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New Type of Exact Solvability and of a Hidden Nonlinear Dynamical Symmetry in Anharmonic Oscillators

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Schrödinger bound-state problem in D dimensions is considered for a set of central polynomial potentials containing $2q$ arbitrary coupling constants. Its polynomial (harmonic-oscillator-like, quasi-exact, terminating) bound-state solutions of degree N are sought at an $(q+1)$-plet of exceptional couplings/energies, the values of which comply with (the same number of) termination conditions. We revealed certain hidden regularities in these coupled polynomial equations and in their roots. A particularly impressive simplification of their pattern occurred at the very large spatial dimensions $D \gg 1$ where all the "multi-spectra" of exceptional couplings/energies proved equidistant. In this way, one generalizes one of the key features of the elementary harmonic oscillators to (presumably, all) non-vanishing integers $q > 0$.

1 Introduction: quasi-exact terminating solutions

The never-ending story of the search for exact bound-state solutions started with the very emergence of quantum mechanics. Its part which pays attention to the polynomial central potentials $V(r)$ in the ordinary differential "radial" Schrödinger equation

$$-\psi''(r) + \frac{\ell(\ell+1)}{r^2}\,\psi(r) + V(r)\,\psi(r) = E\,\psi(r), \qquad \psi \in L_2(0,\infty) \tag{1}$$

is not much younger. Indeed, the elementary nineteen-century mathematics proves sufficient for the construction of $\psi(r)$ from (1) in analytic form with, say, a power-series ansatz for components $A(r)$ and $B(r)$ in

$$\psi(r) = r^{\ell+1}\,A(r)\,e^{B(r)}. \tag{2}$$

The most common harmonic-oscillator model $V^{(\mathrm{HO})}(r) = \omega^2 r^2$ provides a particularly appealing illustration of such an approach because the semiclassical exponent $B^{(\mathrm{HO})}(r) = -\frac{1}{2}\omega\, r^2$ in equation (2) describes the correct asymptotic decrease of $\psi^{(\mathrm{HO})}(r)$ while the Taylor series for $A^{(\mathrm{HO})}(r)$ degenerates to a polynomial as well.

A broad family of polynomial potentials admits a similar specification of their "asymptotically optimal" polynomial exponents $B(r)$. *Vice versa*, for all the "canonical" polynomial WKB-like exponents

$$B^{(\mathrm{WKB})}(r) = \frac{1}{2}\alpha_0 r^2 + \frac{1}{4}\alpha_1 r^4 + \cdots + \frac{1}{2q+2}\alpha_q r^{2q+2} \tag{3}$$

and for all the "canonical" power-series choices of the ansatz (2),

$$\psi(r) = \sum_{n=0}^{\infty} h_n\, r^{2n+\ell+1} \exp\left[-B_{\mathrm{WKB}}(r)\right] \tag{4}$$

potentials may be polynomials with $2q+1$ *arbitrary* couplings,

$$V(r) = V^{[q]}(r) = g_0\, r^2 + g_1\, r^4 + \cdots + g_{2q}\, r^{4q+2} = \left[\Omega^{(q)}(r)\right]^2 r^2 + S^{(q)}(r). \tag{5}$$

The first, asymptotically dominating auxiliary factor

$$\Omega^{(q)}(r) = \alpha_0 + \alpha_1 r^2 + \cdots + \alpha_q\, r^{2q} \tag{6}$$

is determined precisely by the $(q+1)$-plet of the WKB-related free parameters while

$$S^{(q)}(r) = G_0 r^2 + G_1 r^4 + \cdots + G_{q-1} r^{2q}$$

carries just the asymptotically less relevant information about the full force $V(r)$ at any $q \geq 1$. All the relevant details may be found in our older review of the related, so called Hill-determinant bound-state method [1].

