Proceedings

of the Fifth International Conference

SYMMETRY

in Nonlinear Mathematical Physics

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Institute of Mathematics

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of the Fifth International Conference

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Цей том "Праць Інституту математики НАН України" є збірником статей учасників П'ятої міжнародної конференції "Симетрія у нелінійній математичній фізиці". Збірник складається з трьох частин, кожна з яких видана окремою книгою.

Дане видання є першою частиною і включає праці, присвячені застосуванню теоретико-групових методів у математичній фізиці та теорії диференціальних рівнянь, а також питанням інтегровності.

Розраховано на наукових працівників, аспірантів, які цікавляться новими тенденціями симетрійного аналізу та його застосувань.

Symmetry in Nonlinear Mathematical Physics // Proceedings of Institute of Mathematics of NAS of Ukraine. — V. 50. — Part 1. — Kyiv: Institute of Mathematics of NAS of Ukraine, 2004 / Editors: A.G. Nikitin, V.M. Boyko, R.O. Popovych, I.A. Yehorchenko. — 524 p.

This volume of the Proceedings of Institute of Mathematics of NAS of Ukraine includes papers of participants of the Fifth International Conference "Symmetry in Nonlinear Mathematical Physics". The collection consists of three parts which are published as separate issues.

This issue is the first part which contains papers devoted to applications of group-theoretical methods in mathematical physics and differential equations, and to the integrability problems.

The book may be useful for researchers and post graduate students who are interested in modern trends in symmetry analysis and its applications.

Редактори: *А.Г. Нікітін, В.М. Бойко, Р.О. Попович, І.А. Єгорченко*

Editors: *A.G. Nikitin, V.M. Boyko, R.O. Popovych, I.A. Yehorchenko*

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Preface

Specialisation of science long ago resulted in that researchers in different fields practically lost ability to understand each other. Even mathematics that permeates absolutely all serious theoretical studies tends to increasing divergence of its various branches, including those that quite recently were the single whole.

On the other side there is increased universal penetration of methods and ideas generated in different fields. One of such concepts applicable in a vast multitude of areas of modern science is symmetry. And symmetry is the main link for the papers included into the present Proceedings.

Symmetry is a multi-faceted concept. It was discovered in ancient times and is utilised extensively in arts, natural sciences, mathematics, industry, crafts and design etc. Even in mathematics, in fact we have a large variety of different kinds of symmetries. And our conference that gathered many experts from rather different fields may serve as a proof to this fact.

The Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" was traditionally organised by the Institute of Mathematics of the National Academy of Sciences of Ukraine. It continues the series of meetings started in 1995 by Professor Wilhelm Fushchych. We are very pleased to see that popularity of these meetings increases with each new conference.

The Fifth Conference was the most representative with more than 250 participants from 38 countries presented their talks, the majority of which were included into these Proceedings. The titles of the conference talks not presented for publication are given at the end of Part 3.

The Proceedings are published in three parts and include 209 papers. A significant proportion of the papers deal with integrability as well as mainstream subjects in the Lie theory application in mathematical physics that is analysis of symmetries of differential equations, construction of invariants, symmetry reductions and construction of exact solutions. These papers are collected mostly in Part 1. The papers on supersymmetry and application of symmetry in physics are presented in Part 2. Papers on algebra, group and representation theory, together with various symmetry-related problems were included into Part 3.

We believe that these Proceedings will be a worthy summary of the successful conference, and contain valuable input into development of ideas and methods related to studies and applications of symmetries. We would wish them to serve as an inspiration for future research for all readers.

Anatoly NIKITIN

March, 2004

Fifth International Conference

SYMMETRY IN NONLINEAR MATHEMATICAL PHYSICS

June 23–29, 2003, Kyiv, Ukraine

Organized by

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June 23–29, 2003, Kyiv, Ukraine

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* Integrable and superintegrable systems
* Symbolic computations in symmetry analysis
* Dynamical systems, solitons
* Supersymmetry and its generalizations
* Quantum field theory
* Lie groups and algebras, representation theory and special functions
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Conference Address

Institute of Mathematics
National Academy of Sciences of Ukraine
3 Tereshchenkivska Street
01601 Kyiv 4, Ukraine

Web-page: www.imath.kiev.ua/˜appmath/conf.html

E-mail: appmath@imath.kiev.ua

Fax: +38 044 235 20 10

Phone: +38 044 234 63 22

We invite everybody to participate in the next Conference planned for June 20–26, 2005.

Fifth International Conference

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and for those donated to the library of the Institute of Mathematics.

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List of Participants

1. **Dmitry L. ABRAROV** (Computing Center of Academy of Science, Moscow, RUSSIA)
   e-mail: abrarov@ccas.ru
2. **Faridon AMDJADI** (Glasgow Caledonian University, Glasgow, UK)
   e-mail: F.Amdjadi@gcal.ac.uk
3. **Andrij ANDREYTSEV** (Kherson State Technical University, Kherson, UKRAINE)
   e-mail: appmath@imath.kiev.ua
4. **Alexander Val. ANTONIOUK** (Institute of Mathematics of NAS of Ukraine,
   Kyiv, UKRAINE) e-mail: antoniouk@imath.kiev.ua
5. **Alexandra Vict. ANTONIOUK** (Institute of Mathematics of NAS of Ukraine,
   Kyiv, UKRAINE) e-mail: antoniouk@imath.kiev.ua
6. **Asao ARAI** (Hokkaido University, Sapporo, JAPAN)
   e-mail: arai@math.sci.hokudai.ac.jp
7. **Leon ARRIOLA** (University of Wisconsin, Whitewater, USA)
   e-mail: arriolal@uww.edu, arriola@t7.lanl.gov
8. **Senol ATAOGLU** (Istanbul Technical University, Istanbul, TURKEY)
   e-mail: ataoglu@itu.edu.tr
9. **Lucas BADOWSKI** (Institute of Physics, A. Mickiewicz University, Poznan, POLAND)
   e-mail: punio@webska.pl
10. **Dumitru BALEANU** (Cankaya University, Ankara, TURKEY)
    e-mail: dumitru@cankaya.edu.tr
11. **Tetyana BARANNYK** (Poltava State Pedagogical University, Poltava, UKRAINE)
    e-mail: slpoltva@e-mail.pl.ua
12. **Peter BASARAB-HORWATH** (Linköping University, SWEDEN)
    e-mail: pehor@mai.liu.se
13. **Orest BATSULA** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE)
    e-mail: mmtpitp@gluk.org
14. **Yuri BAZILEVICH** (Institute of Continuing Special Education, Pridniprovsk State University
    of Civil Engineering, Dnipropetrovsk, UKRAINE) e-mail: bazilevich@ua.fm
15. **Jules BECKERS** (University of Liege, BELGIUM)
    e-mail: Jules.Beckers@ulg.ac.be
16. **Eugene BELOKOLOS** (Institute of Magnetism of NAS of Ukraine, Kyiv, UKRAINE)
    e-mail: bel@imag.kiev.ua
17. **Carl M. BENDER** (Washington University, St Louis, MO, USA)
    e-mail: cmb@wustl.edu
18. **Donald L. BENNETT** (Brookes Institute for Advanced Studies, Hellerup, DENMARK)
    e-mail: bennett@nbivms.nbi.dk
19. **Yuri BEREST** (Cornell University, Ithaca, NY, USA)
    e-mail: berest@polygon.math.cornell.edu
20. **Yuriy BERKELA** (Carpathian Biosphere Reserve, Rakhiv,, UKRAINE)
    e-mail: yuri@rakhiv.ukrtel.net
21. **Yuri BESPALOV** (Bogolyubov Institute for Theoretical Physics of NAS of Ukraine,
    Kyiv, UKRAINE) e-mail: mmtpitp@gluk.org
22. **Ayse Humeyra BILGE** (Istanbul Technical University, Istanbul, TURKEY)
    e-mail: bilge@itu.edu.tr
23. **Nicoleta BILA** (Industrial Mathematics Institute, Johannes Kepler University, Linz, AUSTRIA)
    e-mail: bila@indmath.uni-linz.ac.at
24. **Maciej BŁASZAK** (Institute of Physics, A. Mickiewicz University, Poznan, POLAND)
    e-mail: blaszakm@amu.edu.pl
25. **George BOGOSLOVSKY** (Skobeltsyn Institute of Nuclear Physics, Moscow State University,
    Moscow, RUSSIA) e-mail: bogoslov@theory.sinp.msu.ru

26. **Andrey BONDARENKO** (Kyiv Taras Shevchenko University, Kyiv, UKRAINE)
e-mail: bondarenkoa@ukr.net
27. **Vitalij BONDARENKO** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: vit-bond@imath.kiev.ua
28. **Luis J. BOYA** (Universidad de Zaragoza, SPAIN)
e-mail: luisjo@unizar.es
29. **Vyacheslav BOYKO** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: boyko@imath.kiev.ua
30. **Kirill A. BRONNIKOV** (Center of the Gravitation and Fundamental Metrology, VNIIMS, Moscow, RUSSIA) e-mail: kb@rgs.mccme.ru
31. **Paolo BUDINICH** (International School for Advanced Studies, Trieste, ITALY)
e-mail: fit@ictp.trieste.it
32. **Ivan BURBAN** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE)
e-mail: mmtpitp@bitp.kiev.ua
33. **Georgy I. BURDE** (Ben-Gurion University, Jacob Blaustein Institute for Desert Research, ISRAEL) e-mail: georg@bgumail.bgu.ac.il
34. **Vladimir BURSKII** (Inst. Appl. Math. Mech. of NAS of Ukraine, Donetsk, UKRAINE)
e-mail: burskii@iamm.ac.donetsk.ua
35. **Robert CARROLL** (University of Illinois, Urbana, USA)
e-mail: rcarroll@math.uiuc.edu
36. **Jen-Hsu CHANG** (Chung-Cheng Institute of Technology, National Defense University, Dashi, Tau-Yuan County, TAIWAN) e-mail: jhchang@ccit.edu.tw
37. **Claudia CHANU** (University of Turin, Torino, ITALY)
e-mail: chanu@dm.unito.it
38. **Yury CHAPOVSKY** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: yc@imath.kiev.ua
39. **Leonid CHEKHOV** (Steklov Mathematical Institute, Moscow, RUSSIA)
e-mail: chekhov@mi.ras.ru
40. **Roman CHERNIHA** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: cherniha@imath.kiev.ua
41. **Alexei F. CHEVIAKOV** (Moscow State Engineering-Physics Institute, Moscow, RUSSIA and Queen's University, Kingston, CANADA)
e-mail: afs_canada@mail.ru, alexch@mast.queensu.ca
42. **Margarita CHICHIKINA** (Moscow State University, Moscow, RUSSIA)
e-mail: chich@goa.bog.msu.ru
43. **Vladimir CHIRICALOV** (Kyiv Taras Shevchenko National University, Kyiv, UKRAINE)
e-mail: cva@skif.kiev.ua
44. **Giampaolo CICOGNA** (Universita degli Studi di Pisa, Pisa, ITALY)
e-mail: cicogna@df.unipi.it
45. **Howard S. COHL** (University of Exeter, Exeter, UK)
e-mail: hhcohl@shell42.com
46. **Heinz Otto CORDES** (University of California, Berkeley, USA)
e-mail: cordes@math.berkeley.edu
47. **Alexei DALETSKII** (Nottingham Trent University, Nottingham, UK)
e-mail: alexei.daletskii@ntu.ac.uk
48. **Pantelis A. DAMIANOU** (University of Cyprus, Nicosia, CYPRUS)
e-mail: damianou@ucy.ac.cy
49. **Marc DE MONTIGNY** (University of Alberta, Edmonton, CANADA)
e-mail: montigny@phys.ualberta.ca
50. **Dmitry DEMSKOI** (Orel State University, Orel, RUSSIA)
e-mail: demskoy@iname.com, demskoy@zaoksem.tula.net
51. **Oleg DERZHKO** (Institute for Condensed Matter Physics NASU, L'viv, UKRAINE)
e-mail: derzhko@icmp.lviv.ua

52. **Stylianos DIMAS** (University of Patras, Rio Patras, GREECE)
e-mail: spawn@math.upatras.gr
53. **Heinz-Dietrich DOEBNER** (Technical University of Clausthal, GERMANY)
e-mail: asi@pt.tu-clausthal.de
54. **Valerii DRYUMA** (Institute of Mathematics & Informatics AS Moldova, Kishinev, MOLDOVA)
e-mail: valery@gala.moldova.su, cainar@mail.md
55. **Vladislav DUBROVSKY** (Novosibirsk State Technical University, Novosibirks, RUSSIA)
e-mail: dubrovsky@online.nsk.su
56. **Brian EDGAR** (Linköping University, Linköping, SWEDEN)
e-mail: bredg@mai.liu.se
57. **Homayoon ESHRAGHI** (Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran, IRAN) e-mail: eshraghi@theory.ipm.ac.ir
58. **Nikolai FADEEV** (Joint Institute for Nuclear Research, Dubna, RUSSIA)
e-mail: fadeev@sunse.jinr.ru
59. **Vasyl FEDORCHUK** (Pedagogical Academy, Institute of Mathematics, Krakow, POLAND and Pidstryhach Institute of Applied Problems of Mechanics and Mathematics of NAS of Ukraine, Lviv, UKRAINE) e-mail: fedorchuk@ap.krakow.pl
60. **Volodymyr FEDORCHUK** (Ivan Franko Lviv National University, Lviv, UKRAINE)
e-mail: fedorchukv@ukr.net
61. **Dmitri FINKELSHTEIN** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: fdl@ln.ua
62. **Domenico FIORENZA** (Universita' degli Studi di Roma "la Sapienza", Roma, ITALY)
e-mail: fiorenza@mat.uniroma1.it
63. **José GAITE** (Instituto de Matemáticas y Física Fundamental, Madrid, SPAIN)
e-mail: gaite@laeff.esa.es
64. **Alexander GALKIN** (Kyiv Taras Shevchenko National University, Kyiv, UKRAINE)
e-mail: galkin@unicyb.kiev.ua
65. **Alexandre GAVRILIK** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE)
e-mail: omgavr@bitp.kiev.ua
66. **Victor GERSHUN** (Kharkiv Institute of Physics and Technology, Kharkiv, UKRAINE)
e-mail: gershun@kipt.kharkov.ua
67. **Alex GIACOMINI** (University of Trento, Trento, ITALY)
e-mail: giacomin@science.unitn.it
68. **Nikolaj GLAZUNOV** (Glushkov Institute of Cybernetics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: glanm@yahoo.com, glanm@d105.icyb.kiev.ua
69. **Sergey V. GOLOVIN** (Lavrentyev Institute of Hydrodynamics, Novosibirsk, RUSSIA)
e-mail: sergey@hydro.nsc.ru
70. **Barbara GOŁUBOWSKA** (Institute of Fundamental Technological Research PAS, Warsaw, POLAND) e-mail: bgolub@ippt.gov.pl
71. **Georgi GRAHOVSKI** (Universita degli studi di Milano, Milano, ITALY and Institute for Nuclear Research and Nuclear Energy, Sofia, BULGARIA) e-mail: grah@inrne.bas.bg
72. **Alfred Michel GRUNDLAND** (Centre de Recherches Mathematiques, Universite de Montreal, Montreal, CANADA) e-mail: grundlan@crm.umontreal.ca
73. **Yurdahan GÜLER** (Cankaya University, Ankara, TURKEY)
e-mail: Yurdahan@cankaya.edu.tr
74. **Faruk GÜNGÖR** (Istanbul Technical University, Maslak-Istanbul, TURKEY)
e-mail: gungorf@itu.edu.tr
75. **Uwe GÜNTHER** (Research Centre Rossendorf Inc., Institute of Safety Research, Dresden, GERMANY) e-mail: u.guenther@fz-rossendorf.de
76. **B. Kent HARRISON** (Brigham Young University, Provo, Utah, USA)
e-mail: bkentharrison@comcast.net, bkh@byu.edu
77. **Klaus HASSELMANN** (Max Planck Institute of Meteorology, Hamburg, GERMANY)
e-mail: klaus.hasselmann@dkrz.de

78. **Malte HENKEL** (Laboratoire de Physique des Materiaux, Universite Henri Poincare Nancy I, Vandoeuvre les Nancy Cedex, FRANCE) e-mail: henkel@lpm.u-nancy.fr
79. **Oksana HENTOSH** (Institute for Applied Problems of Mechanics and Mathematics of NAS of Ukraine, Lviv, UKRAINE) e-mail: hentoks@ua.fm
80. **Brigitte HILLER** (Centro de Fisica Teorica, Universidade de Coimbra, Coimbra, PORTUGAL) e-mail: brigitte@teor.fis.uc.pt
81. **Luuk HOEVENAARS** (University of Twente, Enschede, THE NETHERLANDS) e-mail: L.K.Hoevenaars@math.utwente.nl
82. **Petro HOLOD** (National University of "Kyiv-Mohyla Academy" and Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE) e-mail: holod@g.com.ua
83. **Peter HORETZKY** (University of Vienna, St. Polten, AUSTRIA) e-mail: horetzky@aon.at
84. **Kostyantyn ILYENKO** (Institute for Radiophysics and Electronics (IRE), NAS of Ukraine, Kharkiv, UKRAINE) e-mail: kost@ire.kharkov.ua
85. **Vladimir INOZEMTSEV** (Joint Institute for Nuclear Research, Dubna, RUSSIA) e-mail: inozv@thsun1.jinr.ru
86. **Nikolai IORGOV** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE) e-mail: iorgov@aep1.bitp.kiev.ua
87. **Nataliya IVANOVA** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: ivanova@imath.kiev.ua
88. **Anatolii P. IVASHIN** (National Science Center "Kharkiv Institute of Physics & Technology", Kharkiv, UKRAINE) e-mail: ivashin@kipt.kharkov.ua
89. **Anvar KACIMOV** (Sultan Qaboos University, Al-Khod, SULTANATE of OMAN) e-mail: anvar@squ.edu.om
90. **Vyacheslav S. KALNITSKY** (Saint-Petersburg State University, Saint-Petersburg, RUSSIA) e-mail: skalnitsky@hotmail.com
91. **Alexander KALYUZHNYI** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: kalyuz@imath.kiev.ua
92. **Dar'ya KARPENKO** (Donetsk Physical and Technical Institute of NAS of Ukraine, Donetsk, UKRAINE) e-mail: ph_d@ukr.net
93. **Magda KASSEM** (Zagazig University, Zagazig, EGYPT) e-mail: mkassem@thewayout.net
94. **Jaime KELLER** (Universidad Nacional Autonoma de Mexico, MEXICO and Technical University of Vienna, AUSTRIA) e-mail: keller@servidor.unam.mx, keller@cms.tuwien.ac.at
95. **Paul H.M. KERSTEN** (University of Twente, The NETHERLANDS) e-mail: kerstenphm@math.utwente.nl
96. **Mykhailo V. KHOMA** (Uzhgorod National University, Uzhgorod, UKRAINE) e-mail: khoma@uzhgorod.ukrsat.com
97. **Iosif KHRIPLOVICH** (Budker Institute of Nuclear Physics, Novosibirsk, RUSSIA) e-mail: khriplovich@inp.nsk.su
98. **Andriy P. KIRILYUK** (Institute of Metal Physics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: kiril@metfiz.freenet.kiev.ua
99. **Vladimir KISIL** (University of Leeds, UK and Odessa National University, Odessa, UKRAINE) e-mail: kisilv@maths.leeds.ac.uk
100. **Frieder KLEEFELD** (Centro de Fisica das Interaccoes Fundamentais (CFIF), Instituto Superior Tecnico, Lisbon, PORTUGAL) e-mail: kleefeld@cfif.ist.utl.pt
101. **Anatoliy KLIMYK** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE and Instituto de Matematicas, Morelos, MEXICO) e-mail: aklimyk@bitp.kiev.ua, anatoliy@matcuer.unam.mx
102. **Yuliya KLOCHKO** (Kharkiv Karazin's National University, Kharkiv, UKRAINE) e-mail: klochko@vil.com.ua
103. **Mehmet KOCA** (Sultan Qaboos University, Muscat, OMAN) e-mail: kocam@squ.edu.om

104. **Mariya KOLEVA** (Institute of Catalysis, Bulgarian Academy of Science, Sofia, BULGARIA)
e-mail: mkoleva@bas.bg
105. **Eser KORCUK** (Gaziantep University, Gaziantep, TURKEY)
e-mail: korcuk@gantep.edu.tr
106. **Andrii KOROSTIL** (Institute of Magnetism of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: amk@imag.kiev.ua
107. **Dmitry KOROTKIN** (Concordia University, Montreal, CANADA)
e-mail: korotkin@mathstat.concordia.ca
108. **Alyona KOROVNICHENKO** (University of Notre-Dame, Notre-Dame, USA)
e-mail: okorovni@nd.edu
109. **Gennadii KOTEL'NIKOV** (Russian Research Centre Kurchatov Institute, Moscow, RUSSIA)
e-mail: kga@electronics.kiae.ru
110. **Anatoly KOTIKOV** (Karlsruhe Universitat, Karlsruhe, GERMANYand Joint Institute for Nuclear Research, Dubna, RUSSIA)
e-mail: kotikov@particle.uni-karlsruhe.de, kotikov@thsun1.jinr.ru
111. **Lubomir KOVACHEV** (Institute of Electronics, Bulgarian Academy of Sciences, Sofia, BULGARIA) e-mail: lkovach@ie.bas.bg
112. **Vasyl KOVALCHUK** (Institute for Fundamental Technological Problems, Warsaw, POLAND)
e-mail: vkoval@ippt.gov.pl
113. **Vladimir F. KOVALEV** (Institute for Mathematical Modelling RAS, Moscow, RUSSIA)
e-mail: kovalev@imamod.ru
114. **Iosif KRASIL'SHCHIK** (Independent University of Moscow, Moscow, RUSSIA)
e-mail: josephk@diffiety.ac.ru
115. **Valentyn KUCHERYAVY** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE)
e-mail: mmtpitp@bitp.kiev.ua
116. **Ekaterina P. KUROCHKINA** (Institute of Thermophysics, Novosibirsk, RUSSIA)
e-mail: kurochkina@itp.nsc.ru
117. **Sengül KURU** (Ankara University, Ankara, TURKEY)
e-mail: kuru@science.ankara.edu.tr
118. **Gennady A. KUZ'MIN** (Institute of Thermophysics, SB RAS, Novosibirsk, RUSSIA)
e-mail: kuzmin@itp.nsc.ru
119. **Anatolii A. KYRYCHENKO** (Kyiv National University of Building and Architecture, Kyiv, UKRAINE) e-mail: akyrych@imath.kiev.ua
120. **Victor LAHNO** (Poltava State Pedagogical University, Poltava, UKRAINE)
e-mail: laggo@poltava.bank.gov.ua, lvi@pdpu.septor.net.ua
121. **Larisa V. LAPERASHVILI** (Institute of Theor. and Exper. Physics, Moscow, RUSSIA)
e-mail: laper@heron.itep.ru
122. **Peter LEACH** (University of Natal, Durban, SOUTH AFRICA)
e-mail: leachp@nu.ac.za, leach@math.aegean.gr
123. **Victor LEHENKYI** (Institute of Mathematical Machines and Systems Problems of NAS of Ukraine, Kyiv, UKRAINE) e-mail: victor@imath.kiev.ua
124. **Alexei LOPATIN** (National Academy of Management, Kyiv, UKRAINE)
e-mail: lopatin@carrier.kiev.ua
125. **Igor LOUTSENKO** (SISSA, Trieste, ITALY)
e-mail: loutseni@fm.sissa.it
126. **Maxim LUTFULLIN** (Poltava State Pedagogical University, Poltava, UKRAINE)
e-mail: MWL@pdpu.septor.net.ua
127. **Olena MAGDA** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: magda@imath.kiev.ua
128. **Tatjana MAISTRENKO** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: maistrenko@imath.kiev.ua
129. **Alexander MAKARENKO** (National Technical University of Ukraine (KPI), Kyiv, UKRAINE)
e-mail: makalex@mmsa.ntu-kpi.kiev.ua

130. **Eberhard MALKOWSKY** (University of Nis, Nis, SERBIA and MONTENEGRO)
e-mail: ema@BankerInter.net
131. **Yuri MALYUTA** (Institute for Nuclear Research, Kyiv, UKRAINE)
e-mail: malyuta@mail.kar.net
132. **Michal MARVAN** (Mathematical Institute, Silesian University in Opava,
Opava, CZECH REPUBLIC) e-mail: Michal.Marvan@math.slu.cz
133. **Oliver MASPFUHL** (Institut de Mathematiques, Ecole polytechnique federale de Lausanne,
Lausanne, SWITZERLAND) e-mail: oliver.maspfuhl@epfl.ch
134. **Roman MATSYUK** (Institute for Applied Problems in Mechanics and Mathematics,
L'viv, UKRAINE) e-mail: matsyuk@lms.lviv.ua
135. **Ernesto MATUTE** (Universidad de Santiago de Chile, Santiago, CHILE)
e-mail: ematute@lauca.usach.cl
136. **Anton MELLIT** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: mellit@imath.kiev.ua
137. **Antonino MESSINA** (Universita di Palermo, Palermo, ITALY)
e-mail: messina@fisica.unipa.it
138. **Alexander MIKHAILOV** (University of Leeds, Leeds, UK and Landau Institute
for Theoretical Physics, Moscow, RUSSIA) e-mail: sashamik@maths.leeds.ac.uk
139. **Benedetto MILITELLO** (Universita di Palermo, Palermo, ITALY)
e-mail: bdmilite@fisica.unipa.it
140. **Yukitaka MINESAKI** (Kyoto University, Kyoto, JAPAN)
e-mail: minesaki@amp.i.kyoto-u.ac.jp
141. **Masahiko MIYAMOTO** (University of Tsukuba, Tsukuba, JAPAN)
e-mail: miyamoto@math.tsukuba.ac.jp
142. **Oleg MOROZOV** (Snezhinsk Physical and Technical Academy, Chelyabinsk Region, RUSSIA)
e-mail: morozov@sfti.snz.ru
143. **Pavel MOROZOV** (Moscow State University, Moscow, RUSSIA)
e-mail: morozov_pv@rambler.ru
144. **Sibusiso MOYO** (Durban Institute of Technology, Durban, SOUTH AFRICA)
e-mail: moyos@dit.ac.za
145. **Olena MUL** (Institute of Computer Information Technologies, Ternopil Academy of National
Economy, Ternopil, UKRAINE) e-mail: elena_mul@lycos.com, elena_mul@novocom.te.ua
146. **Maryna NESTERENKO** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: maryna@imath.kiev.ua
147. **Anatoly NIKITIN** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: nikitin@imath.kiev.ua
148. **Dmitri NIKSHYCH** (University of New Hampshire, Durham, USA)
e-mail: nikshych@cisunix.unh.edu
149. **Andrey NOVIKOV-BORODIN** (Institute for Nuclear Research, Moscow, RUSSIA)
e-mail: novikov@inr.troitsk.ru
150. **Alexei NURMAGAMBETOV** (Institute for Theoretical Physics, NSC KIPT,
Kharkiv, UKRAINE) e-mail: ajn@kipt.kharkov.ua
151. **Tatiana OBIKHOD** (Institute for Nuclear Research, Kyiv, UKRAINE)
e-mail: malyuta@mail.kar.net
152. **Andrzej OKNINSKI** (Politechnika Swietokrzyska, Kielce, POLAND)
e-mail: fizao@tu.kielce.pl
153. **Vjacheslav OLYEYNIK** (Odessa National University, Odessa, UKRAINE)
e-mail: ovp@dtp.odessa.ua
154. **Teoman ÖZER** (Istanbul Technical University, Istanbul, TURKEY)
e-mail: tozer@itu.edu.tr
155. **Stanislav PAKULIAK** (Joint Institute for Nuclear Research, Dubna, RUSSIA)
e-mail: pakuliak@thsun1.jinr.ru

156. **Ragavachariar PARTHASARATHY** (The Institute of Mathematical Sciences, Chennai, INDIA) e-mail: sarathy@imsc.res.in
157. **Jiri PATERA** (Centre de Recherches Mathematiques, Universite de Montreal, CANADA) e-mail: patera@crm.umontreal.ca
158. **Anatoli PAVLYUK** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE) e-mail: a_pavlyuk@ukr.net
159. **Alexei V. PENSKOI** (Centre de Recherches Mathematiques, Universite de Montreal, CANADA) e-mail: penskoi@crm.umontreal.ca
160. **Serge PERRINE** (France Telecom Research and Development, FRANCE) e-mail: serge.perrine@francetecom.fr
161. **Irina PIROZHENKO** (Joint Institute for Nuclear Research, Dubna, Moscow region, RUSSIA) e-mail: pirozhen@thsun1.jinr.ru
162. **Andrei PIRYATINSKI** (Los Alamos National Laboratory, Los Alamos, USA) e-mail: apiryat@lanl.gov
163. **Marta POCHYNAIKO** (National University "Lviv Polytechnic", Lviv, UKRAINE) e-mail: matmod@franko.lviv.ua
164. **Nataly POPOVA** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: popova_n@yahoo.com
165. **Roman POPOVYCH** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: rop@imath.kiev.ua
166. **Stanislav POPOVYCH** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: stascon@i.com.ua
167. **Daniil PROSKURIN** (Kyiv Taras Shevchenko University, Kyiv, UKRAINE) e-mail: prosk@unicyb.kiev.ua
168. **Anatoliy PRYKARPATSKY** (University of Science and Technology, Kraków, POLAND and Institute APMM of NAS of Ukraine, Lviv, UKRAINE) e-mail: prykanat@cyberagl.com, pryk.anat@ua.fm
169. **Oleksandr PRYLYPKO** (Zhytomyr State Technological University, Zhytomyr, UKRAINE) e-mail: prilipko@user.ziet.zhitomir.ua
170. **Vyacheslav RABANOVICH** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: slavik@imath.kiev.ua
171. **Maido RAHULA** (Tartu University, Tartu, ESTONIA) e-mail: Maido.Rahula@math.ut.ee
172. **Olexandr RASIN** (Dnipropetrovsk National University, Dnipropetrovsk, UKRAINE) e-mail: sasha_rasin@rambler.ru
173. **Michel RAUSCH de TRAUBENBERG** (University of Strasbourg I, Strasbourg, FRANCE) e-mail: rausch@lpt1.u-strasbg.fr
174. **Vasyl' V. RUBISH** (Uzhgorod National University, Uzhgorod, UKRAINE) e-mail: vrubish@univ.uzhgorod.ua
175. **Tatiana RYABUKHA** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: vyrtum@imath.kiev.ua
176. **Mustafa SAIT-AMETOV** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: mustafa@imath.kiev.ua
177. **Yevgeniy Ye. SAMOILENKO** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: sam7@imath.kiev.ua
178. **Yurii S. SAMOILENKO** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE) e-mail: yurii_sam@imath.kiev.ua
179. **Valeriy Hr. SAMOYLENKO** (Kyiv National Taras Shevchenko University, Kyiv, UKRAINE) e-mail: vsam@univ.kiev.ua, svhr@mechmat.univ.kiev.ua
180. **Maria SAVINA** (Joint Institute for Nuclear Research, Dubna, RUSSIA) e-mail: savina@lhe.jinr.ru
181. **Nobuyuki SAWADO** (Tokyo University of Science, JAPAN) e-mail: sawado@ph.noda.tus.ac.jp

182. **Andrew SEMENOV** (Institute of Physics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: sem@iop.kiev.ua
183. **Artur SERGYEYEV** (Ben-Gurion University, Jacob Blaustein Institute for Desert Research, ISRAEL) e-mail: asergeev@bgumail.bgu.ac.il, Artur.Sergyeyev@math.slu.cz
184. **Alexander SHAPOVALOV** (Tomsk State University, Tomsk, RUSSIA)
e-mail: shpv@phys.tsu.ru
185. **Noriko SHIIKI** (Tokyo University of Science, JAPAN)
e-mail: norikoshiiki@mail.goo.ne.jp
186. **Kazunari SHIMA** (Saitama Institute of Technology, JAPAN)
e-mail: shima@sit.ac.jp
187. **Igor SHIROKOV** (Omsk State University, Omsk, RUSSIA)
e-mail: shirokov@univer.omsk.su
188. **Vasilisa SHRAMCHENKO** (Concordia University, Montreal, CANADA)
e-mail: vasilisa@mathstat.concordia.ca
189. **Oleg SHVEDOV** (Moscow State University, Moscow, RUSSIA)
e-mail: shvedov@qs.phys.msu.su
190. **Douglas SINGLETON** (California State University, Fresno, USA)
e-mail: dougs@csufresno.edu
191. **Yurii SITENKO** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE)
e-mail: yusitenko@bitp.kiev.ua
192. **Wolodymyr SKRYPNIK** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: skrypnik@imath.kiev.ua
193. **Taras SKRYPNYK** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE)
e-mail: tskrypnyk@imath.kiev.ua
194. **Leonid SLAD** (Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow, RUSSIA) e-mail: slad@theory.sinp.msu.ru
195. **Jan SŁAWIANOWSKI** (Institute of Fundamental Technological Research PAS, Warsaw, POLAND) e-mail: jslawian@ippt.gov.pl
196. **Roman SMIRNOV** (Dalhousie University and University of Waterloo, CANADA)
e-mail: smirnov@mathstat.dal.ca
197. **Olga N. SOBOLEVA** (Institute of Computational Mathematics and Mathematical Geophysics, Novosibirsk, RUSSIA) e-mail: olga@nmsf.sscc.ru
198. **Andrey SOKOLOV** (Institute of Physics, St. Petersburg State University, Sankt-Petersburg, RUSSIA) e-mail: sokolov@mph.phys.spbu.ru
199. **Christodoulos SOPHOCLEOUS** (University of Cyprus, Nicosia, CYPRUS)
e-mail: christod@ucy.ac.cy
200. **Vyacheslav SOROKA** (Kharkiv Institute of Physics and Technology, Kharkiv, UKRAINE)
e-mail: vsoroka@kipt.kharkov.ua
201. **Stanislav SPICHAK** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: spichak@imath.kiev.ua
202. **Valeriy STOGNIY** (National Technical University of Ukraine "Kyiv Polytechnic Institute", Kyiv, UKRAINE) e-mail: valeriy_stogniy@mail.ru
203. **Alexander STOLIN** (Chalmers University of Gothenburg, SWEDEN)
e-mail: astolin@math.chalmers.se
204. **Alexander STRELETS** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: sav@imath.kiev.ua
205. **Vladimir STUKOPIN** (Don State Technical University, Rostov-na-Donu, RUSSIA)
e-mail: stukopin@math.rsu.ru
206. **Yuriy SYDORENKO** (Franko Lviv National University, Lviv, UKRAINE)
e-mail: matmod@franko.lviv.ua
207. **Błazej SZABLIKOWSKI** (Institute of Physics, A. Mickiewicz University, Poznan, POLAND)
e-mail: bszablik@amu.edu.pl

208. **Volodymyr TARANOV** (Institute for Nuclear Research of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: vtaranov@kinr.kiev.ua, vbtaranov@netscape.net
209. **Piergiulio TEMPESTA** (Centre de Recherches Mathematiques, Universite de Montreal, Montreal (Quebec), CANADA) e-mail: Tempesta@CRM.UMontreal.ca
210. **Mikhail TIMOSHIN** (Samara State Pedagogical University, Samara, RUSSIA)
e-mail: midvolga@mail.ru
211. **Volodymyr TKACHUK** (Ivan Franko National University of Lviv, Lviv, UKRAINE)
e-mail: tkachuk@ktf.franko.lviv.ua
212. **Kouichi TODA** (Toyama Prefectural University, Imizu, JAPAN)
e-mail: kouichi@yukawa.kyoto-u.ac.jp
213. **Delfim F.M. TORRES** (University of Aveiro, Aveiro, PORTUGAL)
e-mail: delfim@mat.ua.pt
214. **Violeta TRETYNYK** (Internat. Science and Technology University, Kyiv, Ukraine)
e-mail: vtretynyk@yahoo.com
215. **Takayuki TSUCHIDA** (University of Tokyo, Tokyo, JAPAN)
e-mail: tsuchida@poisson.ms.u-tokyo.ac.jp
216. **Victor TSUKANOV** (Kharkiv Phys. & Tech. Institute, Kharkiv, UKRAINE)
e-mail: v_tsuk@ic.kharkov.ua
217. **Ivan TSYFRA** (Institute of Geophysics of NAS of Ukraine, Kyiv, UKRAINE
and Institute of Mathematics, University of Bialystok, POLAND)
e-mail: tsyfra@igph.kiev.ua, tsyfra@math.uwb.edu.pl
218. **Krzysztof URBANOWSKI** (Uniwersytet Zielonogorski, Zielona Gora, POLAND)
e-mail: K.Urbanowski@proton.if.zu.zgora.pl, K.Urbanowski@if.zu.zgora.pl
219. **Constantin USENKO** (Kyiv National Shevchenko University, Kyiv, UKRAINE)
e-mail: usenko@phys.univ.kiev.ua
220. **Dmytro UVAROV** (NSC Kharkiv Institute of Physics and Technology, Kharkiv, UKRAINE)
e-mail: uvarov@kipt.kharkov.ua
221. **Vyacheslav O. VAKHNENKO** (Institute of Geophysics, Kyiv, UKRAINE)
e-mail: vakhnenko@bitp.kiev.ua
222. **Galliano VALENT** (Centre de Physique Theorique, Marseille and Laboratoire
de Physique Theorique et des Hautes Energies, Paris, FRANCE)
e-mail: gvalent@cpt.univ-mrs.fr, gal.valent@wanadoo.fr
223. **Vladimir VASILCHUK** (B. Verkin Institute for Low Temperature Physics and Engineering, Kharkiv, UKRAINE) e-mail: vasilchuk@ilt.kharkov.ua
224. **Dmitri VASSILIEV** (University of Bath, Bath, UK)
e-mail: D.Vassiliev@bath.ac.uk
225. **Vesna VELIČKOVIĆ** (University of Nis, Nis, SERBIA and MONTENEGRO)
e-mail: vvesna@Bankerinter.net
226. **Alexander VERBOVETSKY** (Independent University of Moscow, Moscow, RUSSIA)
e-mail: verbovet@mccme.ru
227. **Sergey VERNOV** (Institute of Nuclear Physics, Moscow State University, RUSSIA)
e-mail: svernov@theory.sinp.msu.ru
228. **Anca VISINESCU** (National Institute for Physics and Nuclear Engineering,
Bucharest, ROMANIA) e-mail: avisin@theory.nipne.ro, avisin@ifin.nipne.ro
229. **Mihai VISINESCU** (National Institute for Physics and Nuclear Engineering,
Bucharest, ROMANIA) e-mail: mvisin@theory.nipne.ro, mvisin@ifin.nipne.ro
230. **Raffaele VITOLO** (Universita' di Lecce, Lecce, ITALY)
e-mail: Raffaele.Vitolo@unile.it
231. **Vsevolod VLADIMIROV** (University of Science and Technology, Kraków, POLAND)
e-mail: vladimir@mat.agh.edu.pl, vsan@rambler.ru, vsan@mail.com
232. **Mariya VLASENKO** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: mariyka@imath.kiev.ua

233. **Jörg VOLKMANN** (University of Ulm, GERMANY)
e-mail: volk@physik.uni-ulm.de

234. **Alla VOROBYOVA** (Mykolayiv Petro Mogyla State Humanities University, UKRAINE)
e-mail: alla@mksat.net, leifura@kma.mk.ua

235. **Andrij VUS** (Franko Lviv National University, Lviv, Ukraine)
e-mail: andrij_vus@franko.lviv.ua

236. **Qinghai WANG** (Washington University, St. Louis, MO, USA)
e-mail: qwang@hbar.wustl.edu

237. **Jaroslaw WAWRZYCKI** (Institute of Nuclear Physics, Krakow, POLAND)
e-mail: waw@alf.ifj.edu.pl, jwaw@th.if.uj.edu.pl

238. **Nik WEAVER** (Washington University, St. Louis, USA)
e-mail: nweaver@math.wustl.edu

239. **Thomas WOLF** (Brock University, Ontario, CANADA)
e-mail: twolf@brocku.ca

240. **Alexander YAKHNO** (Univercity of Guadalajara, Jalisco, MEXICO)
e-mail: ay223627@cucei.udg.mx

241. **Robert M. YAMALEEV** (Facultad de Estudios Superiores UNAM, MEXICO
and Joint Institute for Nuclear Research, Dubna, RUSSIA) e-mail: iamaleev@servidor.unam.mx

242. **Chao YANG** (Goddard Retired Alumni Association, Greenbelt, USA)
e-mail: chaoyyang@netscape.net

243. **Vitaliy YATSENKO** (Institute of Space Research NASU and NSAU, Kyiv, UKRAINE)
e-mail: fond@pirogov.kiev.ua

244. **Irina YEHORCHENKO** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: iyegorch@imath.kiev.ua

245. **Oksana YERMOLAYEVA** (SISSA, Trieste, ITALY)
e-mail: yermola@fm.sissa.it, yermolae@crm.umontreal.ca

246. **Ivan YURYK** (Ukrainian State University of Food Technologies, Kyiv, UKRAINE)
e-mail: appmath@imath.kiev.ua

247. **Wojtek J. ZAKRZEWSKI** (University of Durham, Durham, UK)
e-mail: W.J.Zakrzewski@durham.ac.uk

248. **Zygmunt Jacek ZAWISTOWSKI** (Institute of Fundamental Technological Research,
Polish Academy of Sciences, Warsaw, POLAND) e-mail: zzawist@ippt.gov.pl

249. **Maxim ZAVODOVSKY** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE)
e-mail: mzv@imath.kiev.ua

250. **Alexander ZHALIJ** (Institute of Mathematics of NAS of Ukraine, Kyiv, UKRAINE
and Centre de Recherches Mathematiques, Universite de Montreal, CANADA)
e-mail: zhalij@crm.umontreal.ca, zhaliy@imath.kiev.ua

251. **Alexei ZHEDANOV** (Donetsk Institute for Physics and Technology, Donetsk, UKRAINE)
e-mail: zhedanov@kinetic.ac.donetsk.ua

252. **Miloslav ZNOJIL** (Nulear Physics Institute, Rez, CZECH REPUBLIC)
e-mail: znojil@ujf.cas.cz

253. **Alexander ZOLOTARYUK** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE)
e-mail: azolo@gluk.org

254. **Yaroslav ZOLOTARYUK** (Bogolyubov Institute for Theoretical Physics, Kyiv, UKRAINE)
e-mail: yzolo@bitp.kiev.ua

255. **Konstantin ZUEV** (Moscow State University, Moscow, RUSSIA)
e-mail: zuev_km@mail333.com

256. **Vladimir ZVEREV** (Ural State Technical University, Ekaterinburg, RUSSIA)
e-mail: zverev@dpt.ustu.ru

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Symmetry of Differential Equations

Symmetry in Nonlinear Mathematical Physics

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Construction of Exact Solutions of Diffusion Equation

Anatoly F. BARANNYK † *and Ivan I. YURYK* ‡

† *Instytut Matematyki, Pomorska Akademia Pedagogiczna, 22b Arciszewskiego Str., 76-200 Słupsk, Poland*

‡ *Ukrainian State University of Food Technologies, 68 Volodymyrs'ka Str., 01033 Kyiv, Ukraine*

New exact solutions for the heat equation $u_t = u_{xx} + \lambda u^{\frac{n-1}{2}} + \varepsilon u^{\frac{n+1}{2}} + cu + du^{\frac{3-n}{2}}$ are found.

1 Introduction

The nonlinear diffusion equations

$$u_t = u_{xx} + b(u)u_x + c(u), \tag{1}$$

where $u = u(t, x)$, $b(u)$, $c(u)$ are smooth functions, and subscripts denote derivatives with respect to the corresponding variables, play fundamental role in the modelling of various processes of heat conduction, reaction-diffusions, in mathematical biology, and also in many other spheres. Equation (1) generalizes a great number of known nonlinear evolution equations. Thus in particular cases equation (1) is regarded as classical Burgers equation

$$u_t = u_{xx} + \lambda_1 u u_x, \tag{2}$$

and also as Kolmogorov–Piskunov equation

$$u_t = u_{xx} + f(u), \tag{3}$$

where $f(u)$ is a sufficiently smooth function.

It is well known, that nonlocal Cole–Hopf transformation

$$u = -2\mu \frac{v_x}{v}, \qquad v = v(t, x) \tag{4}$$

makes every solution of linear heat conduction equation

$$v_t - \mu v_{xx} = 0 \tag{5}$$

correspond to the particular solution of the equation (2). By these means, the transformation (4) reduces the problem of construction of solutions of the equation (2) to the construction of solutions of linear equation (5).

The conditional symmetries equation (3) was investigated in [1, 2]. The operators of conditional symmetry of this equation were found only in the case when $f(u)$ is a polynomial of 3rd order. All known solutions of the equation (3) that were successfully obtained by means of conditional symmetry operators, have such representation

$$u = k \frac{z_x}{z}, \tag{6}$$

where k is constant, and $z = z(t, x)$ is some arbitrary function.

It should be noticed that group classification of equations (1) was done in [3]. The results concerning conditional symmetries of equation (1) are mentioned in [4]. In [5] wide classes of exact solutions of equation (3) are represented.

A goal of this article is to construct the exact solutions of equation

$$u_t = u_{xx} + \lambda u^{\frac{n-1}{2}} u_x + \varepsilon u^{\frac{n+1}{2}} + cu + du^{\frac{3-n}{2}}, \tag{7}$$

where $\varepsilon = \pm 1$, c, d are arbitrary real numbers. In case $n = 3$ we use transformation (3). The transformation (6) is particular case of transformation

$$u = k\left(\frac{z_x}{z}\right)^{\frac{2}{n-1}}, \tag{8}$$

that is effective in case of arbitrary n. By using the transformation (8) we reduce the problem of constructing of exact solutions of equation (7) to the problem of finding function $z = z(t, x)$. As will be mentioned further, function $z = z(t, x)$ is a solution of system of ordinary differential equations that is easily solved in many cases.

2 The exact solutions of equation (7) for $n = 3$

We construct the exact solutions of equation (7) for $n = 3$

$$u_t = u_{xx} + \lambda u u_x + \varepsilon u^2 + cu + d. \tag{9}$$

By substituting (6) into (9), we obtain

$$(kz_{xt} - kz_{xxx} - ckz_x - dz)z^2 + kz_x(-z_t + (3 - \lambda k)z_{xx} - \varepsilon k z_x)z + (-2k + \lambda k^2) = 0.$$

The variable z will be determined from the conditions of zero expressions at z and z^2 simultaneously. As a result we have

$$kz_{xt} - kz_{xxxx} - ckz_x - dz = 0, \tag{10}$$
$$-z_t + (3 - \lambda k)z_{xx} - \varepsilon k z_x = 0, \tag{11}$$
$$2 - \lambda k = 0. \tag{12}$$

From equation (11) and (12) we find

$$z_t = z_{xx} - \varepsilon k z_x. \tag{13}$$

By substituting into equation (10), we obtain

$$\varepsilon k^2 z_{xx} + ckz_x + dz = 0. \tag{14}$$

Thus, the system (10)–(12) is equivalent to the system (12)–(14). This system can be easily solved. The type of solution depends on roots of characteristic equation

$$\varepsilon k^2 r^2 + ckr + d = 0, \tag{15}$$

that corresponds to linear equation (14). The roots of characteristic equation (15) are $\frac{1}{k}m_1$, $\frac{1}{k}m_2$, where m_1 and m_2 are the roots quadratic equation

$$\varepsilon r^2 + cr + d = 0. \tag{16}$$

We consider tree cases.

a) The roots are real and different. The general solution of equation (14) has such form

$$z = \mu_1(t)\exp\left(\frac{1}{k}m_1 x\right) + \mu_2(t)\exp\left(\frac{1}{k}m_1 x\right),$$

where $\mu_1(t)$, $\mu_2(t)$ are functions of t which have to be found. By using equation (13), we obtain

$$\frac{d\mu_1}{dt} = \frac{1}{k^2}m_1^2\mu_1 - \varepsilon m_1\mu_1, \qquad \frac{d\mu_2}{dt} = \frac{1}{k^2}m_2^2\mu_2 - \varepsilon m_2\mu_2.$$

So,

$$\mu_1 = k_1\exp\left[\left(\frac{1}{k^2}m_1^2 - \varepsilon m_1\right)t\right], \qquad \mu_2 = k_2\exp\left[\left(\frac{1}{k^2}m_2^2 - \varepsilon m_2\right)t\right],$$

and

$$z = k_1\exp\left[\frac{1}{k}m_1 x + \left(\frac{1}{k^2}m_1^2 - \varepsilon m_1\right)t\right] + k_2\exp\left[\frac{1}{k}m_2 x + \left(\frac{1}{k^2}m_2^2 - \varepsilon m_2\right)t\right].$$

Thus we have found such solution of equation (2)

$$u = \frac{k_1 m_1\exp\left[\frac{\lambda}{2}m_1 x + \left(\frac{\lambda^2}{4}m_1^2 - \varepsilon m_1\right)t\right] + k_2 m_2\exp\left[\frac{\lambda}{2}m_2 x + \left(\frac{\lambda^2}{4}m_2^2 - \varepsilon m_2\right)t\right]}{k_1\exp\left[\frac{\lambda}{2}m_1 x + \left(\frac{\lambda^2}{4}m_1^2 - \varepsilon m_1\right)t\right] + k_2\exp\left[\frac{\lambda}{2}m_2 x + \left(\frac{\lambda^2}{4}m_2^2 - \varepsilon m_2\right)t\right]},$$

where k_1, k_2 are arbitrary real constants, that are not equal to zero simultaneously,

$$m_1 = \frac{-c_1 + \sqrt{c^2 - 4\varepsilon d}}{2\varepsilon}, \qquad m_2 = \frac{-c_1 - \sqrt{c^2 - 4\varepsilon d}}{2\varepsilon}.$$

b) The roots are complex numbers. Let $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$. In this case we obtain

$$z = \exp\left[\frac{\lambda}{2}dx + \left(\frac{\lambda^2}{4}(\alpha^2 - \beta^2) - \varepsilon\alpha\right)t\right] \times \left\{k_1\cos\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right] + k_2\sin\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right]\right\},$$

where k_1, k_2 are arbitrary real numbers, that are not equal to zero simultaneously. Thus, the solution of equation (9) has such form

$$u = \frac{\alpha k_1 + \beta k_2)\cos\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right] + (\alpha k_2 - \beta k_1)\sin\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right]}{k_1\cos\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right] + k_2\sin\left[\frac{\lambda}{2}\beta x + \left(\frac{\lambda^2}{2}\alpha\beta - \varepsilon\beta\right)t\right]},$$

where

$$\alpha + i\beta = \frac{-c + \sqrt{c^2 - 4\varepsilon d}}{2\varepsilon}, \qquad \alpha - i\beta = \frac{-c - \sqrt{c^2 - 4\varepsilon d}}{2\varepsilon}.$$

c) The roots are equal. In this case $c^2 - 4\varepsilon d = 0$, and thus $d = \frac{c^2}{4\varepsilon}$, $m_1 = m_2 = -\frac{c}{2\varepsilon}$. Function z has such form

$$z = \exp\left[-\frac{\varepsilon\lambda c}{4}x + \left(\frac{\lambda^2 c^2}{16} + \frac{c}{2}\right)t\right]\left\{k_1 x + \left(-\frac{\varepsilon\lambda c}{2}k_1 - \frac{2\varepsilon}{\lambda}k_1\right)t + k_0\right\},$$

where k_1, k_2 are arbitrary real numbers, that are not equal to zero simultaneously. We obtain such solution of equation (9)

$$u = \frac{2}{\lambda}\frac{-\frac{\varepsilon\lambda c}{4}\left\{k_1 x + \left(-\frac{\varepsilon\lambda c}{2}k_1 - \frac{2\varepsilon}{\lambda}k_1\right)t + k_0\right\} + k_1}{k_1 x + \left(-\frac{\varepsilon\lambda c}{2}k_1 - \frac{2\varepsilon}{\lambda}k_1\right)t + k_0}.$$

In particular, if $c = 0$ then the solution has such form

$$u = \frac{2k_1}{\lambda k_1 x - 2\varepsilon k_1 t + \lambda k_0}.$$

Let us consider two separate cases.

a) $c = 0$, $\varepsilon < 0$. The equation (16) has two real and different roots

$$m_1 = \sqrt{-\varepsilon d}, \qquad m_2 = -\sqrt{-\varepsilon d}.$$

Thus the solution of equation (9) has form

$$u = \frac{k_1 m_1 \exp\left[\frac{\lambda m_1}{2}x + \left(-\frac{\varepsilon\lambda^2 d}{4} + \varepsilon m_2\right)t\right] + k_2 m_2 \exp\left[\frac{\lambda m_1}{2}x + \left(-\frac{\varepsilon\lambda^2 d}{4} + \varepsilon m_1\right)t\right]}{k_1 \exp\left[\frac{\lambda m_1}{2}x + \left(-\frac{\varepsilon\lambda^2 d}{4} + \varepsilon m_2\right)t\right] + k_2 \exp\left[\frac{\lambda m_2}{2}x + \left(-\frac{\varepsilon\lambda^2 d}{4} + \varepsilon m_1\right)t\right]}.$$

b) $c = 0$, $\varepsilon d > 0$. The equation (16) has two complex roots

$$m_1 = i\sqrt{\varepsilon d}, \qquad m_2 = -i\sqrt{\varepsilon d}.$$

Thus the solution of equation (9) has form

$$u = \frac{k_2\sqrt{\varepsilon d}\cos\left[\frac{\lambda}{2}\sqrt{\varepsilon d}x - \varepsilon\sqrt{\varepsilon d}t\right] - k_1\sqrt{\varepsilon d}\sin\left[\frac{\lambda}{2}\sqrt{\varepsilon d}x - \varepsilon\sqrt{\varepsilon d}t\right]}{k_1\cos\left[\frac{\lambda}{2}\sqrt{\varepsilon d}x - \varepsilon\sqrt{\varepsilon d}t\right] + k_2\sqrt{\varepsilon d}\sin\left[\frac{\lambda}{2}\sqrt{\varepsilon d}x - \varepsilon\sqrt{\varepsilon d}t\right]}.$$

3 The exact solutions of equation (7) for arbitrary n

Let us consider equation (7) for arbitrary n and for $d = e$

$$u_t = u_{xx} + \lambda u^{\frac{n-1}{2}}u_x + \varepsilon u^{\frac{n+1}{2}} + cu. \tag{17}$$

Substituting (8) into equation (17), we obtain such system for finding variable z

$$\frac{2}{n-1}z_x z_{xt} - \frac{2(3-n)}{(n-1)^2}z_{xx}^2 - \frac{2}{n-1}z_x z_{xx} - cz_x^2 = 0, \tag{18}$$

$$z_t = \left(\frac{n+3}{n-1} - \lambda k^{\frac{n-1}{2}}\right)z_{xx} - \varepsilon\frac{n-1}{2}k^{\frac{n-1}{2}}z_x, \tag{19}$$

$$k^{\frac{n-1}{2}} = \frac{n+1}{\lambda(n-1)}. \tag{20}$$

Solving system (18)–(20), we find such solution for equation (17)

$$u = \left[\frac{n+1}{\lambda(n-1)}\right]^{\frac{2}{n-1}} \frac{\exp\left[-\frac{2\varepsilon\lambda c}{n+1}x + \left(\frac{4\lambda^2 c^2}{(n+1)^2} + c\right)t\right]}{\left\{-\frac{\varepsilon(n+1)}{\lambda(n-1)c}\exp\left[-\frac{\varepsilon\lambda c(n-1)}{n+1}x + \left(\frac{2\lambda^2 c^2}{(n+1)^2} + \frac{c}{2}\right)(n-1)t\right] + c_1\right\}^{\frac{2}{n-1}}},$$

where c, c_2 are arbitrary constants.

This approach can be used for finding the exact solutions of more general type of reaction-diffusion equation that will be done in the next articles.

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Solitary Wave Solutions for Heat Equations

Tetyana A. BARANNYK † *and Anatoly G. NIKITIN* ‡

‡ *Poltava State Pedagogical University, 2 Ostrogradsky Str., 36003 Poltava, Ukraine*
E-mail: *slpoltva@e-mail.pl.ua*

‡ *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *nikitin@imath.kiev.ua*

We present exact solitary wave solutions for a class of nonlinear heat equations and for coupled system of heat equations.

1 Introduction

A great number of mathematical models in thermomechanics, chemistry, biology, ecology, etc., are formulated using nonlinear heat equations. Some of these models admit exact solutions which play fundamental role for theory of the related equations and have large application values, refer, e.g., to monograph [1].

There are well-known regular approaches for searching exact solutions of partial differential equations (see, e.g., monographs [2–4]) which make it possible to find specific changes of dependent and independent variables (Ansätze) reducing equations of interest to simpler ones. Usually the possibility of such reduction is caused by the symmetry (classical Lie [2], conditional [3], Lie–Bäcklund [4], etc.) of the related equations.

In the present paper we find exact solutions for the nonlinear heat equations

$$u_t - u_{xx} = f(u) \tag{1}$$

and for coupled systems of heat equations

$$\begin{aligned} &u_t - u_{xx} = f(u,v),\\ &v_t - v_{xx} = g(u,v), \end{aligned} \tag{2}$$

where $f(u)$, $f(u,v)$ and $g(u,v)$ are functions of dependent variables specified in the following. To find these solutions we use the "universal" Ansatz of the following general form

$$u = (z_x)^k \varphi(z) \tag{3}$$

and

$$v = (z_x)^k \xi(z), \tag{4}$$

where the new variable z is a function of t and x.

The Ansätze (3), (4) appear to be very effective tools for reduction of equations (1), (2) and many others, refer e.g. to [4, 5]. In particular, they enable making a wide class of reductions caused by classical Lie symmetries and conditional symmetries as well [4].

In this paper we use Ansätze (3), (4) to find special exact solutions of equations (1), (2), namely, solitary wave solutions. We present also some other solutions which appear in our analysis. To achieve our goals we select special classes of equations (1), (2) which can be reduced using the mentioned Ansätze.

2 Reduction of equations (1) with cubic nonlinearity

We start with equations (1) with the cubic nonlinearities, i.e.,

$$u_t - u_{xx} = -2u^3. \tag{5}$$

Exact solutions for (5) as well as for equation (1) with cubic polynomial nonlinearity were found in [7, 8] using the conditional symmetry approach. Here we find these solutions directly using Ansatz (3). The results presented in this section will be used to construct solutions for the system of heat equations (2).

Using Ansatz (3) with $k = 1$ we find

$$\begin{aligned} &u_t = z_{tx}\varphi + z_x z_t \varphi_z, \qquad u_x = z_{xx}\varphi + z_x^2\varphi_z, \\ &u_{xx} = z_{xxx}\varphi + 3z_x z_{xx}\varphi_z + z_x^3\varphi_{zz}, \qquad u^3 = z_x^3\varphi^3. \end{aligned} \tag{6}$$

Substituting (6) into (5) and equating coefficients for φ and φ' we come to the following reduced equations

$$z_{tx} - z_{xxx} = 0, \qquad z_t - 3z_{xx} = 0, \tag{7}$$

$$\varphi_{zz} = 2\varphi^3. \tag{8}$$

Up to constant multiplier and shifts of t and x solution of (7) have the form

$$z = 6t + x^2. \tag{9}$$

Solutions of equations (8) can be expressed via Jacobi elliptic functions. In paper [5] we present an infinite set of such solutions (and solutions for equation (1) with cubic polynomial nonlinearities as well) in explicit form.

Thus the Ansatz (3) makes it possible to find exact solutions for equation (5) in the form

$$u = 2x\varphi\left(6t + x^2\right),$$

where $\varphi(z)$ is an elliptic function satisfying (8).

In the following Section we generalize the reduction scheme (5)–(9) to find exact solutions for systems of heat equations (2).

3 Reduction and exact solutions for equations (2)

Consider a subclass of equations (2) where f and g are homogeneous functions of order 3, i.e.,

$$f(\lambda u, \lambda v) = \lambda^3 f(u, v), \qquad g(\lambda u, \lambda v) = \lambda^3 g(u, v). \tag{10}$$

To find exact solutions of these equations we need nothing but a direct generalization of the procedure described in the previous Section. Indeed, all such equations can be reduced to systems of ordinary differential equations using the Ansätze (3), (4) for $k = 1$. In analogy with (5)–(9) we obtain

$$\begin{aligned} &u = 2x\varphi\left(6t + x^2\right), \qquad v = 2x\xi\left(6t + x^2\right), \\ &\varphi'' = f(\varphi, \xi), \qquad \xi'' = g(\varphi, \xi). \end{aligned} \tag{11}$$

Thus to find exact solutions for system (2), (10) it is sufficient to solve the system of ordinary differential equations (11). For some particular form of functions f and g this system can be integrated in quadratures. Here we consider three examples.

1. Let $f = 2\left(u^3 - 3u^2v + 2uv^2\right)$, $g = 0$. Then equations (11) admit the following particular solutions: $\varphi = 1 - \tanh(z)$, $\xi = 1$. Using them we come to the following solutions for the related equations (2)

$$u = 2x\left(1 - \tanh\left(x^2 + 6t\right)\right), \qquad v = 2x. \tag{12}$$

For any finite interval $x_0 < x < x_1$ solutions (12) are bounded. Moreover, the pattern u has a solitary wave sharp. The plot of function u (12) is given in Fig. 1.

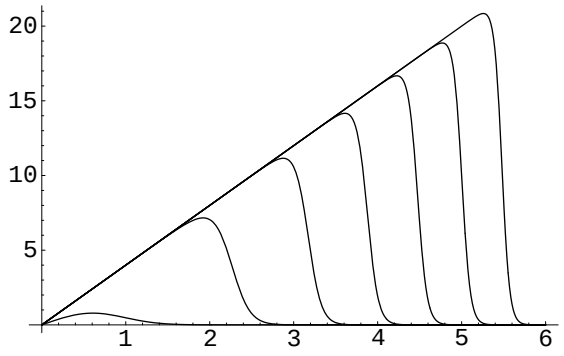

Figure 1. Solutions u (12) for $t = 0, -1/6, -1/3, -1/2, -3/2, -5/6, -1$.

We see that when the time variable t runs from $t = -1$ to $t = 0$ our solitary wave moves from the point $x = 5.5$ to $x = 0.6$ and its amplitude decreases.

2. Consider now the system (2) with nonlinearities of the following general form

$$\begin{aligned} f &= \frac{\varepsilon}{2}\left(n^2u^{n+3}v^{-n} - n(2n+m)v^m u^{3-m}\right), \\ g &= \frac{\varepsilon}{2}\left((n+2)^2u^{n+2}v^{1-n} - (n+2)(2n+m+2)v^{m+1}u^{2-m}\right), \end{aligned} \tag{13}$$

where $\varepsilon = \pm 1$, m and n are arbitrary parameters satisfying $m + n \neq 0$, $n \neq 0,\ -2$.

The Ansatz (4) reduces (2), (13) to the following system of ordinary equations

$$\begin{aligned} \varphi'' &= \varepsilon\left(n^2\varphi^{n+3}\xi^{-n} - n(2n+m)\varphi^m\xi^{-m+3}\right), \\ \xi'' &= \varepsilon\left((n+2)^2\varphi^{n+2}\xi^{1-n} - (n+2)(2n+m+2)\xi^{m+1}\varphi^{2-m}\right) \end{aligned}$$

whose solutions for $\varepsilon = -1$ and $\varepsilon = 1$, are expressed via trigonometric and hyperbolic functions respectively. Thus for $\varepsilon = -1$ we come to the following exact solutions for (2), (13)

$$u = \frac{2\mu x}{n+m}\left[\cos\left(\mu\left(6t + x^2\right)\right)\right]^{-\frac{n}{n+m}}, \qquad v = \frac{2\mu x}{n+m}\left[\cos\left(\mu\left(6t + x^2\right)\right)\right]^{-\frac{(2+n)}{n+m}}. \tag{14}$$

For $\frac{n}{n+m} < 0$ and $\frac{2+n}{n+m} < 0$ solutions (14) are bounded on any finite interval $x_0 \leq x \leq x_1$. The plots of these solutions for $n = 2$, $m = -1$ are given in Figs. 2, 3.

If $\varepsilon = 1$ then the related solutions can be found in the form

$$u = \frac{2\mu x}{n+m}\left[\frac{1}{\mathrm{ch}(\mu(6t + x^2))}\right]^{\frac{n}{n+m}}, \qquad v = \frac{2\mu x}{n+m}\left[\frac{1}{\mathrm{ch}(\mu(6t + x^2))}\right]^{\frac{n+2}{n+m}}. \tag{15}$$

For $\frac{n}{n+m} > 0$ and $\frac{n+2}{n+m} > 0$ formulae (15) present solitary waves. We notice that solutions (15) are not of plane wave type and their amplitudes decrease if t goes from $-t_0$ to 0 ($t_0 > 0$). The plots of these solutions for $n = 1$, $m = 0$ are given in Figs. 4 and 5.

Like in the case of solutions (12), when t runs from $t = -1$ to $t = 0$, our solitary waves move from the point $x = 2.3$ to $x \approx 0.7$ and their amplitudes decrease. When variable t becomes positive and grows the amplitudes tend to zero.

In plots given by Figs. 2–5 the value of amplitudes decrease with increasing of time variable.

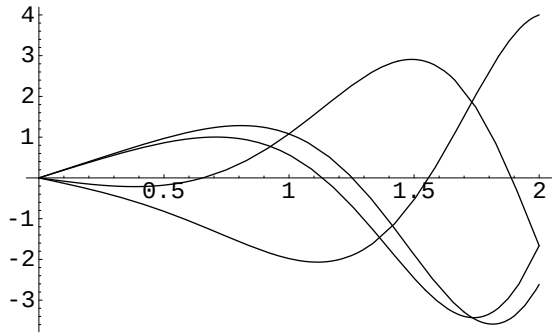

Figure 2. Solutions u (14) for $t = 0, -1/3, -3/2, -1$.

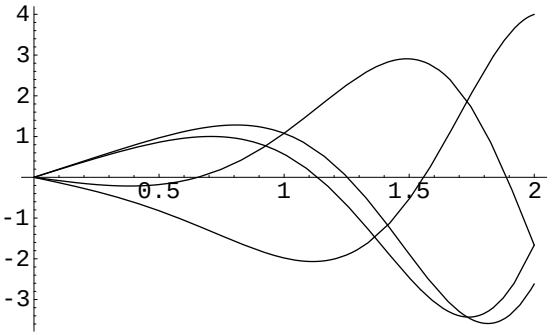

Figure 3. Solutions v (14) for $t = 0, -1/3, -3/2, -1$.

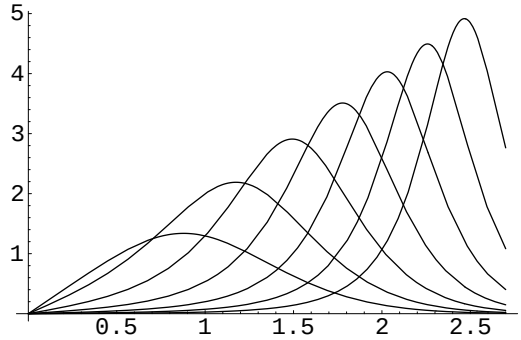

Figure 4. Solutions u (14) for $n = 1$, $m = 0$, and $t = 0, -1/6, -1/3, -1/2, -3/2, -5/6, -1$.

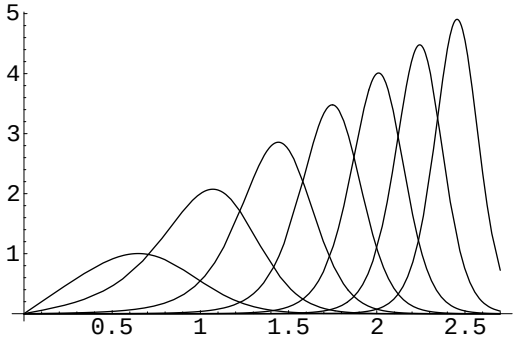

Figure 5. Solutions v (14) for $n = 1$, $m = 0$, $t = 0, -1/3, -3/2, -1$.

4 Solitary wave solutions for equation (1)

Let us return to equation (1) and choose the nonlinearity $f(u)$ in the form

$$f(u) = -\frac{n+1}{n-1}u^n + \lambda_1 u + \lambda_2 u\frac{n+1}{2} + \lambda_3 u^{\frac{3-n}{2}} + \lambda_4 u^{2-n}, \tag{16}$$

where n is an arbitrary parameter.

Reaction-diffusion equations of the general form (1), (16) can be effectively reduced using Ansatz (3) [5]. Choosing $\varphi = z^{-k}$, $k = \frac{2}{n-1}$ we transform (1), (16) to the form

$$z\left(z_x z_{xx} - z_x z_{xxx} - (k-1)z_{xx}^2 - \lambda_4 z^2 - \lambda_3 z_x z\right) = z_x^2(z_x - (2k+1)z_{xx} + \lambda_1 z + \lambda_2 z_x). \tag{17}$$

Equation (17) is homogeneous in dependent variable and is more convenient for searching for exact solutions than the initial equation (1), (16).

Let us search for solutions of (17) in the form $z(t,x) = U(\xi)$ where $\xi = \mu t + x$ and μ is an arbitrary (nonzero) constant. Then we come to the following ordinary differential equation for U

$$\begin{aligned} &U[U'(\mu U'' - U''' - \lambda_3 U) - \lambda_4 U^2 - (k-1)(U'')^2] \\ &\quad = (U')^2[(\mu + \lambda_2)U' + \lambda_1 U - (2k+1)U''], \end{aligned} \tag{18}$$

where $U' = \frac{dU}{d\xi}$.

Using the idea of Fan [9] we express U in the form

$$U = \nu_0 + \nu_1\varphi + \nu_2\varphi^2 + \cdots, \tag{19}$$

where $\nu_0, \nu_1, \dots$ are constants and φ satisfies the equation of the following general form

$$\varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + \cdots}. \tag{20}$$

In order (19) to be compatible with (18) we have to equate separately the terms which include odd and even powers of the square root given by (20). As a result we obtain

$$U'(\mu UU'' - \lambda_3 U^2) = (U')^3(\mu + \lambda_2), \tag{21}$$
$$U\big(U'U''' + \lambda_4 U^2 + (k-1)(U'')^2\big) = (U')^2((2k+1)U'' - \lambda_1 U). \tag{22}$$

Dividing any term in (21) by $\mu U^2 U'$ we come to the Riccati equation

$$Y' - \frac{\lambda_2}{\mu}Y^2 = \frac{\lambda_3}{\mu}$$

for $Y = \frac{U'}{U}$, whose general solutions are

$$Y = \sqrt{\frac{-\lambda_3}{\lambda_2}}\tanh\left(\frac{\sqrt{-\lambda_2\lambda_3}}{\mu}\xi + C\right),$$
$$Y = \sqrt{\frac{-\lambda_3}{\lambda_2}}\left(\tanh\left(\frac{\sqrt{-\lambda_2\lambda_3}}{\mu}\xi + C\right)\right)^{-1}, \qquad \text{if} \quad \lambda_2\lambda_3 < 0, \tag{23}$$
$$Y = \sqrt{\frac{\lambda_3}{\lambda_2}}\tan\left(\frac{\sqrt{\lambda_2\lambda_3}}{\mu}\xi + C\right), \qquad \text{if} \quad \lambda_2\lambda_3 > 0, \tag{24}$$
$$Y = -\frac{\mu}{\lambda_2(\xi + C)}, \qquad \text{if} \quad \lambda_3 = 0, \tag{25}$$

where C is the integration constant.

Solutions (23), (24) and (25) are compatible with (22) provided

$$\mu = -\lambda_2, \qquad \lambda_1 = -k\frac{\lambda_3}{\lambda_2}, \qquad \lambda_4 = (1-k)\left(\frac{\lambda_3}{\lambda_2}\right)^2, \qquad \lambda_2\lambda_3 \neq 0$$

and

$$\lambda_1 = \lambda_4 = 0, \qquad \lambda_3 = 0$$

respectively. Using variables

$$\tau = \frac{2}{(n-1)^2}t, \qquad y = \frac{\sqrt{2}}{n-1}x, \qquad \sigma = -\lambda_2(n-1), \qquad \nu = \frac{\lambda_3}{\lambda_2}$$

we can rewrite the related equation (1), (16) as follows:

$$u_\tau - u_{yy} = \left(1 + \nu u^{1-n}\right)\left(-(n+1)u^n + \nu(n-3)u + \sigma u^{\frac{n+1}{2}}\right). \tag{26}$$

The corresponding solutions for equation (26) have the following form

$$u = (-\nu)^{\frac{1}{n-1}}\left(\tanh\left(b\left(y - \frac{\sigma}{\sqrt{2}}t\right) + C\right)\right)^{\frac{2}{n-1}}, \tag{27}$$
$$u = (-\nu)^{\frac{1}{n-1}}\left(\tanh\left(b\left(y - \frac{\sigma}{\sqrt{2}}t\right) + C\right)\right)^{\frac{2}{1-n}}, \tag{28}$$

where $\nu < 0$ and $b = (n-1)\sqrt{\frac{-\nu}{2}}$,

$$u = (\nu)^{\frac{1}{n-1}} \left(\tan\left(b\left(y - \frac{\sigma}{\sqrt{2}} t \right) + C \right) \right)^{\frac{2}{n-1}}, \tag{29}$$

where $\nu > 0$ and $b = (n-1)\sqrt{\frac{\nu}{2}}$, and

$$u = 2^{n-1} \left((n-1)\left(y - \frac{\sigma}{\sqrt{2}} \tau + C \right) \right)^{\frac{2}{1-n}} \tag{30}$$

if $\nu = 0$.

For $\frac{2}{n-2} > 1$ formula (27) presents solitary wave solutions propagating with the velocity $\frac{\sigma}{\sqrt{2}}$. In the case $n = 2$ we come to the bell-shaped solitary wave.

If $\frac{2}{n-2} < -1$ then equation (26) again admits solitary wave solutions which are given by relation (28). In contrast with (14) they are bell-shaped plane waves.

5 Discussion

We present a collection of solitary wave solutions for nonlinear heat equations. All these solutions are obtained using the Ansatz (3), (4).

It is possible to show that all reductions used in the present paper are caused by the conditional symmetry of the related equations (1), (2). Instead of investigation of conditional symmetries we prefer the direct use of the universal Anzätse which seams to be the easiest way to find equations admitting exact solutions of the desired solitary wave type.

Nevertheless the very existence of extended classes of exact solutions for systems of heat equations shows that the problem of classification of conditional symmetries for these systems is quite interesting. We plane to consider this problem elsewhere.

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The Structure of Lie Algebras and the Classification Problem for Partial Differential Equations

Peter BASARAB-HORWATH [†], *Victor LAHNO* [‡] *and Olena MAGDA* [§]

[†] *Department of Mathematics, Linköping University, S-58183 Linköping, Sweden*
E-mail: *pehor@mai.liu.se*

[‡] *Poltava State Pedagogical University, 2 Ostrograds'kogo Str., 36003 Poltava, Ukraine*
E-mail: *lvi@pdpu.septor.net.ua, laggo@poltava.bank.gov.ua*

[§] *Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *magda@imath.kiev.ua*

We have solved completely the problem of the description of quasi-linear evolution differential equations in two independent variables that are invariant under $so(3)$ and $sl(2,\mathbb{R})$ Lie algebras.

1 Introduction

The basic idea is to combine the standard Lie algorithm for point symmetries with the equivalence group of the given type of equation in order to give a classification of evolution equations in some canonical form [1].

We consider third-order evolution equations of the type

$$u_t = F(t,x,u,u_x,u_{xx})u_{xxx} + G(t,x,u,u_x,u_{xx}). \tag{1}$$

Hereafter $u = u(t,x)$, F, G are sufficiently smooth functions of the corresponding arguments, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$, $F \neq 0$.

The first step of the algorithm is the determination of the most general form of the infinitesimal symmetry operator admitted by the PDE (1). To this end, we use Lie's method and look for a symmetry generator in the form

$$Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u, \tag{2}$$

where τ, ξ, η are arbitrary, real-valued smooth functions defined in some subspace of the space $V = X \otimes \mathbb{R}^1$ of the independent variables $X = \langle t,x\rangle$ and the dependent variable $\mathbb{R}^1 = \langle u\rangle$.

As a result, we find that the operator (2) generates a one-parameter symmetry group of equation (1) if

$$\varphi^t - [\tau F_t + \xi F_x + \eta F_u + \varphi^x F_{u_x} + \varphi^{xx} F_{u_{xx}}]u_{xxx} - \varphi^{xxx}F - \tau G_t - \xi G_x - \eta G_u - \varphi^x G_{u_x} - \varphi^{xx} G_{u_{xx}}\Big|_{u_t = Fu_{xxx}+G} = 0,$$

where

$$\begin{aligned}
&\varphi^t = D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
&\varphi^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
&\varphi^{xx} = D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),\\
&\varphi^{xxx} = D_x(\varphi^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi)
\end{aligned}$$

and D_t, D_x are operators of total differentiation in t and x respectively.

We then find the following results:

Theorem 1. *The symmetry group of the equation* (1) *is generated by the infinitesimal operators of the form*

$$Q = \tau(t)\,\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\,\partial_u, \tag{3}$$

where τ, ξ, η *are real-valued functions that satisfy the system*

$$\begin{aligned}
&(\tau_t - 3\xi_x - 3u_x\xi_u)F + \tau F_t + \xi F_x + \eta F_{u_x} + (\eta_x + u_x(\eta_u - \xi_x) - u_x^2\xi_u)F_{u_x} + (\eta_{xx}\\
&\qquad + u_x(2\eta_{xu} - \xi_{xx}) + u_x^2(\eta_{uu} - 2\xi_{xu}) - u_x^3\xi_{uu} + u_{xx}(\eta_u - 2\xi_x) - 3u_xu_{xx}\xi_u)F_{u_{xx}} = 0,\\
&-\eta_t + u_x\xi_t - (\eta_u - \tau_t + u_x\xi_u)G + F(\eta_{xxx} + u_x(3\eta_{xxu} - \xi_{xxx}) + 3u_x^2(\eta_{xuu} - \xi_{xxu})\\
&\qquad + u_x^3(\eta_{uuu} - 3\xi_{xuu}) - u_x^4\xi_{uuu} + 3u_{xx}(\eta_{xu} - \xi_{xx}) + 3u_xu_{xx}(\eta_{uu} - 3\xi_{xu})\\
&\qquad - 6u_{xx}u_x^2\xi_{uu} - 3u_{xx}^2\xi_u) + \tau G_t + \xi G_x + \eta G_u + (\eta_x + u_x(\eta_u - \xi_x) - u_x^2\xi_u)G_{u_x}\\
&\qquad + (\eta_{xx} + u_x(2\eta_{xu} - \xi_{xx}) + u_x^2(\eta_{uu} - 2\xi_{xu})\\
&\qquad - u_x^3\xi_{uu} + u_{xx}(\eta_u - 2\xi_x) - 3u_xu_{xx}\xi_u)G_{u_{xx}} = 0.
\end{aligned} \tag{4}$$

As one can see from equation (4) unless F and G are functions of one variable, it is well-night impossible to make the usual step of splitting this equation in terms of powers of the derivative u_x. This is where it is necessary to add a further element into the argument. The idea is simple: given that we are not able to derive the Lie invariance algebra by first obtaining the defining equations form (4), we then begin by specifying a Lie algebra and then requiring that it be symmetry algebra of (1). Given a Lie algebra, we then look at the possible representations of this Lie algebra within the class of operators of the form given in (2), and it is in this step that we make use of the equivalence group of equation (1). This gives us canonical representations of the symmetry algebra candidates. The final step is to calculate the allowed forms for the function F and G for a given canonical representation of our chosen Lie algebra. This procedure yields canonical forms of evolution equations which are inequivalent under point transformations of the equivalence group of equation (1). Then, for each such canonical evolution equation, one can calculate the maximal symmetry algebra.

The above method requires a list of inequivalent Lie algebra presentations (in terms of commutation relations). All simple (and therefore all semi-simple) finite-dimensional real and complex Lie algebras have been classified [2,3]. However, the list of solvable Lie algebras is far from complete, and as far as we can ascertain, they are given in the work of Morozov and Mubarakzyanov and Turkowski [4–10]. The work of Morozov and Mubarakzyanov are, as far as we are aware, not translated from the original Russian. Here, the solvable Lie algebras up to and including dimension six are classified. Our method is constructive in the sense that we are able to use the low-dimensional Lie algebras up to dimension five in order to give our complete point symmetry classification, and we conclude that no non-linear evolution equation of the form (1) has a point-symmetry invariance algebra of dimension greater that five. We note that a classification in terms of contact symmetries has been given by Magadeev [11].

2 Some previous work

The problem of group classification of such equations was discussed by many authors (see for instance [1,12–18]). Work on classification using the equivalence group has been done by Torrisi and his co-workers [20–22]. The methods of Torrisi et al are based on the infinitesimal representation of the equivalence transformations, in contrast to our approach which involves finite forms of the equivalence transformations.

3 Results

In this section we give some results and show how the equivalence group of the equation plays its role in the classification of representation of a given Lie algebra. First, we begin with the description of the equivalence group of equation (1).

Lemma 1. *The maximal equivalence group $\mathcal{E}$ of equation* (1) *reads as*

$$\bar{t} = T(t), \qquad \bar{x} = X(t,x,u), \qquad v = U(t,x,u), \tag{5}$$

where $\dot{T} \neq 0$, $\frac{D(X,U)}{D(x,u)} \neq 0$.

