

A Comparison of Methods for Confidence Intervals

A.Bukin
Budker INP, 630090 Novosibirsk, Russia

Comparisons are carried out of the confidence intervals constructed with Neyman's frequentist method and with the $\Delta L = 1/2$ likelihood method, using the example of low-statistics life time estimates.

1. P.D.F. FOR LIFE TIME ESTIMATORS

For a given value τ of the true life time, the P.D.F. of a measurement is

$$\frac{dW}{dt} = \frac{1}{\tau} \exp\left(-\frac{t}{\tau}\right),$$

and so for an experiment with n measurements

$$dW = \frac{1}{\tau^n} \cdot \prod_{k=1}^n dt_k \cdot \exp\left(-\frac{t_k}{\tau}\right). \quad (1)$$

The negative log likelihood function is

$$L = n \ln \tau + \frac{1}{\tau} \cdot \sum_{k=1}^n t_k. \quad (2)$$

The maximum likelihood estimator of the lifetime can easily be found minimizing L

$$\hat{\tau} = \frac{1}{n} \sum_{k=1}^n t_k; \quad \min L = L_0 = n + n \ln \hat{\tau}, \quad (3)$$

so the probability (1) can be transformed to

$$\frac{dW}{d\hat{\tau}} = \frac{1}{(n-1)!} \cdot \left(\frac{n\hat{\tau}}{\tau}\right)^{n-1} \cdot \frac{n}{\tau} \cdot \exp\left(-\frac{n\hat{\tau}}{\tau}\right). \quad (4)$$

Given some true value τ then for any algorithm that defines a confidence interval $\hat{\tau}_{-\Delta\tau(-)}^{+\Delta\tau(+)}$ we can evaluate the coverage P :

$$P = \frac{1}{(n-1)!} \cdot \frac{n}{\tau} \cdot \int_{\hat{\tau}_1}^{\hat{\tau}_2} \left(\frac{n\hat{\tau}}{\tau}\right)^{n-1} \cdot \exp\left(-\frac{n\hat{\tau}}{\tau}\right) d\hat{\tau}, \quad (5)$$

where $\hat{\tau}_1 + \Delta\tau^{(+)} = \tau$; $\hat{\tau}_2 - \Delta\tau^{(-)} = \tau$.

2. LIKELIHOOD FUNCTION CONFIDENCE INTERVAL

The conventional Likelihood function method for finding a 68% confidence interval [1, 2] is to find the values of τ for which

$$\Delta L = L - L_0 = \frac{1}{2}.$$

In our case

$$\Delta L = n \cdot \left(\frac{\hat{\tau}}{\tau} - 1 + \ln \frac{\tau}{\hat{\tau}}\right) = \frac{1}{2}. \quad (6)$$

For example, for $n = 5$ the limits are

$$[\tau_1, \tau_2] = [0.6595\hat{\tau}, 1.6212\hat{\tau}]. \quad (7)$$

The coverage of this interval, from Equation (5), is

$$\frac{1}{4!} \cdot \frac{5}{\tau} \cdot \int_{\hat{\tau}_1}^{\hat{\tau}_2} \left(\frac{5\hat{\tau}}{\tau}\right)^4 \cdot \exp\left(-\frac{5\hat{\tau}}{\tau}\right) d\hat{\tau} = 0.6747,$$

where the integration limits, corresponding to (7), are

$$[\hat{\tau}_1, \hat{\tau}_2] = [0.6168\tau, 1.5163\tau].$$

The coverage is close to, but significantly different from, the nominal value of 0.6827.

Examples of confidence intervals obtained by this means are shown in Table I, as the values in parentheses. The 95% confidence interval was obtained using the rule $\Delta L = 2$, and 90% upper limit using a one side interval for which

$$\Delta L = [\text{erf}^{-1}(2 \cdot 0.9 - 1)]^2 \approx 0.821.$$

The coverage given by such intervals is shown in Fig. 1, evaluated using a Monte Carlo method.

