Using α-stable Distributions to Model the $P(D)$ Distribution of Point Sources in CMB Sky Maps

D. Herranz and E.E. Kuruoğlu
Istituto di Scienza e Tecnologie dell'Informazione. CNR, via G. Moruzzi 1, 56124 Pisa, Italy
L. Toffolatti
Departamento de Física, Universidad de Oviedo, c/ Calvo Sotelo s/n, 33007 Oviedo, Spain

We present a new approach to the statistical study and modeling of point source counts in astronomical images. The approach is based on the theory of α-stable distributions. We show that the non-Gaussian distribution of the intensity fluctuations produced by a generic point source population – whose number counts follow a simple power law – belongs to the α-stable family of distributions. With the α-stable model it is possible to totally describe the non-Gaussian distribution with a few parameters which are closely related to the parameters describing the source counts. Using statistical tools available in the signal processing literature, we show how to estimate these parameters in an easy and fast way. Then we apply the method to Cosmic Microwave Background (CMB) observations where point sources appear as superimposed to the cosmological signal as well as the instrumental noise, and propose a method to statistically disentangle these contributions. In the case of the Planck mission, our technique is able to determine the parameters of the dominant point source populations with relative errors $< 5\%$ for the 30 GHz and 857 GHz channels. The formalism and methods presented here can be useful also for other experiments in other frequency ranges such as X-rays or radio Astronomy.

1. INTRODUCTION

The study of the fluctuations in the Cosmic Microwave Background (CMB) radiation has become one of the milestones of modern cosmology not only for the relevance of the study of the CMB anisotropies in itself but also for the unique opportunity it provides for the study of the physical sources (foregrounds) that are superimposed to the CMB radiation. Among the different foregrounds that appear in CMB observations, extragalactic point sources (EPS) are specially difficult to deal with. While the brightest EPS can be individually detected in CMB sky maps, the vast majority of them are faint and remain unresolved, creating a diffuse foreground that is frequently referred as 'confusion noise'. Due to the intrinsic diversity of the galaxies that contribute to this 'confusion noise', it is impossible to establish a single spectral behavior characterizing it, thus hampering the performance of classical component separation methods that use multi-frequency observations to separate the different foregrounds. Therefore, the study of the EPS foreground is a difficult task in CMB Astronomy.

There are two traditional ways of determining the main statistical properties of the EPS population. One possibility is to detect the brightest point sources in a given data set, e.g. using a linear filter to detect them (see for example [12]), and then obtain parameters such as the number counts as a function of the observed flux. This approach is limited by the current low sensitivity of detectors at CMB frequencies. The other possibility is to directly study the statistical properties of the confusion noise distribution which, in general, is mixed with the signal coming from CMB and the other foregrounds plus instrumental noise. It is a well-known fact that the intensity distribution given by unresolved sources is strongly non-Gaussian and shows long positive tails. The statistical study of the EPS confusion noise is generally performed using statistical indicators such as the moments up to a certain degree (see, e.g., [7, 8]). Anyway, the lack of an analytical form for the probability density function (*pdf*) makes it difficult to determine the optimal statistics for this kind of studies. In particular, it is not clear which moments are necessary to characterize the distribution.

In this contribution, we will focus in the application of a relatively novel formalism, the α-stable distributions, to model the *pdf* of the intensity fluctuations due to point extragalactic sources. α-stable distributions are known to be very efficient in modeling random processes with long non-Gaussian tails. Moreover, we will show that the *pdf* of the confusion noise generated by EPS whose number counts follow a simple power law, observed with a filled-aperture instrument, must follow exactly an α-stable distribution. The great advantage is that α-stable distributions are completely described by a small number of parameters instead of an infinite number of moments. Optimal techniques already existent in the signal processing literature are easy to adapt to directly extract the main parameters of the source distribution (namely, the slope of the number counts power law and its normalization) using straightforward statistical estimators specifically designed to deal with α-stable distributions. Finally, the methods can be generalized for dealing with mixtures of signals, as is the case when the EPS population is added to CMB signal and instrumental noise.