Due to the one-to-one correspondence $g_{2q} = {\alpha_q}^2$, $g_{2q-1} = g_{2q-1}(\alpha_q, \alpha_{q-1}) = 2\,\alpha_{q-1}\,\alpha_q$, ... etc (or, in opposite direction, $\alpha_q = \sqrt{g_{2q}} > 0$, $\alpha_{q-1} = g_{2q-1}/(2\alpha_q)$ etc), we may work with both the old and new couplings. Moreover, using the trivial changes of variables in our differential equation (1) ($r^2 = x$ etc, with all details described again thoroughly in the above-mentioned review [1]), the canonical potential (5) generates the whole series of its mathematical equivalents,

$$U^{[q]}(x) = f_0\, x^{-1} + f_1\, x + f_2\, x^2 + \cdots + f_{2q}\, x^{2q}, \qquad f_{2q} > 0, \tag{7}$$

$$W^{[q]}(z) = h_0\, z^{-3/2} + h_1\, z^{-1} + \cdots + h_{2q-1}\, z^{q-3/2} + h_{2q}\, z^{q-1}, \qquad h_{2q} > 0, \tag{8}$$

etc. Thus, the well known one-to-one mapping between harmonic oscillator and Coulombic spectra of bound states exemplifies the transition from (5) to (7) at $q = 0$. Similarly, we shall not distinguish, at any $q \geq 0$, between the wave functions pertaining to the symmetric well (5) and to its descendants (7) or (8).

Returning to the simplest $q = 0$ models, let us emphasize that they are extremely exceptional, possessing

- *all* their wave functions in terminating Taylor-series form (note that their factors $A^{(\mathrm{HO})}(r)$ are Laguerre *polynomials*);
- *all* their energies in closed form (note that the HO set forms an *equidistant* family).

As a consequence, one should not be surprised by the existence of numerous symmetries (and even supersymmetries [2]) in the underlying Hamiltonians at $q = 0$.

At $q \neq 1$, many (though not all) of these symmetries become hidden or lost (see the monograph [3] for a wealth of details). Still, one reveals that the exceptional polynomial solutions exist in the form

$$\psi(r) = \sum_{n=0}^{N-1} h_n^{(N)}\, r^{2n+\ell+1} \exp\left[-B_{\mathrm{WKB}}(r)\right] \tag{9}$$

at all the finite integers $N \geq 1$ and $q \geq 1$ (see the review of this point in our recent paper [4]).

2 Schrödinger equation at large ℓ

For our canonical potential (5), the use of the quasi-exact solution ansatz (9) converts the differential equation (1) in algebraic recurrences

$$\left(\begin{array}{ccccccc} B_0 & C_0 & & & & & \\ A_1^{(1)} & B_1 & C_1 & & & & \\ \vdots & & \ddots & \ddots & & & \\ A_q^{(q)} & \ldots & A_q^{(1)} & B_q & C_q & & \\ & A_{q+1}^{(q)} & \ldots & A_{q+1}^{(1)} & B_{q+1} & C_{q+1} & \\ & & \ddots & & & \ddots & \ddots \end{array} \right) \left(\begin{array}{c} h_0^{(N)} \\ h_1^{(N)} \\ \vdots \\ h_{N-1}^{(N)} \\ 0 \\ \vdots \end{array} \right) = 0 \tag{10}$$

with coefficients

$$
\begin{aligned}
&C_n = (2n+2)\,(2n+2\ell+3), \qquad B_n = E - \alpha_0\,(4n+2\ell+3),\\
&A_n^{(1)} = -\alpha_1\,(4n+2\ell+1) + \alpha_0^2 - g_0, \qquad A_n^{(2)} = -\alpha_2\,(4n+2\ell-1) + 2\alpha_0\alpha_1 - g_1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\\
&A_n^{(q)} = -\alpha_q\,(4n+2\ell+3-2q) + (\alpha_0\alpha_{q-1} + \alpha_1\alpha_{q-2} + \cdots + \alpha_{q-1}\alpha_0) - g_{q-1},\\
&n = 0, 1, \ldots .
\end{aligned}
\tag{11}
$$

They form a finite set of algebraic equations which can hardly be solved non-numerically at the generic q and N. In most cases, people only pay attention to their very first "square-matrix" special case at $q = 1$ [3].

In what follows, let us admit an arbitrary pair of integers q and N and, for simplification, accept merely the assumption that the spatial dimension D is very large. In the other words, on the basis of the well known formula

$$
\ell = \frac{D-3}{2}, \frac{D-1}{2}, \frac{D+1}{2}, \frac{D+3}{2}, \ldots
\tag{12}
$$

we postulate that these numbers are *all* very large, $\ell \gg 1$. This is a key assumption of our forthcoming considerations, inspired by the well known fact that for *any* potential $V(r)$, the practical solution of radial Schrödinger equations is easier in the domain of the large angular momenta (a deeper explanation may be found, say, in the randomly selected paper [5] or in many other relevant papers with citations listed therein).

