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Extended Relativistic Dynamics with n-degree Characteristic Polynomial

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It is shown that the relativistic equations of motion can be decomposed into two Newtonian equations. The mapping between momenta of the system of Newtonian and relativistic equations is given by Vieta's formulae of a quadratic polynomial – the *characteristic polynomial* of the relativistic dynamics. In the present contribution we extend this scheme to obtain an extension of the relativistic mechanics. We suggest extended dynamic equations, which, with respect to the definite set of evolution parameters, are decomposed into three (n) Newtonian equations. The corresponding mapping is built on the basis of Vieta's formulae of n-degree polynomial. This polynomial we define as the *characteristic polynomial* of the extended mechanics. In the polar representation the solutions of the dynamic equations are given by Jacobi and Weierstrass (hyper) elliptic functions.

1 Introduction

The *principle of relativity* does not necessarily imply the quadratic metric in the space-time plane. The Lorentz-invariance may be violated in the same way as it has been violated the Galilei-invariance. A violation, rather, will be stipulated with the increasing of the energy of motion. That is to say, the relativistic mechanics is not consummate step of the extension of the Newtonian mechanics. The former can be extended in the same manner, namely, by involving the energy conservation law into the system of dynamic equations.

In this paper we present one faithful way of extension of the relativistic equations of motion. First of all (Section 2) we show that the relativistic equations of motion can be decomposed into two Newtonian equations with different evolution parameters. For this decomposition we use the Vieta's mapping of a quadratic polynomial. The last is defined as a characteristic polynomial of the relativistic dynamics. Further (Section 3), we extend the relativistic dynamics in such a way that the extended equations of motion will decomposed into three Newtonian equations. The dynamic equations of the extended mechanics are constructed by using Vieta's mapping of the characteristic cubic polynomial. The solutions in the polar representation are given by Jacobi and Weierstrass elliptic functions. In Section 4, the scheme elaborated in Section 3 generalized in order to obtain an extended mechanics with n-degree characteristic polynomial.

2 Decomposition of the relativistic equation into two Newtonian equations

Consider a motion of the relativistic particle under potential field $V(r)$. With respect to the proper-time these equations are written as

$$\frac{d\vec{P}}{d\tau} = \frac{1}{mc}\vec{E}\ P_0, \qquad \frac{dP_0}{d\tau} = \frac{1}{mc}(\vec{E}\cdot\vec{P}), \qquad \vec{E} = -\vec{\nabla}V(r), \tag{1}$$

$$\frac{d\vec{r}}{d\tau}=\frac{\vec{P}}{m}, \qquad \frac{dt}{d\tau}=\frac{P_0}{mc}, \tag{2}$$

These equations imply the first integral of motion:

$$P_0^2-P^2=M^2c^2. \tag{3}$$

Let us represent the relationship (3) in the following factorized form

$$c^2P^2=c^2P_0^2-m^2c^4=\left(cP_0-mc^2\right)\left(cP_0+mc^2\right). \tag{4}$$

In the non-relativistic limit

$$cP_0-mc^2\to\frac{p^2}{2m}.$$

Introduce two kinetic energies by definition

$$\frac{p^2}{2m}:=\left(cP_0-Mc^2\right), \qquad \frac{q^2}{2\mu}:=\left(cP_0+Mc^2\right), \tag{5}$$

Here we use the following units: p has a dimension of a momentum, q and μ have a dimension of an energy.

In these terms the square of momentum P in (4) is represented as a product of two kinetic energies

$$c^2P^2=\frac{p^2}{2m}\frac{q^2}{2\mu}. \tag{6}$$

The linear combinations of the formulae (5) give the following expressions for P_0 and Mc^2:

$$cP_0=\frac{1}{2}\left(\frac{q^2}{2\mu}+\frac{p^2}{2m}\right), \qquad Mc^2=\frac{1}{2}\left(\frac{q^2}{2\mu}-\frac{p^2}{2m}\right). \tag{7}$$