2. SOURCE COUNTS AND $P(D)$ DISTRIBUTION

Let us consider a population of EPS whose differential number counts can be described in a power law form: $n(S) = kS^{-\eta}$, $S > 0$, where η is the *slope* of the differential counts power law, k is called its *normalization* and S is the observed flux. The sources are now observed with an instrument whose angular response is $f(\theta, \phi)$. Let us now define the *deflection* D as the fluctuation field that is observed, that is $D = I - \langle I \rangle$, where I is the intensity at a given point (time) and $\langle I \rangle$ is its average value. The characteristic function $\psi(w)$ of the deflection probability distribution $P(D)$ was studied in [1] among others and, after some straightforward calculations, can be expressed as

$$\psi(w) = \exp\left\{ i\mu w - \gamma \left|w\right|^\alpha \left[1 + i\beta \mathrm{sgn}(w) \tan\left(\frac{\alpha\pi}{2}\right)\right]\right\}, \quad (1)$$

where the parameters α, β, γ and μ relate to the physical parameters of the EPS distribution and of the detector through

$$\beta = \frac{1}{\pi}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right) = 1, \quad (2)$$

$$\alpha = \eta - 1, \quad \gamma = \frac{\pi^{3/2} k \Omega_e}{2^{\alpha+1}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha+2}{2}\right)\sin\left(\frac{\alpha\pi}{2}\right)}, \quad (3)$$

$$\mu = \frac{k\Omega_e}{1-\alpha}\lim_{a\to 0^+} a^{1-\alpha}, \quad (4)$$

and where $\Omega_e = \int \left[f(\theta,\phi)\right]^{\eta-1} d\Omega$ is a geometrical factor called *effective beam solid angle*. The second equality in eq. (2) is due to the properties of the Γ function but we keep β as a 'variable' in eq. (1) for reasons that will be clear in the next section. The previous equations are valid for $1 < \eta < 3$ and can be obtained from eq. (8) in [1] just by rearranging terms (except for a $[2\pi]^\alpha$ term that corresponds to a different choice of the normalization of the beam and that is not relevant). The utility of expressing the characteristic function of the $P(D)$ in this way will be clear in the next section.

Equation (1) has an important drawback: to obtain the *pdf* of the deflections, $P(D)$, it is necessary to make the inverse Fourier transform of $\psi(w)$ which, in general, cannot be evaluated analytically. Although it can be performed numerically, the computational cost can be high if many different realizations are needed for a particular task. This has hampered the study of the $P(D)$ in the past but, as we will show in the following, it is not necessary to work with the *pdf* in all the cases. As we will show in the following sections, the study of the characteristic function itself can be insightful enough.

3. α-STABLE DISTRIBUTIONS

In section 2 we have reformulated the expression for $\psi(w)$ given by [1] so that it appears as in eqs. (1) to (4). The reason for doing so is that eq. (1) has exactly the same expression than the characteristic function of a family of distributions called in the statistical signal processing literature α*-stable* distributions. In fact, α-stable distributions are *defined* by characteristic functions such as the one in eq. (1). In this section we will very briefly overview the main properties of this kind of distributions. For a detailed description of α-stable distributions and its mathematical foundation, see [9].

The α-stable are a family of distributions that include the Gaussian distribution as a particular case. The α-stable distributions furnish tractable examples of impulsive behavior (i. e. the presence of heavy non-Gaussian tails in the *pdf*). While in a general case the full description of a non-Gaussian distribution requires the knowledge of all the cumulants of the distribution, in the case of α-stable distributions the distribution is uniquely described by means of only four parameters μ, α, β and γ. Among these parameters, α is a measure of the degree of impulsivity (non-Gaussianity) of the distribution: lower values of α corresponding to more non-Gaussian cases. The parameter β gives an idea of the asymmetry of the distribution, $\beta = 0$ corresponding to symmetric distributions whereas $\beta = \pm 1$ indicates maximum asymmetry. The parameter γ indicates the dispersion of the distribution around its maximum. The parameter μ is a simple shift in the position of the maximum.

When $\alpha = 2$, the α-stable corresponds to a Gaussian distribution with dispersion $\gamma = \sigma_g^2/2$. Then, α-stable distributions include as a particular case the Gaussian distribution, and yet they are able to describe a wider range of cases where non-Gaussianity and long tails of the distribution are present. The α-stable distributions can be shown to be the limit distribution of natural noise processes under realistic assumptions pertaining to their generation mechanism and propagation conditions ([6]).