3 Terminating solutions at large ℓ

In our present very specific context of the incomplete exact solvability, we should not be misled by the observation that virtually all the contemporary $\ell \gg 1$ calculations are based on the perturbation expansions using the "most natural" artificial expansion parameter $1/\ell$. Rather, we shall follow our older paper [6] (on the $q = 2$ partial solvability) as our most relevant guidance in what follows, having in mind the use of a generalized expansion parameter $1/\ell^{\text{const}}$.

In its spirit, our first step will consist in a re-scaling of our over-complete linear set (10), $Q(E)h = 0$, in accord with the simple rule

$$
h_n^{(N)} = p_n/\mu^n, \qquad \mu = \mu(D) = \left(\frac{D}{2\alpha_q}\right)^{1/(q+1)} .
\tag{13}
$$

In this way, all the elements of our non-square band-matrix "Hamiltonian" $Q(E)$ become tremendously simplified in the leading order in $D \gg 1$. In effect [4], we then have to solve the much easier algebraic problem with N columns and $N + q - 1$ rows,

$$
\left(
\begin{array}{cccccc}
s_1 & 1 & & & & \\
s_2 & s_1 & 2 & & & \\
\vdots & & \ddots & \ddots & & \\
s_q & \vdots & & s_1 & N-2 & \\
N-1 & s_q & & & s_1 & N-1 \\
 & N-2 & s_q & & \vdots & s_1 \\
 & & \ddots & \ddots & & \vdots \\
 & & & 2 & s_q & s_{q-1} \\
 & & & & 1 & s_q
\end{array}
\right)
\left(
\begin{array}{c}
p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1}
\end{array}
\right) = 0,
\tag{14}
$$

where we merely re-scaled the energy E $(= -g_{-1})$ and couplings $\{g_0, \ldots, g_{q-2}\}$ in linear manner,

$$g_{k-2} = -\alpha_{k-1} D - \frac{\tau}{\mu^{k-1}}\, s_k, \qquad k = 1, 2, \ldots, q, \qquad \tau = \left(2^{q+2}\, D^q\, \alpha_q\right)^{1/(q+1)}. \tag{15}$$

At this stage of development, one does not see any perceivable progress yet. The required "multi-spectrum" of q different "multi-eigenvalues" $s_1, \ldots, s_q$ seems obtainable only by purely numerical means at all the larger q or N.

4 A brief summary of the known non-numerical results

Besides the well known non-numerical $q = 0$ solutions of the harmonic oscillator, also the first nontrivial $q = 1$ case does not need to be discussed too thoroughly. One just solves the *linear algebraic* eigenvalue problem with spectrum which proves equidistant in the limit $D \to \infty$ (see [7]). For inspiration, let us briefly return to this $q = 1$ model in more detail: In equation (14), the unknown quantities $s = s_1$ represent either the energies of the sextic oscillator of equation (5) or, *mutatis mutandis*, the charges of the spiked and shifted harmonic oscillator [3], the values of which form a finite set,

$$s_1 = N-1, N-3, N-5, \ldots, -N+3, -N+1, \qquad q = 1. \tag{16}$$

We may also very quickly recollect the next $q = 2$ case where the solution of our problem has been found and discussed thoroughly and with direct reference to the quartic oscillator potential (7) in 1999 [6]. The first nontrivial form of our equation (14) has been solved there in closed form for so many values of N that the results could be extrapolated to all $N = 1, 2, \ldots$. In particular, the resulting energies were shown there to form the multiplets

$$\begin{aligned} &s_1 = s_2 = N-1, N-4, N-7, \ldots, -K+2, \qquad N = 2K, \\ &s_1 = s_2 = N-1, N-4, N-7, \ldots, -K, \qquad N = 2K+1, \qquad q = 2 \end{aligned} \tag{17}$$

i.e., $s = N+2-3j$, $j = 1, 2, \ldots, [(N+1)/2]$ at any wave-function degree $N = 1, 2, \ldots$. In the next step of development, an "optimal" calculation method has been discovered in our subsequent study in 2003 [4]. There, we succeeded in the re-interpretation and re-calculation of all the above $q = 2$ energies as special *real* roots selected out of "hidden-symmetric" complex triplets $E_m = s^{1/3} e^{2\pi m/3}$, $m = 1, 2, 3$. The intermediate, auxiliary variables s were produced again, numerically, as roots of a set of polynomials $s^6 - 7s^3 - 8 = 0$ $(N = 3)$, $s^{10} - 27s^7 + 27s^4 - 729s = 0$ $(N - 4)$ etc.