The next step is to give a canonical form (linearization) for a vector field, with respect to the allowed transformations (5). This is given in the following result:

Lemma 2. *There are changes of variables* (5)*, that reduce an operator* (3) *to one of the operators below:*

$$Q = \partial_t, \tag{6}$$
$$Q = \partial_x. \tag{7}$$

Proof. Making the change of variables (5) transforms operator (3) to the following one:

$$Q \to Q' = \tau\dot{T}\partial_{\bar{t}} + (\tau X_t + \xi X_x + \eta X_u)\partial_{\bar{x}} + (\tau U_t + \xi U_x + \eta U_u)\partial_v. \tag{8}$$

Suppose $\tau \neq 0$. Then, choosing in (5) the function T to be a solution of the equation $\dot{T} = \tau^{-1}$ and the functions X and U to be independent fundamental solutions of the first-order PDE

$$\tau Y_t + \xi Y_x + \eta Y_u = 0, \qquad Y = Y(t,x,u),$$

we find that the operator (8) takes the form $Q' = \partial_{\bar{t}}$.

Now suppose $a = 0$. Then $\xi^2 + \eta^2 \neq 0$. If $b \neq 0$, then choosing in (5) a particular solution of PDE $\xi X_x + \eta X_u = 1$ as the function X and a fundamental solution of PDE $\xi U_x + \eta U_u = 0$ as the function U, we transform (8) to become $Q' = \partial_{\bar{x}}$.

If $\xi = 0$, $\eta \neq 0$, then making the change of variables (5) with $\bar{t} = t$, $\bar{x} = u$, $v = x$, we again get the case $\xi \neq 0$.

By the direct calculation we can verify that there is no transformation from $\mathcal{E}$, that reduce operator (6) to the form (7). ■

Example 1. $sl(2,\mathbb{R})$. Now we give an example of the calculations involved. We do this using the Lie algebra $sl(2,\mathbb{R})$ which is the Lie algebra with basis $\langle Q_1, Q_2, Q_3\rangle$ satisfying the commutation relations

$$[Q_1, Q_2] = 2Q_2, \qquad [Q_1, Q_3] = -2Q_3, \qquad [Q_2, Q_3] = Q_1. \tag{9}$$

First, we take one operator from the basis and put it into canonical form. So let us choose Q_3 for this purpose (this is done for purely practical reasons.) By Lemma 2, we can choose Q_3 in one of two canonical forms: $Q_3 = \partial_t$ or $Q_3 = \partial_x$.

We consider in detail the case $Q_3 = \partial_t$. With this choice of Q_3 we proceed to find the allowable forms for Q_3 or Q_3. So put

$$Q_1 = \tau(t)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u.$$

The second commutation relation in (9) then implies that

$$\dot{\tau} = 2, \qquad \dot{\xi} = \dot{\eta} = 0.$$

Therefore, $\tau(t) = 2t + c$, $c = \text{const}$ and ξ, η are independent of t. So we take

$$Q_1 = 2t\partial_t + \xi(x,u)\partial_x + \eta(x,u)\partial_u.$$

The next step is to find a canonical form for Q_1 under the equivalence transformations (5). However, we must now use only those equivalence transformations of (5) which preserve the form $Q_3 \to \bar{Q}_3$ with

$$\bar{Q}_3 = \dot{T}\partial_{\bar{t}} + X_t\partial_{\bar{x}} + U_t\partial_v = \partial_{\bar{t}}$$

which yields $\dot{T} = 1$, $X_t = U_t = 0$. Thus we take $T(t) = t$, $X = X(x,u)$, $U = U(x,u)$. Under this type of transformation we find

$$Q_1 \to \bar{Q}_1 = 2\bar{t}\partial_{\bar{t}} + (\xi X_x + \eta X_u)\partial_{\bar{x}} + (\xi U_x + \eta U_u)\partial_v.$$

We now choose X and U so that

$$\xi X_x + \eta X_u = X, \qquad \xi U_x + \eta U_u = 0.$$

This gives us the canonical form

$$\bar{Q}_1 = 2\bar{t}\partial_{\bar{t}} + \bar{x}\partial_{\bar{x}}.$$

This means that we can, up to an equivalence transformation of equation (1), take

$$Q_3 = \partial_t, \qquad Q_1 = 2t\partial_t + x\partial_x.$$

Finally, we need to determine Q_2. We put

$$Q_2 = \alpha(t)\partial_t + \beta(t,x,u)\partial_x + \gamma(t,x,u)\partial_u.$$

The commutation relation $[Q_2, Q_3] = Q_1$ gives

$$\alpha(t) = -t^2 + c, \qquad \beta = -xt + b(x,u), \qquad \gamma = \gamma(x,u).$$

Thus we may take

$$Q_2 = -t^2\partial_t + (b(x,u) - xt)\partial_x + \gamma(x,u)\partial_u.$$

Then we use the relation $[Q_1, Q_2] = 2Q_2$. From this we obtain

$$b(x,u) = m(u)x^3, \qquad \gamma(x,u) = n(u)x^2$$

and we have

$$Q_2 = -t^2\partial_t + \left(m(u)x^3 - xt\right)\partial_x + n(u)x^2\partial_u.$$

All that remains to be done is to find a canonical form for Q_2. We do this using equivalence transformations (5) which leave invariant the form of Q_1, Q_3. These are given by

$$T(t) = t, \qquad X(tx,u) = q(u)x, \qquad U = p(u)$$

with $q(u) \neq 0$ and $\dot{p}(u) \neq 0$. Under this transformation we find

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \left(\frac{q(u)m(u) + n(u)\dot{q}(u)}{q^3(u)}\bar{x}^3 - \bar{x}\bar{t}\right)\partial_{\bar{x}} + \frac{n(u)\dot{p}(u)}{q^2(u)}\bar{x}^2\partial_v.$$

There are two cases: $n(u) \neq 0$ and $n(u) = 0$. If $n(u) \neq 0$ then we may choose $p(u)$ and $q(u)$ such that

$$\frac{q(u)m(u) + n(u)\dot{q}(u)}{q^3(u)} = 1, \qquad \frac{n(u)\dot{p}(u)}{q^2(u)} = 1$$

which give us

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \big(\bar{x}^3 - \bar{x}\bar{t}\big)\partial_{\bar{x}} + \bar{x}^2\partial_v.$$

If, however, $n(u) = 0$ then we have

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \left(\frac{q(u)m(u) + n(u)\dot{q}(u)}{q^3(u)}\bar{x}^3 - \bar{x}\bar{t}\right)\partial_{\bar{x}}.$$

If now $m(u) > 0$ we may choose $m(u) = q^2(u)$ and we find

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \big(\bar{x}^3 - \bar{x}\bar{t}\big)\partial_{\bar{x}}.$$

If $m(u) < 0$ then we choose $m(u) = -q^2(u)$ and we find

$$\bar{Q}_2 = -\bar{t}^2\partial_{\bar{t}} + \big(-\bar{x}^3 - \bar{x}\bar{t}\big)\partial_{\bar{x}}.$$

Finally, we note that $t \to -t, \quad x \to x, \quad u \to u$ is an equivalence transformation of equation (1), and the last two canonical forms for Q_2 are equivalent under this transformation. Summarizing this calculation, we find that the algebra $sl(2,\mathbb{R})$ has four canonical forms with $Q_3 = \partial_t$:

$$\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t - tx\partial_x + x^2\partial_u,\ \partial_t\rangle,$$
$$\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t + x(x^2 - t)\partial_x,\ \partial_t\rangle,$$
$$\langle 2t\partial_t,\ -t^2\partial_t,\ \partial_t\rangle,$$
$$\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t - xt\partial_x,\ \partial_t\rangle,$$

and these are inequivalent under the equivalence group given by (5). The canonical form $Q_3 = \partial_x$ gives rise to a similar calculation, and three inequivalent representations for the Lie algebra $sl(2,\mathbb{R})$ are found. Other six realizations of the algebra $sl(2,\mathbb{R})$ be able admitted by PDEs of the form (1).

These results are summarized in the following:

Theorem 2. *There exist six inequivalent realizations of the algebra $sl(2,\mathbb{R})$ by operators* (3)*, which are admitted by PDEs of the form* (1)

$$\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t - tx\partial_x + x^2\partial_u,\ \partial_t\rangle, \tag{10}$$
$$\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t + x\big(x^2 - t\big)\partial_x,\ \partial_t\rangle, \tag{11}$$
$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u,\ \partial_x\rangle, \tag{12}$$
$$\langle 2x\partial_x - u\partial_u,\ \big(u^{-4} - x^2\big)\partial_x + xu\partial_u,\ \partial_x\rangle, \tag{13}$$
$$\langle 2x\partial_x - u\partial_u,\ -\big(u^{-4} + x^2\big)\partial_x + xu\partial_u,\ \partial_x\rangle, \tag{14}$$
$$\langle 2x\partial_x,\ -x^2\partial_x,\ \partial_x\rangle. \tag{15}$$

The forms of the functions F, G determining the corresponding invariant equations are given in Table 1, where $\tilde{F}$, $\tilde{G}$ are arbitrary functions.

Table 1.

$sl(2,\mathbb{R})$	F	G
(10)	$x\tilde{F}(\omega,v)$, $\omega=2u-xu_1$, $v=2u-x^2u_{xx}$	$\frac{u_x^2}{4}+x^{-2}\tilde{G}(\omega,v)$
(11)	$x^{-3}u_x^{-4}\tilde{F}(\sigma,u)$, $\sigma=u_{xx}u_x^{-2}+3x^{-1}u_x^{-1}$	$-\frac{x^{-2}\omega}{4}+x^{-2}\left(\frac{9\sigma}{\omega^2}-\frac{12}{\omega^3}\right)\tilde{F}(u,\sigma)+\frac{1}{x^2\omega}\tilde{G}(u,\delta)$, $\omega=xu_x$, $\delta=u_{xx}u_x^{-2}-3x^{-1}u_x^{-1}$, $\sigma=u_{xx}^{-1}u_x^2-\frac{xu_x}{3}$
(12)	$u^{-6}\tilde{F}(t,\omega)$, $\omega=u^{-5}u_{xx}-2u^{-6}u_x^2$	$u\tilde{G}(t,\omega)+(12u^{-9}u_x^3-9u^{-8}u_xu_{xx})\tilde{F}(t,\omega)$
(13)	$\frac{u^{-6}}{(1+4\omega^2)^{\frac{3}{2}}}\tilde{F}(t,\sigma)$, $\sigma=\frac{2v-10\omega^2-1}{2(1+4\omega^2)^{\frac{3}{2}}}$, $\omega=u^{-3}u_x$, $v=u^{-5}u_{xx}$	$u\tilde{G}(t,\sigma)\sqrt{1+4\omega^2}-u\Big[12\sigma^2\omega\sqrt{1+4\omega^2}+21\sigma\omega+\frac{15}{2}\frac{\omega+6\omega^3}{(1+4\omega^2)^{\frac{3}{2}}}\Big]\tilde{F}(t,\sigma)$
(14)	$\frac{u^{-6}}{(4\omega^2-1)^{\frac{3}{2}}}\tilde{F}(t,\sigma_1)$ if $4\omega^2>1$, $\omega=u^{-3}u_x$, $v=u^{-5}u_{xx}$, $\sigma_1=\frac{2v-10\omega^2-1}{2(4\omega^2-1)^{\frac{3}{2}}}$	$u\tilde{G}(t,\sigma_1)\sqrt{4\omega^2-1}-u\Big[12\sigma_1^2\omega\sqrt{4\omega^2-1}+21\sigma_1\omega+\frac{15}{2}\frac{6\omega^3-\omega}{(1-4\omega^2)^{\frac{3}{2}}}\Big]\tilde{F}(t,\sigma)$
	$\frac{u^{-6}}{(1-4\omega^2)^{\frac{3}{2}}}\tilde{F}(t,\sigma_2)$ if $4\omega^2<1$, $\omega=u^{-3}u_x$, $v=u^{-5}u_{xx}$, $\sigma_2=\frac{2v-10\omega^2-1}{2(1-4\omega^2)^{\frac{3}{2}}}$	$u\tilde{G}(t,\sigma_2)\sqrt{1-4\omega^2}-u\Big[12\sigma_2^2\omega\sqrt{1-4\omega^2}+21\sigma_2\omega+\frac{15}{2}\frac{6\omega^3-\omega}{(1-4\omega^2)^{\frac{3}{2}}}\Big]\tilde{F}(t,\sigma_2)$
(15)	$u_x^{-3}\tilde{F}(t,u)$	$-\frac{3}{2}\frac{u_{xx}^2}{u_x^4}\tilde{F}(t,u)+\tilde{G}(t,u)$

Theorem 3. *There exists only one realization of the algebra so*(3) *by operators of the form* (2) *which is an invariance algebra of* (1)*:*

$$\langle\partial_x,\ \tan u\sin x\partial_x+\cos x\partial_u,\ \tan u\cos x\partial_x-\sin x\partial_u\rangle. \tag{16}$$

Furthermore, the most general form of the functions F, G allowing for PDE (1) *to be invariant under the above realization is given by*

$$F=\frac{\sec^3 u}{(1+\omega^2)^{3/2}}\tilde{F}(t,\psi),$$
$$G=\left[9\omega\psi\tan u-3\omega\psi^2(1+\omega^2)^{1/2}+\frac{\omega(1+2\omega^2)}{(\omega^2+1)^{3/2}}-\frac{\omega(5+6\omega^2)\tan^2 u}{(\omega^2+1)^{3/2}}\right]\tilde{F}(t,\psi)$$
$$+(\omega^2+1)^{1/2}\tilde{G}(t,\psi),$$

where we have used the notation

$$\omega=u_x\sec u,\qquad \psi=\frac{u_{xx}\sec^2 u+(1+2\omega^2)\tan u}{(1+\omega^2)^{3/2}}.$$

Provided the function $\tilde{G}$ is arbitrary, the realization (16) *is the maximal symmetry algebra of the corresponding equation* (1)*.*

Our approach is, as we have demonstrated, a combination of the Lie point symmetry analysis and an exploitation of the equivalence group of the equation to give possible canonical forms for the various Lie algebras. Each representation of the Lie algebra is then tested as a symmetry algebra, and is discarded if it gives no result (by which we mean, amongst other things, that the equation admitting a given representation as symmetry algebra must have $F\neq 0$ in (1)).

The present method gives a complete point-symmetry classification of (1), so that any evolution equation of the form is necessarily point-equivalent under the transformations (5) to one of the canonical forms for equation (1).

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Nonlocal Symmetry and Integrable Classes of Abel Equation

Vyacheslav M. BOYKO

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *boyko@imath.kiev.ua*

We suggest an approach for description of integrable cases of the Abel equations using the procedure of increasing the order and equivalence transformations for the induced second-order equations.

A diversity of methods were developed to date for finding solutions of nonlinear ordinary differential equations (ODE). Everybody who encounters integration of a particular ODE uses, as a rule, the accumulated databases (or reference books) of the classes of ODE and methods for their integration (e.g. [12, 19]). But if an ODE does not belong to any of the described classes then it does not mean that there is no approaches for finding solutions of this ODE in the closed form.

The symmetry approach is one of the most algorithmic approaches for integration and lowering of the order of ODE that admit a certain nontrivial symmetry (see e.g. Lie's book [13], the books [10, 17, 18] and review papers [10, 25]). In the framework of the symmetry approach (and its modifications) it is possible to obtain many of the known classes of integrable ODE. However, the needs of the applications stimulate new research into development of new methods for construction of ODE solutions in the closed form. The papers [2–10, 14–16, 18–25] may give an idea of current developments and directions of research in the field of symmetry (algebraic) methods for investigation of ODE.

In this paper we study Abel equations of the first and the second kind [1, 12, 19]

$$\dot p = p^3 f_4(y) + p^2 f_3(y) + p f_2(y) + f_1(y), \tag{1}$$

$$\dot p (p + f_0(y)) = p^3 f_4(y) + p^2 f_3(y) + p f_2(y) + f_1(y), \tag{2}$$

where $p = p(y)$, $\dot p = \frac{dp}{dy}$, f_i, $i = 0, \dots, 4$, are arbitrary smooth functions (with f_1, f_2, f_3, f_4 not identically vanishing simultaneously). Equations (1), (2) along with the Riccati equation are among the "simplest" nonlinear first-order ODE that have extensive applications. At the same time the problem of description of integrable classes of these equations stays within the focus of current research, and was previously considered in many papers (see e.g. [5–8, 16, 19, 20, 22–24].

Note that the Abel equations of the first and the second kind (1), (2) are related with each other by a local change of variables (namely, the equation (2) can be reduced to the form (1) by means of the change of variables $p = 1/v(y) - f_0$). Besides, the well-known Riccati equation is a partial case of equation (1).

The problem of finding Lie symmetries for the first-order ODE is equivalent to finding solutions for these equations, and for this reason the direct application of the Lie method is complicated in the general case. On of the well-known approaches in the cases when for a given ODE it is not feasible (or not effective) to apply the Lie method directly, is increasing of the order of the ODE under consideration (in particular, to obtain a second-order ODE related to the respective ODE by a change of variables). For examples of utilisation of such approach we can refer to papers [2–6, 9, 14–16]. In such cases, if the "induced" equation of a higher order admits a non-trivial Lie symmetry (that generated a non-local symmetry for the initial equation), we can speak of so-called hidden symmetries for an initial equation (for more details see [2–4]).

Further we will consider the following second-order ODE

$$\ddot{y} = \dot{y}^4 f_4(y) + \dot{y}^3 f_3(y) + \dot{y}^2 f_2(y) + \dot{y} f_1(y), \tag{3}$$

$$\ddot{y}(\dot{y} + f_0(y)) = \dot{y}^4 f_4(y) + \dot{y}^3 f_3(y) + \dot{y}^2 f_2(y) + \dot{y} f_1(y), \tag{4}$$

where $y = y(x)$, $\dot{y} = \frac{dy}{dx}$, $\ddot{y} = \frac{d^2y}{dx^2}$, related to the Abel equations (1) and (2).

The substitution $\dot{y} = p(y)$ reduces equations (3) and (4) respectively to the Abel equations (1) and (2) (reduction of the order for equations (3) and (4)). Such reduction is induced by the Lie operator $X_1 = \partial_x$ (that corresponds to invariance of equations (3) and (4) with respect to translations by the variable x). This is exactly the fact that explains why we consider equations (3) and (4).

In the case when (3) or (4) are invariant with respect to another operator (that is when (3) or (4) admit two-dimensional Lie algebras), then equations (3) and (4) are integrable in the framework of the Lie approach. And in this way we can obtain exact solutions of the equations (1) and (2) respectively.

Further we will consider only the equation (4) (since equations (1)–(4) are interconnected – see Remark 3). Let (4) admit a two-dimensional Lie algebra

$$L = \langle X_1, X_2 \rangle, \qquad X_1 = \partial_x, \qquad X_2 = \xi(x,y)\partial_x + \eta(x,y)\partial_y. \tag{5}$$

We will consider a problem of description of inequivalent equations (4) that are invariant with respect to two-dimensional Lie algebras of the form (5) (non-equivalent realisations of the operator X_2 in the algebra (5) will determine canonical representatives for equation (4)).

It is well-known that any two-dimensional Lie algebra in the general case, by means of choosing the basis operators X_1 and X_2 in an appropriate manner, may be reduced to four nonequivalent cases (see e.g. [13,17]). In the framework of our problem additional cases arise as we have fixed the form of the operator X_1.

So, it is quite straightforward to show that equation (4) may admit a two-dimensional Lie algebra (5) only of one of the following types:

1. $[X_1, X_2] = 0, \quad \operatorname{rank} L = 1;$
2. $[X_1, X_2] = 0, \quad \operatorname{rank} L = 2;$
3. $[X_1, X_2] = X_1, \quad \operatorname{rank} L = 1;$
4. $[X_1, X_2] = X_1, \quad \operatorname{rank} L = 2;$
5. $[X_1, X_2] = X_2, \quad \operatorname{rank} L = 1;$
6. $[X_1, X_2] = X_2, \quad \operatorname{rank} L = 2.$ (6)

Further, utilising classification of two-dimensional algebras (6), we obtain that equation (4) may admit only the following realisations of two-dimensional Lie algebras (5):

1. $X_1 = \partial_x, \qquad X_2 = \xi(y)\partial_x, \qquad \xi(y) \not\equiv \text{const};$
2. $X_1 = \partial_x, \qquad X_2 = \xi(y)\partial_x + \eta(y)\partial_y,$
   $\xi(y) \not\equiv \text{const or } \xi(y) \equiv 0, \quad \eta(y) \neq 0;$
3. $X_1 = \partial_x, \qquad X_2 = (x + \xi(y))\partial_x, \qquad \xi(y) \not\equiv \text{const or } \xi(y) \equiv 0;$
4. $X_1 = \partial_x, \qquad X_2 = (x + \xi(y))\partial_x + \eta(y)\partial_y,$
   $\xi(y) \not\equiv \text{const or } \xi(y) \equiv 0, \quad \eta(y) \neq 0;$
5. $X_1 = \partial_x, \qquad X_2 = e^x \xi(y)\partial_x, \qquad \xi(y) \not\equiv 0;$
6. $X_1 = \partial_x, \qquad X_2 = e^x(\xi(y)\partial_x + \eta(y)\partial_y), \qquad \eta(y) \neq 0.$ (7)

It is clear that using these realisations we can describe equations of the form (4) that are invariant with respect to two-dimensional Lie algebras (similarly as we have discussed in [22]). However, this way is too cumbersome, and thus obtained types of equations (4) will be quite complicated (functions f_i, $i = 0, \dots, 4$ in (4) will be expressed through coefficients of the operator X_2 from realisations (7)).

It is straightforward to show that the most general transformations that preserve the form of the operator X_1 we look as follows:

$$t = x + \omega(y), \qquad u = g(y), \tag{8}$$

where $\omega(y)$, $g(y)$ are arbitrary smooth functions, $g(y) \not\equiv \text{const}$.

After substitution (8) equation (4) takes the form

$$\begin{aligned}\ddot{u}((1-\omega' f_0)\dot{u} + f_0 g')g'^2 = {} & \big(f_4 - \omega''(1-\omega' f_0) - \omega' f_3 + \omega'^2 f_2 - \omega'^3 f_1\big)\dot{u}^4 \\ & + \big(g' f_3 - \omega'' g' f_0 + g''(1-\omega' f_0) - 2\omega' g' f_2 + 3\omega'^2 g' f_1\big)\dot{u}^3 \\ & + \big(g'^2 f_2 + g'' g'^2 f_0 - 3\omega' g'^2 f_1\big)\dot{u}^2 + f_1 g'^3 \dot{u},\end{aligned} \tag{9}$$

where $\omega' = \frac{d\omega}{dy}$, $\omega'' = \frac{d^2\omega}{dy^2}$, $g' = \frac{dg}{dy}$, $g'' = \frac{d^2 g}{dy^2}$ (in addition in (9) all functions of the variable y should be expressed as functions of the variable u).

With $(1 - \omega' f_0) \not\equiv 0$ equation (9) belongs again to the class of equations (4).

Remark 1. With $(1 - \omega' f_0) \equiv 0$ after the substitution (8), equation (4) is transformed to the equation (3), that is reduced to the Abel equation of the first kind (1).

Remark 2. It is possible to regard that $(1 - \omega' f_0) \not\equiv 0$ for the equation (4) as a result of the substitution (8) (we attain that by combination of transformations (8)).

Thus (8) are equivalence transformations for (4), and, besides, these transformations preserve the form of the operator $X_1 = \partial_x$ in the algebra (5).

Remark 3. So, the transformations (8) are equivalence transformations for the class of equations (3)–(4). Moreover, if we prolongate these transformations for $\dot{u} = p$ then these transformations form an equivalence transformation group for (1)–(2).

Thus, by means of transformations (8), realisations (7) of the algebra (5) may be reduced to the simplest canonical form. The transformations (8) in that process will not take us out of the class of equations (4).

By means of transformations (8) the realisations (7) of two-dimensional Lie algebras (5) admitted for equation (4) are reduced to the following canonical realisations:

$$\begin{aligned}
&1.\ X_1 = \partial_x, && X_2 = y\partial_x;\\
&2.\ X_1 = \partial_x, && X_2 = \partial_y;\\
&3.\ X_1 = \partial_x, && X_2 = x\partial_x;\\
&4.\ X_1 = \partial_x, && X_2 = x\partial_x + y\partial_y;\\
&5.\ X_1 = \partial_x, && X_2 = e^x\partial_x;\\
&6.\ X_1 = \partial_x, && X_2 = e^x(\partial_x + \partial_y).
\end{aligned} \tag{10}$$

In accordance to (10) we obtain the following integrable cases for equation (4) that are non-equivalent with respect to (8):

1. $\ddot{y} = \alpha(y)\dot{y}^3$;

2. $\ddot{y}(\dot{y}+e)=d\dot{y}^4+c\dot{y}^3+b\dot{y}^2+a\dot{y}$;

3. $\ddot{y}=\alpha(y)\dot{y}^2$;

4. $y\ddot{y}(\dot{y}+e)=d\dot{y}^4+c\dot{y}^3+b\dot{y}^2+a\dot{y}$;

5. $\ddot{y}(\dot{y}+\beta(y))=\alpha(y)\dot{y}^3+(1-\alpha(y)\beta(y))\dot{y}^2-\beta(y)\dot{y}$;

6. *a)* $f_0=0$:

$$\ddot{y}=de^y\dot{y}^3+(-3de^y+c)\dot{y}^2+(-de^y-(2c+1)+be^{-y})\dot{y}$$
$$+(-de^y+(c+1)-be^{-y}+ae^{-2y});$$

b) $f_0\neq 0$:

$$\ddot{y}(\dot{y}+\alpha(y))=-\dot{y}^3+(1-\alpha(y))\dot{y}^2+\alpha(y)\dot{y}, \tag{11}$$

where $\alpha(y)$, $\beta(y)$ are arbitrary smooth functions, a, b, c, d, e are constants.

The case 6a in (11) may be simplified by means of the substitution $t=x$, $u=e^y$ (see (8) and (9)).

Equations (11) determine non-equivalent cases of the form (4) that admit two-dimensional algebras (10) up to equivalence transformations (8).

Thus, summarising the above, we come to the following scheme for integration of the Abel equation (2):

- we increase the order of equation (2), considering a second-order equation (4);
- if a corresponding equation (4) admits a two-dimensional Lie algebra, then we reduce this algebra to one of the canonical forms (10), and thus the equation is reduced to the respective canonical forms (11);
- we integrate the canonical form (11);
- making reverse changes of variables we obtain the solution of the Abel equation (2).

It is obvious from the above that there is an alternative way for generation of new integrable cases of the Abel equation based on utilisation of the relation between the Abel equations of the first and the second kind, and relation between the equations (3) and (4) by means of the transformations (8). Thus, starting from some integrable Abel equation (that is of such equation for which the solution is known) it is possible to obtain new integrable cases of the Abel equations (solutions of these equations will be related through transformations (8)). It would be possible to use for this purpose even the well-known Riccati equation that is a partial case of the equation (1) (for generation of integrable Riccati equations an approach that is proposed in [20] may be used).

We hope that new results for classification of integrable classes of ODE may be obtained also using our classification of inequivalent realizations of real low-dimensional Lie algebras [21].

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On Equivariant Boundary Value Problems

Vladimir P. BURSKII

Institute of Applied Mathematics and Mechanics, 74 Luxemburg Str., 83114 Donetsk, Ukraine
E-mail: *burskii@iamm.ac.donetsk.ua*

We describe an investigation method of general equivariant boundary value problem for PDE of general form in a domain and show how it works for the case of simplest group $O(n, \mathbb{R})$.

1 Solvability conditions of equivariant expansion

Let $\Omega \subset \mathbb{R}^n$ be a domain, $\mathcal{L} = \sum\limits_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be some arbitrary differential operation with smooth coefficients $a_\alpha(x)$, $\mathcal{L}^+$ be a formally adjoint differential operation. Let L_0, L_0^+ be minimal operators (i.e., for example, $D(L_0)$ is the clozure of $C_0^\infty(\Omega)$ in the norm of the graph $\|u\|_L^2 = \|u\|_{L_2(\Omega)}^2 + \|Lu\|_{L_2(\Omega)}^2$), and L, L^+ be maximal expansions of $\mathcal{L}, \mathcal{L}^+$ in the space $L_2(\Omega)$ respectively (i.e. $L = (L_0^+)^*$, $L^+ = (L_0)^*$), $\tilde{L} = L|_{D(\tilde{L})}$ where $D(\tilde{L})$ is the clozure of $C^\infty(\bar{\Omega})$ in the norm of the graph $\|u\|_L$ and it is analogous for $\tilde{L}^+$.

M.Yo. Vishik introduced the conditions:

V_1) the operator L_0: $D(L_0) \to L_2(\Omega)$ has a continuous left-inverse,

V_2) the operator L_0^+: $D(L_0^+) \to L_2(\Omega)$ has a continuous left-inverse

and proved that

1) these conditions are necessary and sufficient for the existence of a solvable expansion L_B: $D(L_B) \to L_2(\Omega)$, (that is $D(L_0) \subset D(L_B)$, $\exists\, L_B^{-1}$: $L_2(\Omega) \to D(L_B)$);

2) under conditions V_1), V_2) for any solvable expansion L_B the following decomposition of the domain $D(L)$ is valid: $D(L) = D(L_0) + \ker L + B$, and L: $B \to \ker L^+$ is an isomorphism.

Let G be some Lie group, smoothly acting in the closed domain $\bar{\Omega}$. It means, that there is a group of diffeomorphisms U_g: $\bar{\Omega} \ni x \to g \cdot x = U_g(x) \in \bar{\Omega}$ of domain $\bar{\Omega}$ onto itself, group, smoothly depending on an element of G, and mapping $g \to U_g$ is a homomorphism of groups. Thus the contraction of diffeomorphisms U_g on boundary $\partial\Omega$ induces a smooth action of group G on boundary $\partial\Omega$.

The action of group G on domain $\bar{\Omega}$ generates a representation of the group G in function spaces: $(gu)(x) = u(g^{-1}x)$ (homomorphism of group G into group of converted operators). Such representation is induced on spaces $C_0^\infty(\Omega)$, $C^\infty(\Omega)$, $H^m(\Omega)$, $H^{-m}(\Omega)$, $\mathcal{D}'(\Omega)$, $H^{(m)}(\Omega)$, $H^{(-m)}(\Omega)$ and others. Let the differential operation $\mathcal{L}$ be invariant with respect to the action of group G, that is $g(\mathcal{L}u) = \mathcal{L}(gu)$. Then spaces $D(L)$, $D(L_0)$, $C(L)$, $\ker L$ are invariant with respect to the action of the group.

If the action of group preserves the volume of the domain Ω then the scalar product in the space $L_2(\Omega)$ is invariant with respect to the action of group G, and consequently the representation of the group G is unitary in this space. In this case the operation $\mathcal{L}^+$ is also invariant with respect to an action of group G, the spaces $D(L^+)$, $D(L_0^+)$, $C(L^+)$, $\ker L^+$ are invariant.

Boundary value problem

$$Lu = f, \qquad \Gamma u \in B, \tag{1}$$

generated by a subspace $B \subset C(L)$ of the boundary space $C(L) = D(L)/D(L_0)$ we shall name G-**invariant**, if the space B is invariant with respect to the indicated action of group G. A G-invariant boundary value problem we will name **equivariant**, if it is clear what group acts.

If the group G is compact (and is continuous), then, as it is well known, the Hilbert space of representation is decomposed in the direct sum of finite-dimensional invariant subspaces, in which the irreducible representations of group G are induced. And if the group is also commutative, the irreducible representations are one-dimensional.

Let space of a representation of the group G be the boundary space $C(L)$. For the case of compact group we have decompositions

$$C(L) = \sum_{k=0}^{\infty} \oplus \tilde{C}^k, \qquad C(\ker L) = \sum_{k=0}^{\infty} \oplus C^k(\ker L), \qquad B = \sum_{k=0}^{\infty} \oplus B^k.$$

If our G-invariant boundary value problem is well-posed, the decompositions in the direct sum $C(L) = C(\ker L) \oplus B$ imply decompositions in the direct sum $C^k := C^k(\ker L) \oplus B^k = \sum\limits_l \tilde{C}^{k_l}$ with finite-dimensional projectors Π^k: $C^k \to C^k(\ker L)$ along B^k and now check of the well-posedness of a G-invariant boundary value problem can be reduced to check of two properties:

1) $C^k(\ker L) \cap B^k = 0$;

2) $\exists\ \kappa > 0, \quad \forall\ k, \quad \|\Pi^k\|_{C^k} < \kappa.$

Below we shall study a spectrum of an operator of a general well-posed equivariant boundary value problem for the Poisson equation in a disk and in a ball, detecting cases of violation of the well-posedness of the problem, which are expressed in violation of property 1). Thus the fulfilment of property 2) will be assured by the assumed property of the well-posedness of this problem for the Poisson equation.

2 Equivariant boundary value problems for the Helmholtz equation in a disk

Let us conduct evaluations on check of two properties of the well-posedness of a general equivariant boundary value problem in a simplest case. As the group we will choose group of rotations of the plane $SO(2, \mathbb{R})$. It is compact commutative group.

Let us consider the problem (1), where $\mathcal{L} = \Delta$ and L is the maximum operator generated by the Laplace operator Δ, invariant with respect to the action of rotations group, domain $\Omega = K = \{x \in \mathbb{R}^2 \,|\, |x| < 1\}$ is the disk. Let us remark, that here we have $L = \tilde{L}$, i.e. each function from $D(L)$ can be approximated by smooth functions. Let us assume that this boundary value problem is G-invariant and is well-posed.

And we study such boundary value problem for the Helmholtz equation

$$L_\lambda v = \Delta v + \lambda^2 v = g, \qquad \Gamma v \in B,$$

where λ is a complex number. But in the beginning we study the boundary space $C(L_\lambda)$ of the Helmholtz operator and its subspace $C(\ker L_\lambda)$.

Boundary space consists of some pairs of functions $(u|_{\partial\Omega}, u'_\nu|_{\partial\Omega}) \in H^{-1/2}(\partial K) \times H^{-3/2}(\partial K)$, therefore a general boundary condition must have the form

$$Au|_{\partial K} + Bu'_\nu|_{\partial K} = 0$$

with some operators A, B. The G-invariance of this condition means the commutativity of operators A and B with all the representation operators. But, as it is well-known, a rotation invariant linear operator has a form of the convolution with a function. Therefore we will consider boundary value problems of the type:

$$\alpha * u|_{\partial K} - \beta * u'_\nu|_{\partial K} = 0,$$

where $\alpha = \sum \alpha_k e^{ik\tau}$; $\beta = \sum \beta_k e^{ik\tau}$ are functions on the boundary ∂K, $*$ is the convolution on ∂K: $\alpha * \psi = \sum \alpha_k \psi_k e^{ik\tau}$.

The boundary condition for Fourier coefficients of functions $u|_{\partial\Omega}$, $u'_\nu|_{\partial\Omega}$ from the space B can be written in the form

$$\forall\, k \in \mathbb{Z}, \qquad \alpha_k a_k + \beta_k b_k = 0. \tag{2}$$

Let us designate by C^k an image of an enclosure I_k: $\mathbb{C}^2 \to C(L)$ acting by a rule I_k: $(a,b) \to (ae^{ik\tau}, be^{ik\tau})$. The boundary problem sets a subspace B of the space $C(L)$, which, as we see, intersects each space C^k in a straight line. The well-posedness of our problem, i.e. expansion in the direct sum $C(L_\lambda) = B \oplus C(\ker L_\lambda)$, means now that

$$\exists\, A > 0, \quad \forall\, k \in \mathbb{Z}, \quad |\sin(B^k, C^k(\ker L_\lambda))| > A,$$

i.e.

$$\forall\, k, \quad \frac{|\beta_k \lambda J'_k(\lambda) - \alpha_k J_k(\lambda)|}{\sqrt{|\lambda J'_k(\lambda)|^2 + |J_k(\lambda)|^2}} > A > C \quad \text{at} \quad \lambda \neq 0$$

and

$$\frac{|k\beta_k - \alpha_k|}{\sqrt{k^2+1}} > A > 0 \quad \text{at} \quad \lambda = 0.$$

Proposition 1. *The problem* (2)*, which is well-posed for the equation* $\Delta u = g$*, is well-posed for the equation* $\Delta u + \lambda^2 u = g$ *at* $\lambda \neq 0$ *if and only if the following condition holds*

$$\forall\, k, \quad |k\beta_k J^1_k(\lambda) - \alpha_k J^2_k(\lambda)| \neq 0.$$

Proposition 2. *Spectrum of the operator of well-posed boundary value problem* (2) *for the equation* $\Delta u = g$ *is a set* $\bigcup\limits_k \Sigma_k$*, where* Σ_k *is the set of proper values of a form* $-\lambda^2$ *and* λ *runs all zeros of the equation*

$$\beta_k \lambda J'_k(\lambda) - \alpha_k J_k(\lambda) = 0 \quad \textit{at} \quad \lambda \neq 0. \tag{3}$$

Proposition 3. *Spectrum of the operator of well-posed boundary value problem* (2) *for the equation* $\Delta u = g$ *is finite-to-one.*

Proposition 4. *Every well-posed* G*-invariant boundary value problem for the Poisson equation is quite correct, i.e. its solving operator is compact.*

Propositions concerning the same equation on n-dimensional ball have similar formulations but the equation (3) has the following form:

$$\beta^1_l \lambda J'_{\nu+l}(\lambda) - \beta^1_l \nu J_{\nu+l}(\lambda) - \alpha^1_l J_{\nu+l}(\lambda) = 0,$$

where $\nu = \frac{n}{2} - 1$, $l \in \mathbb{N} \cup 0$.

And the corresponding equivariant boundary value problem must have the following form:

$$u|_{\partial\Omega} * \alpha + u'_\nu|_{\partial\Omega} * \beta = 0,$$

where $\alpha = \sum\limits_{l=0}^{\infty} \sum\limits_k \alpha^k_l H^k_l$, $\beta = \sum\limits_{l=0}^{\infty} \sum\limits_k \beta^k_l H^k_l$ are functions on sphere S^{n-1}, which are decomposed in Fourier series, $*$ is convolution on $\partial\Omega$: $\psi * \alpha = \sum\limits_{l=0}^{\infty} \sum\limits_k \psi^k_l \alpha^1_l H^k_l$, what means, in particular, that we can omit tesseral (not zonal) parts in decomposition α and β.

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On the Deformations of Dorfman's and Sokolov's Operators

Jen-Hsu CHANG

Department of General Courses, Chung-Cheng Institute of Technology, National Defense University, Tau-Yuan County, Taiwan
E-mail: *jhchang@ccit.edu.tw*

We deform the Dorfman's and Sokolov's Hamiltonian operators by the quasi-Miura transformation coming from the topological field theory and investigate the deformed operators.

1 Introduction

The Dorfman's and Sokolov's Hamiltonian operators are defined respectively as [2,11] ($D = \partial_x$)

$$J = D\frac{1}{v_x}D\frac{1}{v_x}D, \tag{1}$$

$$S = v_x D^{-1} v_x, \tag{2}$$

which are Hamiltonian operators (or $J^{-1} = D^{-1}v_x D^{-1} v_x D^{-1}$ and $S^{-1} = \frac{1}{v_x}D\frac{1}{v_x}$ are symplectic operators). The Dorfman's operator J (or J^{-1}) and the Sokolov's operator S are related to integrable equations as follows.

- The Riemann hierarchy

$$v_{t_n} = v^n v_x = S\delta H_n = \frac{1}{(n+1)(2n+1)}K\delta H_{n+1} = \frac{1}{(n+1)(n+2)}D\delta H_{n+2}$$
$$= \frac{1}{(n+1)(n+2)(n+3)(n+4)}J\delta H_{n+4}, \tag{3}$$

where

$$K = Dv + vD, \qquad H_n = \int v^n dx, \qquad n = 1, 2, 3\ldots,$$

and δ is the variational derivative. When $n = 1$, it is called the Riemann equation or dispersionless KdV equation. We notice that it seems that the Riemann hierarchy (3) is a quater-Hamiltonian system. But one can show that S and J is not compatible, i.e., $S + \lambda J$ are not a Hamiltonian operator for any $\lambda \neq 0$ (see below).

- The Schwarzian KdV equation [10,13]

$$v_t = v_{xxx} - \frac{3}{2}\frac{v_{xx}^2}{v_x} = v_x\{v, x\} = S\delta H_1 = J^{-1}\delta H_2, \tag{4}$$

where $\{v, x\}$ is the Schwartz derivative and

$$H_1 = \frac{1}{2}\int \left(v_x^{-2}v_{xx}^2\right) dx, \qquad H_2 = \frac{1}{2}\int \left(-v_x^{-2}v_{xxx}^2 + \frac{3}{4}v_x^{-4}v_{xx}^4\right) dx.$$

Remark 1. It is not difficult to verify that J^{-1} is also a Hamiltonian operator and, then, J is also a symplectic operator; however, $S^{-1} = \frac{1}{v_x}D\frac{1}{v_x}$ is <u>not</u> a Hamiltonian operator and, then, S is not a symplectic operator.

Next, to deform the operators J and S, we use the free energy in topological field theory of the famous KdV equation

$$u_t = uu_x + \frac{\epsilon^2}{12}u_{xxx} \tag{5}$$

to construct the quasi-Miura transformation as follows. The free energy F of KdV equation (5) in TFT has the form ($F_0 = \frac{1}{6}v^3$)

$$F = \frac{1}{6}v^3 + \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g \left(v; v_x, v_{xx}, v_{xxx}, \ldots, v^{(3g-2)} \right).$$

Let

$$\begin{aligned} \triangle F &= \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g \left(v; v_x, v_{xx}, v_{xxx}, \ldots, v^{(3g-2)} \right) \\ &= F_1(v; v_x) + \epsilon^2 F_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx}) \\ &\quad + \epsilon^4 F_3 \left(v; v_x, v_{xx}, v_{xxx}, v_{xxxx}, \ldots, v^{(7)} \right) + \cdots . \end{aligned}$$

The $\triangle F$ will satisfy the loop equation [4, p. 151]

$$\begin{aligned} &\sum_{r\geq 0} \frac{\partial \triangle F}{\partial v^{(r)}} \partial_x^r \frac{1}{v-\lambda} + \sum_{r\geq 1} \frac{\partial \triangle F}{\partial v^{(r)}} \sum_{k=1}^{r} \left(\begin{array}{c} r \\ k \end{array} \right) \partial_x^{k-1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{r-k+1} \frac{1}{\sqrt{v-\lambda}} \\ &\quad = \frac{1}{16\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2} \\ &\quad + \frac{\epsilon^2}{2} \sum_{k,l\geq 0} \left[\frac{\partial^2 \triangle F}{\partial v^{(k)} \partial v^{(l)}} + \frac{\partial \triangle F}{\partial v^{(k)}} \frac{\partial \triangle F}{\partial v^{(l)}} \right] \partial_x^{k+1} \frac{1}{\sqrt{v-\lambda}} \partial_x^{l+1} \frac{1}{\sqrt{v-\lambda}} \\ &\quad - \frac{\epsilon^2}{16} \sum_{k\geq 0} \frac{\partial \triangle F}{\partial v^{(k)}} \partial_x^{k+2} \frac{1}{(v-\lambda)^2}. \end{aligned} \tag{6}$$

Then we can determine F_1, F_2, F_3, ... recursively by substituting $\triangle F$ into equation (6). For F_1, one obtains

$$\frac{1}{v-\lambda}\frac{\partial F_1}{\partial v} - \frac{3}{2}\frac{v_x}{(v-\lambda)^2}\frac{\partial F_1}{\partial v_x} = \frac{1}{16\lambda^2} - \frac{1}{16(v-\lambda)^2} - \frac{\kappa_0}{\lambda^2}.$$

From this, we have

$$\kappa_0 = \frac{1}{16}, \qquad F_1 = \frac{1}{24}\log v_x.$$

For the next terms $F_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx})$, it can be similarly computed and the result is

$$F_2 = \frac{v_{xxxx}}{1152 v_x^2} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xx}^3}{360 v_x^4}.$$

Now, one can define the quasi-Miura transformation as

$$\begin{aligned} u &= v + \epsilon^2 (\triangle F)_{xx} = v + \epsilon^2 (F_1)_{xx} + \epsilon^4 (F_2)_{xx} + \cdots \\ &= v + \frac{\epsilon^2}{24}(\log v_x)_{xx} + \epsilon^4 \left(\frac{v_{xxxx}}{1152 v_x^2} - \frac{7 v_{xx} v_{xxx}}{1920 v_x^3} + \frac{v_{xx}^3}{360 v_x^4} \right)_{xx} + \cdots . \end{aligned} \tag{7}$$

One remarks that Miura-type transformation means the coefficients of ϵ are homogeneous polynomials in the derivatives $v_x, v_{xx}, \ldots, v^{(m)}$ [4, p. 37], [5] and "quasi" means the ones of ϵ are quasi-homogeneous rational functions in the derivatives, too [4, p. 109] (see also [12]).

The truncated quasi-Miura transformation

$$u = v + \sum_{n=1}^{g} \epsilon^{2n} \left[F_n \left(v; v_x, v_{xx}, \ldots, v^{(3g-2)} \right) \right]_{xx} \tag{8}$$

has the basic property [4, p. 117] that it reduces the Magri–Poisson pencil [6] of KdV equation (5)

$$\{u(x), u(y)\}_\lambda = [u(x) - \lambda] D\delta(x-y) + \frac{1}{2} u_x(x)\delta(x-y) + \frac{\epsilon^2}{8} D^3\delta(x-y) \tag{9}$$

to the Poisson pencil of the Riemann hierarchy (3):

$$\{v(x), v(y)\}_\lambda = [v(x) - \lambda] D\delta(x-y) + \frac{1}{2} v_x(x)\delta(x-y) + O\left(\epsilon^{2g+2}\right). \tag{10}$$

One can also say that the truncated quasi-Miura transformation (8) deforms the KdV equation (5) to the Riemann equation $v_t = vv_x$ up to $O\big(\epsilon^{2g+2}\big)$.

Remark 2. A simple calculation shows that, under the transformation $u = \frac{\epsilon^2}{4}\{m, x\}$, the KdV equation (5) is transformed into the Schwarzian KdV equation

$$m_t = \frac{\epsilon^2}{12} m_x \{m, x\} = \frac{\epsilon^2}{12} \left(m_{xxx} - \frac{3}{2} \frac{m_{xx}^2}{m_x} \right).$$

Furthermore, after a direct calculation, one can see that the Magri Poisson bracket

$$K(\epsilon) = \{u(x), u(y)\} = u(x) D\delta(x-y) + \frac{1}{2} u_x(x)\delta(x-y) + \frac{\epsilon^2}{8} D^3\delta(x-y) \tag{11}$$

is transformed into the Dorfman's symplectic operator J^{-1} $(m = v)$

$$\{m(x), m(y)\} = -\frac{\epsilon^2}{8} D^{-1} m_x D^{-1} m_x D^{-1} \delta(x-y).$$

Now, a natural question arises: under the truncated quasi-Miura transformation (8), are the deformed Dorfman's operator $J(\epsilon)$ and Sokolov's operator $S(\epsilon)$ still Hamiltonian operators up to $O\big(\epsilon^{2g+2}\big)$? For simplicity, we consider only the case $g = 1$, i.e.,

$$u = v + \frac{\epsilon^2}{24} (\log v_x)_{xx} + O\left(\epsilon^4\right) \tag{12}$$

or

$$v = u - \frac{\epsilon^2}{24} (\log u_x)_{xx} + O\left(\epsilon^4\right). \tag{13}$$

The answer is true for the Dorfman's operator $J(\epsilon)$ but it is false for the Sokolov's operator $S(\epsilon)$. It is the purpose of this article.

2 Deformations under quasi-Miura transformation

In the new "u-coordinate", J and S will be given by the operators

$$J(\epsilon) = M^* D \frac{1}{u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}} D \frac{1}{u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}} DM + O\left(\epsilon^4\right), \tag{14}$$

$$S(\epsilon) = M^* \left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) D^{-1} \left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) M + O\left(\epsilon^4\right), \tag{15}$$

where

$$M = 1 - \frac{\epsilon^2}{24} D \frac{1}{u_x} D^2, \qquad M^* = 1 + \frac{\epsilon^2}{24} D^2 \frac{1}{u_x} D, \tag{16}$$

M^* being the adjoint operator of M. Then we have the following

Theorem 1. 1. *$J(\epsilon)$ is a Hamiltonian operator up to $O\left(\epsilon^4\right)$.* 2. *$S(\epsilon)$ is <u>not</u> a Hamiltonian operator up to $O\left(\epsilon^4\right)$.*

Proof. 1. The fact that $J(\epsilon)$ is a skew-adjoint (or $J^*(\epsilon) = -J(\epsilon)$) differential operator (up to $O\left(\epsilon^4\right)$) follows immediately from (14). Rather than prove the Poisson form [7] of the Jacobi identity for $J(\epsilon)$, it is simpler to prove that the symplectic two-form

$$\Omega_J(\epsilon) = \int \{du \wedge J(\epsilon)^{-1} du\} dx + O\left(\epsilon^4\right)$$

is closed [8,9]: $d\Omega_J(\epsilon) = O\left(\epsilon^4\right)$.

A simple calculation can yield

$$\begin{aligned}
J(\epsilon)^{-1} &= \left(1 + \frac{\epsilon^2}{24} D \frac{1}{u_x} D^2\right) D^{-1} \left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) D^{-1} \\
&\quad \times \left(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}\right) D^{-1} \left(1 - \frac{\epsilon^2}{24} D^2 \frac{1}{u_x} D\right) \\
&= \left(D^{-1} u_x - \frac{\epsilon^2}{24} D^{-1}(\log u_x)_{xxx} + \frac{\epsilon^2}{24} D \frac{1}{u_x} D u_x\right) D^{-1} \\
&\quad \times \left(u_x D^{-1} - \frac{\epsilon^2}{24}(\log u_x)_{xxx} D^{-1} - \frac{\epsilon^2}{24} u_x D \frac{1}{u_x} D\right) + O\left(\epsilon^4\right) \\
&= D^{-1} u_x D^{-1} u_x D^{-1} + \frac{\epsilon^2}{24} \left[D \frac{1}{u_x} D u_x D^{-1} u_x D^{-1} - D^{-1}(\log u_x)_{xxx} D^{-1} u_x D^{-1}\right. \\
&\quad \left. - D^{-1} u_x D^{-1} u_x D \frac{1}{u_x} D - D^{-1} u_x D^{-1}(\log u_x)_{xxx} D^{-1}\right] + O\left(\epsilon^4\right) \\
&= D^{-1} u_x D^{-1} u_x D^{-1} \\
&\quad + \frac{\epsilon^2}{24} \left[D u_x D^{-1} - D^{-1} u_x D + (\log u_x)_x u_x D^{-1} + D^{-1}(\log u_x)_x u_x\right] + O\left(\epsilon^4\right).
\end{aligned}$$

Let ψ denote the potential function for u, i.e., $u = \psi_x$. Thus, formally,

$$D_x^{-1}(du) = d\psi$$

and hence, after a series of integration by parts, one has

$$\begin{aligned}
\Omega_J(\epsilon) = \int \left\{ \left[\left(D^{-1} d\left(\frac{\psi_x^2}{2}\right)\right) \wedge d\left(\frac{\psi_x^2}{2}\right) - \psi_x d\psi \wedge d\left(\frac{\psi_x^2}{2}\right)\right] \right. \\
\left. + \frac{\epsilon^2}{24} \left[2\psi_{xx} d\psi \wedge d\psi_{xx} + 2\psi_{xxx} d\psi_x \wedge d\psi\right] \right\} dx + O\left(\epsilon^4\right).
\end{aligned}$$

So

$$d\Omega_J(\epsilon) = \int \left\{ 0 + \frac{\epsilon^2}{12} \left[d\psi_{xxx} \wedge d\psi_x \wedge d\psi \right] \right\} dx + O\left(\epsilon^4\right)$$
$$= \frac{\epsilon^2}{12} \int \left\{ (d\psi_{xx} \wedge d\psi_x \wedge d\psi)_x \right\} dx + O(\epsilon^4) = O\left(\epsilon^4\right).$$

This completes the proof of (1).

2. The skew-adjoint property of the deformed Sokolov's operator $S(\epsilon)$ (15) is obvious. To see whether $S(\epsilon)$ is Hamiltonian operator or not, we must check whether $S(\epsilon)$ satisfies the Jacobi identity up to $O\left(\epsilon^4\right)$. Following [7, 8], we introduce the arbitrary basis of tangent vector Θ, which is then conveniently manipulated according to the rules of exterior calculus. The Jacobi identity is given by the compact expression

$$P(\epsilon) \wedge \delta I = O\left(\epsilon^4\right) \quad \text{(mod. div.)}, \tag{17}$$

where $P(\epsilon) = S(\epsilon)\Theta$, $I = \frac{1}{2}\Theta \wedge P(\epsilon)$ and δ denotes the variational derivative. The vanishing of the tri-vector (17) modulo a divergence is equivalent to the satisfaction of the Jacobi identity.

After a tedious calculation, one can obtain

$$S(\epsilon) = M^* \left(u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx} \right) D^{-1} \left(u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx} \right) M + O\left(\epsilon^4\right)$$
$$= \left[u_x + \frac{\epsilon^2}{24} \left(D^3 + D^2 (\log u_x)_x - (\log u_x)_{xxx} \right) \right] D^{-1}$$
$$\times \left[u_x - \frac{\epsilon^2}{24} \left(D^3 - (\log u_x)_x D^2 + (\log u_x)_{xxx} \right) \right] + O\left(\epsilon^4\right)$$
$$= u_x D^{-1} u_x + \frac{\epsilon^2}{24} \left[D^2 u_x + D^2 (\log u_x)_x D^{-1} u_x - (\log u_x)_{xxx} D^{-1} u_x - u_x D^2 \right.$$
$$\left. + u_x D^{-1} (\log u_x)_x D^2 - u_x D^{-1} (\log u_x)_{xxx} \right] + O\left(\epsilon^4\right)$$
$$= u_x D^{-1} u_x + \frac{\epsilon^2}{24} \left[D^2 u_x - u_x D^2 + (\log u_x)_x D u_x + u_x D (\log u_x)_x \right] + O\left(\epsilon^4\right)$$
$$= u_x D^{-1} u_x + \frac{\epsilon^2}{12} [D u_{xx} + u_{xx} D] + O\left(\epsilon^4\right).$$

So

$$P(\epsilon) = S(\epsilon)\Theta = u_x D^{-1} (u_x \Theta) + \frac{\epsilon^2}{12} [2 u_{xx} \Theta_x + u_{xxx} \Theta] + O\left(\epsilon^4\right).$$

Hence

$$I = \frac{1}{2} \Theta \wedge P(\epsilon) = \frac{1}{2} u_x \Theta \wedge D^{-1} (u_x \Theta) + \frac{\epsilon^2}{12} u_{xx} \Theta \wedge \Theta_x + O\left(\epsilon^4\right)$$

and then

$$\delta I = -\frac{1}{2} [\Theta \wedge D^{-1} (u_x \Theta)]_x - \frac{1}{2} u_x \Theta \wedge D^{-1} (\Theta_x) + \frac{\epsilon^2}{12} [\Theta \wedge \Theta_x]_{xx} + O\left(\epsilon^4\right)$$
$$= -\frac{1}{2} \Theta_x \wedge D^{-1} (u_x \Theta) + \frac{\epsilon^2}{12} [\Theta \wedge \Theta_x]_{xx} + O\left(\epsilon^4\right).$$

Finally,

$$\begin{aligned}
P(\epsilon)\wedge\delta I &= \left\{u_xD^{-1}(u_x\Theta)+\frac{\epsilon^2}{12}[2u_{xx}\Theta_x+u_{xxx}\Theta]\right\}\\
&\quad\wedge\left\{-\frac{1}{2}\Theta_x\wedge D^{-1}(u_x\Theta)+\frac{\epsilon^2}{12}[\Theta\wedge\Theta_x]_{xx}\right\}+O\left(\epsilon^4\right)\\
&= 0+\frac{\epsilon^2}{12}\Big\{-\frac{1}{2}u_{xxx}\Theta\wedge\Theta_x\wedge D^{-1}(u_x\Theta)+u_{xxx}D^{-1}(u_x\Theta)\wedge\Theta\wedge\Theta_x\\
&\quad+3u_{xx}u_x\Theta\wedge\Theta\wedge\Theta_x+u_x^2\Theta_x\wedge\Theta\wedge\Theta_x\Big\}+O\left(\epsilon^4\right)\\
&= 0+\frac{\epsilon^2}{24}u_{xxx}\Theta\wedge\Theta_x\wedge D^{-1}(u_x\Theta),
\end{aligned}$$

which can be easily checked that it cannot be expressed as a total divergence. So $S(\epsilon)$ cannot satisfy the Jacobi identity and therefore $S(\epsilon)$ is not a Hamiltonian operator. This completes the proof of (2). ■

Remark 3. Using the technics of the last proof, one can show that J and S is <u>not</u> compatible. Since J and S are Hamiltonian operators, what we are going to do is show that [7, 8]

$$\tilde{Q}(\Theta)\wedge\delta R+Q(\Theta)\wedge\delta\tilde{R}\neq 0\quad(\text{mod.}\quad\text{div.}),$$

where

$$\begin{aligned}
&Q(\Theta)=v_xD^{-1}(v_x\Theta),\qquad R=\frac{1}{2}\Theta\wedge Q(\Theta),\\
&\tilde{Q}(\Theta)=\left(\frac{1}{v_x}\left(\frac{\Theta_x}{v_x}\right)_x\right)_x,\qquad \tilde{R}=\frac{1}{2}\Theta\wedge\tilde{Q}(\Theta)=-\frac{1}{2v_x^2}\Theta_x\wedge\Theta_{xx}.
\end{aligned}$$

Then

$$\delta R=\frac{-1}{2}[\Theta\wedge D^{-1}(v_x\Theta)]_x-\frac{1}{2}v_x\Theta\wedge D^{-1}(\Theta_x)=\frac{-1}{2}\Theta_x\wedge D^{-1}(v_x\Theta)$$

and

$$\delta\tilde{R}=-\left(\frac{1}{v_x^3}\Theta_x\wedge\Theta_{xx}\right)_x.$$

Hence

$$\begin{aligned}
&\tilde{Q}(\Theta)\wedge\delta R+Q(\Theta)\wedge\delta\tilde{R}\\
&\quad=\left(\frac{1}{v_x}\left(\frac{\Theta_x}{v_x}\right)_x\right)_x\wedge\left(\frac{-1}{2}\Theta_x\wedge D^{-1}(v_x\Theta)\right)-v_xD^{-1}(v_x\Theta)\wedge\left(\frac{1}{v_x^3}\Theta_x\wedge\Theta_{xx}\right)_x\\
&\quad=\frac{1}{2}\frac{1}{v_x}\left(\frac{\Theta_x}{v_x}\right)_x\wedge\left[\Theta_{xx}\wedge D^{-1}(v_x\Theta)+v_x\Theta_x\wedge\Theta\right]\\
&\qquad+\left[v_{xx}D^{-1}(v_x\Theta)+v_x^2\Theta\right]\wedge\left(\frac{1}{v_x^3}\Theta_x\wedge\Theta_{xx}\right)\\
&\quad=\frac{1}{2v_x}\Theta_{xx}\wedge\Theta_x\wedge\Theta-\frac{v_{xx}}{2v_x^3}\Theta_x\wedge\Theta_{xx}\wedge D^{-1}(v_x\Theta)\\
&\qquad+\frac{v_{xx}}{v_x^3}D^{-1}(v_x\Theta)\wedge\Theta_x\wedge\Theta_{xx}+\frac{1}{v_x}\Theta\wedge\Theta_x\wedge\Theta_{xx}\\
&\quad=\frac{1}{2v_x}\Theta\wedge\Theta_x\wedge\Theta_{xx}+\frac{v_{xx}}{2v_x^3}\Theta_x\wedge\Theta_{xx}\wedge D^{-1}(v_x\Theta)\\
&\quad\neq 0\quad(\text{mod. div.}),
\end{aligned}$$

as required.

3 Concluding remarks

• That $J(\epsilon)$ is a Hamiltonian operator (up to $O(\epsilon^4)$) is proved in [1]. We give another proof here, which remarkably simplifies the proof given in [1].
• We notice that all the deformed operators $J(\epsilon)$ (14), $D(\epsilon)$ $\left(= D + O\left(\epsilon^4\right)\right)$, $K(\epsilon)$ (11) under the quasi-Miura transformation (7) are Hamiltonian operators (up to $O\left(\epsilon^4\right)$). That the deformed Sokolov's operator $S(\epsilon)$ is not Hamiltonian is a little surprising that means that the Poisson bracket of the Hamiltonians $H_m(u;\epsilon)$, $H_n(u;\epsilon)$ for $S(\epsilon)$

$$\{H_m(u;\epsilon), H_n(u;\epsilon)\}_{S(\epsilon)}$$

will not be $O\left(\epsilon^4\right)$ but $O\left(\epsilon^2\right)$, i.e., it cannot be a conserved quantity of the Riemann hierarchy (3).

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Non-Lie Reductions of Nonlinear Reaction-Diffusion Systems with Variable Diffusivities

Roman CHERNIHA

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv, Ukraine
E-mail: *cherniha@imath.kiev.ua*

Non-Lie reductions to systems of first-order ordinary differential equations are performed for a class of systems of two quasilinear reaction-diffusion equations having variable diffusivities. Moreover, families of exact solutions of a diffusive Lotka–Volterra type system are constructed.

1 Introduction

In this communication, semilinear systems of two reaction-diffusion (RD) equations of the form

$$\begin{aligned} U_t &= d_1(U^{\alpha_1}U_x)_x + U(a_1 + b_1U^{\alpha_1}) + U^{1-\alpha_1}(h_1 + c_1V^{\alpha_2}),\\ V_t &= d_2(V^{\alpha_2}V_x)_x + V(a_2 + b_2V^{\alpha_2}) + V^{1-\alpha_2}(h_2 + c_2U^{\alpha_1}) \end{aligned} \tag{1}$$

are considered. Here the subscripts t and x to the functions U and V denote differentiation with respect to these variables and all the coefficients are constants. The communication is organized as follows. In Section 2, a set of new non-Lie ansätze are constructed which reduce RD system (1) to systems of four or more ordinary differential equations. It is shown that these ansätze cannot be obtained using Lie symmetries of this system. In Section 3, families of exact solutions of the RD system

$$\begin{aligned} U_t &= (UU_x)_x + U(a_1 + b_1U) + h_1 + c_1V,\\ V_t &= (VV_x)_x + V(a_2 + b_2V) + h_2 + c_2U. \end{aligned} \tag{2}$$

One can note that system (2) is a system of Lotka–Volterra type, with variable (degenerate) diffusivities (see, e.g. [1]), in which the standard terms c_1UV and c_2UV are replaced by the terms $h_1 + c_1V$ and $h_2 + c_2U$, respectively. Hereafter this RD system will be referred as the degenerate diffusive Lotka–Volterra system (DDLV system).

2 New non-Lie ansätze for the RD system (1)

This section is devoted to non-Lie reductions of RD systems with power-law nonlinearities of the form (1). Hereafter I assume that *both* equations contain diffusion coefficients with variable diffusivities and that the system is coupled, i.e., $d_1d_2\alpha_1\alpha_2 \neq 0$, and $c_1 \neq 0$ or $c_2 \neq 0$.

It is easily checked that the linear substitution

$$U = d_1^{-1/\alpha_1}U^*, \qquad V = d_2^{-1/\alpha_2}V^* \tag{3}$$

reduces (1) to the same form with $d_1 = d_2 = 1$; therefore, without losing generality, I can consider DDLV systems of the form

$$\begin{aligned} U_t &= (U^{\alpha_1}U_x)_x + U(a_1 + b_1U^{\alpha_1}) + U^{1-\alpha_1}(h_1 + c_1V^{\alpha_2}),\\ V_t &= (V^{\alpha_2}V_x)_x + V(a_2 + b_2V^{\alpha_2}) + V^{1-\alpha_2}(h_2 + c_2U^{\alpha_1}). \end{aligned} \tag{4}$$

According to the results of group classification for systems of two reaction-diffusion equations with variable diffusivities [2,3], system (4) is invariant only with respect to the trivial Lie algebra with the basic operators

$$P_t = \partial_t, \qquad P_x = \partial_x \tag{5}$$

if the coefficients are arbitrary constants. However, there are the following special cases leading to non-trivial algebras of invariance: $b_1 = b_2 = h_1 = h_2 = 0$ (see case 6 of Table 1 in [3]); $b_1 = b_2 = h_1 = h_2 = 0$ and $\alpha_1 = \alpha_2 = -4/3$ (see case 9 of Table 1 in [3]); $b_1 = b_2 \neq 0$, $h_1 = h_2 = 0$ and $\alpha_1 = \alpha_2 = -4/3$ (see cases 10 and 12 of Table 1 in [3]). In particular, the DDLV system (4) admits only the trivial Lie algebra (5) if $h_1 \neq 0$ or $h_2 \neq 0$.

With $U = u^{1/\alpha_1}$, $V = v^{1/\alpha_2}$ this system is reduced to the form

$$\begin{aligned} u_t &= uu_{xx} + \frac{1}{\alpha_1}u_x^2 + \alpha_1\left[h_1 + a_1u + b_1u^2 + c_1v\right], \\ v_t &= vv_{xx} + \frac{1}{\alpha_2}v_x^2 + \alpha_2\left[h_2 + a_2v + b_2v^2 + c_2u\right], \end{aligned} \tag{6}$$

where $u(t,x)$ and $v(t,x)$ are new unknown functions. One observes that system (6) contains only quadratic nonlinearities. Several new approaches were recently suggested to find exact solutions of single evolution equation with quadratic nonlinearities (see [6–10] and references cited therein). Those approaches lead to so called non-Lie ansätze which cannot be found using the classical Lie method and the non-classical Bluman–Cole method [11] (it was, however, shown in [12] that some solutions constructed by those ansätze can be found by a modification of the Bluman–Cole method).

However, it is not easy to apply those approaches for finding exact solutions of systems of equations. To my knowledge there are only a few papers devoted to the case of evolution systems [13, 10, 14].

It turns out that it is possible to construct non-Lie ansätze and to interpret the relevant exact solutions of the system (6) using the notion of additional generating conditions [10, 15].

Consider an additional generating condition of the following non-coupled system of third order ordinary differential equations (ODEs)

$$\beta_1(t)\frac{du}{dx} + \beta_2(t)\frac{d^2u}{dx^2} + \frac{d^3u}{dx^3} = 0, \qquad \beta_1(t)\frac{dv}{dx} + \beta_2(t)\frac{d^2v}{dx^2} + \frac{d^3v}{dx^3} = 0, \tag{7}$$

where $\beta_1(t)$ and $\beta_2(t)$ are arbitrary smooth functions and the variable t is considered as a parameter. Depending on the coefficients, the solution to the linear ODE system (7) can take the following forms:

$$u = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2, \qquad v = \psi_0(t) + \psi_1(t)x + \psi_2(t)x^2 \tag{8}$$

if $\beta_1 = \beta_2 = 0$;

$$u = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)\exp(\gamma(t)x), \qquad v = \psi_0(t) + \psi_1(t)x + \psi_2(t)\exp(\gamma(t)x) \tag{9}$$

if $\beta_1 = 0$;

$$\begin{aligned} u &= \varphi_0(t) + \varphi_1(t)\exp(\gamma_1(t)x) + \varphi_2(t)\exp(\gamma_2(t)x), \\ v &= \psi_0(t) + \psi_1(t)\exp(\gamma_1(t)x) + \psi_2(t)\exp(\gamma_2(t)x) \end{aligned} \tag{10}$$

if $\gamma_{1,2}(t) = \frac{1}{2}\left(\pm\sqrt{D} - \beta_2\right)$, $D = \beta_2^2 - 4\beta_1 > 0$ and $\gamma_1 \neq \gamma_2$;

$$u = \varphi_0(t) + \exp\left(-\frac{\beta_2 x}{2}\right)\left[\varphi_1(t)\cos\left(\frac{\sqrt{-D}}{2}x\right) + \varphi_2(t)\sin\left(\frac{\sqrt{-D}}{2}x\right)\right],$$

$$v=\psi_0(t)+\exp\left(-\frac{\beta_2 x}{2}\right)\left[\psi_1(t)\cos\left(\frac{\sqrt{-D}}{2}x\right)+\psi_2(t)\sin\left(\frac{\sqrt{-D}}{2}x\right)\right] \tag{11}$$

if $D<0$; and, finally,

$$\begin{aligned} u&=\varphi_0(t)+\varphi_1(t)\exp(\gamma(t)x)+x\varphi_2(t)\exp(\gamma(t)x),\\ v&=\psi_0(t)+\psi_1(t)\exp(\gamma(t)x)+x\psi_2(t)\exp(\gamma(t)x) \end{aligned} \tag{12}$$

if $D=0$, i.e. $\gamma_1=\gamma_2=\gamma\neq 0$.

Let me consider relations (8)–(12) as a chain of ansätze for the system (6). It is important to note that each ansatz contains 6 yet-to-be determined functions φ_i, ψ_i, $i=0,1,2$. This enables us to reduce the given system of PDEs to a nonlinear system of first-order ODEs for the unknown functions φ_i and ψ_i (γ, β_1 and β_2 can provide additional unknowns, depending on the ansatz in question). Indeed, calculating the derivatives U_t, U_x, U_{xx} with the help of ansatz (8) and substituting them into system (6), one obtains an expression, which can be splitted into separate parts for the powers of x. Simultaneously the restriction on coefficients

$$b_1=b_2=0 \tag{13}$$

is required (otherwise $\varphi_1=\psi_1=\varphi_2=\psi_2=0$) and then a correctly-specified ODE system for the functions φ_i and ψ_i has the form

$$\begin{aligned} \dot{\varphi_0}&=2\varphi_0\varphi_2+\frac{1}{\alpha_1}\varphi_1^2+\alpha_1(h_1+a_1\varphi_0+c_1\psi_0),\\ \dot{\psi_0}&=2\psi_0\psi_2+\frac{1}{\alpha_2}\psi_1^2+\alpha_2(h_2+a_2\psi_0+c_2\varphi_0),\\ \dot{\varphi_1}&=\frac{2\alpha_1+4}{\alpha_1}\varphi_1\varphi_2+\alpha_1(a_1\varphi_1+c_1\psi_1),\\ \dot{\psi_1}&=\frac{2\alpha_2+4}{\alpha_2}\psi_1\psi_2+\alpha_2(a_2\psi_1+c_2\varphi_1),\\ \dot{\varphi_2}&=\frac{2\alpha_1+4}{\alpha_1}\varphi_2^2+\alpha_1(a_1\varphi_2+c_1\psi_2),\\ \dot{\psi_2}&=\frac{2\alpha_2+4}{\alpha_2}\psi_2^2+\alpha_2(a_2\psi_2+c_2\varphi_2). \end{aligned} \tag{14}$$

Finally, given any solution of the ODE system (14), one constructs an exact solution of DDLV system (4), (13) of the form

$$U=\left(\varphi_0+\varphi_1 x+\varphi_2 x^2\right)^{1/\alpha_1},\qquad V=\left(\psi_0+\psi_1 x+\psi_2 x^2\right)^{1/\alpha_2}. \tag{15}$$

In the case of ansatz (9), two different subcases are obtained, namely: (i) $\alpha_1\neq -1$ and $\alpha_2\neq -1$; (ii) $\alpha_1=\alpha_2=-1$.

The first subcase leads to the requirement $\varphi_1=\psi_1=0$ and then the ansatz reduces to a particular case of the ansatz (10) that will be considered below. Consider the second subcase. In a similar way to the above calculation for the ansatz (8), I arrive at the coefficient restrictions (13) and also $a_1+c_1=a_2+c_2$, together with the following correctly-specified ODE system for the functions γ, φ_i and ψ_i

$$\begin{aligned} &\dot{\gamma}=\varphi_1\gamma^2,\qquad \psi_1=\varphi_1,\qquad \dot{\varphi_1}=-(a_1+c_1)\varphi_1,\\ &\dot{\varphi_0}=-\left(h_1+a_1\varphi_0+c_1\psi_0+\varphi_1^2\right),\\ &\dot{\psi_0}=-\left(h_2+a_2\psi_0+c_2\varphi_0+\psi_1^2\right),\\ &\dot{\varphi_2}=\gamma^2\varphi_0\varphi_2-2\gamma\varphi_1\varphi_2-(a_1\varphi_2+c_1\psi_2), \end{aligned}$$

$$\dot{\psi_2} = \gamma^2\psi_0\psi_2 - 2\gamma\psi_1\psi_2 - (a_2\psi_2 + c_2\varphi_2). \tag{16}$$

Thus, any solution of (16) generates an exact solution of the form

$$U = (\varphi_0 + \varphi_1 x + \varphi_2 \exp\gamma x)^{-1}, \qquad V = (\psi_0 + \psi_1 x + \psi_2 \exp\gamma x)^{-1} \tag{17}$$

of the DDLV system

$$\begin{aligned} U_t &= \left(U^{-1}U_x\right)_x + U(a_1 + h_1U + c_1\,U/V), \\ V_t &= \left(V^{-1}V_x\right)_x + V(a_2 + h_2V + c_2\,V/U), \end{aligned} \tag{18}$$

where $a_1 + c_1 = a_2 + c_2$.

Analogously, the ansätze (10) and (11) have been analysed and the two different subcases listed above were found too. Considering the subcase (i), I have obtained the restriction on coefficients

$$\gamma_1 = -\gamma_2 = \sqrt{-\frac{b_1\alpha_1^2}{\alpha_1+1}} = \sqrt{-\frac{b_2\alpha_2^2}{\alpha_2+1}} \equiv \gamma \tag{19}$$

for both ansätze. To find real solutions ansatz (10) can be applied only for $\frac{b_k}{\alpha_k+1} < 0$, $k = 1, 2$ while ansatz (11) works only for $\frac{b_k}{\alpha_k+1} > 0$, $k = 1, 2$. The relevant ODE systems for finding the functions φ_i and ψ_i have the form

$$\begin{aligned} \dot{\varphi_0} &= \alpha_1 b_1\varphi_0^2 + \alpha_1(h_1 + a_1\varphi_0 + c_1\psi_0) - \frac{4}{\alpha_1}\gamma^2\varphi_1\varphi_2, \\ \dot{\psi_0} &= \alpha_2 b_2\psi_0^2 + \alpha_2(h_2 + a_2\psi_0 + c_2\varphi_0) - \frac{4}{\alpha_2}\gamma^2\psi_1\psi_2, \\ \dot{\varphi_1} &= (2\alpha_1 b_1 + \gamma^2)\varphi_0\varphi_1 + \alpha_1(a_1\varphi_1 + c_1\psi_1), \\ \dot{\psi_1} &= (2\alpha_2 b_2 + \gamma^2)\psi_0\psi_1 + \alpha_2(a_2\psi_1 + c_2\varphi_1), \\ \dot{\varphi_2} &= (2\alpha_1 b_1 + \gamma^2)\varphi_0\varphi_2 + \alpha_1(a_1\varphi_2 + c_1\psi_2), \\ \dot{\psi_2} &= (2\alpha_2 b_2 + \gamma^2)\psi_0\psi_2 + \alpha_2(a_2\psi_2 + c_2\varphi_2) \end{aligned} \tag{20}$$

and

$$\begin{aligned} \dot{\varphi_0} &= \alpha_1 b_1\varphi_0^2 + \alpha_1(h_1 + a_1\varphi_0 + c_1\psi_0) - \frac{\gamma^2}{\alpha_1}\left(\varphi_1^2 + \varphi_2^2\right), \\ \dot{\psi_0} &= \alpha_2 b_2\psi_0^2 + \alpha_2(h_2 + a_2\psi_0 + c_2\varphi_0) - \frac{\gamma^2}{\alpha_2}\left(\psi_1^2 + \psi_2^2\right), \\ \dot{\varphi_1} &= (2b_1\alpha_1 + \gamma^2)\varphi_0\varphi_1 + \alpha_1(a_1\varphi_1 + c_1\psi_1), \\ \dot{\psi_1} &= (2b_2\alpha_2 + \gamma^2)\psi_0\psi_1 + \alpha_2(a_2\psi_1 + c_2\varphi_1), \\ \dot{\varphi_2} &= (2b_1\alpha_1 + \gamma^2)\varphi_0\varphi_2 + \alpha_1(a_1\varphi_2 + c_1\psi_2), \\ \dot{\psi_2} &= (2b_2\alpha_2 + \gamma^2)\psi_0\psi_2 + \alpha_2(a_2\psi_2 + c_2\varphi_2) \end{aligned} \tag{21}$$

respectively. The resulting exact solutions of the DDLV system (4) take the form

$$\begin{aligned} U &= \left[\varphi_0(t) + \varphi_1(t)\exp(-\gamma x) + \varphi_2(t)\exp(\gamma x)\right]^{1/\alpha_1}, \\ V &= \left[\psi_0(t) + \psi_1(t)\exp(-\gamma x) + \psi_2(t)\exp(\gamma x)\right]^{1/\alpha_2} \end{aligned} \tag{22}$$

and

$$U = \left[\varphi_0(t) + \varphi_1(t)\cos(|\gamma|x) + \varphi_2(t)\sin(|\gamma|x)\right]^{1/\alpha_1},$$

$$V = \left[\psi_0(t) + \psi_1(t)\cos(|\gamma|x) + \psi_2(t)\sin(|\gamma|x)\right]^{1/\alpha_2}. \tag{23}$$

In the subcase (ii), the restrictions

$$\gamma_1 = -\gamma_2 = \gamma \in \mathbb{R}, \qquad b_1 = b_2 = 0 \tag{24}$$

arise instead of (19). The corresponding systems of ODEs coincide with (20) and (21), in which one needs to set $b_k = 0$, $\alpha_k = -1$, $k = 1, 2$.

Finally, I have established that ansatz (12) can be applied only in the case $\varphi_2 = \psi_2 = 0$, i.e. if one is reduced to a particular case of (10).

I have also investigated the direct generalization of (8) of the form

$$u = \varphi_0 + \varphi_1 x + \cdots + \varphi_{m_1} x^{m_1}, \qquad v = \psi_0 + \psi_1 x + \cdots + \psi_{m_2} x^{m_2}. \tag{25}$$

It can be shown that ansatz (25) reduces system (6) to a correctly-specified ODE system only in the cases $m_1 = m_2 = 3$ and $m_1 = m_2 = 4$ (otherwise $\varphi_i = \psi_i = 0$, $i > 2$). The corresponding restrictions on the coefficients are as follows: $\alpha_1 = \alpha_2 = -\frac{3}{2}$, $b_1 = b_2 = 0$ for $m_1 = m_2 = 3$ and $\alpha_1 = \alpha_2 = -\frac{4}{3}$, $b_1 = b_2 = 0$ for $m_1 = m_2 = 4$. The relevant ODE systems are omitted here but one can find them in [4].

It should be noted that both exponents of power nonlinearities $\alpha = -\frac{3}{2}$ and $\alpha = -\frac{4}{3}$ were earlier found (see [16, 17] and [9]) to provide reductions of the single RD equation

$$U_t = (U^\alpha U_x)_x + a_1 U + c_1 U^{1-\alpha}. \tag{26}$$

3 Exact solutions of the DDLV system (2)

Consider system (2) and assume $b_1 b_2 \neq 0$ and $c_1 c_2 \neq 0$, i.e. the system contains quadratic nonlinearities in the reaction terms and the equations are coupled. Taking into account [2, 3], one easily observes that under these typical restrictions on coefficients, the DDLV system (2) is invariant only with respect to the trivial algebra (5). So, one can find in that manner only plane wave solutions of the form

$$U = \varphi(\omega), \qquad V = \psi(\omega), \qquad \omega = k_1 x - k_0 t, \tag{27}$$

where $k_0, k_1 \in \mathbb{R}$ and the functions φ and ψ are solutions of the reduced ODE system

$$\begin{aligned} &k_1^2(\varphi\varphi_\omega)_\omega + k_0\varphi_\omega + \varphi(a_1 + b_1\varphi) + h_1 + c_1\psi = 0,\\ &k_1^2(\psi\psi_\omega)_\omega + k_0\psi_\omega + \psi(a_2 + b_2\psi) + h_2 + c_2\varphi = 0. \end{aligned} \tag{28}$$

The ODE system (28) is not integrable for $k_1 \neq 0$, and only particular solutions can be find (some of them are presented below). The case $k_1 = 0$ of course leads to solutions which do not depend on the space variable.

It turns out that a much wider class of exact solutions of the DDLV system (2) can be constructed using the non-Lie reductions presented in Section 3. Consider the ODE system (20) with $\alpha_1 = \alpha_2 = 1$. As it was shown above, this system is obtained from system (2) using the ansatz

$$\begin{aligned} &U = \varphi_0(t) + \varphi_1(t)\exp(-\gamma x) + \varphi_2(t)\exp(\gamma x),\\ &V = \psi_0(t) + \psi_1(t)\exp(-\gamma x) + \psi_2(t)\exp(\gamma x), \end{aligned} \tag{29}$$

where $\gamma = \sqrt{\frac{b}{2}}$, $b = -b_1 = -b_2 > 0$.