3. BAYESIAN CONFIDENCE INTERVAL

For comparison we can estimate a Bayesian confidence interval for the same example of $n = 5$. In the Bayesian approach [3-5], the likelihood function is considered to be a probability density for the true parameter τ . Assuming a flat prior distribution for τ this is

$$\frac{dW}{d\tau} = P(\tau) \sim \mathcal{L} = \frac{1}{\tau^n} \cdot \exp\left(-\frac{n\hat{\tau}}{\tau}\right).$$

After normalization (for $n \geq 2$) this becomes:

$$P(\tau) = \frac{(n\hat{\tau})^{n-1}}{(n-2)! \cdot \tau^n} \cdot \exp\left(-\frac{n\hat{\tau}}{\tau}\right),$$

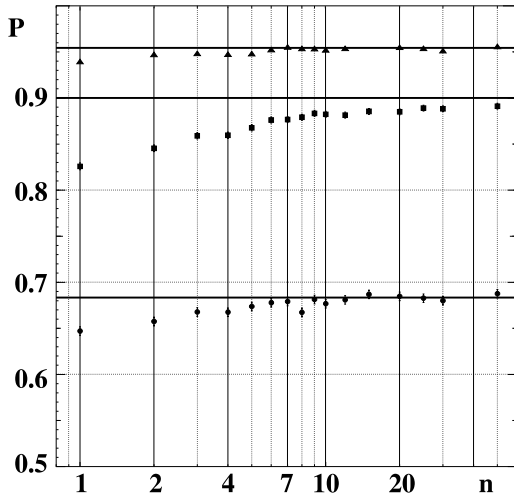


Figure 1: Coverage for likelihood function confidence intervals, evaluated by Monte Carlo. Statistical errors are shown when they exceed the size of polymarker. \blacktriangle — 95% Conf.Interv., \blacksquare — 90% Upper limit, \bullet — 68% Conf.Interv.

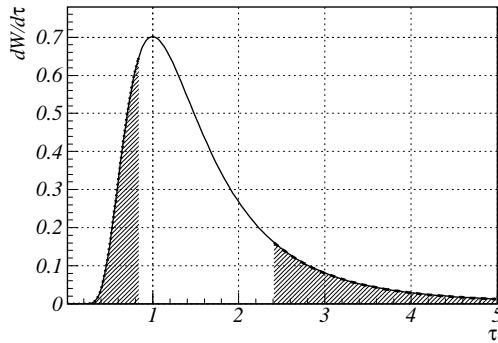


Figure 2: Probability density function for the true value of the parameter τ in a Bayesian approach (Equation (8)) with $n = 5$ and $\hat{\tau} = 1$. The shaded regions are the 16% “tails”.

which for $n = 5$ gives

$$\int_0^{\tau} dW = \left[1 + \frac{5\hat{\tau}}{\tau} + \frac{1}{2} \left(\frac{5\hat{\tau}}{\tau} \right)^2 + \frac{1}{6} \left(\frac{5\hat{\tau}}{\tau} \right)^3 \right] \cdot e^{-\frac{5\hat{\tau}}{\tau}}. \quad (8)$$

The 68% central confidence region for this distribution is (see Fig. 2):

$$\tau = \hat{\tau} \cdot \left(1_{-0.1552}^{+1.3974} \right)$$

The coverage of this region is actually not 68.27% but 64.31%.

4. NEYMAN'S CONFIDENCE INTERVAL

Neyman [5–7] proposed a frequentist construction of a confidence zone (or confidence belt) as follows

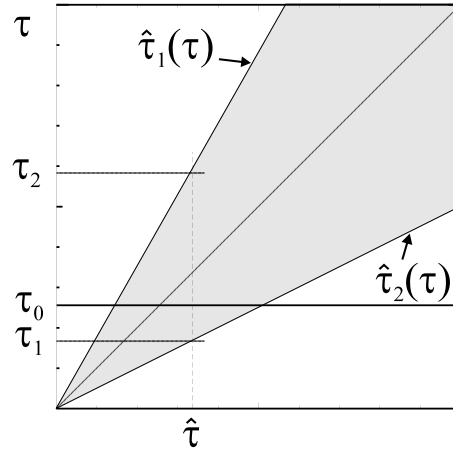


Figure 3: Illustration of the construction of a confidence zone (or confidence belt).