In the same paper, the complete solution of the next problem with $q = 3$ has been offered. The very similar sequence of the secular polynomials $F[s^4]$ has been obtained there, with $F[s^4] = s^9 - 12s^5 - 64s = 0$ at $N = 3$, with $F[s^4] = s^{16} - 68s^{12} + \cdots + 50625 = 0$ at $N = 4$ etc. This gives the result

$$s_2 = N-1, N-5, N-9, \ldots, N+3-4\left[\frac{N+1}{2}\right], \qquad q = 3, \tag{18}$$

the presentation of which proves hindered by the occurrence of the other two independent eigenvalues s_1 and s_3. Incidentally, the latter quantities coincide and may be specified by a closed formula. For our present purposes it is sufficient to elucidate the N-dependence of the resulting multi-spectrum via its first few examples,

$$\left|\begin{array}{c} s_1\\ s_2\\ s_3 \end{array}\right| = \left|\begin{array}{r} -2\\ 2\\ -2 \end{array}\right|\begin{array}{r} 0\\ 2\\ 0 \end{array}\left|\begin{array}{r} 2\\ 2\\ 2 \end{array}\right|\begin{array}{r} 0\\ -2\\ 0 \end{array}\right|, \qquad N=3, \tag{19}$$

$$\left|\begin{array}{c} s_1\\ s_2\\ s_3 \end{array}\right| = \left|\begin{array}{r|r|r|r|r|r} -3 & -1 & 1 & 3 & -1 & 1\\ 3 & 3 & 3 & 3 & -1 & -1\\ -3 & -1 & 1 & 3 & -1 & 1 \end{array}\right|, \qquad N=4, \tag{20}$$

$$\left|\begin{array}{c} s_1\\ s_2\\ s_3 \end{array}\right| = \left|\begin{array}{r|r|r|r|r|r|r|r|r} -4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 0\\ 4 & 4 & 4 & 4 & 4 & 0 & 0 & 0 & -4\\ -4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 0 \end{array}\right|, \qquad N=5 \tag{21}$$

etc. As long as we solved equation (14) for a sufficiently long series of "dimensions" N, we succeeded in determining the general, extrapolated pattern for s. The paper itself should be consulted for more details since the latter spectrum proves to have an impressively compact representation in integer arithmetic, with $s_2 = N+3-4j$, $j = 1, 2, \ldots, [(N+1)/2]$ etc.

At the time of its derivation, this feature looked ephemeral as definitely failing to hold at the next degree $q = 4$ of the potential. At the same time, although the computer-assisted solution of equation (14) ceased to be feasible, the $q = 4$ problem looked extremely interesting as long as it involves not only a less appealing polynomial of the symmetric well (5) of the eighteen degree, but also much more interesting octic-polynomial anharmonic oscillator (7) and, first of all, the phenomenologically most important case of the asymptotically cubic force (8).

5 Brand new result: The case of $q = 4$

Before a thorough description of our present continuation of the systematic and efficiently computerized symbolic-manipulation study we should re-emphasize that at any $q \geq 2$, our algebraic set of $N+q-1$ equations (14) is nonlinear. It is formed by the sums of the one- and two-term products of the $N+q$ unknown quantities. In the latter role we selected the $N-1$ arbitrarily normalized Taylor coefficients p_j and the q multi-eigenvalues $s_1, s_2, \ldots, s_q$.