It can be noted that the ODE system (20) admits essential simplification for $\varphi_2 = \psi_2 = 0$, reducing to

$$\dot{\varphi}_0 = -b\varphi_0^2 + h_1 + a_1\varphi_0 + c_1\psi_0, \qquad \dot{\psi}_0 = -b\psi_0^2 + h_2 + a_2\psi_0 + c_2\varphi_0,$$
$$\dot{\varphi}_1 = -\frac{3}{2}b\varphi_0\varphi_1 + a_1\varphi_1 + c_1\psi_1, \qquad \dot{\psi}_1 = -\frac{3}{2}b\psi_0\psi_1 + a_2\psi_1 + c_2\varphi_1. \tag{30}$$

Now one observes that the first two equations of (30) decouple. Formally speaking, this subsystem has four steady-state solutions. I assume that there is at least one real solution among them, say (U_0, V_0). In other words, (U_0, V_0) is a real solution of the system of algebraic equations

$$bU_0^2 = h_1 + a_1U_0 + c_1V_0, \qquad bV_0^2 = h_2 + a_2V_0 + c_2U_0. \tag{31}$$

Substituting $\varphi_0 = U_0$, $\psi_0 = V_0$ into the last two equation of (30), one arrives at a linear ODE system

$$\dot{\varphi}_1 = \left(a_1 - \frac{3}{2}bU_0\right)\varphi_1 + c_1\psi_1, \qquad \dot{\psi}_1 = c_2\varphi_1 + \left(a_2 - \frac{3}{2}bV_0\right)\psi_1. \tag{32}$$

According to the classical theory of linear ODE systems, the form of the general solution of (32) depends on $\triangle_1 = [(a_1 - a_2) + \frac{3}{2}b(V_0 - U_0)]^2 + 4c_1c_2$. So I obtain the following general solutions of (32):

if $\triangle_1 > 0$ then

$$\varphi_1 = e_1c_1\exp(s_1t) + e_2\left(s_2 - a_2 + \frac{3}{2}bV_0\right)\exp(s_2t),$$
$$\psi_1 = e_1\left(s_1 - a_1 + \frac{3}{2}bU_0\right)\exp(s_1t) + e_2c_2\exp(s_2t), \tag{33}$$

where $s_{1,2} = \frac{1}{2}\left[a_1 + a_2 - \frac{3}{2}b(U_0 + V_0) \pm \sqrt{\triangle_1}\right]$;

if $\triangle_1 < 0$ then

$$\varphi_1 = c_1\exp(pt)(e_1\sin(qt) + e_2\cos(qt)),$$
$$\psi_1 = \exp(pt)\left[\left(e_1\left(p - a_1 + \frac{3}{2}bU_0\right) - e_2q\right)\sin(qt) + \left(e_1q + e_2\left(p - a_1 + \frac{3}{2}bU_0\right)\right)\cos(qt)\right], \tag{34}$$

where $p = \frac{1}{2}\Big(a_1 + a_2 - \frac{3}{2}b(U_0 + V_0)\Big)$, $q = \frac{1}{2}\sqrt{-\triangle_1}$;

if $\triangle_1 = 0$ then

$$\varphi_1 = (e_1c_1t + c_2)\exp(st),$$
$$\psi_1 = \left[e_1(1 \pm \sqrt{-c_1c_2}t) \pm e_2\sqrt{\frac{-c_2}{c_1}}\right]\exp(st)m, \tag{35}$$

where $s = \frac{1}{2}\left[a_1 + a_2 - \frac{3}{2}b(U_0 + V_0)\right]$. Here e_1 and e_2 are arbitrary constants.

Thus, three two-parameter families of exact solutions of the DDLV system

$$U_t = (UU_x)_x + U(a_1 - bU) + h_1 + c_1V,$$
$$V_t = (VV_x)_x + V(a_2 - bV) + h_2 + c_2U \tag{36}$$

of the form

$$U = U_0 + \varphi_1(t)\exp\left(-\sqrt{\frac{b}{2}}x\right), \qquad V = V_0 + \psi_1(t)\exp\left(-\sqrt{\frac{b}{2}}x\right) \tag{37}$$

can be constructed. Of course, each family arises only if the coefficients of the DDLV system (36) satisfy the relevant conditions on $\triangle_1$.

It should be stressed that any exact solution (37) with $\varphi_1(t)$ and $\psi_1(t)$ given by (33)–(35) is a non-Lie solution of the DDLV system (36) if $e_1e_2 \neq 0$. In fact, it is easily seen that this solution cannot be reduced to the form (27), i.e. it cannot be obtained using the Lie symmetries of the DDLV system (36). However, one observes that all solutions obtained, except the solution generated by formulas (37) and (34), are reduced to the form (27) if $e_1 = 0$. In other words, these two-parameter families of solutions contain as subclasses one-parameter families of Lie solutions.

Some other families of exact solutions of DDLV (2) are constructed in [4].

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Bogoyavlenskij Symmetries of Isotropic and Anisotropic MHD Equilibria as Lie Point Transformations

Alexei F. CHEVIAKOV

Department of Mathematics, Queen's University, Kingston, Ontario K7L 3N6, Canada
E-mail: *alexch@mast.queensu.ca*

Isotropic Magnetohydrodynamic (MHD) Equilibrium Equations and generalized Anisotropic Magnetohydrodynamic (Chew–Goldberger–Low, CGL) Equilibrium Equations possess infinite-dimensional groups of intrinsic symmetries. We show that certain non-trivial Lie point transformations (that can be obtained by direct application of the general Lie group analysis method) are equivalent to Bogoyavlenskij symmetries in both isotropic and anisotropic cases.

1 Introduction

Among all descriptions of plasma as a continuous medium, two single-fluid models are used most frequently. They are the isotropic magnetohydrodynamics (MHD) equations [1] and the anisotropic CGL (Chew–Goldberger–Low) magnetohydrodynamics equations [2]. Both of them are generalizations of Navier–Stokes system onto the case of conducting fluids; they are derived from Boltzmann and Maxwell equations under different isotropy assumptions.

The isotropic MHD approximation employs scalar pressure p and is valid when the mean free path of plasma particles is much smaller than the typical scale of the problem, so that the picture is maintained nearly isotropic via frequent collisions.

On the other hand, when the mean free path for particle collisions is long compared to Larmor radius, for instance, in strongly magnetized or rarified plasmas, the CGL approximation should be used. In this model, the pressure is a tensor with two different components: the pressure along the magnetic field $p_{\|}$ and in the transverse direction $p_{\perp}$. In the limit $p_{\perp} = p_{\|} = p$, CGL and MHD models coincide.

The applications of these models include but are not limited to the problem of controlled thermonuclear fusion, astrophysical applications (star formation, solar activity, astrophysical jets) and terrestrial applications (laboratory and industrial plasmas, ball lightning models). The relevant references are [1, 3–9].

Similarly to the Navier–Stokes gas dynamics equations, both MHD and CGL models are essentially nonlinear, and there are no methods so far for solving corresponding general initial/boundary value problems. There also is a lack of physically relevant particular analytical solutions that could model specific phenomena.

In recent papers [10, 11] Bogoyavlenskij introduced new symmetry transforms of the ideal MHD equilibrium equations. In certain classes of plasma configurations, Bogoyavlenskij symmetries break geometrical symmetry, thus giving rise to important classes of non-symmetric MHD equilibrium solutions.

In [12], we have shown that anisotropic CGL plasma equilibria possess similar topology-dependent infinite-dimensional symmetries, which generalize Bogoyavlenskij symmetries.

It is shown that Bogoyavlenskij symmetries for isotropic plasmas and generalized Bogoyavlenskij symmetries for anisotropic plasmas are equivalent to in particular Lie groups of point transformations, which are found independently using the classical Lie group analysis method.

The Lie symmetry method [13] used in this work finds Lie groups of point symmetries of partial differential equations. Lie transformations are used to build particular solutions of the system under consideration, to reduce the order and to obtain invariants. Self-similar solutions constructed from Lie symmetries often have direct physical significance. Many appropriate examples can be found in [14].

We remark, however, that not all symmetries of a given system can be found by the Lie method, but only continuous symmetries that have one-parametric Lie group structure.

In Section 2 of this paper, the isotropic and anisotropic plasma equilibrium equations are described, together with their most important properties, including their Bogoyavlenskij symmetries.

In Sections 3 and 4, the general Lie point symmetry method is described, and an equivalence correspondence is established between Bogoyavlenskij symmetries for isotropic and anisotropic plasmas and certain infinite-dimensional Lie point symmetry groups.

2 MHD and CGL equilibrium equations

The equilibrium states of isotropic moving plasmas are described by the system of MHD equilibrium equations, which under the assumptions of infinite conductivity and negligible viscosity has the form [1]

$$\rho \boldsymbol{V} \times \operatorname{curl} \boldsymbol{V} - \frac{1}{\mu} \boldsymbol{B} \times \operatorname{curl} \boldsymbol{B} - \operatorname{grad} P - \rho \operatorname{grad} \frac{\boldsymbol{V}^2}{2} = 0, \tag{1}$$

$$\operatorname{div} \rho \boldsymbol{V} = 0, \qquad \operatorname{curl} (\boldsymbol{V} \times \boldsymbol{B}) = 0, \qquad \operatorname{div} \boldsymbol{B} = 0. \tag{2}$$

Here $\boldsymbol{V}$ is plasma velocity; $\boldsymbol{B}$ is the vector of the magnetic field induction; ρ, plasma density; P, plasma pressure; and μ, magnetic permeability coefficient.

In the case of incompressible plasma, the equation

$$\operatorname{div} \boldsymbol{V} = 0 \tag{3}$$

is added to the above system. In this paper we restrict our consideration to incompressible plasmas.

It is known [10, 11, 15, 16] that all compact incompressible MHD equilibrium configurations, except the Beltrami case $\operatorname{curl} \boldsymbol{B} = \alpha \boldsymbol{B}$, $\alpha = \text{const}$, have two-dimensional magnetic surfaces – the vector fields $\boldsymbol{B}$ and $\boldsymbol{V}$ are in every point tangent to magnetic surfaces. The magnetic surfaces may not exist for unbounded incompressible MHD equilibrium configurations with $\boldsymbol{V} \parallel \boldsymbol{B}$.

In the case when plasma Larmor radius is small compared to characteristic dimensions of the system, the set of equations was found by Chew, Goldberger and Low [2]. The anisotropic equilibrium equations are:

$$\rho \boldsymbol{V} \times \operatorname{curl} \boldsymbol{V} - \frac{1}{\mu} \boldsymbol{B} \times \operatorname{curl} \boldsymbol{B} = \operatorname{div} \mathbb{P} + \rho \operatorname{grad} \frac{\boldsymbol{V}^2}{2}, \tag{4}$$

$$\operatorname{div} \rho \boldsymbol{V} = 0, \qquad \operatorname{curl} (\boldsymbol{V} \times \boldsymbol{B}) = 0, \qquad \operatorname{div} \boldsymbol{B} = 0, \tag{5}$$

where $\mathbb{P}$ is the pressure tensor with two independent components:

$$\mathbb{P} = \mathbb{I} p_\perp + \frac{p_\parallel - p_\perp}{\boldsymbol{B}^2} (\boldsymbol{B}\boldsymbol{B}). \tag{6}$$

Here $\mathbb{I}$ is a unit tensor.

For this system to be closed, one needs to add to it two equations of state. In this paper we will consider incompressible CGL plasmas: $\operatorname{div} \boldsymbol{V} = 0$.

Using vector calculus identities, the pressure tensor divergence may be rewritten in the form

$$\operatorname{div}\mathbb{P} = \operatorname{grad} p_\perp + \tau \operatorname{curl}\boldsymbol{B}\times\boldsymbol{B} + \tau \operatorname{grad}\frac{\boldsymbol{B}^2}{2} + \boldsymbol{B}(\boldsymbol{B}\cdot\operatorname{grad}\tau), \tag{7}$$

$$\tau = \frac{p_\parallel - p_\perp}{\boldsymbol{B}^2}. \tag{8}$$

Hence the system (4), (5) rewrites as

$$\rho\boldsymbol{V}\times\operatorname{curl}\boldsymbol{V} - \left(\frac{1}{\mu} - \tau\right)\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B} = \operatorname{grad} p_\perp + \rho\operatorname{grad}\frac{\boldsymbol{V}^2}{2} + \tau\operatorname{grad}\frac{\boldsymbol{B}^2}{2} + \boldsymbol{B}(\boldsymbol{B}\cdot\operatorname{grad}\tau), \tag{9}$$

$$\operatorname{div}\boldsymbol{V} = 0, \qquad \operatorname{div}\boldsymbol{B} = 0, \qquad \operatorname{curl}(\boldsymbol{V}\times\boldsymbol{B}) = 0. \tag{10}$$

Recently Bogoyavlenskij [10, 11] found that the ideal MHD equilibrium equations (1)–(3) possess the following symmetries.

Let $\{\boldsymbol{V}(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r}), P(\boldsymbol{r}), \rho(\boldsymbol{r})\}$ be a solution of (1)–(3), where the density $\rho(\boldsymbol{r})$ is constant on both magnetic field lines and streamlines. Then $\{\boldsymbol{V}_1(\boldsymbol{r}), \boldsymbol{B}_1(\boldsymbol{r}), P_1(\boldsymbol{r}), \rho_1(\boldsymbol{r})\}$ is also a solution, where

$$\boldsymbol{V}_1 = \frac{b(\boldsymbol{r})}{m(\boldsymbol{r})\sqrt{\mu\rho}}\boldsymbol{B} + \frac{a(\boldsymbol{r})}{m(\boldsymbol{r})}\boldsymbol{V}, \qquad \boldsymbol{B}_1 = a(\boldsymbol{r})\boldsymbol{B} + b(\boldsymbol{r})\sqrt{\mu\rho}\,\boldsymbol{V}, \tag{11}$$

$$\rho_1 = m^2(\boldsymbol{r})\rho, \qquad P_1 = CP + \left(C\boldsymbol{B}^2 - \boldsymbol{B}_1^2\right)/(2\mu).$$

Here

$$a^2(\boldsymbol{r}) - b^2(\boldsymbol{r}) = C = \text{const},$$

and $a(\boldsymbol{r})$, $b(\boldsymbol{r})$, $c(\boldsymbol{r})$ are functions constant on both magnetic field lines and streamlines (i.e. on magnetic surfaces $\Psi = \text{const}$, when they exist).

These transformations form an infinite-dimensional Abelian group [11]

$$G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2, \tag{12}$$

where R^+ is a multiplicative group of positive numbers, and A_m is an additive Abelian group of smooth functions in $\mathbb{R}^3$ that are constant on magnetic surfaces. The group G_m has eight connected components.

In [12] we have found that a similar group of transformations is present in the case of anisotropic (CGL) incompressible plasma equilibria: let $\{\boldsymbol{V}(\boldsymbol{r}), \boldsymbol{B}(\boldsymbol{r}), p_\perp(\boldsymbol{r}), p_\parallel(\boldsymbol{r}), \rho(\boldsymbol{r})\}$ be a solution of the CGL system (9), (10), where the density $\rho(\boldsymbol{r})$ and the anisotropy factor $\tau(\boldsymbol{r})$ (8) are constant on both magnetic field lines and streamlines. Then $\{\boldsymbol{V_1}(\boldsymbol{r}), \boldsymbol{B_1}(\boldsymbol{r}), p_{\perp 1}(\boldsymbol{r}), p_{\parallel 1}(\boldsymbol{r}), \rho_1(\boldsymbol{r})\}$ is also a solution, where

$$\rho_1 = m^2(\boldsymbol{r})\rho,$$

$$\boldsymbol{V}_1 = \frac{b(\boldsymbol{r})\sqrt{1/\mu - \tau}}{m(\boldsymbol{r})\sqrt{\rho}}\boldsymbol{B} + \frac{a(\boldsymbol{r})}{m(\boldsymbol{r})}\boldsymbol{V}, \qquad \boldsymbol{B}_1 = \frac{a(\boldsymbol{r})}{n(\boldsymbol{r})}\boldsymbol{B} + \frac{b(\boldsymbol{r})\sqrt{\rho}}{n(\boldsymbol{r})\sqrt{1/\mu - \tau}}\boldsymbol{V},$$

$$p_{\perp 1} = Cp_\perp + \frac{(C\boldsymbol{B}^2 - \boldsymbol{B}_1^2)}{2\mu},$$

$$p_{\parallel 1} = p_\parallel n^2(\boldsymbol{r})\frac{\boldsymbol{B}_1^2}{\boldsymbol{B}^2} + p_\perp\left(C - n^2(\boldsymbol{r})\frac{\boldsymbol{B}_1^2}{\boldsymbol{B}^2}\right) + \frac{(C\boldsymbol{B}^2 + \boldsymbol{B}_1{}^2(1 - 2n^2(\boldsymbol{r})))}{2\mu}. \tag{13}$$

Here

$$a^2(\boldsymbol{r}) - b^2(\boldsymbol{r}) = C = \text{const},$$

and $a(\boldsymbol{r})$, $b(\boldsymbol{r})$, $m(\boldsymbol{r})$, $n(\boldsymbol{r})$ are functions constant on both magnetic field lines and streamlines.

For these symmetries, the anisotropy factor $\tau(\boldsymbol{r})$ is transformed as follows:

$$\tau_1 \equiv \frac{p_{\parallel 1} - p_{\perp 1}}{\boldsymbol{B_1}^2} = \frac{1}{\mu} - n^2(\boldsymbol{r}) \left(\frac{1}{\mu} - \tau \right). \tag{14}$$

The transformations for anisotropic case form an Abelian Lie group $G = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ with sixteen connected components.

3 Lie group formalism for the MHD equilibrium equations

A solution to a system of l first-order partial differential equations

$$\begin{aligned}
&\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}) = 0, \\
&\boldsymbol{E} = (E^1, \ldots, E^l), \qquad \boldsymbol{x} = (x^1, \ldots, x^n) \in X, \qquad \boldsymbol{u} = (u^1, \ldots, u^m) \in U, \\
&\underset{1}{\boldsymbol{u}} = \left(\frac{\partial u^j}{\partial x^i} \middle| \, i = 1, \ldots, n; \; j = 1, \ldots, m \right) \in U_1
\end{aligned} \tag{15}$$

represents a manifold Ω in $(m+n)$-dimensional space $X \times U$, which corresponds to a manifold Ω^1 in $(m + n + mn)$-dimensional prolonged (jet) space $X \times U \times U_1$ of dependent and independent variables together with partial derivatives [13].

Studying ideal MHD equilibria, one should take into account the generally the plasma domain is spanned by nested 2-dimensional *magnetic surfaces* – surfaces on which magnetic field lines and plasma streamlines lie [16].

The Lie method of seeking one-parametric groups of transformations that map solutions of (15) into solutions consists in finding the Lie algebra of vector fields tangent to the solution manifold Ω^1 in the jet space. These vector fields serve as infinitesimal generators for a Lie symmetry group with representation

$$\begin{aligned}
&(x')^i = f^i(\boldsymbol{x}, \boldsymbol{u}, a) \qquad (i = 1, \ldots, n), \\
&(u')^j = g^j(\boldsymbol{x}, \boldsymbol{u}, a) \qquad (j = 1, \ldots, m),
\end{aligned} \tag{16}$$

and have the form

$$\boldsymbol{v} = \sum_i \xi^i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x^i} + \sum_k \eta^k(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u^k} + \sum_{i,\,k} \xi_i^k(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}) \frac{\partial}{\partial u_i^k}. \tag{17}$$

Components of these tangent vector fields are expressed through the group representation as follows:

$$\begin{aligned}
&\xi^i(\boldsymbol{x}, \boldsymbol{u}) = \frac{\partial f^i(\boldsymbol{x}, \boldsymbol{u}, a)}{\partial a}\Big|_{a=0}, \qquad \eta^j(\boldsymbol{x}, \boldsymbol{u}) = \frac{\partial g^j(\boldsymbol{x}, \boldsymbol{u}, a)}{\partial a}\Big|_{a=0}, \\
&i = 1, \ldots, n, \qquad j = 1, \ldots, m.
\end{aligned} \tag{18}$$

The variables ξ_i^k in (17) are the coordinates of the prolonged tangent vector field corresponding to the derivatives u_i^k:

$$\xi_i^j(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}}) = D_i \eta^j - \sum_{k=1}^{n} u_k^j D_i \xi^k, \qquad D_i \equiv \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} u_i^j \frac{\partial}{\partial u^j}. \tag{19}$$

The relation (19) defines an isomorphism between tangent vector fields (17) and infinitesimal operators

$$X = \sum_i \xi^i(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial x^i} + \sum_k \eta^k(\boldsymbol{x}, \boldsymbol{u}) \frac{\partial}{\partial u^k}. \tag{20}$$

The explicit reconstruction of the transformations (16) from a generator (17) is done by solving the initial value problem

$$\begin{aligned} &\frac{\partial f^i(a)}{\partial a} = \xi^i(\boldsymbol{f}, \boldsymbol{g}), && \frac{\partial g^k(a)}{\partial a} = \eta^k(\boldsymbol{f}, \boldsymbol{g}), \\ &f^i(0) = x^i, && g^k(0) = u^k. \end{aligned} \tag{21}$$

To find all Lie group generators admissible by the original system (15), one needs to solve the determining equations

$$\boldsymbol{v}\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}})\Big|_{\boldsymbol{E}(\boldsymbol{x}, \boldsymbol{u}, \underset{1}{\boldsymbol{u}})=0} = 0. \tag{22}$$

All l determining equations (22) are linear partial differential equations with respect to $m+n$ unknown functions (18) of $m+n$ variables.

4 Connection between Bogoyavlenskij symmetries and Lie transformations

In this section the equivalence is established between Bogoyavlenskij symmetries (11), (13) for isotropic and anisotropic cases and certain infinite-dimensional Lie group transformations.

Theorem 1 shows that the application of the Lie group formalism to MHD equilibria (1), (2) yields certain Lie point transformations, some of which depend on arbitrary functions.

Theorem 2 proves that these Lie point transformations are equivalent to the groups G_m of Bogoyavlenskij symmetries (11).

Theorems 3 and 4 contain similar statements for Anisotropic (CGL) plasma equilibria.

Theorem 1. *Consider the incompressible MHD equilibrium system of equations* (1)–(3)*, where the density $\rho(\boldsymbol{r})$ is constant on both magnetic field lines and streamlines. This system admits the infinitesimal operators*

$$X^{(1)} = M(\boldsymbol{r}) \left(\sum_{k=1}^{3} \frac{B_k}{\mu\rho} \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} V_k \frac{\partial}{\partial B_k} - \frac{1}{\mu} (\boldsymbol{V} \cdot \boldsymbol{B}) \frac{\partial}{\partial P} \right), \tag{23}$$

$$X^{(2)} = \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2P \frac{\partial}{\partial P}, \tag{24}$$

$$X^{(3)} = N(\boldsymbol{r}) \left(2\rho \frac{\partial}{\partial \rho} - \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} \right), \tag{25}$$

$$X^{(4)} = \frac{\partial}{\partial P}. \tag{26}$$

These operators form a basis of the Lie algebra of infinitesimal operators in the class of Lie point transformations $\{\boldsymbol{x}' = \boldsymbol{x},\ \boldsymbol{u}' = \boldsymbol{g}(\boldsymbol{u}, a)\}$. Here $M(\boldsymbol{r})$, $N(\boldsymbol{r})$ are arbitrary smooth functions constant on both magnetic field lines and streamlines.

The proof of Theorem 1 is given in [17]. All infinitesimal operators are obtained by direct differentiation of Bogoyavlenskij symmetries with respect to a properly chosen parameter.

Remark 1. Let us explicitly write down the transformations contained in the infinitesimal operators (23)–(26). According to the reconstruction procedure (21), for the operator (23), we have

$$\rho_1 = \rho, \qquad \boldsymbol{x}_1 = \boldsymbol{x},$$

and solve the linear initial value problem

$$\begin{aligned} &\frac{\partial \boldsymbol{V}_1}{\partial \tau} = \boldsymbol{B}_1 \frac{M(\boldsymbol{r})}{\mu\rho}, \qquad \frac{\partial \boldsymbol{B}_1}{\partial \tau} = \boldsymbol{V}_1 M(\boldsymbol{r}), \qquad \frac{\partial P_1}{\partial \tau} = -\frac{M(\boldsymbol{r})}{\mu}(\boldsymbol{V}_1 \cdot \boldsymbol{B}_1), \\ &\boldsymbol{V}_1(\tau = 0) = \boldsymbol{V}, \qquad \boldsymbol{B}_1(\tau = 0) = \boldsymbol{B}, \qquad P_1(\tau = 0) = P. \end{aligned} \tag{27}$$

The solution is

$$\begin{aligned} &\boldsymbol{B}_1 = \cosh\left(\frac{M(\boldsymbol{r})\tau}{\sqrt{\mu\rho}}\right)\boldsymbol{B} + \sinh\left(\frac{M(\boldsymbol{r})\tau}{\sqrt{\mu\rho}}\right)\sqrt{\mu\rho}\,\boldsymbol{V}, \\ &\boldsymbol{V}_1 = \sinh\left(\frac{M(\boldsymbol{r})\tau}{\sqrt{\mu\rho}}\right)\frac{\boldsymbol{B}}{\sqrt{\mu\rho}} + \cosh\left(\frac{M(\boldsymbol{r})\tau}{\sqrt{\mu\rho}}\right)\boldsymbol{V}, \\ &P_1 = P + (\boldsymbol{B}^2 - \boldsymbol{B}_1^2)/(2\mu), \qquad \rho_1 = \rho. \end{aligned} \tag{28}$$

The infinitesimal operator (23) thus contains the possibility of "mixing" the components of the vector fields $\boldsymbol{B}$ and $\boldsymbol{V}$ of the original solution into a new solution.

The same way by solving a corresponding initial value problem (21) we find that transformations contained in the operator (24) are scalings

$$\rho_1 = \rho, \qquad \boldsymbol{B}_1 = \exp(\tau)\boldsymbol{B}, \qquad \boldsymbol{V}_1 = \exp(\tau)\boldsymbol{V}, \qquad P_1 = \exp(2\tau)P; \tag{29}$$

the operator (25) corresponds to infinite-dimensional scalings

$$\rho_1 = \exp(2N(\boldsymbol{r})\tau)\rho, \qquad \boldsymbol{B}_1 = \boldsymbol{B}, \qquad \boldsymbol{V}_1 = \exp(-N(\boldsymbol{r})\tau)\boldsymbol{V}, \qquad P_1 = P; \tag{30}$$

the operator (26) – to translations

$$\rho_1 = \rho, \qquad \boldsymbol{B}_1 = \boldsymbol{B}, \qquad \boldsymbol{V}_1 = \boldsymbol{V}, \qquad P_1 = P + \tau. \tag{31}$$

Theorem 2. *Lie point transformations* (28)–(30) *are equivalent to the group G_m of Bogoyavlenskij transformations* (11).

The proof of Theorem 2 can also be found in [17].

Theorems 3, 4 establish connection of the infinite-dimensional symmetries of anisotropic plasmas with Lie symmetry method.

Theorem 3. *Consider the incompressible anisotropic CGL equilibrium system of equations* (9)–(10)*, where the density $\rho(\boldsymbol{r})$ and anisotropy factor $\tau(\boldsymbol{r})$ are constant on both magnetic field lines and streamlines. This system admits the infinitesimal operators*

$$X^{(1)} = M(\boldsymbol{r})\left(\sum_{k=1}^{3} B_k \frac{1/\mu - \tau}{\rho}\frac{\partial}{\partial V_k} + \sum_{k=1}^{3} V_k \frac{\partial}{\partial B_k} - \frac{1}{\mu}(\boldsymbol{V}\cdot\boldsymbol{B})\frac{\partial}{\partial p_\perp}\right), \tag{32}$$

$$X^{(2)} = \sum_{k=1}^{3} V_k \frac{\partial}{\partial V_k} + \sum_{k=1}^{3} B_k \frac{\partial}{\partial B_k} + 2P\frac{\partial}{\partial p_\perp}, \tag{33}$$

$$X^{(3)} = N(\boldsymbol{r})\left(2\rho\frac{\partial}{\partial\rho} - \sum_{k=1}^{3} V_k\frac{\partial}{\partial V_k}\right), \tag{34}$$

$$X^{(4)} = L(\boldsymbol{r})\left(2(1/\mu-\tau)\frac{\partial}{\partial\tau} - \sum_{k=1}^{3} B_k\frac{\partial}{\partial B_k} + \frac{B^2}{\mu}\frac{\partial}{\partial p_\perp}\right), \tag{35}$$

$$X^{(5)} = \frac{\partial}{\partial p_\perp}. \tag{36}$$

These operators form a basis of the Lie algebra of infinitesimal operators in the class of Lie point transformations $\{\boldsymbol{x}' = \boldsymbol{x},\ \boldsymbol{u}' = \boldsymbol{g}(\boldsymbol{u}, a)\}$. *Here* $L(\boldsymbol{r})$, $M(\boldsymbol{r})$, $N(\boldsymbol{r})$ *are arbitrary smooth functions constant on both magnetic field lines and streamlines.*

This theorem is proven exactly the same way as Theorem 1, by replacing the arbitrary functions of the transformations (13), (14) with exponents containing a parameter and differentiating by it.

The analog of Theorem 2 is also true for incompressible CGL plasma equilibria:

Theorem 4. *Lie point transformations defined by* (32)–(36) *are equivalent to the group* G *of Bogoyavlenskij transformations* (13), (14).

To prove this theorem, one needs to follow the sequence of steps of proof of Theorem 2, replacing MHD equilibrium infinitesimal operators by operators (32)–(36).

5 Summary

The isotropic and anisotropic plasma equilibrium equations (1), (2), (4), (5) in the incompressible case (3) possess infinite-dimensional families of intrinsic symmetries (11), (13), which are the richest known classes of transformations for these equations. These symmetries form Abelian groups $G_m = A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2$ and $G = A_m \oplus A_m \oplus A_m \oplus R^+ \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$ with eight and sixteen connected components respectively.

It is important that both of these groups of symmetries are implied by the Lie point transformations of these equations, and are thus obtainable from the standard procedure of Lie group analysis, which is applicable to any system of PDEs with sufficiently smooth coefficients.

The Lie procedure in application to MHD equilibria is described in Section 3. Every solution to a system of PDEs with n variables and m unknown functions represents a manifold Ω^1 in $(m+n+mn)$-dimensional jet space $X\times U\times U_1$ of independent and dependent variables $\boldsymbol{x}$, $\boldsymbol{u}$ and partial derivatives u_i^k (15). The Lie procedure consists in finding vector fields $\boldsymbol{v}$ (17) tangent to Ω^1. These vector fields serve as infinitesimal transformation group generators. Their components ξ^i, η^j (18) are functions of all independent and dependent variables. The equations (22) for determining the tangent vector field components are the conditions of invariance of the solution manifold Ω^1 under the action of $\boldsymbol{v}$.

To find the above Bogoyavlenskij symmetries for MHD and CGL equilibria, one must take into account the fact that in the general case ideal plasma domain is spanned by nested 2-dimensional magnetic surfaces – surfaces tangent to plasma velocity and magnetic field [16]. The arbitrary functions of Bogoyavlenskij symmetries are functions constant on such magnetic surfaces.

The operators (23)–(26) admissible by incompressible MHD equilibria form a basis of the Lie algebra of infinitesimal operators corresponding to the subgroup $\{\boldsymbol{x}' = \boldsymbol{x},\ \boldsymbol{u}' = \boldsymbol{g}(\boldsymbol{u}, a)\}$ of the group (16) of all Lie point transformations.

The same statement is true for the operators (32)–(36) and incompressible MHD equilibrium equations.

Theorems 2 and 4 show that the transformations generated by infinitesimal operators from Theorems 1, 3 are equivalent to the groups G_m, G of Bogoyavlenskij symmetries for MHD and CGL equilibria respectively.

This result illustrates that the general Lie approach of analyzing systems of partial differential equations is capable of revealing highly non-trivial intrinsic transformations that may have great importance in applications, as is the case for Bogoyavlenskij symmetries.

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A Discussion on the Different Notions of Symmetry of Differential Equations

Giampaolo CICOGNA

Dipartimento di Fisica "E. Fermi" dell'Università and I.N.F.N. Sez. di Pisa,
Via Buonarroti 2, Ed. B-C, I–56127 – Pisa, Italy
E-mail: *cicogna@df.unipi.it*

A discussion is presented, within a simple unifying scheme, about different types of symmetry of PDE's, with the introduction and a precise characterization of the notions of "standard" and "weak" conditional symmetries, together with their relationship with exact and partial symmetries. An extensive use of "symmetry-adapted" variables is made; some clarifying examples are also provided.

1 Introduction

This paper is essentially a presentation of a unifying and comprehensive scheme, where several different notions of symmetry for differential problems may be considered and compared. In particular, this approach will permit the introduction of "subtler" notions of conditional symmetries (or "nonclassical symmetries ") [1–5], with a clear distinction and characterization of these symmetries and of other related concepts, including the more recently introduced notions of "partial symmetries" [6] (see also [7, 8]), and of "hidden symmetries" (see e.g. [9, 10]).

For simplicity, we will consider here only the case of partial differential equations (PDE)

$$\Delta \equiv \Delta_a\big(x, u^{(m)}\big) = 0 \qquad (a = 1, \dots, \nu) \tag{1}$$

for the q functions $u_\alpha = u_\alpha(x)$ of the p variables x_i (as usual, $u^{(m)}$ denotes the functions u_α together with their x derivatives up to the order m), and only "geometrical" or Lie-point symmetries, i.e. symmetries generated by vector fields of the form (sum over repeated indices)

$$X = \xi_i(x, u)\frac{\partial}{\partial x_i} + \phi_\alpha(x, u)\frac{\partial}{\partial u_\alpha} \tag{2}$$

although the relevant results could be suitably extended also to other types of symmetries, as generalized or Bäcklund, potential or nonlocal symmetries, whose importance is well known and also recently further emphasized (cf. e.g. [11–14]).

2 Exact symmetries

Let us start with the basic and standard definition, with the usual nondegeneracy and regularity assumptions tacitly understood (see [15–19]):

Definition 1. A system of PDE $\Delta_a\big(x, u^{(m)}\big) = 0$ is said to admit the Lie-point symmetry generated by the vector field X (or to be *symmetric* under X) if the following condition

$$X^*(\Delta)|_{\Delta=0} = 0 \tag{3}$$

is satisfied, or – equivalently (at least under mild hypotheses) – if there are functions $G = G_{ab}(x, u^{(m)})$ such that

$$(X^*(\Delta))_a = G_{ab}\,\Delta_b. \tag{4}$$

We simply denote by X^* the "appropriate" prolongation of X for the equation at hand, or – alternatively – its infinite prolongation (indeed, only a finite number of terms will appear in calculations).

Let us also give this other definition:

Definition 2. A system of PDE as before is said to be *invariant* under X if

$$X^*(\Delta) = 0. \tag{5}$$

For instance, the Laplace equation $u_{xx} + u_{yy} = 0$ is *invariant* under the rotation symmetry $X = y\partial/\partial x - x\partial/\partial y$; the heat equation $u_t = u_{xx}$ is *symmetric* but *not* invariant under

$$X = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$$

indeed one has $X^*(u_t - u_{xx}) = -x(u_t - u_{xx})$.

We then have:

Theorem 1. *Let $\Delta = 0$ be a nondegenerate system of PDE's, symmetric under a projectable vector field X, according to Definition 1. Then, there are new $p+q$ variables s, z and v, with $s \in \mathbb{R}$, $z \in \mathbb{R}^{p-1}$ and $v \equiv (v_1(s,z), \dots, v_q(s,z))$, and a new system of PDE's, say $K = 0$, with $K_a = S_{ab}(s, z, v^{(m)})\, \widetilde{\Delta}_b(s, z, v^{(m)})$ (where $v^{(m)}$ stands for $v(s,z)$ and its derivatives with respect to s and z, and $\widetilde{\Delta} = \widetilde{\Delta}(s, z, v^{(m)})$ is Δ when expressed in terms of the new variables s, z, v), which is locally* equivalent *to the initial system and is* invariant *(as in Definition 2) under the symmetry $X = \partial/\partial s$, i.e. $K_a = K_a(z, v^{(m)})$.*

Proof. (a sketch) Given the symmetry X, one has to introduce "canonical variables" (or symmetry-adapted variables) $s, z \equiv (z_1, \dots, z_{p-1})$, which are defined by

$$Xs \equiv \xi_i \frac{\partial s}{\partial x_i} + \phi_\alpha \frac{\partial s}{\partial u_\alpha} = 1, \qquad Xz_k = 0 \qquad (k = 1, \dots, p-1).$$

Using the method of characteristics, one also finds the q dependent variables $v = v_\alpha(s, z)$; once written in these coordinates, the symmetry field and all its prolongations are simply given by

$$\widetilde{X} = \widetilde{X}^* = \frac{\partial}{\partial s} \tag{6}$$

whereas the symmetry condition (Definition 1) becomes $\frac{\partial \widetilde{\Delta}}{\partial s}\Big|_{\widetilde{\Delta}=0} = 0$, or

$$\frac{\partial}{\partial s}\widetilde{\Delta}_a = G_{ab}\widetilde{\Delta}_b. \tag{7}$$

It is not difficult to show (cf. [20]) that for any $\widetilde{\Delta}_a$ satisfying the system (7) there are smooth locally invertible functions S_{ab} such that the combinations $K_a := S_{ab}\widetilde{\Delta}_b$ are independent of s, as claimed. We have assumed here for convenience that the vector fields X are "projectable", or – more explicitly – that the functions ξ in (2) do not depend on u (as happens in most cases in the study of PDE's) in order to simplify calculations in the introduction of the canonical coordinates, and to get a more direct relationship between symmetries and symmetry-invariant solutions (for a discussion on this point, cf. [21]). ■

See [22] for a result in an analogous problem, although with different aim (i.e., constructing equations invariant under a given Lie algebra). It should be also emphasized that the result in Theorem 1 is not the same as (but is related to and includes in particular) the well known result concerning the reduction of the given equations to X-invariant equations for the invariant

variables $w(z)$: indeed, introducing the new "symmetry-adapted" variables s, z and $v(s,z)$, we have transformed the equations into *equivalent* equations for $v(s,z)$. If one now *further* assumes that $\partial v/\partial s = 0$, i.e. if one looks for the $X-$invariant solutions where $v = w(z)$, then the equations $K_a = 0$ become a system of equations

$$K_a^{(0)}\big(z, w^{(m)}\big) = 0 \tag{8}$$

involving only the variables z and functions depending only on z (see [23] for a general discussion on the reduction procedure).

3 "Standard" and "weak" conditional symmetries, and related notions

Let us now consider the case of non-exact symmetries. A fundamental and largely comprehensive notion has been introduced by Fushchych [24]: let us say that X is a *conditional symmetry of the equation* $\Delta = 0$ *in the sense of Fushchych* if there is a supplementary equation $E = 0$ such that X is an exact symmetry of the system $\Delta = E = 0$.

The simplest and more common case is obtained choosing as supplementary equation the "side condition" or "invariant surface condition"

$$X_Q u \equiv \xi_i \frac{\partial u}{\partial x_i} - \phi = 0, \tag{9}$$

where X_Q is the symmetry written in "evolutionary form" [16]: this corresponds to the usual (properly called) conditional symmetry (CS) (also called Q-conditional symmetry), and the above condition indicates that we are looking precisely for solutions which are *invariant* under X.

To avoid unessential complications with notations, we will consider from now on only the case of a single PDE $\Delta = 0$ for a single unknown function $u(x)$. The extension to more general cases is in principle completely straightforward.

It is known that the above definition of CS suffers from some intrinsic difficulties, essentially due to the necessity of introducing and dealing with the differential consequences of (9) (for a discussion of this point, see e.g. [16, 25, 26], and [27–29] for a more complete definition). Related to these difficulties is the quite embarrassing sentence by Olver and Rosenau [25] (see also [26]), which says – essentially – that, given any differential equation, *any* vector field X is a CS, and *any* solution of the equation is an invariant solution under some X.

To clarify this point, we will introduce a subtler definition of CS. This will be made resorting once again to the canonical coordinates s, z, $v = v(s,z)$, introduced in the proof of Theorem 1. First of all, in these coordinates the invariance condition $X_Q v = 0$ becomes

$$\frac{\partial v}{\partial s} = 0 \tag{10}$$

and the condition of CS takes the simple form (let us now retain for simplicity the same notation Δ, instead of $\widetilde{\Delta}$, also in the new coordinates)

$$\frac{\partial \Delta}{\partial s}\Big|_{\Sigma} = 0 \tag{11}$$

here Σ stands for the set of the simultaneous solutions of $\Delta = 0$ and $v_s = \partial v/\partial s = 0$, together with the derivatives of v_s with respect to all the variables s and z_k. Introducing the global notation $v_s^{(\ell)}$ to indicate v_s, v_{ss}, v_{sz_k} etc., we shall say that $X = \partial/\partial s$ is a CS in *standard sense* if the equation takes the form

$$\Delta = R\big(s, z, v^{(m)}\big) K\big(z, v^{(m)}\big) + \sum_{\ell} \Theta_\ell\big(s, z, v^{(m)}\big)\, v_s^{(\ell)} = 0, \tag{12}$$

where the point to be emphasized is that K does not depend *explicitly* on s, and R, K do not contain $v_s^{(\ell)}$. It is then clear that, if one now looks for solutions of $\Delta = 0$ which are independent on s, i.e. such that $v_s^{(\ell)} = 0$, or of the form $v = w(z)$, then equation (12) becomes a "reduced" equation $K^{(0)}(z, w^{(m)}) = 0$, just as in the exact symmetry case.

But this is clearly only a special case. Indeed, the equation $\Delta = 0$ may also take the form

$$\Delta = \sum_{r=1}^{\sigma} s^{r-1} K_r\big(z, v^{(m)}\big) + \sum_{\ell} \Theta_\ell\big(s, z, v^{(m)}\big)\, v_s^{(\ell)} = 0, \tag{13}$$

where the part not containing $v_s^{(\ell)}$ is a polynomial in the variable s, with coefficients K_r not depending explicitly on s, or also – more in general (with some different regrouping of the terms containing s into linearly and functionally independent terms R_r)

$$\Delta = \sum_{r=1}^{\sigma} R_r\big(s, z, v^{(m)}\big)\, K_r\big(z, v^{(m)}\big) + \sum_{\ell} \Theta_\ell\big(s, z, v^{(m)}\big)\, v_s^{(\ell)} = 0. \tag{14}$$

In this case, if one looks for X-invariant solutions $w(z)$ of $\Delta = 0$, one is faced with the system of reduced equations (not containing s nor functions of s)

$$K_r^{(0)}\big(z, w^{(m)}\big) = 0, \qquad r = 1, \dots, \sigma. \tag{15}$$

Assume that this system admits some solution (it is known that the existence of invariant solutions is by no means guaranteed in general, neither for "standard" CS, nor for "exact" Lie symmetries), we will say that X is *weak CS of order* σ.

We now see that the set of the solutions of the above system can be characterized equivalently as the set of the solutions of the system

$$\Delta = 0, \quad \frac{\partial \Delta}{\partial s} = 0, \quad \dots, \quad \frac{\partial^{\sigma-1}\Delta}{\partial s^{\sigma-1}} = 0, \quad v_s^{(\ell)} = 0. \tag{16}$$

Coming back to the original coordinates x, u, the set of conditions (16) becomes

$$\Delta = \Delta^{(1)} = \dots = \Delta^{(\sigma-1)} = 0, \qquad X_Q u = 0, \tag{17}$$

where

$$\Delta^{(1)} := X^*(\Delta), \qquad \Delta^{(2)} := X^*\big(\Delta^{(1)}\big), \qquad \dots \tag{18}$$

(as already pointed out, also the differential consequences of $X_Q u = 0$ must be taken into account), and a CS of order σ can be characterized by the condition

$$X^*(\Delta)|_{\Sigma_\sigma} = 0, \tag{19}$$

where Σ_σ is the set (if not empty, of course) of the solutions of the system (17).

We can summarize our discussion in the following form.

Proposition 1. *Given a PDE $\Delta = 0$, a projectable vector field X is a "standard" conditional symmetry for the equation if it is a symmetry for the system*

$$\Delta = 0, \qquad X_Q u = 0$$

and this corresponds to the existence of a reduced equation in $p-1$ independent variables, which – if admits solutions – gives X-invariant solutions of $\Delta = 0$. A vector field X is a "weak" CS (of order σ) if it is a symmetry of the system

$$\Delta = 0, \quad \Delta^{(1)} := X^*(\Delta) = 0, \quad \Delta^{(2)} := X^*\big(\Delta^{(1)}\big) = 0, \quad \dots, \quad \Delta^{(\sigma-1)} = 0, \quad X_Q u = 0$$

and this corresponds to the existence of a system of σ reduced equations, which – if admits solutions – gives $X-$invariant solutions of $\Delta = 0$. Introducing $X-$adapted variables s, z, such that $Xs = 1$, $Xz = 0$, the PDE has the form (12) *in the case of standard CS, or* (14) *in the case of weak CS.*

If one neglects the invariance condition $X_Q u = 0$, one is actually dealing with the case of *partial* symmetries. Indeed (see [6–8]), X is precisely a partial symmetry of order σ if X is a symmetry of the system

$$\Delta = \Delta^{(1)} = \cdots = \Delta^{(\sigma-1)} = 0. \tag{20}$$

If this is the case, X maps one into another the solutions of the system (20), which is then a "symmetric set of solutions of $\Delta = 0$" [7]. In particular, if in this set there are some solutions which are left fixed by X, then X is also a CS (either standard or weak) of $\Delta = 0$.

We can then rephrase the Olver–Rosenau statement [25] in the form:

Proposition 2. *Any vector field X is either an exact, or a standard CS, or a weak CS. Similarly, any X is either an exact or a partial symmetry.*

It is well known that the set of the solutions which can be obtained in this way may be empty or contain only trivial solutions (e.g., $u = \text{const}$): it is clear that the choice of good candidates for these "non-exact" symmetry generators should be guided by some reasonable criterion and motivated guess. It is also clear that all the notions of non-exact symmetries considered above can be viewed as special cases of CS in the sense of Fushchych.

In all the above discussion, we have considered the case of a single vector field X; clearly, the situation becomes richer and richer if more than one vector field is taken into consideration. First of all, the reduction procedure itself must be adapted and refined when the given equation admits an algebra of symmetries of dimension larger than 1 (possibly infinite): for a recent discussion see [30]. Secondly, for instance, it can happen that the reduced equations (8) or (15) may admit some new symmetry Y not shared by the original equation $\Delta = 0$: this is (essentially) the case of "hidden symmetries" [9,10]. Different reduction procedures have been also proposed, based on the introduction of multiple suitable differential constraints: see, e.g., [11,31–33], and also [17].

4 Examples

We will give here some simple examples, to illustrate the properties of the different types of symmetries introduced above, and the different solutions that can be obtained accordingly.

Example 1. Consider the equation, proposed by Popovych [29]

$$u_t + u_{xx} - u + t(u_x - u) = 0, \qquad u = u(x,t).$$

The vector field $X = \partial/\partial t$ is not an exact nor a standard CS, but is a weak CS, indeed the system of equations (15) (here $s = t$) becomes $u_{xx} = u$, $u_x = u$, with solution $u = c\exp(x)$. The same vector field $X = \partial/\partial t$ is a weak CS also for this variation of the above equation:

$$u_t - u_{tt} + u_{xx} - u + t(u_x - u) = 0$$

with the same solution as above. But X is now also a partial symmetry: indeed, the equation $\Delta^{(1)} = 0$ is now $u_x - u = 0$, and combining it with $\Delta = 0$ we find the more general solution $u = c\exp(x) + c_1\exp(x+t)$. Considering this other variation of the Popovych example

$$t^2(u_t - u) + u_{xx} - u + t(u_x - u) = 0$$

here $X = \partial/\partial t$ is only a partial symmetry, leading to the solution $u = c\exp(x+t)$ (strictly speaking, it is also a weak CS, but producing only the trivial solution $u = 0$!).

Example 2. It is well known that the Korteweg-de Vries equation

$$u_t + u_{xxx} + uu_x = 0, \qquad u = u(x,t)$$

does not admit (standard) CS, apart from its exact symmetries. There are however weak CS; e.g. the scaling

$$X = 2x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$$

is an exact symmetry for the system $\Delta = 0$, $\Delta^{(1)} := X^*(\Delta) = 0$, $X_Q u = 0$, which means that this is a weak CS, giving the scaling-invariant solution $u = x/t$. But also, if we neglect the invariance condition $X_Q u = 0$, we obtain the (clearly larger) symmetric set of solutions $u = (x + c_1)/(t + c_2)$, showing that the above X is also a partial symmetry.

Example 3. The symmetry properties of the Boussinesq equation

$$u_{tt} + u_{xxxx} + uu_{xx} + u_x^2 = 0, \qquad u = u(x,t) \tag{21}$$

have been the object of several papers (see e.g. [11, 34, 35]). For what concerns standard CS, writing the general vector field in the form

$$X = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{22}$$

a complete list of CS has been given both for the case $\tau \neq 0$ (and therefore, without any restriction, $\tau = 1$) [34] and for the case $\tau = 0$ [35, 36]; it has been also shown that the invariant solutions under these CS are precisely those found by means of the "direct method", which is not based on (but clearly related to) symmetry properties [34, 35, 37].

To complete the analysis, one can also look for symmetries with $\xi = 0$. It is not difficult to verify that no standard CS of this form is admitted. There are however weak CS: an example is

$$X = \frac{\partial}{\partial t} + \left(\frac{1}{t^2} - \frac{2u}{t}\right)\frac{\partial}{\partial u} \tag{23}$$

one obtains from this: $s = t$, $z = x$ and $u(x,t) = t^{-1} + t^{-2}v(x,t)$, giving

$$vv_{xx} + v_x^2 + 6v + t(v_{xx} + 2) + t^2 v_{xxxx} - 4tv_t + t^2 v_{tt} = 0 \tag{24}$$

which is precisely of the form (14) (the role of s is played here by t). Looking indeed for solutions with $v = w(x)$, one gets a system of three ODE's

$$ww_{xx} + w_x^2 + 6w = 0, \qquad w_{xx} + 2 = 0, \qquad w_{xxxx} = 0$$

(cf. (16)), admitting the common solution $w = -x^2$ and giving the (quite elementary) solution $u = 1/t - x^2/t^2$ of the Boussinesq equation.

Another example of weak CS for the Boussinesq equation is the following

$$X = t^2\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - \left(2x + \frac{10}{3}t^3\right)\frac{\partial}{\partial u} \tag{25}$$

now $s = t$, $z = x - t^3/3$ and $u = -2sz - s^4 + v(s,z)$. The additional equations $\Delta^{(1)} := X^*(\Delta) = 0$ etc.: now become

$$\Delta^{(1)} = -10t - 3u_x - 2tu_{xt} - \frac{5}{3}t^3 u_{xx} - xu_{xx} = 0, \qquad \Delta^{(2)} = 2 + u_{xt} + t^2 u_{xx} = 0 \tag{26}$$

and taking into account also the invariance condition $X_Q u = 0$, we easily conclude that this is a weak CS of order $\sigma = 3$ and obtain the solution

$$u(x,t) = -\frac{t^4}{3} - 2tx - \frac{12}{(x - t^3/3)^2} \tag{27}$$

If instead we do *not* impose the invariance condition $X_Q u = 0$ and solve the three equations (21), (26), we find, in addition to the invariant solution (27), also the following family of solutions $u(x,t) = -t^4/3 + c_1 t - 2tx + c_2$, showing that the above symmetry is also a partial symmetry.

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On the Differential First-Order Invariants of the Non-Splitting Subgroups of the Poincaré Group $P(1,4)$

Vasyl M. FEDORCHUK † *and Volodymyr I. FEDORCHUK* ‡

† *Pedagogical Academy, Institute of Mathematics, Podchorążych 2, 30-084 Kraków, Poland; Pidstryhach Institute of Applied Problems of Mechanics and Mathematics of NAS of Ukraine, 3b Naukova Str., 79601 Lviv, Ukraine*
E-mail: *fedorchuk@ap.krakow.pl*

‡ *Franko Lviv National University, 1 Universytets'ka Str., 79000 Lviv, Ukraine*
E-mail: *fedorchukv@ukr.net*

The differential first-order invariants for all non-splitting subgroups of the generalized Poincaré group $P(1,4)$ are constructed.

1 Introduction

The differential invariants play an important role in geometry, group analysis of differential equations, theoretical and mathematical physics, mechanics, gas dynamics, etc. They have been studied in many works (see, for example [1–14]). In particular, the papers [12–14] are devoted to the construction of the first-order differential invariants for the splitting subgroups [15–17] of the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group is very useful in the theoretical and mathematical physics [18–20].

The objective of this paper is to present some of the new results concerning the differential first-order invariants for the non-splitting subgroups [17, 21, 22] of the group $P(1,4)$.

For this purpose, we have to consider the Lie algebra of the group $P(1,4)$.

2 The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras

The Lie algebra of the group $P(1,4)$ is generated by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$
$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in the following: $M'_{\mu\nu} = iM_{\mu\nu}$.

Let us consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial x_4}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

Next, we will use the following basis elements:

$$G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0} \quad (a = 1, 2, 3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \qquad X_k = P'_k \quad (k = 1, 2, 3), \qquad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

We used the method proposed in [23] to study the subgroup structure of the group $P(1,4)$.

Non-conjugate subgroups of the group $P(1,4)$ have been described in [15–17, 21, 22]. The conjugation was considered under the group $P(1,4)$.

From the results obtained it follows that the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [20]. The Lie algebra of the group $\widetilde{G}(1,3)$ has the following basis elements: $L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4$.

3 The differential first-order invariants of the non-splitting subgroups of the group $P(1,4)$

The functional bases of the differential first-order invariants have been constructed for all non-splitting subgroups of the group $P(1,4)$. Since it is impossible to present here all the results obtained, only some of them are given in this section.

Below, for some of the non-splitting subalgebras of the Lie algebra of the group $P(1,4)$ we write their respective basis elements and corresponding functional basis of the differential invariants.

One-dimensional subalgebras

1. $\langle P_3 + X_1 \rangle$,

$$J_1 = x_2, \quad J_2 = x_0 + x_4, \quad J_3 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_4 = x_1 + \frac{x_3}{x_0 + x_4}, \quad J_5 = u,$$
$$J_6 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_0 - u_4,$$
$$J_{10} = u_0^2 - u_3^2 - u_4^2, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3, 4;$$

2. $\langle G + cX_1,\ c < 0 \rangle$,

$$J_1 = x_1 - c\ln(x_0 + x_4), \quad J_2 = x_2, \quad J_3 = x_3, \quad J_4 = (x_0^2 - x_4^2)^{1/2}, \quad J_5 = u,$$
$$J_6 = (x_0 + x_4)(u_0 + u_4), \quad J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_3, \quad J_{10} = u_0^2 - u_4^2;$$

3. $\langle L_3 - P_3 + \alpha_0 X_0,\ \alpha_0 < 0 \rangle$,

$$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_0 + x_4)^2 + 2\alpha_0 x_3, \quad J_3 = \alpha_0 \arctan\frac{x_1}{x_2} - x_0 - x_4,$$
$$J_4 = 2(x_0 + x_4)^3 + 6\alpha_0 x_3(x_0 + x_4) + 3\alpha_0^2(x_0 - x_4), \quad J_5 = u, \quad J_6 = x_1u_2 - x_2u_1,$$
$$J_7 = x_0 + x_4 - \alpha_0\frac{u_3}{u_0 - u_4}, \quad J_8 = u_0 - u_4, \quad J_9 = u_1^2 + u_2^2, \quad J_{10} = u_0^2 - u_3^2 - u_4^2;$$

4. $\langle L_3 + \tilde{d}(X_0 + X_4),\ \tilde{d} < 0 \rangle$,

$$J_1 = x_3, \quad J_2 = x_4, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = \tilde{d}\arctan\frac{x_1}{x_2} - x_0, \quad J_5 = u,$$
$$J_6 = x_1u_2 - x_2u_1, \quad J_7 = u_0, \quad J_8 = u_3, \quad J_9 = u_4, \quad J_{10} = u_1^2 + u_2^2.$$

Two-dimensional subalgebras

1. $\langle L_3 - X_4,\ P_3 + hX_0,\ h > 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = (x_0 + x_4)^2 - 2hx_3,$$
$$J_3 = 2(x_0 + x_4)^3 - 3h(2x_3 - h)(x_0 + x_4) + 3h^2 \arctan\frac{x_1}{x_2} - 6h^2 x_4, \quad J_4 = u,$$
$$J_5 = x_1u_2 - x_2u_1, \quad J_6 = (x_0 + x_4) + \frac{hu_3}{u_0 - u_4}, \quad J_7 = u_0 - u_4, \quad J_8 = u_1^2 + u_2^2,$$
$$J_9 = u_0^2 - u_3^2 - u_4^2;$$

2. $\langle G + aX_3,\ L_3,\ a < 0\rangle$,

$$J_1 = x_3 - a\ln(x_0 + x_4), \quad J_2 = \left(x_0^2 - x_4^2\right)^{1/2}, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = u,$$
$$J_5 = x_1u_2 - x_2u_1, \quad J_6 = (x_0 + x_4)(u_0 + u_4), \quad J_7 = u_3, \quad J_8 = u_0^2 - u_4^2, \quad J_9 = u_1^2 + u_2^2;$$

3. $\langle G + aX_1,\ P_3,\ a < 0\rangle$,

$$J_1 = x_1 - a\ln(x_0 + x_4), \quad J_2 = x_2, \quad J_3 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_4 = u, \quad J_5 = \frac{x_0 + x_4}{u_0 - u_4},$$
$$J_6 = \frac{u_0 - u_4}{x_0 + x_4}x_3 + u_3, \quad J_7 = u_1, \quad J_8 = u_2, \quad J_9 = u_0^2 - u_3^2 - u_4^2;$$

4. $\langle G,\ L_3 + dX_3,\ d < 0\rangle$,

$$J_1 = x_3 + d\arctan\frac{x_1}{x_2}, \quad J_2 = \left(x_0^2 - x_4^2\right)^{1/2}, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = u,$$
$$J_5 = x_1u_2 - x_2u_1, \quad J_6 = (x_0 + x_4)(u_0 + u_4), \quad J_7 = u_3, \quad J_8 = u_0^2 - u_4^2, \quad J_9 = u_1^2 + u_2^2.$$

Three-dimensional subalgebras

1. $\langle L_3 + cG + bX_3,\ P_1,\ P_2,\ c > 0,\ b < 0\rangle$,

$$J_1 = cx_3 - b\ln(x_0 + x_4), \quad J_2 = \left(x_0^2 - x_1^2 - x_2^2 - x_4^2\right)^{1/2}, \quad J_3 = u,$$
$$J_4 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_5 = \left(x_1 + \frac{x_0 + x_4}{u_0 - u_4}u_1\right)^2 + \left(x_2 + \frac{x_0 + x_4}{u_0 - u_4}u_2\right)^2,$$
$$J_6 = c\arctan\left(\frac{u_1(x_0 + x_4) + x_1(u_0 - u_4)}{u_2(x_0 + x_4) + x_2(u_0 - u_4)}\right) + \ln(x_0 + x_4), \quad J_7 = u_3,$$
$$J_8 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

2. $\langle L_3 + eG + \kappa_3X_3,\ X_0,\ X_4,\ e > 0,\ \kappa_3 < 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_2 - x_2u_1, \quad J_4 = \ln(u_0 + u_4) - e\arctan\frac{x_1}{x_2},$$
$$J_5 = \kappa_3\ln(u_0 + u_4) + ex_3, \quad J_6 = u_3, \quad J_7 = u_0^2 - u_4^2, \quad J_8 = u_1^2 + u_2^2;$$

3. $\langle L_3 + d_3X_3,\ X_0,\ X_4,\ d_3 < 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_2u_1 - x_1u_2, \quad J_4 = d_3\arctan\frac{x_1}{x_2} + x_3,$$
$$J_5 = u_0, \quad J_6 = u_3, \quad J_7 = u_4, \quad J_8 = u_1^2 + u_2^2;$$

4. $\langle L_3 + dX_3,\ P_3 + X_0,\ X_4,\ d < 0\rangle$,

$$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = 2d\arctan\frac{x_1}{x_2} - (x_0 + x_4)^2 + 2x_3, \quad J_3 = u, \quad J_4 = x_1u_2 - x_2u_1,$$

$$J_5=(x_0+x_4)(u_0-u_4)+u_3,\quad J_6=u_0-u_4,\quad J_7=u_1^2+u_2^2,\quad J_8=u_0^2-u_3^2-u_4^2.$$

Four-dimensional subalgebras

1. $\langle G+a_3X_3,\ L_3+d_3X_3,\ P_1,\ P_2,\ a_3<0,\ d_3<0\rangle$,

$$J_1=\left(x_0^2-x_1^2-x_2^2-x_4^2\right)^{1/2},\quad J_2=u,\quad J_3=\frac{x_0+x_4}{u_0-u_4},$$
$$J_4=\left(x_1+u_1\frac{x_0+x_4}{u_0-u_4}\right)^2+\left(x_2+u_2\frac{x_0+x_4}{u_0-u_4}\right)^2,$$
$$J_5=x_3-a_3\ln(x_0+x_4)+d_3\arctan\left(\frac{(x_0+x_4)u_1+(u_0-u_4)x_1}{(x_0+x_4)u_2+(u_0-u_4)x_2}\right),$$
$$J_6=u_3,\quad J_7=u_0^2-u_1^2-u_2^2-u_4^2;$$

2. $\langle G+a_3X_3,\ L_3+d_3X_3,\ P_3,\ X_4,\ a_3<0,\ d_3<0\rangle$,

$$J_1=\left(x_1^2+x_2^2\right)^{1/2},\quad J_2=u,\quad J_3=\frac{x_0+x_4}{u_0-u_4},\quad J_4=x_1u_2-x_2u_1,$$
$$J_5=d_3\arctan\frac{x_1}{x_2}+x_3-a_3\ln(u_0-u_4)+u_3\frac{x_0+x_4}{u_0-u_4},\quad J_6=u_1^2+u_2^2,\quad J_7=u_0^2-u_3^2-u_4^2;$$

3. $\langle L_3-X_4,\ P_1,\ P_2,\ X_3\rangle$,

$$J_1=x_0+x_4,\quad J_2=u,\quad J_3=\left(x_1+u_1\frac{x_0+x_4}{u_0-u_4}\right)^2+\left(x_2+u_2\frac{x_0+x_4}{u_0-u_4}\right)^2,$$
$$J_4=x_0^2-x_1^2-x_2^2-x_4^2+(x_0+x_4)\arctan\left(\frac{x_1(u_0-u_4)+u_1(x_0+x_4)}{x_2(u_0-u_4)+u_2(x_0+x_4)}\right),$$
$$J_5=u_3,\quad J_6=u_0-u_4,\quad J_7=u_0^2-u_1^2-u_2^2-u_4^2;$$

4. $\langle L_3-P_3+\alpha_0X_0,\ X_1,\ X_2,\ X_4,\ \alpha_0<0\rangle$,

$$J_1=2x_3\alpha_0+(x_0+x_4)^2,\quad J_2=u,\quad J_3=\alpha_0\frac{u_3}{u_0-u_4}-x_0-x_4,\quad J_4=u_0-u_4,$$
$$J_5=u_1^2+u_2^2,\quad J_6=u_0^2-u_3^2-u_4^2,\quad J_7=\arctan\frac{u_1}{u_2}-\frac{u_3}{u_0-u_4}.$$

Five-dimensional subalgebras

1. $\langle L_3,\ P_1+X_2,\ P_2-X_1,\ P_3+\tilde h_3X_3,\ X_4\rangle$,

$$J_1=x_0+x_4,\quad J_2=u,\quad J_3=(u_0-u_4)x_3+(x_0+x_4)u_3-u_3\tilde h_3,$$
$$J_4=(x_1(u_0-u_4)+u_2+u_1(x_0+x_4))^2+(x_2(u_0-u_4)-u_1+u_2(x_0+x_4))^2,$$
$$J_5=u_0-u_4,\quad J_6=u_0^2-u_1^2-u_2^2-u_3^2-u_4^2;$$

2. $\langle G+a_2X_2+a_3X_3,\ P_1,\ P_2,\ X_1,\ X_4,\ a_2<0,\ a_3<0\rangle$,

$$J_1=x_3-a_3\ln(x_0+x_4),\quad J_2=u,\quad J_3=\frac{x_0+x_4}{u_0-u_4},$$
$$J_4=x_2-a_2\ln(x_0+x_4)+\frac{x_0+x_4}{u_0-u_4}u_2,\quad J_5=u_3,\quad J_6=u_0^2-u_1^2-u_2^2-u_4^2;$$

3. $\langle L_3+dG+\alpha_3X_3,\ P_3,\ X_1,\ X_2,\ X_4,\ d>0,\ \alpha_3<0\rangle$,

$$J_1=u,\quad J_2=\frac{x_0+x_4}{u_0-u_4},\quad J_3=d\arctan\frac{u_2}{u_1}-\ln(x_0+x_4),$$

$$J_4 = dx_3 - \alpha_3 \ln(x_0 + x_4) + du_3 \frac{x_0 + x_4}{u_0 - u_4}, \quad J_5 = u_1^2 + u_2^2, \quad J_6 = u_0^2 - u_3^2 - u_4^2;$$

4. $\langle L_3 + d_3 X_3,\ P_1,\ P_2,\ P_3,\ X_4,\ d_3 < 0 \rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = \left(\frac{x_1}{x_0 + x_4} + \frac{u_1}{u_0 - u_4} \right)^2 + \left(\frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4} \right)^2,$$
$$J_4 = x_3 + \frac{x_0 + x_4}{u_0 - u_4} u_3 - d_3 \arctan \left(\frac{x_2(u_0 - u_4) + u_2(x_0 + x_4)}{x_1(u_0 - u_4) + u_1(x_0 + x_4)} \right),$$
$$J_5 = u_0 - u_4, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2.$$

Six-dimensional subalgebras

1. $\langle G,\ L_3 + dX_3,\ P_1,\ P_2,\ P_3,\ X_4,\ d < 0 \rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = \left(\frac{u_0 - u_4}{x_0 + x_4} x_1 + u_1 \right)^2 + \left(\frac{u_0 - u_4}{x_0 + x_4} x_2 + u_2 \right)^2,$$
$$J_4 = x_3 + \frac{x_0 + x_4}{u_0 - u_4} u_3 + d \arctan \left(\frac{x_1(u_0 - u_4) + u_1(x_0 + x_4)}{x_2(u_0 - u_4) + u_2(x_0 + x_4)} \right),$$
$$J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

2. $\langle G + \tilde{a}_2 X_2,\ P_1,\ P_2,\ P_3,\ X_1,\ X_4,\ \tilde{a}_2 < 0 \rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = x_2 + \frac{x_0 + x_4}{u_0 - u_4} u_2 - \tilde{a}_2 \ln(x_0 + x_4),$$
$$J_4 = x_3 + u_3 \frac{x_0 + x_4}{u_0 - u_4}, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

3. $\langle L_3 - P_3 + \alpha_0 X_0,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ \alpha_0 < 0 \rangle$,

$$J_1 = 2\alpha_0 x_3 + (x_0 + x_4)^2, \quad J_2 = u, \quad J_3 = \alpha_0 u_3 - (x_0 + x_4)(u_0 - u_4),$$
$$J_4 = u_0 - u_4, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

4. $\langle G + aX_3,\ L_3 + dX_3,\ X_0,\ X_1,\ X_2,\ X_4,\ a < 0,\ d < 0 \rangle$,

$$J_1 = u, \quad J_2 = x_3 - d \arctan \frac{u_2}{u_1} + a \ln(u_0 + u_4), \quad J_3 = u_3,$$
$$J_4 = u_0^2 - u_4^2, \quad J_5 = u_1^2 + u_2^2.$$

Seven-dimensional subalgebras

1. $\langle P_1 + X_0,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4 \rangle$,

$$J_1 = u, \quad J_2 = u_1 + (x_0 + x_4)(u_0 - u_4), \quad J_3 = u_0 - u_4, \quad J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

2. $\langle L_3 + d_3 X_3,\ P_1,\ P_2,\ P_3 + X_0,\ X_1,\ X_2,\ X_4,\ d_3 < 0 \rangle$,

$$J_1 = u, \quad J_2 = (x_0 + x_4)(u_0 - u_4) + u_3, \quad J_3 = u_0 - u_4, \quad J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

3. $\langle G + a_3 X_3,\ L_3,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ a_3 < 0 \rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = x_3 - a_3 \ln(u_0 - u_4), \quad J_4 = u_3, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_4^2;$$

4. $\langle G,\ L_3 + d_3 X_3,\ P_1,\ P_2,\ X_1,\ X_2,\ X_4,\ d_3 < 0 \rangle$,

$$J_1 = u, \quad J_2 = \frac{u_0 - u_4}{x_0 + x_4}, \quad J_3 = u_3, \quad J_4 = u_0^2 - u_1^2 - u_2^2 - u_4^2.$$

Eight-dimensional subalgebras

1. $\langle G + aX_3,\ L_3,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_4,\ a < 0\rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = a\ln(u_0 - u_4) - \frac{x_0 + x_4}{u_0 - u_4}u_3 - x_3,$$
$$J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

2. $\langle P_1,\ P_2,\ P_3 + X_0,\ L_3 + \beta X_0,\ X_1,\ X_2,\ X_3,\ X_4,\ \beta < 0\rangle$,

$$J_1 = u, \quad J_2 = u_0 - u_4, \quad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

3. $\langle G,\ L_3 + bX_3,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_4,\ b < 0\rangle$,

$$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

4. $\langle L_3,\ P_1,\ P_2,\ P_3 + X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,

$$J_1 = u, \quad J_2 = x_0 + x_4 + \frac{u_3}{u_0 - u_4}, \quad J_3 = u_0 - u_4,$$
$$J_4 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2.$$

The results obtained can be used, for example, to construct the first-order differential equations in the space $M(1,4) \times R(u)$ with non-trivial symmetry groups. In particular, among them, there will be equations, which are invariant under the non-splitting subgroups of the Poincaré group $P(1,3)$, as well as equations which are invariant under the non-splitting subgroups of the extended Galilei group $\widetilde{G}(1,3)$.

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On Symmetry Reduction of Some Classes of the First-Order Differential Equations in the Space $M(1,4) \times R(u)$

Volodymyr I. FEDORCHUK

Franko Lviv National University, 1 Universytets'ka Str., 79000 Lviv, Ukraine
E-mail: *fedorchukv@ukr.net*

The symmetry reduction of some classes of the first-order differential equations (in the space $M(1,4)\times R(u)$), invariant under splitting subgroups of the generalized Poincaré group $P(1,4)$, to differential equations with fewer independent variables are done.

1 Introduction

It is well known that among equations, important for theoretical and mathematical physics, there are also ones which have nontrivial symmetry groups. In the space $M(1,4) \times R(u)$ we have the linear and nonlinear wave equations and the Dirac equation. Here, and in what follows, $R(u)$ is the number axis of the dependent variable u. These equations are invariant under the generalized Poincaré group $P(1,4)$ (see, for example, [1–3]). The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in the theoretical and mathematical physics [1–3]. The group $P(1,4)$ has many subgroups used in the theoretical physics [4–6]. Among these subgroups there are the Poincaré group $P(1,3)$ and the extended Galilei group $\widetilde{G}(1,3)$ (see also [7]). Thus, the results obtained with the use of the subgroup structure of the group $P(1,4)$ will be useful in the relativistic and non-relativistic physics. Therefore, it is important from the physical and mathematical points of view, that we are able to construct, in the space $M(1,4) \times R(u)$, new differential equations invariant under continuous subgroups of the generalized Poincaré group $P(1,4)$. The paper [8] is devoted to the construction of the first-order differential equations invariant under splitting subgroups of the group $P(1,4)$, defined in the space $M(1,4) \times R(u)$.

In the present paper we continue to study this type of equations. We concentrate our attention on the symmetry reduction of the first-order differential equations, invariant under splitting subgroups of the group $P(1,4)$, to differential equations with fewer independent variables.

Our paper is organized as follows: in Section 2 we introduce some notations and results concerning the Lie algebra of the group $P(1,4)$, which are used in the next chapter. Section 3 presents our main results.

2 The Lie algebra of the group $P(1,4)$ and its subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P'_μ $(\mu = 0,1,2,3,4)$, which satisfy the commutation relations

$$\begin{aligned}
&\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,\\
&\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},
\end{aligned}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

We consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial x_4}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

Below, we will use the following basis elements:

$$G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0} \quad (a = 1,2,3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \qquad X_k = P'_k \quad (k = 1,2,3), \qquad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

For the study of the subgroup structure of the group $P(1,4)$ we used the method proposed in [9]. Continuous subgroups of the group $P(1,4)$ have been described in [4–6].

3 The first-order differential equations in the space $M(1,4) \times R(u)$

In this section we present some of the new results concerning the first-order differential equations in the space $M(1,4) \times R(u)$.

The equations in the space $M(1,4) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$, can be written in the following form (see, for example [10–12]):

$$F(J_1, J_2, \ldots, J_t) = 0,$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, \ldots, J_t\}$ is a functional basis of the first-order differential invariants of splitting subgroups of the group $P(1,4)$.

The classes of the first-order differential equations in the space $M(1,4) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$, are constructed. Some of the results obtained have been presented in [8]. To study these classes of the differential equations, we used only their symmetry properties.

Taking into account the functional bases of the invariants of some splitting subgroups of the group $P(1,4)$, we have constructed the ansatzes, which reduce majority of the equations obtained to differential equations with fewer independent variables. The corresponding symmetry reduction has been done. Among the reduced equations there are one-, two-, three- and four-dimensional ones.

It is impossible to present here all the results obtained. Therefore, we will give a short review of the results concerning the symmetry reduction of the considered classes of differential equations to classes of ODEs.

Let us consider the classes of the first-order differential equations, which are constructed with the use of the functional bases of the first-order differential invariants containing two usual invariants. One of them always is u, since it is one of the invariants of the group $P(1,4)$. On the base of these two invariants we have constructed the ansatzes, which reduce the classes considered to classes of ODEs. The corresponding symmetry reduction is done. Among the considered classes there are ones which are invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$. The Lie algebra of the group $\widetilde{G}(1,3)$ is generated by the following basis elements: L_1, L_2, L_3, P_1, P_2, P_3, X_0, X_1, X_2, X_3, X_4. We have 32 such classes of differential equations. From

among them, 17 classes have been reduced to the classes of functional equations. From these functional equations we can find solutions for the corresponding classes of differential equations.

Below, for some splitting subgroups of the group $P(1,4)$, we write the basis elements of their respective Lie algebras, the corresponding arguments $J_1, J_2, \ldots, J_t$ of the function F, the ansatzes obtained, as well as the reduced equation for each case.

1. $\langle P_1, P_2, P_3, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = u_1 + x_1\frac{u_0 - u_4}{x_0 + x_4}, \quad J_4 = u_2 + x_2\frac{u_0 - u_4}{x_0 + x_4},$$
$$J_5 = \frac{u_0 - u_4}{x_0 + x_4}x_3 + u_3, \quad J_6 = u_0 - u_4, \quad J_7 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
$$u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F\left(\omega, \varphi, 0, 0, 0, 0, 0\right) = 0; \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3, 4;$$

2. $\langle P_1, P_2, X_3, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = u_1(x_0 + x_4) + x_1(u_0 - u_4),$$
$$J_4 = u_2(x_0 + x_4) + x_2(u_0 - u_4), \quad J_5 = u_3, \quad J_6 = u_0 - u_4,$$
$$J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F\left(\omega, \varphi, 0, 0, 0, 0, 0\right) = 0;$$

3. $\langle P_3, X_1, X_2, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_4 = u_1,$$
$$J_5 = u_2, \quad J_6 = u_0 - u_4, \quad J_7 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4;$$
$$F\left(\omega, \varphi, 0, 0, 0, 0, 0\right) = 0;$$

4. $\langle L_3, P_1, P_2, P_3, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = \frac{u_0 - u_4}{x_0 + x_4}x_3 + u_3, \quad J_4 = \left(u_1 + x_1\frac{u_0 - u_4}{x_0 + x_4}\right)^2 +$$
$$+ \left(u_2 + x_2\frac{u_0 - u_4}{x_0 + x_4}\right)^2, \quad J_5 = u_0 - u_4, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
$$u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F\left(\omega, \varphi, 0, 0, 0, 0\right) = 0;$$

5. $\langle L_3, P_1, P_2, X_3, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = \left(u_1 + x_1\frac{u_0 - u_4}{x_0 + x_4}\right)^2 + \left(u_2 + x_2\frac{u_0 - u_4}{x_0 + x_4}\right)^2,$$
$$J_4 = u_3, \quad J_5 = u_0 - u_4, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4;$$
$$F\left(\omega, \varphi, 0, 0, 0, 0\right) = 0;$$

6. $\langle L_3, P_3, X_1, X_2, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3,$$
$$J_4 = u_0 - u_4, \quad J_5 = u_1^2 + u_2^2, \quad J_6 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4;$$
$$F\left(\omega, \varphi, 0, 0, 0, 0\right) = 0;$$

7. $\langle L_3 - P_3, P_1, P_2, X_1, X_2, X_4\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = (x_0 + x_4)u_3 + (u_0 - u_4)x_3, \quad J_4 = u_0 - u_4,$$
$$J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = x_0 + x_4; \quad F\left(\omega, \varphi, 0, 0, 0\right) = 0.$$

However, among the classes of the first-order differential equations, which are invariant under the splitting subgroups of the group $\widetilde{G}(1,3)$, there exist classes, which are reduced to classes of ODEs. We obtained 15 such classes. Let us present some of them.

1. $\langle L_3, X_0, X_3, X_4\rangle$,

$$J_1=\left(x_1^2+x_2^2\right)^{1/2},\quad J_2=u,\quad J_3=x_1u_2-x_2u_1,\quad J_4=u_0,\quad J_5=u_3,\quad J_6=u_4,$$
$$J_7=u_1^2+u_2^2;\quad u=\varphi(\omega),\quad \omega=\left(x_1^2+x_2^2\right)^{1/2};\quad F\left(\omega,\varphi,0,0,0,0,\left(\varphi'\right)^2\right)=0;$$

2. $\langle X_1, X_2, X_3, X_4\rangle$,

$$J_1=x_0+x_4,\quad J_2=u,\quad J_3=u_0,\quad J_4=u_1,\quad J_5=u_2,\quad J_6=u_3,\quad J_7=u_4;$$
$$u=\varphi(\omega),\quad \omega=x_0+x_4;\quad F\left(\omega,\varphi,\varphi',0,0,0,\varphi'\right)=0;$$

3. $\langle L_1, L_2, L_3, X_0, X_4\rangle$,

$$J_1=\left(x_1^2+x_2^2+x_3^2\right)^{1/2},\quad J_2=u,\quad J_3=x_1u_1+x_2u_2+x_3u_3,\quad J_4=u_0,\quad J_5=u_4,$$
$$J_6=u_1^2+u_2^2+u_3^2;\quad u=\varphi(\omega),\quad \omega=\left(x_1^2+x_2^2+x_3^2\right)^{1/2};\quad F\left(\omega,\varphi,\omega\varphi',0,0,\left(\varphi'\right)^2\right)=0;$$

4. $\langle L_1, L_2, L_3, P_1, P_2, P_3, X_4\rangle$,

$$J_1=x_0+x_4,\quad J_2=u,\quad J_3=(u_0-u_4)(x_0^2-x_1^2-x_2^2-x_3^2-x_4^2)-2(x_0u_0+x_1u_1$$
$$+x_2u_2+x_3u_3+x_4u_4)(x_0+x_4),\quad J_4=u_0-u_4,\quad J_5=u_0^2-u_1^2-u_2^2-u_3^2-u_4^2;$$
$$u=\varphi(\omega),\quad \omega=x_0+x_4;\quad F\left(\omega,\varphi,-2\omega^2\varphi',0,0\right)=0.$$

The remainder of the considered classes of the first-order differential equations are invariant under the splitting subgroups of the group $P(1,4)$, which do not belong to the splitting subgroups of the group $\widetilde{G}(1,3)$. We have 43 such classes. All these classes are reduced to ODEs. Now, we give some examples of this type of reduction.