The expressions for P and P_0 from (6) and (7) substitute into projection of (1), (2), on the direction of momentum

$$\vec{n}=\frac{\vec{P}}{P}.$$

On making equal the expressions at p and q we come to the following evolution equations for p and q [1]:

$$\text{(a)}\quad \frac{dp}{d\tau}=-(\vec{n}\cdot\vec{\nabla})V(r)\frac{q}{\mu}, \qquad \text{(b)}\quad \frac{dq}{d\tau}=-(\vec{n}\cdot\vec{\nabla})V(r)\frac{p}{m}, \qquad \text{(c)}\quad \frac{d\vec{r}}{d\tau}=\vec{n}\frac{p}{m}\frac{q}{\mu}, \tag{8}$$

with the first integral

$$\frac{q^2}{2\mu}-\frac{p^2}{2m}=2Mc^2.$$

Now, let us exchange the parameter of evolution τ in (8) by a new time-like parameter of evolution $dt_p=d\tau\frac{q}{\mu}$. With respect to the new time-parameter (8) are reduced into Newtonian equations

$$\frac{dp}{dt_p}=-(\vec{n}\cdot\vec{\nabla})V(r), \qquad \frac{dr}{dt_p}=\frac{p}{m}. \tag{9}$$

In a same manner we obtain another Newtonian equation:

$$\frac{d}{d(ct_q)}q = -(\vec{n}\cdot\vec{\nabla})V(r), \qquad \frac{dr}{d(ct_q)} = \frac{q}{\mu}, \qquad d\tau\frac{p}{m} = d(ct_q). \tag{10}$$

Thus, (8) are decomposed into two Newtonian equations of motion: (9) and (10) with different evolution parameters (times).

Equations (1), (2) can be written in the *polar representation*:

$$P = Mc\sinh(\phi), \qquad P_0 = Mc\cosh(\phi),$$

where ϕ obeys the equation

$$\frac{d\phi}{d\tau} = \frac{e}{mc}(\vec{E}\cdot\vec{n}).$$

In this representation for p and q we obtain

$$\frac{p}{\sqrt{2m}} = \sqrt{2M}c\sinh\left(\frac{\phi}{2}\right), \qquad \frac{q}{\sqrt{2\mu}} = \sqrt{2M}c\cosh\left(\frac{\phi}{2}\right).$$

Renormalize the values p, q by

$$p = p\frac{1}{\sqrt{2m}}, \qquad q = q\frac{1}{\sqrt{2\mu}}.$$

Within these designations the formulae (6) and (7) are written as

$$c^2P^2 = p^2q^2, \qquad 2cP_0 = \left(q^2 + p^2\right),$$

which are nothing else than Vieta's formulae for the quadratic equation

$$X^2 - 2cP_0X + c^2P^2 = 0.$$

This polynomial is the *characteristic polynomial* of the relativistic mechanics [2].

3 Extended relativistic mechanics with cubic characteristic polynomial

In the previous section we have shown that the relativistic equations of motion can be decomposed into two Newtonian equations. In this section we seek such an extension of the relativistic equations of motion which in a similar manner can be decomposed into three Newtonian equations.

Let us start from the set of three Newtonian equations written with respect to different evolution parameters ($c = 1$):

$$\frac{dp}{dt_p} = -(\vec{n}\cdot\vec{E}), \qquad \frac{dq}{dt_q} = -(\vec{n}\cdot\vec{E}), \qquad \frac{dh}{dt_h} = -(\vec{n}\cdot\vec{E}). \tag{11}$$

In order to obtain this set of Newtonian equations from unique system of equations we should extend (8) in the following way

$$\frac{dp}{ds} = (\vec{n}\cdot\vec{E})\frac{q}{\mu}\frac{h}{\mu_h}, \qquad \frac{dq}{ds} = (\vec{n}\cdot\vec{E})\frac{p}{m}\frac{h}{\mu_h}, \qquad \frac{dh}{ds} = (\vec{n}\cdot\vec{E})\frac{p}{m}\frac{q}{\mu}, \tag{12}$$

$$\frac{dr}{ds} = \frac{p}{m}\frac{q}{\mu}\frac{h}{\mu_h}. \tag{13}$$

It is easily seen that these equations are reduced into (8) by exchanging the evolution parameter ds by $d\tau = \frac{h}{\mu_h}ds$. From (12) we obtain three Newtonian equations (12) by using the following formulae for the evolution parameters

$$d\tau\frac{qh}{\mu\mu_h} = dt_p, \qquad d\tau\frac{hp}{m\mu_h} = dt_q, \qquad d\tau\frac{pq}{m\mu} = dt_h.$$