In order to convey the feeling of what happens when one chooses the different strategies of the usual elimination, we may start experimenting at $N = 1$ an find that the only real solution is trivial, $s_1 = s_2 = s_3 = s_4 = 0$. The solution at $N = 2$ is also unambiguous. Once we abbreviate $s_1 = t$, $s_2 = r$, $s_3 = \tilde{r}$, $s_4 = \tilde{t}$, we may fix the norm by setting $p_1 = 1$ and proceed, recurrently, in an upwards direction in (14). This gives $p_0 = -\tilde{t}$ while $\tilde{t}^5 = 1$, providing finally the unique real root $\tilde{t} = 1$ and, subsequently, full solution with $\tilde{r} = r = t = 1$.

We have seen in ref. [8] that the similar construction is also feasible at $N = 3$. Proceeding in an upward-downward-symmetric recurrent manner we now normalize $p_1 = 1$ and infer that $p_0 = -1/t$ while $p_2 = -1/\tilde{t}$. Next we abbreviate $t\,\tilde{t} = \xi$ and re-write the remaining four lines of equation (14) in the following form,

$$r\tilde{t}/t = \xi - 2, \qquad \tilde{r}\tilde{t}/t^2 = \xi - 3, \qquad rt/\tilde{t}^2 = \xi - 3, \qquad \tilde{r}t/\tilde{t} = \xi - 2. \tag{22}$$

The ratio of the two odd or two even lines eliminates r or $\tilde{r}$, respectively, and we get the same quantity $(\xi-2)/(\xi-3)$. Its next-step elimination gives the desired simplification $\tilde{t}^5 = t^5$, with the only real solution $\tilde{t} = t$. Then the first and last line of equation (22) define easily $r = r(\xi)$ and $\tilde{r} = \tilde{r}(\xi)$ while, finally, the appropriate insertions in one of the middle lines results in the "secular" equation

$$\mathcal{P}(t) = t^3 - t^2 - 3\,t + 2 = 0 \tag{23}$$

with the following three real roots,

$$t_1 = 2, \qquad t_{2,3} = \frac{1}{2}\left(-1 \pm \sqrt{5}\right) \tag{24}$$

(cf. also Table 1 below).

Table 1. Columns of energy-roots $s_4 = s_4(N) = \frac{1}{2}\left(P_N \pm \sqrt{5} \cdot Q\right)$ at $q = 4$.

	P_N													Q
$N =$	1	2	3	4	5	6	7	8	9	10	11	12	…	*all* N
	0	2	4	6	8	10	12	14	16	18	20	22	…	0
					−2	0	2	4	6	8	10	12	…	0
									−4	−2	0	2	…	0
													…	⋮
			−1	1	3	5	7	9	11	13	15	17	…	1
							−3	−1	1	3	5	7	…	1
											−5	−3	…	1
					−2	0	2	4	6	8	10	12	…	2
									−4	−2	0	2	…	2
													…	⋮
							−3	−1	1	3	5	7	…	3
											−5	−3	…	3
									−4	−2	0	2	…	4
													…	⋮
											−5	−3	…	5
													…	⋮
total #	1	1	3	3	6	6	10	10	15	15	21	21	…	⋮

The elimination of the unknowns becomes almost prohibitively tedious from $N = 5$ on. The comparatively high complexity of the (necessarily, computerized) reduction of our multi-polynomial problem (14) to the single polynomial "secular" equation $\mathcal{P}(s) = 0$ is accompanied by an extremely quick growth of the degree of our secular polynomials with N. At the same time, there exists an empirically observed fact [4] that, paradoxically, the Gröbner-based solution of the next $q = 5$ problem is in fact more easy than its $q = 4$ predecessor. This underlines the key importance of the revealed "missing pattern" in the $q = 4$ roots as presented here in Table 1.

For compensation, the impression produced by the high degree of our polynomials $\mathcal{P}(s)$ is again strongly weakened when we notice that these functions depend in effect just on the powers of the new auxiliary variable $z = s^{q+1}$. This has several consequences. Firstly, we see that even if all the auxiliary roots z themselves were real, the final number of the complex roots s would still be much higher than that of their real and, hence, "physically acceptable" partners. Secondly, the formidable task of the search for the real roots in the closed form did not prove to be as prohibitively difficult as it might have appeared at first sight.