1. $\langle G, P_1, P_2, P_3\rangle$,

$$J_1=(x_0^2-x_1^2-x_2^2-x_3^2-x_4^2)^{1/2},\quad J_2=u,\quad J_3=\frac{u_0-u_4}{x_0+x_4},\quad J_4=x_1+u_1\frac{x_0+x_4}{u_0-u_4},$$
$$J_5=x_2+u_2\frac{x_0+x_4}{u_0-u_4},\quad J_6=x_3+\frac{x_0+x_4}{u_0-u_4}u_3,\quad J_7=u_0^2-u_1^2-u_2^2-u_3^2-u_4^2;$$
$$u=\varphi(\omega),\quad \omega=(x_0^2-x_1^2-x_2^2-x_3^2-x_4^2)^{1/2};\quad F\left(\omega,\varphi,\frac{\varphi'}{\omega},0,0,0,\left(\varphi'\right)^2\right)=0;$$

2. $\langle G, P_3, L_3, X_4\rangle$,

$$J_1=\left(x_1^2+x_2^2\right)^{1/2},\quad J_2=u,\quad J_3=x_1u_2-x_2u_1,\quad J_4=\frac{u_0-u_4}{x_0+x_4},\quad J_5=\frac{u_0-u_4}{x_0+x_4}x_3+u_3,$$
$$J_6=u_1^2+u_2^2,\quad J_7=u_0^2-u_3^2-u_4^2;\quad u=\varphi(\omega),\quad \omega=\left(x_1^2+x_2^2\right)^{1/2};$$
$$F\left(\omega,\varphi,0,0,0,\left(\varphi'\right)^2,0\right)=0;$$

3. $\langle G, P_1, P_2, X_3\rangle$,

$$J_1=\left(x_0^2-x_1^2-x_2^2-x_4^2\right)^{1/2},\quad J_2=u,\quad J_3=\frac{u_0-u_4}{x_0+x_4},\quad J_4=x_1+u_1\frac{x_0+x_4}{u_0-u_4},$$
$$J_5=x_2+u_2\frac{x_0+x_4}{u_0-u_4},\quad J_6=u_3,\quad J_7=u_0^2-u_1^2-u_2^2-u_4^2;\quad u=\varphi(\omega),$$

$\omega = \left(x_0^2 - x_1^2 - x_2^2 - x_4^2\right)^{1/2}; \quad F\left(\omega, \varphi, \dfrac{\varphi'}{\omega}, 0, 0, 0, \left(\varphi'\right)^2\right) = 0;$

4. $\langle G, L_3, X_3, X_4\rangle$,

$J_1 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_2 - x_2u_1, \quad J_4 = (x_0 + x_4)(u_0 + u_4),$
$J_5 = u_3, \quad J_6 = u_1^2 + u_2^2, \quad J_7 = u_0^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2\right)^{1/2};$
$F\left(\omega, \varphi, 0, 0, 0, \left(\varphi'\right)^2, 0\right) = 0;$

5. $\langle G, P_3, X_1, X_2\rangle$,

$J_1 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = \dfrac{u_0 - u_4}{x_0 + x_4}, \quad J_4 = \dfrac{u_0 - u_4}{x_0 + x_4}x_3 + u_3,$
$J_5 = u_1, \quad J_6 = u_2, \quad J_7 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2};$
$F\left(\omega, \varphi, \dfrac{\varphi'}{\omega}, 0, 0, 0, \left(\varphi'\right)^2\right) = 0;$

6. $\langle G, X_1, X_2, X_3\rangle$,

$J_1 = \left(x_0^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = (x_0 + x_4)(u_0 + u_4), \quad J_4 = u_1, \quad J_5 = u_2,$
$J_6 = u_3, \quad J_7 = u_0^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_0^2 - x_4^2\right)^{1/2};$
$F\left(\omega, \varphi, \omega\varphi', 0, 0, 0, \left(\varphi'\right)^2\right) = 0;$

7. $\langle P_3 + C_3 + 2L_3, X_1, X_2, X_0 + X_4\rangle$,

$J_1 = \left(x_3^2 + x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_3u_4 - x_4u_3, \quad J_4 = x_3u_2 - x_4u_1, \quad J_5 = u_0,$
$J_6 = u_1^2 + u_2^2, \quad J_7 = u_3^2 + u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_3^2 + x_4^2\right)^{1/2};$
$F\left(\omega, \varphi, 0, 0, 0, 0, \left(\varphi'\right)^2\right) = 0;$

8. $\langle L_1 + \dfrac{1}{2}(P_1 + C_1), L_2 + \dfrac{1}{2}(P_2 + C_2), L_3 + \dfrac{1}{2}(P_3 + C_3), X_0 + X_4\rangle$,

$J_1 = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4,$
$J_4 = x_1u_2 - x_2u_1 + x_4u_3 - x_3u_4, \quad J_5 = x_2u_3 + x_4u_1 - x_1u_4 - x_3u_2, \quad J_6 = u_0,$
$J_7 = u_1^2 + u_2^2 + u_3^2 + u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2};$
$F\left(\omega, \varphi, \omega\varphi', 0, 0, 0, \left(\varphi'\right)^2\right) = 0;$

9. $\langle G, L_1, L_2, L_3, X_4\rangle$,

$J_1 = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = (x_0 + x_4)(u_0 + u_4), \quad J_4 = x_1u_1 + x_2u_2 + x_3u_3,$
$J_5 = u_0^2 - u_4^2, \quad J_6 = u_1^2 + u_2^2 + u_3^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2};$
$F\left(\omega, \varphi, 0, \omega\varphi', 0, \left(\varphi'\right)^2\right) = 0;$

10. $\langle G, P_3, C_3, X_1, X_2\rangle$,

$J_1 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_0u_0 + x_3u_3 + x_4u_4, \quad J_4 = u_1, \quad J_5 = u_2,$

$$J_6 = u_0^2 - u_3^2 - u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}; \quad F\left(\omega, \varphi, \omega\varphi', 0, 0, \left(\varphi'\right)^2\right) = 0;$$

11. $\langle L_1, L_2, L_3, P_1 + C_1, P_2 + C_2, P_3 + C_3, X_0 + X_4\rangle$,

$$J_1 = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4, \quad J_4 = u_0,$$
$$J_5 = u_1^2 + u_2^2 + u_3^2 + u_4^2; \quad u = \varphi(\omega), \quad \omega = \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{1/2};$$
$$F\left(\omega, \varphi, \omega\varphi', 0, \left(\varphi'\right)^2\right) = 0;$$

12. $\langle G, P_3, C_3, L_3, X_1, X_2\rangle$,

$$J_1 = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}, \quad J_2 = u, \quad J_3 = x_0u_0 + x_3u_3 + x_4u_4, \quad J_4 = u_1^2 + u_2^2,$$
$$J_5 = u_0^2 - u_3^2 - u_4^2\,; \quad u = \varphi(\omega), \quad \omega = \left(x_0^2 - x_3^2 - x_4^2\right)^{1/2}; \quad F\left(\omega, \varphi, \omega\varphi', 0, \left(\varphi'\right)^2\right) = 0;$$

13. $\langle G, L_1, L_2, L_3, P_1, P_2, P_3\rangle$,

$$J_1 = (x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)^{1/2}, \quad J_2 = u, \quad J_3 = \frac{u_0 - u_4}{x_0 + x_4},$$
$$J_4 = x_0u_0 + x_1u_1 + x_2u_2 + x_3u_3 + x_4u_4, \quad J_5 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
$$u = \varphi(\omega), \quad \omega = (x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2)^{1/2}; \quad F\left(\omega, \varphi, \frac{\varphi'}{\omega}, \omega\varphi', \left(\varphi'\right)^2\right) = 0.$$

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Symmetry of Stochastic Equations

Giuseppe GAETA

Dipartimento di Matematica, Università di Milano, v. Saldini 50, I–20133 Milano, Italy
E-mail: *gaeta@mat.unimi.it*

We discuss symmetries of Ito equations, and the relations of these with symmetries of the associated Fokker–Planck equation.

Symmetry methods are by now recognized as one of the main tools to attack *deterministic* differential equations (both ODEs and PDEs); see e.g. [3, 8, 15, 20, 21, 24, 25]. The situation is quite different for what concerns *stochastic* differential equations [1,12,13,16,19]: here, symmetry considerations are of course quite widely used by theoretical physicists (see e.g. [4, 14, 17] in the context of KPZ theory), but a rigorous and general theory comparable to the one developed for deterministic equation is still lacking, and the attention of the community working on symmetry methods should maybe be called to this.

We would like to quote here early work by Misawa [18], mainly concerned with conservation laws for stochastic systems, and later on the work by Arnold and Imkeller [1,2] for normal forms of stochastic equations.

In the following I will report on some work I have done – to a large extent in collaboration with N. Rodríguez Quintero – on symmetries of stochastic (Ito) equations, and how these compare with the symmetries of the associated diffusion (Fokker–Planck) equations [5, 7, 22, 23]; further details can be found in [9, 11].

It should be stressed that in this note an Ito equation will be seen as describing a one-particle stochastic process only. The same equation does also describe N-particles processes, or a random dynamical system [1], and in this case there will be further symmetries beyond those admitted in the frame of the one-particle processes interpretation, as discussed in [9].

1 Projectable symmetries

We assume the reader to be fairly familiar with the concept of symmetry of a deterministic differential equation, and the methods to characterize these; so we will just recall some formula of use in the following, mainly to fix notation, identify the class of transformations we are going to consider, and compare with the situation for stochastic equations.

Let us first consider the case of systems of first order ODEs in $\mathbb{R}^n$,

$$\dot{x}^i - f^i(x,t) = 0. \tag{1}$$

The general form of Lie-point vector fields is in this case, writing $\partial_i = \partial/\partial x^i$, $X = \tau(x,t)\partial_t + \xi^i(x,t)\partial_i$. Such a vector field, however, is somehow too general: indeed, we would like the change on t not to depend on the value assumed by the solution $x(t)$; this is particularly relevant when we think of the ODE as describing not the motion of a single particle in $\mathbb{R}^n$, but that of an assembly of particles. We would thus like to keep the special role of time, i.e. require $\tau = \tau(t)$; in other words, we will consider

$$X_0 = \tau(t)\partial_t + \xi^i(x,t)\partial_i. \tag{2}$$

A similar discussion holds for symmetries of PDEs, first or higher order. Here we will deal only with scalar PDEs, i.e. equations for a single dependent variable $u = u(x,t)$. In this case

we would write in general $X = \tau(x,t,u)\partial_t + \xi^i(x,t,u)\partial_i + \phi(x,t,u)\partial_u$. On physical terms, it is preferable to consider only transformations such that the independent variables are transformed independently of the values assumed by the dependent ones, i.e. such that τ and the ξ^i do not depend on u; also, here we are primarily interested in evolution equations, and in order to guarantee that the time keeps its distinguished role we should also ask that τ does not depend on x. Thus, in the end we want to consider transformations of the form

$$X = \tau(t)\frac{\partial}{\partial t} + \xi^i(x,t)\frac{\partial}{\partial x^i} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{3}$$

Definition 1. The vector fields – and in particular symmetry generators – of the form (2), (3) will be called "fiber-preserving", or simply **projectable**.

Let us now consider in particular a system of ODEs, which we write in the form (1). The determining equations for the symmetry generators of this are easily determined (these should be compared to the determining equations for stochastic differential equations discussed in the next section). We will use, here and below, the notation

$$\{f,\xi\}^i := \ (f^j \cdot \partial_j)\xi^i - (\xi^j \cdot \partial_j)f^i. \tag{4}$$

Lemma 1. *The projectable symmetries of* (2) *are given by vector fields* X_0 *as above with coefficients satisfying*

$$\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i = 0. \tag{5}$$

For given functions $f^i(x,t)$ the partial differential equations (5) always [20, 24] have non-trivial solutions $\tau(t)$ and $\xi^i(x,t)$; for example we can fix $\tau(t)$ and compute $\xi^i(x,t)$. Although an infinite number of symmetries of (5) exist, in general there is no constructive way to find them.

For further detail on symmetries of (deterministic) ODEs and PDEs, and applications, the reader is referred to [3, 8, 15, 20, 21, 24, 25].

2 Stochastic differential equations

Let us turn to stochastic equations [1, 6, 12, 13, 16, 19], and let us consider an Ito equation[1]

$$dx^i = f^i(x,t)\,dt + \sigma^i_k(x,t)\,dw^k, \tag{6}$$

where f and σ are smooth functions, $\sigma(x,t)$ is a nonzero matrix and the w^k are independent homogeneous standard Wiener processes, so that

$$\langle |w^i(t) - w^j(s)|^2 \rangle = \delta^{ij}(t-s).$$

The equation (6) should be seen as a map from the vector Wiener process $\boldsymbol{w}(t) = \{w^1(t), \ldots, w^n(t)\}$ to the stochastic process undergone by $\{x^1(t), \ldots, x^n(t)\}$, and its meaning is precisely that of defining the vector stochastic process $\boldsymbol{x}(t)$.

If we consider a function $y = \Phi(x)$, its evolution under the Ito equation (6) is described (again in terms of a stochastic equation) by the *Ito formula*:

$$\begin{aligned} dy^i &= \frac{\partial \Phi^i}{\partial x^j}\,dx^j + \frac{1}{2}\frac{\partial^2 \Phi^i}{\partial x^j \partial x^k}\,dx^j dx^k \\ &= \left[f^j \partial_j \Phi^i + (1/2)\partial^2_{jm}\Phi^i(\sigma\sigma^T)^{jm}\right]dt + \left[(\partial_j \Phi^i)\sigma^{jk}\right]dw^k. \end{aligned} \tag{7}$$

[1] It is well known that any Ito equation is equivalent to a Stratonovich equation; for the Ito equation (6), the associated Stratonovich equation is $dx^i = b^i(x,t)dt + \sigma^i_k(x,t) \circ dw^k$, with $b^i = f^i - (1/2)[\sigma(\partial\sigma^T/\partial x))]$. We will stick to the Ito formulation in all of this note.

The Fokker–Planck equation

$$\partial_t u = -\partial_i \left(f^i u\right) + \frac{1}{2}\,\partial^2_{ij}\left[(\sigma\sigma^T)^{ij} u\right] \tag{8}$$

is associated to the Ito equation (6) and describes the evolution of the probability measure $\rho = u(x,t)$ under the stochastic process described by (6). Equations (6) and (8) contain the same *statistical* information [1, 12, 13, 16, 19] as far as one-particle processes are concerned (but not if we think of the Ito equation as describing a random dynamical system, see [1,9]), provided σ satisfies the **non-degeneracy condition**

$$A := \frac{1}{2}\sigma\,\sigma^T \neq 0$$

which we will assume throughout this note.

We stress that we are interested in equation (8) as far as it describes the time evolution of the probability measure $\rho = u(x,t)$ under the stochastic process (6). It is obvious that for this interpretation $u(x,t)$ should be non-negative and subject to the condition

$$\int_{-\infty}^{+\infty} u(x,t)\, dx^1 \cdots dx^n = 1. \tag{9}$$

This is relevant in connection with the allowed transformations in the $(x,t;u)$ space: only transformations preserving this normalization do represent symmetries of the Fokker–Planck equation compatible with its probabilistic interpretation[2], and one should expect a correspondence between symmetries of the Ito equation and this subclass – rather than all – of the symmetries of the Fokker–Planck equation.

Another simple but relevant point which should be stressed is the following: different Ito equations which have the same f but different matrices σ can provide the same term $\sigma\sigma^T$ and thus the same Fokker–Planck equation.

A simple example is provided e.g. by σ orthogonal: in this case we have by definition $\sigma\sigma^T = I$, so that all the Ito equations with the same f and any orthogonal matrix σ give the same Fokker–Planck equation[3]. Similarly, σ and $\widetilde{\sigma} = \sigma B$, with B any orthogonal matrix, will give the same Fokker–Planck equation (and conversely σ and $\widetilde{\sigma}$ give the same Fokker–Planck – with the same f – only if there is an orthogonal matrix B such that the above relation is satisfied.

Thus if we consider a continuous variation of σ, say $\sigma + \varepsilon\gamma$, in the Ito equation, the associated Fokker–Planck equation remains unchanged provided $(\sigma+\varepsilon\gamma)(\sigma+\varepsilon\gamma)^T = \sigma\sigma^T$, which at order ε is simply

$$\sigma\gamma^T + \gamma\sigma^T = 0. \tag{10}$$

3 Symmetries of stochastic ODEs

3.1 Spatial symmetries

We will at first consider symmetries not acting on the time variable t, i.e. $X_0 = \xi^i(x,t)\partial_i$. Thus, we consider a near-identity change of coordinates, passing from x to y via

$$x^i \to y^i = x^i + \varepsilon\xi^i(x,t). \tag{11}$$

[2] In other words, we should set the FP equation in the function space $\mathcal{F}$ defined by (9), and thus accept only transformations mapping $\mathcal{F}$ to itself.

[3] The one-point stochastic processes described by two different Ito equations having the same associated Fokker–Planck equation have the same statistical properties (the probability measures evolve in the same way), but *are* different: the same realization of the Wiener process $\boldsymbol{w}(t)$ leads to different sample paths.

Using the Ito formula, we easily check that at first order in ε, this transformation maps the Ito equation (6) into a new Ito equation

$$dy^i = \widetilde{f}^i(y,t)\,dt + \widetilde{\sigma}^i_k(y,t)dw^k, \tag{12}$$

where we have explicitly

$$\begin{aligned}\widetilde{f}^i &= f^i + \varepsilon\,[\partial_t\xi^i + f^j\partial_j\xi^i - \xi^j\partial_j f^i + (1/2)(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i],\\ \widetilde{\sigma}^i_k &= \sigma^i_k + \varepsilon\,[\sigma^j_k\partial_j\xi^i - \xi^j\partial_j\sigma^i_k].\end{aligned} \tag{13}$$

When (11) maps (6) into itself, i.e. when (12) coincides with (6) [up to terms $o(\varepsilon)$], we say that $X_0 = \xi^i(x,t)\partial_i$ is a (Lie-point) spatial **symmetry** of (6). Thus, Lie-point spatial symmetries of (6) are identified by the vanishing of terms $O(\varepsilon)$ in (13). In the following statement, we use the notations introduced above for A and $\{\cdot,\cdot\}$.

Proposition 1. *The (generators of) Lie-point spatial symmetries of the Ito equation* (6) *are given by vector fields $X_0 = \xi^i(x,t)\partial^i_x$ with coefficients ξ^i satisfying the* determining equations for spatial symmetries *of an Ito equation:*

$$\begin{aligned}&\partial_t\xi^i + \{f,\xi\}^i + (1/2)(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i = 0,\\ &\{\sigma_k,\xi\}^i = 0.\end{aligned} \tag{14}$$

These are $n + n^2$ equations for the $n+1$ functions ξ^i, τ; thus for $n > 1$ they have some solution only in very exceptional cases. This should not be surprising, as symmetry is a non-generic property.

Note that if $\sigma = \sigma(t)$ does not depend on the spatial variables, the second equation of (14) reduces to $\sigma^j_k\partial_j\xi^i = 0$, which in turns implies vanishing of the term $(\sigma\sigma^T)^{jk}\partial^2_{jk}\xi^i$, which in this case can be rewritten as $\sigma^k_p\partial_k(\sigma^j_p\partial_j\xi^i)$. Thus, (14) are in this case equivalent to the determining equations for symmetries of the deterministic part of the Ito equation [see (5) and recall now we are assuming $\tau = 0$] with the additional condition $\sigma^j_k\partial_j\xi^i = 0$.

3.2 Spatio-temporal symmetries

We could of course consider transformations acting also on t: in this case the t transformation would also affect the processes $w(t)$, and some extra care should be paid. We will not go over the details (which consist in studying how $w(t)$ is changed and include this in the change undergone by σ), and just report the result obtained in [11], with the same notation as above.

Proposition 2. *The projectable vector field $X_0 = \tau(t)\partial_t + \xi^i(x,t)\partial^i_x$ is a symmetry generator for the Ito equation* (6) *if and only if the coefficients $\{\tau,\xi\}^i$ satisfy the* full determining equations *for projectable symmetries of an Ito equation:*

$$\begin{aligned}&\partial_t\left(\xi^i - \tau f^i\right) + \{f,\xi\}^i - A^{jk}\partial^2_{jk}\xi^i = 0,\\ &\{\sigma_k,\xi\}^i - \tau\partial_t\sigma^i_k - (1/2)\sigma^i_k\partial_t\tau = 0.\end{aligned} \tag{15}$$

Note that the symmetries which are linear in x, i.e. such that $\xi^i(t;\boldsymbol{x}) = M^i_j(t)x^j$, are given by the same equations as for symmetries of the deterministic part of the Ito equation (6), i.e. (5), plus the additional condition $M^i_j\sigma^j_k = M^j_p x^p\partial_j\sigma^i_k + \tau\partial_t\sigma^i_k + (1/2)\sigma^i_k\partial_t\tau$.

A relevant case in applications is the one where $f^i = f^i(\boldsymbol{x})$ and $\sigma^i_k = \sigma^i_k(t)$ or even $\sigma^i_k = \text{const} = S^i_k$; this correspond to an autonomous dynamical system subject to a noise which depends only on t or even a constant noise. In this case (14) can be discussed quite completely. Indeed, with σ independent of the spatial coordinates, the second of these reads $2\sigma^j_k\partial_j\xi^i =$

$2\tau\partial_t\sigma^i_k + \sigma^i_k\partial_t\tau$. As the r.h.s. depends only on t, by differentiating with respect to x^m we get the equation $\sigma^j_k(\partial^2\xi^i/\partial x^j\partial x^m) = 0$: i.e., the (symmetric) matrix of second derivatives of ξ^i, H^i_{jm} must be such that $\sigma H^i = H^i\sigma = 0$. Notice that in particular, if $\sigma^i_k(t) = \lambda_{(i)}(t)\delta^i_k$ with all $\lambda_{(i)} \not\equiv 0$ (or however if σ^{-1} exists), this means that ξ can be at most linear in the x.

4 Symmetries of the Fokker–Planck equation

We will now discuss the determining equations for projectable symmetry generators of the Fokker–Planck equation (8) in arbitrary spatial dimensions; general symmetries of the Fokker–Planck equations (with some limitations on σ) in one and two space dimensions have been completely classified [5, 7, 23].

It will be convenient to rewrite the Fokker–Planck equation as

$$u_t + A^{ij}\,\partial^2_{ij}u + B^i\,\partial_i u + Cu = 0; \tag{16}$$

the coefficients A, B, C depend on x and t only and are given explicitly by[4]

$$A = -(1/2)\sigma\sigma^T, \qquad B^i = f^i + 2\partial_j A^{ij}, \qquad C(x,t) = (\partial_i \cdot f^i) + \partial^2_{ij}A^{ij}. \tag{17}$$

We consider a projectable vector field of the form (3). Computing the second prolongation of this and applying it to the equation, we obtain the determining equations for symmetries of the Fokker–Planck equation.

It follows by general results on Lie-point symmetries of linear equations [8, 20], or by an elementary explicit computation, that:

Lemma 2. *The projectable symmetries of the Fokker–Planck equation* (16) *are given by vector fields in the form* (8) *with* $\phi = \alpha(x,t) + \beta(x,t)u$.

Again by linearity, we will have "trivial" symmetries $X_\alpha = \alpha(x,t)\partial_u$, with $\alpha(x,t)$ an arbitrary solution to the FP equation itself; this is just expressing the linear superposition principle [20]. Note that in order to preserve the normalization (9), we should moreover require $\int \alpha(x,t)dx = 0$, which rules out nontrivial non-negative solutions.

We will thus from now on focus on other, i.e. "nontrivial", symmetries only, and take $\alpha(x,t) = 0$.

Proposition 3. *The nontrivial projectable symmetries of the Fokker–Planck equation* (16) *are given by vector fields in the form* (3) *with* $\phi = \beta(x,t)u$, *where* τ, ξ^i, β *satisfy the* determining equations

$$\begin{aligned}
&\partial_t(\tau A^{ik}) + (\xi^m\partial_m A^{ik} - A^{im}\partial_m\xi^k - A^{mk}\partial_m\xi^i) = 0,\\
&\partial_t(\tau B^i) - \left[\xi^i_t + B^m\partial_m\xi^i - \xi^m\partial_m B^i\right] + (A^{ik}\partial_k\beta + A^{mi}\partial_m\beta) - A^{mk}\partial^2_{mk}\xi^i = 0,\\
&\partial_t(\tau C) + \beta_t + A^{ik}\partial^2_{ik}\beta + B^i\partial_i\beta + \xi^m\partial_m C = 0.
\end{aligned} \tag{18}$$

It should be stressed that in the computations leading to (18) we have not required preservation of the normalization $\int u\,dx = 1$; when we require this, it turns out that

Lemma 3. *The nontrivial projectable symmetries of the Fokker–Planck equation* (16), *in the form* (3) *with* $\phi = \beta(x,t)u$, *preserve the normalization* (9) *if and only if*

$$\beta = -\mathrm{div}(\xi). \tag{19}$$

[4] Recall that our non-degeneracy assumption guarantees $A \neq 0$.

In other words, this lemma guarantees that if we are looking for the (nontrivial) symmetries of a Fokker–Planck equation compatible with its probabilistic interpretation, we do not have to deal with the general form (3) of the symmetry vector field, but we can use instead the ansatz

$$X = \tau(t)\partial_t + \xi^i(x,t)\partial_i - [\mathrm{div}(\xi)u]\partial_u; \tag{20}$$

this substantially simplifies the analysis of symmetries.

5 Symmetries of the Ito versus symmetries of the associated FP equation

We are specially interested in discussing how the symmetries of the partial differential equation (16) and those of the symmetries of the system of stochastic ODEs (6) are related.

In order to do this, we should express the coefficients A, B, C in terms of f and σ; indeed $A = -(1/2)\sigma\sigma^T$, $B^i = f^i + 2\partial_k A^{ik}$, and $C = \partial_i f^i + \partial^2_{ik} A^{ik}$. With the latter two of these and with some manipulations, see [11], (18) reads

$$\begin{aligned}
&\partial_t(\tau A^{ik}) + \left(\xi^m \partial_m A^{ik} - A^{im}\partial_m \xi^k - A^{km}\partial_m \xi^i\right) = 0,\\
&\left[\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i - A^{mk}\partial^2_{mk}\xi^i\right] - 2\left[A^{ik}\partial_k\beta + \ A^{im}\partial^2_{mk}\xi^k\right] = 0,\\
&\left[\partial_t + f^i\partial_i - A^{ik}\partial^2_{ik}\right][\beta + \partial_m\xi^m] = 0.
\end{aligned} \tag{21}$$

It is convenient to introduce the shorthand notations

$$\begin{aligned}
&\Gamma^k_j = \sigma^m_j\partial_m\xi^k - \xi^m\partial_m\sigma^k_j - \tau\partial_t\sigma^k_j - (1/2)\sigma^k_j\partial_t\tau,\\
&\Lambda^i = -\left[\partial_t(\xi^i - \tau f^i) + \{f,\xi\}^i - A^{mk}\partial^2_{mk}\xi^i\right].
\end{aligned} \tag{22}$$

With these, (21) is rewritten as

$$\begin{aligned}
&\sigma^i_j\Gamma^k_s\delta^{js} + \sigma^k_j\Gamma^i_s\delta^{js} = 0,\\
&\Lambda^i + 2\left[A^{ik}\partial_k\beta + \ A^{im}\partial^2_{mk}\xi^k\right] = 0,\\
&\left[\partial_t + f^i\partial_i - A^{ik}\partial^2_{ik}\right][\beta + \partial_m\xi^m] = 0.
\end{aligned} \tag{23}$$

It should be stressed that with the notation just introduced, the determining equations (15) for symmetries of the Ito equation (6) read simply

$$\Lambda^i = 0, \qquad \Gamma^i_j = 0. \tag{24}$$

With this notation, the comparison of symmetries for the Ito and the associated Fokker–Planck equation is quite a simple matter.

5.1 From the Ito to the FP symmetries

We will first investigate if symmetries of an Ito equation result in symmetries of the associated Fokker–Planck equation.

We first note that for $\Gamma = 0$, the first of (23) is satisfied. As for the second of (23), when $\Lambda^i = 0$ this reduces to

$$A^{ik}\partial_k\beta = -A^{im}\partial^2_{mk}\xi^k. \tag{25}$$

Obviously β is not present in symmetries of the Ito equation, so we can choose it as to satisfy the third of (23) and (25): for this it suffices to choose

$$\beta = -\partial_m \xi^m + c_0 = -\mathrm{div}(\xi) + c_0. \tag{26}$$

We know from Lemma 2 that actually we have to take $c_0 = 0$, and that this is actually the only choice of β providing us with a normalization-preserving symmetry of the FP. Our findings are summarized as follows:

Proposition 4. *Let $X_0 = \tau\partial_t + \xi^i\partial_i$ be a symmetry of the Ito equation* (6)*. Then X_0 extends to a unique normalization preserving symmetry $X := X_0 - [(\mathrm{div}\xi)u]\partial_u$ of the associated Fokker–Planck equation.*

5.2 From the FP to the Ito symmetries

Let us now consider the converse question, i.e. if and when a symmetry of the FP equation associated to an Ito equation can be projected to a symmetry of the Ito equation itself.

As the probabilistic interpretation of the FP requires the normalization condition (9), we expect only symmetries identified by Lemma 3 can provide symmetries of related Ito equations.

Moreover, as recalled above, there are transformations which map an Ito equation into a different one with the same statistical properties and thus the same associated FP equation: these would be symmetries of the FP but not of the Ito equation. Thus, not all the symmetries of the FP should be expected to produce symmetries of the Ito equation.

By Lemma 3, we assume $\beta = -\mathrm{div}(\xi)$; this guarantees that the third of (23) holds, and also that the second of (23) reduces to $\Lambda^i = 0$. Hence we only have to discuss the relation between the first of (23) and the second of (24). These two equations can be rewritten, respectively, as

$$\sigma\Gamma^T + \Gamma\sigma^T = 0 \qquad \text{and} \qquad \Gamma = 0. \tag{27}$$

The second of these implies the first, but the converse is not true: hence there is *not* a complete equivalence.

This just corresponds to a given FP equation corresponding to different Ito equations: the transformations with $\Gamma \neq 0$ but $\sigma\Gamma^T + \Gamma\sigma^T = 0$ will be precisely those which map an Ito equation E_0 into a different Ito equation E_1 which has the *same* Fokker–Planck associated equation, see (10).

Proposition 5. *Let $X = \tau(t)\partial_t + \xi^i(x,t)\partial_i - [(\mathrm{div}\xi)u]\partial_u$ be a symmetry of the Fokker–Planck equation associated to the Ito equation* (6)*. Then X maps the Ito equation into a (generally, different) Ito equation with the same statistical properties; X is a symmetry of the Ito equation* (6) *if and only if Γ defined in* (22) *satisfies $\Gamma^i_k = 0$.*

We may note that from the above system (24), restricting to the case where $f(t,x) = f(x)$ and $\sigma(t,x) = \sigma(x)$, we recover the results of Cicogna and Vitali [5] for the one-dimensional setting. From (22) it is also possible to recover the results of Shtelen and Stogny [23] for the two-dimensional Kramers equation, as well as the results of Finkel [7] for certain two-dimensional FP equations.

5.3 Examples

Example 1. As the first example, we consider the case $f(t,x) = 0$, $\sigma(t,x) = \sigma_0 = \mathrm{const} \neq 0$, i.e. the equation

$$dx = \sigma_0\, dw(t) \tag{28}$$

which represents a free particle subject to constant noise. The corresponding Fokker–Planck equation is simply the heat equation $u_t = (\sigma_0^2/2)u_{xx}$. The symmetries of the heat equation (other than the trivial ones $v_\alpha = \alpha(x,t)\partial_u$ with $\alpha_t = \alpha_{xx}$) are well-known to be [20, 24]

$$v_1 = \partial_t, \qquad v_2 = \partial_x, \qquad v_3 = u\partial_u, \qquad v_4 = \sigma_0^2 t\partial_x - \sigma_0 xu\partial_u,$$
$$v_5 = 2t\partial_t + x\partial_x, \qquad v_6 = t^2\partial_t + xt\partial_x - (1/2)(t + x^2/\sigma_0^2)u\partial_u.$$

Of these, v_1, v_2 and v_5 (which do not act on u) are also symmetries of the Ito equation (28), as is easily checked using (14). Notice that (19) is satisfied for these, and is not satisfied for v_3, v_4 and v_6. The vector fields v_1, v_2 and v_5 do actually span the symmetry algebra of (28).

Example 2. As an example in two space dimensions [with coordinates $(x_1, x_2) = (x, y)$], we choose

$$dx = y\,dt,$$
$$dy = -k^2 y\,dt + \sqrt{2k^2}\,dw(t) \tag{29}$$

with k^2 a positive constant.

The corresponding Fokker–Planck equation is the Kramers equation

$$u_t = k^2 u_{yy} - yu_x + k^2 yu_y + k^2 u; \tag{30}$$

the symmetries of this were studied in [23] and, apart from the trivial ones v_α, are

$$v_1 = \partial_t, \qquad v_2 = \partial_x, \qquad v_3 = e^{-k^2 t}\,[k^{-2}\partial_x - \partial_y], \qquad v_4 = u\partial_u,$$
$$v_5 = t\partial_x + \partial_y - (1/2)(y + k^2 x)u\partial_u, \qquad v_6 = e^{k^2 t}\,[k^{-2}\partial_x + \partial_y - yu\partial_u].$$

Here v_1, v_2 and v_3 satisfy (19), while for v_4, v_5 and v_6 this is violated. According to our definition, the symmetries of the equations (29) are again v_1, v_2 and v_3, as easily checked.

Example 3. We consider now an example where the correspondence between normalization-preserving symmetries of the FP equation and symmetries of the Ito equation is not complete, i.e. the two-dimensional Ito system (with zero drift)

$$dx^1 = \cos(t)dw^1 - \sin(t)dw^2,$$
$$dx^2 = \sin(t)dw^1 + \cos(t)dw^2; \tag{31}$$

the corresponding Fokker–Planck equation is now just the two dimensional heat equation $u_t = (1/2)\triangle u$.

It is now immediate to check that the vector field $X_0 = \partial_t$ is a symmetry of the FP equation, but *not* a symmetry of (31). Obviously, the case of any orthogonal σ with $\partial_t \sigma_k^i \not\equiv 0$ will be exactly the same.

Example 4. Consider n uncoupled equations for equal "Langevin harmonic oscillators" subject to independent stochastic noises [16, 22]; this system is described by the Ito system

$$dx^i = -x^i\,dt + \sqrt{2s_i}\,dw^i, \qquad i = 1, 2, \ldots, n \tag{32}$$

(no sum on i) where we assume all the s_i are strictly positive; the corresponding Fokker–Planck equation is

$$\partial_t u = \sum_{i=1}^{n} [s_i \partial_{ii}^2 u + x^i \partial_i u + u].$$

In this case we get the symmetries ($i = 1, 2, \dots, n$)

$$v_1 = \partial_t, \qquad v_2 = e^{-2t}\left[\partial_t - \sum_{i=1}^{n} x^i \partial_i + nu\partial_u\right], \qquad v_{q_i} = e^{-t}\partial_i.$$

Example 5. Finally, we consider now the n-dimensional nonlinear case

$$dx^i = -(1 - \lambda\|x\|^2)x^i\, dt + dw^i, \tag{33}$$

where $\|x\|$ is the norm of the vector $\boldsymbol{x}$ and $\lambda \neq 0$. By inserting $f^i = -x^i(1-\lambda\|x\|^2)$ and $\sigma^i_j = \delta_{ij}$ in (24) we obtain, from the second of these, $\xi^i = h^i(t) + (\tau_t/2)x^i$. Inserting this into the first of (24) and isolating the coefficients of different powers of x, we get that the only symmetry is given by $v = \partial_t$.

Note that we got no rotation symmetry; this is because within the class of transformations we are considering we can rotate the vector x, but *not* the vector Wiener process $w(t)$; transformations allowing to rotate $w(t)$ as well will be considered in the next section.

6 W-symmetries

We will now consider symmetries involving not only the spatial and time variables (x, t), but also the vector Wiener processes $w(t)$ entering in the n-dimensional Ito equation (6). We will specifically consider infinitesimal transformations of the form

$$\begin{aligned} &x^i \to y^i = x^i + \varepsilon\xi^i(x,t), \\ &t \to s = t + \varepsilon\tau(t), \\ &w^i \to z^i = w^i + \varepsilon\mu^i(w,t). \end{aligned} \tag{34}$$

We also call symmetry generators of this form, "*W-symmetries*".

Note that this is not the most general possible form of a transformation for the variables involved in (1); some words on this restriction are in order here.

The restriction on τ and ξ is the same as considered above. Moreover, we only allow "internal" transformations of w, i.e. they cannot depend on $x(t)$. As we think of the stochastic process $w(t)$ to be independent of the evolution of the $x(t)$, we like its transformation not to depend on the latter. Finally, we have allowed the transformation on the spatial coordinates $x(t)$ to depend on x and t, but not on the w; this means that we do not want to consider transformations of the spatial coordinates which depend on the realization of the stochastic process $w(t)$. This again is somewhat a natural requirement in physical terms[5].

In order to compute how (34) acts on the Ito equation (6), we use the Ito formula (7). Moreover, due to the form of (34), we can adopt a "two-steps procedure", see [9]: the transformations of the (x, t) variables and of the stochastic process $w(t)$ do not interfere with each other. It turns out that some strong limitation on the functions μ^i arise.

Lemma 4. *The infinitesimal transformation $w \to z = w + \varepsilon\mu(w, t)$ maps the vector Wiener process $w(t)$ into a vector Wiener process $z(t)$ if and only if $\mu = Bw$, with B a real antisymmetric matrix. Equivalently, if and only if $z = Mw$ with M orthogonal.*

Note that as B is antisymmetric matrix, it vanishes in dimension one: for one-dimensional Ito equations we have no new symmetries with respect to those discussed above.

[5]It should be however stressed that in this way we are also discarding the transformations needed to obtain *normal forms* of stochastic differential equations, see [1, 2] and references therein.

Proposition 6. *Under the infinitesimal transformation* (34)*, with B a real antisymmetric matrix, the Ito equation* (6) *is changed into a (generally, different) Ito equation. The Ito equation* (6) *is invariant under* (34) *if and only if τ, ξ, μ satisfy the determining equations for W-symmetries:*

$$\partial_t \xi^i + \{f, \xi\}^i - \partial_t(\tau f^i) + A^{jk}\partial^2_{jk}\xi^i = 0,$$
$$\{\sigma_k, \xi\}^i - \tau\partial_t\sigma^i_k - (1/2)(\partial_t\tau)\sigma^i_k - \sigma^i_p B^p_k = 0. \tag{35}$$

If we just consider a transformation $w \to z = Mw$, with M orthogonal as required by Lemma 4, the Fokker–Planck equations associated to the Ito equation and to the transformed one will be the same. Thus all the discussion conducted in Section 5 about the relations betwen symmetries of an Ito equation and of the associated FP equation also applies to transformations of the form (34)[6].

Example 6. Let us consider again the two-dimensional Ito equation (29). If we look at the second set of determining equations and single out the one for $i = 2$ and $k = 1$, we have immediately that $B = 0$, i.e. in this case we have no new symmetry by allowing transformation of w.

Example 7. Let us consider again the two-dimensional Ito system (31) We will write $\xi^1 = \alpha$, $\xi^2 = \beta$, and $B_{12} = -B_{21} = b$. The most general solution to the determining equations turns out to be

$$\alpha = [\tau_t/2 - \sin(2t)\tau]x + [b + \cos(2t)\tau]y + \alpha_0(t),$$
$$\beta = -[b + \tau]x + [\tau_t/2]y + \beta_0(t).$$

By setting $\tau = \alpha_0 = \beta_0 = 0$, we obtain a new symmetry, corresponding to $\alpha = by$ and $\beta = -bx$ with b arbitrary, i.e. given by

$$X = \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \left(w^2\frac{\partial}{\partial w^1} - w^1\frac{\partial}{\partial w^2}\right).$$

Note this is nothing else than a simultaneous (and identical) rotation in the (x, y) and in the (w_1, w_2) planes.

Example 8. We will now consider again the case of n uncoupled equal "Langevin harmonic oscillators", see (32) above.

If we discard symmetries with $B = 0$ (i.e. those obtained above), we are still left with a nontrivial possibility, i.e. B an arbitrary (real, antisymmetric) constant matrix and $\xi^i = C^i{}_k x^k$, where $C^i{}_k = (s_i/s_k)B^i_k$ (no sum on i, k here).

The meaning of this result is obvious: we can act on this system by an arbitrary $SO(n)$ rotation in the w space and a related rotation in the x space.

If we assume $s_1 = s_2 = \cdots = s_n = s$, we have indeed $C = B$ and the rotations in w and x spaces do just coincide. For general $s_1, \ldots, s_n$, the (s_i/s_k) factors relating the C and B matrices are also easily understood: we could rescale each of the x^i by a factor $\sqrt{2s_i}$, $x^i = \sqrt{2s_i}y^i$ (no sum on i), arriving at the manifestly rotationally invariant n-dimensional Ito equation $dy^i = -y^i dt + dw^i$.

Example 9. Let us consider again (33). We easily check that it is rotationally invariant with a W-symmetry: take $\tau = 0$ and $\xi^i = B^i_k x^k$, so that

$$\{\xi, f\}^i = \{Bx, (1 - \lambda||x||^2)x\} = -2\lambda x^i\left(x^j B^j_k x^k\right) = 0,$$

the last equality following from $B = -B^T$. That is, (33) is indeed symmetric under simultaneous identical rotations of the x and w vectors.

[6]Needless to say, for more general transformations, for which the transformations of the (x, t) variables and of the vector Wiener process $w(t)$ do interfere with each other, this would not be the case.

7 Discrete symmetries

We will now briefly consider, for the sake of completeness, *discrete* symmetries of stochastic differential equations. Similarly to what happens in the deterministic case (see e.g. [10]), the resulting determining equations are in general too difficult to be attacked except for very simple classes of transformations (e.g. reflections, or however linear ones). On the other side, they can be used to check if a given discrete transformation is a symmetry of a given stochastic differential equation; and/or to determine the stochastic differential equations which admit a given discrete transformation as a symmetry.

We consider again an Ito equation of the form (6); here we will not consider transformations acting on the t coordinate, and will thus limit to consider the change of coordinates in the x and w spaces given by

$$y^i = \phi^i(x,t), \qquad w^k(t) = R^k_{\ p}(t) z^p(t). \tag{36}$$

As required by Lemma 4, we take $R \in O(n)$. We write as usual $A = -(1/2)\sigma\sigma^T$.

With the Ito formula we have at once that y obeys the Ito equation

$$dy^i = \left[\frac{\partial \phi^i}{\partial x^j} f^j - A^{jk} \frac{\partial^2 \phi^i}{\partial x^j \partial x^k} + \frac{\partial \phi^i}{\partial t} \right] dt + \left[\frac{\partial \phi^i}{\partial x^j} \sigma^j_{\ p} R^p_{\ k} \right] dz^k. \tag{37}$$

Proposition 7. *The transformation* (34) *is a symmetry of the Ito equation* (6) *if and only if* ϕ, R *satisfy the* determining equations for discrete symmetries *of an Ito equation:*

$$\begin{aligned} &\frac{\partial \phi^i(x,t)}{\partial x^j} f^j(x,t) + A^{jk}(x,t) \frac{\partial^2 \phi^i(x,t)}{\partial x^j \partial x^k} + \frac{\partial \phi^i(x,t)}{\partial t} \ = \ f^i\left(\phi(x,t),t\right), \\ &\frac{\partial \phi^i(x,t)}{\partial x^j} \sigma^j_{\ p}(x,t) R^p_{\ k} = \sigma^i_{\ k}\left(\phi(x,t),t\right). \end{aligned} \tag{38}$$

Example 10. The simplest case of discrete transformation is provided by $\phi^i(x,t) = -x^i$, $R = \pm I$. In these cases, (38) reduce to

$$f^i(x,t) = -f^i(-x,t), \qquad \sigma(x,t) = \mp\sigma(-x,t). \tag{39}$$

Example 11. For the n independent "Langevin oscillators", see (32), it is immediate to check that (39) is satisfied (for $R = -I$). More generally, (38) read now (no sum over i, sum over j when repeated)

$$a_{(j)} \frac{\partial \phi^i}{\partial x_j} - \frac{s^2_{(j)}}{2} \frac{\partial^2 \phi^i}{\partial x^j \partial x^j} = a_{(i)} \phi^i, \qquad s_{(j)} \frac{\partial \phi^i}{\partial x_j} R^j_{\ k} = s_{(i)} \delta^i_{\ k}. \tag{40}$$

Example 12. For equations with $\sigma^i_{\ k}(x,t) = s_0 \delta^i_{\ k}$ (with s_0 a real constant), the second of (38) always reduces to $(\partial \phi^i / \partial x^j) = (R^{-1})^i_{\ j}$. But this implies ϕ must be a linear function of the x, $\phi^i(x,t) = L^i_{\ j}(t) x^j$ with $L \in O(n)$ an orthogonal matrix ($L = R^{-1}$). Thus in this case we are always reduced to the simple equation $L^i_{\ j} f^j(x,t) = f^i(Lx,t)$.

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Group Foliation of Euler Equations in Nonstationary Rotationally Symmetrical Case

Sergey V. GOLOVIN

Lavrentyev Institute of Hydrodynamics, 15 Lavrentyev Ave., 630090 Novosibirsk, Russia
E-mail: *sergey@hydro.nsc.ru*

Euler equations for rotationally symmetrical motions of ideal fluid are considered. Basis of differential invariants for infinite-dimensional part of admitted group is calculated. The basis is used for construction of group foliation of Euler equations. Both automorphic and resolving systems are completed to involution. The resolving part of group foliation inherits finite-dimensional part of group, admitted by Euler equations. It allows us to construct invariant and partially invariant solutions of resolving system. The original functions are restored then by means of integration of automorphic system. Example of such construction is provided.

1 Introduction

The present paper can be regarded as development of the author's previous work published in [1]. Objects of investigations are differential invariants of infinite-dimensional groups, which appear as admissible groups of continuous transformations for various hydrodynamical systems. The main goal is utilization of infinite-dimensional part of admitted groups for construction of new exact solutions and obtaining new information about the observed systems. Here we use an approach based on so-called group foliation (or group stratification) of the system of PDEs with respect to admitted group. This approach proved to be useful in several examples [2, 4–7]. Here we construct a group foliation of Euler equations describing rotationally symmetrical motions of ideal fluid. The base of group foliation is an infinite-dimensional group, which involves two arbitrary functions of time. In comparison with cited articles we observe not a single equation but a system of equations with 4 unknown functions. It causes the main complexity of present work.

Description of classical theory of differential invariants one can find in the books [2,3]. All the necessary information concerned automorphic system and algorithm for construction of group foliation are presented there also. The problem of group foliation construction of given system of equations was first set by Sophus Lie [8]. The group G, admitted by some system of equations E transforms any solution U: $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ into solution again. Thus, in the set of all solutions of system E the equivalence relation appears: two solutions U and U' are equivalent if they are connected by the group G transformation: $U' = TU$, $T \in G$. Each equivalence class is the orbit of one of the solutions under all possible transformations of group G. The problem of group foliation is formulated as follows: for a given system of equations E and a given group G, being admitted by system E, it is required to form a system of equations, which would describe an orbit of any solution (system AG) and a system, which would give an assemblage of all orbit of different solutions (system RE). System AG is named as *automorphic* and has a property that any its solution belongs to the orbit of one solution, i.e. any solution obtained from any other by the action of group G. On the contrary, the *resolving* system RE does not admit the original group and, thus, distinguishes the orbits of different solutions.

In present work we use Ovsiannikov's algorithm [2] for construction of group foliation. This algorithm sufficiently uses the basis of differential invariants for investigated group.

2 Preliminary information

We consider the Euler equations, which describe rotationally symmetrical motions of ideal fluid. Rotational symmetry means that in the cylindrical coordinate system (r, θ, z) all functions are assumed to be independent of the polar angle θ. Thus the system under consideration is the following:

$$\begin{aligned} &u_t + uu_r + wu_z + p_r = \frac{v^2}{r}, \\ &v_t + uv_r + wv_z = -\frac{uv}{r}, \\ &w_t + uw_r + ww_z + p_z = 0, \\ &u_r + \frac{1}{r}u + w_z = 0. \end{aligned} \tag{1}$$

Here u, v and w are velocity components which correspond to (r, θ, z) axes; p is a pressure. The admitted group for system (1) is known [9]. Its finite-dimensional part L_4 is generated by the following operators

$$\begin{aligned} &X_1 = \partial_t, \qquad X_2 = -\frac{1}{r^2 v}\partial_v + \frac{1}{r^2}\partial_p, \qquad X_3 = t\partial_t + r\partial_r + z\partial_z, \\ &X_4 = 2t\partial_t + r\partial_r + z\partial_z - u\partial_u - v\partial_v - w\partial_w - 2p\partial_p. \end{aligned} \tag{2}$$

Operators of infinite-dimensional part L_∞ of admitted group depend on two arbitrary functions of time:

$$Z_f = f(t)\partial_z + \dot{f}(t)\partial_w - \ddot{f}(t)z\partial_p, \qquad H_\varphi = \varphi(t)\partial_p. \tag{3}$$

Further we construct the group foliation of the system (1) with respect to an infinite-dimensional group L_∞ (3).

3 The representation of the solution

The first step is the calculation of the basis of differential invariants and operators of invariant differentiation of the group L_∞.

Lemma 1. *A basis of differential invariants of the group L_∞ can be chosen as follows:*

$$t, \quad r, \quad u, \quad v, \quad W_0 = w_t + ww_z + p_z, \quad W_1 = w_r, \quad W_2 = w_z, \quad P = p_r. \tag{4}$$

Operators of invariant differentiation are

$$\delta_0 = D_t + wD_z, \qquad \delta_1 = D_r, \qquad \delta_2 = D_z. \tag{5}$$

The commutative relations between δ_i are the following:

$$[\delta_0, \delta_1] = -W_1\delta_2, \qquad [\delta_0, \delta_2] = -W_2\delta_2, \qquad [\delta_1, \delta_2] = 0. \tag{6}$$

Proof. The proof is the same as in [1]. ■

As it follows from general theory the Euler equations (1) can be rewritten as a relation between differential invariants only. Since action of operators of invariant differentiation on differential invariant provides differential invariants of higher order we have the following system

$$\begin{aligned} &\delta_0 u + u\delta_1 u + P = \frac{v^2}{r}, \qquad \delta_0 v + u\delta_1 v = -\frac{uv}{r}, \\ &W_0 + uW_1 = 0, \qquad \delta_1 u + \frac{u}{r} + W_2 = 0, \end{aligned} \tag{7}$$

which is equivalent to the original equations (1). The system (7) provides the first part of relations between differential invariants.

According to the general algorithm [2] we have to choose 3 (the same number as the number of original independent variables) of the differential invariants (4) to be the new independent variables. The remaining 5 invariants from the basis further will be regarded as functions of the 3 chosen independent variables.

It is convenient to choose invariants t, r, u as the new independent variables. This choice is valid for all solutions where $u_z \neq 0$. The case $u_z = 0$ will be investigated separately. Due to the third equation of (7) the representation of solution for the group foliation has the following form:

$$w_t + ww_z + p_z = -uW_1(t,r,u), \qquad w_r = W_1(t,r,u), \qquad w_z = W_2(t,r,u),$$
$$p_r = P(t,r,u), \qquad v = V(t,r,u). \tag{8}$$

For given functions V, W_1, W_2 and P equations (7) and (8) provide an overdetermined system of nonlinear PDEs. It is a part of automorphic system AS of group foliation. Compatibility conditions of system (8) which are equations only for invariant functions V, W_1, W_2 and P give a resolving system RS of the group foliation. For convenience we use further the terminology of partially differential equations. Namely, we call functions V, W_1, W_2, P 'invariant' and function u a 'superfluous'.

4 Compatibility conditions

Here we investigate compatibility conditions of the overdetermined system (7), (8). Note that first and fourth equations (7) give the expression for derivatives of the superfluous function u:

$$\delta_0 u = \frac{u^2+V^2}{r} + uW_2 - P,$$
$$u_r = -\frac{u}{r} - W_2. \tag{9}$$

Substituting a representation of v from (8) into the second equation (7) by virtue of (9) we obtain an equation only for invariant functions

$$V_t + uV_r + \left(\frac{1}{r}V^2 - P\right)V_u = -\frac{u}{r}V. \tag{10}$$

Cross-differentiating of the second and third equations of (8) allows us to eliminate function w providing

$$W_{1u}u_z = W_{2r} - W_{2u}\left(\frac{u}{r} + W_2\right). \tag{11}$$

Then we differentiate the first equation of (8) with respect to r and subtract $\delta_0(w_r)$ and $(p_r)_z$, obtained from the second and fourth equations (8). Using equations (9) and commutative relations (6) we find

$$P_u u_z = -W_{1t} + \frac{u}{r}W_1 - uW_{1r} - W_{1u}\left(\frac{1}{r}V^2 - P\right). \tag{12}$$

Elimination of the derivative u_z from relations (11), (12) provides an additional equation for invariant functions only

$$W_{2r}P_u - W_{2u}\left(\frac{u}{r} + W_2\right)P_u + W_{1t}W_{1u} - \frac{u}{r}W_1W_{1u}$$
$$+ W_{1u}^2\left(\frac{1}{r}V^2 - P\right) + uW_{1r}W_{1u} = 0. \tag{13}$$

Next we have to derive a compatibility conditions for the superfluous function u. Hereafter we suppose that $W_{1u} \neq 0$. Case $W_{1u} = 0$ by virtue of (8) corresponds to separation of variables $w = f(t,r) + g(t,z)$. This case is not considered here. Note only that analysis of the particular case $w = f(t,r)$ serves as a topic for paper [10].

According to the above propositions all derivatives of the superfluous function u are determined from relations (9) and (11). Its cross-differentiation taking into consideration commutative relations (6) gives three equations, which involve first-order derivatives of u. Elimination of these derivatives gives three more equations for invariant functions only. Due to a big amount of calculation we present only the result:

$$
\begin{aligned}
&W_1 W_{1u}^{-1}\left(W_{2r} - \left(\frac{u}{r} + W_2\right) W_{2u}\right) = W_{2t} + uW_{2r} + \frac{2}{r}VV_r - P_r - \frac{1}{r}P \qquad (14)\\
&\quad + \left(\frac{1}{r}V^2 - P\right) W_{2u} - \frac{2u}{r}W_2 + W_2 P_u - \frac{2}{r}W_2 V V_u - \frac{2u}{r^2}VV_u + \frac{u}{r}P_u - W_2^2 - \frac{2u^2}{r^2},\\
&-W_{2tr} + \left(\frac{u}{r} + W_2\right) W_{2tu} - \left(\frac{u^2+V^2}{r} + uW_2 - P\right) W_{2ru}\\
&\quad + \left(\frac{u}{r} + W_2\right)\left(\frac{u^2+V^2}{r} + uW_2 - P\right) W_{2uu}\\
&\quad + \left(W_{2t} + uW_{2r} - \frac{2}{r}V\left(\frac{u}{r} + W_2\right) V_u - \left(\frac{u}{r} + W_2\right)\left(\frac{u}{r} - P_u\right)\right) W_{2u}\\
&\quad + \left(\frac{2u}{r} + \frac{2}{r}VV_u - P_u\right) W_{2r} + \left(\frac{1}{r}V^2 - P\right)\left(W_{2u}^2 + \frac{1}{r}W_{2u}\right)\\
&\quad = W_{1u}^{-1}\left(-W_{2r} + \left(\frac{u}{r} + W_2\right) W_{2u}\right)\left(W_{1tu} + \left(\frac{u^2+V^2}{r} + uW_2 - P\right) W_{1uu}\right), \qquad (15)\\
&W_{2rr} - 2\left(\frac{u}{r} + W_2\right) W_{2ru} + \left(\frac{u}{r} + W_2\right)^2 W_{2uu} + \frac{1}{r}W_{2r} + \frac{u}{r^2}W_{2u}\\
&\quad = W_{1u}^{-1}\left(-W_{2r} + \left(\frac{u}{r} + W_2\right) W_{2u}\right)\left(-W_{1ru} + \left(\frac{u}{r} + W_2\right) W_{1uu}\right). \qquad (16)
\end{aligned}
$$

Note that equation (16) involves only functions W_i and their derivatives. System of equations (10), (13), (14)–(16) forms an overdetermined resolving system of 5 equations for 4 functions V, W_1, W_2, P. It is necessary to investigate its compatibility.

Lemma 2. *System of equations* (10), (13), (14)–(16) *is consistent and in involution. Arbitrariness of its general solution is 4 functions of 2 arguments and 9 functions of 1 argument.*

Proof. To proof the lemma we check the Cartan criterion [11]. Having the goal to obtain second-order system of equations we differentiate all first-order equations (10), (13), (14) with respect to t, r, u. The complete second-order system (together with equations (15), (16)) consists of 11 equations including $4C_4^2 = 24$ second-order derivatives of invariant functions. Let us denote that system as E_1. The matrix of coefficients of the second derivatives in E_1 happens to be non-degenerate. We calculate the auxiliary constant $\tau_0 = 24 - 11 = 13$. Next we add the following relations to E_1

$$
V_{ti} = 0, \qquad W_{1ti} = 0, \qquad W_{2ti} = 0, \qquad P_{ti} = 0, \qquad i = t, r, u \qquad (17)
$$

(all second derivatives involving t-differentiation vanish). An extended system E_2 included 23 equations. Rank of matrix constructed from coefficients of second derivatives equals 20. Thus $\tau_1 = 24 - 20 = 4$. Further we add to E_2 the relations

$$
V_{ri} = 0, \qquad W_{1ri} = 0, \qquad W_{2ri} = 0, \qquad P_{ri} = 0, \qquad i = t, r, u \qquad (18)
$$

(all second derivatives containing r derivatives vanish). The obtained system contains 31 equations. Rank of matrix of coefficients of the second derivatives equals 24. Thus $\tau_2 = 24 - 24 = 0$.

The data obtained allows to calculate Cartan characters $\sigma_1 = \tau_0 - \tau_1 = 9$, $\sigma_2 = \tau_1 - \tau_2 = 4$, $\sigma_3 = \tau_2 = 0$. The Cartan number is $Q = \tau_0 + \tau_1 + \tau_2 = 17$. To check the Cartan criterion the third prolongation of the initial second-order system E_1 is required. Cartan criterion is satisfied if the number of 'free' third-order derivatives in prolonged system coincides with Cartan number Q. Actually, the number of all third derivatives of invariant functions is $4C_5^3 = 40$. Prolongation of E_1 allows to calculate the rank of matrix of coefficients of third derivatives. It is equal to 23. Hence $40 - 23 = 17 = Q$, i.e. the Cartan criterion for system (10), (13), (14)–(16) is satisfied. General solution of that system is determined with arbitrariness in σ_i functions of i variables. ■

Summarizing all obtained above we can formulate the following theorem.

Theorem 1. *The group foliation of equations* (1) *with respect to infinite-dimensional group* (3) *in the regular case* $u_z \neq 0$, $w_{rz} \neq 0$ *consists of resolving system* (10), (13), (14)–(16) *for invariant functions* V, W_1, W_2, P *and automorphic system* (8), (9), (11).

5 Special case $u_z = 0$

Here we observe the special case specified in the title of the paragraph. In this case variables t, r, u cannot be chosen as new independent variables for group foliation as it was done above. However it is possible to describe all 'lost' solutions. Actually, the consequence of stated condition is $u = u(t, r)$. From the last equation (1) by integration we obtain that function w linearly depends on z

$$w = -\left(u_r + \frac{u}{r}\right) z + w_0(t, r). \tag{19}$$

We substitute representation (19) into the third equation (1). After integration with respect to z we obtain pressure p to be a quadratic polynomial of variable z:

$$\begin{aligned} &p = a(t,r)z^2 - 2b(t,r)z + p_0(t,r),\\ &a(t,r) = u_{tr} + uu_{rr} + \frac{1}{r}u_t - u_r^2 - \frac{1}{r}uu_r - \frac{2}{r^2}u^2,\\ &b(t,r) = w_{0t} + uw_{0r} - \left(u_r + \frac{1}{r}u\right) w_0. \end{aligned} \tag{20}$$

Substitution of (20) into the first equation (1) allows us to express v^2 as a quadratic polynomial of variable z:

$$v^2 = r\left(u_t + uu_r + a_r z^2 - 2b_r z + p_{0r}\right). \tag{21}$$

Finally, we rewrite the second equation of (1) in the form

$$\left(v^2\right)_t + u\left(v^2\right)_r + w\left(v^2\right)_z = -\frac{2uv^2}{r}. \tag{22}$$

Substituting (19), (21) into (22) we obtain the equation which has a special form: quadratic polynomial of z equals zero. Splitting with respect to z provides a system of 3 equations for 3 unknown functions: u, w_0 and p_0. Solutions of that well-defined system completely determine an investigated class of solutions of Euler equations. It should be noted that these solutions with $u_z = 0$ were retrieved in Pukhnachov's work [12] on the base of the so-called 'heuristic' method.

Remark 1. The only unexamined case left is $w_{rz} = 0$. Its investigation is complicated by the fact that equations (1) do not split after substitution of function w. The higher-order prolongation of the obtained system is required for its completion to involution.

6 Admitted group

The resolving system RS inherits finite-dimensional part of L_4 of the group, admitted by equations (1). Action of L_4 in the space of differential invariants (4) is generated by the following operators:

$$Y_1 = \partial_t, \qquad Y_2 = -\frac{1}{r^2 V}\partial_V - \frac{2}{r^3}\partial_P, \qquad Y_3 = t\partial_t + r\partial_r - W_1\partial_{W_1} - W_2\partial_{W_2} - P\partial_P,$$
$$Y_4 = 2t\partial_t + r\partial_r - u\partial_u - v\partial_v - 2W_1\partial_{W_1} - 2W_2\partial_{W_2} - 3P\partial_P. \tag{23}$$

Group (23) can be used for construction of invariant and partially invariant solutions of the resolving system RS. The velocity components u, v, w and pressure p are restored then by means of integration of automorphic system AS. The optimal system of subalgebras [13, 14] ΘL_4 can be easily constructed. Its representatives are written below.

dim = 4

1. $\{Y_1,\ Y_2,\ Y_3,\ Y_4\}$.

dim = 3

1. $\{Y_1,\ Y_3,\ Y_4\}$, 2. $\{Y_2,\ Y_3,\ Y_4\}$, 3. $\{Y_1,\ Y_2,\ \alpha Y_3 + \beta Y_4;\ \alpha^2 + \beta^2 = 1\}$.

dim = 2

1. $\{Y_1,\ Y_2\}$, 2. $\{Y_3,\ Y_4\}$, 3. $\{Y_1,\ Y_3 + \alpha Y_4\}$, 4. $\{Y_1 + \alpha Y_2,\ 2Y_3 + Y_4\}$,
5. $\{Y_2,\ Y_3 + \alpha Y_4\}$, 6. $\{Y_1,\ \alpha Y_2 + Y_4\}$, 7. $\{Y_2,\ Y_4\}$.

dim = 1

1. $\{Y_1 + \alpha Y_2\}$, 2. $\{Y_2\}$, 3. $\{Y_3 + \alpha Y_4\}$, 4. $\{Y_1 + 2Y_3 - Y_4\}$, 5. $\{\alpha Y_2 + Y_4\}$.

Here $\{Y_1, \ldots, Y_k\}$ denotes a Lie algebra generated by basic operators $Y_1, \ldots, Y_k$; α, β are arbitrary constants.

7 Example of a partially invariant solution

Here we provide an example of a partially invariant solution for resolving system RS. We use the whole 4-dimensional group with generators (23) admitted by the resolving system RS. Invariants of L_4 in the space $\mathbb{R}^7(t, r, u, V, W_1, W_2, P)$ are

$$\frac{rW_1}{u}, \qquad \frac{rW_2}{u}, \qquad \frac{r}{u^2}\left(P - \frac{V^2}{r}\right). \tag{24}$$

All the invariant functions except for V could be expressed from invariants (24). One can construct partially invariant solution of rank 0 and defect 1. The representation of solution of resolving system RS is

$$W_1 = \frac{au}{r}, \qquad W_2 = \frac{bu}{r}, \qquad P = \frac{V^2}{r} + \frac{p_0 u^2}{r}, \qquad a, b, p_0 = \text{const}. \tag{25}$$

The non-invariant superfluous function V supposed to depend on all the variables

$$V = V(t, r, u). \tag{26}$$

The representation of solution (25), (26) is to be substituted into resolving system RS. In accordance with general theory an overdetermined system for function $V(t,r,u)$ and constants a, b, p_0 will appear. This system of 5 nonlinear equations should be completed to involution. It is consistent since it has a trivial solution. However the careful investigation of all cases, which arise during this system's compatibility analysis, is quite complicated.

To avoid this intricate process we start directly from the automorphic system AS given by formulas (8), (9), (11). We substitute the representation of solution (25), (26) into automorphic system AS and repeat all steps which were previously done to obtain the resolving system RS. Now it is simpler since invariant functions have a specific form.

Remark 2. The described algorithm provides a so-called differentially invariant solution. It can be used without any prior information about group foliation of the system (1) starting directly from optimal system of subgroups for admitted group.

Implementation of the algorithm gives the following. The solution is reduced to

$$\begin{aligned} &u = ru(t,z), \qquad v^2 = \frac{k}{r^2} - r^2u^2(t,z),\\ &w = w(t,z), \qquad p = -\frac{k}{2r^2} + p(t,z), \qquad k = \text{const}. \end{aligned} \tag{27}$$

Note that $k = 0$ modulo to Y_2 transformation. Functions $u(t,z)$, $w(t,z)$, $p(t,z)$ satisfy the system

$$u_t + wu_z = -2u^2, \qquad w_z = -2u, \qquad w_t + ww_z + p_z = 0. \tag{28}$$

Equations (28) are reduced to one key equation for function $w(t,z)$ only

$$w_{tz} + ww_{zz} = w_z^2. \tag{29}$$

Equation (29) can be an object of separate investigation. It admits infinite-dimensional group of contact transformation with the following generators:

$$\begin{aligned} &\partial_t, \qquad t\partial_t + z\partial_z, \qquad z\partial_z + w\partial_w, \qquad t^2\partial_t - tz\partial_z - (z+3tw)\partial_w,\\ &h_{w_z}\partial_z + (w_zh_{w_z} - h)\partial_w - h_t\partial_{w_t}. \end{aligned} \tag{30}$$

Here $h = h(t,w_z)$ is an arbitrary function, which satisfies the linear equation

$$h_{tw_z} + (w_z)^2h_{w_zw_z} - w_zh_{w_z} + h = 0. \tag{31}$$

Apparently it indicates that equation (29) is linearizable.

8 Conclusion

We constructed a group foliation of Euler ideal liquid equations in the rotationally symmetrical case. For group foliation we used an infinite-dimensional part of the admitted group. The resolving system RS of group foliation consists of 5 equations (two of them are second-order) for 4 sought functions of 3 variables. This system is in involution. The arbitrariness of its general solution is determined. For any solution of resolving system one can restore original functions by solving a consistent automorphic system AS.

The resolving system inherits finite-dimensional part of initial admitted group. We construct a partially invariant solution for resolving system constructed on the base of inherited transformations. This solution can be observed as a differentially invariant solution for the original Euler equations.

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Invariant and Conditionally Invariant Solutions of Magnetohydrodynamic Equations in (3 + 1) Dimensions

A.M. GRUNDLAND † *and P. PICARD* ‡

† *Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada*
E-mail: *grundlan@crm.umontreal.ca*

‡ *Département de Physique, Université de Montréal, C.P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada*
E-mail: *picard@crm.umontreal.ca*

The symmetry reduction method is used to obtain invariant and conditionally invariant solutions of magnetohydrodynamic equations in (3+1) dimensions. Detail description of the procedure for constructing such solutions is presented. These solutions represent simple and double Riemann waves.

1 Introduction

We consider the magnetohydrodynamic (MHD) equations in (3+1) dimensions which can be written in the matrix evolutionary form

$$\mathrm{u}_t + A^1(\mathrm{u})\mathrm{u}_x + A^2(\mathrm{u})\mathrm{u}_y + A^3(\mathrm{u})\mathrm{u}_z = 0, \qquad \nabla\cdot\vec{H} = 0, \tag{1}$$

where the 8 by 8 matrix functions A^1, A^2 and A^3 have the form

$$A^1 = \begin{pmatrix} u & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u & \kappa p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\rho & u & 0 & 0 & 0 & H_2/\rho & H_3/\rho \\ 0 & 0 & 0 & u & 0 & 0 & -H_1/\rho & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & H_1/\rho \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & H_2 & -H_1 & 0 & 0 & u & 0 \\ 0 & 0 & H_3 & 0 & -H_1 & 0 & 0 & u \end{pmatrix}, \tag{2}$$

$$A^2 = \begin{pmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v & \kappa p & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 0 & -H_2/\rho & 0 & 0 \\ 0 & 1/\rho & 0 & v & 0 & H_1/\rho & 0 & H_3/\rho \\ 0 & 0 & 0 & 0 & v & 0 & 0 & -H_2/\rho \\ 0 & 0 & -H_2 & H_1 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & 0 & H_3 & -H_2 & 0 & 0 & v \end{pmatrix},$$

and

$$A^3 = \begin{pmatrix} w & 0 & 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & w & 0 & 0 & \kappa p & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & -H_3/\rho & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 & -H_3/\rho & 0 \\ 0 & 1/\rho & 0 & 0 & w & H_1/\rho & H_2/\rho & 0 \\ 0 & 0 & -H_3 & 0 & H_1 & w & 0 & 0 \\ 0 & 0 & 0 & -H_3 & H_2 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{pmatrix}. \tag{3}$$

Here ρ and p represent the density and the pressure of the fluid, respectively; κ is an adiabatic exponent; $\vec{\mathrm{v}} = (u, v, w)$ and $\vec{H} = (H_1, H_2, H_3)$ represent the velocity of the fluid and the magnetic field, respectively. This system describes an ideal nonstationary isentropic flow of a compressible conductive fluid (with negligible dissipative effect, and the electrical conductivity of the fluid is assumed to be infinitely large) placed in magnetic fluid $\vec{H}$. The MHD system of equations (1) is composed of nine equations involving eight unknown functions $\mathrm{u} = (\rho, p, \vec{\mathrm{v}}, \vec{H}) \in \mathbb{R}^8$.

The aim of this paper is to analyse two special cases of point symmetry of MHD equations (1) that describe the nonlinear propagation and superposition of waves. The methodological approach accepted here is based on the Lie group methods of infinitesimal transformations [1, 2]. For a comprehensive review of this subject see references [3–5]. Examples of invariant and conditionally invariant solutions of system (1) are presented, and some physical and mathematical consequences are discussed.

2 Conditionally invariant solutions and multiples waves

Consider a properly determined homogeneous quasilinear hyperbolic system of first order partial differential equations in p independent variable $\mathrm{x} = (x^1, \dots, x^p)$ and q dependent variables $\mathrm{u} = (u^1, \dots, u^q)$ of the form

$$A^i(\mathrm{u})\mathrm{u}_i = 0, \qquad i = 1, \dots, p, \tag{4}$$

where $A^1, \dots, A^p$ are $q \times q$ matrix functions of u. A vector λ is said to be a *wave vector* of the system (4) if there exists a non zero function $\lambda(\mathrm{u}) = (\lambda_1(\mathrm{u}), \dots, \lambda_p(\mathrm{u}))$ such that $\ker(\lambda_i A^i) \neq 0$. The function r of x and u is said to be a *Riemann invariant* associated to the wave vector λ if the following relation holds

$$r(\mathrm{x}, \mathrm{u}) = \lambda_i(\mathrm{u})x^i. \tag{5}$$

Now let us fix $1 \leq k \leq p$ linearly independent wave vectors $\lambda^1, \dots, \lambda^k$ with corresponding Riemann invariants

$$r^1(\mathrm{x}, \mathrm{u}) = \lambda^1_i(\mathrm{u})x^i, \dots, r^k(\mathrm{x}, \mathrm{u}) = \lambda^k_i(\mathrm{u})x^i. \tag{6}$$

The equation

$$\mathrm{u} = f\big(r^1(\mathrm{x}, \mathrm{u}), \dots, r^k(\mathrm{x}, \mathrm{u})\big) \tag{7}$$

defines a unique function u(x) on a neighborhood of $\mathrm{x} = 0$, for any function $f : \mathbb{R}^k \to \mathbb{R}^q$, and the Jacobian matrix is given by

$$\frac{\partial u^\alpha}{\partial x^i} = \big(\phi^{-1}(\mathrm{x})\big)^l_j \lambda^j_i\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial f^\alpha}{\partial r^l}\big(r(\mathrm{x}, \mathrm{u})\big), \qquad l, j = 1, \dots, k,\ i = 1, \dots, p,\ \alpha = 1, \dots, q, \tag{8}$$

where the $k \times k$ matrix ϕ takes the form

$$\big(\phi\,(\mathrm{x})\big)^l_j = \delta^l_j - \frac{\partial r^l}{\partial u^\alpha}\,\frac{\partial f^\alpha}{\partial r^j}\,\big(r\,(\mathrm{x},\mathrm{u})\big). \tag{9}$$

Note that the rank of u(x) is at most equal to k. If the vector-valued function

$$\xi_a(\mathrm{u}) = \big(\xi^1_a(\mathrm{u}),\ldots,\xi^p_a(\mathrm{u})\big)^T, \qquad a = 1,\ldots,p-k \tag{10}$$

satisfies the orthogonality condition

$$\lambda^s_i \cdot \xi^i_a = 0, \qquad s = 1,\ldots,k, \quad a = 1,\ldots,p-k \tag{11}$$

then we get

$$\xi^i_a\big(u^\alpha(\mathrm{x})\big)\frac{\partial u^\alpha}{\partial x^i} = 0. \tag{12}$$

Therefore $u^1(\mathrm{x}),\ldots,u^q(\mathrm{x})$ are invariants of the vector field

$$\xi^i_a\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial}{\partial x^i}, \qquad a = 1,\ldots,p-k \tag{13}$$

on $\mathbb{R}^p$. Hence the graph of a solution Γ_f is invariant under the vector field

$$X_a = \xi^i_a(\mathrm{u})\frac{\partial}{\partial x^i}, \qquad a = 1,\ldots,p-k \tag{14}$$

on $\mathbb{R}^p \times \mathbb{R}^q$. Conversely, if u(x) is a q-component function defined on a neighborhood of $\mathrm{x} = 0$ such that the graph Γ_f is invariant under all vector fields (14) with the property (11), then u(x) is the solution of (7) for some function f, because $r^1,\ldots,r^k,u^1,\ldots,u^q$ constitute a complete set of invariants of the Abelian algebra of such vector fields (14). This geometrically characterizes the solutions of the equations of the form (7). If u(x) is the solution of (7) then

$$A^i\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial \mathrm{u}}{\partial x^i}(\mathrm{x}) = \big(\phi^{-1}(\mathrm{x})\big)^l_j\lambda^j_i\big(\mathrm{u}(\mathrm{x})\big)A^i\big(\mathrm{u}(\mathrm{x})\big)\frac{\partial f}{\partial r^l}\big(r\big(\mathrm{x},\mathrm{u}(\mathrm{x})\big)\big) = 0. \tag{15}$$

Now we choose an appropriate coordinate system on space $\mathbb{R}^p \times \mathbb{R}^q$ in order to rectify the vector fields (14). We assume that the k by k matrix

$$\Lambda = \big(\lambda^i_j\big), \qquad 1 \le i,\, j \le k \tag{16}$$

is invertible. The independent vector fields

$$X_{k+1} = \frac{\partial}{\partial x^{k+1}} - \sum_{i,j=1}^{k}\big(\Lambda^{-1}\big)^j_i\lambda^i_{k+1}\frac{\partial}{\partial x^j}, \quad \ldots, \quad X_p = \frac{\partial}{\partial x^p} - \sum_{i,j=1}^{k}\big(\Lambda^{-1}\big)^j_i\lambda^i_p\frac{\partial}{\partial x^j} \tag{17}$$

have the required form (14) with the orthogonality property (7). If we change the independent and dependent variables as follows

$$\overline{x}^1 = r^1(\mathrm{x},\mathrm{u}),\ \ldots,\ \overline{x}^k = r^k(\mathrm{x},\mathrm{u}),\ \overline{x}^{k+1} = x^{k+1},\ \ldots,\ \overline{x}^p = r^p,\ \overline{u}^1 = u^1,\ \ldots,\ \overline{u}^q = u^q \tag{18}$$

then the vector fields (17) take rectified form

$$X_{k+1} = \frac{\partial}{\partial \overline{x}^{k+1}}, \quad \ldots, \quad X_p = \frac{\partial}{\partial \overline{x}^{\,p}}. \tag{19}$$

The graph of the solution Γ_f invariant under $X_{\kappa+1}, \dots, X_p$ is defined by equations of the form

$$\overline{\mathrm{u}}(\overline{\mathrm{x}}) = f\big(\overline{x}^1, \dots, \overline{x}^k\big), \tag{20}$$

where $f : \mathbb{R}^k \to \mathbb{R}^q$ is an arbitrary function. This implies that expression (20) is the general solution of the invariance conditions

$$\overline{\mathrm{u}}_{\overline{k+1}} = 0, \quad \dots, \quad \overline{\mathrm{u}}_{\overline{p}} = 0. \tag{21}$$

The system (4) is described in the coordinates $(\overline{\mathrm{x}}, \overline{\mathrm{u}})$ by

$$\overline{A}^{\,i}\big(\overline{\mathrm{x}}, \overline{\mathrm{u}}, \overline{\mathrm{u}}_{\overline{\mathrm{x}}}\big)\overline{\mathrm{u}}_{\overline{i}} = 0, \tag{22}$$

where,

$$\overline{A}^{\,1} = \frac{Dr^1}{Dx^i}\,A^i, \ \dots, \ \overline{A}^{\,k} = \frac{Dr^k}{Dx^i}A^i, \quad \overline{A}^{\,k+1} = A^{k+1}, \ \dots, \ \overline{A}^{\,p} = A^p. \tag{23}$$

Now, we have

$$\left.\frac{Dr^i}{Dx^i}\right|_{\overline{\mathrm{u}}_{\overline{k+1}},\dots,\overline{\mathrm{u}}_{\overline{p}}=0} = \big(\Phi^{-1}\big)^i_{\ j}\lambda^j_i, \quad \text{where} \quad \big(\Phi^{-1}\big)^i_{\ j} = \delta^i_j - \frac{\partial r^i}{\partial \overline{u}^{\,\alpha}}\overline{u}^\alpha_j, \quad i,j = 1,\dots,k. \tag{24}$$

Adding the invariance conditions (21) to the system (22) we obtain the reduced system of the form

$$\sum_{i,j=1}^{k}\sum_{i=1}^{p}\big(\Phi^{-1}\big)^i_{\ j}\lambda^j_i A^i\overline{\mathrm{u}}_{\overline{i}} = 0, \qquad \overline{\mathrm{u}}_{\overline{k+1}} = 0, \ \dots, \ \overline{\mathrm{u}}_{\overline{p}} = 0, \tag{25}$$

which must then be solved.

3 Double Alfvén-entropic wave

Using a version of the conditional symmetry method presented above we construct an example of a double Alfvén-entropic wave. According to the proposed method we consider the case where two wave vectors λ^1 and λ^2 corresponding to the Alfvén and entropic wave vectors are given by

$$\lambda^1 = \big(\delta_A - (\vec{\lambda}^1 \cdot \vec{\mathrm{v}}), \vec{\lambda}^1\big), \quad \lambda^2 = \big(-(\vec{\lambda}^2 \cdot \vec{\mathrm{v}}), \vec{\lambda}^2\big), \quad \delta_A = \varepsilon\frac{(\vec{H}\cdot\vec{\lambda}^1)}{\sqrt{\rho}}, \quad \varepsilon = \pm 1, \tag{26}$$

$$\vec{\lambda}^1 = (\cos\varphi, \sin\varphi, 0), \quad \vec{\lambda}^2 = (\cos\theta, \sin\theta, 0). \tag{27}$$

We consider additionally two vectors ξ_1, ξ_2 which are orthogonal to the given vectors λ^1 and λ^2. We look for the graph of the double wave solution which is invariant under these vector fields

$$\begin{aligned} X_1 &= \sin(\theta-\varphi)\frac{\partial}{\partial t} - [\delta_A\sin\theta + u\sin(\varphi-\theta)]\frac{\partial}{\partial x} + [\delta_A\cos\theta + v\sin(\theta-\varphi)]\frac{\partial}{\partial y}, \\ X_2 &= \frac{\partial}{\partial z}. \end{aligned} \tag{28}$$

Note that the vector fields X_1 and X_2 form an Abelian distribution on the space $\mathcal{M} \subset \mathbb{R}^4 \times \mathbb{R}^8$, i.e. $[X_1, X_2] = 0$. This requirement constitutes the necessary and sufficient conditions for the existence of a local curvilinear system of coordinates on the integral surface spanned by the fields

X_1 and X_2 in the space $\mathcal{M}$. Hence the unknown functions can be written in the parametric form $\mathrm{u} = f(r,s)$ where r and s are the Riemann invariants of the vector fields X_1 and X_2

$$r = \vec{\lambda}^2 \cdot \vec{\mathrm{x}} - (\vec{\lambda}^2 \cdot \vec{\mathrm{v}}), \qquad s = \vec{\lambda}^1 \cdot \vec{\mathrm{x}} + \big(\delta_A - (\vec{\lambda}^1 \cdot \vec{\mathrm{v}})\big)t. \tag{29}$$

Substituting the unknown functions in term of r and s into the MHD equations (1) we obtain the following solution

$$\begin{aligned} &\rho = \rho(r), \qquad p(r) + \frac{1}{2}\mathcal{H}^2(r) = p_0, \qquad \vec{\mathrm{v}} = \frac{\varepsilon \vec{H}}{\sqrt{\rho(r)}} + w(r)\vec{e}_3, \\ &\vec{H} = \mathcal{H}(r)[\cos\varphi(s)\vec{e}_1 + \sin\varphi(s)\vec{e}_2], \qquad p_0 \in \mathbb{R}, \qquad \varepsilon = \pm 1, \end{aligned} \tag{30}$$

where ρ, p, w, $\mathcal{H}$ and φ are arbitrary functions of their arguments

$$r = y\cos\varphi - x\sin\varphi, \qquad s = x\cos\varphi + y\sin\varphi, \tag{31}$$

and $\vec{e}_i$, $i = 1,2,3$ denote the constant unit vectors along x, y, z axes of $\mathbb{R}^3$. Note that the rank of the solution determined by equations (30) and (31) is equal two. The Alfvén-entropic double wave describes a stationary incompressible fluid flow. We have an equilibrium between hydrodynamic and magnetic pressures resulting from the entropic wave. We notice that the density ρ and the pressure p are unchanged by the Alfvén wave. They do however depend on the Riemann invariant r, which is related to the entropic wave.