So far we have shown that the $P(D)$ distribution, originated by an EPS power law whose number counts follow a power law, is a member of the α-stable family of distributions. Moreover, from eq. (2) we know that for this case $\beta = 1$ (i.e. the distribution shows a tail only to positive values), so we need to know only three parameters α, γ and μ to have a full statistical knowledge of the $P(D)$ distribution. As already said, μ is a simple shift in the position of the maximum and its value does not affect the shape of the $P(D)$. Therefore, the knowledge of only two statistical parameters, α and γ, gives us a complete description of the statistics of the $P(D)$ distribution. From eq. (3) we can see that the values of α and γ depend on the beam of the experiment, that is usually well-known, and the EPS number counts parameters k and η. Hence, if we are able to determine α and γ we will determine as well the properties of the EPS number counts.

The α-stable distributions have been thoroughly studied in the signal processing literature. In particular, a great effort has been devoted to design statistically optimal methods to estimate the α-stable parameters α and γ (see for example [5]). As a result of this effort, in the signal processing literature there is a plethora of available methods to perform statistical inference on α-stable environments. In this work we will focus on the application of existent techniques for α-stable parameter extraction in order to obtain optimal estimators of the parameters describing the differential number counts of the EPS population, namely the slope η and the normalization k.

3.1. Point source parameter extraction using α-stable distributions

As mentioned in the introduction, most statistical studies of the $P(D)$ have been performed in the past by calculating moments of the observed deflections and then fitting them to theoretical models. However, it can be proved that these 'classical' methods based on the study of integer order moments of the distribution (variance, skewness, etc) have bad convergence properties in α-stable environments. The intuitive reason for this is that the moments of order ≥ 2 of the $P(D)$ are not well defined (except for the case $\alpha = 2$), as can be seen from eq. (1) just by remembering that the moments of the distribution are related to the derivatives of the characteristic function through

$$i^n M_n = \left[\frac{d^n\psi}{dw^n}\right]_{w=0}, \qquad (5)$$

where M_n is the moment of order n. Therefore, integer moment-based methods are not reliable as a means to learn information on the EPS population from the $P(D)$.

Fortunately, from eq. (3) the values of the EPS power law k and η can be directly obtained if we are able to estimate first the α-stable parameters α and γ. Over the past years a number of efficient estimators for the parameters of α-stable distributions have been developed. In [5] several groups of techniques to determine the parameters α and γ are described, for example the so-called *logarithmic moments estimators* and the *fractional-lower order moments estimators*. These methods exploit the fact that while integer order moments are not well defined for this kind of distribution, it is possible to calculate non-integer order moments that are well defined from the mathematical point of view. By means of the mentioned estimators, given an image of the sky containing EPS confusion noise it is easy to estimate the parameters α and γ (and conversely k and η) in an unbiased and efficient way.

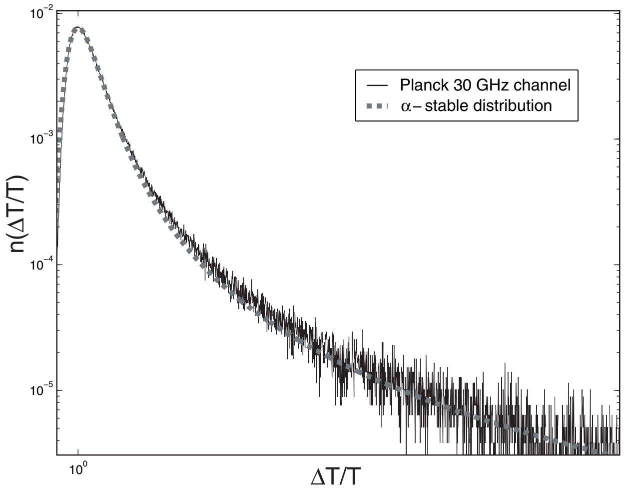

Figure 1: $P(D)$ distribution of the EPS at the 30 GHz channel (solid line) and its corresponding α-stable model (gray dotted line).

A complete description of these estimators and their performance is beyond the scope of this work, and can be found in [5]. The application of these methods to the study of EPS have been studied in profundity in [3].