(see Figure 3):

1. One obtains functions $\hat{\tau}_1(\tau)$ and $\hat{\tau}_2(\tau)$ of the true parameter τ such that

$$\int_0^{\hat{\tau}_1(\tau)} \frac{dW(\hat{\tau};\tau)}{d\hat{\tau}} d\hat{\tau} = \frac{1-\beta}{2};$$

$$\int_{\hat{\tau}_2(\tau)}^{\infty} \frac{dW(\hat{\tau};\tau)}{d\hat{\tau}} d\hat{\tau} = \frac{1-\beta}{2},$$

where β is the confidence level required, here $\beta = 0.6827$. For $n = 5$ these are simply $\hat{\tau}_1(\tau) = 0.568\tau$, $\hat{\tau}_2(\tau) = 1.433\tau$, as shown in Figure 3.

2. One defines the inverse functions

$$\tau_1(\hat{\tau}) = \hat{\tau}_2^{-1}(\hat{\tau}); \quad \tau_2(\hat{\tau}) = \hat{\tau}_1^{-1}(\hat{\tau})$$

which, for a given value of $\hat{\tau}$, define the borders of the confidence interval for τ , with coverage β .

In our example, there are $\tau_1(\hat{\tau}) = 0.698\hat{\tau}$, $\tau_2(\hat{\tau}) = 1.760\hat{\tau}$.

Thus the result of a lifetime experiment of this type can be written

$$\tau = \hat{\tau} \cdot \left(1_{-0.302}^{+0.760} \right).$$

The coverage evaluated is 0.6826 — the difference of 0.0001 is purely due to rounding errors.

Table I shows these intervals for several values of n , with the likelihood approximation shown in parentheses for comparison.

Table II compares the coverage of all three methods for the $n = 5$ case.

Table I Lifetime confidence intervals obtained by Neyman’s method for various values of n , the number of measurements, and confidence levels.

n	68% C.L.		95% C.L.		90% C.L. upper limit
	$\frac{\Delta\tau^{(-)}}{\hat{\tau}}$	$\frac{\Delta\tau^{(+)}}{\hat{\tau}}$	$\frac{\Delta\tau^{(-)}}{\hat{\tau}}$	$\frac{\Delta\tau^{(+)}}{\hat{\tau}}$	
1	0.457 (0.576)	4.789 (2.314)	0.736 (0.778)	42.45 (18.06)	9.49 $\hat{\tau}$ (8.49 $\hat{\tau}$)
2	0.394 (0.469)	1.824 (1.228)	0.648 (0.682)	7.690 (5.305)	3.76 $\hat{\tau}$ (2.76 $\hat{\tau}$)
3	0.353 (0.410)	1.194 (0.894)	0.592 (0.621)	4.031 (3.164)	
4	0.324 (0.370)	0.918 (0.725)	0.551 (0.576)	2.781 (2.314)	
5	0.302 (0.341)	0.760 (0.621)	0.519 (0.541)	2.159 (1.858)	
6	0.284 (0.318)	0.657 (0.550)	0.492 (0.513)	1.786 (1.571)	
7	0.270 (0.299)	0.584 (0.497)	0.470 (0.489)	1.538 (1.374)	
8	0.257 (0.284)	0.529 (0.456)	0.452 (0.469)	1.359 (1.228)	
9	0.247 (0.271)	0.486 (0.423)	0.435 (0.451)	1.225 (1.116)	
10	0.237 (0.260)	0.451 (0.396)	0.421 (0.436)	1.119 (1.027)	
20	0.182 (0.194)	0.285 (0.261)	0.331 (0.341)	0.654 (0.621)	
50	0.124 (0.129)	0.164 (0.156)	0.232 (0.237)	0.356 (0.346)	

Table II Coverage of all three methods for $n = 5$

Method	Negative error $\Delta\tau^{(-)}/\hat{\tau}$	Positive error $\Delta\tau^{(+)}/\hat{\tau}$	Coverage, %
Likelihood	0.341	0.621	67.47
Bayesian	0.155	1.397	64.31
Neyman’s	0.302	0.760	68.26

5. CONCLUSION

- Neyman’s method for confidence intervals provides exact coverage, by construction.
- The intervals from $\Delta L = 1/2$ agree well with the Neyman intervals for large n , but differ for small n , as can be seen in Table I. In such cases they undercover, i.e. the interval is smaller than the true one.
- Bayesian confidence intervals give very different results, and can undercover or overcover.

References

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