4 Spherically invariant solution

We intend now to discuss the construction of nonstationary spherically symmetric solution of the MHD equations (1). For the purpose of this investigation we limit ourselves to the reduction obtained from the subalgebra

$$g_1 = \operatorname{span}\big\{J_1, J_2, J_3\big\}, \tag{32}$$

where the infinitesimal generators are of the form

$$\begin{aligned} J_1 &= z\partial_y - y\partial_z + w\partial_v - v\partial_w + H_3\partial_{H_2} - H_2\partial_{H_3}, \\ J_2 &= x\partial_z - z\partial_x + u\partial_w - w\partial_u + H_1\partial_{H_3} - H_3\partial_{H_1}, \\ J_3 &= y\partial_x - x\partial_y + v\partial_u - u\partial_v + H_2\partial_{H_1} - H_1\partial_{H_2}. \end{aligned} \tag{33}$$

Note that for the subalgebra g_1 the transversality condition is not satisfied

$$\operatorname{rank}\{\Omega_1\} < \operatorname{rank}\{\Omega_2\}, \tag{34}$$

where matrices Ω_1 and Ω_2 are of the form

$$\Omega_1 = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} 0 & z & -y & 0 & w & -v & 0 & H_3 & -H_2 \\ -z & 0 & x & -w & 0 & u & -H_3 & 0 & H_1 \\ y & -x & 0 & v & -u & 0 & H_2 & -H_1 & 0 \end{pmatrix}, \tag{35}$$

and in principle the classical symmetry reduction method cannot be applied. However, there exists a certain domain $\mathcal{D}$ given by

$$u^j = x^j f(t,x,y,z), \qquad H_j = x^j g(t,x,y,z), \qquad j = 1,2,3 \tag{36}$$

for which the rank condition is fulfilled, i.e.,

$$\operatorname{rank}\{\Omega_1\}\big|_{\mathcal{D}} = \operatorname{rank}\{\Omega_2\}\big|_{\mathcal{D}}. \tag{37}$$

In this case the subgroup G_1 acts regularly and transversally and invariant solutions can be constructed [2]. The corresponding invariants of subalgebra g_1 are

$$\begin{aligned} &t, \quad s=\sqrt{x^2+y^2+z^2}, \quad \Phi^1=\sqrt{u^2+v^2+w^2}, \quad \Phi^2=xu+yv+zw,\\ &\Phi^3=\sqrt{H_1^2+H_2^2+H_3^2}, \quad \Phi^4=xH_1+yH_2+zH_3, \quad \Phi^5=\rho, \quad \Phi^6=p. \end{aligned} \tag{38}$$

Note that the condition

$$\operatorname{rank}\left(\frac{\partial\Phi^i}{\partial u^j}\right)=7, \qquad i=1,\dots,6, \quad j=1,\dots,8 \tag{39}$$

implies that the dimension of the graph of the solution Γ_f is not preserved by the action of the symmetry group G_1. Therefore, we cannot obtain invariant solutions from this set of invariants by the classical symmetry reduction method. This is due to the fact that the defect structure δ of the solution with respect the the group G_1 is equal to one, because the rank of the characteristic matrix Q associated to the subalgebra g_1 is equal to one

$$\delta=\operatorname{rank}\left(\mathrm{Q}\big(\mathrm{x},u^{(1)}\big)\right)=1. \tag{40}$$

The graph of the solution sweeps out an orbit of dimension $\dim(G_1\Gamma_f)=p+\delta=5$. The equations determining the orbit of the graph of the solution are given by

$$\begin{aligned} &\rho=R(t,s), \qquad p=A(t,s), \qquad v=\frac{(V-xu)-zw}{y},\\ &w=\frac{z(V-xu)+\varepsilon y\sqrt{[U^2-u^2](y^2+z^2)-(xu-V)^2}}{y^2+z^2}, \qquad H_2=\frac{(X-xH_1)-zH_3}{y^2+z^2},\\ &H_3=\frac{z(Y-xH_1)+\varepsilon y\sqrt{[X^2-H_1^2](y^2+z^2)-(xH_1-Y)^2}}{y^2+z^2}, \qquad \varepsilon=\pm1, \end{aligned} \tag{41}$$

where u and H_1 are arbitrary functions of (t,x,y,z); $\Phi^2=V(t,s)$, $\Phi^3=X(t,s)$, $\Phi^4=Y(t,s)$, $\Phi^5=R(t,s)$ and $\Phi^6=A(t,s)$ are arbitrary functions of their arguments. Assuming that conditions (36) and $p=A_0\rho^\kappa$ hold and substituting (41) into the MHD equations (1) we find the reduced system

$$\begin{aligned} &\frac{\partial R}{\partial s}+sf\frac{\partial R}{\partial s}+\left(3f+s\frac{\partial f}{\partial s}\right)R=0,\\ &\frac{\partial f}{\partial t}+\left(f+s\frac{\partial f}{\partial s}\right)f+\kappa\frac{A_0}{s}R^{(\kappa-2)}\frac{\partial R}{\partial s}=0,\\ &3g+s\frac{\partial g}{\partial s}=0, \qquad A_0\in\mathbb{R}. \end{aligned} \tag{42}$$

Using the method of separation of variables we can solve system (42). We obtain the following invariant solution

$$\begin{aligned} &\rho=\rho_o\left[\frac{s^2}{2\alpha}\frac{d^2\alpha}{dt^2}-\frac{s^2}{\alpha^2}\left(\frac{d\alpha}{dt}\right)^2+C_1\alpha^{3(\kappa-1)}\right]^{1/(\kappa-1)},\\ &p=A_o\rho^\kappa, \qquad u^j=-\frac{x^j}{\alpha}\frac{d\alpha}{dt}, \qquad H_j=\frac{\alpha}{s^3}x^j, \end{aligned} \tag{43}$$

where

$$\rho_0 = \left(\frac{\kappa-1}{\kappa A_0}\right)^{1/(\kappa-1)}, \qquad \alpha(t) = C_2 \exp\left[\int \eta(t')\,dt'\right], \qquad C_1, C_2 \in \mathbb{R}, \quad j = 1,2,3. \tag{44}$$

The function η satisfies the ordinary differential equation

$$\frac{d^2\eta}{dt^2} - (1+3\kappa)\eta\frac{d\eta}{dt} + (3\kappa-1)\eta^3 = 0 \tag{45}$$

which can be solved as follows. First, we introduce the change of variable

$$\zeta(\eta) = \frac{d\eta}{dt}, \tag{46}$$

so that the equation (45) can be rewritten as

$$\zeta\frac{d\zeta}{d\eta} - (1+3\kappa)\eta\zeta + (3\kappa-1)\eta^3 = 0. \tag{47}$$

Using the second change of variable

$$\zeta(\eta) = \eta^2\chi(\tau), \qquad \text{where} \quad \tau = \ln\eta, \tag{48}$$

we solve (47) to finally obtain

$$\tau = -\int \frac{\chi d\chi}{2\chi^2 - (1+3\kappa)\chi + (3\kappa-1)} + \tau_0, \qquad \tau_0 \in \mathbb{R}. \tag{49}$$

The solution (43) represents a nonstationary compressible irrotational fluid flow for which, according Kelvin's Theorem, the circulation around a fluid element is preserved. Note that $\vec{F}_m \cdot \vec{\mathrm{v}} = 0$ which indicates that the interaction effect between the hydrodynamic and magnetic expressions in the MHD equations can be decoupled.

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Symmetry Classification of KdV-Type Nonlinear Evolution Equations

Faruk GÜNGÖR [†], *Victor LAHNO* [‡] *and Renat ZHDANOV* [§]

[†] *Department of Mathematics, Faculty of Science and Letters, Istanbul Technical University, 80626 Istanbul, Turkey*
E-mail: *gungorf@itu.edu.tr*

[‡] *Poltava State Pedagogical University, 2 Ostrograds'kogo Str., 36003 Poltava, Ukraine*
E-mail: *lvi@pdpu.septor.net.ua, laggo@poltava.bank.gov.ua*

[§] *Institute of Mathematics of NAS Ukraine, 3 Tereschenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *renat@imath.kiev.ua*

This paper presents a short summary of results on classification of a class of third-order nonlinear evolution equations with respect to Lie point symmetries. The approach we take is a suitable combination of the concept of equivalence group with the usual infinitesimal technique of Lie based on several results on the structure of low-dimensional Lie algebras.

1 Introduction

We classify third-order KdV-type nonlinear evolution equations containing an arbitrary element F of the form

$$u_t = u_{xxx} + F(x, t, u, u_x, u_{xx}), \tag{1}$$

where F is a smooth real-valued function of its arguments that admits nontrivial symmetry groups.

We adopt the approach applied to the classification of several partial differential equations such as the most general second order quasi-linear heat-conductivity equation and the Schrödinger equation [1–3]. The approach is basically a combination of Lie's infinitesimal method and the use of equivalence transformations and the classification theory of abstract finite-dimensional Lie algebras.

We implement this classification in an algorithmic way in the following steps. The first step of the procedure is to find the most general form of the symmetry group of the equation under study using the standard Lie algorithm [4–6]. This is equivalent to solving a first order system of linear PDEs. In effect, the coefficients of the infinitesimal generator will satisfy certain relations connecting them with the arbitrary term and its derivatives. We then use the direct method (or, equivalently, infinitesimal one) to construct the equivalence group of the equation.

In the second step, we classify realizations of finite-dimensional Lie algebras by vector fields of the above form up to equivalence transformations. To this end, we use various results on the structure of abstract Lie algebras [7–11]. In the last step, after transforming symmetry generators to canonical forms, we proceed to classifying equations that admit nontrivial symmetries. We do this by inserting these generators into the symmetry condition and solving for F. With the increasing dimension of the realized algebras, the form of the arbitrary element will be restricted. Typically we continue this procedure until the arbitrary term figuring in the equation is completely specified. At this stage, the obtained equations are expected to depend on arbitrary parameters at most and it is easily checked by Lie algorithm whether the symmetry algebras obtained can be maximal among all those leaving invariant the equation under study.

The full details of the group classification of (1) can be found in [12].

2 Determining equations and equivalence transformations

Lie algebra of the symmetry group of equation (1) is realized by vector fields of the form

$$X = \tau(x,t,u)\partial_t + \xi(x,t,u)\partial_x + \phi(x,t,u)\partial_u. \tag{2}$$

Applying the standard Lie algorithm we have

Proposition 1. *The symmetry group of the nonlinear equation* (1) *for an arbitrary (fixed) function F is generated by the vector field*

$$X = \tau(t)\partial_t + \left(\frac{\dot\tau}{3}x + \rho(t)\right)\partial_x + \phi(x,t,u)\partial_u, \tag{3}$$

where the functions $\tau(t)$, $\rho(t)$ and $\phi(x,t,u)$ satisfy the determining equation

$$\begin{aligned} &-3u_x\dot\rho - xu_x\ddot\tau - 9u_xu_{xx}\phi_{uu} - 3{u_x}^3\phi_{uuu} \\ &\quad + 3\phi_t - 9u_{xx}\phi_{xu} - 9{u_x}^2\phi_{xuu} - 9u_x\phi_{xxu} - 3\phi_{xxx} + 3\left(\phi_u - \dot\tau\right)F \\ &\quad + \left(2u_{xx}\dot\tau - 3u_{xx}\phi_u - 3{u_x}^2\phi_{uu} - 6u_x\phi_{xu} - 3\phi_{xx}\right)F_{u_{xx}} \\ &\quad + \left(u_x\dot\tau - 3u_x\phi_u - 3\phi_x\right)F_{u_x} - 3\phi F_u - 3\tau F_t - (3\rho + x\dot\tau)F_x = 0. \end{aligned} \tag{4}$$

Here the dot over a symbol stands for time derivative.

Next we construct the local group of point transformations preserving the form of the equation but possibly changing the function F. We require that equation (1) be preserved by local point transformations

$$\tilde t = T(x,t,u), \qquad \tilde x = Y(x,t,u), \qquad \tilde u = U(x,t,u), \tag{5}$$

where

$$\frac{D(T,Y,U)}{D(t,x,u)} \neq 0.$$

This requirement constrains the form of the transformation and we obtain the following assertion

Proposition 2. *The maximal equivalence group $\mathcal{E}$ has the form*

$$\tilde t = T(t), \qquad \tilde x = \dot T^{1/3}x + Y(t), \qquad \tilde u = U(x,t,u), \tag{6}$$

where $\dot T \neq 0$, $U_u \neq 0$.

Proposition 3. *Vector field* (3) *is equivalent within a point transformation of the form* (5) *to one of the following vector fields:*

$$X = \partial_t, \qquad X = \partial_x, \qquad X = \partial_u. \tag{7}$$

Proof. Transformation (6) transform vector field (3) into

$$\begin{aligned} X \to \tilde X = \tau(t)\dot T(t)\partial_{\tilde t} &+ \left[\frac{1}{3}(\tau\dot T^{-1}\ddot T + \dot\tau)(\tilde x - Y) + \tau\dot Y + \rho\dot T^{1/3}\right]\partial_{\tilde x} \\ &+ \left[\tau U_t + \left(\frac{1}{3}\dot\tau x + \rho\right)U_x + \phi U_u\right]\partial_{\tilde u}. \end{aligned} \tag{8}$$

There are two cases to consider:

i) $\phi = 0$. Choose $U = U(u)$ so that we have

$$\tilde{X} = \tau(t)\dot{T}(t)\partial_{\tilde{t}} + \left[\frac{1}{3}(\tau\dot{T}^{-1}\ddot{T} + \dot{\tau})(\tilde{x} - Y) + \tau\dot{Y} + \rho\dot{T}^{1/3}\right]\partial_{\tilde{x}} + \phi U_u \partial_{\tilde{u}}. \tag{9}$$

Now if $\tau = 0$, then $\rho \neq 0$ (otherwise X would be zero) we choose $T(t)$ to satisfy

$$\dot{T} = \rho^{-3}.$$

In this case $\tilde{X}$ is transformed into $\partial_{\tilde{x}}$.

If $\tau \neq 0$, then we choose T and Y to satisfy

$$\dot{T} = \tau^{-1}, \qquad \tau\dot{Y} + \rho\dot{T}^{1/3} = 0.$$

With this choice of T and Y vector field $\tilde{X}$ is transformed into $\partial_{\tilde{t}}$.

ii) $\phi \neq 0$. If $\tau = \rho = 0$ then we can choose U to satisfy $\phi U_u = 1$ so that we have $\tilde{X} = \partial_{\tilde{u}}$. Otherwise, U can be chosen to satisfy

$$\tau U_t + \left(\frac{1}{3}\dot{\tau}x + \rho\right)U_x + \phi U_u = 0.$$

Hence we recover Case i.

Summing up, the vector field (3) is equivalent, up to equivalence under $\mathcal{E}$, to one of the three standard vector fields ∂_x, ∂_t, ∂_u. This completes the proof. ■

3 Classification of equations invariant under semi-simple algebras and algebras having nontrivial Levi decompositions

We start out by classifying equations (1) that admit Lie algebras isomorphic to the Lie algebras having nontrivial Levi decomposition. To achieve this, we construct equations admitting semi-simple Lie algebras as invariance algebra.

The lowest order semi-simple Lie algebras are isomorphic to one of the following three-dimensional algebras:

$$\begin{aligned}
&\mathrm{sl}(2,\mathbb{R}): \quad [X_1, X_3] = -2X_2, \quad [X_1, X_2] = X_1, \quad [X_2, X_3] = X_3;\\
&\mathrm{so}(3): \quad [X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.
\end{aligned}$$

Using these commutation relations one can show that there is no realization of so(3) in terms of vector fields (3). We conclude that we cannot have so(3)-invariant equation of the form (1).

Theorem 1. *There exist no realizations of the algebra* so(3) *in terms of vector fields* (3)*. Hence no equation of the form* (1) *is invariant under* so(3) *algebra.*

There are three inequivalent realizations of the algebra $\mathrm{sl}(2,\mathbb{R})$ by operators of the form (2)

$$\begin{aligned}
&\left\{\partial_t, t\partial_t + \frac{1}{3}x\partial_x, -t^2\partial_t - \frac{2}{3}tx\partial_x\right\},\\
&\left\{\partial_t, t\partial_t + \frac{1}{3}x\partial_x, -t^2\partial_t - \frac{2}{3}tx\partial_x - x^3\partial_u\right\},\\
&\left\{\partial_u, u\partial_u, -u^2\partial_u\right\}.
\end{aligned}$$

Inserting the coefficients of basis operators of the above realizations of the algebra $\mathrm{sl}(2,\mathbb{R})$ into invariance relation (4) and solving and analyzing for their maximality we arrive at the following two theorems.

Theorem 2. *The class of PDEs* (1) *contains two inequivalent equations whose invariance algebra are semi-simple* ($\mathrm{sl}(2,\mathbb{R})$)

$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 + u_x G(x,t);$$
$$u_t = u_{xxx} - x^{-3}\left[2xu_x + \frac{1}{9}x^2u_x^2 - G(\omega_1,\omega_2)\right], \qquad \omega_1 = 3u - xu_x, \quad \omega_2 = 6u - x^2u_{xx}.$$

The maximal invariance algebras of the above equations for arbitrary G are

$$\mathrm{sl}^1(2,\mathbb{R}) = \left\{\partial_u, u\partial_u, -u^2\partial_u\right\};$$
$$\mathrm{sl}^2(2,\mathbb{R}) = \left\{\partial_t, t\partial_t + \frac{1}{3}x\partial_x, -t^2\partial_t - \frac{2}{3}tx\partial_x - x^3\partial_u\right\}.$$

Theorem 3. *Nonlinear equation* (1) *whose invariance algebra is isomorphic to a Lie algebra having non-trivial Levi decomposition is represented by one of the following equations:*

$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 + u_x\tilde{G}(x), \quad \mathrm{sl}^1(2,\mathbb{R}) \oplus \{\partial_t\}; \tag{10}$$
$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 + \lambda x^{-2}u_x, \quad \lambda \neq 0, \quad \mathrm{sl}^1(2,\mathbb{R}) \oplus \left\{\partial_t, t\partial_t + \frac{1}{3}x\partial_x\right\}; \tag{11}$$
$$u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2, \quad \mathrm{sl}^1(2,\mathbb{R}) \oplus \left\{\partial_t, \partial_x, t\partial_t + \frac{1}{3}x\partial_x\right\}; \tag{12}$$
$$u_t = u_{xxx} - 2x^{-2}u_x - \frac{1}{9}x^{-1}u_x^2 + x^{-3}\tilde{G}(\sigma), \quad \sigma = x^2u_{xx} - 2xu_x, \quad \mathrm{sl}^2(2,\mathbb{R}) \oplus \{\partial_u\}, \tag{13}$$

where $\tilde{G}$ is an arbitrary function of x or σ. Moreover, the associated symmetry algebras are maximal.

4 Equations invariant under solvable algebras

4.1 Equations with one-dimensional symmetry algebras

Given an F we assume that equation (1) is invariant under one-dimensional symmetry groups whose canonical forms are generated by translational vector fields ∂_t, ∂_x and ∂_u (Proposition 3):

$$A_{1,1}: \ X_1 = \partial_t, \qquad A_{1,2}: \ X_1 = \partial_x, \qquad A_{1,3}: \ X_1 = \partial_u. \tag{14}$$

The corresponding invariant equations will have the form

$$A_{1,1}: \quad u_t = u_{xxx} + F(x,u,u_x,u_{xx}), \tag{15a}$$
$$A_{1,2}: \quad u_t = u_{xxx} + F(t,u,u_x,u_{xx}), \tag{15b}$$
$$A_{1,3}: \quad u_t = u_{xxx} + F(x,t,u_x,u_{xx}). \tag{15c}$$

4.2 Equations with two-dimensional symmetry algebras

There are two isomorphy classes of two-dimensional Lie algebras, Abelian and non-Abelian satisfying the commutation relations $[X_1,X_2] = \kappa X_2$, $\kappa = 0, 1$. We denote them by $A_{2,1}$ and $A_{2,2}$:

$$A_{2.1}: \ [X_1,X_2] = 0, \qquad A_{2.2}: \ [X_1,X_2] = X_2. \tag{16}$$

We construct all inequivalent realizations of the algebras $A_{2,1}$, $A_{2,2}$ in terms of infinitesimal operators (3) and obtain nine classes of two-dimensional algebras altogether. The representatives of equivalence classes of invariant equations are again found by substituting the coefficients in the symmetry condition (4). By doing this, the form of F in (15) is further restricted and the number of variables of F is reduced from four to three.

Theorem 4. *There exist nine classes of two-dimensional symmetry algebras admitted by equation* (1)*. Those algebras and functions F are represented by*

$$
\begin{aligned}
&A^1_{2,1}:\ X_1=\partial_t,\quad X_2=\partial_x,\quad F=F(u,u_x,u_{xx}),\\
&A^2_{2,1}:\ X_1=\partial_t,\quad X_2=\partial_u,\quad F=F(x,u_x,u_{xx}),\\
&A^3_{2,1}:\ X_1=\partial_x,\quad X_2=\alpha(t)\partial_x+\partial_u,\quad F=-\dot\alpha u u_x+\tilde F(t,u_x,u_{xx}),\\
&A^4_{2,1}:\ X_1=\partial_u,\quad X_2=g(x,t)\partial_u,\quad g_x\neq\text{const},\\
&\qquad F=(g_t-g_{xxx})g_x^{-1}u_x+\tilde F(x,t,\omega),\quad \omega=g_{xx}u_x-g_xu_{xx},\\
&A^1_{2,2}:\ X_1=\partial_t,\quad X_2=-t\partial_t-\frac{x}{3}\partial_x,\\
&\qquad F=x^{-3}\tilde F(u,\omega_1,\omega_2),\quad \omega_1=xu_x,\quad \omega_2=x^2u_{xx},\\
&A^2_{2,2}:\ X_1=-3t\partial_t-x\partial_x,\quad X_2=\partial_x,\\
&\qquad F=t^{-1}\tilde F(u,\omega_1,\omega_2),\quad \omega_1=t^{1/3}u_x,\quad \omega_2=t^{2/3}u_{xx},\\
&A^3_{2,2}:\ X_1=-u\partial_u,\quad X_2=\partial_u,\quad F=u_x\tilde F(x,t,\omega),\quad \omega=u_x^{-1}u_{xx},\\
&A^4_{2,2}:\ X_1=\partial_x-u\partial_u,\quad X_2=\partial_u,\\
&\qquad F=e^{-x}\tilde F(t,\omega_1,\omega_2),\quad \omega_1=e^xu_x,\quad \omega_2=e^xu_{xx},\\
&A^5_{2,2}:\ X_1=\partial_t-u\partial_u,\quad X_2=\partial_u,\\
&\qquad F=u_x\tilde F(x,\omega_1,\omega_2),\quad \omega_1=e^tu_x,\quad \omega_2=e^tu_{xx}.
\end{aligned}
$$

4.3 Equations with three-dimensional symmetry algebras

Two classes of decomposable three-dimensional Lie algebras exist over $\mathbb{R}$. Six classes of indecomposable three-dimensional Lie algebras exist over $\mathbb{R}$.

The decomposable three-dimensional algebras are represented by

$$A_{3,1}=3A_1=A_1\oplus A_2\oplus A_3$$

with $[X_i,X_j]=0$ for $i,j=1,2,3$ and

$$A_{3,2}=A_{2,2}\oplus A_1$$

with $[X_1,X_2]=X_2$, $[X_1,X_3]=0$, $[X_2,X_3]=0$.

A solvable three-dimensional Lie algebra has a two-dimensional Abelian ideal. We choose X_1 and X_2 as basis elements of the ideal. The commutation relations can be represented in the matrix notation

$$\begin{pmatrix}[X_1,X_3]\\ [X_2,X_3]\end{pmatrix}=J\begin{pmatrix}X_1\\ X_2\end{pmatrix},\qquad [X_1,X_2]=0,$$

where $J\in\mathbb{M}^{2\times2}$ can be chosen in its Jordan canonical form.

Over $\mathbb{R}$ we have

$$
\begin{aligned}
&A_{3.3}:\ J=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad A_{3.4}:\ J=\begin{pmatrix}1&0\\1&1\end{pmatrix},\qquad A_{3.5}:\ J=\begin{pmatrix}1&0\\0&1\end{pmatrix},\\
&A_{3.6}:J=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad A_{3.7}:\ J=\begin{pmatrix}1&0\\0&q\end{pmatrix},\quad 0<|q|<1,\\
&A_{3.8}:\ J=\begin{pmatrix}0&-1\\1&0\end{pmatrix},\qquad A_{3.9}:\ J=\begin{pmatrix}q&-1\\1&q\end{pmatrix},\quad q>0.
\end{aligned}
$$

Two of them, $A_{3.7}$ and $A_{3.9}$ depend on a continuous parameter q. Once algebras in the above list have been realized we can proceed to construct invariant equations (see [12] for details). The final result on three-dimensional symmetry algebras can be summed up as a theorem.

Theorem 5. *There are thirty-eight inequivalent three-dimensional solvable symmetry algebras admitted by equation* (1).

The explicit forms of those algebras as well as the associated invariant equations are given in [12].

5 Equations with four-dimensional solvable algebras

Ten isomorphism classes of decomposable four-dimensional Lie algebras and twelve decomposable ones exist. Omitting the details we present the result as a theorem. For the forms of inequivalent realizations of four-dimensional algebras and the corresponding invariant equations the reader is referred to Ref. [12].

Theorem 6. *There exist fifty-two inequivalent four-dimensional symmetry algebras admitted by equation* (1).

Again, the realizations of these algebras and the corresponding functions F are too long to present here. The interested reader should consult Ref. [12].

Finally we mention that we performed group classification of the most general third-order linear PDE. To our surprise we established that the nontrivial symmetry group is four-dimensional at most. However, as confirmed by the results of this work, there are nonlinear equations with larger symmetry groups than that of the linear equation. This situation is not the case for second-order PDEs.

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Applications of Symmetry to General Relativity

B. Kent HARRISON

Physics and Astronomy Department, Brigham Young University, Provo, Utah 84602, USA
E-mail: *bkentharrison@comcast.net, bkh@byu.edu*

The equations of general relativity are highly nonlinear partial differential equations and require special techniques to solve exactly. Symmetry considerations have sometimes helped to find solutions. Three examples will be considered here. The first one considers a scaling symmetry in the equations for perfect fluid with a spherical symmetry, which is of historical importance in understanding neutron stars and black holes. The second involves symmetries in the Ernst equation for stationary axially symmetric fields, which are related to the methods worked out in the 1970s for finding solutions to that equation. The third example is an ongoing investigation into critical gravitational collapse, a topic of current interest, using a symmetry of the equations and other analytic techniques.

1 Introduction

The Einstein equations of general relativity are, in general, second order highly nonlinear partial differential equations. It has been necessary to employ many different techniques in order to obtain exact solutions of these equations, and many are now known. I will discuss here a few examples of the use of symmetry in such investigations. First I will present some cases of historical interest, and then I will treat, more extensively, a current ongoing research problem.

2 Spherically symmetric configurations

My first example has to do with spherically symmetric gravitational fields for a perfect fluid. The first solution of the vacuum Einstein's equations to be found was the Schwarzschild solution,

$$ds^2 = -(1-2m/r)dt^2 + (1-2m/r)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

which is the spherically symmetric solution for a point mass m and which later proved the foundation for the discovery of black holes. In this equation $G = c = 1$. For a perfect fluid, the equations for pressure P, density ρ, and mass m out to radius r take the Oppenheimer–Volkoff form [1],

$$dP/dr = -(P+\rho)\left(m + 4\pi r^3 P\right)/r(r-2m), \tag{1}$$

$$dm/dr = 4\pi r^2 \rho \tag{2}$$

in which there is assumed an equation of state $P = P(\rho)$. For high densities we expect the equation of state to take the form $P = k\rho$, where k is a constant, which takes the value $1/3$ for a Fermi gas of neutrons, protons, and electrons. In this case, one can see that equations (1) and (2) admit a scaling symmetry. If we write $a = m/r$ and $b = 4\pi r^2\rho/3m$ and take $k = 1/3$, these equations become [2]

$$rda/dr = a(3b-1),$$
$$rdb/dr = b[3 - 3b - 4a(3+b)/(1-2a)].$$

Division of the second of these by the first now gives a first order equation for $b(a)$. If $x = 4\pi\rho_0 r^2/3$, where ρ_0 is the density at $r = 0$, then the boundary condition near $r = 0$ ($x = 0$) is $a = x + \cdots$ and $b = 1 - 8x/5 + \cdots = 1 - 8a/5 + \cdots$. Phase plane analysis of this equation shows that there is a stable focus and saddle points. It is clear that $2a < 1$ always, ensuring that there is not a black hole event horizon outside the star. Numerical solution (done here with MAPLE) gives Fig. 1, where the focus is obvious.

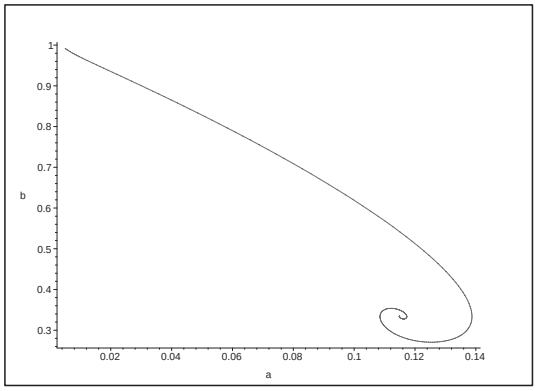

Figure 1. b vs. a for large density.

Density decreases as one moves out through the star and one must use a different equation of state, which means that the a–b trajectory must leave the focus and move down close to the a-axis, ending at the point where the pressure equals zero at the outer edge of the star. At that point one can determine the radius and mass of the star. But the oscillation at the focus is reflected in the final values. The total mass of the star is an oscillatory function of the central density and there is a maximum mass in the neighborhood of 1.4 solar masses, the Chandrasekhar limit. Thus for masses greater than that there will be collapse. There are two families of stable equilibria, white dwarfs and neutron stars, and this points the way toward the existence of collapsed configurations or black holes [2].

3 Axially symmetric configurations

For my second example, I consider stationary (rotating) axially symmetric fields, which include the Schwarzschild solution and the Kerr solution for a spinning point mass. The standard approach uses the metric

$$ds^2 = -f(dt + \omega d\phi)^2 + \rho^2 f^{-1} d\phi^2 + h f^{-1}\left(d\rho^2 + dz^2\right),$$

where f, ω, and h are functions of ρ and z. We can define a new, conjugate variable ψ, often called the twist potential, by (subscripts denote derivatives):

$$\omega_\rho = \rho f^{-2}\phi_z, \qquad \omega_z = -\rho f^{-2}\psi_\rho.$$

The field equations now are

$$\begin{aligned} f(f_{\rho\rho} + f_\rho/\rho + f_{zz}) &= f_\rho^2 + f_z^2 - \psi_\rho^2 - \psi_z^2, \\ f(\psi_{\rho\rho} + \psi_\rho/\rho + \psi_{zz}) &= 2f_\rho\psi_\rho + 2f_z\psi_z \end{aligned} \tag{3}$$

or, where $E = f + i\psi$ (the Ernst potential),

$$\Re(E)\nabla^2 E = (\nabla E)^2,$$

the Ernst equation. One can easily calculate the symmetries of equations (3). (Reference [3] has a similar calculation for the Einstein–Maxwell equations.) There are five symmetries: two on the independent variables alone,

$$\partial_z, \qquad \rho\partial_\rho + z\partial_z \tag{4}$$

and three on the dependent variables alone,

$$\partial_\psi, \qquad f\partial_f + \psi\partial_\psi, \qquad f\psi\partial_f + (1/2)\left(\psi^2 - f^2\right)\partial_\psi. \tag{5}$$

The symmetries (4), in particular the second (scale) symmetry, can be used to look for special solutions of the Ernst equation (which probably have little physical significance.) The symmetries (5), combined and exponentiated, yield the expression

$$E' = (aE + ib)/(icE + d)$$

the Ehlers transformation [4, 5], where a, b, c, and d are real constants. If E is a solution of the Ernst equation, then E' is also. Certain combinations of these variables and their derivatives are potentials for the Ernst equation. W. Kinnersley and his colleagues, through extensive research in the 1970s, pursued relations involving these potentials and higher symmetries of the Ernst equation in the jet space, eventually leading to a method of generating new solutions of the Ernst equation [6]. Alternate methods include the use of Bäcklund transformations, found by the present author and by G. Neugebauer [7, 8], and several other techniques found in the late 1970s.

4 Critical gravitational collapse

The third, and main, example is that of critical gravitational collapse. This work is still unfinished, but several interesting results have been obtained. About ten years ago M. Choptuik published a paper considering a massless scalar field in spherically symmetric gravity [9]. The aim was to investigate possible collapse to a black hole. He assumed a metric as follows:

$$ds^2 = -\alpha^2(r,t)dt^2 + a^2(r,t)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

with a time variable suitable near the origin $T_0 = \int_0^t \alpha(0,t)dt$. If $\phi(r,t)$ is the scalar field and where

$$\Phi = \phi_r, \qquad \Pi = a\phi_t/\alpha,$$

the Einstein equations are

$$\begin{aligned}
&\Phi_t = (\alpha\Pi/a)_r, \qquad \Pi_t = r^{-2}\left(r^2\alpha\Phi/a\right)_r, \\
&\alpha_r/\alpha - a_r/a + \left(1 - a^2\right)/r = 0, \\
&a_r/a + \left(a^2 - 1\right)/(2r) - 2\pi r\left(\Phi^2 + \Pi^2\right) = 0, \\
&a_t/\alpha = 4\pi r\Pi\Phi,
\end{aligned} \tag{6}$$

where subscripts denote derivatives. Choptuik does not use the last equation, since it is consistent with the others in the sense that calculation of the cross derivative of a from the last two equations, with use of the other equations, gives the same value. By analogy to the Schwarzschild case, he takes $a \to \infty$ to be the criterion for a black hole.

4.1 Results from numerical integration

Choptuik integrated the equations numerically, scaling the integration steps carefully, and found several interesting features. (1) For some values of a parameter p in the initial conditions for the scalar field black holes form, but for other values they do not. There is a critical parameter, p^*, for which black holes first form. (2) The precisely critical behavior, in which $p = p^*$, is *universal*, holding for all types of initial data. He uses variables $X = \sqrt{(2\pi)}r\Phi/a$, $Y = \sqrt{(2\pi)}r\Pi/a$, $\rho = \ln(kr)$, $\tau = \ln[k(T_0^* - T_0)]$, and finds the existence of the universal critical solution X^*, Y^*, which occurs in the neighborhood of $a = \infty$ and $r = 0$. (3) There is a scaling behavior. If $Z =$ either X or Y, then this is $Z^*(\rho-\Delta, \tau-\Delta) = Z^*(\rho, \tau)$, or $Z^*(\rho-\tau, \tau-\Delta) = Z^*(\rho-\tau, \tau)$. Thus there is a periodicity in τ with period Δ, where Δ is a universal constant, approximately equal to 3.43, independent of initial conditions. This indicates that the critical solution is somehow dependent only on the original equations, in which there is no natural length or time parameter; Δ thus should naturally come out of the equations. Furthermore, the mass M of the black hole is given by a power law $M = c|p - p^*|^\gamma$, where γ is another universal constant, approximately equal to 0.37, similar to critical phenomena in thermodynamics.

4.2 Symmetry treatment of equations

The scaling behavior suggests that an analytical approach, using scaled variables, might be useful. Choptuik already made the observation that the original equations (6) are invariant under the scaling $r \to kr$, $t \to kt$. We first define

$$A = \sqrt{(2\pi)}\Phi, \qquad B = \sqrt{(2\pi)}\Pi, \qquad C = \alpha/a, \qquad \sigma = \sqrt{(2\pi)}\phi, \qquad U = a^2.$$

Then the basic equations become

$$\begin{aligned} &A = \sigma_r, \qquad B = \sigma_t/C, \\ &A_t = (BC)_r, \qquad B_t = (AC)_r + 2AC/r, \qquad C_r/C = (U-1)/r, \\ &U_r/U + (U-1)/r = 2r\left(A^2 + B^2\right), \qquad U_t/U = 4rABC. \end{aligned} \tag{7}$$

We calculate the symmetries of equations (7) and find that the following are invariant variables: $\xi = rh(t)$, U, $F = rA$, $G = rB$, and $P = Ch^2/h'$, where $h(t)$ is an arbitrary function of t. In terms of Choptuik's variables, we have $F = aX$ and $G = aY$. We note that α^2, which is $-g_{00}$, is $(aC)^2$, which has a factor $\left(h'/h^2\right)^2$. So if we define a new time coordinate t_0 with $dt_0 = -h'dt/h^2$, we see that $h = 1/t_0$. We could make this definition of h, but it is more convenient to define it in terms of Choptuik's function $T_0(t)$ as $h^{-1}(t) = T_0^* - T_0(t)$, where the star indicates critical value. (This then gives the convenient boundary condition $aP = 1$ at r, or $\xi = 0$.) We now express the equations in terms of the variables $\xi = rh(t)$ and $v = t$. (We introduce a new name for t to avoid complications with partial derivative notation.) For any function H, we have $H_r = hH_\xi$ and $H_t = H_v + \xi(h'/h)H_\xi$. We also find it useful to define $P = -\xi Q$, $z = \ln\xi$, $\tau = \text{const} - \ln h$. We now note that τ is the same as Choptuik's τ and $z = \rho - \tau$, so we can look for the periodicity with period 3.43. In terms of derivatives with respect to z and τ (which do not appear explicitly), the equations take the following nice form

$$\begin{aligned} &F_\tau = (F + QG)_z, \qquad G_\tau = (G + QF)_z + 2QF, \qquad Q_z = Q(U-2), \\ &U_z/U = 2F^2 + 2G^2 + 1 - U, \qquad U_\tau/U = 4QFG + 2F^2 + 2G^2 + 1 - U. \end{aligned} \tag{8}$$

4.3 Approaches to solution

We now comment on several avenues of approach. (1) If we assume no τ dependence, then the last equation of (8) gives

$$U = 1 + 2F^2 + 2G^2 + 4QFG,$$

and the first equation gives

$$F = k - QG,$$

where k is a constant. The fourth equation is satisfied automatically. We get

$$Q_z = Q(U - 2)$$

and

$$G_z = Q\left(1 - Q^2\right)^{-1}[U(2QG - k) - 2QG]$$

so we can write an equation for $G(Q)$:

$$dG/dQ = \left(1 - Q^2\right)^{-1}(U - 2)^{-1}[U(2QG - k) - 2QG], \tag{9}$$

where U is given above in terms of G and Q. Equation (9) for $G(Q)$ can be put in various forms. However, I will not pursue those here, since we really want to search for the periodicity in τ.

In our search for periodic behavior in τ, we note that it can be exhibited either in linear equations resembling simple harmonic oscillator equations – which would have variable amplitude – or in nonlinear cases with a limit cycle. The nonlinear character of the equations suggests that the latter is the case; indeed, C. Gundlach in a review paper [10], sketches qualitative limit cycles – which, however, may actually change to nonperiodic behavior.

(2) In the Schwarzschild solution, $\alpha^2 = 1/a^2 = 1 - 2m/r$, for constant geometrized mass m. That suggests that near that solution, we might suspect that $\alpha \propto 1/a$, or that $C \propto \alpha/a \propto 1/a^2 = 1/U$. For U large, then, C (or P or Q) should be small. So let us consider an expansion of equations (8) in powers of Q. We first define $V = UQ$, which satisfies

$$V_z = V\left(2F^2 + 2G^2 - 1\right),$$

with

$$Q_z = V - 2Q.$$

We ignore the U_τ equation. Expand G, F, V:

$$G = a + bQ + \cdots, \qquad F = c + dQ + \cdots, \qquad V = f + gQ + \cdots,$$

where $a, b, \ldots, g$ are functions of τ. We get, from the expansion,

$$\dot{c} = f(a + d), \qquad \dot{d} = (a + d)(g - 2), \qquad \dot{a} = f(b + c),$$
$$\dot{b} = (b + c)(g - 2) + 2c, \qquad g = 2a^2 + 2c^2 - 1,$$

where the dot represents differentiation with respect to τ. There is another equation for g which we ignore. We will also assume f to be a constant. We solve the first and third equations for d and b and substitute them and g into the second and fourth equations. We get

$$\ddot{a} = \dot{a}\left(2a^2 + 2c^2 - 3\right) + f(\dot{c} + 2c),$$
$$\ddot{c} = \dot{c}\left(2a^2 + 2c^2 - 3\right) + f\dot{a}.$$

We integrate these with MAPLE. For $f = 20$, we get oscillatory behavior with a period of about 2.5, but not a perfect limit cycle in a and c (Fig. 2). For $f = 200$, we are closer (Fig. 3). For $f = 225$, we get what appears to be a perfect limit cycle, with period about 1.8 (Figs. 4 and 5).

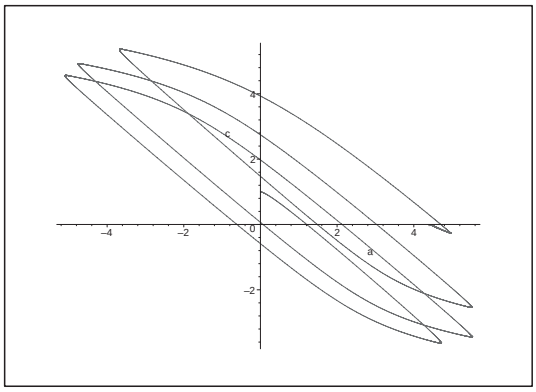

Figure 2. c vs. a for $f = 20$.

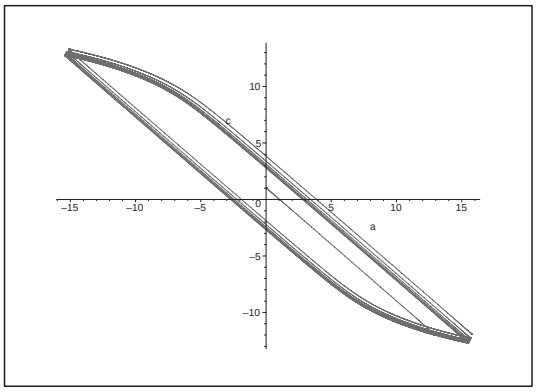

Figure 3. c vs. a for $f = 200$.

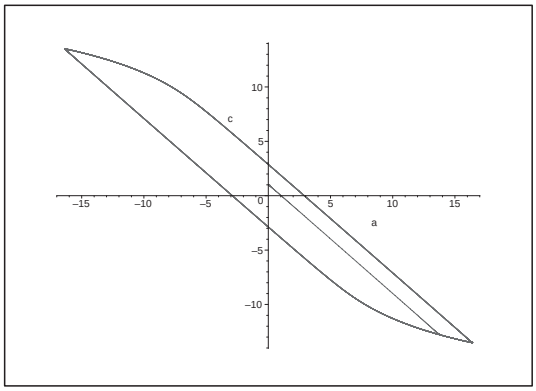

Figure 4. c vs. a for $f = 225$.

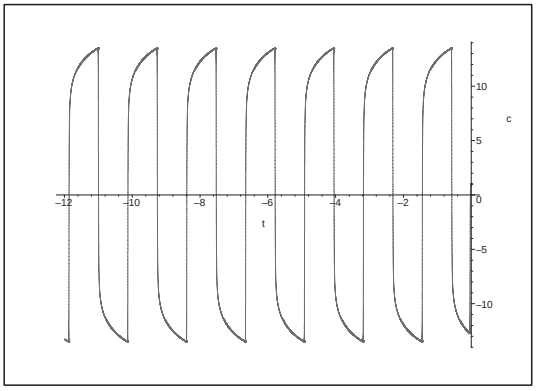

Figure 5. c vs. τ (shown as t) for $f = 225$.

There are several problems associated with this calculation. We ignored the U_τ equation. A closely related matter is that we ignored any τ dependence in Q. We also ignored the second equation for g, which appears to be inconsistent with other equations, and we assumed f to be constant. There is no particular obvious reason why the value of 225 for f should be special, and the period obtained is not 3.4 but is roughly half that. Nevertheless, the periodic behavior of a and c strongly suggests that we may be on the right track.

(3) In order to clear up some of the problems of the last treatment, we take our functions F, etc., to be functions of $Q(z, \tau)$ and v $(= \tau)$. Then for any function H, we have $H_z = H_Q Q_z = H_Q Q(U - 2)$ and $H_\tau = H_Q Q_\tau + H_v$. If we put

$$\beta = 2F^2 + 2G^2 + 1 - U,$$

then from the U equations we can find

$$U_Q = \beta U/[Q(U - 2)]$$

and

$$Q_\tau = Q(U - 2)(4QFG + \beta - U_v/U)/\beta.$$

We substitute for Q_τ in the two equations for F and G and get rather messy equations involving Q, F, G, and the derivatives of F and G with respect to v and Q. We expand F and G in powers of Q, with coefficients as functions of v. We write

$$U = a/Q + b + \cdots, \qquad F = d + eQ + \cdots, \qquad G = h + mQ + \cdots.$$

We find, where the dot indicates $d/dv = d/d\tau$,

$$b = 2d^2 + 2h^2 - 1, \qquad \dot{d} = ha, \qquad \dot{h} = da,$$

and equations for e and m,

$$a\dot{e} + e\dot{a} = ha(b-2) + 2ma^2,$$
$$a\dot{m} + m\dot{a} = dab + 2ea^2.$$

Thus d, for example, obeys

$$\ddot{d} - \dot{a}\dot{d}/a - a^2 d = 0,$$

indicating exponential instead of oscillatory behavior. Inspection of the equations indicates that the change results precisely from the inclusion of Q_τ.

(4) We now note that it is more natural to consider an expansion in $1/U$ than in Q, since U is the variable considered to be large near a black hole. Furthermore, it is very convenient to do so since we have expressions for both U_z and U_τ (8). For any $H(z,\tau)$, we convert it to $H(U,v)$, where $v = \tau$. This gives

$$H_z = H_U U_z = H_U U\left(2F^2 + 2G^2 + 1 - U\right)$$

and

$$H_\tau = H_v + H_U U_\tau = H_v + H_U U\left(4QFG + 2F^2 + 2G^2 + 1 - U\right).$$

The equation for Q_z now becomes

$$Q_U U\left(2F^2 + 2G^2 + 1 - U\right) = Q(U-2).$$

This expression can be used to replace Q_U in the F and G equations when it occurs. Those equations now become

$$F_v + 4QUFGF_U = QU\left(2F^2 + 2G^2 + 1 - U\right)G_U + QG(U-2),$$
$$G_v + 4QUFGG_\tau = QU\left(2F^2 + 2G^2 + 1 - U\right)F_U + QFU.$$

It will be noted that Q appears as a multiplier in all terms except the v derivatives, and U appears in most terms. We define $R = QU$ and rewrite the equations as

$$R_U/R = \left(2F^2 + 2G^2 - 1\right)/[U\left(2F^2 + 2G^2 + 1 - U\right)],$$
$$F_v/R = \left(2F^2 + 2G^2 + 1 - U\right)G_U - 4FGF_U + G(1 - 2/U),$$
$$G_v/R = \left(2F^2 + 2G^2 + 1 - U\right)F_U - 4FGG_U + F.$$

We see that the integration of the first equation gives an additive function $\ln b(v)$ in $\ln R$, or a multiplicative function $b(v)$ in R. This can be removed in the second and third equations simply by redefining the time variable as $w = \int b(v)dv$. This removes the ambiguity in the v $(= \tau)$ dependence of Q $(= R/U)$. Inspection now shows that it is advantageous to define $x = F + G$ and $y = F - G$. The equations are now simplified to

$$R_U/R = \left(x^2 + y^2 - 1\right)/[U\left(x^2 + y^2 + 1 - U\right)],$$
$$x_v/R = 2y^2 x_U + (1-U)\left(x_U - x/U\right) + y/U, \tag{10}$$
$$y_v/R = -2x^2 y_U - (1-U)\left(y_U - y/U\right) - x/U.$$

These are still exact. If they could be solved, then x, y, and R could be substituted into the U_z equation to find the z dependence of U, then the τ $(= v)$ dependence could be found from the U_τ equation. The obvious approach now is to expand in powers of $1/U$. Unfortunately, this does not give oscillatory behavior for x and y, but exponential behavior instead, just as in the expansion for small Q. Variations of approach do not solve the problem. It may be that we have an essential singularity, or to say it another way, that we should assume x and y involve, say, $\exp(-U)$. It is not clear how to investigate this possibility.

We try to solve the x and y equations (10) by a brute force approach. We assume

$$x = \phi(w)S(U), \qquad y = \psi(w)Y(U),$$

substitute into (10), and assume that ϕ and ψ satisfy equations that are purely functions of w, as follows (prime indicates d/dw):

$$\phi' = a\psi^2\phi + b\phi + c\psi,$$
$$\psi' = d\phi^2\psi + f\psi + g\phi,$$

where a, b, c, d, f, g are constants. We substitute these equations into (10) and equate coefficients, thus obtaining six equations for the two functions S and Y. If we assume $x^2 + y^2$ large, we get $R = U$, and then we find some consistency in these six equations, provided that $Y = cS$, $gc = -1$, $a = -dc^2$, and $f = -b$. If we solve for d, f, and g and substitute in the second equation, we find

$$\psi' = -(a/c^2)\phi^2\psi - b\psi - \phi/c.$$

If we let $\phi = c/\sqrt{a}\phi_0$ and $\psi = 1/\sqrt{a}\psi_0$, we see that we may scale a and c away. So just put $a = c = 1$, and we get

$$\phi' = \psi^2\phi + b\phi + \psi,$$
$$\psi' = -\phi^2\psi - b\psi - \phi. \tag{11}$$

We get inconsistent equations for $S(U)$. One of these is

$$S_U/S = (1 - U + b)/[U(1 - U)],$$

yielding

$$S = kU[U/(U - 1)]^b$$

for some constant k. The other is

$$2SS_U = 1/U,$$

yielding $S^2 = \ln U +$ const. We ignore the second equation and note that for large U, the first gives approximately $S = kU$.

We explore the ϕ, ψ equations (11). In the phase plane of ϕ and ψ the origin is a critical point, a center. We can actually solve (11) in the phase plane exactly, finding

$$\phi^2 + \psi^2 + 2b\ln(1 + \phi\psi) = \lambda^2,$$

where λ is a constant. If $b = 0$ this is exactly a circle. Integration of the time equations (Figs. 6, 7) shows that we get oscillations in w and a limit cycle, which is nearly circular for small b. λ measures the amplitude. For small b and $\lambda = 1$ (fixed by the initial conditions), the period is about 7.2. Calculation from the exact solution indicates that the period should

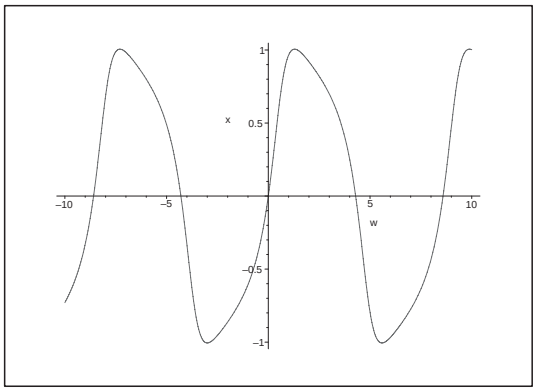

Figure 6. ϕ (shown as x) vs. t for $b = 0.1$.

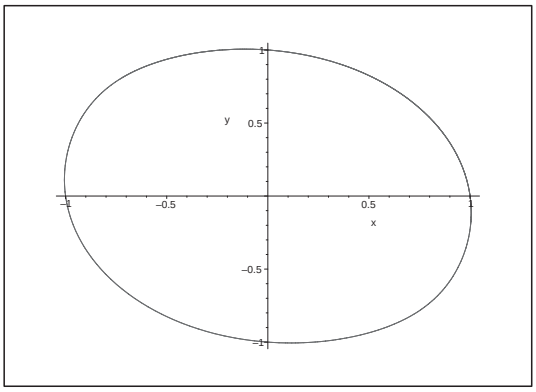

Figure 7. ψ (shown as y) vs. ϕ (shown as x) for $b = 0.1$.

be about 2π. The cubic terms in (10) appear to be proportional to both U and λ, suggesting that λ scales with $1/U$. These oscillations are for constant U. We really want constant z, the original scaled variable. Attempts to include the time dependence of U produce the exponential behavior seen before.

Another approach proceeds as follows. Let x and y in equation (10) be functions of w only and write, temporarily, $\alpha = x^2 + y^2$. Treat α as a constant and integrate the R equation. We get, exactly,

$$R = (1 - (\alpha + 1)/U)^{\beta},$$

where

$$\beta = (1 - \alpha)/(1 + \alpha).$$

Expansion for large U gives approximately

$$R = 1 + (\alpha - 1)/U.$$

Put $1/U = b$, which is now to be treated as a parameter, and put this into the x and y equations in (10) along with the expression for α, keep terms of order b, and get:

$$x_w = x + b\left[\left(x^2 + y^2 - 2\right) + y\right],$$
$$y_w = -y - b\left[\left(x^2 + y^2 - 2\right) + x\right].$$

These equations have interesting behavior. If $x^2 + y^2$ is small, the equations may be treated as linear. We find then that the origin in (x, y) phase space is a center if $1 > b > 1/3$. In that region, if we start with small x and y they remain small and we get linear equations, with period depending on b, and arbitrary amplitude. For $b > 1$, x and y grow and we get a nonlinear oscillation. For large enough b, the period is approximately proportional to $15/b$ and the maximum amplitude of x and y is unity (Figs. 8, 9).

Instead of using the approximate value of R, we can use the exact value (being careful to take the absolute value of the quantity in parentheses before raising to the power. Then

$$x_w = R[x + b(y - x)],$$
$$y_w = -R[y + b(x - y)].$$

Integration of these equations gives some very nice limit cycles, but the period varies with the choice of b and of the initial conditions (Figs. 10–12).

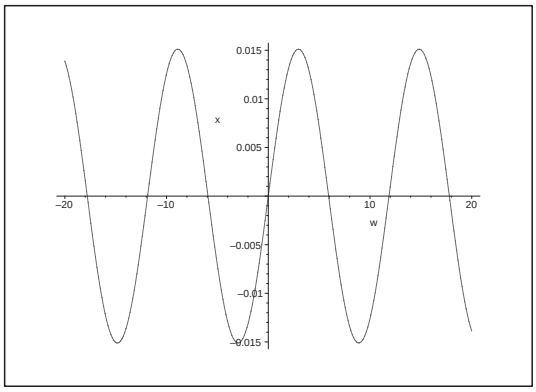

Figure 8. x vs. w, $b = 0.8$, approximate R.

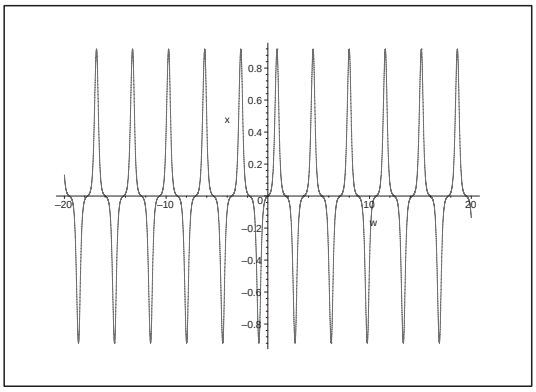

Figure 9. x vs. w, $b = 5$, approximate R.

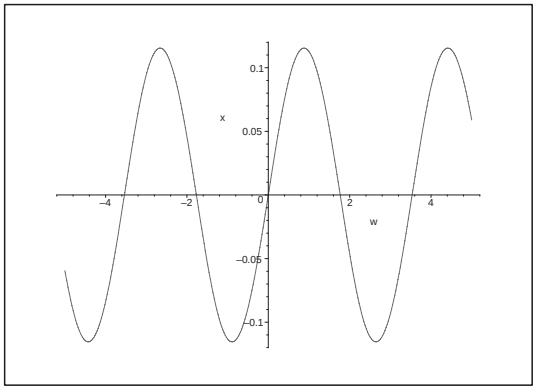

Figure 10. x vs. w, $b = 2$, $y_0 = 0.1$, exact R.

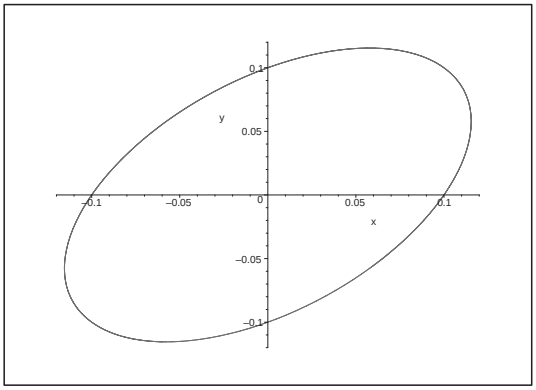

Figure 11. y vs. x, $b = 2$, $y_0 = 0.1$, exact R.

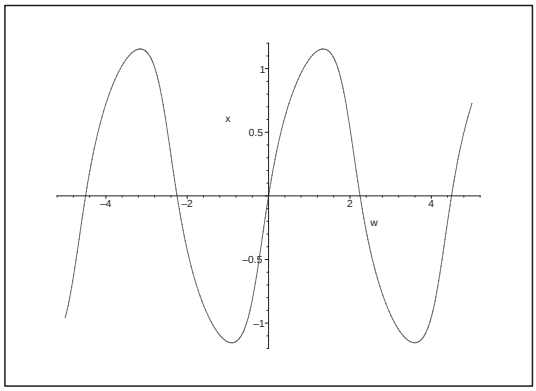

Figure 12. x vs. w, $b = 2$, $y_0 = 1$, exact R.

4.4 Summary of results

In summary, we note several features in this treatment. First, the original equations have been expressed with τ and the scaled variable z as independent variables. Second, in the τ-independent case, the equations may be reduced to a single first order equation. The significance of this case is still unclear. Third, changing the independent variables to $w\ (= \tau)$ and U separates the equations into three equations for R, x, and y and two for the derivatives of U – a convenient form. Fourth, oscillatory behavior is suggested in four different ways. The amplitude in these may scale as $1/U$. Each way has unacceptable assumptions. Trying to remove the assumptions produces exponential behavior. It may be that, as indicated by Gundlach [10] and as mentioned

earlier, there is an epoch of oscillation followed by an epoch of final nonoscillatory behavior, which might be exponential. One would think, however, that there still should be a way of finding the period $\Delta = 3.43$ analytically. This has not yet been done successfully.

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The author expresses appreciation to Eric W. Hirschmann of the Brigham Young University Department of Physics and Astronomy, who brought this problem to his attention and with whom a number of useful discussions have been held.

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Invariants for Evolution Equations

Nail H. IBRAGIMOV † *and Christodoulos SOPHOCLEOUS* ‡

† *Centre ALGA: Advances in Lie Group Analysis, Department of Health, Science and Mathematics, Blekinge Institute of Technology, 371 79 Karlskrona, Sweden*
E-mail: *nib@bth.se*

‡ *Department of Mathematics and Statistics, University of Cyprus, CY 1678 Nicosia, Cyprus*
E-mail: *christod@ucy.ac.cy*

In the spirit of the recent work of Ibragimov [1] who adopted the infinitesimal method for calculating invariants of families of differential equations using the equivalence groups, we apply the method to evolution type equations of the form $u_t = f(x,u)u_{xx} + g(x,u,u_x)$. We show that the equivalence Lie algebra admitted by this equation has two functionally independent differential invariants of the second order.

1 Introduction

We consider evolution equations of the form

$$u_t = f(x,u)u_{xx} + g(x,u,u_x). \tag{1}$$

A number of many special cases in this class of equations have been successfully used to model physical problems. Such example is the nonlinear diffusion equation $u_t = [D(u)u_x]_x$. Group properties of this equation were studied by Ovsiannikov [2]. Other examples of such equations that appear in the literature are $u_t = [g(x)D(u)u_x]_x$, $u_t = [g(x)D(u)u_x]_x - K(u)u_x$, $u_t = (u^n)_{xx} + g(x)u^m + f(x)u^s u_x$, etc.

It can be shown that equations (1) admit equivalence transformation

$$x' = P(x), \qquad t' = c_1 t + c_2, \qquad u' = R(x,u) \tag{2}$$

with

$$f' = \frac{P_x^2 f}{c_1}, \qquad g' = \frac{P_x R_u g + (P_{xx}R_u u_x + P_{xx}R_x - 2P_x R_{ux}u_x - P_x R_{uu}u_x^2 - P_x R_{xx})f}{c_1 P_x}.$$

If we set $P(x) = x + \epsilon\phi(x)$ and $R(x,u) = u + \epsilon\psi(x,u)$, we can write the above transformations in infinitesimal form. That is, in the form

$$Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial u_x} + \mu\frac{\partial}{\partial f} + \nu\frac{\partial}{\partial g}, \tag{3}$$

where ξ^1, ξ^2 and η depend on t, x and u, while μ and ν depend on t, x, u, u_x, f and g, and ζ is given by $\zeta = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2)$. The operator D_x is the total derivative with respect to x.

We deduce that the class of equations (1) has an infinite continuous group of equivalence transformations generated by the infinite-dimensional Lie algebra which is spanned by the ope-

rators:

$$Y_1 = \frac{\partial}{\partial t}, \qquad Y_2 = t\frac{\partial}{\partial t} - f\frac{\partial}{\partial f} - g\frac{\partial}{\partial g},$$
$$Y_\phi = \phi(x)\frac{\partial}{\partial x} - \phi' u_x \frac{\partial}{\partial u_x} + 2\phi' f\frac{\partial}{\partial f} + \phi'' f u_x \frac{\partial}{\partial g},$$
$$Y_\psi = \psi(x,u)\frac{\partial}{\partial u} + (\psi_x + \psi_u u_x)\frac{\partial}{\partial u_x} + \left[\psi_u g - (\psi_{uu}u_x^2 + 2\psi_{xu}u_x + \psi_{xx})f\right]\frac{\partial}{\partial g}. \tag{4}$$

In this paper we calculate differential invariants of equivalence transformations of equations (1) by using the infinitesimal method for calculations of invariants of families of equations developed in [3]. In the following three sections we consider the problem of classifying differential invariants of equations (1) of zero, first and second order.

2 Differential invariants of order zero

We search for invariants of order zero. That is, invariants of the form

$$J = J(t, x, u, u_x, f, g).$$

We apply the invariant test $Y(J) = 0$ to the operators Y_1, Y_2, Y_ϕ and Y_ψ and using the fact that $\phi(x)$ and $\psi(x,u)$ are arbitrary functions, we obtain $J = \text{const}$. Hence, equations (1) do not admit differential invariants of order zero.

3 Differential invariants of first order

In order to determine differential invariants of the first order,

$$J = J(t, x, u, u_x, f, g, f_x, f_u, g_x, g_u, g_{u_x})$$

we need to consider the first prolongation of Y,

$$Y^{(1)} = Y + \mu^x\frac{\partial}{\partial f_x} + \mu^u\frac{\partial}{\partial f_u} + \nu^x\frac{\partial}{\partial g_x} + \nu^u\frac{\partial}{\partial g_u} + \nu^{u_x}\frac{\partial}{\partial g_{u_x}},$$

where

$$\mu^k = \tilde{D}_k(\mu) - f_x\tilde{D}_k(\xi^2) - f_u\tilde{D}_k(\eta), \qquad k = x,\ u,$$
$$\nu^k = \tilde{D}_k(\nu) - g_x\tilde{D}_k(\xi^2) - g_u\tilde{D}_k(\eta) - g_{u_x}\tilde{D}_k(\zeta), \qquad k = x,\ u,\ u_x, \tag{5}$$

where $\tilde{D}_x$, $\tilde{D}_u$ and $\tilde{D}_{u_x}$ denote the total derivatives with respect to x, u and u_x:

$$\tilde{D}_x = \frac{\partial}{\partial x} + f_x\frac{\partial}{\partial f} + g_x\frac{\partial}{\partial g} + f_{xx}\frac{\partial}{\partial f_x} + f_{xu}\frac{\partial}{\partial f_u} + g_{xx}\frac{\partial}{\partial g_x} + g_{xu}\frac{\partial}{\partial g_u} + g_{xu_x}\frac{\partial}{\partial g_{u_x}} + \cdots,$$
$$\tilde{D}_u = \frac{\partial}{\partial u} + f_u\frac{\partial}{\partial f} + g_u\frac{\partial}{\partial g} + f_{xu}\frac{\partial}{\partial f_x} + f_{uu}\frac{\partial}{\partial f_u} + g_{xu}\frac{\partial}{\partial g_x} + g_{uu}\frac{\partial}{\partial g_u} + g_{uu_x}\frac{\partial}{\partial g_{u_x}} + \cdots,$$
$$\tilde{D}_{u_x} = \frac{\partial}{\partial u_x} + f_{u_x}\frac{\partial}{\partial f} + g_{u_x}\frac{\partial}{\partial g} + f_{xu_x}\frac{\partial}{\partial f_x} + f_{uu_x}\frac{\partial}{\partial f_u} + g_{xu_x}\frac{\partial}{\partial g_x} + g_{uu_x}\frac{\partial}{\partial g_u} + g_{u_xu_x}\frac{\partial}{\partial g_{u_x}} + \cdots. \tag{6}$$

Using the formulae (5) and (6) we obtain the first extension of the generators Y_1, Y_2, Y_ϕ, Y_ψ given by equations (4):

$$
\begin{aligned}
&Y_1^{(1)} = Y_1, \qquad Y_2^{(1)} = Y_2 - f_x\frac{\partial}{\partial f_x} - f_u\frac{\partial}{\partial f_u} - g_x\frac{\partial}{\partial g_x} - g_u\frac{\partial}{\partial g_u} - g_{u_x}\frac{\partial}{\partial g_{u_x}},\\
&Y_\phi^{(1)} = Y_\phi + (2\phi''f + \phi' f_x)\frac{\partial}{\partial f_x} + 2\phi' f_u\frac{\partial}{\partial f_u}\\
&\qquad + (u_x\phi'''f + u_x\phi'' f_x - g_x\phi' + \phi'' u_x g_{u_x})\frac{\partial}{\partial g_x} + u_x\phi'' f_u\frac{\partial}{\partial g_u} + (\phi'' f + \phi' g_{u_x})\frac{\partial}{\partial g_{u_x}},\\
&Y_\psi^{(1)} = Y_\psi - f_u\psi_x\frac{\partial}{\partial f_x} - f_u\psi_u\frac{\partial}{\partial f_u} + [\psi_{xu} - (\psi_{xuu}u_x^2 + 2\psi_{xxu}u_x + \psi_{xxx})f\\
&\qquad - (\psi_{uu}u_x^2 + 2\psi_{xu}u_x + \psi_{xx})f_x + \psi_u g_x - \psi_x g_u - (\psi_{xx} + \psi_{xu}u_x)g_{u_x}]\frac{\partial}{\partial g_x}\\
&\qquad + [\psi_{uu}g - (\psi_{uuu}u_x^2 + 2\psi_{xuu}u_x + \psi_{xxu}) - (\psi_{uu}u_x^2 + 2\psi xuu_x + \psi_{xx})f_u\\
&\qquad - (\psi_{uu}u_x + \psi_{xu})g_{u_x}]\frac{\partial}{\partial g_u} - 2(\psi_{uu}u_x + \psi_{xu})f\frac{\partial}{\partial g_{u_x}}. \qquad (7)
\end{aligned}
$$

We note that $Y_1^{(n)} = Y_1$. Hence for any order of differential invariants $J_t = 0$.

Now from the differential invariant test $Y^{(1)}(J) = 0$, we get three identities

$$E_2 = Y_2^{(1)}(J) = 0, \qquad E_\phi = Y_\phi^{(1)}(J) = 0, \qquad E_\psi = Y_\psi^{(1)}(J) = 0. \qquad (8)$$

Since $\phi(x)$ and $\psi(x,u)$ are arbitrary functions, coefficients of ϕ in $E_\phi = 0$ and ψ in $E_\psi = 0$ give $J_x = J_u = 0$. Now coefficients of ψ_{uuu}, ψ_{xuu}, ψ_{uu} and ψ_{xu} give $J_{g_u} = J_{g_x} = J_g = J_{g_{u_x}} = 0$. Coefficient of ϕ'' in $E_\phi = 0$ gives $J_{f_x} = 0$ and coefficient of ψ_x in $E_\psi = 0$ gives $J_{u_x} = 0$. Hence, $J = J(f, f_u)$ and equations (8) read

$$E_1 = -\left(f\frac{\partial J}{\partial f} + f_u\frac{\partial J}{\partial f_u}\right) = 0, \qquad E_\phi = 2\left(f\frac{\partial J}{\partial f} + f_u\frac{\partial J}{\partial f_u}\right)\phi' = 0, \qquad E_\psi = f_u\frac{\partial J}{\partial f_u}\psi_u = 0.$$

If $f_u \neq 0$ from the above relations we deduce that $J_{f_u} = J_f = 0$ and therefore equations (1) do not admit differential invariant of the first order. However the equation

$$f_u = 0 \qquad (9)$$

is invariant under the group which is spanned by (7). That is,

$$Y_1^{(1)}(f_u)|_{f_u=0} = 0, \qquad Y_2^{(1)}(f_u)|_{f_u=0} = 0, \qquad Y_\phi^{(1)}(f_u)|_{f_u=0} = 0, \qquad Y_\psi^{(1)}(f_u)|_{f_u=0} = 0.$$

4 Differential invariants of second order

Now we determine differential invariants that depend on the second derivatives of f and g. Here we need to calculate the second prolongation of (4). As in the previous case it is straightforward to deduce that $J_t = J_x = J_u = 0$. Hence,

$$J = J(u_x, f, g, f_x, f_u, g_x, g_u, g_{u_x}, f_{xx}, f_{xu}, f_{uu}, g_{xx}, g_{xu}, g_{xu_x}, g_{uu}, g_{uu_x}, g_{u_x u_x}).$$

Now the second prolongation of (4) reads

$$
\begin{aligned}
Y_2^{(2)} &= Y_2^{(1)} - f_{xx}\frac{\partial}{\partial f_{xx}} - f_{xu}\frac{\partial}{\partial f_{xu}} - f_{uu}\frac{\partial}{\partial f_{uu}} - g_{xx}\frac{\partial}{\partial g_{xx}} - g_{xu}\frac{\partial}{\partial g_{xu}} - g_{xu_x}\frac{\partial}{\partial g_{xu_x}} \\
&\quad - g_{uu}\frac{\partial}{\partial g_{uu}} - g_{uu_x}\frac{\partial}{\partial g_{uu_x}} - g_{u_xu_x}\frac{\partial}{\partial g_{u_xu_x}}, \\
Y_\phi^{(2)} &= Y_\phi^{(1)} + (2\phi'''f + 3\phi''f_x)\frac{\partial}{\partial f_{xx}} + (2\phi''f_u + \phi' f_{xu})\frac{\partial}{\partial f_{xu}} + 2\phi' f_{uu}\frac{\partial}{\partial f_{uu}} \\
&\quad + (u_x\phi^{(iv)}f + \cdots)\frac{\partial}{\partial g_{xx}} + (u_x\phi'''f_u + \cdots)\frac{\partial}{\partial g_{xu}} \\
&\quad + (\phi'''f + \phi''g_{u_x} + \phi''f_x + u_x\phi''g_{u_xu_x})\frac{\partial}{\partial g_{xu_x}} + u_x\phi''f_{uu}\frac{\partial}{\partial g_{uu}} \\
&\quad + (\phi''f_u + \phi' g_{uu_x})\frac{\partial}{\partial g_{uu_x}} + 2\phi' g_{u_xu_x}\frac{\partial}{\partial g_{u_xu_x}}, \\
Y_\psi^{(2)} &= Y_\psi^{(1)} - (\psi_{xu}f_u + 2\psi_x f_{xu})\frac{\partial}{\partial f_{xx}} - (\psi_{xu}f_u + \psi_u f_{xu} + \psi_x f_{uu})\frac{\partial}{\partial f_{xu}} \\
&\quad - (f_u\psi_{uu} + 2f_{uu}\psi_u)\frac{\partial}{\partial f_{uu}} + (-\psi_{xxxx}f + \cdots)\frac{\partial}{\partial g_{xx}} + (-\psi_{xxxu}f + \cdots)\frac{\partial}{\partial g_{xu}} \\
&\quad + (-\psi_{uuuu}fu_x^2 + \cdots)\frac{\partial}{\partial g_{uu}} - [2\psi_{xuu}u_xf + 2\psi_{xxu}f + 2\psi_{uu}f_xu_x + 2\psi_{xu}f_x + \psi_x g_{uu_x} \\
&\quad + (\psi_{xx} + \psi_{xu}u_x)g_{u_xu_x}]\frac{\partial}{\partial g_{xu_x}} - [2\psi_{uuu}u_xf + 2\psi_{xuu}f + 2\psi_{uu}f_uu_x + 2\psi_{xu}f_u \\
&\quad + \psi_u g_{uu_x} + (\psi_{xu} + \psi_{uu}u_x)g_{u_xu_x}]\frac{\partial}{\partial g_{uu_x}} - (2\psi_{uu}f + \psi_u g_{u_xu_x})\frac{\partial}{\partial g_{u_xu_x}}. \qquad (10)
\end{aligned}
$$

The invariant test produces three identities

$$
E_2 = Y_2^{(2)}(J) = 0, \qquad E_\phi = Y_\phi^{(2)}(J) = 0, \qquad E_\psi = Y_\psi^{(2)}(J) = 0. \qquad (11)
$$

Coefficients of ψ_{xxxx}, ψ_{xxxu}, ψ_{uuuu} and ψ_{xxx} in $E_\psi = 0$ give $J_{g_{xx}} = J_{g_{xu}} = J_{g_{uu}} = J_{g_x} = 0$. Hence,

$$
J = J(u_x, f, g, f_x, f_u, g_u, g_{u_x}, f_{xx}, f_{xu}, f_{uu}, g_{xu_x}, g_{uu_x}, g_{u_xu_x}).
$$

Equation $E_2 = 0$ now reads

$$
\begin{aligned}
&fJ_f + gJ_g + f_xJ_{f_x} + f_uJ_{f_u} + g_uJ_{g_u} + g_{u_x}J_{g_{u_x}} + f_{xx}J_{f_{xx}} + f_{xu}J_{f_{xu}} + f_{uu}J_{f_{uu}} \\
&\quad + g_{xu_x}J_{g_{xu_x}} + g_{uu_x}J_{g_{uu_x}} + g_{u_xu_x}J_{g_{u_xu_x}} = 0.
\end{aligned}
$$

From this first order partial differential equation we get 12 integrals

$$
\begin{aligned}
&p_1 = \frac{f}{g}, \quad p_2 = \frac{f_x}{f}, \quad p_3 = \frac{f_u}{f}, \quad p_4 = \frac{g_u}{g}, \quad p_5 = \frac{g_{u_x}}{g}, \quad p_6 = \frac{f_{xx}}{f}, \\
&p_7 = \frac{f_{xu}}{f}, \quad p_8 = \frac{f_{uu}}{f}, \quad p_9 = \frac{g_{xu_x}}{g}, \quad p_{10} = \frac{g_{uu_x}}{g}, \quad p_{11} = \frac{g_{u_xu_x}}{g}, \quad p_{12} = u_x. \qquad (12)
\end{aligned}
$$

Coefficient of ψ_{xxu} in $E_\psi = 0$ gives

$$
J_{p_4} + 2J_{p_9} = 0,
$$

where we have used the new variables p_i. The above relation reduces the integrals by one:

$$
p_1, \quad p_2, \quad p_3, \quad p_5, \quad p_6, \quad p_7, \quad p_8, \quad p_{10}, \quad p_{11}, \quad p_{12}, \quad q_4 = 2p_4 - p_9. \qquad (13)
$$

From the coefficient of ψ_{xuu} in $E_\psi = 0$ we get

$$J_{p_{10}} + p_{12} J_{q_4} = 0$$

and therefore we have the following 10 integrals

$$p_1, \quad p_2, \quad p_3, \quad p_5, \quad p_6, \quad p_7, \quad p_8, \quad p_{11}, \quad p_{12}, \quad r_4 = q_4 - p_{12} p_{10}. \tag{14}$$

Coefficient of ψ_x in $E_\psi = 0$ gives

$$J_{p_{12}} - p_3 J_{p_2} - 2p_7 J_{p_6} - p_8 J_{p_7} = 0$$

which produces the integrals

$$p_1, \quad p_3, \quad p_5, \quad p_8, \quad p_{11}, \quad r_4, \quad q_2 = p_2 + p_3 p_{12}, \quad q_6 = p_7^2 - p_6 p_8, \quad q_7 = p_7 + p_8 p_{12}. \tag{15}$$

Coefficient of ψ_{xx} in $E_\psi = 0$ produces

$$p_1^2 J_{p_1} + p_1 p_{11} J_{p_{11}} + p_1 p_5 J_{p_5} + p_3 p_8 J_{q_6} + (p_{11} + p_1 r_4 - 2p_1 p_3) J_{r_4} = 0$$

which implies the integrals

$$p_3, \quad p_8, \quad q_2, \quad q_7, \quad q_5 = \frac{p_5}{p_1}, \quad q_{11} = \frac{p_{11}}{p_1}, \quad r_6 = q_6 + \frac{p_3 p_8}{p_1}, \quad \mu_4 = \frac{r_4}{p_1} + \frac{p_{11}}{p_1^2} - 2\frac{p_3}{p_1}. \tag{16}$$

We take the coefficient of ψ_{xu} in $E_\psi = 0$,

$$2J_{q_5} + p_3 J_{q_7} + 2p_3 q_7 J_{r_6} + 2(q_5 - q_2) J_{\mu_4} = 0.$$

We obtain the integrals

$$p_3, \quad p_8, \quad q_2, \quad q_{11}, \quad r_5 = q_5^2 - 2q_2 q_5 - 2\mu_4, \quad r_7 = 2q_7 - p_3 q_5, \quad \mu_6 = q_7^2 - r_6. \tag{17}$$

Coefficient of ψ_{uu} in $E_\psi = 0$ gives

$$p_3 \mu_6 J_{\mu_6} + 2p_8 J_{q_{11}} + p_3 p_8 J_{p_8} = 0$$

which produces the integrals

$$p_3, \quad q_2, \quad r_5, \quad r_7, \quad r_{11} = p_3 q_{11} - 2p_8, \quad \lambda_6 = \frac{\mu_6}{p_8}. \tag{18}$$

Coefficient of ψ_u in $E_\psi = 0$ gives the equation

$$p_3 J_{p_3} + 2r_{11} J_{r_{11}} + r_7 J_{r_7} = 0$$

from which we get the solutions

$$q_2, \quad r_5, \quad \lambda_6, \quad \mu_7 = \frac{r_7}{p_3}, \quad \mu_{11} = \frac{r_{11}}{p_3^2}. \tag{19}$$

Solutions (19) satisfy $E_\psi = 0$ for any arbitrary function $\psi(x, u)$. Now we use the identity $E_\phi = 0$. Coefficient of ϕ''' gives

$$J_{r_5} + J_{\lambda_6} = 0$$

and therefore we have

$$q_2, \quad \mu_7, \quad \mu_{11}, \quad \mu_5 = r_5 - \lambda_6. \tag{20}$$

Coefficient of ϕ'' in $E_\phi = 0$ produces the equation

$$2J_{q_2} + 3J_{\mu_7} - 3q_2 J_{\mu_5} = 0$$

which gives the integrals

$$\mu_{11}, \quad \lambda_7 = 2\mu_7 - 3q_2, \quad \lambda_5 = 4\mu_5 + 3q_2^2. \tag{21}$$

Finally equation $E_\phi = 0$ reads

$$(2\lambda_5 J_{\lambda_5} + \lambda_7 J_{\lambda_7})\phi' = 0.$$

Hence we obtain the solutions

$$J_1 = \mu_{11}, \qquad J_2 = \frac{\lambda_5}{\lambda_7^2}. \tag{22}$$

Now, using the sequence of integrals (12)–(22), we can write the forms of J_1 and J_2 in terms of the original variables, u_x, f, g, f_x, f_u, g_u, g_{u_x}, f_{xx}, f_{xu}, f_{uu}, g_{xu_x}, g_{uu_x}, $g_{u_xu_x}$. We therefore conclude that equation (1) has two invariants of the second order:

$$J_1 = \frac{f_u g_{u_xu_x} - 2ff_{uu}}{f_u^2}, \tag{23}$$

$$\begin{aligned} J_2 = f_u^2(&-4u_x^2 ff_{uu} - 8u_x ff_{xu} - 4ff_{xx} - 16fg_u \\ &+ 8u_x fg_{uu_x} + 8fg_{xu_x} + 3u_x^2 f_u^2 + 6u_x f_x f_u + 20gf_u - 8u_x f_u g_{u_x} + 3f_x^2 - 8f_x g_{u_x} \\ &- 8gg_{u_xu_x} + 4g_{u_x}^2)/(4u_x ff_{uu} + 4ff_{xu} - 3u_x f_u^2 - 3f_x f_u - 2f_u g_{u_x})^2. \end{aligned} \tag{24}$$

In addition to the invariant equation $f_u = 0$ (equation (9)) that we found in the previous section, here we have also the following three invariant equations:

$$f_u g_{u_xu_x} - 2ff_{uu} = 0, \tag{25}$$

$$\begin{aligned} &-4u_x^2 ff_{uu} - 8u_x ff_{xu} - 4ff_{xx} - 16fg_u + 8u_x fg_{uu_x} + 8fg_{xu_x} + 3u_x^2 f_u^2 \\ &\quad + 6u_x f_x f_u + 20gf_u - 8u_x f_u g_{u_x} + 3f_x^2 - 8f_x g_{u_x} - 8gg_{u_xu_x} + 4g_{u_x}^2 = 0, \end{aligned} \tag{26}$$

$$4u_x ff_{uu} + 4ff_{xu} - 3u_x f_u^2 - 3f_x f_u - 2f_u g_{u_x} = 0. \tag{27}$$

To show this we need to apply the second prolongation (10) of (4) to these equations. That is, we have to show that

$$Y_2^{(2)}(\phi)|_{\phi=0} = 0, \qquad Y_\phi^{(2)}(\phi)|_{\phi=0} = 0, \qquad Y_\psi^{(2)}(\phi)|_{\phi=0} = 0,$$

where ϕ is the left hand side of equations (25), (26) and (27).