Now, let us construct analogous mapping as we have built in the previous section, but for the triplet of variables $\{p^2, q^2, h^2\}$. The expression for the momentum we obtain from (13)

$$P = p\frac{h}{\mu_h}\frac{q}{\mu}. \tag{14}$$

In the polar representation momenta p, q, h are represented by Jacobi elliptic functions of imaginary argument:

$$p = \sqrt{m\mu}\,\mathrm{sc}\,(\phi, k), \qquad q = \mu\,\mathrm{nc}\,(\phi, k), \qquad h = \mu_h\,\mathrm{dc}\,(\phi, k),$$

where $k = \mu/\mu_h$, and the "angle" ϕ obeys the equation

$$\frac{d\phi}{ds} = (\vec{n}\cdot\vec{E})\frac{1}{\sqrt{m\mu}}.$$

In the sequel we shall use the set of dimensionless variables. For that purpose define

$$p_3 := \frac{1}{\sqrt{k}}\frac{h}{\mu_h}, \qquad p_2 := \frac{q}{\mu}, \qquad p_1 := \frac{p}{\sqrt{m\mu}}.$$

Then (12) in the polar representation are written in the following symmetric form

$$\frac{1}{\sqrt{k}}\frac{dp_1}{d\phi} = p_2p_3, \qquad \frac{1}{\sqrt{k}}\frac{dp_2}{d\phi} = p_3p_1, \qquad \frac{1}{\sqrt{k}}\frac{dp_3}{d\phi} = p_1p_2. \tag{15}$$

Correspondingly, re-define (14) by

$$P := \frac{1}{\sqrt{k}}\frac{1}{\sqrt{m\mu}}\,P = p_1p_2p_3. \tag{16}$$

Evolution equations for the squares p_1^2, p_2^2, p_3^2 are derived from (15):

$$\frac{1}{\sqrt{k}}\frac{dp_k^2}{d\phi} = 2p_1p_2p_3 = 2P, \qquad k = 1, 2, 3. \tag{17}$$

By using these equations calculate derivative of P with respect to ϕ. In the process of evaluation we shall introduce additional variables P_1, P_2 as algebraic functions of p_1^2, p_2^2, p_3^2. The first equation is:

$$\frac{1}{\sqrt{k}}\frac{dP}{d\phi} = 2P_2, \tag{18}$$

where

$$P_2 = \frac{1}{2}\left(p_1^2p_2^2 + p_2^2p_3^2 + p_3^2p_1^2\right). \tag{19}$$

Further, by evaluating the derivative of P_2, we get

$$\frac{1}{\sqrt{k}}\frac{dP_2}{d\phi} = 6PP_1, \tag{20}$$

where

$$P_1 = \frac{1}{3}\left(p_1^2 + p_2^2 + p_3^2\right). \tag{21}$$

The next differentiation of P_1 brings up the system of equations for the set of variables $\{P, P_1, P_2\}$:

$$\frac{1}{\sqrt{k}}\frac{dP_1}{d\phi} = 2P. \tag{22}$$

Notice that the mapping given by formulae (16), (19), (21) is nothing else than Vieta's formulae for the cubic polynomial

$$X^3 - 3P_1X^2 + 2P_2X - P^2 = 0.$$

Notice, (18), (20), (22) admit two first integrals of motion which do not depend of the potential function $V(r)$:

$$R_1 = -2P_2 + 3P_1^2, \qquad R_0 = P_1^3 - R_1P_1 - P^2.$$

From equations (18), (20), (22) we obtain

$$\frac{1}{\sqrt{k}}\frac{dP_1}{d\phi} = \sqrt{4P_1^3 - 4R_1P_1 - 4R_0}.$$

The integral of this equation is given by the Weierstrass elliptic integral

$$\sqrt{k}\phi = \int \frac{dy}{\sqrt{4y^3 - 4R_1y - 4R_0}}.$$

Consequently, the solution $y = P_1(\sqrt{k}\phi; 4R_1, 4R_0)$ is expressed via Weierstrass elliptic function [4].

4 Extended relativistic mechanics with n-degree characteristic polynomial

The theory we built in the previous sections can be extended to the case of mechanics with n-degree characteristic polynomial.

Consider the set of n Newtonian equations written with respect to different evolution parameters ($c = 1$):

$$\frac{dp}{dt_k} = -(\vec{n}\cdot\vec{E}).$$

By working in a similar manner as it has been in the previous section we come to the following equations of motion

$$\frac{dp}{ds} = (\vec{n}\cdot\vec{E})\frac{q}{\mu}\prod_{l=1}^{n-2}\frac{p_l}{\mu_l}, \qquad \frac{dq}{ds} = (\vec{n}\cdot\vec{E})\frac{p}{m}\prod_{l=1}^{n-2}\frac{p_l}{\mu_l},$$

$$\frac{dp_k}{ds} = (\vec{n} \cdot \vec{E}) \frac{q}{\mu} \frac{p}{m} \prod_{(l \neq k)}^{n-2} \frac{p_l}{\mu_l}, \qquad k = 1, 2, \ldots, n-2, \tag{23}$$

$$\frac{dx}{ds} = \frac{p}{m} \frac{q}{\mu} \prod_{l=1}^{n-2} \frac{p_l}{\mu_l}. \tag{24}$$