When the α-stable distribution generated by the EPS is 'corrupted' with another signal (for example CMB) the problem becomes much more complicated, since the resultant distribution of the mixture is not a pure α-stable but the convolution of the α-stable *pdf* with the *pdf* of the other signals (in the case of the CMB, a Gaussian). In that case, the estimators mentioned above are not any longer optimal and it is necessary to use directly the expression for the characteristic function. As an example, let us consider the case in which the α-stable signal is mixed with a Gaussian component (as in the case of the mixture of CMB, EPS and instrumental noise). In that case, the resultant characteristic function will be similar to the one in eq. (1) but an additional term $\sigma^2 w^2/2$ will appear inside the exponential (σ^2 is the variance of the Gaussian component). It is possible to simplify the characteristic function using a simple transformation of the data called centro-symmetrization ([5]). If $\{X_i,\ i = 1, \ldots, N\}$ is the sequence of data, we can generate a centro-symmetrized sequence of data $\{X_i^S\}$ just by making $X_j^S = X_{2j} - X_{2j-1}$ for all $j = 1, \ldots, N/2$. In that case, after centro-symmetrization the new characteristic function can be proven to be:

$$\psi_{mix}^S(w) = \exp\left[-2\gamma\,|w|^\alpha - \sigma^2 w^2\right], \qquad (6)$$

where σ is the dispersion of the Gaussian component, that may be either a known value or one of the parameters to estimate, and the superscript S stands for centro-symmetrization. The parameter extraction in the case of mixtures with characteristic functions such as in eq. (6) has been studied by [4]. One possibility to perform the estimation is to mimimize the distance

$$D_\Theta \equiv \int_{-\infty}^{\infty} \left|\hat{\psi}_N(w) - \psi_\Theta(w)\right|^2 W(w)dw, \qquad (7)$$

with respect to the set of parameters $\Theta = \{\alpha, \gamma, \sigma\}$, where $\hat{\psi}_N(w)$ is the empiric characteristic function $\hat{\psi}_N(w) \equiv \frac{1}{N}\sum_{m=1}^{N} e^{iwx(m)}$ and $W(w)$ is an appropriate weighting function [4]. For example, the choice $W(w) = \exp(-w^2)$ allows the integral (7) to be solved by means of Gauss-Hermite quadratures, which is computationally convenient.

4. APPLICATION TO PLANCK SIMULATED DATA

To test the ideas shortly reviewed in the last sections, we performed realistic simulations of the sky at the nine frequencies that will be covered by Planck, containing CMB, EPS and instrumental noise at the Planck expected levels. The simulations cover the whole sky and use the HEALPix (Hierarchical, Equal Area and iso-latitude) pixelisation scheme. For each channel, the resolution and the beam size correspond to the technical specifications of the mission. CMB emission has been simulated assuming a flat ΛCDM Universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ The C_l's were generated with the CMBFAST code ([10]). The EPS simulations were done using the point source model given by [11] (hereafter, T98 model) for each one of the Planck channels. Finally, Gaussian white noise was added to each channel using the expected noise levels for Planck.

Note that the EPS in the model by [11] do not follow a pure, single power law distribution as presented in section 2. However, the power law is a good approximation for the true behaviour of the number counts over a significant range of fluxes. In [2] the authors have shown that the α-stable model is a good representation of the $P(D)$ generated T98 model in all the Planck channels, specially the lower and higher frequency ones. As an example, in fig. 1, the $P(D)$ of the EPS in the 30 GHz Planck channel is compared with an α-stable model whose parameters have been determined from the T98 simulated sky using the logarithmic estimators mentioned in section 3.1. The model fits the data almost perfectly.

From the simulations containing the mixture of CMB, EPS and instrumental noise we tried to estimate the parameters of the EPS number counts η and k using the α-stable model and minimizing the distance (7) between the empiric characteristic function of the simulated data and the model (eq:charfmix). This gives us the estimates of the α-stable parameters

Table I Results for the most significant Planck channels. The values of k are expressed in $10^{-5} \times \mathrm{Jy}^{\alpha}$ pixel^{-2} units. Values of σ are expressed in $10^{-5} \times \Delta T/T$ units (thermodinamic temperature). The subscript f refers to the best-fit values of the EPS simulations, whereas the subscript e makes reference to values estimated by mimizing the distance (7).

