead of performing an angular average over the solid angle, we might have determined the expected values and variances of the Hubble rate and the deceleration parameter as a function of the direction of observation. However, since the variance is dominated by the infrared part of the spectrum, we expect that the local anisotropies of the variance are as small as  $(\nabla^2 \varphi / a^2 \mathcal{H}^2)$ .

Finally, one should extend our computation to values of the physical redshift larger than unity. Indeed,

another, albeit indirect, evidence of the presence of a Dark Energy fluid, is the measurement of the location of the first Doppler peak of the CMB anisotropies. Since the theoretical prediction for such a location depends, in an unperturbed Universe, on the total energy density parameter  $\Omega_0$ , from its measurement one may infer that  $\Omega_0$  is very close to unity. Since Dark Matter amounts to only 30% of the critical energy density, it is concluded that a consistent fraction of the energy density of the Universe is made of Dark Energy. However, this deduction holds in an unperturbed Universe only. On the contrary, one should compute the location of the first Doppler peak in a real perturbed Universe. This amounts to computing the angular diameter distance  $d_A$  subtended by the sound horizon on the last scattering surface. However, since the angular diameter distance is related to the luminosity distance by the *reciprocity relation*  $d_A = d_L / (1 + z)^2$  at any order in perturbation theory, it suffices to determine the luminosity distance. The expression (24) for the luminosity distance–redshift relation holds for any redshift and any direction of observation and is therefore suitable to perform such a task. Furthermore, extending our computation (for instance to higher redshift) is required to compare the theoretical predictions of a real perturbed Universe to other physical observables supporting the Dark Energy picture in the unperturbed cosmology, such as



the Integrated Sachs-Wolfe (ISW) effect and the transition from the decelerating to the accelerating phase at redshifts of order unity [16]. In this respect we notice that the contribution to the variance of physical observables, as  $\frac{\sqrt{\text{Var}[\langle \tilde{q} \rangle_\Omega]}}{q_0}$ , increases with time and therefore both the ISW effect and the transition from the decelerating to the accelerating phase might well be consistent with a perturbed flat matter-dominated Universe. We also point out that the generic expression (??) may be used to estimate the influence of lensing on the brightness of supernovae sources. Finally, let us reiterate once more that a deeper understanding of the physical significance of long wavelength perturbations is certainly needed.

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