Let P be momentum of the particle. Then from (24) it follows

$$P = p \frac{q}{\mu} \prod_{l=1}^{n-2} \frac{p_l}{\mu_l}. \tag{25}$$

In the sequel we shall use renormalized values

$$p := \frac{p}{\sqrt{m\mu}}, \qquad q := \frac{q}{\mu}, \qquad p_l = \frac{p_l}{\sqrt{k_l \mu_l}}, \qquad l = 1, \ldots, n-2.$$

Then, in the polar representation (23) will be written as

$$\begin{aligned} &\frac{dp}{d\phi} = q \prod_{l=1}^{n-2} p_l, \qquad \frac{dq}{d\phi} = p \prod_{l=1}^{n-2} p_l, \\ &\frac{dp_k}{d\phi} = pq \prod_{(l \neq k)}^{n-2} p_l, \qquad k = 1, 2, \ldots, n-2. \\ &\frac{d\phi}{ds} = -k (\vec{n} \cdot \vec{E}) \frac{1}{\sqrt{m\mu}}, \end{aligned} \tag{26}$$

where $k = \prod_{l=1}^{n-2} \sqrt{\mu_l}$.

It is convenient to use the following set of designations $p_1 := p$, $p_2 := q$, $p_k := p_{k-2}$, $k = 3, \ldots, n$. The formula (25) is written now as

$$P^2 = \prod_{k=1}^{n} p_k^2. \tag{27}$$

The other dynamic variable is defined by:

$$P_1 = \frac{1}{n} \sum_{k=1}^{n} p_k^2. \tag{28}$$

The values P^2, P_1 are the first members of a mapping which we are seeking. These formulae are nothing else than the Vieta's formulae for the polynomial:

$$X^n + \sum_{k=1}^{n} (-1)^k (n-k+1) P_k X^{n-k} = 0, \qquad \text{with} \quad P_n = P^2. \tag{29}$$

The roots of this polynomial are given by the set of momenta p_k^2 $(k = 1, \ldots, n)$, whereas the coefficients of the polynomial will be used in the capacity of dynamic variables of the extended dynamics.

The evolution equations for the squares of the momenta are given by

$$\frac{dp_k^2}{ds} = (\vec{E} \cdot \vec{P}) \frac{1}{m}, \qquad k = 1, \ldots, n.$$

By using the mapping given by formulae (27), (28) and formulae Vieta of the equation (29) from equations (26) we deduce the following evolution equations for the values $P_1, P_2, \ldots, P_n$:

$$\frac{dP_{n-k}}{ds} = (\vec{E} \cdot \vec{P}) P_{n-k-1} \frac{k+2}{m}, \qquad k = 0, \ldots, n-1, \qquad \text{with} \quad P_0 = \frac{1}{n+1}. \tag{30}$$

These equations admit $(n-1)$ invariants no containing the potential $V(r)$. These invariants are given by algebraic expressions of $P_1, P_2, \ldots, P_n$. It is not easy to construct these expressions, therefore let us outline the method of construction of these invariants. The method is given by the following theorem [3]

Theorem 1. *The coefficients of the polynomial equation of n-degree*

$$Y^n + R_1 Y^{n-2} + \cdots + R_k Y^{n-k+1} + \cdots + R_{n-2} Y + R_0 = 0,$$

which is obtained from the polynomial equation (29) *by replacing X with $X = Y + P_1$, are invariants of equations of motion* (30).

In this contribution, we did not touch the problem of Extended Lorentz-kinematics. Here, let us only notice that due to the addition theorem for the elliptic functions still is possible to formulate *a group of transformation*. However, for higher order elliptic functions an addition theorem is not more valid. In this aspect the Extended Relativistic Mechanics with cubic characteristic polynomial is the last step of possible extensions via elliptic functions.

In the conclusion let us mention about some connection of this contribution with the results of [5], where hyperelliptic Lax representation for the generalized model of Euler's top has been done. In [2] we have shown that the relativistic oscillator model formally coincides with the model of Euler's top, whereas the oscillator models of the extended relativistic dynamics (see, for example, [3]), give generalizations of the Euler's top model. It is expected that the mathematical tool developed by Skrypnyk will be helpful within the mechanics elements of which we have presented in this contribution.

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