We make the following remarks: If equation (1) is such that

1. All four equations (9), (25)–(27) hold, then it has no invariants. We note that if (9) holds, then equations (25) and (27) are satisfied.
2. Equation (9) holds, then it has one invariant, $J_2 = 0$.
3. Equations (26) and (27) hold, then it has one invariant, J_1.
4. Equation (25) holds (but not (9)), then it has two invariants, $J_1 = 0$, J_2.
5. Equation (26) holds, then it has two invariants, J_1, $J_2 = 0$.
6. Equation (27) holds, then it has two invariants, J_1, $J_2' = \frac{1}{J_2} = 0$.

Finally, we make a comment on the invariant equation $f_u = 0$. From this relation we deduce that when two equations of the form (1) are connected by a point transformation, the corresponding functions $f(x,u)$ must both depend on u, or both do not depend on u. From this we can deduce that there exists **no** point transformation that maps an equation of the form (1) with $f_u \neq 0$ to the linear heat equation $u_t = u_{xx}$. In general, an equation of the form (1) with $f_u \neq 0$ cannot be linearised by a point transformation.

Example. We consider the integrable equation

$$u_t = u^2 u_{xx} \tag{28}$$

and the class of equations

$$u_t = u^n u_{xx} + g(x,u,u_x). \tag{29}$$

Both of these equations are special forms of (1). Setting $f = u^2$ and $g = 0$ into equations (23) and (24) we find that equation (28) has invariants $J_1 = -1$ and $J_2 = 1$. From (23) and (24) we deduce that equation (29) has invariants $J_1 = -1$ and $J_2 = 1$ if it is of the form

$$u_t = u^n u_{xx} + \frac{1}{2}(n-2)u^{n-1}u_x^2 + k(x)u^n u_x + \frac{2}{n}\frac{\mathrm{d}k}{\mathrm{d}x}u^{n+1} + h(x)u^{\frac{3n+4}{4}}. \tag{30}$$

Now if we consider transformation (2), it can be shown that the most general form of (30) (and consequently of (29)) that can be linked with (28) is

$$u_t = u^n u_{xx} + \frac{1}{2}(n-2)u^{n-1}u_x^2 + k(x)u^n u_x + \frac{2}{n}\frac{\mathrm{d}k}{\mathrm{d}x}u^{n+1}. \tag{31}$$

In fact, it can be shown that the transformation

$$x \mapsto \int \mathrm{e}^{\int k(x)\mathrm{d}x}\mathrm{d}x, \qquad t \mapsto t, \qquad u \mapsto \mathrm{e}^{\int k(x)\mathrm{d}x}u^{\frac{n}{2}}$$

maps (28) into (31).

5 Remarks

We have shown that the class of equations (1) has no differential invariants of order zero and order one. We have determined two functionally independent differential invariants of second order. In order to produce higher order invariants, we need to follow the procedure as above by considering higher order prolongations, or alternatively we can introduce the idea of invariant differentiation. Details about invariant differentiation can be found in the book of Ibragimov [3].

We note that for the invariants (23) and (24) we need to have $f_u \neq 0$. Hence in the case where $f_u = 0$, that is equation $u_t = f(x)u_{xx} + g(x,u,u_x)$, needs to be considered separately. However, by introducing a new space variable $\xi = \int \frac{1}{f(x)}\mathrm{d}x$, this latter equation takes the form $u_t = u_{\xi\xi} + h(\xi,u,u_\xi)$. The problem of classification of differential invariants for the class of equations $u_t = u_{xx} + g(x,u,u_x)$ will be considered in a separate article in the near future.

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Conservation Laws and Potential Systems of Diffusion-Convection Equations

Nataliya M. IVANOVA

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., 01601 Kyiv-4, Ukraine
E-mail: *ivanova@imath.kiev.ua*

We investigate conservation laws of diffusion-convection equations to construct first-order potential systems corresponding to these equations. We do two iterations of the construction procedure, looking, in the second step, for the first-order conservation laws of the potential systems obtained in the first step.

In her famous paper [14] Emmy Noether proved that each Noether symmetry associated with a Lagrangian generates a conservation law. (Modern treatment of relationship between components of Noether conserved vector and Lie–Bäcklund operators which are Noether symmetries was adduced in [10,18]. Kara *et al* [12,13] constructed conservation laws of some classes of PDEs with two independent variables using nonlocal symmetries.) V.A. Dorodnitsyn and S.R. Svirshchevskii [8] (see also [9, Chapter 10]) completely studied the conservation laws for reaction-diffusion equations

$$u_t = (d(u)u_x)_x + q(u).$$

In [1] a detailed discussion of the local nature of conservation laws derived from Noether's theorem is presented.

A *conservation law* of a system of PDEs $\Delta(x, u_{(r)}) = 0$ is a divergence expression $\operatorname{div} F = 0$ which vanishes for all solutions of these system. Here $x = (x_1, \dots, x_n)$, $u = (u^1, \dots, u^m)$. $F = (F^1, \dots, F^n)$, where $F^i = F^i(x, u_{(r)})$, is a conserved vector of this conservation law $u_{(r)}$ is the set of all the partial derivatives of the function u with respect to x of order no greater than r, including u as the derivative of zero order. We call the maximal order of derivatives explicitly appear in f as the *order* of the conservation law $\operatorname{div} F = 0$.

The conservation laws are closely connected with non-local (potential) symmetries. A system of PDEs may admit symmetries of such sort when some of the equations can be written in a conserved form. After introducing potentials for PDEs written in the conserved form as additional dependent variables, we obtain a new (potential) system of PDEs. Any local invariance transformation of the obtained system induces a symmetry of the initial system. If transformations of some of the "non-potential" variables explicitly depend on potentials, this symmetry is a non-local (potential) symmetry of the initial system. More details about potential symmetries and their applications can be found in [4–6].

In this paper we consider conservation laws and potential systems for nonlinear diffusion-convection equations of the form

$$u_t = (d(u)u_x)_x + k(u)u_x, \tag{1}$$

that have a number of applications in physics (see for instance [2, 3, 11]). Here $d = d(u)$ and $k = k(u)$ are arbitrary smooth functions of u, $d(u) \neq 0$.

The complete and strong group classification of (1) was presented in [15] (see also references therein for more details about symmetry analysis of classes intersecting class (1)). The conservation laws associated with first-order potential systems for equations from class (1) with $k = 0$ were considered in [7]. However, some of conservation laws and potential systems were

missing there since the authors did not investigate some necessary differential consequences of considered systems.

Note 1. If a local transformation connects two PDEs then under the action of this transformation a conservation law of the first of these equations is transformed into a conservation law of the second equation, i.e. the equivalence transformation establishes a one-to-one correspondence between conservation laws of these equations. So, we can consider a problem of investigation of conservation laws with respect to the equivalence group of the initial class.

Taking into account Note 1 we start our investigation of potential systems and conservation laws for the class (1) from finding equivalence transformations. Using the classical Lie approach, we find the Lie invariance algebra of equivalence group $G^{\rm equiv}$ for class (1).

Theorem 1 (see [15,16]). *The Lie algebra of the equivalence group $G^{\rm equiv}$ for the class* (1) *is*

$$A^{\rm equiv}=\langle\partial_t,\ \partial_x,\ \partial_u,\ t\partial_t-d\partial_d-k\partial_k,\ x\partial_x+2d\partial_d+k\partial_k,\ u\partial_u,\ t\partial_x-\partial_k\rangle.$$

For class (1) there also exists a nontrivial group of discrete equivalence transformations generated by three involutive transformations of the coupled alternation of signs in the sets of variables $\{t,d,k\}$, $\{x,k\}$ and $\{u\}$. It can be proved by the direct method that $G^{\rm equiv}$ coincides with the group generated by the both continuous and discrete above transformations.

Theorem 2 (see [15,16]). *Any transformation from $G^{\rm equiv}$ has the form*

$$\tilde t=\varepsilon_4 t+\varepsilon_1,\quad \tilde x=\varepsilon_5 x+\varepsilon_7 t+\varepsilon_2,\quad \tilde u=\varepsilon_6 u+\varepsilon_3,\quad \tilde d=\varepsilon_4^{-1}\varepsilon_5^2 d,\quad \tilde k=\varepsilon_4^{-1}\varepsilon_5 k-\varepsilon_7,$$

where $\varepsilon_1,\ \dots,\ \varepsilon_7$ are arbitrary constants, $\varepsilon_4\varepsilon_5\varepsilon_6\neq 0$.

First we search the first-order conservation laws for equations from class (1) in the form

$$D_tF(t,x,u,u_t,u_x)+D_xG(t,x,u,u_t,u_x)=0, \tag{2}$$

where D_t and D_x are the operators of the total differentiation with respect to t and x correspondingly, namely, $D_t=\partial_t+u_t\partial_u+\cdots$, $D_x=\partial_x+u_x\partial_u+\cdots$. Although this is the simplest class of conservation laws, it allows us to deduce first-order potential systems

$$v_x=F,\qquad v_t=-G. \tag{3}$$

Such systems as a rule admit a lot of nontrivial symmetries and so they are of a great interest.

Definition 1. Two conservation laws $D_tF+D_xG=0$ and $D_tF'+D_xG'=0$ are *equivalent* if

$$F'=F+D_xH,\qquad G'=G+D_tH,$$

where H is an arbitrary function of the same variables as the functions F and G.

Theorem 3. *A complete set of inequivalent (with respect to the above equivalence relation) conservation laws of the form* (2) *and potential systems* (3) *for equations from class* (1) *is exhausted by the simple numbered cases of Table 1.*

Proof. Let us substitute expression for u_{xx} deduced from (1) into equation (2) and decompose obtained equation with respect to u_{tt} and u_{tx}. As a result we have

$$F=F^0(t,x,u,u_x),\qquad G=-F^0_{u_x}u_t+G^0(t,x,u,u_x).$$

Table 1. Conservation laws and potential systems.

N	d	k	F	G	Potential system
1	$\forall$	$\forall$	u	$-du_x-\int k$	$v_x=u,\ v_t=du_x+\int k$
1.1	$\forall$	d	e^xv	$-e^x\int d$	$v_x=u,\ w_x=e^xv,\ w_t=e^x\int d$
1.2	$\forall$	$\int d+ud$	e^v	$-e^v\int d$	$v_x=u,\ w_x=e^v,\ w_t=e^v\int d$
1.3	$\forall$	0	v	$-\int d$	$v_x=u,\ w_x=v,\ w_t=\int d$
1.4	u^{-2}	u^{-2}	σe^x	$\sigma_vu^{-1}e^x$	$v_x=u,\ w_x=\sigma e^x,\ w_t=-\sigma_vu^{-1}e^x$
1.5	u^{-2}	0	σ	σ_vu^{-1}	$v_x=u,\ w_x=\sigma,\ w_t=-\sigma_vu^{-1}$
1.6	1	0	αv	$\alpha_xv-\alpha u$	$v_x=u,\ w_x=\alpha v,\ w_t=\alpha u-\alpha_xv$
1.7	1	$2u$	αe^v	$\alpha_xe^v-\alpha ue^v$	$v_x=u,\ w_x=\alpha e^v,\ w_t=\alpha ue^v-\alpha_xe^v$
2	$\forall$	d	$(e^x+\varepsilon)u$	$-(e^x+\varepsilon)du_x-\varepsilon\int d$	$v_x=(e^x+\varepsilon)u,\ v_t=(e^x+\varepsilon)du_x+\varepsilon\int d$
2.1	u^{-2}	u^{-2}	$\dfrac{e^x}{(e^x+\varepsilon)^2}v$	$\dfrac{e^x}{(e^x+\varepsilon)u}$	$v_x=(e^x+\varepsilon)u,\ w_x=\dfrac{e^x}{(e^x+\varepsilon)^2}v,$ $w_t=-\dfrac{e^x}{(e^x+\varepsilon)u}$
3	$\forall$	0	xu	$\int d-xdu_x$	$v_x=xu,\ v_t=xdu_x-\int d$
3.1	$\forall$	0	$x^{-2}v$	$-x^{-1}\int d$	$v_x=xu,\ w_x=x^{-2}v,\ w_t=x^{-1}\int d$
3.2	1	0	$\alpha x^{-1}v$	$(\alpha_xx^{-1}+\alpha x^{-2})v-\alpha u$	$v_x=xu,\ w_x=\alpha x^{-1}v,$ $w_t=\alpha u-(\alpha_xx^{-1}+\alpha x^{-2})v$
4	1	0	αu	$\alpha_xu-\alpha u_x$	$v_x=\alpha u,\ v_t=\alpha u_x-\alpha_xu$
4.1	1	0	$\left(\dfrac{\beta}{\alpha}\right)_x v$	$-\alpha\left(\dfrac{\beta}{\alpha}\right)_x u-\left(\dfrac{\beta}{\alpha}\right)_t v$	$v_x=\alpha u,$ $w_x=\left(\dfrac{\beta}{\alpha}\right)_x v,\quad w_t=\alpha\left(\dfrac{\beta}{\alpha}\right)_x u+\left(\dfrac{\beta}{\alpha}\right)_t v$

Here $\varepsilon\in\{0,\pm1\}$, $\int d=\int d(u)du$, $\int k=\int k(u)du$; $\alpha(t,x)$, $\beta(t,x)$ and $\sigma(t,v)$ are arbitrary solutions of the linear heat equation ($\alpha_t+\alpha_{xx}=0$, $\beta_t+\beta_{xx}=0$, $\sigma_t+\sigma_{vv}=0$).

Taking into account the latter equations and splitting the rest of equation (2) step by step with respect to the powers of u_t and u_x, we obtain a system of PDEs on functions F^0 and G^0. Integrating these equations results in

$$F=D_x\varphi-\psi u,\qquad G=\chi-D_t\varphi+d\psi u_x+\psi\textstyle\int k(u)du-\psi_x\int d(u)du,$$
$$\psi_t+d\psi_{xx}-k\psi_x=0, \tag{4}$$

where $\varphi=\varphi(t,x,u)$, $\chi=\chi(t)$, $\psi=\psi(t,x)$. With respect to the equivalence relation of conservation laws we can assume that $\varphi=\chi=0$. Solving classifying equation (4) for different values of d and k we obtain Cases 1, 2, 3 and 4 of Table 1. ■

Let us consider first-order conservation laws of potential systems 1–4 from Table 1 which have the form

$$D_tF(t,x,u,v,u_t,u_x,v_t,v_x)+D_xG(t,x,u,v,u_t,u_x,v_t,v_x)=0. \tag{5}$$

These laws indeed nonlocal conservation laws for equations from class (1).

Lemma 1. *Any conservation law of form* (5) *for any from systems 1–4 from Table 1 is equivalent to the law with the conserved density F and the flux G that are independent of the derivatives u_t, u_x, v_t and v_x.*

Theorem 4. *A complete set of inequivalent (with respect to the above equivalence relation) conservation laws of the form* (5) *for equations from class* (1) *is presented in Table 1 with a double numeration of cases.*

Note 2. To prove Theorem 4 we use all functionally independent differential consequences of correspondent potential systems. In Table 1 for the double numerated potential systems we omit equations containing v_t since they are only differential consequences of equations of these systems.

Note that potential symmetries arising for equations (1) from Case 1 and 1.4 of Table 1 were found by C. Sophocleous [17]. The complete group classification of potential system corresponding to Case 1 of Table 1 was carried out in [16].

We plan to continue investigations of the subject. For the class under consideration we plan to perform complete classification of potential and nonclassical (conditional) symmetries. We also plan to study conservation laws of the more general structure (the high-order conservation laws and conservation laws with pseudopotentials).

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On Realizations of the Conformal Algebra and Nonlinear Invariant Equations in Low-Dimensional Space-Time

Victor LAHNO

Poltava State Pedagogical University, 2 Ostrograds'kogo Str., 36003 Poltava, Ukraine
E-mail: *lvi@pdpu.septor.net.ua, laggo@poltava.bank.gov.ua*

This paper devoted to description of conformally-invariant PDE in low-dimensional spaces.

1 Introduction

This talk is devoted to the question, "What is the most general partial differential equation invariant under a given local Lie group?"

Principal object of our consideration is the second-order partial differential equation

$$F(x, u, \underset{1}{u}, \underset{2}{u}) = 0 \tag{1}$$

invariant under the conformal group $C(1,n)$ and its subgroups – the Poincaré group $P(1,n)$ and extended Poincaré group $\tilde{P}(1,n)$ acting as Lie transformation groups in the space $V = \mathbb{R}(1,n) \times \mathbb{R}^r$ of independent and dependent variables, where $\mathbb{R}(1,n)$ is the pseudo-Euclidean space with the metric tensor

$$g_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta = 0, \\ 0, & \alpha \neq \beta, \\ -1, & \alpha = \beta = 1, 2, \ldots, n. \end{cases} \tag{2}$$

The Lie algorithm requires using for construction of invariant equations not local transformations groups, but the respective Lie algebras of symmetry operators (realizations of Lie algebras in the class of first-order linear differential operators). In our case these linear operators have the form

$$Q = \xi_\mu(x,u)\partial_{x_\mu} + \eta_k(x,u)\partial_{u_k}, \qquad \mu = 0, 1, \ldots, n, \quad k = 1, 2, \ldots, r. \tag{3}$$

A set of $1 + n + C^2_{1+n}$ differential operators P_μ, $J_{\alpha\beta} = -J_{\beta\alpha}$, $\mu, \alpha, \beta = 0, 1, \ldots, n$ of the form (3) satisfying the commutation relations

$$\begin{aligned} &[P_\alpha, P_\beta] = 0, \qquad [P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta, \\ &[J_{\alpha\beta}, J_{\mu\nu}] = g_{\alpha\nu}J_{\beta\mu} + g_{\beta\mu}J_{\alpha\nu} - g_{\alpha\mu}J_{\beta\nu} - g_{\beta\nu}J_{\alpha\mu} \end{aligned} \tag{4}$$

is called a realization of the Poincaré algebra $p(1,n)$.

A set of $2 + n + C^2_{n+1}$ differential operators $P_\mu, J_{\alpha\beta}, D$ of the form (3) satisfying the commutation relations (4) and

$$[D, J_{\alpha\beta}] = 0, \qquad [P_\alpha, D] = P_\alpha \tag{5}$$

is called a realization the extended Poincaré algebra $\tilde{p}(n,m)$.

A set of $3+2n+C^2_{1+n}$ differential operators P_μ, $J_{\alpha\beta}$, D, K_μ of the form (3) satisfying the commutation relations (4), (5) and

$$[K_\alpha, K_\beta]=0, \qquad [K_\alpha, J_{\beta\gamma}]=g_{\alpha\beta}K_\gamma - g_{\alpha\gamma}K_\beta,$$
$$[P_\alpha, K_\beta]=2(g_{\alpha\beta}D - J_{\alpha\beta}), \qquad [D, K_\alpha]=K_\alpha$$

is called a realization of the conformal algebra $c(1,n)$.

The following realization of the conformal algebra $c(1,n)$ is well-known in many problems of mathematical physics:

$$P_\mu=\partial_{x_\mu}, \qquad J_{\mu\nu}=g_{\mu\gamma}x_\gamma\partial_{x_\nu}-g_{\nu\gamma}x_\gamma\partial_{x_\mu}, \qquad D=x_\mu\partial_{x_\mu}+\epsilon u_k\partial_{u_k}, \quad \epsilon=0,1,$$
$$K_\mu=2g_{\mu\nu}x_\nu D-(g_{\alpha\beta}x_\alpha x_\beta)\partial_{x_\mu}, \qquad \alpha,\beta,\mu,\nu,\gamma=0,1,\dots,n, \quad k=1,\dots,r. \tag{6}$$

In the following we call this realization the standard realization of the conformal algebra.

For the realization (6) in the case of arbitrary n and r the problem of description of equations of the form (1) invariant with respect to the Poincaré group, the extended Poincaré group and the conformal group was completely solved by W.I. Fushchych and I.A. Yehorchenko in [1].

However, the following question will be natural: do other realizations of the algebras $p(1,n)$, $\tilde{p}(1,n)$ and $c(1,n)$ in the class of operators (3) exist that would be different from the standard? In this respect the problem of description of non-equivalent realizations for the algebras $p(1,n)$, $\tilde{p}(1,n)$, $c(1,n)$ in the class of operators (3) will be relevant. W. Fushchych, R. Zhdanov and V. Lahno [2] showed that in the case when translation operators have the form (6)

$$P_\mu=\partial_{x_\mu}, \qquad \mu=0,1,\dots,n,$$

and $V=\mathbb{R}(1,n)\times\mathbb{R}^1$, $n\geq 3$, realizations of the Poincaré algebra $p(1,n)$ and of the extended Poincaré algebra $\tilde{p}(1,n)$ are equivalent to the standard realization (6). The conformal algebra $c(1,n)$, besides the standard realization (6), has also the following one:

$$P_\mu=\partial_{x_\mu}, \qquad J_{\mu\nu}=g_{\mu\gamma}x_\gamma\partial_{x_\nu}-g_{\nu\gamma}x_\gamma\partial_{x_\mu},$$
$$D=x_\mu\partial_{x_\mu}+u\partial_u, \qquad K_\alpha=2g_{\alpha\beta}x_\beta D-\left(g_{\mu\nu}x_\mu x_\nu\pm u^2\right)\partial_{x_\alpha}.$$

Note, this realization is realized on the set of solutions of the eikonal equation

$$g_{\mu\nu}u_{x_\mu}u_{x_\nu}\pm 1=0,$$

and on the set of solutions of d'Alembert–eikonal system

$$g_{\mu\nu}u_{x_\mu}u_{x_\nu}\pm 1=0, \qquad g_{\mu\nu}u_{x_\mu x_\nu}\pm nu^{-1}=0.$$

Realizations for the spaces $V=\mathbb{R}(1,1)\times\mathbb{R}^1$ and $V=\mathbb{R}(1,2)\times\mathbb{R}^1$ were studied in more detail. Below we will consider these results in particular.

2 $\tilde{P}(1,1)$-, $\tilde{P}(1,1)$- and $C(1,1)$-invariant equations

Here $V=\mathbb{R}(1,1)\times\mathbb{R}^1$, $\mathbb{R}(1,1)=\langle t,x\rangle$, $\mathbb{R}^1=\langle u\rangle$, $u=u(t,x)$. G. Rideau and P. Winternitz [3] considered the issue of description of equations

$$F(t,x,u,u_t,u_x,u_{tt},u_{tx},u_{xx})=0, \tag{7}$$

that are invariant with respect to the groups $P(1,1)$, $\tilde{P}(1,1)$ and $C(1,1)$. For this purpose they first described realizations of the Lie algebras of these groups assuming that

$$P_0=\partial_t, \qquad P_1=\partial_x. \tag{8}$$

Their results can be summarized by the following lists:

1. Inequivalent realizations of the Poincaré algebra:

$$p^1(1,1):\{P_0=\partial_t,\ P_1=\partial_x,\ J_{01}=x\partial_x+t\partial_x\};$$
$$p^2(1,1):\{P_0=\partial_t,\ P_1=\partial_x,\ J_{01}=x\partial_t+t\partial_x+u\partial_u\}.$$

2. Inequivalent realizations of the extended Poincaré algebra:

$$\tilde{p}^1(1,1):\left\{p^1(1,1),D=t\partial_t+x\partial_x\right\};$$
$$\tilde{p}^2(1,1):\left\{p^1(1,1),D=t\partial_t+x\partial_x+u\partial_u\right\};$$
$$\tilde{p}^3(1,1):\left\{p^2(1,1),D=\left(t+au+bu^{-1}\right)\partial_t+(x+au-bu^{-1})\partial_x+\lambda u\partial_u\right\},$$

where $\lambda\in\mathbb{R}$, $(a,b)=(0,0)$ or $\lambda=1$, $(a,b)=(1,0)$ or $\lambda=-1$, $(a,b)=(0,1)$.

3. Inequivalent realizations of the conformal algebra:

$$c^1(1,1):\left\{\tilde{p}^1(1,1),\ K_0=\left(t^2+x^2\right)\partial_t+2tx\partial_x,\ K_1=-\left(t^2+x^2\right)\partial_x-2tx\partial_t\right\};$$
$$c^2(1,1):\left\{\tilde{p}^2(1,1),\ K_0=\left(t^2+x^2+au^2\right)\partial_t+2tx\partial_x+2tu\partial_u,\right.$$
$$\left.K_1=-\left(t^2+x^2+au^2\right)\partial_x-2tx\partial_t-2xu\partial_u,\ a=0,1,-1\right\};$$
$$c^3(1,1):\left\{\tilde{p}^3(1,1)\ (\lambda\in\mathbb{R},\ a=b=0),\ K_0=\left(t^2+x^2+cu^2+du^{-2}\right)\partial_t\right.$$
$$\left.+\left(2tx+cu^2-du^{-2}\right)\partial_x+\left(2u(x+\lambda t)+eu^2+k\right)\partial_u,\ K_1=-[J_{01},K_0]\right\},$$

where if $\lambda\in\mathbb{R}$, $\lambda\neq\pm1$, then $c=d=e=k=0$; if $\lambda=1$, then $d=k=0$, $c=\pm1$, $e\in\mathbb{R}$ or $c=0$, $e=0,\pm1$; if $\lambda=-1$, then $c=e=0$, $d=\pm1$, $k\in\mathbb{R}$ or $d=0$, $k=0,1$.

The realizations $p^2(1,1)$, $\tilde{p}^3(1,1)$, $c^2(1,1)$ $(a\neq0)$ and $c^3(1,1)$ are non-equivalent to the standard realizations (6).

Use of non-standard realizations allowed G. Rideau and P. Winternitz to add to the list of equations obtained by W. Fushchych and I. Yehorchenko, new invariant equations. We [4] give up the condition (8) and set the problem of description of all inequivalent realizations of the algebras $p(1,1)$, $\tilde{p}(1,1)$, $c(1,1)$. As a result, we obtained one new realization of the algebra $p(1,1)$

$$p^3(1,1):\left\{P_0=\partial_t,\ P_1=x\partial_t,\ J_{01}=xt\partial_t+\left(x^2-1\right)\partial_x\right\}$$

and two new realizations of the algebra $\tilde{p}(1,1)$:

$$p^4(1,1):\left\{p^3(1,1),\ D=t\partial_t\right\};$$
$$\tilde{p}^5(1,1):\left\{p^3(1,1),\ D=t\partial_t+u\partial_u\right\}.$$

The $p^3(1,1)$-invariant equation of the form (7) will have following form:

$$\Phi(\mathcal{J}_1,\ldots,\mathcal{J}_5)=0, \tag{9}$$

where

$$\mathcal{J}_1=u,\qquad \mathcal{J}_2=u_t^2\left(x^2-1\right),\qquad \mathcal{J}_3=u_{tt}\left(x^2-1\right),$$
$$\mathcal{J}_4=\left(x^2-1\right)^2\left(u_xu_{tt}-u_tu_{tx}\right)-x\left(x^2-1\right)u_t^2,$$
$$\mathcal{J}_5=\left(x^2-1\right)\left(u_{tt}u_{xx}-u_{tx}^2\right)+2x\left(x^2-1\right)^2\left(u_xu_{tt}-u_tu_{tx}\right)-x^2\left(x^2-1\right)u_t^2.$$

$\tilde{p}^4(1,1)$- and $\tilde{p}^5(1,1)$-invariant equations have the form

$$\bar{\Phi}(\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_4)=0, \tag{10}$$

where

$$\Sigma_1 = \mathcal{J}_1, \qquad \Sigma_2 = \mathcal{J}_2^{-1}\mathcal{J}_3, \qquad \Sigma_3 = \mathcal{J}_2^{-2}\mathcal{J}_4, \qquad \Sigma_4 = \mathcal{J}_2^{-2}\mathcal{J}_5, \quad \text{for} \quad \tilde{p}^4(1,1),$$

and

$$\Sigma_1 = \mathcal{J}_1\mathcal{J}_3, \qquad \Sigma_2 = \mathcal{J}_2, \qquad \Sigma_3 = \mathcal{J}_4, \qquad \Sigma_4 = \mathcal{J}_5, \quad \text{for} \quad \tilde{p}^5(1,1).$$

Notice that (9) and (10) include equations which are generations of the well-known Monge–Amperé equations.

Summing up the above, we see that for second-order scalar differential equations in the two-dimensional space-time we may consider the problem of construction of the general equation invariant with respect to the groups $P(1,1)$, $\tilde{P}(1,1)$, $C(1,1)$ to be resolved completely.

3 $P(1,2)$-, $\tilde{P}(1,2)$- and $C(1,2)$-invariant equations

Let us consider in more detail the results obtained for the equation

$$F(x_\mu, u, u_{x_\mu}, u_{x_\mu x_\nu}) = 0, \qquad \mu,\nu = 0,1,2. \tag{11}$$

As we said above, W. Fushchych and I. Yegorchenko gave full description of equations of the form (11), that are invariant with respect to the groups $P(1,2)$, $\tilde{P}(1,2)$ and $C(1,2)$, generated by the standard realizations of the algebras $p(1,2)$, $\tilde{p}(1,2)$, $c(1,2)$:

$$\begin{aligned}
&P_\mu = \partial_{x_\mu}, \qquad J_{\mu\nu} = g_{\mu\gamma}\partial_{x_\nu} - g_{\mu\gamma}x_\gamma\partial_{x_\mu},\\
&D = x_\mu\partial_{x_\mu} + \epsilon u\partial_u, \quad \epsilon = 0,1, \qquad K_\mu = 2g_{\mu\nu}x_\nu D - (g_{\alpha\beta}x_\alpha x_\beta)\partial_{x_\mu},\\
&g_{\mu\nu} = \begin{cases} 1, & \mu=\nu=0,\\ 0, & \mu\neq\nu, \\ -1, & \mu=\nu=1,2, \end{cases} \qquad \alpha,\beta,\gamma,\mu,\nu = 0,1,2.
\end{aligned} \tag{12}$$

I. Yehorchenko [5] found a new realization of the Poincaré group $p(1,2)$:

$$\begin{aligned}
&P_\mu = P_{x_\mu}, \qquad J_{01} = x_0\partial_{x_1} + x_1\partial_{x_0} + \sin u\partial_u,\\
&J_{02} = x_0\partial_{x_2} + x_2\partial_{x_0} + \cos u\partial_u, \qquad J_{12} = x_2\partial_{x_1} - x_1\partial_{x_2} + \partial_u.
\end{aligned} \tag{13}$$

Later W. Fushchych, R. Zhdanov and V. Lahno [2] showed that in the case $P_\mu = \partial_{x_\mu}$ ($\mu = 0,1,2$) non-equivalent realizations of the algebra $p(1,2)$ are exhausted by realizations (12) and (13) and found new realizations of the algebras $\tilde{p}(1,2)$ and $c(1,2)$ as extensions of the realizations (13) of the algebra $p(1,2)$:

$$\begin{aligned}
&P_\mu, J_{\mu\nu} \ \text{ have the form (13)}, \qquad D = x_\mu\partial_{x_\mu},\\
&K_0 = 2x_0D - (g_{\mu\nu}x_\mu x_\nu)\partial_{x_0} - 2(x_2\cos u + x_1\sin u)\partial_u,\\
&K_1 = -2x_1D - (g_{\mu\nu}x_\mu x_\nu)\partial_{x_1} - 2(x_0\sin u - x_2)\partial_u,\\
&K_2 = -2x_2D - (g_{\mu\nu}x_\mu x_\nu)\partial_{x_2} - 2(x_0\cos u + x_1)\partial_u.
\end{aligned}$$

These results were later re-discovered in a somehow new form by F. Güngor [6].

I. Yehorchenko [7] also obtained a general form of the equation (11), that is invariant with respect to the realization (13) of the algebra $p(1,2)$.

We aim to obtain the complete description of inequivalent realizations of the algebras $p(1,2)$, $\tilde{p}(1,2)$, $c(1,2)$, that might be invariance algebras for equation of the form (11). The list of results of our calculations [8] is given below. There are four new cases.

New realizations of the algebra $p(1,2)$

Case 1. Let the operators P_μ have the form

$$P_0 = \partial_{x_0}, \qquad P_1 = \partial_{x_1}, \qquad P_2 = x_2\partial_{x_0} + u\partial_{x_1}.$$

Then there is only one class of inequivalent realizations of the algebra $p(1,2)$, whose operators $J_{\mu\nu}$ can be taken as follows:

$$J_{01} = x_1\partial_{x_0} + x_0\partial_{x_1} + u\partial_{x_2} + x_2\partial_u, \qquad J_{02} = x_0x_2\partial_{x_0} + ux_0\partial_{x_1} + \left(x_2^2 - 1\right)\partial_{x_2} + ux_2\partial_u,$$
$$J_{12} = -x_1x_2\partial_{x_0} - x_1u\partial_{x_1} - x_2u\partial_{x_2} - \left(1 + u^2\right)\partial_u.$$

Case 2. Let the operators P_μ have the form

$$P_0 = \partial_{x_0}, \qquad P_1 = \partial_{x_1}, \qquad P_2 = x_2\partial_{x_0} + \varphi\partial_{x_1}, \quad \varphi = \sqrt{x_2^2 - 1}, \quad |x_2| > 1.$$

Then there are four classes of inequivalent realizations of the algebra $p(1,2)$. Their operators $J_{\mu\nu}$ can be taken in the following form:

$$J_{01} = x_1\partial_{x_0} + x_0\partial_{x_1} + \varphi\partial_{x_2}, \qquad J_{02} = (x_0x_2 + a)\partial_{x_0} + (x_0\varphi + b)\partial_{x_1} + \varphi^2\partial_{x_2} + q\partial_u,$$
$$J_{12} = (-x_1x_2 + \alpha)\partial_{x_0} + (-x_1\varphi + \beta)\partial_{x_1} - x_2\varphi\partial_{x_2} + p\partial_u.$$

Here a, b, α, β, q, p take the following values:

1. $a = \beta = \lambda = \text{const},\ \alpha = b = -\epsilon e^{-2u},\ \epsilon = 0, 1,\ q = -x_2,\ p = \varphi;$
2. $a = \beta = \lambda_1\left[\dfrac{u}{1-u^2} + \dfrac{1}{2}\ln\left|\dfrac{1+u}{1-u}\right|\right] + \lambda_2,$
   $\alpha = b = \lambda_1[1-u^2]^{-1},\ q = \varphi - x_2u,\ p = \varphi u - x_2,\ \lambda_1, \lambda_2 = \text{const};$
3. $a = -\beta = \lambda x_2\varphi,\ b = \lambda x_2^2,\ \alpha = -\lambda\varphi^2,\ q = p = 0,\ \lambda = \text{const};$
4. $a = -\beta = x_2\varphi u,\ b = x_2^2u,\ \alpha = -\varphi^2u,\ q = p = 0.$

Case 3. Let the operators P_μ have the form

$$P_0 = \partial_{x_0}, \qquad P_1 = x_1\partial_{x_0}, \qquad P_2 = \psi\partial_{x_0}, \qquad \psi = \pm\sqrt{1 - x_1^2}, \quad |x_1| < 1.$$

Then there are three classes of inequivalent realizations of the algebra $p(1,2)$. Their operators $J_{\mu\nu}$ can be taken in the following form:

$$J_{01} = (x_0x_1 + B\psi)\partial_{x_0} - \psi^2\partial_{x_1} + (Cx_1 + D\psi)\partial_{x_2} + A\psi\partial_u,$$
$$J_{02} = (x_0\psi - x_1B)\partial_{x_0} + x_1\psi\partial_{x_1} + (C\psi - x_1D)\partial_{x_2} - Ax_1\partial_u,$$
$$J_{12} = \psi\partial_{x_1}.$$

The variables A, B, C, D take the following values:

1. $A = B = C = D = 0;$
2. $A = \sqrt{|x_2|}g(u),\ B = x_2,\ C = 2x_2,\ D = x_2\sqrt{|x_2|}f(u);$
3. $A = x_2f(u),\ B = 0,\ C = x_2,\ D = x_2^2g(u).$

Here f and g are some sufficiently smooth real–valued functions.

Case 4. Finally, let the operators P_μ have the form

$$P_0 = \partial_{x_0}, \qquad P_1 = x_1\partial_{x_0}, \qquad P_2 = x_2\partial_{x_0}.$$

Then there are two classes of inequivalent realizations of the algebra $p(1,2)$. Their operators $J_{\mu\nu}$ can be taken in the following form:

$$J_{12}=x_2\partial_{x_1}-x_1\partial_{x_2},\qquad J_{01}=(x_0x_1+x_2\theta)\partial_{x_0}+\left(x_1^2-1\right)\partial_{x_1}+x_1x_2\partial_{x_2}+x_2\rho\partial_u,$$
$$J_{02}=(x_0x_2-x_1\theta)\partial_{x_0}+x_1x_2\partial_{x_1}+\left(x_2^2-1\right)\partial_{x_2}-x_1\rho\partial_u.$$

The variables θ, ρ take the following values:

1. $\theta=f(u)\left(1-\omega^{-1}\right),\ \omega=x_1^2+x_2^2,\ \rho=0;$
2. $\theta=0,\ \rho=\omega^{-1}\sqrt{|\omega-1|},\ \omega=x_1^2+x_2^2.$

Here f is some sufficiently smooth real-valued function.

Notice that the obtained realizations can be extended to the realizations of the algebra $\tilde{p}(1,2)$ and cannot be extended to the realizations of the algebra $c(1,2)$.

Further application of the obtained realizations for construction of invariant equations results in considerable technical difficulties. For the realization

$$P_0=\partial_{x_0},\qquad P_1=x_1\partial_{x_0},\qquad P_2=x_2\partial_{x_0},\qquad J_{12}=x_2\partial_{x_1}-x_1\partial_{x_2},$$
$$J_{01}=x_0x_1\partial_{x_0}+\left(x_1^2-1\right)\partial_{x_1}+x_1x_2\partial_{x_2},\qquad J_{02}=x_0x_2\partial_{x_0}+x_1x_2\partial_{x_1}+\left(x_2^2-1\right)\partial_{x_2}$$

the search for functions $\mathcal{J}_1$, $\mathcal{J}_2$, …, $\mathcal{J}_7$ in the respective invariant equation

$$F(\mathcal{J}_1,\dots,\mathcal{J}_7)=0$$

showed that they have an extremely cumbersome form. Below we adduce four of the seven functions:

$$\mathcal{J}_1=u,\qquad \mathcal{J}_2=u_{x_0}^2u_{x_0x_0},\qquad \mathcal{J}_3=(\Sigma_1-1)u_{x_0}^2,$$
$$\mathcal{J}_4=\Sigma_1^{-1}\left[(1-\Sigma_1)^3(2\Sigma_1\Sigma_2u_{x_0}^2+(1-\Sigma_1)\Sigma_2^2+\Sigma_3^2)+u_{x_0}^4(\Sigma_1-1)^2\Sigma_1^2\right],$$

where

$$\Sigma_1=x_1^2+x_2^2,\qquad \Sigma_2=x_1(u_{x_1}u_{x_0x_0}-u_{x_0}u_{x_0x_1})+x_2(u_{x_2}u_{x_0x_0}-u_{x_0}u_{x_0x_2}),$$
$$\Sigma_3=x_2(u_{x_1}u_{x_0x_0}-u_{x_0}u_{x_0x_1})-x_1(u_{x_2}u_{x_0x_0}-u_{x_0}u_{x_0x_2}).$$

We believe that in this respect a constructive problem may be presented by a problem of description of conformally invariant equations of the considered form using the basis of differential invariants for the standard realization found by W. Fushchych and I. Yehorchenko realization and realization (12) of the algebra $p(1,2)$.

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Application of Symmetry and Symmetry Analyses to Systems of First-Order Equations Arising from Mathematical Modelling in Epidemiology

P.G.L. LEACH † *and K. ANDRIOPOULOS* ‡

† *School of Mathematical and Statistical Sciences, University of Natal, Durban 4041, South Africa*
E-mail: *leachp@nu.ac.za*

‡ *Department of Mathematics, National and Capodistrian University of Athens, Panepistimioupolis, Ilisia, 15771 Athens, Greece*
E-mail: *kand@aegean.gr*

We examine the integrability, in terms of the Painlevé singularity analysis and of the Lie symmetry analysis, of systems of nonlinear first-order ordinary differential equations that arise in the particular area of Mathematical Modelling known as Rational Epidemiology. These analyses are presented as being complementary to the standard analysis using the methods of Dynamical Systems. The importance to obtain complete understanding of evolution of epidemics – one need think only of the potential for devastation by HIV-AIDS in African countries or the more recent threat posed by SARS – demands that all possible approaches of analysis be used. The concept of decomposed systems is introduced as illustrating some rather attractive mathematical properties of certain classes of systems of equations arising in Mathematical Modelling. Such systems allow the mathematician some opportunity to enjoy Mathematics even in the context of quite strict applications.

1 Introduction

There are four standard approaches to the analysis of a system of nonlinear first-order ordinary differential equations of the type which arise frequently in Mathematical Modelling. The approaches comprise numerical computation, dynamical systems, singularity analysis and symmetry analysis, all of which possess extensive literatures.

This work is concerned with Mathematical Modelling with particular emphasis on Rational Epidemiology. In particular our specialisation is to systems of first-order ordinary differential equations, i.e. compartmental models, in contrast to systems of partial differential equations. Thus we stress the evolutionary aspect of an epidemic in contrast to the dispersion in space, which happens when one uses partial differential equations. Obviously we can consider only a subset, and a simple one at that, of the class of models which can be investigated, but one can argue a case for the commencement with the simple. Here we concentrate upon the singularity and symmetry aspects of models. There are already many who have devoted much effort and ingenuity to the application of the methods of Dynamical Systems to and/or the determination of numerical solutions of these models.

A particular subset, the recently christened 'decomposed' systems, attracts our attention for the mathematical relief provided in the midst of the human – equally animal or vegetable should the system to be so modelled – misery underlying our studies.

There are some general and particular reasons for our interest. To general reasons are the need for sophisticated Mathematical Modelling in an environment of finite resources and rampant consumerism and the need to determine whether we live in the midst of potentially chaotic

systems. In the particular case of Rational Epidemiology there is a need to develop the mathematical models and their analyses to the fullest extent of our ability. In this modern era we have the burden of crowding and widespread intermingling of diverse population groups with the consequent dangers of the catastrophic spreading of deadly diseases. Even in the slower times the catastrophe of a pandemic could make its mark for many centuries. In the context of the African Continent the spectre of depopulation consequent upon HIV-AIDS is a reality. This provides a case study for mathematical modelling which in the magnitude of its implications is probably a worthy successor of the Black Death that devastated Europe 700 years ago.

The structure of this paper is as follows. In Sections 2, 3 and 4 we examine certain specific epidemiological models in terms of the symmetry and/or singularity analyses. In Section 5 we introduce the notion of 'decomposition' and indicate some of the properties of a particular example of a decomposed system. We conclude with some remarks and observations in Section 6.

2 A model for the transmission of HIV-AIDS

The model [24],

$$\dot{u}_1 = -\frac{\beta c u_1 u_2}{u_1+u_2+u_3} - \mu u_1, \qquad \dot{u}_2 = \frac{\beta c u_1 u_2}{u_1+u_2+u_3} - (\mu+\nu)\, u_2, \qquad \dot{u}_3 = \nu u_2 - \alpha u_3, \quad (1)$$

has been quite successful in replicating results reported in studies of HIV transmission in the San Francisco area [12, 27]. The variables $u_i(t)$, $i = 1, 3$ represent that part of population which is HIV negative, is HIV positive and has AIDS respectively. The parameter μ is the death rate from normal causes, α the death rate from AIDS and ν the rate at which HIV positives develop AIDS. The parameters β and c represent the rates of infection and change of partner respectively.

By inspection one notes that the system (1) possesses the two Lie point symmetries

$$\Gamma_1 = \partial_t \qquad \text{and} \qquad \Gamma_2 = u_1\partial_{u_1} + u_2\partial_{u_2} + u_3\partial_{u_3}$$

with the Abelian algebra $2A_1$.

The presence of the symmetry Γ_2, representing homogeneity in the dependent variables, precludes the possibility that the system (1) possess the Painlevé Property.

These two symmetries are insufficient for integrability in the sense of Lie. A third symmetry, which would constitute a three-dimensional solvable Lie algebra with these two, is required.

As a system of first-order equations (1) has an infinite number of Lie point symmetries. Infinity is not a satisfactory number. We use the second part of the method of reduction of order [18–20] to replace the system of three first-order equations by a system of one second-order equation and one first-order equation.

In the analysis of this system, obtained by the elimination of u_3 from (1a), one finds that the system is separable and linearisable in the case that $\alpha = \mu + \beta c$. The symmetry

$$\Gamma_3 = \exp\left[-(\mu+\nu)t\right]\partial_{u_2} + \frac{u_1+u_3}{u_2}\exp\left[-(\mu+\nu)t\right]\partial_{u_3}$$

provides the third element of the required solvable algebra.

The explicit solution to the system (1) is found to be

$$u_1 = \frac{e^{\nu t}c_2}{e^{\mu t}\left[e^{\nu t}\left(\beta\, c-\nu\right)c_1 + e^{\beta\, c\, t}\beta\, c\,\right]},$$

$$u_2 = \frac{\left(\beta\, c-\nu\right)\int \frac{e^{\beta\, c\, t+2\nu t}}{\left(e^{\beta\, c\, t}\beta\, c + e^{\nu t}\beta\, c\, c_1 - e^{\nu t}c_1\nu\right)^2}\,\mathrm{d}\, t}{\beta\, c\, c_2 + c_3} e^{\mu t+\nu t},$$

$$u_3 = \frac{\left[e^{\nu t}\left(\beta\, c-\nu\right)c_1 + e^{\beta\, c\, t}\nu\right]c_3}{e^{\beta\, c\, t+\mu t+\nu t}\left(\beta\, c-\nu\right)} + \frac{e^{\nu t}\, c_2}{e^{\mu t}\left[e^{\nu t}\left(\beta\, c-\nu\right)c_1 + e^{\beta\, c\, t}\beta\, c\,\right]}$$
$$+ \frac{\beta\, c\, c_2\left[e^{\nu t}\left(\beta\, c-\nu\right)c_1 + e^{\beta\, c\, t}\nu\right]\int e^{\beta\, c\, t+2\nu t}\left(e^{\beta\, c\, t}\beta\, c\, + e^{\nu t}\beta\, c\, c_1 - e^{\nu t}c_1\nu\right)^2 \mathrm{d}\, t}{\exp[\beta\, c\, t + \mu t + \nu t]}$$

for general values of the parameters subject to the constraint $\alpha = \mu + \beta c$.

There is a considerable simplification if in addition $\beta c = 2\nu$. The failure of the system to possess the Painlevé Property is quite manifest in this case for the solution is given in closed-form as

$$\begin{aligned}
u_1 &= \frac{c_1}{e^{\mu t}\left(2e^{\nu t} + c_1 c_2\right)},\\
u_2 &= \left[2e^{\nu t}\log\left(2e^{\nu t} + c_1 c_2\right)c_1 - 2e^{\nu t}c_1 + 4e^{\nu t}c_3\right.\\
&\quad \left. + \log\left(2e^{\nu t} + c_1 c_2\right)c_1^2 c_2 + 2c_1 c_2 c_3\right] / \left[2e^{\mu t+\nu t}\left(2e^{\nu t} + c_1 c_2\right)\right],\\
u_3 &= \left[e^{\nu t}\log\left(2e^{\nu t} + c_1 c_2\right)c_1 - 2e^{\nu t}c_1 + 2e^{\nu t}c_3\right.\\
&\quad \left. + \log\left(2e^{\nu t} + c_1 c_2\right)c_1^2 c_2 + 2c_1 c_2 c_3\right] / \left[2e^{\mu t+2\nu t}\right].
\end{aligned}$$

One notes with interest that the constraint, $\alpha = \mu + \beta c$, which leads to the integrability of the system (1) in the sense of Lie, fits quite well with observational data accumulated over the last twenty years in the United States.

This coincidence of a relationship between the parameters of a system leading to better properties in terms of integrability and the practical realisation of that relationship in terms of the modelling of reality is intriguing and is sufficient to prompt one to pursue the possibility of a similar coincidence in other models.

3 The classic $S - I - R$ model

The model introduced by Kermack and McKendrick [13] to describe the effects of the Black Death in the seventeenth century is

$$\dot{S} = -rSI, \qquad \dot{I} = rSI - aI, \qquad \dot{R} = aI, \tag{2}$$

where S is the proportion of the population susceptible to the infection, I the proportion of population infected by the infection and R that proportion of population removed from consideration either through recovery with immunity or death. The parameters r and a represent the rate of infection and the rate of removal respectively.

When we add the three equations comprising (2), we have

$$(S + I + R)^{\cdot} = 0,$$

i.e. $\dot{N} = 0$, where $N = S + I + R$.

Obviously $\dot{N} = 0 \implies N = \text{const} = 1$.

The parameter-free form of the system (2) is

$$\dot{x} = -xy, \qquad \dot{y} = xy - y, \qquad \dot{z} = y, \tag{3}$$

where

$$S = \frac{a}{r}x, \qquad I = \frac{a}{r}y, \qquad R = \frac{a}{r}z, \qquad t = \frac{t}{r},$$

with the obvious summation $(x + y + z)^{\cdot} = 0$.

The Painlevé analysis of the system (3) fails immediately since the exponents of the leading order terms are not all negative. An attempt to rescue the Painlevé analysis by the dropping of the third member of (3) is encouraging since the exponent of the leading order term of the remaining two variables is -1 and the resonances are at ± 1. Unfortunately the system fails the test for consistency. Consequently a logarithmic term must be introduced and this destroys the integrability in the sense of Painlevé.

In the system (3) the variable z is ignorable and the system is a candidate for the application of the technique of reduction of order. The quotients (3a)/(3c) and (3b)/(3c) are

$$\frac{\mathrm{d}x}{\mathrm{d}z} = -x \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}z} = x - 1. \tag{4}$$

We ignore the trivial route to the solution of the system (4) and persist with the spirit of the method of reduction of order to reduce the pair of first-order equations to a single second-order equation through the elimination of the dependent variable x. We obtain

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \frac{\mathrm{d}y}{\mathrm{d}z} = 1, \tag{5}$$

which is a linear second-order ordinary differential equation in $y(z)$, indeed, it is a quasi-first-order equation in the first derivative of the function.

The integration of (5) and hence (4) is trivial. However, in the case of (3c) we have

$$\dot{z} = A + z - B\mathrm{e}^{-z},$$

where A and B are the constants of integration introduced at the solution of (5), the quadrature of which in closed-form is not possible.

We note that in the reduction of the system (3) to (5) we have reduced the nonlinear system to a linear system and also that (5), as a linear second-order ordinary differential equation, possesses eight Lie point symmetries.

4 Some curious facets of gonorrhoea

A somewhat simplified model [16] for the sexually transmitted disease gonorrhoea is described by the system

$$\dot{S} = -sSI^* + aI, \qquad \dot{I} = rSI^* - aI, \qquad \dot{S}^* = -r^*S^*I + a^*I^*, \qquad \dot{I}^* = r^*S^*I - a^*I^*, \tag{6}$$

in which the lower case letters are the rate constants and the symbols for susceptibles and infectives are obvious. The population is divided into two groups, male and female, and each of these two groups is divided into susceptibles and infectives. The females are distinguished by the possession of an asterisk. The model assumes free mixing which is generally not a valid assumption for sexually transmitted diseases. However, as one is informed, the properties of gonorrhoea are such to permit this assumption. The simplification in the model is that, since gonorrhoea does not confer immunity and so a person who has been treated returns to the classes of susceptibles, the time of treatment, during which one hopes the interaction between infectives and susceptibles is suspended, is regarded as negligible. The populations of males and females are assumed to be constant.

There are three conservation laws, *videlicet*

$$\dot{N} = 0, \qquad \dot{N}^* = 0 \qquad \text{and} \qquad \dot{P} = 0, \tag{7}$$

where $N = S + I$, $N^* = S^* + I^*$ and $P = N + N^*$.

The leading order behaviour is given by

$$S = -\frac{1}{r^*}\tau^{-1}, \qquad I = \frac{1}{r^*}\tau^{-1}, \qquad S^* = -\frac{1}{r}\tau^{-1}, \qquad I^* = \frac{1}{r}\tau^{-1} \tag{8}$$

and the resonances occur at -1 and 1(3).

The consistency of the system is determined by the substitution of

$$S = a_i\tau^{i-1}, \qquad I = b_i\tau^{i-1}, \qquad S^* = c_i\tau^{i-1}, \qquad I^* = d_i\tau^{i-1}$$

into the full system (6) with the leading order terms as given in (8). Fortunately the location of the resonance at $+1$ makes the calculation fairly easy. At the positive resonance we obtain the system

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{r}{r^*} \\ -1 & 0 & 0 & \frac{r}{r^*} \\ 0 & -\frac{r^*}{r} & 1 & 0 \\ 0 & \frac{r^*}{r} & -1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} -\frac{a}{r^*} \\ \frac{a}{r^*} \\ -\frac{a^*}{r} \\ \frac{a^*}{r} \end{bmatrix}.$$

Performance of the row operations $R_2' = R_2 + R_1$ and $R_4' = R_4 + R_3$ leads to the simpler system

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{r}{r^*} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{r^*}{r} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} -\frac{a}{r^*} \\ 0 \\ -\frac{a^*}{r} \\ 0 \end{bmatrix}$$

in which the consistency is self-evident.

We have the explicit relations

$$a_1 = \frac{1}{r^*}\left(a + rd_1\right) \qquad \text{and} \qquad c_1 = \frac{1}{r}\left(a^* + r^*b_1\right). \tag{9}$$

There are two arbitrary constants introduced at the resonance, i.e. the geometric multiplicity of the eigenvector is two. However, the algebraic multiplicity of the eigenvalue is three.

The Laurent expansion contains only three arbitrary constants – the two introduced in (9) and the location of the movable pole – so that it does not represent the general solution of the system (6).

To obtain the general solution one must introduce a logarithmic term at the resonance and this destroys the analytic nature of the solution.

We continue our analysis of the system (6) by transforming the system of first-order equations to a scalar higher-order equation. From (7) we obtain

$$I = N - S \qquad \text{and} \qquad I^* = N^* - S^*. \tag{10}$$

With (10) equations (6b) and (6d) may be written as

$$-\dot{S} = rS(N^* - S^*) - a(N - S), \tag{11}$$

$$-\dot{S}^* = r^* S^* (N - S) - a^* (N^* - S^*). \tag{12}$$

We substitute for S^* from (11) into (12) to obtain a single second-order equation for S, *videlicet*

$$\begin{aligned} &S\ddot{S} - \dot{S}^2 - r^* S^2 \dot{S} + \left(r^* N + a^*\right) S\dot{S} + aN\dot{S} - r^* \left(N^* r + a\right) S^3 \\ &\quad + \left(NN^* r r^* + 2Nar^* + aa^*\right) S^2 - aN \left(r^* N + a^*\right) S = 0. \end{aligned} \tag{13}$$

The first three terms in (13) are dominant. The exponent of the leading order term is -1 and the resonances are at ± 1. To establish that there is consistency when the coefficient of the resonant term firstly enters we examine the coefficients in (summation on repeated indices is implied)

$$\begin{aligned} &(i-1)(i-2)a_i a_j \tau^{i+j-4} - (i-1)(j-1)a_i a_j \tau^{i+j-4} - (i-1) r^* a_i a_j a_k \tau^{i+j+k-4} \\ &\quad + \left(r^* N + a^*\right)(i-1) a_i a_j \tau^{i+j-3} + aN(i-1) a_i \tau^{i-2} - r^* \left(rN^* + a\right) a_i a_j a_k \tau^{i+j+k-3} \\ &\quad + \left(NN^* r r^* + 2aNr^* + aa^*\right) a_i a_j \tau^{i+j-2} - aA \left(r^* N + a^*\right) a_i \tau^{i-1} = 0. \end{aligned}$$

The coefficient of τ^{-4} gives

$$2a_0^2 - a_2^2 + r^* a_0^3 = 0 \implies a_0 = -\frac{1}{r^*} \tag{14}$$

and the coefficient of τ^{-3} gives

$$2a_0 a_1 + r^* a_0 (2a_2 a_1) - (r^* N + a^*) a_0^2 - r^* (rN^* + a) a_0^3 = 0. \tag{15}$$

When the value of a_0 from (14) is substituted into (15), the coefficient of a_1 is identically zero as one expects. The terms remaining impose the condition

$$r^* N + a = rN^* + a^*. \tag{16}$$

Subject to the constraint (16) on the parameters in (13) the latter equation has an analytic solution for $S(t)$. It follows from (11) that $S^*(t)$ is also analytic and from (10) that $I(t)$ and $I^*(t)$ are also analytic.

This analysis simply reinforces the conclusion reached above that the three-parameter solution obtained for the original system, (6), is analytic away from the movable polelike singularity.

This type of integrability, which occurs for specific values of the first integrals of the base system determined by the relationship (16), is something of a generalisation of the integrability which occurs when an integral takes a particular value, i.e. in the context of a configurational invariant [8, 23].

The solution obtained is not the general solution of (6) since it lacks the requisite four arbitrary constants of integration. However, we have demonstrated the existence of an analytic solution of (6) containing three arbitrary constants of integration. Such solutions are not unknown [11, p. 355].

In the past this type of solution has been interpreted as indicating that the system is at least integrable on a surface in the space of initial conditions [5, 22]. This interpretation has not been universally accepted [4]. Since the inference of an analytic solution has been the basis for the interpretation in the past, there has been room for disagreement over the interpretation. However, in this instance the interpretation is based upon explicit demonstration of the analytic nature of the solutions for each of the component functions of the system.

One instance does not prove a universal fact. On the other hand one instance does disprove the denial of the possibility.

The system (6) and the scalar equation (13) have the obvious Lie symmetry, ∂_t, which reflects their autonomy. Even with the constraint (16) there appears to be no other point symmetries. This is a little curious as one would expect some additional symmetry to match the better quality of the system in terms of integrability. Perhaps this is to be found as either a generalised or a nonlocal symmetry.

5 Decomposed systems

There are several standard models for the rate of change of a population which is of sufficient magnitude to be treated as a continuum. They are

$$\textbf{1}.\quad \frac{\mathrm{d}N}{\mathrm{d}t} = 0 \quad \Leftrightarrow \quad N(t) = N_0. \tag{17}$$

This is the trivial model and applies if one is considering a population over a period of time sufficiently small for the normal means of increase and decrease of population not to have any effect. The famous SIR model of Kermack and McKendrick [13], *videlicet*

$$\dot{S} = -rSI, \qquad \dot{I} = rSI - aI, \qquad \dot{R} = aI, \tag{18}$$

is an example for, if we sum the three equations, we obtain

$$(S + I + R)^{\cdot} = 0$$

which is just (17). One recalls that the system (18) was used to model the Great Plague of 1666-7 in London and did it very well.

$$\textbf{2}.\quad \frac{\mathrm{d}N}{\mathrm{d}t} = K \quad \Leftrightarrow \quad N(t) = N_0 + Kt. \tag{19}$$

The constant rate of increase of the population described by (19) enables one to do a little more than with (17) without a serious increase in mathematical difficulty. An integrable model of this genre is the $S - I - R$ system

$$\dot{S} = -\beta SI - \mu S + \gamma I + \mu K, \qquad \dot{I} = \beta SI - (\mu + \gamma)I, \qquad \dot{R} = \mu(S + I). \tag{20}$$

The sum of the three equations in the system (20) gives

$$(S + I + R)^{\cdot} = \mu K$$

which is precisely the form of (19).

$$\textbf{3}.\quad \frac{\mathrm{d}N}{\mathrm{d}t} = \sigma N \quad \Leftrightarrow \quad N(t) = N_0 \exp[\sigma t]. \tag{21}$$

This is the famous model publicised by the Englishman, Thomas Malthus, in 1798 [17]. An integrable model of this genre is the $S - I - S$ system [21]

$$\dot{S} = -\beta SI - \mu S + \gamma I + \mu K, \tag{22}$$

$$\dot{I} = \beta SI - (\mu + \gamma)I, \tag{23}$$

i.e. (20) when the third equation of the system is ignored in the analysis. We sum these two equations to obtain

$$(S + I)^{\cdot} = \mu K - \mu(S + I)$$

which is precisely of the type (21) if we set $S + I - K = N$.

$$\textbf{4}.\quad \frac{\mathrm{d}N}{\mathrm{d}t} = \sigma N \left(1 - \frac{N}{C}\right) \quad \Leftrightarrow \quad N(t) = \frac{CN_0}{N_0 + (C - N_0)\exp[-\sigma t]} \tag{24}$$

in which the additional parameter C is of the nature of a 'carrying capacity'. This variation was introduced in 1838 [25] by Verhulst to obviate the excesses to which the model of Malthus led.

These four models may be regarded as successive approximations of the same phenomenon. For a short period of time one could treat the population as a constant. For a longer period of time a constant rate of increase is not unreasonable. The next model is to assume that the rate is proportional to the existing population and finally one must take into account the limitations of the environment in which the population is growing by adding the additional term due to Verhulst.

Because these different models have validity for different regimes it is possible to find a mixture of growth rates in a population containing varied components. An early example of this is found in an extension to the model proposed by Volterra [26,3] of a predator-prey system of two types of fish in the Adriatic. The model of Volterra is

$$\dot{N} = aN - bPN, \qquad \dot{P} = -cP + dPN. \tag{25}$$

A similar model, connected to chemical reactions, was proposed by Lotka [15] about the same time.

The bilinear term PN is typical of Lotka–Volterra models. In this model the prey N is taken to have Malthusian growth with the implicit assumption that the predator P consumes sufficient of the prey for the Malthusian growth not to test the limits of the carrying capacity of the local environment. If this be not the case, the Malthusian term should be replaced by a logistic term – for some reason generally Verhulst is forgotten in the naming – so that (25a) is replaced by

$$\dot{N} = aN\left(1 - \frac{N}{C}\right) - bPN.$$

Since the Malthusian term in (25b) gives a reduction in the population of the predator, there is no need to introduce a logistic term here.

The four model equations, *videlicet* (17), (19), (21) and (24), are trivially integrable and possess the Painlevé Property.

The systems (18), (20) and (22)–(23) have the property that their sums are simply (17), (19) and (21) respectively.

The same behaviour is not found with the Lotka–Volterra model (25) except for unexpectedly agreeable values of the parameters.

Systems such as the $S - I - R$ model represented by (18) which can be summed to give a single scalar equation in one variable are called 'decomposed systems' [2] and may be viewed as decomposition of the single equation, (17) in the case of (18), according to some rule.

Under which rules of decomposition are the decomposed systems integrable?

The general decomposition cannot be expected to be integrable. We consider an example [6, 7, 14] which has its origin in the operator Yang–Baxter equations of Mathematical Physics, but we do so in reverse. The two-dimensional system

$$\dot{r} = 2rw, \qquad \dot{w} = r^2 + w^2 \tag{26}$$

can be considered to be the decomposition of the Riccati equation

$$\dot{z} = z^2, \tag{27}$$

where $z = r + w$. A further decomposition is

$$\dot{p} = 2pw + \lambda qw, \qquad \dot{q} = 2qw - \lambda wp, \qquad \dot{w} = (p+q)^2 + w^2, \tag{28}$$

where $r = p + q$.

As a Riccati equation (27) possesses the Painlevé Property and is integrable in terms of analytic functions. The first decomposed system (26) is equally integrable. The second decomposed system (28) is integrable in the sense that the dependent variables can be written as explicit expressions of the independent variable, but the system does not possess the Painlevé Property since its solution is not in terms of analytic functions.

A whole class of problems has been excluded from even the possibility of consideration here. We are concerned with populations with a rate of change which is either independent of the population or proportional in some sense to the total population. In some populations, microbes come to mind, this is not unreasonable as microbes become capable of replication rather quickly. On the other hand populations comprising more complicated species can be divided into the three classes of nonreproductive due to immaturity, reproductive and nonreproductive due to – shall we politely put it – postmaturity.

This class of population lies outside the class of decomposible systems considered here.

Even models for which the reproductive aspect is not of major relevance can still exhibit various ways to be decomposed from a simpler system. A model for HIV infection in a homosexual population,

$$\begin{aligned} \dot{X} &= B - \mu X - \lambda c X, \qquad \dot{Y} = \lambda c X - (\nu + \mu) Y, \\ \dot{A} &= p\nu Y - (d + \mu) A, \qquad \dot{Z} = (1 - p)\nu Y - \mu Z, \end{aligned} \tag{29}$$

sums to give the rate of change of the total population as

$$\dot{N} = B - \mu N - dA,$$

where B represents the recruitment rate of susceptibles into the population, μ is the natural rate of demise and d the additional rate of demise of the AIDS-afflicted portion of the population A.

In the absence of AIDS the system (29) fits into the scheme of decomposition. In the absence of AIDS perhaps the population being modelled would not attract so much interest.

Our present interest in decomposible systems is motivated by the existence of decomposed systems as models of actual phenomena, in particular epidemiology.

Decomposed systems entered the literature from a more mathematical direction. In 2002 Imai and Hirata [9] published their investigations of the existence of autonomous symmetries of the form

$$\Gamma = \phi_i(x) \partial_{x_i}$$

for the autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{g}(x),$$

where $\mathbf{g}(x)$ and the coefficient functions $\phi_i(x)$ are analytic in the neighbourhood of some fixed point. Their interest was in the determination of integrable systems through an algebraic approach. As applications they considered n-dimensional homogeneous, that is to say quadratic, Lotka–Volterra systems of the form

$$\dot{x} = x_i \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, n. \tag{30}$$

Imai and Hirata considered some very specific examples of (30) which they termed the ladder system and the generalised ladder system [10].

The n-dimensional ladder system is the system (30) with the specific coefficient matrix

$$A = (a_{ij}) = \begin{bmatrix} 1 & 0 & \cdots & -n+2 \\ 2 & 1 & \cdots & -n+3 \\ \vdots & \vdots & & \vdots \\ n & n-1 & \cdots & 1 \end{bmatrix},$$

i.e. the elements of the matrix A are given by

$$a_{ij} = a_i + 1 - a_j \qquad \text{and} \qquad a_{i+1,j} - a_{ij} = 1.$$

The ladder system has certain general properties [1].

Proposition 1.

$$\det A = \begin{cases} 1, & n = 1, 2, \\ 0, & n > 2. \end{cases}$$

Corollary. *The rank of the matrix A is two for $n > 2$.*

Proposition 2. *The n-dimensional ladder system can be considered as the simplest Riccati equation for $\sum_{i=1}^{n} x_i$.*

Proposition 3. *The ratio of two successive solutions of the ladder system is*

$$\frac{x_{i+1}}{x_i} = \frac{K_i}{t - t_0}.$$

Proposition 4. *The solution of the n-dimensional ladder system is given by*

$$x_{i+1} = x_1 \frac{\prod_{j=1}^{i} K_j}{(t - t_0)^i}, \tag{31}$$

where

$$x_1 = -\frac{(t - t_0)^{n-2}}{\sum_{j=0}^{n-1} \left(\prod_{k=0}^{j} K_k \right) (t - t_0)^{n-1-j}}. \tag{32}$$

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Towards the Group Classification of Control Systems

Victor I. LEHENKYI † *and Joachim RUDOLPH* ‡

† *Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *victor@imath.kiev.ua*

‡ *Institut für Regelungs- und Steuerungstheorie, TU Dresden,*
Mommsenstr. 13, D-01062 Dresden, Germany
E-mail: *Rudolph@erss11.et.tu-dresden.de*

We consider the group classification problem for the class of control systems which is described by systems of ordinary differential equations, and we give the full group classification for the second-order control systems.

1 Introduction

Recently there has been quite a few publications dealing with the symmetries of control systems. Nevertheless, most of them deal with particular symmetry problems for concrete control systems. Others in some, perhaps hidden, form attempt to solve the problem of finding symmetries for the "general case". It means that the problem is formulated for control systems of the form

$$\dot{x}^i = f^i(t,x,u), \qquad i=\overline{1,n}, \quad t\in\mathbb{T}\subset\mathbb{R}_+, \quad u\in\mathbb{U}^*\subset\mathbb{R}^r, \quad x\in\mathbb{X}^*\subset\mathbb{R}^n \tag{1}$$

for arbitrary $\{f^i(\cdot), n, r\}$. Symmetry conditions for the coefficients τ, ξ^i, φ^j of the infinitesimal generator of the form

$$X=\tau(t,x,u)\frac{\partial}{\partial t}+\xi^i(t,x,u)\frac{\partial}{\partial x^i}+\varphi^j(t,x,u)\frac{\partial}{\partial u^j} \tag{2}$$

read as

$$Xf^i-X_0\xi^i+f^iX_0\tau=0, \qquad U_j\xi^i+f^iU_j\tau=0, \qquad i=\overline{1,n}, \quad j=\overline{1,r}, \tag{3}$$

where

$$X_0=\frac{\partial}{\partial t}+f^i\frac{\partial}{\partial x^i}, \qquad U_j=\frac{\partial}{\partial u^j}. \tag{4}$$

Up to now there is no recipe for solving system (3) with arbitrary functions $f^i(\cdot)$ and arbitrary numbers (n,r). So, we need to fix something. Following L. Ovsiannikov's approach, we will describe our systems by fixing (n,r) for arbitrary $f^i(\cdot)$ and, hence, we will consider a classical group classification problem for some class of control systems. In the present paper we consider the case $n=2$, but before that we give some remarks on a definition of equivalence of control systems.

2 Feedback equivalence and essential controls of control systems

Definition 1. System (1) and any system which can be derived from (1) via a diffeomorphism $\hat{t}=\hat{t}(t,x), \hat{x}=\hat{x}(t,x), \hat{u}=\hat{u}(t,x,u)$ are called feedback equivalent control systems.

Based on this definition we can define also the notion of so called "essential controls", a natural generalization of the notion of "essential parameters" (see, for example, [2]).

Definition 2. Controls u^j are called essential ones if there are no controls v^k $(k < r)$, for which $f^i(t,x,u) = \hat{f}^i(t,x,v)$.

Using Definitions 1 and 2 we can give the following proposition.

Proposition 1. *Any control system with r essential controls is feedback equivalent to the system of the (canonical) form*

$$\begin{aligned}
&\dot{x}^1 = f^1(t,x,u),\\
&\dot{x}^2 = f^2(t,x,u),\\
&\cdots\cdots\cdots\cdots\\
&\dot{x}^{n-r} = f^{n-r}(t,x,u),\\
&\dot{x}^{n-r+1} = u^1,\\
&\dot{x}^{n-r+2} = u^2,\\
&\cdots\cdots\cdots\cdots\\
&\dot{x}^n = u^r.
\end{aligned} \tag{5}$$

It follows from Proposition 1 that there is only one canonical one-dimensional $(n = 1)$ control system $\dot{x} = u$, so the group classification of this system is trivial. Thus, we start from the case $n = 2$.

3 Canonical forms for second-order control systems

Obviously (see Proposition 1), there exist only two canonical forms for second-order control systems. One of them (with 2 controls) reads

$$\begin{aligned}
&\dot{x}^1 = u^1,\\
&\dot{x}^2 = u^2,
\end{aligned} \tag{6}$$

and the second one (with a single control) looks like

$$\begin{aligned}
&\dot{x}^1 = F\left(t, x^1, x^2, u\right),\\
&\dot{x}^2 = u.
\end{aligned} \tag{7}$$

Of course, we ignore the case $u^j = 0$. System (6) does not contain any arbitrary elements and its symmetry generator is given by the direct sum $Y = Y_1 \oplus Y_2 \oplus Y_3$ with

$$\begin{aligned}
&Y_1 = \tau\partial_t - Y_0\tau\left(u^1\partial_{u^1} + u^2\partial_{u^2}\right),\\
&Y_2 = \xi^1\partial_{x^1} + \left(Y_0\xi^1\right)\partial_{u^1}, \qquad Y_3 = \xi^2\partial_{x^2} + \left(Y_0\xi^2\right)\partial_{u^2},
\end{aligned} \tag{8}$$

with $\tau = \tau(t,x)$, $\xi^i = \xi^i(t,x)$ arbitrary functions and $Y_0 = \partial_t + u^1\partial_{u^1} + u^2\partial_{u^2}$.

4 Controllability

Controllability (strong accessibility) is known to be directly related to the existence of invariants (first integrals). As we can see below, controllability plays an important role in the group properties of control systems.

Definition 3. We call system (7) (locally) uncontrollable, if (locally) all its solutions (trajectories) lie in an invariant manifold of the form $\omega\left(t, x^1, x^2\right) = C$.

Thus, the existence of such a manifold means that there exists a function ω depending on (t, x^1, x^2) such that $\dot{\omega} = 0$ (i.e., ω is invariant under the action of the input u).

We will now discuss for which choice of $F\left(t, x^1, x^2, u\right)$ in (7) there exist such invariants. To this end we interpret the invariant $\omega\left(t, x^1, x^2\right) = C$ as a nontrivial solution of the system of partial differential equations

$$\frac{\partial \omega}{\partial t} + F\left(t, x^1, x^2, u\right)\frac{\partial \omega}{\partial x^1} + u\frac{\partial \omega}{\partial x^2} = 0, \qquad \frac{\partial \omega}{\partial u} = 0, \tag{9}$$

where the first equation means that ω must be a first integral of system (7), and the second condition means that this first integral must be independent of u. If we denote

$$X_0 = \frac{\partial}{\partial t} + F\left(t, x^1, x^2, u\right)\frac{\partial}{\partial x^1} + u\frac{\partial}{\partial x^2}, \qquad U = \frac{\partial}{\partial u}, \tag{10}$$

system (9) takes the form

$$X_0\omega = 0, \qquad U\omega = 0. \tag{11}$$

In order to look for the functionally independent solutions of system (11), we need to calculate the involutive closure [3] for a distribution of vector fields given as (10). We get the Lie brackets

$$\begin{aligned} &X_1 = [U, X_0] = F_u\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \qquad X_2 = [U, X_1] = F_{uu}\frac{\partial}{\partial x^1}, \\ &X_3 = [X_0, X_1] = \left(F_{ut} + FF_{ux^1} + uF_{ux^2} - F_uF_{x^1} - F_{x^2}\right)\frac{\partial}{\partial x^1}. \end{aligned} \tag{12}$$

System (7) admits a first integral if for the above vector fields the two conditions $U \wedge X_0 \wedge X_1 \wedge X_2 = 0$ and $U \wedge X_0 \wedge X_1 \wedge X_3 = 0$ are fulfilled simultaneously. The first condition reads

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & F & u \\ 0 & 0 & F_u & 1 \\ 0 & 0 & F_{uu} & 0 \end{vmatrix} = F_{uu} = 0, \tag{13}$$

the second is

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & F & u \\ 0 & 0 & F_u & 1 \\ 0 & 0 & F_{ut} + FF_{ux^1} + uF_{ux^2} - F_uF_{x^1} - F_{x^2} & 0 \end{vmatrix} = 0. \tag{14}$$

Therefore, system (7) is uncontrollable if and only if the function F satisfies the following two conditions:

$$F_{uu} = 0, \qquad F_{ut} + FF_{ux^1} + uF_{ux^2} - F_uF_{x^1} - F_{x^2} = 0. \tag{15}$$

One direct implication of these results is the following.

Corollary 1. *System* (7) *is uncontrollable if and only if* $F = \alpha\left(t, x^1, x^2\right)u + \beta\left(t, x^1, x^2\right)$ *and the coefficients* α *and* β *satisfy*

$$\alpha\beta_{x^1} - \beta\alpha_{x^1} - \alpha_t + \beta_{x^2} = 0.$$

Example 1. The linear system

$$\dot{x}^1 = a_1 x^1 + a_2 x^2 + bu,$$
$$\dot{x}^2 = u$$

will be uncontrollable if and only if the condition $ba_1 + a_2 = 0$ holds true. Indeed, elimination of u leads us to the Pfaffian equation

$$dx^1 - bdx^2 + a_1 \left(bx^2 - x^1\right) dt = 0,$$

which has solutions of the form

$$\omega(t, x^1, x^2) = \left(bx^2 - x^1\right) e^{-a_1 t} = C.$$

5 Group classification for system (7)

As we analyze a whole class of systems of the form (7), with arbitrary function F, there naturally occurs a problem of group classification [4]: For the class of systems (7) find the kernel of basic groups GE_0 and all specializations of the arbitrary function F which allow us to extend the basic groups.

Notice that in accordance with the terminology of the book [4], the kernel of basic groups is the symmetry group admitted by system (7) for all specializations of the function F. Therefore, in our case the kernel of basic groups is empty (i.e., $GE_0 = \{0\}$), because the function F depends on all variables.

Hence, we start our analysis deriving the so-called *determining equations*. In order to determine the coefficients $\tau\left(t, x^1, x^2, u\right)$, $\xi^i\left(t, x^1, x^2, u\right)$, $\varphi\left(t, x^1, x^2, u\right)$ of the infinitesimal symmetry operator

$$X = \tau \frac{\partial}{\partial t} + \xi^1 \frac{\partial}{\partial x^1} + \xi^2 \frac{\partial}{\partial x^2} + \varphi \frac{\partial}{\partial u} \tag{16}$$

one needs to solve the system

$$Xf^i - X_0 \xi^i + f^i X_0 \tau = 0, \qquad U\xi^i + f^i U\tau = 0, \qquad i = 1, 2, \tag{17}$$

where $f^1 = F$, $f^2 = u$ (see also [5]). Substituting

$$\xi^i = f^i \tau + \hat{\xi}^i \tag{18}$$

in (17) we get

$$\hat{X} f^i - X_0 \hat{\xi}^i = 0 \qquad U\hat{\xi}^i + U(f^i)\tau = 0, \qquad i = 1, 2, \tag{19}$$

where

$$\hat{X} = \hat{\xi}^1 \frac{\partial}{\partial x^1} + \hat{\xi}^2 \frac{\partial}{\partial x^2} + \varphi \frac{\partial}{\partial u}.$$

If in (19) we replace f^2 by u we obtain

$$\varphi = X_0 \hat{\xi}^2, \qquad \tau = -U\hat{\xi}^2. \tag{20}$$

Now substituting (τ, φ) in (17) and replacing f^1 by F we get

$$\hat{\xi}^1 F_{x^1} + \hat{\xi}^2 F_{x^2} + F_u X_0 \hat{\xi}^2 - X_0 \hat{\xi}^1 = 0, \qquad U\hat{\xi}^1 - F_u U\hat{\xi}^2 = 0. \tag{21}$$

If we introduce

$$\sigma = \hat{\xi}^1 - F_u \hat{\xi}^2, \tag{22}$$

from the second equation of (21) we get

$$F_{uu}\hat{\xi}^2 + \sigma_u = 0. \tag{23}$$

Assuming $F_{uu} \neq 0$, solve (22) and (23) for $(\hat{\xi}^1, \hat{\xi}^2)$:

$$\hat{\xi}^1 = \sigma - F_u F_{uu}^{-1}\sigma_u, \qquad \hat{\xi}^2 = -F_{uu}^{-1}\sigma_u. \tag{24}$$

Substituting (24) in the first equation of the system (21), we finally obtain

$$F_{uu}(\sigma_t + F\sigma_{x^1} + u\sigma_{x^2}) + (F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2})\sigma_u = F_{uu}F_{x^1}\sigma. \tag{25}$$

Using some substitutions, we reduced our problem to the single equation (25) for the function σ. In accordance with the terminology of [6], this function is called the generating function of symmetries. Indeed, if we are able to solve equation (25) for a particular specialization of the function F then all coefficients can be calculated via

$$\begin{aligned}
&\tau = U\left(F_{uu}^{-1}\sigma_u\right),\\
&\xi^1 = \sigma + FU\left(F_{uu}^{-1}\sigma_u\right) - F_u F_{uu}^{-1}\sigma_u,\\
&\xi^2 = uU\left(F_{uu}^{-1}\sigma_u\right) - F_{uu}^{-1}\sigma_u,\\
&\varphi = -X_0\left(F_{uu}^{-1}\sigma_u\right),
\end{aligned}$$

where $U = \partial_u$. Equation (25) is a quasi-linear inhomogeneous partial differential equation. The corresponding system of characteristics for this equation reads as

$$\begin{aligned}
&\dot{x}^1 = F(t, x^1, x^2, u),\\
&\dot{x}^2 = u,\\
&\dot{u} = F_{uu}^{-1}(F_u F_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2}),\\
&\dot{\sigma} = F_{x^1}\sigma.
\end{aligned} \tag{26}$$

Thus, the problem of calculation of the point symmetries for (open loop) second-order control systems is reduced to the problem of integration of (closed loop) system (26). Conversely, if we know some symmetries (coefficients τ, ξ^1, ξ^2), we are able to give some first integral of system (26). Indeed, converting formulas (18), (22), (24), we have

$$\sigma = \xi^1 - F_u\xi^2 + (uF_u - F)\tau. \tag{27}$$

Remark 1. With system (7) is associated the Pfaffian system $I = \{\omega^1, \omega^2\}$, where

$$\omega^1 = dx^1 - Fdt, \qquad \omega^2 = dx^2 - udt \tag{28}$$

on a subset of $\mathbb{R}^4$. With I is associated the so-called derived flag [1]. The first derived system $I^{(1)}$ includes the single one-form ω:

$$I^{(1)} = \{\omega = dx^1 + (uF_u - F)dt - F_u dx^2\}. \tag{29}$$

With this we are in a position to define the formula (27) in the intrinsic form

$$\sigma = X \rfloor \omega. \tag{30}$$

where ω is the generator of the first derived flag $I^{(1)}$, $\rfloor$ the inner product, and X the symmetry generator (16).