ν (GHz)	η_f	η_e	k_f	k_e	$\sigma_{CMB+n,f}$	$\sigma_{CMB+n,e}$
30	2.26	2.24	5.72	3.83	3.14	3.16
857	2.63	2.71	6.34	5.22	1940	1940

α and γ that can be used in turn to estimate η and k using eq. (eq:gamma). Results are shown in table I for the two most significant Planck channels from the point of view of the EPS: the 30 GHz channel, that is dominated by flat-spectrum radio sources, and the 857 GHz channel, that is dominated by dusty far-IR galaxies. These two channels are very useful to study both kinds of EPS populations in a frequency range where their properties are not well known. Table I shows that our method is able to estimate the parameters of the number counts of the dominating EPS populations with significant accuracy. In particular the slope η is determined with relative errors lower than 5%.

5. CONCLUSIONS

In this work we have introduced the formalism of α-stable distributions as a useful tool for the statistical modelling of the intensity fluctuations due to point sources in astronomical images. We have shown that when the number counts of the sources follow a power law the characteristic function of the resultant distribution is exactly an α-stable one. The α-stable model allows us to describe the $P(D)$ distribution with a few parameters directly related to the parameters of the number counts law and to design statistically optimal and fast estimators to extract these parameters of the EPS populations from $P(D)$ distribution alone or mixed with other astrophysical sources whose *pdf* is known. We have proposed a method to extract the relevant information of the EPS population and the CMB plus noise joint variance in the Planck sky maps using the empirical characteristic function We have applied our technique to *realistic* Planck simulations containing CMB, instrumental noise and extragalactic point sources. The technique succeeds in extracting the α-stable parameters of the EPS distribution as well as the variance of the CMB plus noise contribution in the most relevant frequency channels of Planck, the 30 GHz one (at which the EPS are dominated by radio galaxies) and the 857 GHz one (dominated by dusty galaxies). The method uses all the information in the data, taking into account bright sources as well as very faint ones which contribute to the confusion noise.

The method presented here could also be applied to different fields in Astronomy, including the X-ray background and radio Astronomy. The application to these fields is now under study.

Acknowledgments

DH and LT acknowledge partial financial support from the European Community's Human Potential Programme under contract HPRN-CT-2000-00124 CMBNET. This work has used the software package HEALPix (Hierarchical, Equal Area and iso-latitude pixelisation of the sphere, http://www.eso.org/science/healpix), developed by K.M. Gorski, E. F. Hivon, B. D. Wandelt, J. Banday, F. K. Hansen and M. Bartelmann.

References

[1] Condon, J.J., 1974, ApJ, 188, 279.
[2] Herranz, D., Kuruoğlu, E.E. & Toffolatti, L., A&A submitted (astro-ph/0307114).
[3] Herranz, D., Kuruoğlu, E.E. & Toffolatti, L., 2003, *'Using α-stable distributions to model the point source population in CMB sky maps'*, Technical Report 2003-TR-16, ISTI-CNR.
[4] Ilow, J. & Hatzinakos, D., 1998, Signal Processing, 65, 199.
[5] Kuruoğlu, E.E., 2001, IEEE Trans. Signal Proc., 49, 2192.
[6] Nikias, C.L. & Shao, M., *'Signal Processing with α-Stable Distributions and Applications'*, Wiley & Sons, 1995.
[7] Pierpaoli, E., 2003, ApJ submitted (astro-ph/0301563).
[8] Rubiño-Martín, J.A. & Sunyaev, R.A., 2003, MNRAS submitted (astro-ph/0211430).
[9] Samorodnitsky, G. & Taqqu, M.S., *'Stable non-Gaussian random processes: stochastic models with infinite variance'*, New York, NY: Chapman & Hall, 1994.
[10] Seljak, U. & Zaldarriaga, M, 1996, ApJ, 469, 437.
[11] Toffolatti, L., Argüeso, F., De Zotti, G., Mazzei, P., Franceschini, A., Danese, L. and Burigana, C. 1998, MNRAS, 297, 117.
[12] Vielva, P., Martínez-González, E., Gallegos, J., Toffolatti, L. & Sanz, J. L., 2003, MNRAS, 344, 89.