In Table 1 summarizing the results of our $q = 4$ construction, a climax of our present effort is perceived in an absolute regularity of all its items. The pattern of extrapolation of these results beyond their boundaries set by the computer is already fully obvious,

$$\begin{aligned} & s = s_4(N) = \frac{1}{2}\left(P_N \pm \sqrt{5}\cdot Q\right), \\ & P(N) = P(N)_{(j,k)} = 2N + 13 - 5j - 10k, \qquad Q = Q_{(j,k)} = j - 1, \\ & j, k = 1, 2, \ldots, \qquad 2j + 4k \le N + 5 \end{aligned} \tag{25}$$

and does not seem to create any doubts and/or unanswered questions. With respect to the non-doubling of the $j = 1$ (i.e., $Q = 0$) roots, the elementary formula

$$\text{total}\,\# = \left(\begin{array}{c} K+1 \\ 2 \end{array}\right), \qquad K = \left[\frac{N+1}{2}\right], \tag{26}$$

also expresses the total number of the separate items in each column of Table 1, i.e., of the real energy roots at each fixed N.

6 Summary and outlook

We reported the progress achieved in the field where the quasi-exact solutions are sought for the radial equations where the potentials are "next-to-most-common". Our main result is that we were able to construct the energies for the class of the $q = 4$ models which involves the important and very popular cubic and octic anharmonic oscillators.

Our main task lied in the necessity of making the form of our exact and polynomial wave functions $\psi(r)$ closed and explicit for *all* their integer degrees $N = 1, 2, \ldots$. The main difficulty in this direction emerges from the fact that the dimensions of the matrices we need (or degrees $\mathcal{N}(N)$ of the "effective" secular polynomials) seem to grow extremely quickly with N. Unfortunately, we did not find any regularity in the series $\mathcal{N}(5) = 70$, $\mathcal{N}(6) = 126$, $\mathcal{N}(7) = 210$, $\mathcal{N}(8) = 330$, $\mathcal{N}(9) = 495$, $\mathcal{N}(10) = 715$ etc.

Our task was quite challenging formally, and we must admit that we did not even expect that its solution could appear very soon. Our biggest surprise occurred in the form of the explicit factorizability of all the polynomials $\mathcal{P}(s)$ over the (sometimes called "surdic") field of the quasi-complex numbers $a + b\sqrt{5}$ with rational coefficients.

One cannot resist to re-emphasize here that after a certain suitable re-numbering and re-grouping of levels, the spectrum of our "solvable" $q = 4$ couplings/energies remains expressible directly in terms of integers. Such a type of a generalized equidistance re-emerges also in the next, $q = 5$ case (i.e., for the class of potentials involving the square-root-power-series form of the quartic oscillator, etc). The analysis of $q = 5$ already lies beyond the scope of our present study. Even in the purely formal setting, it lies on the very boundary of the capacity of the computers and software which are at our disposal at present. We were still able to factorize the corresponding effective secular polynomials at a few N in ref. [4], and we obtained the regular recipe $s_5 = N-1, N-2, N-3, \ldots, -N+1$ there. One feels how this achievement was formidable since at $N = 7$, the extreme coefficient c in the secular polynomial $F(s) = s^{127} - 60071 s^{121} + \cdots + cs$ possesses as many as 72 decimal digits and, hence, looks like a candidate for being placed in the Guiness' book of records in the factorization context.

In conclusion, let us point out that our results sample a nice mathematics in interplay with a useful physics. Thus, in physics, the equidistance and representation of the energies in integer arithmetics in the $D \to \infty$ limit will enable us to work, in any "realistic" dimension $D < \infty$, with perturbation theory *without rounding errors.* In mathematics, the ease of the factorization of polynomials almost certainly reflects a hidden symmetry of the Schrödinger equation, but in the light of the nonlinearity of its present “algebraization”, we still do not dare to predict any form of its possible “explicit manifestation” in the future.

Besides that “new horizon”, let us also stress once more that our present study has been motivated by the disturbing paradox (revealed in [8]) that “phenomenologically the simplest” cubic oscillator (such that $V(x) \approx x^3$ for $x \gg 1$) belongs, in terms of mathematics, among “the most difficult” examples when its incomplete but exact $D \gg 1$ solvability is concerned. In this sense, we described here a resolution of this paradox, showing that the existence of the elementary and exact wave functions $\psi(x)$ in the large-D regime *and for any degree N* is admitted not only by the standard anharmonic Schrödinger equation with $q = 2$ (involving the quartic potentials) but also by its cubic analogue with $q = 4$.

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