The conditions (15) for uncontrollability play the role of the classification conditions for equation (25), and if they are met the symmetry algebra is wider (see also [5] for further details).

Let us consider some examples.

Example 2. The linear time-varying system

$$\dot{x}^1 = tu + x^2,$$
$$\dot{x}^2 = u$$

is uncontrollable. Condition (15) is fulfilled and the system has a first integral

$$x^1 - tx^2 = C.$$

The symmetry algebra for this system is spanned by the vector fields

$$X_1 = \gamma(x^1 - tx^2)\partial_{x^1},$$
$$X_2 = -\sigma_u\partial_t + \left(t\sigma - \left(tu + x^2\right)\sigma_u\right)\partial_{x^1} + (\sigma - u\sigma_u)\partial_{x^2} + \left(\sigma_{tu} + \left(tu + x^2\right)\sigma_{ux^1} + u\sigma_{ux^2}\right)\partial_u,$$

where $\gamma\left(x^1 - tx^2\right)$ and $\sigma\left(t, x^1, x^2, u\right)$ are arbitrary functions of the given arguments.

Example 3. For the system in Brunovský normal form

$$\dot{x}^1 = x^2,$$
$$\dot{x}^2 = u$$

one has $F_{uu} = 0$. Thus, in accordance with (25), $\sigma_u = 0$ and $\sigma = \sigma\left(t, x^1, x^2\right)$. The infinitesimal symmetry operator takes the form

$$X = -\sigma_{x^2}\partial_t + \left(\sigma - x^2\sigma_{x^2}\right)\partial_{x^1} + \left(\sigma_t + x^2\sigma_{x^1}\right)\partial_{x^2} + X_0^2(\sigma)\partial_u,$$

where $X_0 = \partial_t + x^2\partial_{x^1} + u\partial_{x^2}$, $\sigma = \sigma\left(t, x^1, x^2\right)$.

Example 4. For the nonlinear system

$$\dot{x}^1 = u^2,$$
$$\dot{x}^2 = u$$

the equation $F_uF_{x^1} + F_{x^2} - F_{tu} - FF_{ux^1} - uF_{ux^2} = 0$ is fulfilled, so (from (25)) it follows that $X_0\sigma = 0$, and $\sigma = \sigma\left(u, x^1 - tu^2, x^2 - tu\right)$. The symmetry operator is

$$X = \sigma_{uu}\partial_t + \left(u^2\sigma_{uu} - 2u\sigma_u + \sigma\right)\partial_{x^1} + (u\sigma_{uu} - \sigma_u)\partial_{x^2} - \left(\sigma_{tu} + u^2\sigma_{x^1u} + u\sigma_{x^2u}\right)\partial_u.$$

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On Integrable Systems of ODEs

Maxim W. LUTFULLIN

Poltava State Pedagogical University, 2 Ostrogradsky Str., 36003 Poltava, Ukraine
E-mail: *MWL@pdpu.septor.net.ua*

We classify normal systems of three and four first-order ordinary differential equations (ODEs) that are invariant with respect to three- and four-dimensional solvable real Lie algebras respectively, and therefore can be integrated by the Lie method.

Description of the most general form of systems of differential equations of some type that are invariant with respect to the algebras of a fixed dimension is one of the main problems of group analysis. The standard Lie algorithm leads to classification of such systems if all the realizations of these algebras in Lie vector fields are known [5]. For instance, in [1] the second-order partial differential equations that are invariant under the standard realizations of the Euclid, Poincaré, Galilei, conformal and projective algebras were obtained. New realizations of the above algebras was constructed and a partial differential equations invariant with respect to these realizations were described in [2, 3, 7] (see also references therein).

A necessary step for searching realizations of Lie algebras is classification of these algebras, i.e. classification of possible commutative relations between basic elements. As for solvable Lie algebras, classification was completely made for only low-dimensional algebras. All possible non-isomorphic complex Lie algebras of dimension $N \leq 4$ were listed by S. Lie himself. G.M. Mubarakzyanov was the first who got the complete and suitable for application classification of the real Lie algebras of dimension $N \leq 4$ [4]. In paper [6] that are based on his classification realizations of these algebras in the spaces of the arbitrary finite number of the variables have been classified.

In this paper we construct the systems of the ODEs that are invariant with respect to three- and four-dimensional solvable real Lie algebras using results of [6]. The numeration of algebras and realizations corresponds to that paper: $R(A, N)$ denotes the Nth realization of the algebra A (see Tables 2–4) in [6].

Consider systems of ODEs of the form

$$F_k(t, x, \dot{x}) = 0, \tag{1}$$

where F_k are smooth functions, $k = 1, 2, 3$ for three-dimensional Lie algebras and $k = 1, \ldots, 4$ for four-dimensional ones. We will imply that the variable t is independent, and $x = (x_1, x_2, \ldots, x_k)$ variables are dependent. We denote derivatives with dots over the symbols $\dot{x}_i = \frac{dx_i}{dt}$, $\dot{x} = (\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_k)$.

Let us note that invariant systems obtained by various choices of the independent variable in basis elements of realizations are equivalent to each other. The variables are chosen to simplify calculations in a such way that the canonic forms of realizations would not contain the operator of differentiation with respect to the independent variable. A simplest general form of invariant systems corresponds to this choice. An arbitrary realization of an m-dimensional Lie algebra ($m \leq 4$) is equivalent to a realization into which no more than $m + 1$ first variables explicitly enter. (There is exception is only for the realizations of algebra $4A_1$ with rank 2 and 3, namely $R(4A_1, 2)$, $R(4A_1, 5)$). Under the choice of the respective coordinates and the independent variable ($t = x_{m+1}$), all the variables x_i and $\dot{x}_i$, $i \in \{m+2, \ldots, n\}$ will be invariants. Therefore, we search the differential invariants of realizations of three- and four-dimensional solvable Lie algebras in the space with three (four) dependent variables.

At first we construct the systems of three ODEs that are invariant with respect to realizations of three-dimensional solvable Lie algebras (see Table 2 [6]). Let $x_4 = t$ be an independent variable and x_1, x_2, x_3 be functions of t.

We find the functional bases of the first-order differential invariants for every realization of three-dimensional solvable Lie algebras. The corresponding calculations are not given here as there are quite cumbersome. Using the constructed sets of differential invariants of Lie algebras we obtain the exhaustive list of inequivalent (with respect to arbitrary regular changes of variables) systems of first-order ODEs that are invariant under three-dimensional solvable real Lie algebras.

By the direct verification we make sure that such systems may be reduced to the normal forms if the rank of the corresponding realization equals three.

Theorem 1. *Let system of the three ordinary first-order ODEs be invariant with respect to a realization of a real three-dimensional solvable Lie algebra with rank* 3*. Then the system is integrable in quadratures by means of the Lie method and using the same transformation of variables, which reduce the realization to one of the realizations given in Table* 2 *of* [6]*, we can transform the system to one of the following normal forms:*

$R(3A_1, 1)$ $\quad \dot{x}_1 = f_1(t), \quad \dot{x}_2 = f_2(t), \quad \dot{x}_3 = f_3(t);$

$R(A_{2.1} \oplus A_1, 1)$ $\quad \dot{x}_1 = f_1(t)e^{x_3}, \quad \dot{x}_2 = f_2(t), \quad \dot{x}_3 = f_3(t);$

$R(A_{3.1}, 1)$ $\quad \dot{x}_1 = f_1(t) + x_3\, f_2(t), \quad \dot{x}_2 = f_2(t), \quad \dot{x}_3 = f_3(t);$

$R(A_{3.2}, 1)$ $\quad \dot{x}_1 = [f_1(t) + x_3 f_2(t)]e^{x_3}, \ \dot{x}_2 = f_2(t)e^{x_3}, \ \dot{x}_3 = f_3(t);$

$R(A_{3.3}, 1)$ $\quad \dot{x}_1 = f_1(t)e^{x_3}, \quad \dot{x}_2 = f_2(t)e^{x_3}, \quad \dot{x}_3 = f_3(t);$

$R(A_{3.4}, 1)$ $\quad \dot{x}_1 = f_1(t)e^{x_3}, \quad \dot{x}_2 = f_2(t)e^{ax_3}, \quad \dot{x}_3 = f_3(t);$

$R(A_{3.5}, 1)$ $\quad \dot{x}_1 = f_1(t)e^{bx_3}\sin(x_3 + f_2(t)), \ \dot{x}_2 = f_1(t)e^{bx_3}\cos(x_3 + f_2(t)), \ \dot{x}_3 = f_3(t).$

Further we carry out the similar construction of differential invariants of realizations of four-dimensional solvable real Lie algebras, assuming $x_5 = t$ is the independent variable and x_1, x_2, x_3, x_4 are functions of t. We choose only the realizations that exist in the space of variables x_1, x_2, x_3, x_4, $x_5 = t$.

As to systems of four ODEs which are invariant with respect to the four-dimensional solvable real Lie algebras, in a similar way we obtain the following theorem.

Theorem 2. *Let system of the four ordinary first-order ODEs be invariant with respect to a realization of a real four-dimensional solvable Lie algebra with rank* 4*. Then the system is integrable in quadratures with Lie's method and using the same transformation of variables which reduce the realization to one of the realizations given in Tables* 3 *and* 4 *of* [6] *we can transform the system to one of the following normal forms:*

$R(A_{4.1}, 1)$ $\quad \dot{x}_1 = f_1(t) + x_4 f_2(t) + \dfrac{1}{2} f_3(t) {x_4}^2, \ \dot{x}_2 = f_2(t) + x_4\, f_1(t),$

$\dot{x}_3 = f_3(t), \quad \dot{x}_4 = f_4(t);$

$R(A_{4.2}, 1)$ $\quad \dot{x}_1 = f_1(t)e^{qx_4}, \ \dot{x}_2 = [x_4 + f_2(t)]f_3(t)e^{x_4}, \ \dot{x}_3 = f_3(t)e^{x_4}, \ \dot{x}_4 = f_4(t);$

$R(A_{4.3}, 1)$ $\quad \dot{x}_1 = f_1(t)e^{x_4}, \quad \dot{x}_2 = x_4\, f_3(t) + f_2(t), \ \dot{x}_3 = f_3(t), \quad \dot{x}_4 = f_4(t);$

$R(A_{4.4}, 1)$ $\quad \dot{x}_1 = [\dfrac{1}{2}{x_4}^2 + x_4\, f_2(t) + f_1(t)]f_3(t)e^{x_4},$

$\dot{x}_2 = [x_4 + f_2(t)]f_3(t)e^{x_4}, \ \dot{x}_3 = f_3(t)e^{x_4}, \quad \dot{x}_4 = f_4(t);$

$R(A_{4.5},1)$ $\dot{x}_1 = f_1(t)e^{ax_4},\ \dot{x}_2 = f_2(t)f_1{}^{\frac{b}{a}}(t)\,x_4^b,\ \dot{x}_3 = f_3(t)f_1{}^{\frac{c}{a}}(t)\,x_4^c,\ \dot{x}_4 = f_4(t);$

$R(A_{4.6},1)$ $\dot{x}_1 = f_1(t)e^{qx_4},\quad \dot{x}_2 = -f_3(t)e^{2px_4}\sin(f_2(t)+x_4),$

$\dot{x}_3 = f_3(t)e^{2px_4}\cos(f_2(t)+x_4),\quad \dot{x}_4 = f_4(t);$

$R(A_{4.7},1)$ $\dot{x}_1 = [f_1(t)e^{x_4} + x_3(f_2(t)+x_4)]\,f_3(t)e^{x_4},$

$\dot{x}_2 = [f_2(t)+x_4]\,f_3(t)e^{x_4},\ \dot{x}_3 = f_3(t)e^{x_4},\quad \dot{x}_4 = f_4(t);$

$R(A_{4.8},1)$ $\dot{x}_1 = [f_1(t)e^{qx_4}+x_3]f_2(t)e^{x_4},\ \dot{x}_2 = f_2(t)e^{x_4},\ \dot{x}_3 = f_3(t)e^{qx_4},\ \dot{x}_4 = f_4(t);$

$R(A_{4.9},1)$ $\dot{x}_1 = [(f_1(t)+ x_3f_2(t)\sin(x_4+f_3(t))]e^{2qx_4},\ \dot{x}_2 = f_2(t)e^{2qx_4}\sin(x_4+f_3(t)),$

$\dot{x}_3 = f_3(t)e^{2qx_4}\cos(x_4+f_3(t)),\ \dot{x}_4 = f_4(t);$

$R(A_{4.10},1)$ $\dot{x}_1 = f_1(t)e^{2x_3}\sin(x_4+f_2(t)),$

$\dot{x}_2 = f_1(t)e^{2x_3}\cos(x_4+f_2(t)),\quad \dot{x}_3 = f_3(t),\quad \dot{x}_4 = f_4(t);$

$R(4\,A_1,1)$ $\dot{x}_1 = f_1(t),\quad \dot{x}_2 = f_2(t),\quad \dot{x}_3 = f_3(t),\quad \dot{x}_4 = f_4(t);$

$R(A_{2.1}\oplus 2A_1,1)$ $\dot{x}_1 = f_1(t)e^{x_4},\quad \dot{x}_2 = f_2(t),\quad \dot{x}_3 = f_3(t),\quad \dot{x}_4 = f_4(t);$

$R(A_{2.1}\oplus A_{2.1},1)$ $\dot{x}_1 = f_1(t)e^{x_3},\ \dot{x}_2 = f_2(t)e^{x_4},\ \dot{x}_3 = f_3(t),\ \dot{x}_4 = f_4(t);$

$R(A_{3.1}\oplus A_1,1)$ $\dot{x}_1 = f_1(t)+x_4f_3(t),\ \dot{x}_2 = f_2(t),\ \dot{x}_3 = f_3(t),\ \dot{x}_4 = f_4(t);$

$R(A_{3.2}\oplus A_1,1)$ $\dot{x}_1 = [x_3+f_1(t)]f_2(t)e^{x_3},\ \dot{x}_2 = f_2(t)e^{x_3},\ \dot{x}_3 = f_3(t),\ \dot{x}_4 = f_4(t);$

$R(A_{3.3}\oplus A_1,1)$ $\dot{x}_1 = f_1(t)e^{x_3},\ \dot{x}_2 = f_2(t)e^{x_3},\ \dot{x}_3 = f_3(t),\ \dot{x}_4 = f_4(t);$

$R(A_{3.4}\oplus A_1,1)$ $\dot{x}_1 = f_1(t)e^{x_3},\ \dot{x}_2 = f_2(t)e^{ax_3},\ \dot{x}_3 = f_3(t),\ \dot{x}_4 = f_4(t);$

$R(A_{3.5}\oplus A_1,1)$ $\dot{x}_1 = f_1(t)e^{2bx_3}\sin(x_3+f_1(t)),$

$\dot{x}_2 = f_1(t)e^{2bx_3}\cos(x_3+f_1(t)),\ \dot{x}_3 = f_3(t),\ \dot{x}_4 = f_4(t).$

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Recursion Operators for Vacuum Einstein Equations with Symmetries

Michal MARVAN

Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, Czech Republic
E-mail: *Michal.Marvan@math.slu.cz*

Direct and inverse recursion operator is derived for the vacuum Einstein equations for metrics with two commuting Killing vectors that are orthogonal to a foliation by 2-dimensional leaves.

1 Introduction

In the past decade, inverse recursion operators became the subject of a number of papers [7, 8, 14, 15]. In particular, the work of Guthrie [7] opened a new perspective on recursion operators by essentially identifying them with auto-Bäcklund transformations for linearized equation [18].

It is known for a long time that recursion operators of integrable systems are obtainable from their Lax pairs (see [6] and references therein) and ZCR's (see [10, 24]). However, recently it became clear that zero curvature representations are related much closer to inverse recursion operators than to their 'direct' counterparts [19, 20]. Examples that have been already published elsewhere include the Korteweg–de Vries and Tzitzéica equation [19] and the stationary Nizhnik–Novikov–Veselov equation [20]. In the present paper, the methods of [19, 20] are applied to equations of General Relativity.

2 Recursion operators

Let $\mathcal{E} = \{F^l = 0\}$ be a system of PDE's in unknown functions u^k of two independent variables x, y. We assume that F^l are functions of x, y, u^k and a finite number of the derivatives

$$u^k_{ij} = \frac{\partial^{i+j} u^k}{\partial x^i \partial y^j} \qquad \bigl(u^k_{00} = u^k\bigr).$$

Consider the infinite-dimensional jet space J^∞ with local coordinates x, y, u^k_{ij} along with the commuting vector fields

$$D_x = \frac{\partial}{\partial x} + \sum_{ij} u^k_{i+1,j} \frac{\partial}{\partial u^k_{ij}}, \qquad D_y = \frac{\partial}{\partial y} + \sum_{ij} u^k_{i,j+1} \frac{\partial}{\partial u^k_{ij}},$$

called *total derivatives*. The submanifold E determined by equations $F^l = 0$ and their differential consequences $D_x F^l = 0$, $D_y F^l = 0$, $D_x^2 F^l = 0$, $D_x D_y F^l = 0$, $D_y^2 F^l = 0$, …, is called the *equation manifold* (and is an underlying space of the diffiety structure [12] employed in [19]). In this context, infinitesimal symmetries (more precisely, their generating functions) are functions U^k defined on E such that

$$\sum_{k,i,j} \frac{\partial F^l}{\partial u^k_{ij}} D_x^i D_y^j U^k = 0.$$

It is then natural to consider the jet space with coordinates x, y, u^k_{ij}, U^k_{ij}, and denote

$$LF^l = \sum_{k,i,j} \frac{\partial F^l}{\partial u^k_{ij}} U^k_{ij}. \tag{1}$$

The system $L\mathcal{E} := \{F^l = 0,\ LF^l = 0\}$ on unknowns u^k, U^k will be called the *linearized equation*. Now, Guthrie's recursion operators [7] may be interpreted as auto-Bäcklund transformations of the linearized equation $L\mathcal{E}$ that keep variables u^k unchanged [19].

In now standard formalism [21], recursion operators are pseudodifferential operators, characterized by the occurrence of inverses of total derivatives D_x^{-1}. Under Guthrie's approach, $p = D_x^{-1} f$ is introduced as an auxiliary nonlocal variable satisfying

$$p_x = f, \qquad p_y = g, \tag{2}$$

provided such a g exists, and it actually does without known exception; see Sergyeyev [23] for a proof in the case of evolution systems. Thus, p is a potential of a conservation law $f\,dx + g\,dy$ of the linearized equation $L\mathcal{E}$.

For the inverse recursion operators, the nonlocalities tend to be genuinely non-Abelian pseudopotentials related to a zero curvature representation of the system in question. Let $\mathfrak{g}$ be a matrix Lie algebra. Let $\alpha = A\,dx + B\,dy$ be a $\mathfrak{g}$-valued zero curvature representation (ZCR) for the system $\mathcal{E}$. This means that A, B are $\mathfrak{g}$-valued functions on the equation submanifold E and $D_y A - D_x B + [A, B] = 0$ holds on E. Let us introduce the associated pseudopotential P as a $\mathfrak{g}$-valued solution of the compatible system

$$P_x = [A, P] + LA, \qquad P_y = [B, P] + LB. \tag{3}$$

A recursion operator R is then a linear operator in U^k and P such that $U' = R(U, P)$ solves the linearized system $L\mathcal{E}$ whenever U does and P satisfies (3) (see [19,20]). In this way, the inverse recursion operator can be found without previous knowledge of the direct recursion operator. A remarkable aspect of this approach is that $R(U, P)$ tends to be a very simple expression.

For the above scheme to work, it is not necessary that the ZCR α a priori depends on the "spectral parameter". However, if R is a recursion operator related to the ZCR α as above, then $(R^{-1} + \mu\,\mathrm{Id})^{-1}$ is another recursion operator, associated with a ZCR α_μ which depends on μ.

3 The results

We consider vacuum Einstein equations for a space-time with two commuting Killing vectors that are orthogonal to a foliation by 2-dimensional surfaces [4,5]. Our presentation will be restricted to the case when both Killing vectors are space-like. The case when one of the Killing vectors is time-like is equivalent to ours via an appropriate complex transformation of coordinates.

As is well known, there exist coordinates x, y, z^1, z^2 such that the metric in question can be written in the form $ds^2 = 2f(x, y)\,dx\,dy + g_{ij}(x, y)\,dz^i\,dz^j$ (the Lewis [13] metric). The vacuum Einstein equations essentially reduce to

$$\left(\sqrt{\det g}\, g_x g^{-1}\right)_y + \left(\sqrt{\det g}\, g_y g^{-1}\right)_x = 0, \tag{4}$$

while f can be obtained by quadrature. Using the standard normalization $\det g = (x + y)^2$ compatible with equation (4), we parametrize g as follows: $g_{11} = (x + y)/u$, $g_{12} = (x + y)v/u$, $g_{22} = (x + y)(u^2 + v^2)/u$. Equation (4) then becomes

$$u_{xy} = \frac{u_x u_y - v_x v_y}{u} - \frac{1}{2}\frac{u_x + u_y}{x + y}, \qquad v_{xy} = \frac{v_x u_y + u_x v_y}{u} - \frac{1}{2}\frac{v_x + v_y}{x + y}. \tag{5}$$

As is well known, equation (5) has a ZCR and a Bäcklund transformation [1–3,9,16,17,22]. The ZCR reads

$$A=\frac{1}{2}\begin{pmatrix}-(\theta+1)u_x/u & (\theta+1)v_x/u^2\\ (\theta-1)v_x & (\theta+1)u_x/u\end{pmatrix},\qquad B=\frac{1}{2\theta}\begin{pmatrix}-(\theta+1)u_y/u & (\theta+1)v_y/u^2\\ (-\theta+1)v_y & (\theta+1)u_y/u\end{pmatrix},$$

where $\theta=\sqrt{(\mu+y)/(\mu-x)}$, μ being the spectral parameter.

The main result of this paper, obtained by the methods of [19,20], is as follows: If nonlocal variables p_{11}, p_{12}, p_{21} satisfy

$$\begin{aligned}
p_{11,x}&=-\frac{\theta-1}{2}v_xp_{12}+\frac{\theta+1}{2}\frac{v_x}{u^2}p_{21}-\frac{\theta+1}{2}\frac{1}{u}U_x+\frac{\theta+1}{2}\frac{u_x}{u^2}U,\\
p_{12,x}&=-(\theta+1)\frac{v_x}{u^2}p_{11}-(\theta+1)\frac{u_x}{u}p_{12}-(\theta+1)\frac{v_x}{u^3}U+\frac{\theta+1}{2}\frac{1}{u^2}V_x,\\
p_{21,x}&=(\theta-1)v_xp_{11}+(\theta+1)\frac{u_x}{u}p_{21}+\frac{\theta-1}{2}V_x,\\
p_{11,y}&=\frac{\theta-1}{2\theta}v_yp_{12}+\frac{\theta+1}{2\theta}\frac{v_y}{u^2}p_{21}+\frac{\theta+1}{2\theta}\frac{u_y}{u^2}U-\frac{\theta+1}{2\theta}\frac{1}{u}U_y,\\
p_{12,y}&=-\frac{\theta+1}{\theta}\frac{v_y}{u^2}p_{11}-\frac{\theta+1}{\theta}\frac{u_y}{u}p_{12}-\frac{\theta+1}{\theta}\frac{v_y}{u^3}U+\frac{\theta+1}{2\theta}\frac{1}{u^2}V_y,\\
p_{21,y}&=-\frac{\theta-1}{\theta}v_yp_{11}+\frac{\theta+1}{\theta}\frac{u_y}{u}p_{21}-\frac{\theta-1}{2\theta}V_y,
\end{aligned}\tag{6}$$

then

$$\begin{aligned}
U'&=2\frac{u}{\sqrt{(\mu-x)(\mu+y)}}p_{11}+\frac{1}{\sqrt{(\mu-x)(\mu+y)}}U,\\
V'&=-\frac{u^2}{\sqrt{(\mu-x)(\mu+y)}}p_{12}-\frac{1}{\sqrt{(\mu-x)(\mu+y)}}p_{21}
\end{aligned}\tag{7}$$

is a recursion operator for equation (5), namely, it sends symmetries to symmetries if the latter are viewed as solutions of the linearized system

$$\begin{aligned}
U_{xy}&=\left(\frac{u_y}{u}-\frac{1}{2(x+y)}\right)U_x+\left(\frac{u_x}{u}-\frac{1}{2(x+y)}\right)U_y-\frac{u_xu_y-v_xv_y}{u^2}U-\frac{v_y}{u}V_x-\frac{v_x}{u}V_y,\\
V_{xy}&=\frac{v_y}{u}U_x+\frac{v_x}{u}U_y-\frac{v_xu_y+u_xv_y}{u^2}U+\left(\frac{u_y}{u}-\frac{1}{2(x+y)}\right)V_x+\left(\frac{u_x}{u}-\frac{1}{2(x+y)}\right)V_y.
\end{aligned}$$

The 'direct' recursion operator for this equation seems to be missing in the literature; we can obtain it by inverting the operator (7), the result being

$$\begin{aligned}
U'&=uvp_1-up_2+(y-x)U,\\
V'&=-\tfrac{1}{2}\left(u^2-v^2\right)p_1-vp_2-\tfrac{1}{2}p_3+(y-x)V,
\end{aligned}$$

where p_1, p_2, p_3 satisfy

$$\begin{aligned}
p_{1,x}&=(x+y)\left(-2\frac{v_x}{u^3}U+\frac{1}{u^2}V_x\right),\\
p_{2,x}&=(x+y)\left(-\frac{uu_x+2vv_x}{u^3}U+\frac{1}{u}U_x+\frac{v_x}{u^2}V+\frac{v}{u^2}V_x\right),\\
p_{3,x}&=(x+y)\left(2\frac{(uu_x+vv_x)v}{u^3}U-2\frac{v}{u}U_x-2\frac{uu_x+vv_x}{u^2}V+\frac{u^2-v^2}{u^2}V_x\right),
\end{aligned}$$

$$p_{1,y}=(x+y)\left(2\frac{v_y}{u^3}U-\frac{1}{u^2}V_y\right),$$
$$p_{2,y}=(x+y)\left(\frac{uu_y+2vv_y}{u^3}U-\frac{1}{u}U_y-\frac{v_y}{u^2}V-\frac{v}{u^2}V_y\right),$$
$$p_{3,y}=(x+y)\left(-2\frac{(uu_y+vv_y)v}{u^3}U+2\frac{v}{u}U_y+2\frac{uu_y+vv_y}{u^2}V-\frac{u^2-v^2}{u^2}V_y\right).$$

It is readily seen that p_i are potentials of the linearizations [18] of the three obvious conservation laws of equation (4).

Quite unusually, neither of the recursion operators found generates an infinite series of local symmetries (and no such series is known). The action of our operators on the infinite-dimensional Geroch group of nonlocal symmetries [5, 11] remains to be investigated.

It is convenient to rewrite system (6) in triangular form. To achieve this, we introduce the Riccati pseudopotential q by

$$q_x=\frac{\theta-1}{2}v_xq^2-(\theta+1)\frac{u_x}{u}q-\frac{\theta+1}{2}\frac{v_x}{u^2},\qquad q_y=-\frac{\theta-1}{2\theta}v_yq^2-\frac{\theta+1}{2\theta}\frac{u_y}{u}q-\frac{\theta+1}{2\theta}\frac{v_y}{u^2}$$

and a nonlocal potential r by

$$r_x=(\theta-1)v_xq-(\theta+1)\frac{u_x}{u},\qquad r_y=-\frac{\theta-1}{\theta}v_yq-\frac{\theta+1}{\theta}\frac{u_y}{u}.$$

Then the inverse recursion operator assumes the form

$$U'=\frac{1}{\sqrt{(\mu-x)(\mu+y)}}\left(2uQ-2\frac{uq}{\mathrm{e}^r}R+U\right),$$
$$V'=\frac{1}{\sqrt{(\mu-x)(\mu+y)}}\left(-u^2\mathrm{e}^rP-2u^2qQ+\frac{u^2q^2-1}{\mathrm{e}^r}R\right),$$

where P, Q, R are supposed to satisfy

$$P_x=(\theta+1)\frac{q}{u}\mathrm{e}^{-r}U_x+\left(\frac{\theta+1}{2}\frac{1}{u^2}-\frac{\theta-1}{2}q^2\right)\mathrm{e}^{-r}V_x-(\theta+1)\left(\frac{qu_x}{u^2}+\frac{v_x}{u^3}\right)\mathrm{e}^{-r}U,$$
$$Q_x=-\frac{\theta+1}{2}\frac{1}{u}U_x+\frac{\theta-1}{2}qV_x+\frac{\theta+1}{2}\frac{u_x}{u^2}U-\frac{\theta-1}{2}v_x\mathrm{e}^rP,$$
$$R_x=\frac{\theta-1}{2}\mathrm{e}^rV_x+(\theta-1)v_x\mathrm{e}^rQ,$$
$$P_y=\frac{\theta+1}{\theta}\frac{q}{u}\mathrm{e}^{-r}U_y+\left(\frac{\theta+1}{2\theta}\frac{1}{u^2}+\frac{\theta-1}{2\theta}q^2\right)\mathrm{e}^{-r}V_y-\frac{\theta+1}{\theta}\left(\frac{qu_y}{u^2}+\frac{v_y}{u^3}\right)\mathrm{e}^{-r}U,$$
$$Q_y=-\frac{\theta+1}{2\theta}\frac{1}{u}U_y-\frac{\theta-1}{2\theta}qV_y+\frac{\theta+1}{2\theta}\frac{u_y}{u^2}U+\frac{\theta-1}{2\theta}v_y\mathrm{e}^rP,$$
$$R_y=-\frac{\theta-1}{2\theta}\mathrm{e}^rV_y-\frac{\theta-1}{\theta}v_y\mathrm{e}^rQ.$$

This form of the inverse recursion operator is better adapted to generation of symmetries, which is, however, beyond the scope of this paper.

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Infinite Order Symmetries for Two-Dimensional Separable Schrödinger Equations

W. MILLER †, *E. KALNINS* ‡ *and G. POGOSYAN* §

† *School of Mathematics, University of Minnesota, Minneapolis, Minnesota, 55455, U.S.A.*
E-mail: *miller@ima.umn.edu*

‡ *Department of Mathematics, University of Waikato, Hamilton, New Zealand*
E-mail: *e.kalnins@waikato.ac.nz*

§ *Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 14980 Dubna, Moscow Region, Russia*
E-mail: *pogosyan@thsun1.jinr.dubna.su*

Consider a non-relativistic Hamiltonian operator H in 2 dimensions consisting of a kinetic energy term plus a potential. We show that if the associated Schrödinger eigenvalue equation admits an orthogonal separation of variables, there is a calculus to describe the (in general) infinite-order differential operator symmetries of the Schrödinger equation. The calculus is formal but can be made rigorous when all functions in the eigenvaue equation are analytic. The infinite-order calculus exhibits structure that is not apparent when one studies only finite-order symmetries. The search for finite-order symmetries can then be reposed as one of looking for solutions of a coupled system of PDEs that are polynomial in certain parameters. We go further and extend the calculus to the situation where the Schrödinger equation admits a second-order symmetry operator, not necessarily associated with orthogonal separable coordinates.

1 Introduction

Consider a Schrödinger operator

$$H \equiv L_1 = \Delta_2 + V(\boldsymbol{x}),$$

where Δ_2 is the Laplace–Beltrami operator on a real or complex two-dimensional Riemannian space and $\boldsymbol{x} = (x_1, x_2)$ are orthogonal coordinates on that space such that the Schrödinger equation

$$H\Psi = E\Psi$$

separates multiplicatively in these coordinates. We here consider locally analytic solutions of this equation; application of boundary conditions is a separate step. We will introduce a calculus to describe the differential symmetry operators for this system, even those of infinite order. The question of finding finite order symmetries is reposed as a question of finding solutions of finite systems of PDEs that are polynomial in certain parameters. (See [1] where we introduced infinite-order conformal symmetry operators for the time-dependent Schrödinger equation in one space variable.) We give a number of examples where this method yields finite order differential symmetry operators. These questions as to when a system with two second-order constants of the motion, classical or quantum, (generated by an orthogonal separation of variables) admits additional polynomial constants of the motion are closely related to the concept of superintegrability [2–7]. We also consider all cases where the Schrödinger equation admits a second-order symmetry, not necessarily associated with orthogonal separable coordinates, or even separable coordinates at all. In each instance we construct additional symmetries.

2 Two-dimensional separable systems

Consider the case of orthogonal separable coordinates in a general Riemannian space. In the local separable coordinates $\{x, y\}$ the Schrödinger operator has the form

$$H = L_1 = \frac{1}{f_1(x) + f_2(y)} \left(\partial_x^2 + \partial_y^2 + v_1(x) + v_2(y)\right). \tag{1}$$

and, due to the separability, there is the invariant

$$L_2 = \frac{f_2(y)}{f_1(x) + f_2(y)} \left(\partial_x^2 + v_1(x)\right) - \frac{f_1(x)}{f_1(x) + f_2(y)} \left(\partial_y^2 + v_2(y)\right),$$

i.e., $[L_2, H] = 0$, [9,10]. We have the operator identities

$$f_1(x)H + L_2 = \partial_x^2 + v_1(x), \qquad f_2(y)H - L_2 = \partial_y^2 + v_2(y). \tag{2}$$

We look for a partial differential operator $\tilde{L}(H, L_2, x, y)$ that satisfies

$$[H, \tilde{L}] = 0. \tag{3}$$

and $[L_2, \tilde{L}] \neq 0$. We require that the invariant take the standard form

$$\tilde{L}(H, L_2, x, y) = \sum_{j,k} \left(A_{j,k}(x, y)\partial_{xy} + B_{j,k}(x, y)\partial_x + C_{j,k}(x, y)\partial_y + D_{j,k}(x, y)\right) H^j L_2^k. \tag{4}$$

Note that if the formal operators (4) contained partial derivatives in x and y of orders ≥ 2 we could use the identities (2), recursively, and rearrange terms to achieve the unique standard form (4).

Using the operator identities

$$\partial_x H = H\partial_x - \frac{f_1'}{f_1 + f_2}H + \frac{v_1'}{f_1 + f_2}, \qquad \partial_y H = H\partial_y - \frac{f_2'}{f_1 + f_2}H + \frac{v_2'}{f_1 + f_2},$$
$$\partial_x L_2 = L_2\partial_x - \frac{f_1' f_2}{f_1 + f_2}H + \frac{f_2 v_1'}{f_1 + f_2}, \qquad \partial_y L_2 = L_2\partial_y + \frac{f_1 f_2'}{f_1 + f_2}H - \frac{f_1 v_2'}{f_1 + f_2},$$

we see that

$$\begin{aligned}
&(f_1(x) + f_2(y))[H, A(x, y)\partial_{xy} + B(x, y)\partial_x + C(x, y)\partial_y + D(x, y)] \\
&\quad = (A_{xx} + A_{yy} + 2B_y + 2C_x)\partial_{xy} + (B_{xx} + B_{yy} - 2A_y v_2 + 2D_x - Av_2')\partial_x \\
&\quad + (2A_y f_2 + Af_2')\partial_x H - 2A_y\partial_x L_2 + (C_{xx} + C_{yy} - 2A_x v_1 + 2D_y - Av_1')\partial_y \\
&\quad + (2A_x f_1 + Af_1')\partial_y H + 2A_x\partial_y L_2 + (D_{xx} + D_{yy} - 2B_x v_1 - 2C_y v_2 - Bv_1' - Cv_2') \\
&\quad + (2B_x f_1 + 2C_y f_2 + Bf_1' + Cf_2')H + (2B_x - 2C_y)L_2.
\end{aligned}$$

Thus the condition (3) for the symmetries of (1) is equivalent to the system of equations

$$\partial_{xx}A_{j,k} + \partial_{yy}A_{j,k} + 2\partial_y B_{j,k} + 2\partial_x C_{j,k} = 0, \tag{5}$$
$$\begin{aligned}
&\partial_{xx}B_{j,k} + \partial_{yy}B_{j,k} - 2\partial_y A_{j,k}v_2 + 2\partial_x D_{j,k} - A_{j,k}v_2' \\
&\quad + (2\partial_y A_{j-1,k}f_2 + A_{j-1,k}f_2') - 2\partial_y A_{j,k-1} = 0,
\end{aligned} \tag{6}$$
$$\begin{aligned}
&\partial_{xx}C_{j,k} + \partial_{yy}C_{j,k} - 2\partial_x A_{j,k}v_1 + 2\partial_y D_{j,k} - A_{j,k}v_1' \\
&\quad + (2\partial_x A_{j-1,k}f_1 + A_{j-1,k}f_1') + 2\partial_x A_{j,k-1} = 0,
\end{aligned} \tag{7}$$
$$\partial_{xx}D_{j,k} + \partial_{yy}D_{j,k} - 2\partial_x B_{j,k}v_1 - 2\partial_y C_{j,k}v_2 - B_{j,k}v_1' - C_{j,k}v_2' \tag{8}$$

$$+ (2\partial_x B_{j-1,k} f_1 + 2\partial_y C_{j-1,k} f_2 + B_{j-1,k} f_1' + C_{j-1,k} f_2') + (2\partial_x B_{j,k-1} - 2\partial_y C_{j,k-1}) = 0.$$

Note that condition (4) makes sense for infinite order differential equations, in domains where the functions f_j, v_j are analytic. Indeed, one can consider H, L_2 as parameters in these equations. Then once $\tilde{L}$ is expanded as a convergent power series in these parameters, the terms are reordered so that the powers of the parameters are on the right, before they are replaced by explicit differential operators. Alternatively one can consider the operator $\tilde{L}$ as acting on a simultaneous eigenbasis of the commuting operators H and L_2, in which case the parameters are the eigenvalues. In this view we can write

$$\tilde{L}(H, L_2, x, y) = A(x, y, H, L_2)\partial_{xy} + B(x, y, H, L_2)\partial_x + C(x, y, H, L_2)\partial_y + D(x, y, H, L_2),$$

and consider $\tilde{L}$ as an at most second-order order differential operator in x, y that is analytic in the parameters H, L_2. Then the above system (5), (6), (7), (8) can be written in the more compact form

$$A_{xx} + A_{yy} + 2B_y + 2C_x = 0, \tag{9}$$
$$B_{xx} + B_{yy} - 2A_y v_2 + 2D_x - A v_2' + (2A_y f_2 + A f_2')H - 2A_y L_2 = 0, \tag{10}$$
$$C_{xx} + C_{yy} - 2A_x v_1 + 2D_y - A v_1' + (2A_x f_1 + A f_1')H + 2A_x L_2 = 0, \tag{11}$$
$$D_{xx} + D_{yy} - 2B_x v_1 - 2C_y v_2 - B v_1' - C v_2' + (2B_x f_1 + 2C_y f_2 + B f_1' + C f_2')H + (2B_x - 2C_y)L_2 = 0. \tag{12}$$

and this system has many solutions.

Remark 1. Although our derivation has implicitly assumed analyticity of A, B, C, D in H, L_2 in a neighborhood of $(0, 0)$. It is easy to show that the same equations (9)–(12) arise if these functions are analytic in the neighborhood of any point (H^0, L_2^0).

We start with a very special case

$$A \equiv 0, \quad B = X(x, H, L_2), \quad C = Y(y, H, L_2), \quad D = \tilde{X}(x, H, L_2) + \tilde{Y}(y, H, L_2). \tag{13}$$

Then the above PDEs uncouple into ODEs for X and Y, whose structure we can easily analyse. The separated equation for X can be written in the compact form

$$X''' + 4(v_1 - f_1 H - L_2)X' + 2(v_1' - f_1' H)X = -2P(H, L_2), \qquad \tilde{X} = -\frac{1}{2}X', \tag{14}$$

where the separation constant $P(H, L_2)$ is a given analytic function of its arguments. (We can take it to be a polynomial.) The first equation (14) always has solutions for any f_1, v_1, say continuously differentiable. Thus we can always construct M and it will be analytic in the parameters H, L_2. Further we have the result

Lemma 1. *Let $\Psi_1(H, L_2, x)$, $\Psi_2(H, L_2, x)$ be a basis of solutions for the equation*

$$\left(\frac{d^2}{dx^2} + v_1(x) - f_1(x)H - L_2\right)\Psi(x) = 0.$$

Then $S_1(x) = \Psi_1^2$, $S_2(x) = \Psi_1\Psi_2$, $S_3(x) = \Psi_2^2$ is a basis of solutions for the homogeneous equation

$$S''' + 4(v_1 - f_1 H - L_2)S' + 2(v_1' - f_1' H)S = 0. \tag{15}$$

Thus the general solution of the first equation (14) is a particular solution of this equation plus an arbitrary linear combination of S_1, S_2, S_3.

Similarly, the separation equation for $C = Y$ is

$$Y''' + 4(v_2 - f_2 H + L_2)Y' + 2(v_2' - f_2' H)Y = 2P(H, L_2), \qquad \tilde{Y} = -\frac{1}{2}Y'.$$

Once we have obtained X and Y, then we see that the corresponding operator L_3 commutes with H. Thus we can view L_3 as an infinite order differential symmetry operator for H. In special cases this will be a finite order operator.

It is important to note that (for $P \neq 0$) L_3 is not just a function of H and L_2. Indeed, a straightforward computation yields $[L_2, L_3] = P(H, L_2) \neq 0$. In fact, an analogous constructions to the above but based on the operator L_2 yields an operator L_4 such that $H = L_1$, L_2, L_3, L_4 satisfy the commutation relations

$$[L_1, L_2] = [L_1, L_3] = [L_2, L_4] = [L_3, L_4] = 0, \qquad [L_2, L_3] = [L_1, L_4] = P(L_1, L_2).$$

If we choose $P(L_1, L_2) = I$, the identity operator, then these are the canonical commutation relations.

Example 1. We consider Cartesian coordinates in flat space and assume that the potential is separable in these coordinates. Thus we have $f_1(x) = f_2(y) = \frac{1}{2}$ and

$$H = \partial_x^2 + \partial_y^2 + v_1(x) + v_2(y), \qquad L_2 = \frac{1}{2}(\partial_x^2 + v_1(x)) - \frac{1}{2}(\partial_y^2 + v_2(y)).$$

We look for a 3rd order constant of the motion L_3. Thus we have

$$P(H, L_2) = \sum_{j+k \leq 2} \alpha_{jk} H^j L_2^k$$

and the x-dependent part of the symmetry operator must take the form

$$M = (X_{10}\partial_x + \tilde{X}_{10})H + (X_{01}\partial_x + \tilde{X}_{01})L_2 + (X_{00}\partial_x + \tilde{X}_{00}), \qquad \tilde{X}_{jk} = -\frac{1}{2}X_{jk}'.$$

This leads to the system of equations (labeled by the powers (j, k) of $H^j L_2^k$)

$$\begin{aligned}
&(2,0) \quad X_{10}' = \alpha_{20},\\
&(1,1) \quad X_{01}' + 2X_{10}' = \alpha_{11},\\
&(0,2) \quad 2X_{01}' = \alpha_{02},\\
&(1,0) \quad -\frac{1}{2}X_{10}''' - v_1 X_{10} - 2v_1 X_{10}' + X_{00}' = \alpha_{10},\\
&(0,1) \quad -\frac{1}{2}X_{01}''' - v_1 X_{01} - 2v_1 X_{01}' + 2X_{00}' = \alpha_{01},\\
&(0,0) \quad -\frac{1}{2}X_{00}''' - v_1 X_{00} - 2v_1 X_{00}' = \alpha_{00}.
\end{aligned} \tag{16}$$

These equations are equivalent to

$$\begin{aligned}
&X_{10} = \alpha_{20}x + c_{10}, \qquad X_{01} = \frac{1}{2}\alpha_{02}x + c_{01}, \qquad \alpha_{02} + 4\alpha_{20} = 2\alpha_{11},\\
&X_{00}' = (\alpha_{20}x + c_{10})v_1' + 2\alpha_{20}v_1 + \alpha_{10},\\
&2X_{00}' = \left(\frac{1}{2}\alpha_{02}x + c_{01}\right)v_1' + \alpha_{02}v_1 + \alpha_{01}, \qquad -\frac{1}{2}X_{00}''' - v_1 X_{00} - 2v_1 X_{00}' = \alpha_{00}.
\end{aligned}$$

A very similar computation, with the same polynomial $P(H, L_2)$, yields the possibilities for $v_2(y)$, and the construction of the y-dependent part of the symmetry operator L_3. Then L_3 is a third-order quantum constant of the motion.

3 The general case

Up to now we have only considered the special case (13): $A = 0, B = X(x)$, $C = Y(y)$, $D = \tilde{X}(x) + \tilde{Y}(y)$ of conditions (9)–(12). Let us now consider the case such that $A \equiv 0$, but, otherwise, B, C, D are arbitrary. Then there is a function $G(x, y, H, L_2)$ such that

$$B = -\partial_x G, \qquad C = \partial_y G,$$

and the determining conditions simplify to

1) $G_{xxxy} + G_{xyyy} = 0$,

2) $\frac{1}{2}G_{xxxx} + 2G_{xx}v_1 + G_x v_1' - (2G_{xx}f_1 + G_x f_1')H - 2G_{xx}L_2$
$= \frac{1}{2}G_{yyyy} + 2G_{yy}v_2 + G_y v_2' - (2G_{yy}f_2 + G_y f_2')H + 2G_{yy}L_2.$

The first determining equation means that $G(x, y) = K(x, y) + F(x) + J(y)$ where F, J are arbitrary and K is harmonic: $K_{xx} + K_{yy} = 0$. This representation is unique in K, F, J, up to the addition of the harmonic separable function $\tilde{K}(x, y) = \frac{a}{2}(x^2 - y^2) + bx + cy + d$. Alternatively, we can write

$$G(x, y) = z_1(x + iy) + z_2(x - iy) + F(x) + J(y),$$

where z_1, z_2 are arbitrary analytic functions. Then only condition 2) remains to be satisfied.

Example 2. If we make the ansatz $G = X(x, H, L_2)Y(y, H, L_2)$ then, in addition to the well known angular momentum invariant, we find the following two polynomial invariants:

1. $X = \left(\frac{1}{4} + L_2\right)\cos x + s(1 + \beta H), \qquad Y = \left(\frac{1}{4} + L_2\right)\cosh y + t(1 + \xi H),$
$v_1(x) = 2s\frac{\sin x}{\cos^2 x} + \frac{a_1}{\cos^2 x}, \qquad f_1(x) = -2s\beta\frac{\sin x}{\cos^2 x} + \frac{a_2}{\cos^2 x},$
$v_2(y) = 2t\frac{\sinh y}{\cosh^2 y} + \frac{b_1}{\cosh^2 y}, \qquad f_2(y) = -2t\xi\frac{\sinh y}{\cosh^2 y} + \frac{b_2}{\cosh^2 y},$
$D = -\frac{1}{2}(\frac{1}{4} + L_2)\left(t\cos x(1 + \xi H) + s\cosh y(1 + \beta H)\right).$
2. $\tilde{L} = -2x(y^2 + 4L_2)\partial_x + 2y(x^2 - 4L_2)\partial_y + x^2 - y^2,$
$v_1(x) = \frac{1}{8}x^2 + \frac{a_1}{x^2}, \qquad f_1(x) = \frac{a_2}{x^2}, \qquad v_2(y) = \frac{1}{8}y^2 + \frac{b_1}{y^2}, \qquad f_2(y) = \frac{b_2}{y^2}.$

Example 3. Again we consider the special case of conditions (9)–(12) such that $A \equiv 0$. Now we require

$$G(x, y) = -2\log(X(x) + Y(y)) + \mathcal{F}(x) + \mathcal{J}(y) = K(x, y) + F(x) + J(y),$$

where F, J are arbitrary and K is harmonic. Then the harmonic requirement on K implies that

$$K = -2\log(X + Y) + \tilde{F}(x) + \tilde{J}(y),$$

where

$$(X')^2 = \frac{\alpha}{12}X^4 + \frac{\beta}{3}X^3 + \gamma X^2 + 2\delta X + \phi,$$
$$(Y')^2 = -\frac{\alpha}{12}Y^4 + \frac{\beta}{3}Y^3 - \gamma Y^2 + 2\delta Y - \phi,$$

$$X'' = \frac{\alpha}{6}X^3 + \frac{\beta}{2}X^2 + \gamma X + \delta, \qquad Y'' = -\frac{\alpha}{6}Y^3 + \frac{\beta}{2}Y^2 - \gamma Y + \delta.$$

Further,

$$\tilde{F}(x) = \frac{1}{3}\frac{X'''}{X'}, \qquad \tilde{J}(y) = \frac{1}{3}\frac{Y'''}{Y'},$$

and the metric and potential terms have the solution

$$v_1 - f_1 H = \frac{-\frac{a}{12}X^4 - \frac{b}{3}X^3 + \frac{b_1}{2}X^2 + \eta_1 X + \eta_2}{24(X')^2},$$
$$v_2 - f_2 H = \frac{\frac{a}{12}Y^4 - \frac{b}{3}Y^3 - \frac{b_1}{2}Y^2 + \eta_1 Y - \eta_2}{24(Y')^2}.$$

Here, α, β, γ, δ, ϕ and

$$a = a^{(1)} + a^{(2)}H, \qquad b = b^{(1)} + b^{(2)}H, \qquad b_1 = b_1^{(1)} + b_1^{(2)}H,$$
$$\eta_1 = \eta_1^{(1)} + \eta_1^{(2)}H, \qquad \eta_2 = \eta_2^{(1)} + \eta_2^{(2)}H$$

are parameters. The remaining condition is

$$\frac{1}{2}F'''' + 2F''(v_1 - f_1 H - L_2) + F'(v_1 - f_1' H) - \frac{1}{2}J'''' - 2J''(v_2 - f_2 H - L_2) - J'(v_2' - f_2' H)$$
$$= \frac{1}{36}\left(\frac{a}{2}X^2 + bX - \frac{a}{2}Y^2 + bY\right) + \frac{2}{3}\left(\frac{X'''}{X'}(v_1 - f_1 H) - \frac{Y'''}{Y'}(v_2 - f_2 H)\right)$$
$$+ \tilde{F}'(v_1' - f_1' H) - \tilde{J}'(v_2' - f_2' H).$$

The simplest family of solutions is obtained by setting $F \equiv \tilde{F}$, $J \equiv \tilde{J}$ and $\alpha = \beta = a = b = 0$.

Now we consider the general case of conditions (9)–(12). Then there are two functions $F(x, y, H, L_2)$, $G(x, y, H, L_2)$ such that

$$A = \partial_{xy}F, \qquad B = -\frac{1}{2}\partial_{xyy}F - \partial_x G, \qquad C = -\frac{1}{2}\partial_{xxy}F + \partial_y G,$$

and the determining conditions simplify to

1) $2G_{xyyy} + \frac{1}{2}F_{xyyyyy} + 2F_{xyyy}(v_2 - f_2 H + L_2) + 3F_{xyy}(v_2' - f_2' H) + F_{xy}(v_2'' - f_2'' H)$
$= -2G_{xxxy} + \frac{1}{2}F_{xxxxxy} + 2F_{xxxy}(v_1 - f_1 H - L_2)$
$+ 3F_{xxy}(v_1' - f_1' H) + F_{xy}(v_1'' - f_1'' H),$

2) $\frac{1}{2}F_{xxxxyy} + 2F_{xxyy}(v_1 - f_1 H) + F_{xxy}(v_2' - f_2' H) + \frac{1}{2}G_{xxxx}$
$+ 2G_{xx}(v_1 - f_1 H - L_2) + G_x(v_1' - f_1' H)$
$= -\frac{1}{2}F_{xxyyyy} - 2F_{xxyy}(v_2 - f_2 H) - F_{xyy}(v_1' - f_1' H) + \frac{1}{2}G_{yyyy}$
$+ 2G_{yy}(v_2 - f_2 H + L_2) + G_y(v_2' - f_2' H).$

Example 4. Consider the problem of finding all third-order symmetry operators corresponding to potentials that separate in Cartesian coordinates in Euclidean space. Then $f_1 = f_2 = \frac{1}{2}$. Third-order symmetry operators correspond to $F = F(x, y)$ and $G = g(x, y)L_1 + h(x, y)L_2 + k(x, y)$. Now let $f(x, y) = F_{xy}$, and substitute these expressions into conditions 1) and 2).

Equating coefficients of L_1^2, L_2^2, L_1L_2 in 1) and coefficients of L_1, L_2 in 2), we obtain easily the expressions

$$
\begin{aligned}
f(x,y) &= e_{00} + e_{10}x + e_{01}y + \frac{c}{2}(x^2 - y^2) + e_{11}xy,\\
g(x,y) &= c_{00} + c_{10}x + c_{01}y + c_{20}(x^2 + y^2) + c_{11}xy + c_{30}(x^3 + 3xy^2)\\
&\quad + c_{21}\left(\frac{1}{3}y^3 + x^2y\right) - \frac{c}{12}\left(x^3y + xy^3\right),\\
h(x,y) &= d_{00} + d_{10}x + d_{01}y - 2c_{20}\left(x^2 - y^2\right) + d_{11}xy - 2c_{30}\left(x^3 - 3xy^2\right)\\
&\quad - 2c_{21}\left(x^2y - \frac{1}{3}y^3\right) + \frac{c}{6}(x^3y - xy^3),
\end{aligned}
$$

where e_{ij}, c_{ij}, d_{ij}, c are constants. Equating coefficients of L_1 and L_2 in 1) and integrating, we can get expressions for k_x and k_y. The integrability condition $\partial_x k_y = \partial_y k_x$, together with equating coefficients of the constant term in 2) leads to a system of ordinary differential equations for the potential terms:

$$
\begin{aligned}
&\left(\frac{1}{2}d_{11} - c_{11} - 2c_{21}x + \frac{1}{2}cx^2\right)v_1'' + (-12c_{21} + 3cx)v_1' + 3cv_1 = -\ell_5 - \ell_4 x + \frac{1}{2}\ell_1 x^2,\\
&\left(\frac{1}{2}d_{10} - c_{10} - 4c_{20}x - 6c_{30}x^2\right)v_1'' - (12c_{20} + 36c_{30}x)v_1' - 36c_{30}v_1 = -m_2 - m_1 x + \frac{1}{2}\ell_2 x^2,\\
&(12c_{30} - e_{01} + e_{11}x)v_1'' - 3e_{11}v_1' = -q_1 x + q_3,\\
&(d_{11} - e_{00} - e_{10}x - 4c_{21}x)v_1'' + (-3e_{10} - 4c_{21})v_1' = -q_2 x + q_4,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\frac{1}{2}d_{11} + c_{11} + 12c_{30}y - \frac{1}{6}cy^2)v_2'' + (36c_{30} - \frac{7}{3}cy\right)v_2' - 3cv_2 = -\ell_3 - \ell_2 y - \frac{1}{2}\ell_1 y^2,\\
&\left(\frac{1}{2}d_{01} + c_{01} - 4c_{20}y - 2c_{21}y^2\right)v_2'' + (12c_{20} + 12c_{21}y)v_2' + 12c_{21}v_2 = -m_3 + m_1 y + \frac{1}{2}\ell_4 y^2,\\
&(4c_{21} + e_{10} + e_{11}y)v_2'' + 3e_{11}v_2' = q_1 y + q_2,\\
&\left(e_{00} + e_{01}y - 12c_{30}y - \frac{1}{3}cy^2\right)v_2'' + \left(3e_{01} - 36c_{30} - \frac{2}{3}cy\right)v_2' = -q_3 y - q_4.
\end{aligned}
$$

Here, ℓ_j, m_j, q_j are constants. The strategy is now clear. The f, g, h expressions simply define the higher order (potential-independent) terms in the Euclidean symmetry operators and are easy to understand. For a true third-order symmetry h_x, h_y, g_x, g_y cannot all vanish. For each such choice of terms we can solve explicitly the ODEs for v_1 and v_2. Since the same v_1, v_2 have to satisfy all these equations, this puts restrictions on the constants. Then we verify that each common solution v_1, v_2 actually corresponds to a 3rd order symmetry. A tedious case-by-case procedure leads to all solutions, provided not all of the ODEs for either v_1 or v_2 are vacuous. If, say, the ODEs for v_1 are vacuous, then we must make use of the remaining integrability condition. Equating coefficients of the constant term in 1) leads to a coupled partial differential equation for the potential terms v_1, v_2

$$
\begin{aligned}
0 = \Bigg[&- 2(4c_{20} + 12c_{30}x + 4c_{21}y - cxy)v_2\\
&+ \left(\frac{1}{2}d_{10} + c_{10} + \frac{1}{2}d_{11}y + c_{11}y + 6c_{30}y^2 - \frac{1}{6}cy^3\right)v_1'
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{1}{2}d_{01} + c_{01} + 4c_{20}y + \frac{1}{2}d_{11}x + c_{11}x + 12c_{30}xy + 2c_{21}y^2 - \frac{1}{6}cxy^2\right) v_2' \Bigg] v_1 \\
&+ \Bigg[- \left(4c_{20}x + 6c_{30}x^2 + 4c_{21}xy - \frac{1}{2}cx^2y\right) v_2 \\
&+ \frac{1}{2}\left(\frac{1}{2}d_{10} + c_{10} + \frac{1}{2}d_{11}y + c_{11}y + 6c_{30}y^2 - \frac{1}{6}cy^3\right) v_1 \\
&- \frac{1}{2}\left(\frac{1}{2}d_{01}x + c_{01}x + 4c_{20}xy + \frac{1}{4}d_{11}x^2 + \frac{1}{2}c_{11}x^2 + 6c_{30}x^2y + 2c_{21}xy^2 - \frac{1}{12}cx^2y^2\right) v_2' \\
&- \frac{1}{4}m_2y^2 - \frac{1}{12}y^3\ell_5 - \frac{1}{2}\left(\frac{1}{2}d_{11} - c_{11}\right) V_2 + \frac{1}{2}n_2y + \frac{1}{2}p_2 + e_{01} - cy \Bigg] v_1' \\
&+ \Bigg[2(4c_{20} + 12c_{30}x + 4c_{21}y - cxy)v_1 \\
&+ \left(\frac{1}{2}d_{01} - c_{01} + \frac{1}{2}d_{11}x - c_{11}x - 2c_{21}x^2 + \frac{1}{6}cx^3\right) v_2' \\
&- \left(\frac{1}{2}d_{10} - c_{10} - 4c_{20}x + \frac{1}{2}d_{11}y - c_{11}y - 6c_{30}x^2 - 4c_{21}xy + \frac{1}{2}cx^2y\right) v_1' \Bigg] v_2 \\
&+ \Bigg[\left(4c_{20}y + 12c_{30}xy + 2c_{21}y^2 - \frac{1}{2}cxy^2\right) v_1 \\
&+ \frac{1}{2}\left(\frac{1}{2}d_{01} - c_{01} + \frac{1}{2}d_{11}x - c_{11}x - 2c_{21}x^2 + \frac{1}{6}cx^3\right) v_2 \\
&- \frac{1}{2}\left(\frac{1}{2}d_{10}y - c_{10}y - 4c_{20}xy + \frac{1}{4}d_{11}y^2 - \frac{1}{2}c_{11}y^2 - 6c_{30}x^2y - 2c_{21}xy^2 + \frac{1}{4}cx^2y^2\right) v_1' \\
&- \frac{1}{4}m_3x^2 - \frac{1}{12}x^3\ell_3 - \frac{1}{2}\left(\frac{1}{2}d_{11} + c_{11}\right) V_1 - \frac{1}{2}n_1x - \frac{1}{2}p_1 + e_{10} + cx \Bigg] v_2' \\
&+ \frac{1}{4}\left(\frac{1}{2}d_{10} + c_{10} + \frac{1}{2}d_{11}y + c_{11}y + 6c_{30}y^2 - \frac{1}{6}cy^3\right) v_1''' \\
&+ \frac{1}{4}\left(\frac{1}{2}d_{01} - c_{01} + \frac{1}{2}d_{11}x - c_{11}x - 2c_{21}x^2 + \frac{1}{6}cx^3\right) v_2'''. \qquad (17)
\end{aligned}
$$

Here, $v_j = V_j'$. This equation is complicated, but if the ODEs for v_1 are vacuous, then v_2 is a second-order polynomial in y and all constants in the above equation must be zero, except, d_{11}, d_{01}, d_{10}, ℓ_3, m_3, n_1, n_2, p_1, p_2. Thus v_1 must satisfy a simple nonlinear ODE and all solutions can be obtained. The details can be found in [12], with a different method. (That paper is focused on 3rd order invariants alone, rather than considering them as a special case of infinite-order invariants as is done here.)

Factorized solutions for the general conditions 1), 2), exist for all spaces and separable potentials.

Theorem 1. *For any v_1, v_2, f_1, f_2 there are always solutions for equations 1), 2) in which $A \not\equiv 0$, $G \equiv 0$ and F factors as $F = \mathcal{X}(x, H, L_2)\mathcal{Y}(y, H, L_2)$, where $\mathcal{X}'\mathcal{Y}' \neq 0$.*

Proof. Making the indicated substitutions, we see that the equations reduce to

$$
1)\ \ \frac{1}{2}\mathcal{X}^{(1)}\mathcal{Y}^{(5)} + 2\mathcal{X}^{(1)}\mathcal{Y}^{(3)}(v_2 - f_2H + L_2) + 3\mathcal{X}^{(1)}\mathcal{Y}^{(2)}(v_2' - f_2'H) + \mathcal{X}^{(1)}\mathcal{Y}^{(1)}(v_2'' - f_2''H)
$$

$$= \frac{1}{2}\mathcal{X}^{(5)}\mathcal{Y}^{(1)} + 2\mathcal{X}^{(3)}\mathcal{Y}^{(1)}(v_1 - f_1H - L_2) + 3\mathcal{X}^{(2)}\mathcal{Y}^{(1)}(v_1' - f_1'H)$$
$$+ \mathcal{X}^{(1)}\mathcal{Y}^{(1)}(v_1'' - f_1''H),$$
$$2)\ \ \frac{1}{2}\mathcal{X}^{(4)}\mathcal{Y}^{(2)} + \frac{1}{2}\mathcal{X}^{(2)}\mathcal{Y}^{(4)} + 2\mathcal{X}^{(2)}\mathcal{Y}^{(2)}(v_1 + v_2 - f_1H - f_2H)$$
$$+ \mathcal{X}^{(2)}\mathcal{Y}^{(1)}(v_2' - f_2'H) + \mathcal{X}^{(1)}\mathcal{Y}^{(2)}(v_1' - f_1'H) = 0.$$

Variables separate in these equations and we have

$$1a)\ \ \frac{1}{2}\mathcal{X}^{(5)} + 2\mathcal{X}^{(3)}(v_1 - f_1H - L_2) + 3\mathcal{X}^{(2)}(v_1' - f_1'H) + \mathcal{X}^{(1)}(v_1'' - f_1''H) = \alpha\mathcal{X}^{(1)},$$
$$1b)\ \ \frac{1}{2}\mathcal{Y}^{(5)} + 2\mathcal{Y}^{(3)}(v_2 - f_2H + L_2) + 3\mathcal{Y}^{(2)}(v_2' - f_2'H) + \mathcal{Y}^{(1)}(v_2'' - f_2''H) = \alpha\mathcal{Y}^{(1)},$$

where α is a constant. Similarly, if $\mathcal{X}^{(2)}\mathcal{Y}^{(2)} \neq 0$ we have the separation equations

$$2a)\ \ \frac{1}{2}\mathcal{X}^{(4)} + 2\mathcal{X}^{(2)}(v_1 - f_1H - L_2) + \mathcal{X}^{(1)}(v_1' - f_1'H) = \beta\mathcal{X}^{(2)},$$
$$2b)\ \ \frac{1}{2}\mathcal{Y}^{(4)} + 2\mathcal{Y}^{(2)}(v_2 - f_2H + L_2) + \mathcal{Y}^{(1)}(v_2' - f_2'H) = -\beta\mathcal{Y}^{(2)},$$

These equations are consistent if

$$A)\ \alpha = \beta = 0 \qquad \text{or} \qquad B)\ \alpha\mathcal{X}' = \beta\mathcal{X}''', \quad \alpha\mathcal{Y}' = -\beta\mathcal{Y}'''.$$

Now set $X = \mathcal{X}'$, $Y = \mathcal{Y}'$. Considering case A) first, we see that we have a solution of equations (9)–(12) whenever $X'Y' \neq 0$ and X and Y satisfy the ordinary differential equations

$$X''' + 4X'(v_1 - f_1H - L_2) + 2X(v_1' - f_1'H) = 0,$$
$$Y''' + 4Y'(v_2 - f_2H + L_2) + 2Y(v_2' - f_2'H) = 0,$$

essentially the same as the third order homogeneous ordinary differential equation (15).

Now suppose that $X'Y' \equiv 0$, but $X' \neq 0$. Then we have $v_2' - f_2'H = 0$ so v_2', f_2' are constants. Further, X satisfies the ordinary differential equation

$$\frac{1}{2}X'''' + 2X''(v_1 - f_1L - L_2) + 3X'(v_1' - f_1'H) + X'(v_1'' - f_1''H) = 0.$$

Finally, if $X = Y = 1$ we have $v_2'' - f_2''H = v_1'' - f_1''H = \alpha(H)$, corresponding to oscillator potentials.

For case B) we can assume $\beta = 1$. Then we find, with $\alpha = a^2$,

$$X(x) = c_1\sinh(ax) + c_2\cosh(ax), \quad f_1(x) = \frac{c_3}{X(x)^2}, \quad v_1(x) = \frac{4L_2 + 2 - a^2}{4} + \frac{c_4}{X(x)^2},$$
$$Y(y) = d_1\sin(ax) + d_2\cos(ax), \quad f_2(y) = \frac{d_3}{Y(y)^2}, \quad v_2(y) = -\frac{4L_2 + 2 - a^2}{4} + \frac{d_4}{Y(y)^2}. \qquad \blacksquare$$

Remark 2. The underlying structure of the solutions of the general equations (9)–(12) is fairly simple. Let $u_1(x, L_2) = u_1[L_2]$, $u_2(x, L_2) = u_2[L_2]$ be a basis of solutions of the separated equation

$$\left(\frac{d^2}{dx^2} + v_1(x) - f_1(x)H - L_2\right)u = 0,$$

and let $w_1(y, L_2)$, $w_2(y, L_2)$ be a basis of solutions of the separated equation

$$\left(\frac{d^2}{dy^2} + v_2(y) - f_2(y)H - L_2\right) w = 0.$$

Then for any parameter $\hat{L}_2$ the operator

$$S(\hat{L}_2) = w_2[\hat{L}_2]u_2[\hat{L}_2]\left(w_1[L_2]u_1[L_2]\partial_{xy} - w_1'[L_2]u_1[L_2]\partial_x - w_1[L_2]u_1'[L_2]\partial_y + w_1'[L_2]u_1'[L_2]\right)$$

is a symmetry operator of L_1 that maps any eigenspace of L_2 into another (generally different) eigenspace. It is not hard to characterize the space spanned by all linear combinations of functions $w_2[\hat{L}_2]u_2[\hat{L}_2]w_1[L_2]u_1[L_2]$ and this gives the equations for A. Similarly we can characterize B, C, and D. Also the solutions of (9)–(12) in the special case $A \equiv 0$ can be obtained by taking limits of the above general solutions. Indeed

$$\begin{aligned}
\tilde{S}(\hat{L}_2) = {} & (\partial_L w_2[\hat{L}_2]u_2[\hat{L}_2] + w_2[\hat{L}_2]\partial_l u_2[\hat{L}_2]) \\
& \times \left(-w_1'[L_2]u_1[L_2]\partial_x - w_1[L_2]u_1'[L_2]\partial_y + w_1'[L_2]u_1'[L_2]\right) \\
& + (\partial_L w_1[\hat{L}_2]u_1[\hat{L}_2] + w_1[\hat{L}_2]\partial_l u_1[\hat{L}_2]) \\
& \times \left(-w_2'[L_2]u_2[L_2]\partial_x - w_2[L_2]u_2'[L_2]\partial_y + w_2'[L_2]u_2'[L_2]\right) \\
& - (\partial_L w_2[\hat{L}_2]u_1[\hat{L}_2] + w_2[\hat{L}_2]\partial_l u_1[\hat{L}_2]) \\
& \times \left(-w_1'[L_2]u_2[L_2]\partial_x - w_1[L_2]u_2'[L_2]\partial_y + w_1'[L_2]u_2'[L_2]\right) \\
& - (\partial_L w_1[\hat{L}_2]u_2[\hat{L}_2] + w_1[\hat{L}_2]\partial_l u_2[\hat{L}_2]) \\
& \times \left(-w_2'[L_2]u_1[L_2]\partial_x - w_2[L_2]u_1'[L_2]\partial_y + w_2'[L_2]u_1'[L_2]\right)
\end{aligned}$$

is a symmetry with the right properties.

4 Lie form and nonorthogonal separation in two dimensions

We know that if a Hamiltonian $H = \sum_{i,j=1}^{2} g^{ij}p_ip_j$ admits a constant of the motion L that is quadratic in the momenta

$$L = \sum_{i,j=1}^{2} a^{ij}p_ip_j, \qquad \{H, L\} = 0$$

and if the roots of the determinant $|a^{ij} - \lambda g^{ij}|$ are distinct, then the eigenforms define new (separable) variables x_1, x_2 and the associated Schrödinger operator can be written in Liouville form

$$H = \frac{1}{f_1(x_1) + f_2(x_2)}\left(\partial_{x_1x_1} + \partial_{x_2x_2} + v_1(x_1) + v_2(x_2)\right).$$

However, it may be that the roots of this determinant are equal. In this case H cannot be put into Liouville form, but rather Lie form, which for a suitable choice of variables (non-separable) is

$$H = \frac{1}{x + B(y)}\left(\partial_{xy} + \frac{1}{2}K(y)\right) + \frac{1}{2}U'(y).$$

The associated quantum constant is $L = \partial_{xx} - 2yH + U(y)$.

How can we calculate a third symmetry? We look for a quantum constant of the form

$$L^o = M(H, L, y)\partial_y + N(H, L, y).$$

Applying the condition $[H, L^o] = 0$, we see that the functions M and N are of the form

$$M = \frac{1}{2H - U'(y)},$$
$$N = -\frac{1}{2}\int \frac{U''K - 2K'H + K'U' + 4B'H^2 + 4B'U'H + B'U'^2}{\sqrt{L + 2yH - U}(2H - U')^2}dy.$$

According to our operational calculus, these relations make sense when the operators are applied to a simultaneous eigenspace of H and L. Note in particular that if we consider the free Hamiltonian then L^o has the particularly simple form

$$L^o = H \int \frac{B'(y)}{\sqrt{L + 2yH}}dy - \partial_y.$$

While this appears to be only a formal expression it makes good sense for a spectral resolution analytic in a neighborhood of $(H_0, L_0) \neq (0, 0)$. Indeed, the following explicit example is informative.

Example 5. Consider the zero potential case $K = U \equiv 0$, $B(y) = y^2$. We can formally evaluate the integral in the expression for L^o by integrating by parts, and then multiply by $3H$ to obtain

$$L^i = 2(L + 2yH)^{1/2}(-L + yH) - 3H\partial_y.$$

This expression can immediately be interpreted as the differential operator

$$\hat{L} = 2\partial_x(-L + yH) - 3\partial_y H = -2\partial_x^3 + \frac{1}{x + y^2}(6y\partial_x^2\partial_y - 3\partial_y^2\partial_x)$$

which can be verified to commute with H. Indeed, if $B(y)$ is a polynomial then, through integration by parts, we can always uncover a symmetry operator of finite order. In this particular example the Hamiltonian also admits a second order symmetry operator

$$\hat{N} = x\partial_x^2 + \frac{3}{4}\partial_y^2 - \frac{3xy + \frac{1}{3}y^3}{x + y^2}\partial_x\partial_y,$$

and $[\hat{N}, L] = \hat{L}$. However, for general polynomial $B(y)$ the corresponding invariant $\hat{L}$ cannot be obtained as a commutator of other finite differential invariants.

It is clear from the method of the example that if one takes $U(y)$ as a constant and $B(y)$ and $K(y)$ as polynomials then, as before, we can generate explicit finite order differential operators that commute with H.

There is one remaining possibility for a quadratic constant of the motion in two dimensions: the constant may be associated with *nonorthogonal* separation of variables. In two dimensions there is only one case: separation in light cone (null) coordinates, [11]. For this case the Hamiltonian takes the form

$$H = \partial_z\partial_{\bar{z}} + f(\bar{z})$$

and there is a first-order symmetry operator ∂_z, so ∂_{zz} is a second-order constant of the motion. In addition there is a quadratic constant

$$L = M\partial_z + \frac{i}{2}\int \bar{z}\frac{df}{d\bar{z}}d\bar{z}.$$

where $M = i(z\partial_z - \bar{z}\partial_{\bar{z}})$.

5 Further work

A particular feature of our approach is that we can replace the partial differential operators L_1, L_2 (that do not commute with the underlying variables x, y) by scalar parameters that commute with all variables. It provides a relatively simple tool to uncover invariants. Encouraged by the theorem above, we expect to be able to characterize the structure of the algebra of infinite dimensional symmetries of L_1. Of particular interest would be conditions that would guarantee that the symmetry conditions had solutions that were polynomials in the parameters.

In future papers we will show that there is a similar calculus to express differential symmetries of the time-independent Schrödinger equation on an n-dimensional Riemannian manifold, when the equation admits R-separation in an orthogonal coordinate system. Further there is a calculus to express differential conformal symmetries of the time-dependent Schrödinger equation on an n-dimensional Riemannian manifold, when the equation admits R-separation. The time-dependent case is particularly interesting because some of its conformal symmetries can be considered as energy-shifting operators for the time-independent equations. These energy-shifting operators are usually of infinite order, but appear first order in our canonical form. They correspond to raising and lowering operators in the equations of mathematical physics.

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Symmetries of Differential Equations and Cartan's Equivalence Method

Oleg I. MOROZOV

Snezhinsk Physical and Technical Academy, Snezhinsk, Chelyabinsk Region, Russia
E-mail: *morozov@sfti.snz.ru*

Applications of Cartan's equivalence method to symmetries of differential equations are considered. The examples include interrelations between the nonlinear acoustics equations, and the solutions of equivalence problems for the classes of linear parabolic equations and nonlinear wave equations.

Introduction

In this work we consider applications of Élie Cartan's structure theory of Lie pseudo-groups to symmetry groups of differential equations. This approach allows to solve equivalence problems for classes of differential equations and to find contact transformations mapping equivalent equations into each other. Also, it allows to obtain all differential invariants of symmetry groups for differential equations without analysis of over-determined systems of partial differential equations.

Definition 1 ([20, 13]). A *pseudo-group* $\mathcal{G}$ on a manifold M is a collection of local diffeomorphisms of M, which is closed under composition *when defined*, contains an identity and is closed under inverse. A *Lie pseudo-group* is a pseudo-group whose diffeomorphisms are local analytic solutions of an involutive system of partial differential equations.

Example 1. The pseudo-group of conformal transformations in $\mathbb{R}^2$ consists of local mappings $(x, y) \mapsto (X, Y)$ satisfying the Cauchy–Riemann equations $X_x = Y_y$, $X_y = -Y_x$.

É. Cartan developed an approach to study Lie pseudo-groups by means of exterior differential forms. He showed that for every Lie pseudo-group $\mathcal{G}$ acting on a manifold M there exists a finite-dimensional Lie group H and a set of 1-forms $\Omega = \{\omega^1, \dots, \omega^s\}$ on a direct product $M \times H$ such that a transformation $\Delta : M \to M$ belongs to $\mathcal{G}$ if and only if the forms Ω are invariant under a lift $\hat{\Delta} : M \times H \to M \times H$. The forms Ω are called a *moving coframe* or *tautological forms* for the pseudo-group $\mathcal{G}$.

Example 1 (continued). For the pseudo-group of conformal transformations in $\mathbb{R}^2$ we can take the moving coframe $\omega^1 = a\,dx - b\,dy$, $\omega^2 = b\,dx + a\,dy$ on $\mathbb{R}^2 \times H$, where H is a Lie group of non-degenerate matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Taking differentials of the tautological forms, we obtain Cartan's *structure equations* for the pseudo-group

$$d\omega^i = A^i_{\alpha k}\,\eta^\alpha \wedge \omega^k + T^i_{jk}\,\omega^j \wedge \omega^k,$$

where $A^i_{\alpha k}$, as usual, are constants, the *torsion coefficients* T^i_{jk} are invariants of the pseudo-group, and η^α are linear combinations of Maurer–Cartan forms for the group H and the forms Ω. Cartan gave an explicit, practical test for involutivity of moving coframes. The test deals with ranks of matrices constructed from $A^i_{\alpha k}$. For non-involutive coframes Cartan's prolongation procedure allows us to obtain an involutive coframe after a finite number of extensions of the group H.

Example 1 (continued). The structure equations for the pseudo-group of conformal transformations are $d\omega^1 = \pi_1 \wedge \omega^1 - \pi^2 \wedge \omega^2$, $d\omega^2 = \pi_2 \wedge \omega^1 + \pi^1 \wedge \omega^2$, where $\pi_1 = (a\,da + b\,db)\,(a^2 + b^2)^{-1} + t_1\,\omega^1 + t_2\,\omega^2$, $\pi_2 = (-b\,da + a\,db)\,(a^2 + b^2)^{-1} - t_2\,\omega^1 + t_1\,\omega^2$, while t_1 and t_2 are free parameters.

Full details of Cartan's method can be found in [1–3, 17].

1 Contact transformations

Using Cartan's method, we can find a moving coframe for the pseudo-group $Cont(J^1(\mathcal{E}))$ of contact transformations on a bundle $J^1(\mathcal{E})$ of first-order jets of sections of $\mathcal{E} = \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. In the local coordinates $(x^i, u^\alpha, p_i^\alpha)$, $i \in \{1, \dots, n\}$, $\alpha \in \{1, \dots, m\}$, a contact transformation $\Delta : J^1(\mathcal{E}) \to J^1(\mathcal{E})$, $\Delta : (x^i, u^\alpha, p_i^\alpha) \mapsto (\overline{x}^i, \overline{u}^\alpha, \overline{p}_i^\alpha)$, is defined by the requirement to map the *contact forms* $\vartheta^\alpha = du^\alpha - p_i^\alpha\,dx^i$ into linear combinations of the contact forms: $\Delta^*\overline{\vartheta}^\alpha = d\overline{u}^\alpha - \overline{p}_i^\alpha\,d\overline{x}^i = \zeta_\beta^\alpha(x, u, p)\,\vartheta^\beta$ for appropriate functions ζ_β^α on $J^1(\mathcal{E})$.

As it is shown in [14], we can take the following moving coframe for $Cont(J^1(\mathcal{E}))$:

$$\Theta^\alpha = a_\beta^\alpha\,(du^\beta - p_j^\beta\,dx^j), \quad \Xi^i = c_\beta^i\,\Theta^\beta + b_j^i\,dx^j, \quad \Sigma_i^\alpha = f_{i\beta}^\alpha\,\Theta^\beta + g_{ij}^\alpha\,\Xi^j + a_\beta^\alpha\,B_i^j\,dp_j^\beta. \tag{1}$$

These forms are defined on $J^1(\mathcal{E}) \times H$, where H is the Lie group of block lower triangular matrices

$$\begin{pmatrix} a_\beta^\alpha & 0 & 0 \\ c_\gamma^i\,a_\beta^\gamma & b_j^i & 0 \\ (f_{i\gamma}^\alpha + g_{ik}^\alpha\,c_\gamma^k)\,a_\beta^\gamma & g_{ik}^\alpha\,b_j^k & a_\beta^\alpha\,B_i^j \end{pmatrix},$$

and the parameters a_β^α, b_j^i, c_β^i, $f_{i\beta}^\alpha$, and g_{ij}^α obey the requirements $\det\left(a_\beta^\alpha\right) \neq 0$, $\det\left(b_j^i\right) \neq 0$, $b_k^i\,B_j^k = \delta_j^i$, and $g_{ij}^\alpha = g_{ji}^\alpha$. It is easy to verify directly that Δ is a contact transformation whenever (1) is invariant under an action of a lift of Δ on $J^1(\mathcal{E}) \times H$. The structure equations for the lifted coframe (1) have the form

$$\begin{aligned} &d\Theta^\alpha = \Phi_\beta^\alpha \wedge \Theta^\beta + \Xi^k \wedge \Sigma_k^\alpha, \qquad d\Xi^i = \Psi_k^i \wedge \Xi^k + \Pi_\gamma^i \wedge \Theta^\gamma, \\ &d\Sigma_i^\alpha = \Phi_\gamma^\alpha \wedge \Sigma_i^\gamma - \Psi_i^k \wedge \Sigma_k^\alpha + \Lambda_{i\beta}^\alpha \wedge \Theta^\beta + \Upsilon_{ij}^\alpha \wedge \Xi^j, \end{aligned} \tag{2}$$

where Φ_β^α, Ψ_k^i, Π_γ^i, $\Lambda_{i\beta}^\alpha$, and Υ_{ij}^α are appropriate 1-forms on $J^1(\mathcal{E}) \times H$. From these equations it follows that the moving coframe (1) is involutive. The structure equations (2) remain unchanged if we make the following change of the forms Φ_β^α, Ψ_k^i, Π_γ^i, $\Lambda_{i\beta}^\alpha$, and Υ_{ij}^α:

$$\begin{aligned} &\Phi_\beta^\alpha \mapsto \Phi_\beta^\alpha + K_{\beta\gamma}^\alpha\,\Theta^\gamma, \qquad \Psi_k^i \mapsto \Psi_k^i + L_{kj}^i\,\Xi^j + M_{k\gamma}^i\,\Theta^\gamma, \\ &\Pi_\gamma^i \mapsto \Pi_\gamma^i + M_{k\gamma}^i\,\Xi^k + N_{\gamma\epsilon}^i\,\Theta^\epsilon, \\ &\Lambda_{i\beta}^\alpha \mapsto \Lambda_{i\beta}^\alpha + P_{i\beta\gamma}^\alpha\,\Theta^\gamma + Q_{i\beta k}^\alpha\,\Xi^k + K_{\gamma\beta}^\alpha\,\Sigma_i^\gamma - M_{i\beta}^k\,\Sigma_k^\alpha, \\ &\Upsilon_{ij}^\alpha \mapsto \Upsilon_{ij}^\alpha + Q_{i\beta j}^\alpha\,\Theta^\beta + R_{ijk}^\alpha\,\Xi^k - L_{ij}^k\,\Sigma_k^\alpha, \end{aligned} \tag{3}$$

where $K_{\beta\gamma}^\alpha$, L_{kj}^i, $M_{k\gamma}^i$, $N_{\gamma\epsilon}^i$, $P_{i\beta\gamma}^\alpha$, $Q_{i\beta k}^\alpha$, and R_{ijk}^α are arbitrary functions on $J^1(\mathcal{E}) \times H$ satisfying the symmetry conditions $K_{\beta\gamma}^\alpha = K_{\beta\gamma}^\alpha$, $L_{kj}^i = L_{jk}^i$, $N_{\gamma\epsilon}^i = N_{\epsilon\gamma}^i$, $P_{i\beta\gamma}^\alpha = P_{i\gamma\beta}^\alpha$, $Q_{i\beta k}^\alpha = Q_{k\beta i}^\alpha$, $R_{ijk}^\alpha = R_{ikj}^\alpha = R_{jik}^\alpha$.

Another approach to construction of 1-forms characterizing contact transformations is presented in [18].

2 Symmetries of differential equations

Symmetry groups of differential equations are sub-groups of pseudo-groups of contact transformations. Therefore Cartan's method of equivalence is applicable to finding tautological forms for these symmetry groups. We briefly outline this approach, [4, 14].

Every partial differential equation (PDE) can be transformed into an equivalent PDE of the first order by adding appropriate new variables, [20, th. 3.3.1], [6, § 17.4]. Therefore we take a PDE $\mathcal{R}$ of the first order. $\mathcal{R}$ is a sub-bundle in $J^1(\mathcal{E})$. Let $Sym(\mathcal{R})$ be the group of contact symmetries for $\mathcal{R}$. It consists of all the contact transformations on $J^1(\mathcal{E})$ mapping $\mathcal{R}$ to itself. Let $\iota : \mathcal{R} \to J^1(\mathcal{E})$ be an embedding. The tautological forms of $Sym(\mathcal{R})$ are restrictions of the lifted coframe (1) on $\mathcal{R}$: $\theta^\alpha = \iota^*\Theta^\alpha$, $\xi^i = \iota^*\Xi^i$, and $\sigma_i^\alpha = \iota^*\Sigma_i^\alpha$ (for brevity we identify the map $\iota \times id : \mathcal{R} \times H \to J^1(\mathcal{E}) \times H$ with $\iota : \mathcal{R} \to J^1(\mathcal{E})$). The forms θ^α, ξ^i, and σ_i^α have some linear dependencies, i.e., there exists a non-trivial set of functions S_α, T_i, and U_α^i on $\mathcal{R} \times H$ such that $S_\alpha\,\theta^\alpha + T_i\,\xi^i + U_\alpha^i\,\sigma_i^\alpha \equiv 0$. These functions are lifted invariants of $Sym(\mathcal{R})$. Setting them equal to appropriate constants allows us to specify some parameters a_β^α, b_j^i, c_β^i, $f_{i\beta}^\alpha$, and g_{ij}^α of the group H as functions of the coordinates on $\mathcal{R}$ and the other group parameters.

After these normalizations, some restrictions of the forms $\phi_\beta^\alpha = \iota^*\Phi_\beta^\alpha$, $\psi_k^i = \iota^*\Psi_k^i$, and $\pi_\beta^i = \iota^*\Pi_\beta^i$, or some their linear combinations, become semi-basic, i.e., they do not include the differentials of the parameters of H. From (3), we have the following statements: (i) if ϕ_β^α is semi-basic, then its coefficients at σ_j^γ and ξ^j are lifted invariants of $Sym(\mathcal{R})$; (ii) if ψ_k^i or π_β^i are semi-basic, then their coefficients at σ_j^γ are lifted invariants of $Sym(\mathcal{R})$. Setting these invariants equal to some constants, we get specifications of some more parameters of H as functions of the coordinates on $\mathcal{R}$ and the other group parameters.

More lifted invariants can appear as essential torsion coefficients in the reduced structure equations

$$d\theta^\alpha = \phi_\beta^\alpha \wedge \theta^\beta + \xi^k \wedge \sigma_k^\alpha, \qquad d\xi^i = \psi_k^i \wedge \xi^k + \pi_\gamma^i \wedge \theta^\gamma,$$
$$d\sigma_i^\alpha = \phi_\gamma^\alpha \wedge \sigma_i^\gamma - \psi_i^k \wedge \sigma_k^\alpha + \lambda_{i\beta}^\alpha \wedge \theta^\beta + \upsilon_{ij}^\alpha \wedge \xi^j.$$

After normalizing these invariants and repeating the process, two outputs are possible. In the first case, the reduced lifted coframe appears to be involutive. Then this coframe is the desired set of tautological forms for $Sym(\mathcal{R})$. In the second case, when the reduced lifted coframe does not satisfy Cartan's test, we should use the procedure of prolongation, [17, ch. 12].

Another approach to find structure equations, but not tautological forms, for symmetry groups of PDEs is given in [12, 13].

3 Interrelations between the Khokhlov–Zabolotskaya equation and the short wave gas dynamics equation

Consider two equations: the Khokhlov–Zabolotskaya equation, [23, ch. 3, § 5.3], [21], in the potential form

$$v_x = u_y, \qquad v_y = u_t + u\,u_x, \tag{4}$$

and the short wave gas dynamics equation, [6, § 23.4], [5],

$$v_x = u_y, \qquad v_y = u_t + (u + x)\,u_x + k\,u. \tag{5}$$

The moving coframe method gives the following structure equations for the symmetry pseudo-group of (4):

$$
\begin{aligned}
&d\theta^1 = \eta_1 \wedge \theta^1 + \xi^1 \wedge \sigma_1^1 + \xi^2 \wedge \sigma_2^1 + \xi^3 \wedge \sigma_3^1,\\
&d\theta^2 = \frac{3}{2}\,\eta_1 \wedge \theta^1 + \eta_3 \wedge \theta^1 + \xi^1 \wedge \sigma_1^2 + \xi^2 \wedge \sigma_3^1 + \xi^3 \wedge \sigma_1^1 + \xi^3 \wedge \sigma_2^1,\\
&d\xi^1 = \eta_2 \wedge \xi^1,\\
&d\xi^2 = -\eta_1 \wedge \xi^1 + \eta_1 \wedge \xi^2 + \eta_2 \wedge \xi^2 - \eta_3 \wedge \xi^3 - \theta^1 \wedge \xi^1,\\
&d\xi^3 = \frac{1}{2}\,\eta_1 \wedge \xi^3 + \eta_2 \wedge \xi^3 - 2\,\eta_3 \wedge \xi^1,\\
&d\sigma_1^1 = \eta_1 \wedge \big(\sigma_1^1 + \sigma_2^1\big) - \eta_2 \wedge \sigma_1^1 + 2\,\eta_3 \wedge \sigma_3^1 + \eta_4 \wedge \xi^3 + \eta_5 \wedge \xi^2 + \eta_6 \wedge \xi^1 - \theta^1 \wedge \sigma_2^1,\\
&d\sigma_2^1 = -\eta_2 \wedge \sigma_2^1 + \eta_5 \wedge \xi^1,\\
&d\sigma_3^1 = \frac{1}{2}\,\eta_1 \wedge \sigma_3^1 - \eta_2 \wedge \sigma_3^1 + \eta_3 \wedge \sigma_2^1 + \eta_4 \wedge \xi^1 + \eta_5 \wedge \xi^3,\\
&d\sigma_1^2 = \eta_1 \wedge \left(\sigma_3^1 + \frac{1}{2}\,\sigma_1^2\right) - \eta_2 \wedge \sigma_1^2 + \eta_3 \wedge \big(3\,\sigma_1^1 + 2\,\sigma_2^1\big) + \eta_4 \wedge \xi^2 + \eta_5 \wedge \xi^3 + \eta_6 \wedge \xi^3\\
&\qquad + \eta_7 \wedge \xi^1 - 3\,\theta^2 \wedge \sigma_2^1,\\
&d\eta_1 = 2\,\xi^1 \wedge \sigma_2^1,\\
&d\eta_2 = -3\,\xi^1 \wedge \sigma_2^1,\\
&d\eta_3 = \frac{1}{2}\,\eta_1 \wedge \eta_3 + \xi^1 \wedge \sigma_3^1 + \xi^3 \wedge \sigma_2^1,\\
&d\eta_4 = \pi_1 \wedge \xi^1 + \pi_2 \wedge \xi^3 + \frac{1}{2}\,\eta_1 \wedge \eta_4 - 2\,\eta_2 \wedge \eta_4 + 3\,\eta_3 \wedge \eta_5 - 3\sigma_2^1 \wedge \sigma_3^1,\\
&d\eta_5 = \pi_2 \wedge \xi^1 - 2\,\eta_2 \wedge \eta_5,\\
&d\eta_6 = \pi_1 \wedge \xi^3 + \pi_2 \wedge \xi^2 + \pi_3 \wedge \xi^1 + 2\,\eta_1 \wedge \eta_5 + \eta_1 \wedge \eta_6 - 2\,\eta_2 \wedge \eta_6 + 4\,\eta_3 \wedge \eta_4 + 6\,\sigma_1^1 \wedge \sigma_2^1,\\
&d\eta_7 = \pi_1 \wedge \xi^2 + \pi_2 \wedge \xi^3 + \pi_3 \wedge \xi^3 + \pi_4 \wedge \xi^1 + 2\,\eta_1 \wedge \eta_4 + \frac{3}{2}\,\eta_1 \wedge \eta_7 - 2\,\eta_2 \wedge \eta_7 + 4\,\eta_3 \wedge \eta_5\\
&\qquad + 5\,\eta_3 \wedge \eta_6 - \eta_4 \wedge \theta^1 + 3\,\eta_5 \wedge \theta^2 + 3\,\sigma_1^1 \wedge \sigma_3^1 - 9\,\sigma_2^1 \wedge \sigma_1^2.
\end{aligned}
\tag{6}
$$

The structure of symmetry pseudo-group of the equation (5) depends on the value of the parameter k. Particularly, if $k = -2$ or $k = -\frac{1}{2}$, the pseudo-group has exactly the same structure equations (6) but with different 1-forms θ^1, θ^2, ξ^1, ξ^2, ξ^3, σ_1^1, σ_2^1, σ_3^1, σ_1^2, η_1, …, η_7, π_1, …, π_4. Therefore, in the cases $k = -2$ or $k = -\frac{1}{2}$ there exist transformations mapping equation (5) into equation (4).

For the other values of k the structure equations for the symmetry pseudo-group of equation (5) have the form

$$
\begin{aligned}
&d\theta^1 = \eta_1 \wedge \theta^1 + \xi^1 \wedge \sigma_1^1 + \xi^2 \wedge \sigma_2^1 + \xi^3 \wedge \sigma_3^1,\\
&d\theta^2 = \frac{3}{2}\,\eta_1 \wedge \theta^1 + \eta_2 \wedge \theta^1 + \xi^1 \wedge \sigma_1^2 + \xi^2 \wedge \sigma_3^1 + \xi^3 \wedge \sigma_1^1 + \xi^3 \wedge \sigma_2^1,\\
&d\xi^1 = 0,\\
&d\xi^2 = \eta_1 \wedge \big(\xi^2 - \xi^1\big) - \eta_2 \wedge \xi^3 - \theta^1 \wedge \xi^1 + E\,\xi^1 \wedge \xi^2,\\
&d\xi^3 = \frac{1}{2}\,\eta_1 \wedge \xi^3 - 2\,\eta_2 \wedge \xi^1 + E\,\xi^1 \wedge \xi^3,\\
&d\sigma_1^1 = \eta_1 \wedge \big(\sigma_1^1 + \sigma_2^1\big) + 2\,\eta_2 \wedge \sigma_3^1 + \eta_3 \wedge \xi^3 + \eta_4 \wedge \xi^1 - \theta^1 \wedge \sigma_2^1 + E\,\xi^2 \wedge \sigma_2^1,\\
&d\sigma_2^1 = 0,
\end{aligned}
$$

$$
\begin{aligned}
d\sigma_3^1 &= \frac{1}{2}\,\eta_1\wedge\sigma_3^1+\eta_2\wedge\sigma_2^1+\eta_3\wedge\xi^1-E\,\xi^1\wedge\sigma_3^1+E\,\xi^3\wedge\sigma_2^1,\\
d\sigma_1^2 &= \eta_1\wedge\sigma_3^1+\frac{3}{2}\,\eta_1\wedge\sigma_1^2+3\,\eta_2\wedge\sigma_1^1+2\,\eta_2\wedge\sigma_2^1+\eta_3\wedge\xi^2+\eta_4\wedge\xi^3+\eta_5\wedge\xi^1\\
&\quad+\frac{1}{9}\,\theta_1\wedge\xi^3-3\,\theta_2\wedge\sigma_2^1+E\,\xi^3\wedge\left(\sigma_1^1+\sigma_2^1\right),\\
d\eta_1 &= 2\,\xi^1\wedge\sigma_2^1,\\
d\eta_2 &= \frac{1}{2}\,\eta_1\wedge\eta_2-\frac{1}{9}\,\xi^1\wedge\xi^3+\xi^1\wedge\sigma_3^1+\xi^3\wedge\sigma_2^1,\\
d\eta_3 &= \pi_1\wedge\xi^1+\frac{1}{2}\,\eta_1\wedge\eta_3-3\,E\,\eta_2\wedge\sigma_2^1+\left(\frac{1}{9}-2\,E^2\right)\,\xi^3\wedge\sigma_2^1-3\,\sigma_2^1\wedge\sigma_3^1,\\
d\eta_4 &= \pi_1\wedge\xi^3+\pi_2\wedge\xi^1+\eta_1\wedge\eta_4-E\,\eta_1\wedge\sigma_2^1+4\,\eta_2\wedge\eta_3+2\,E\,\eta_2\wedge\sigma_3^1-E\,\eta_3\wedge\xi^3\\
&\quad-E\,\theta^1\wedge\sigma_2^1-E^2\,\xi^2\wedge\sigma_2^1+\frac{2}{9}\,\xi^3\wedge\sigma_3^1+3\,\sigma_1^1\wedge\sigma_2^1\\
d\eta_5 &= \pi_1\wedge\xi^2+\pi_2\wedge\xi^3+\pi_3\wedge\xi^1+2\,\eta_1\wedge\eta_3+\frac{3}{2}\,\eta_1\wedge\eta_5+E\,\eta_1\wedge\sigma_3^1+5\,\eta_2\wedge\eta_4\\
&\quad+\frac{2}{9}\,\eta_2\wedge\theta^1-2\,E\,\eta_2\wedge\left(\sigma_1^1+\sigma_2^1\right)-\eta_3\wedge\theta^1-E\,\eta_3\wedge\xi^2-2\,E\,\eta_4\wedge\xi^3-\frac{1}{9}\,E\,\theta^1\wedge\xi^3\\
&\quad+\left(\frac{4}{9}-E^2\right)\,\xi^3\wedge\sigma_1^1+\left(\frac{2}{9}-E^2\right)\,\xi^3\wedge\sigma_2^1+3\,\sigma_1^1\wedge\sigma_3^1-6\,\sigma_2^1\wedge\sigma_1^2,
\end{aligned}
$$

where $E=\frac{1}{3}\,(9\,u_x+4\,k+5)\,(2\,k^2+5\,k+2)^{-1/2}$. This function is an invariant of the pseudo-group. Since $dE=3\,\sigma_2^1$, it follows that all the derived invariants are constants, and the classifying manifold of the pseudo-group, i.e., the manifold parameterized by differential invariants, is a line. Thus for every two equations (5) such that $(k+2)\,(k+2^{-1})\neq 0$ the classifying manifolds of their symmetry pseudo-groups (locally) overlap. So these equations are equivalent under a contact transformation, [17, th. 15.12]. For example, we can take $k=0$ for one of the equations.

The knowledge of invariant 1-forms defining the symmetry pseudo-groups allows us to find the transformations explicitly.

Theorem 1. *If $k=-2$, then the transformation*

$$
\begin{aligned}
&\tilde t=t,\qquad \tilde x=\exp\left(\frac{4}{3}\,t\right)\,\left(x+\frac{1}{6}\,y^2\right),\qquad \tilde y=\exp\left(\frac{2}{3}\,t\right)\,y,\\
&\tilde u=\exp\left(\frac{4}{3}\,t\right)\,\left(u+\frac{1}{3}\,x-\frac{1}{18}\,y^2\right),\qquad \tilde v=\mathrm{e}^{2\,t}\,\left(v-\frac{1}{3}\,y\,u-\frac{2}{9}\,x\,y+\frac{2}{81}\,y^3\right)
\end{aligned}
$$

maps equation (4) *into equation* (5) *written in the tilded variables. In the case $k=-\frac{1}{2}$ the transformation between* (4) *and* (5) *has the form*

$$
\begin{aligned}
&\tilde t=t,\quad \tilde x=\exp\left(\frac{2}{3}\,t\right)\,\left(x+\frac{1}{12}\,y^2\right),\qquad \tilde y=\exp\left(\frac{1}{3}\,t\right)\,y,\\
&\tilde u=\exp\left(\frac{2}{3}\,t\right)\,\left(u-\frac{1}{3}\,x-\frac{1}{18}\,y^2\right),\qquad \tilde v=\mathrm{e}^{t}\,\left(v-\frac{1}{6}\,y\,u-\frac{1}{18}\,x\,y+\frac{1}{324}\,y^3\right).
\end{aligned}
$$

The transformation

$$
\begin{aligned}
&\tilde t=\frac{(3\,\lambda+1)\,(3\,\lambda+2)}{2\,(2\,\lambda+1)}\,t,\qquad \tilde x=\frac{(2\,\lambda+1)\,\mathrm{e}^{2\lambda t}}{2\,(3\lambda+1)(3\lambda+2)}\,(4\,x+\lambda^2\,y),\qquad \tilde y=\mathrm{e}^{\lambda t}\,y,\\
&\tilde u=\frac{(2\lambda+1)\,\mathrm{e}^{2\lambda t}}{2(3\lambda+1)^2(3\lambda+2)^2}\,\left(8(2\lambda+1)u-\lambda(\lambda+1)(4x-(5\lambda+2)y^2)\right),\\
&\tilde v=\frac{(2\lambda+1)^2\mathrm{e}^{3\lambda t}}{(3\lambda+1)^3(3\lambda+2)^2}\,\left(24v-12\lambda(2\lambda+1)(u+(\lambda+1)x)-\lambda^2(\lambda+1)(5\lambda+2)y^3\right),
\end{aligned}
$$

where $\lambda=\frac{2}{9}\left(((k+2)(k+2^{-1}))^{1/2}+k-1\right)$*, maps equation* (5) *with* $k\notin\{-2,-2^{-1}\}$ *into this equation with* $k=0$ *(written in tilded variables).*

4 Equivalence of linear parabolic equations

The similar but more complicate analysis gives the solution of the contact equivalence problem for the class of linear parabolic equations (LPEs), [11, V.3, p. 492–523], [19],

$$u_{xx}=T(t,x)\,u_t+X(t,x)\,u_x+U(t,x)\,u. \tag{7}$$

All these equations have infinite-dimensional symmetry pseudo-groups, whose structure depends on the coefficients of equations. For simplicity of notation we consider the following normal form $u_{xx}=u_t+H(t,x)\,u$, while the results are valid for the original equations as well. The solution of the equivalence problem in terms of the coefficients of equations (7) is given in [15].

Define the functions

$$I=-(H_{xxx})^{1/5},\qquad J_1=-\frac{2}{5}\,H_{xxxx}\,(H_{xxx})^{-6/5},$$

$$J_2=\frac{1}{2}\,\left(2\,I\,(J_{1,x}\,J_{1,tx}-J_{1,t}\,J_{1,xx})-2\,I_t\,J_{1,x}^2+I^2\,J_1\,J_{1,t}\,J_{1,x}\right)\,I^{-3}\,J_{1,x}^{-1},$$

$$\begin{aligned}J_3=-\frac{1}{8}\,&\left(I^4J_1^2J_{1,t}^2-4II_{tt}J_{1,x}^2-8I^2J_{1,x}^2H_{xx}+2I^3J_{1,x}J_{1,t}^2+8I_t^2J_{1,x}^2-2I^3J_1J_{1,t}J_{1,tx}\right.\\&\left.-2I^2J_1I_tJ_{1,t}J_{1,x}+2I^3J_1J_{1,x}J_{1,tt}-2I^5J_1J_2J_{1,t}J_{1,x}\right)\,I^{-6}\,J_{1,x}^{-2},\end{aligned}$$

$$J_4=\left(2II_{tt}J_{1,x}-4I^6J_3J_{1,x}+4I^2J_{1,x}H_{xx}-I^3J_{1,t}+2I^3J_1J_{1,x}H_x-4I_t^2J_{1,x}\right)\,I^{-6}\,J_1^{-1}\,J_{1,x}^{-1},$$

$$L_0=\left(2I^2H_{xx}+I^3J_1H_x+II_{tt}-2I_t^2\right)\,I^{-2}J_{1,t}^{-1},\qquad L_1=(L_{0,x}-I_t)\,I^{-3},$$

$$L_2=\frac{1}{2}\,\left(2IL_{0,t}+IJ_1L_0^2+4I^2H_x-2I^3L_0L_1-4L_0I_t\right)\,I^{-5},$$

$$M_0=-2\,M_{1,t}\,(3J_1M_1+2)^{-2},\quad M_1=\frac{1}{2}\,\left(II_{tt}+2IH_{xx}+I^3J_1H_x-2I_t^2\right)\,I^{-6},$$

$$M_2=(M_{0,x}-I_t)\,I^{-3},\qquad M_3=\frac{1}{8}\left(2IM_{0,t}-2I^3M_0M_2-4M_0I_t+IJ_1M_0^2+4I^2H_x\right)I^{-5}.$$

Then we have

Theorem 2. *The class of LPEs is divided into the five subclasses* $\mathcal{P}_1,\mathcal{P}_2,\dots,\mathcal{P}_5$ *invariant under an action of the pseudo-group of contact transformations:*

$\mathcal{P}_1$ *consists of all LPEs such that* $I=0$*;*

$\mathcal{P}_2$ *consists of all LPES such that* $I\neq0$ *and* $J_{1x}\neq0$*;*

$\mathcal{P}_3$ *consists of all LPEs such that* $I\neq0$*,* $J_{1x}=0$*, and* $J_{1t}\neq0$*;*

$\mathcal{P}_4$ *consists of all LPEs such that* $I\neq0$*,* $J_1=m=\mathrm{const}$*, and* $3\,m\,M_1\neq-2$*;*

$\mathcal{P}_5$ *consists of all LPEs such that* $I\neq0$*,* $J_1=m=\mathrm{const}$*, and* $3\,m\,M_1=-2$*.*

Every equation from the subclass $\mathcal{P}_1$ *is equivalent to the linear heat equation* $u_{xx}=u_t$*.*

Two equations from $\mathcal{P}_2$ *are locally equivalent to each other if and only if they have the same functional dependencies among the differential invariants* J_1*,* J_2*,* J_3*,* J_4*, and their invariant derivatives.*

Two equations from $\mathcal{P}_3$ *are locally equivalent to each other if and only if they have the same functional dependencies among the differential invariants* J_1*,* L_1*,* L_2*, and their invariant derivatives.*

Two equations from $\mathcal{P}_4$ *are locally equivalent to each other if and only if they have the same functional dependencies among the differential invariants* M_1*,* M_2*,* M_3*, and their invariant derivatives.*

Every equation from the subclass $\mathcal{P}_5$ *is locally equivalent to the equation* $u_{xx}=u_t+\tilde m\,x^{-2}\,u$ *provided* $\tilde m=-4/(3\,m^5)$*.*

The expression of the function I in terms of the coefficients of equations (7) and the necessary and sufficient condition for equivalence to the linear heat equation are found in [10]. Some particular cases of invariants – so-called Laplace type invariants – are found in [9]. Numerous examples of equations (7) from the first subclass are given in [10, 22].

5 Equivalence of nonlinear wave equations

We outline the solution of the contact equivalence problem for the following class of nonlinear wave equations

$$u_t = a(x,u)\,v_x, \qquad v_t = b(x,u)\,u_x. \tag{8}$$

As it is shown in [8], these equations are equivalent to the class of equations $w_{tt} = f(x,w_x)\,w_{xx} + g(x,w_x)$ studied in [7]. In [8], the symmetry classification for systems (8) is given in the finite-dimensional cases. In [16], the moving coframe method is applied in both finite-dimensional and infinite-dimensional cases. The results are gathered in the following theorem.

Theorem 3. *Every system from the class of nonlinear wave equations* (8) *is equivalent under a contact transformation to a system from one of the five invariant subclasses* $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$, $\mathcal{Q}_4$, *and* $\mathcal{Q}_5$*:*

$\mathcal{Q}_1$ *consists of all systems* (8) *such that* $(b/a)_x \neq 0$, $(a\,b)_u \neq 0$, *and* $b^{-1}\,(b/a)_x\,(b\,a)_u \neq \text{const}$*;*

$\mathcal{Q}_2$ *consists of all systems* $u_t = a(x,u)\,v_x$, $v_t = a(x,u)\,u_x$ *such that* $(\ln a)_{xu} \neq 0$*;*

$\mathcal{Q}_3$ *consists of all systems* $u_t = a(u)\,v_x$, $v_t = a(u)\,u_x$ *such that*

$$N_1 = a\,\left(a_u^2 a_{uu} + a a_{uu}^2 - a a_u a_{uuu}\right)\,a_u^{-4} \neq \text{const};$$

$\mathcal{Q}_4$ *consists of all systems* $u_t = a(u)\,v_x$, $v_t = a(u)\,u_x$ *such that* $N_1 = \text{const}$*;*

$\mathcal{Q}_5$ *consists of the system* $u_t = v_x$, $v_t = u_x$.

For systems from subclasses $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *the symmetry groups are finite-dimensional, and their dimensions are between 2 and 6.*

For systems from subclasses $\mathcal{Q}_3$, $\mathcal{Q}_4$, *and* $\mathcal{Q}_5$ *the symmetry pseudo-groups are infinite-dimensional. The equations from* $\mathcal{Q}_3$ *and* $\mathcal{Q}_4$ *are linearizable by a hodograph transformation. Two equations from* $\mathcal{Q}_3$ *are equivalent under a contact transformation iff they have the same functional dependencies among the differential invariants* N_1, $N_2 = 2N_{1,u}^{-1}\left(a_{uu}a_u^{-1} - a_u a^{-1}\right) + \left(N_{1,u}^{-1}\right)_u$, *and* $N_3 = \left(4a^2 N_{1,u}^2 a_u^{-2} - 1\right) N_2 + 4N_{1,u}^{-1}\left(a^2 N_{1,u} a_u^{-2}\right)_u$.

Two systems from the subclass $\mathcal{Q}_4$ *are equivalent if and only if they have the same constant value of the invariant* N_1.

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Symmetry Methods Applied to a Mathematical Model of a Tumour of the Brain

Sibusiso MOYO † *and P.G.L. LEACH* ‡

† *Dept. of Math. and Centre Center for Research Systems, Durban Institute of Technology, PO Box 953, Steve Biko Campus, Durban 4000, Republic of South Africa*
E-mail: *moyos@dit.ac.za*

‡ *School of Mathematical and Statistical Sciences, University of Natal, Durban 4041, Republic of South Africa*
E-mail: *leachp@nu.ac.za, leach@math.aegean.gr*

We use symmetry methods to study the model of growth of a tumour of the brain proposed by Wein and Koplow (Mathematical modeling of brain cancer to identify promising combination treatments, *Preprint, D Sloan School of Management, MIT*, 1999). The aim here is also to demonstrate the use of symmetries to obtain more complex and reliable models. We translate some of the results found into properties of the tumour of the brain.

1 Introduction

A brain tumour is a dynamic system in which cancer cells grow and spread eventually killing good cells in the brain by deprivation of space and nutrients. The tumour spreads along the periphery and often dies out in the Centre due to a lack of fuel (oxygen and nutrients from the blood). This behaviour has been compared to that of a fire [10]. To be able to destroy the tumour treatments must be able to move faster than the tumour spreads if the treatment is to destroy effectively the tumour. Tumours are known to grow extremely fast. In reference [15] the tumour growth is assumed to be uniform and we claim that an improvement to the assumption of uniformity can be made. Since there are many aspects that can be considered in this particular problem, we concentrate only on the treatment aspect of the problem.

2 Spatio-temporal model

We recall that brain-cancer cells grow very fast and, at any point in time, only a portion of them are replicating and most cancer treatments only kill cells during this active phase.

This means that, when determining the net tumour-cell kill rates, models need take this constraint into account. A small fraction of tumour cells (about one in a thousand) called clonogenic cells are capable of regrowing the entire tumour. What does this mean? If the tumour is not to grow back after treatment all these cells must be killed. A tumour such as glioblastoma multiforme has many billions of cells and no single treatment presently available is capable of such a high kill rate. Wein's paper [15] demonstrates this. For example a log cell kill rate would kill 90 % of the tumour cells. A two log treatment (such as radiation) would kill 99 % of the tumour cells etc. It is important for the treatment to have the ability to target tumour cells rather than healthy cells and to move through the brain to reach the periphery of the tumour. This does affect how well a particular treatment works. We consider the following model [15], known as the Burgess equation [1],

$$\frac{\partial n(r,t)}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n(r,t)}{\partial r}\right) + pn(r,t) - k_t n(r,t), \tag{1}$$

where

- $n(r,t)$ is the concentration of tumour cells at location r at time t,
- D is the diffusion coefficient (estimated at 0.0013 cm^2 per day for glioblastoma multiforme) which measures the invasiveness of the glioblastoma multiforme cells,
- p is the proliferation rate of the tumour,
- k_t is the (therapy-dependent) killing rate at time t and
- r measures the distance from the Centre (i.e. the origin of glioblastoma multiforme).

Under the rescaling $t = (p - k_t)T$, $R = (|p - k_t|D)^2 r$ we may write (1) in the parameter-free form

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial r^2} + \frac{2}{r}\frac{\partial n}{\partial r} + n, \tag{2}$$

in which we have reverted to lower case variables.

Equations (1) and (2) assume spherical symmetry. Without that assumption we have

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial r^2} + \frac{2}{r}\frac{\partial n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 n}{\partial \theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial n}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 n}{\partial \phi^2} + n \tag{3}$$

in which the dependent variable $n = n(t, r, \theta, \phi)$. This enables us to consider a growth of the tumour which does not possess spherical symmetry and thereby opens the possibility to deal with an initial boundary of the tumour which is not spherical. The assumption that the tumour is spherical on detection is restrictive, but it is a starting point. In this paper we accept this restriction and concentrate on treatment models of this problem and leave the extension to nonspherical symmetry to future work.

3 Lie point symmetries for the differential equation with general $K(x,t)$

Equation (2) may be written in the normal form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \tag{4}$$

by the standard substitution $u(t,x) = rn(t,r)$, $x = r$. Here we consider the more general differential equation for a single cell type given by

$$\frac{\partial^2 u}{\partial x^2} - K(x,t)u - \frac{\partial u}{\partial t} = 0, \tag{5}$$

where $K(x,t)$ describes the temporal profile of the treatment and $K(x,t)u$ is the rate of removal of the tumour cells. In the model in reference [15] the treatment is allowed to be only a function of the time and we modify this to make it a function of both the position of the of the cancerous cells and time. This is a more realistic assumption.

The analysis of (5) for its Lie point symmetries has been established for a long time [4] (see also [5] and [13]). No matter the structure of $K(x,t)$ and there are always the homogeneity and solution symmetries, *videlicet*

$$\Gamma_h = u\partial_u \qquad \text{and} \qquad \Gamma_s = f(t,x)\partial_u, \tag{6}$$

where $f(t,x)$ and is any solution of (5). There are four possibilities for the structure of $K(x,t)$.

- Case I: $K(x,t)$ is an undifferentiated function. There is no symmetry additional to the two generic symmetries in (6).
- Case II: $K(x,t) = K(x)$. There is the additional symmetry

$$\Gamma_1 = \partial_t \tag{7}$$

which reflects the invariant of (5) under time translation. Since $[\Gamma_1, \Gamma_h]_{LB} = 0$, the Lie algebra of the finite symmetries is $2A_1$ Mubarakzyanov classification scheme [6–9].

- Case III: $K(x,t) = N\left(Ax+B\right)^2 + C + M/\left(Ax+B\right)^2$, where A, B, C, M and N are arbitrary constants and M and A are nonzero. This case can be reduced to the equivalence class

$$K_E(x) = \frac{\mu}{x^2} \tag{8}$$

by means of a suitable transformation [12]. With K as in (8) the additional symmetries are

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = 2t\partial_t + x\partial_x, \qquad \Gamma_3 = 4t^2\partial_t + 4tx\partial_x - \left(x^2 - 2t\right)u\partial_u \tag{9}$$

which have the Lie brackets $[\Gamma_1, \Gamma_2] = 2\Gamma_1$, $[\Gamma_2, \Gamma_3] = 2\Gamma_3$ and $[\Gamma_3, \Gamma_1] = -4\Gamma_2$ which is a representation of $sl(2,\mathbb{R})$. Since $[\Gamma_i, \Gamma_h] = 0$, $i = 0, 3$, the algebra of the finite symmetries is $sl(2,\mathbb{R}) \oplus A_1$.

- Case IV: $K(x) = Ax^2 + Bx + C$, where again A, B and C are arbitrary constants. This case can be reduced to the equivalence class

$$K_E = 0 \tag{10}$$

by the same transformation as used in Case III. The additional Lie point symmetries are those in (9) plus

$$\Gamma_4 = \partial_x, \qquad \Gamma_5 = 2t\partial_x - xu\partial_u \tag{11}$$

which have the Lie brackets $[\Gamma_4, \Gamma_5] = -\Gamma_4$. Consequently one needs to introduce Γ_h to close the algebra. Since $[\Gamma_4, \Gamma_h] = 0$ and $[\Gamma_5, \Gamma_h] = 0$, the algebra is $A_{3,1}$, commonly known as the Weyl algebra.

We note that the existence of the exceptional symmetries, Γ_1, $\Gamma_{2\pm}$ and $\Gamma_{3\pm}$, is directly related to the existence of Noether symmetries [11] for a Lagrangian of the form $\frac{1}{2}\left(\dot{x}^2 + 4K\right)$ [3].

Each of these cases, although Case I is trivial, is the representative of an equivalence class under point transformation. The essential point is that the number of Lie point symmetries additional to the generic symmetries, Γ_h and Γ_s, is limited to 0, 1, 3 or 5 for an equation of the form (5). To construct solutions using the symmetries viable forms of $K(x,t)$ are those for which the number of Lie point symmetries is 3 or 5, then we can use the method of construction employed by Lemmer *et al* [3] in the case of the time-dependent Schrödinger equation.

4 Similarity solutions of the Burgess equation

In the original model [15] equation (5) has the form

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - (p - k_t)u = 0, \tag{12}$$

where we recall that the constants p and k_t representative proliferation rate and the killing rate respectively. Equation (12) is of Class IV and has the nontrivial Lie point symmetries

$$\begin{aligned}
&\Gamma_1 = \partial_t, \qquad \Gamma_2 = 2t\partial_t + x\partial_x + 2(p-k_t)tu\partial_u,\\
&\Gamma_3 = 4t^2\partial_t + 4tx\partial_x + \left[4(p-k_t)t^2 - 2t - x^2\right]u\partial_u,\\
&\Gamma_4 = \partial_x, \qquad \Gamma_5 = 2t\partial_x - xu\partial_u,
\end{aligned} \tag{13}$$

where we note the differences in Γ_2 and Γ_3 due to the presence of the term $(p-k_t)u$ (12). Any of these symmetries apart from Γ_1 are suitable symmetries for the construction of similarities solutions and, guided by the example of Lemmer *et al* [3] in a treatment of the time-dependent Schrödinger equation, we choose Γ_4. The invariants of Γ_4 are found from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0} \tag{14}$$

and are

$$v = t \qquad \text{and} \qquad w = u \tag{15}$$

so that we may set

$$u = f(t) \tag{16}$$

and substitute it into (12) to obtain

$$f(t) = \exp\left[(p-k_t)\,t\right], \tag{17}$$

where we omit the multiplicitive constant of integration. Hence

$$u_0 = \exp\left[(p-k_t)\,t\right], \tag{18}$$

where we use the subscript zero to denote the basic solution. We use the property that symmetries map solutions into solutions and apply Γ_5 to the solution surface for u_0, *videlicet*

$$\Sigma_0 = u^{-1}\exp\left[(p-k_t)\,t\right] \tag{19}$$

to obtain

$$u_1 = x\exp\left[(p-k_t)\,t\right]. \tag{20}$$

Similarly we obtain

$$u_2 = \left(2t + x^2\right)\exp\left[(p-k_t)\,t\right], \qquad u_3 = \left(6tx + x^3\right)\exp\left[(p-k_t)\,t\right], \qquad \ldots. \tag{21}$$

Equally we could commence with Γ_5 to obtain a basic solution and use Γ_4 to create further solutions from the solution surface. We find that

$$\begin{aligned}
&v_0 = t^{-1/2}\exp\left[(p-k_t)\,t - \frac{x^2}{4t}\right], \qquad v_1 = -\frac{x}{2t^{3/2}}\exp\left[(p-k_t)\,t - \frac{x^2}{4t}\right],\\
&v_2 = t^{-1/2}\left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)\exp\left[(p-k_t)\,t - \frac{x^2}{4t}\right], \qquad \ldots.
\end{aligned} \tag{22}$$

We note that we can use Γ_2 and Γ_3 to generate solutions in the same way and that they also act as ladder operators on the solutions obtained by the other symmetries.

We recall that intensity of cells is essentially given by $n = u/x$ so that the number of cells within a sphere of radius X is

$$N_X = 4\pi \int_0^X x^2 \mathrm{d}x\, n(x,t) = 4\pi \int_0^X xu(x,t)\,\mathrm{d}x. \tag{23}$$

A feature of the solutions is the critical nature of the value of the expression $p - k_t$. If $p - k_t > 0$, the number of cells increases exponentially with time which is one of the salient properties of glioblastoma multiforme and the reason for its generally fatal outcome. By the time the tumour is large enough to be detected the number of cells is proliferating at a prodigious rate.

In the case that $p - k_t < 0$ the solutions indicate a rapid drop in the number of cells. However, the assumption of a constant killing rate is not a good one. In the case of chemotherapy the efficacy of the chemical decreases with time, usually in an exponential fashion. Thus we should replace k_t with something like $k\exp[-\sigma t]$. The number of nontrivial symmetries is unchanged. They are

$$\begin{aligned}
&\Lambda_1 = \partial_t + ku\exp\left[-\sigma t\right]\partial_u, \qquad \Lambda_2 = 2t\partial_t + x\partial_x + 2t\left[p - k\exp\left[-\sigma t\right]\right]u\partial_u,\\
&\Lambda_3 = 4t^2\partial_t + 4tx\partial_x - \left[2t + x^2 - 4t^2\left(p - k\exp\left[-\sigma t\right]\right)\right]u\partial_u,\\
&\Lambda_4 = \partial_x, \qquad \Lambda_5 = 2t\partial_x - xu\partial_u.
\end{aligned} \tag{24}$$

In the same fashion as above we obtain the two families of solutions

$$\begin{aligned}
&u_0 = \exp\left[pt + \frac{k}{\sigma}\,\mathrm{e}^{-\sigma t}\right], \qquad u_1 = x\exp\left[pt + \frac{k}{\sigma}\,\mathrm{e}^{-\sigma t}\right]\\
&u_2 = \left(2t + x^2\right)\exp\left[pt + \frac{k}{\sigma}\,\mathrm{e}^{-\sigma t}\right], \qquad \ldots,\\
&v_0 = t^{-1/2}\exp\left[pt + \frac{k}{\sigma}\,\mathrm{e}^{-\sigma t} - \frac{x^2}{4t}\right], \qquad v_1 = -\frac{x}{2t^{3/2}}\exp\left[pt + \frac{k}{\sigma}\,\mathrm{e}^{-\sigma t} - \frac{x^2}{4t}\right],\\
&v_2 = t^{-1/2}\left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)\exp\left[pt + \frac{k}{\sigma}\,\mathrm{e}^{-\sigma t} - \frac{x^2}{4t}\right], \qquad \ldots.
\end{aligned}$$

in which we note the first similarity to the solutions listed in (21) and (22) and from which we see that the concentration of the tumor cells increases with time for both series of solutions and eventually decreases with radial distance in the second series. The assumption of a constant kill rate in space is suspect since the inner cells are more crowded and the supply of nutrients is not as great as on the surface of the tumour. From our discussion of the symmetries available for different $K(x,t)$ we can see that there are several possibilities which maintain a rich supply of Lie point symmetries. For example, we take $K(x,t) = px - k$, with both p and k constant for simplicity although they both could be time-dependent. Equation (12) now has the five nontrivial Lie point symmetries

$$\begin{aligned}
&\Delta_1 = \partial_t, \qquad \Delta_2 = 2t\partial_t + \left(x - 3pt^2\right)\partial_x + t\left(3px - 2k - p^2t^2\right)u\partial_u,\\
&\Delta_3 = 4t^2\partial_t + 4t\left(x - pt^2\right)\partial_x - \left(4kt^2 + x^2 + 2t - 6pt^2x\right)u\partial_u,\\
&\Delta_4 = \partial_x + ptu\partial_u, \qquad \Delta_5 = 2t\partial_x + \left(pt^2 - x\right)u\partial_u
\end{aligned} \tag{25}$$

and from Δ_4 and Δ_5 we obtain, in the manner described above, the solutions

$$\begin{aligned}
&u_0 = \exp\left[ptx + \frac{1}{3}p^2t^3 - kt\right], \qquad u_1 = \left(x + pt^2\right)\exp\left[ptx + \frac{1}{3}p^2t^3 - kt\right],\\
&u_2 = \left[2t + \left(x + pt^2\right)^2\right]\exp\left[ptx + \frac{1}{3}p^2t^3 - kt\right], \qquad \ldots,
\end{aligned}$$

$$
\begin{aligned}
v_0 &= t^{-1/2}\exp\left[\frac{1}{2}ptx - kt - \frac{1}{12}p^2t^3 - \frac{x^2}{4t}\right],\\
v_1 &= -\frac{1}{2}t^{-1/2}\left(pt + \frac{x}{t}\right)\exp\left[\frac{1}{2}ptx - kt - \frac{1}{12}p^2t^3 - \frac{x^2}{4t}\right],\\
v_2 &= \frac{1}{4}t^{-1/2}\left[\left(pt + \frac{x}{t}\right)^2 - \frac{2}{t}\right]\exp\left[\frac{1}{2}ptx - kt - \frac{1}{12}p^2t^3 - \frac{x^2}{4t}\right], \qquad \ldots
\end{aligned} \tag{26}
$$

from which we can infer the first set of solutions increases with no bound for both t and x and the second series of solutions tends to zero for large t and x.

5 Conclusion

We conclude that the therapy-dependent killing rate K need not be a function of time only but of both position and time. This is an improvement to the model of Wein & Koplow [15]. The symmetry analysis of the model yields some interesting properties, i.e., the symmetries are ladder operators and can be used to map solutions into solutions of the model. Several aspects of the problem remain unanswered from a mathematical point of view and, as additional therapies are developed, definitions of parameters in the model will be helpful in planning the dose and timing of these therapies which are currently limited by the tolerance of normal brain for XRay-irradiation and hematopeotic tissues for chemotherapies. In addition to this one may ask how much the diffusion constant D varies from one glioma to another and how the cell concentration or cell type contributes independently to D. In his book on Mathematical Biology Murray [10] remarks that the prediction of tumour response to therapy is a goal which mathematical models could help to reach.

In the treatment of HIV-AIDS the use of recombinant viruses has been proposed [14] and numerical simulations suggest that such viruses are very effective in reducing the viral load of the HIV and reasonably effective in restoring the number of healthy cells. Effectively the recombinant virus controls the evolution of the disease although it does not provide a cure. In the case of a tumour such as glioblastoma multiforme the development of a viral therapy could remedy the problem of exponential decay of the therapeutic agent. Now there are two coupled equations, one for the growth of the number of cancer cells and one for the growth of the number of viruses. A model for such a system is a generalisation of the Burgess equation (1) such as

$$
\begin{aligned}
\frac{\partial n_1}{\partial t} &= D_1\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n_1}{\partial r}\right) + (p_1 - k_2n_2)\,n_1,\\
\frac{\partial n_2}{\partial t} &= D_2\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n_2}{\partial r}\right) + (p_2 - k_1n_1)\,n_2
\end{aligned} \tag{27}
$$

in which n_1 and n_2 represent the densities of the tumour cells and viruses, p_1 and p_2 their proliferation rates and k_1 and k_2 the killing rates for cells/virus. Such models are to be the subject of further investigation.

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Differential Invariants of Transformation Groups on the Real Plane

Maryna O. NESTERENKO

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *maryna@imath.kiev.ua*

Using results of A. Gonzalez-Lopez, N. Kamran and P.J. Olver we describe differential invariants and operators of invariant differentiation of transformation groups acting on the real plane.

1 Introduction

Differential invariants emerged as one of the most important tools in investigation of differential equations in the works by S. Lie. In 1884 [3] he proved that any non-singular invariant system of differential equations can be expressed in terms of differential invariants of the corresponding symmetry group. In this paper he also applied differential invariants to integration of ODEs. Differential invariants for all finite-dimensional local transformation groups on a space of two complex variables was described also by S. Lie [4]. A modern treatment of these results was adduced in [6], namely for all non-equivalent realizations of Lie algebras of transformation groups on the complex plane functional bases of differential invariants, operators of invariant differentiation and Lie determinants were constructed. Basic concepts of the theory of differential invariants and review of main results may be found in [7, 8].

In this paper for every realization of Lie algebras of transformation groups acting on the real plane from [2] we construct a functional basis of differential invariants and operators of invariant differentiation. For this purpose we find transformations over complex field that reduce realizations from [2] to realizations from [6]. Such nontrivial transformations are adduced in Table 1 (numbers N_1 and N_2 correspond to numeration of real and complex realizations in [2,6]). Using these transformations and results of [6] we completely describe differential invariants of transformation groups on the real plane. This description is presented in the form of Table 2.

Table 1. Transformations of compleficated real realizations to complex ones.

N_1	Transformations of space variables	Transformations of basis elements	N_2
1	$\tilde{x}=x-iy$, $\tilde{y}=x+iy$	$\tilde{e}_1=\frac{1+i}{2}(e_1+e_2)$, $\tilde{e}_2=\frac{1}{c+i}e_3$, $\tilde{e}_3=\frac{1-i}{2}(e_1-e_2)$	2.7, $k=1$
2	$\tilde{x}=x-iy$, $\tilde{y}=\frac{1}{2iy}$	$\tilde{e}_1=e_1$, $\tilde{e}_2=e_2$, $\tilde{e}_3=e_3$	2.2
3	$\tilde{x}=-\frac{1}{ix+y}$, $\tilde{y}=\frac{ix+y}{1+x^2+y^2}$	$\tilde{e}_1=\frac{1}{2}(ie_2+e_3)$, $\tilde{e}_2=ie_1$, $\tilde{e}_3=\frac{1}{2}(e_3-ie_2)$	2.2
4	$\tilde{x}=\frac{y-ix}{2}$, $\tilde{y}=-\frac{y+ix}{2}$	$\tilde{e}_1=ie_1-e_2$, $\tilde{e}_2=ie_1+e_2$, $\tilde{e}_3=\frac{e_3+ie_4}{2}$, $\tilde{e}_4=\frac{e_3-ie_4}{2}$	2.9, $k=1$
7	$\tilde{x}=y+ix$, $\tilde{y}=y-ix$	$\tilde{e}_1=\frac{e_1+ie_2}{2i}$, $\tilde{e}_2=\frac{e_3-ie_4}{2}$, $\tilde{e}_3=\frac{e_6+ie_5}{2}$, $\tilde{e}_4=\frac{ie_2-e_1}{2i}$, $\tilde{e}_5=\frac{e_3+ie_4}{2}$, $\tilde{e}_6=\frac{e_6-ie_5}{2}$	2.4
17	$\tilde{x}=y$, $\tilde{y}=\frac{1}{x-y}$	$\tilde{e}_1=e_1$, $\tilde{e}_2=e_2$, $\tilde{e}_3=e_3$	2.2
18	$\tilde{x}=x$, $\tilde{y}=\frac{1}{y^2}$	$\tilde{e}_1=e_1$, $\tilde{e}_2=\frac{1}{2}e_2$, $\tilde{e}_3=e_3$	2.1
19	$\tilde{x}=x$, $\tilde{y}=\frac{1}{y}$	$\tilde{e}_1=e_1$, $\tilde{e}_2=e_2$, $\tilde{e}_3=-e_3$, $\tilde{e}_4=e_4$	2.3

Table 2. Differential invariants of transformation groups on the real plane.

N	Realization	Fundamental differential invariants	Operator of invariant differentiation
1	$\partial_x,\ \partial_y,\ (cx+y)\partial_x+(cy-x)\partial_y,\ 0\le c$	$\frac{y''e^{-c\arctan y'}}{P^{3/2}}$	$\frac{e^{-c\arctan y'}}{\sqrt{P}}D_x$
2	$\partial_x,\ x\partial_x+y\partial_y,\ (x^2-y^2)\partial_x+2yx\partial_y$	$\frac{y''y+(y')^2+1}{P^{3/2}}$	$\frac{2y}{\sqrt{P}}D_x$
3	$y\partial_x-x\partial_y,\ (1+x^2-y^2)\partial_x+2xy\partial_y,$ $2xy\partial_x+(1+y^2-x^2)\partial_y$	$\frac{y''(1+x^2+y^2)}{P^{3/2}}+\frac{2(y-xy')}{\sqrt{P}}$	$\frac{1+x^2+y^2}{\sqrt{P}}D_x$
4	$\partial_x,\ \partial_y,\ x\partial_x+y\partial_y,\ y\partial_x-x\partial_y$	$\frac{y'''P}{(y'')^2}-3y'$	$\frac{P}{y''}D_x$
5	$\partial_x,\ \partial_y,\ x\partial_x-y\partial_y,\ y\partial_x,\ x\partial_y$	$\frac{3y''y^{\rm IV}-5(y''')^2}{(y'')^{8/3}}$	$\frac{1}{(y'')^{1/3}}D_x$
6	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ y\partial_x,\ x\partial_y$	$\frac{S_5}{R_4^{3/2}}$	$\frac{y''}{\sqrt{R_4}}D_x$
7	$\partial_x,\ \partial_y,\ x\partial_x+y\partial_y,\ y\partial_x-x\partial_y,$ $(x^2-y^2)\partial_x-2xy\partial_y,\ 2xy\partial_x-(y^2-x^2)\partial_y$	$\frac{U}{Q^3}$	$\frac{P}{\sqrt{Q}}D_x$
8	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ y\partial_x,\ x\partial_y,$ $x^2\partial_x+xy\partial_y,\ xy\partial_x+y^2\partial_y$	$\frac{V_7}{S_5^{8/3}}$	$\frac{y''}{S_5^{1/3}}D_x$
9	∂_x	$F(y)$	D_x
10	$\partial_x,\ x\partial_x$	$F(y),\ \frac{(y')^k}{y^{(k)}}$	D_x^2
11	$\partial_y,\ y\partial_y,\ y^2\partial_y$	$\frac{Q_3}{(y')^2}$	D_x
12	$\partial_x,\ \partial_y,\ x\partial_x+cy\partial_y,\ 0<\lvert c\rvert\le 1$	$c=1,\quad \frac{y'''}{(y'')^2}$ $c\ne 1,\quad y''y'^{\frac{2-c}{c-1}}$	$\frac{1}{y''}D_x$ $(y')^{\frac{1}{c-1}}D_x$
13	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y$	$\frac{y'y'''}{(y'')^2}$	$\frac{y'}{y''}D_x$
14	$\partial_x,\ \partial_y,\ x\partial_x,\ x^2\partial_x$	$\frac{Q_3}{(y')^4}$	$\frac{1}{y'}D_x$
15	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ x^2\partial_x$	$\frac{S_4}{Q_3^{\frac{3}{2}}}$	$\frac{y'}{\sqrt{Q_3}}D_x$
16	$\partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,\ x^2\partial_x,\ y^2\partial_y$	$\frac{U_5}{Q_3^3}$	$\frac{y'}{\sqrt{Q_3}}$
17	$\partial_x+\partial_y,\ x\partial_x+y\partial_y,\ x^2\partial_x+y^2\partial_y$	$\frac{2y'(1+y')+y''(x-y)}{(y')^{3/2}}$	$\frac{x-y}{\sqrt{y'}}D_x$
18	$\partial_x,\ 2x\partial_x+y\partial_y,\ x^2\partial_x+xy\partial_y$	y^3y''	y^2D_x
19	$\partial_x,\ x\partial_x,\ y\partial_y,\ x^2\partial_x+xy\partial_y$	$\frac{3y'y''+yy'''}{\sqrt{y}(y'')^{3/2}}$	$\sqrt{\frac{y}{y''}}D_x$
20	$\partial_y,\ \xi_1(x)\partial_y,\ \dots,\ \xi_r(x)\partial_y,\ 1\le r$	$x,\ \frac{W(y,1,\xi_1,\dots,\xi_r)}{W(1,\xi_1,\dots,\xi_r)}$	D_x
21	$\partial_y,\ y\partial_y,\ \xi_1(x)\partial_y,\ \dots,\ \xi_r(x)\partial_y,\ 1\le r$	$D_x\ln W(y,\xi_1,\dots,\xi_r)$	D_x
22	$\partial_x,\ \eta_1(x)\partial_y,\ \dots,\ \eta_r(x)\partial_y,\ 1\le r$	$\frac{W(y,\eta_1,\dots,\eta_r)}{W(\eta_1,\dots,\eta_r)}$	D_x
23	$\partial_x,\ y\partial_y,\ \eta_1(x)\partial_y,\ \dots,\ \eta_r(x)\partial_y,\ r\ge 1$	$D_x\ln W(y,\eta_1,\dots,\eta_r)$	D_x
24	$\partial_x,\ \partial_y,\ x\partial_x+cy\partial_y,$ $x\partial_y,\dots,x^r\partial_y,\ r\ge 1$	$c\ne r+1,\ (y^{(r+1)})^{\frac{2-c+r}{c-(r+1)}}y^{(r+2)}$ $c=r+1,\ y^{(r+1)},\ \frac{y^{(r+3)}}{(y^{(r+2)})^2}$	$(y^{(r+1)})^{\frac{1}{c-(r+1)}}D_x$ $\frac{1}{y^{(r+2)}}D_x$
25	$\partial_x,\ \partial_y,\ x\partial_y,\ \dots,\ x^{r-1}\partial_y,$ $x\partial_x+(ry+x^r)\partial_y,\ r\ge 1$	$y^{(r+1)}e^{\frac{y^{(r)}}{r!}}$	$e^{\frac{y^{(r)}}{r!}}D_x$
26	$\partial_x,\ \partial_y,\ x\partial_x,\ x\partial_y,\ y\partial_y,$ $x^2\partial_y,\ \dots,\ x^r\partial_y,\ r\ge 1$	$\frac{y^{(r+1)}y^{(r+3)}}{(y^{(r+2)})^2}$	$\frac{y^{(r+1)}}{y^{(r+2)}}D_x$
27	$\partial_x,\ \partial_y,\ 2x\partial_x+ry\partial_y,\ x\partial_y,$ $x^2\partial_x+rxy\partial_y,\ x^2\partial_y,\ \dots,\ x^r\partial_y,\ r\ge 1$	$\frac{Q_{r+3}}{(y^{(r+1)})^{\frac{2(r+4)}{r+2}}}$	$\frac{1}{(y^{(r+1)})^{\frac{2}{r+2}}}D_x$
28	$\partial_x,\ \partial_y,\ x\partial_x,\ x\partial_y,\ y\partial_y,$ $x^2\partial_x+rxy\partial_y,\ x^2\partial_y,\ \dots,\ x^r\partial_y,\ r\ge 1$	$\frac{S_{r+4}}{Q_{r+3}^{3/2}}$	$\frac{y^{(r+1)}}{\sqrt{Q_{r+3}}}D_x$

In Table 2 $W(f_1,\dots,f_k)$ denotes the Wronskian of the functions $f_1(x)$, $\dots$, $f_k(x)$, D_x is the operator of total differentiation. We also use the following notations:

$$S_{k+3}=(k+1)^2\big(y^{(k)}\big)^2y^{(k+3)}-3(k+1)(k+3)y^{(k)}y^{(k+1)}y^{(k+2)}+2(k+2)(k+3)\big(y^{(k+1)}\big)^3,$$
$$Q_{k+2}=(k+1)y^{(k)}y^{(k+2)}-(k+2)\big(y^{(k+1)}\big)^2,\qquad R_4=3y''y^{\mathrm{IV}}-5(y''')^2,$$
$$P=1+(y')^2,\qquad Q=y'''P-3y'(y'')^2,$$
$$U=4y^{\mathrm{V}}P^3Q+10y^{\mathrm{IV}}y''P^3\left(4y'''y'+3(y'')^2\right)-5(y^{\mathrm{IV}})^2P^4+40(y''')^2(y'')^2\left((y')^2-2\right)P^2$$
$$-40(y''')^3y'P^3-180y'''y'(y'')^4\left((y')^2-1\right)P^2-(y'')^6\left(45(6(y')^2+1)-135(y')^4\right),$$
$$U_5=(y')^2\left(Q_3D_x^2Q_3-\frac{5}{4}(D_xQ_3)^2\right)+y'y''Q_3D_xQ_3-\left(2y'y'''-(y'')^2\right)Q_3^2,$$
$$V_7=(y'')^2\left(S_5D_x^2S_5-\frac{7}{6}(D_xS_5)^2\right)+y''y'''S_5D_xS_5-\frac{1}{2}\left(9y''y^{\mathrm{IV}}-7(y''')^2\right)S_5^2.$$

The presented results may be used for group classification of ODEs of any finite order. So, in the near future we plan to review and to generalize results of group classification of third and fourth ODEs that were obtained in [1, 10]. To investigate differential invariants of transformations groups acting in the spaces of more than two variables in similar way we can use classification of realizations of real low-dimensional Lie algebras that was obtained in [9].

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On Symmetry Group Properties of the Benney Equations

Teoman ÖZER

Istanbul Technical University, Faculty of Civil Engineering, Division of Mechanics, 34469 Maslak, Istanbul, Turkey
E-mail: *tozer@itu.edu.tr*

In the present study, we investigate the symmetry groups of Benney equations that are the system of nonlinear integro-differential equations. We first investigate the symmetry groups of the Benney equations by using the method. Then we obtain all reduced forms of the system of integro-differential equations with fewer variables based on symmetry groups; and lastly, we seek a similarity solution to the reduced system of the equations.

1 Introduction

Symmetry group analysis deals with applications of continuous symmetry groups to the system of differential equations in engineering, mathematics, and physics. However, in the case of integro-differential equations (IDE), there is no general method of investigating the Lie symmetry groups of these equations based on the solution of their determining equations. The main difficulty in applying Lie's infinitesimal techniques to these systems is their nonlocality, and the approach used in the classical group theory cannot be applied for the investigation of symmetry groups of IDE.

There are some studies about symmetry groups of integro-differential equations in the literature. Meleshko [1] searches the Lie point symmetries of the one-dimensional visco-elastic equation and gets a classification with respect to the free term and the kernel function. The method is based on the fact that one considers the determining equations on any solution at any point and any time, for example, at initial time t_0. Here, it is very important to have an existence of solution of the Cauchy problem, which allows splitting the determining equations. It is worth to note that the same problem stays in an application of the group analysis to differential equations. For differential equations the Cauchy–Kovalewskaya theorem treats this. One-dimensional nonlocal elasticity and one dimensional visco-elasticity equations seem a similar structure due the type of their integral equations, it is clear that these equations have different characteristics with respect to the mathematical and kinematics aspects. First, one-dimensional nonlocal elasticity is in the form of Fredholm integro-differential equations by Özer [2] and one dimensional visco-elasticity equation is in the form of Volterra integro-differential equation [1]. In addition, these two equations describe two different mechanics behaviors of the solids. It is important to define symmetry group properties of nonlocal elasticity equations in order to understand these different characteristics of the equations based on their symmetry groups. In addition, Özer [3] investigated symmetry group properties of two-dimensional elastodynamics problem of nonlocal continuum mechanics and obtained a classification due to the kernel function and the free term.

Bobylev [4] is another author who studied the symmetry groups of the integro-differential equations. He obtained all symmetry groups of the Boltzmann and examined some invariant solutions of the Boltzmann equation using its Lie point symmetries. This approach in this study is based on the assumption that we restrict the determining equations to the subset of the solution of integro-differential equation determined by the initial conditions. Firstly, the coefficients of the infinitesimal operator are assumed to be locally analytic functions, and then

these coefficients are represented by Taylor series with respect to the independent variable of the equation. The determining equation is decomposed with respect to the different powers of an arbitrary parameter that exists in the initial functions. Finally the determining equations are split to the series of equations by equating the coefficient of every different power of the independent parameter to zero. Symmetry groups are found by the so-obtained equations.

Chetverikov and Kudryavtsev [5] have written an important article on the subject. The method in their study consists in reducing an integro-differential equation to a system of boundary differential equation and in computing symmetries and conservation laws for the system of integro-differential equations. In the study symmetry is considered as a geometric concept. Therefore, in order to define the symmetry of something it is first represented by some geometric model. Then the symmetry is found by considering the transformations of the model. In the case of integro-differential equations, the analogy with differential equations can also be used. In the method the first step is the introduction of the nonlocal variables. In the second step the initial system is transformed to the functional differential equations. Later the generalized jet spaces are defined so that boundary differential equations can be expanded as submanifolds of these spaces. Finally, a geometric theory of boundary differential equations is constructed similar to the geometry of nonlinear differential equations. This method is fundamentally different from our method that we use in the study.

The last work to be mentioned here is done by Taranov [6]. The first step in this approach is to find the symmetries of the equation dependent on finite number of variables. The symmetries of the system of integro-differential equations are obtained by approaching the number of variables to infinity.

2 Symmetry groups of the Benney equations

In this part, we present briefly the general characteristics of the method of investigating the symmetry groups of the Benney equations introduced in the study [7]. The method presents an important opportunity for investigation of symmetry groups and similarity solutions, as well as solutions of some boundary value problems related to other problems, including IDE in engineering and science. We consider a scalar kth-order IDE represented by

$$\prod(x,u,\underset{1}{u},\ldots,\underset{k}{u})+\int dx\hat{\prod}(x,u,\underset{1}{u},\ldots,\underset{k}{u})=0. \tag{1}$$

Then, the infinitesimal transformation which is the invariance criterion of invariance of an integro-differential equation introduced in [8, 9] is written as

$$\underset{k}{X}\prod(x,u,\underset{1}{u},\ldots,\underset{k}{u})+\int dx\left[\underset{k}{X}\hat{\prod}(x,u,\underset{1}{u},\ldots,\underset{k}{u})+\hat{\prod}\sum_i\partial_i\xi_i\right]=0. \tag{2}$$

Roberts [8] and Zawistowski [9] studied the symmetry group properties of the one-dimensional Vlasov–Maxwell equation based on the infinitesimal criterion (2) of integro-differential equations.

Here, we investigate the symmetry groups of the system of integro-differential equations corresponding to the Benney kinetic equations below:

$$\frac{\partial f(x,t,v)}{\partial t}+v\frac{\partial f(x,t,v)}{\partial x}-\frac{\partial f(x,t,v)}{\partial v}\frac{\partial A_0(x,t)}{\partial x}=0,\qquad A_0(x,t)=\int_{-\infty}^{\infty}f(x,v,t)dv, \tag{3}$$

where x and t are the independent variables as spatial coordinate and time, respectively, f is the distribution function, v is the horizontal component of the flow velocity, and A_0 is the function defined as

$$\frac{\partial A_n}{\partial t}+\frac{\partial A_{n+1}}{\partial x}+nA_{n-1}\frac{\partial A_0}{\partial x}=0, \tag{4}$$

where

$$A_n = \int_0^h u^n(x,y,t)dy, \qquad n = 0,1,2,\ldots. \tag{5}$$

For this purpose, we can define the infinitesimal operator for the problem (3) with respect to the definition of the infinitesimal operator for a partial differential equation as

$$X = \xi^x\partial_x + \xi^t\partial_t + \xi^v\partial_v + \eta^f\partial_f + \eta^{A_0}\partial_{A_0}. \tag{6}$$

Here x, v, t are independent variables and f, A_0 are dependent variables of the problem. After the calculations, we obtain a five-parameter Lie group for the Benney equations based on the condition (2) as shown below:

$$\begin{aligned} &\eta^{A_0} = 2a_1A_0, \qquad \eta^f = a_1f, \qquad \xi^v = a_1v + a_3,\\ &\xi^x = (2a_1 - a_4)x + a_3t + a_5, \qquad \xi^t = (a_1 - a_4)t + a_2, \end{aligned} \tag{7}$$

that are equivalent results obtained by Krasnoslobodtsev [10] and Ibragimov [11], where a_1, a_2, a_3, a_4 and a_5 are constants.

3 Reduced forms of the Benney equations

In this part, we present all reduced forms of the Benney equations by using the corresponding symmetry groups. First, we construct the characteristic equations with respect to the expressions (7) as below:

$$\begin{aligned} &\frac{dx}{(2a_1 - a_4)x + a_3t + a_5} = \frac{dt}{(a_1 - a_4)t + a_2} = \frac{dv}{a_1v + a_3} = \frac{df}{a_1f},\\ &\frac{dx}{(2a_1 - a_4)x + a_3t + a_5} = \frac{dt}{(a_1 - a_4)t + a_2} = \frac{dv}{a_1v + a_3} = \frac{dA_0}{2a_1A_0}. \end{aligned} \tag{8}$$

We obtain the general similarity variables and reduced forms of the Benney equations by using the "integrating factor" technique. Then if we substitute the similarity forms to the original equation (3) by using new similarity variables, then we get the new system of IDE with two new similarity variables, namely ξ_1 and ξ_2, as independent variables; and $\tilde{f}$ and $\tilde{A}_0$ as dependent variables.

If we take the conditions as $a_1 \neq 0$, $a_4 \neq 2a_1$, $a_4 \neq a_1$, then we obtain the general reduced form of equation (3) as

$$\begin{aligned} &(-2a_1 + a_4)\xi_1\frac{\partial\tilde{f}(\xi_1,\xi_2)}{\partial\xi_1} - a_1\tilde{f}(\xi_1,\xi_2) + a_1\xi_2\frac{\partial\tilde{f}(\xi_1,\xi_2)}{\partial\xi_2}\\ &\quad + \xi_2\frac{\partial\tilde{f}(\xi_1,\xi_2)}{\partial\xi_1} - \frac{d\tilde{A}_0(\xi_1)}{d\xi_1}\frac{\partial\tilde{f}(\xi_1,\xi_2)}{\partial\xi_2} = 0, \qquad \tilde{A}_0 = \int_{-\infty}^{\infty}\tilde{f}(\xi_1,\xi_2)d\xi_2, \end{aligned} \tag{9}$$

where the first and the second similarity variables and similarity forms are as follows:

$$\begin{aligned} \xi_1 &= x[(a_1 - a_4)t + a_2]^{(-2a_1+a_4)(a_1-a_4)^{-1}}\\ &\quad - \frac{a_3}{a_1}\left[-\frac{a_1a_5 + a_2a_3}{(a_3(2a_1 - a_4)} - t\right][(a_1 - a_4)t + a_2]^{(-2a_1+a_4)(a_1-a_4)^{-1}},\\ \xi_2 &= v[(a_1 - a_4)t + a_2]^{-a_1(a_1-a_4)^{-1}} + \frac{a_3}{a_1}[(a_1 - a_4)t + a_2](-a_1)(a_1 - a_4)^{-1},\\ f(x,v,t) &= [(a_1 - a_4)t + a_2]^{a_1(a_1-a_4)^{-1}}\tilde{f}(\xi_1,\xi_2),\\ A_0(x,t) &= [(a_1 - a_4)t + a_2]^{2a_1(a_1-a_4)^{-1}}\tilde{A}_0(\xi_1). \end{aligned} \tag{10}$$

4 The relations between Benney equations and shallow-water equations and similarity solutions

It may possible to transform the solutions of Benney equations into solutions of nonlinear-shallow water equations

$$\frac{\partial h(x,t)}{\partial t} + u(x,t)\frac{\partial h(x,t)}{\partial x} + h(x,t)\frac{\partial u(x,t)}{\partial x} = 0,$$
$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial h(x,t)}{\partial x} = 0. \tag{11}$$

by using the following expression given by Gibbons [12]:

$$f(x,t,v) = h(x,t)\delta(u(x,t) - v), \tag{12}$$

where $h(x,t)$ and $u(x,t)$ are solutions of the nonlinear shallow-water equations (11) and δ is Dirac's delta function. In addition to this relation, expression (12) gives us the relationships between Benney equations and both shallow-water equations. In the case of reduced Benney equations, we can use the following relation:

$$\tilde{f}(\xi_1,\xi_2) = \tilde{h}(\xi_1)\delta(\tilde{u}(\xi_1) - \xi_2) \tag{13}$$

which have similar forms with relations (12). Then one may obtain a system of ordinary differential equations (ODE) by using the relations (12). In this section, we give an example of the application of symmetry groups.

For this purpose, we seek the solution for the reduced form of the Benney equations below:

$$(\xi_1 + \xi_2)\frac{\partial \tilde{f}(\xi_1,\xi_2)}{\partial \xi_1} - \frac{\partial \tilde{f}(\xi_1,\xi_2)}{\partial \xi_2}\int_{-\infty}^{\infty}\frac{\partial \tilde{f}(\xi_1,\xi_2)}{\partial \xi_1}d\xi_2 = 0, \tag{14}$$

which is a special form of the reduced form. Using the form (13) we get the following system of ODE:

$$\xi_1\tilde{h}'(\xi_1) + \tilde{h}'(\xi_1)\tilde{u}(\xi_1) + \tilde{h}(\xi_1)\tilde{u}'(\xi_1) = 0 \quad \text{and} \quad \tilde{u}(\xi_1)\tilde{u}'(\xi_1) + \xi_1\tilde{u}'(\xi_1) + \tilde{h}'(\xi_1) = 0 \tag{15}$$

from the solutions to the system of ODE in the similarity form,

$$\tilde{h}(\xi_1) = \frac{1}{9}\xi_1^2, \qquad \tilde{u}(\xi_1) = -\frac{2}{3}\xi_1. \tag{16}$$

These solutions correspond to the similarity solution of the classical case (11). One can write the equation (3) due to the solutions (16) in the variables x, t, and v as

$$f(x,t,v) = \frac{1}{9}(1-t)^{-2}(x-1)^2\delta\left(v + \frac{2}{3}(t-1)^{-1}(x-1)\right). \tag{17}$$

5 Conclusions

In this study, we obtained symmetry groups of Benney equations that are in the form of the system of IDE. As mentioned in the introduction, although the symmetry group analysis has many applications for the ODE and PDE in the literature, there are relatively few applications on symmetry group of IDE since there is no general method for solving determining equations. Several solution methods may be offered for investigating the determining equations for symmetry groups of problems including IDE. In the case of Benney equations investigated in the study by Krasnoslobodtsev [10] and Ibragimov et al. [11], symmetry groups of these equations

were investigated based on the canonical multiplication law and the methods of moments, respectively. In our study, we calculated the symmetry groups of Benney equations by using a different method, explained in the study. After calculations, it was found a five-parameter Lie group of transformations of Benney equations. These results are parallel to the results presented in Ibragimov's study based on using the canonical Lie–Bäcklund operators. These results are parallel to the results presented in Ibragimov's study based on using the canonical Lie–Bäcklund operators. We obtained the general reduced forms of Benney equations by using the characteristics based on symmetry groups. Then we obtained the general reduced forms of Benney equations by using the characteristics based on symmetry groups. One of the important advantages of investigating solutions of Benney equations is based on the fact that one may transform these solutions into solutions of the shallow-water equations and Benney equations derived form the two-dimensional Euler equations for inviscid, incompressible fluid, and from the reduced form of the equation. In the study, we showed a similarity solution for a reduced equation, and proved that each solution obtained from solutions of each reduced form of the systems of IDE can be used to obtain similarity solutions of Benney equations.

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Lie Symmetries of (1+1)-Dimensional Cubic Schrödinger Equation with Potential

Roman O. POPOVYCH †, *Nataliya M. IVANOVA* † *and Homayoon ESHRAGHI* ‡

† *Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *rop@imath.kiev.ua, ivanova@imath.kiev.ua*

‡ *Institute for Studies in Theor. Physics and Mathematics, P.O.Box 19395-5531, Tehran, Iran*
E-mail: *eshraghi@theory.ipm.ac.ir*

We perform the complete group classification in the class of cubic Schrödinger equations of the form $i\psi_t + \psi_{xx} + \psi^2\psi^* + V(t,x)\psi = 0$, where V is an arbitrary complex-valued potential depending on t and x. We construct all possible inequivalent potentials for which these equations have non-trivial Lie symmetries using algebraic and compatibility methods simultaneously. Our classification essentially amends earlier works on the subject.

Nonlinear Schrödinger equations (NSchEs) have a number of applications in wave propagation in inhomogeneous media. They arise as a model of plasma phenomena, namely, of different processes in nonlinear and non-uniform dielectric medium and in other branches of physics. Schrödinger equations have been investigated by means of symmetry methods by a number of authors, see e.g. [2–10] and references there. In fact, group classification for Schrödinger equations was first performed by S. Lie. More precisely, his classification [1] of all the linear equations with two independent complex variables contains, in an implicit form, solution of the classification problem for the linear (1+1)-dimensional Schrödinger equations with arbitrary potentials.

In this paper we study a class of NSchEs of the form

$$i\psi_t + \psi_{xx} + \psi^2\psi^* + V\psi = 0, \tag{1}$$

where the potential $V = V(t,x)$ is an arbitrary complex-valued smooth function of the variables t and x. (Here and below subscripts of functions denote differentiation with respect to the corresponding variables.) To find a complete set of inequivalent cases of V admitting extensions of the maximal Lie invariance algebra, we combine the classical Lie approach, studying the algebra generated by all the possible Lie symmetry operators for equations from class (1) (the adjoint representation, the inequivalent one-dimensional subalgebras etc.) and investigation of compatibility of classifying equations. See [2, 3, 11–13] for precise formulation of group classification problems and more details on the used methods.

Finishing excellent series of papers [8–10] on group analysis and exact solutions of NSchEs, in [10] L. Gagnon and P. Winternitz investigated essentially more general class of variable coefficient NSchEs than (1). Unfortunately, we were not able to see a direct and simple way for deducing classifications obtained here from their results.

Theorem 1. *Any operator* $Q = \xi^t\partial_t + \xi^x\partial_x + \eta\partial_\psi + \eta^*\partial_{\psi^*}$ *from the maximal Lie invariance algebra* $A^{\max}(V)$ *of equation* (1) *with arbitrary potential* V *lies in the linear span of operators of the form*

$$D(\xi) = \xi\partial_t + \tfrac{1}{2}\xi_t x\partial_x + \tfrac{1}{8}\xi_{tt}x^2M - \tfrac{1}{2}\xi_t I, \qquad G(\chi) = \chi\partial_x + \tfrac{1}{2}\chi_t xM, \qquad \lambda M. \tag{2}$$

Here $\chi = \chi(t)$, $\xi = \xi(t)$ *and* $\lambda = \lambda(t)$ *are arbitrary smooth functions of* t, $M = i(\psi\partial_\psi - \psi^*\partial_{\psi^*})$, $I = \psi\partial_\psi + \psi^*\partial_{\psi^*}$. *Moreover, the coefficients of* Q *should satisfy the classifying condition*

$$i\eta_{\psi t} + \eta_{\psi xx} + \xi^t V_t + \xi^x V_x + \xi^t_t V = 0. \tag{3}$$

Note 1. The linear span of operators of the form (2) is an (infinite-dimensional) Lie algebra $A^{\cup}$ under the usual Lie bracket of vector fields. Since for any $Q\in A^{\cup}$ where $(\xi^t,\xi^x)\neq(0,0)$ we can find V satisfying condition (3) then $A^{\cup}=\langle\bigcup_V A^{\max}(V)\rangle$. The non-zero commutation relations between the basis elements of $A^{\cup}$ are the following ones:

$$\big[D\big(\xi^1\big),D\big(\xi^2\big)\big]=D\big(\xi^1\xi^2_t-\xi^2\xi^1_t\big),\qquad [D(\xi),G(\chi)]=G\big(\xi\chi_t-\tfrac12\,\xi_t\chi\big),$$
$$[D(\xi),\lambda M]=\xi\lambda_t M,\qquad \big[G\big(\chi^1\big),G\big(\chi^1\big)\big]=\tfrac12\big(\chi^1\chi^2_t-\chi^2\chi^1_t\big)M.$$

We use the notation $\mathrm{Aut}(A^{\cup})$ for the automorphism group acting on $A^{\cup}$, which is generated by all the one-parameter groups corresponding to the adjoint representations of operators of $A^{\cup}$ into $A^{\cup}$ and two discrete transformations $\mathrm{Ad}\,I_x$ and $\mathrm{Ad}\,I_t$ included additionally. The actions of $\mathrm{Ad}\,I_x$ and $\mathrm{Ad}\,I_t$ on the basis elements of $A^{\cup}$ are defined by the formulas $\mathrm{Ad}\,I_x\ G(\chi)=G(-\chi)$ (the other basis operators do not change) and $\mathrm{Ad}\,I_t\ D(\xi)=D(\tilde\xi)$, $\mathrm{Ad}\,I_t\ G(\chi)=G(\tilde\chi)$, $\mathrm{Ad}\,I_t\ \lambda M=\tilde\lambda M$, where $\tilde\xi(t)=-\xi(-t)$, $\tilde\chi(t)=\chi(-t)$ and $\tilde\lambda(t)=-\lambda(-t)$.

Theorem 2. *The Lie algebra of the kernel of maximal Lie invariance groups of equations from class* (1) *is* $A^{\ker}=\langle M\rangle$.

Theorem 3. *The Lie algebra* A^{equiv} *of the equivalence group* G^{equiv} *of the class* (1) *is generated by the operators*

$$D'(\xi)=D(\xi)+\tfrac18\xi_{ttt}x^2(\partial_V+\partial_{V^*})+\tfrac{i}{2}\xi_{tt}(\partial_V-\partial_{V^*})-\xi_t(V\partial_V+V^*\partial_{V^*}),$$
$$G'(\chi)=G(\chi)+\tfrac12\chi_{tt}x(\partial_V+\partial_{V^*}),\qquad M'(\lambda)=\lambda M+\lambda_t(\partial_V+\partial_{V^*}).$$

Therefore, $A^{\mathrm{equiv}}\simeq A^{\cup}$, *and the isomorphism is determined by means of prolongation of operators from* $A^{\cup}$ *to the space* (V,V^*).

Theorem 4. *The equivalence group* G^{equiv} *of the class* (1) *is generated by the family of continuous transformations*

$$\begin{aligned}&\tilde t=T,\quad \tilde x=x\varepsilon\sqrt{T_t}+X,\quad \tilde\psi=\psi\frac{1}{\sqrt{T_t}}\exp\left(\frac{i}{8}\frac{T_{tt}}{T_t}\,x^2+\frac{i}{2}\frac{X_t}{\sqrt{T_t}}\,x+i\Psi\right),\\ &\tilde V=\frac{1}{T_t}\left(V+\frac18\left(\frac{T_{tt}}{T_t}\right)_t x^2+\frac12\left(\frac{X_t}{\sqrt{T_t}}\right)_t x+\frac{i}{4}\frac{T_{tt}}{T_t}-\left(\frac14\frac{T_{tt}}{T_t}\,x+\frac12\frac{X_t}{\sqrt{T_t}}\right)^2+\Psi_t\right),\end{aligned}\tag{4}$$

and two discrete transformations: the space reflection I_x *(*$\tilde t=t$, $\tilde x=-x$, $\tilde\psi=\psi$, $\tilde V=V$*) and the Wigner time reflection* I_t *(*$\tilde t=-t$, $\tilde x=x$, $\tilde\psi=\psi^*$, $\tilde V=V^*$*). Here* T, X *and* Ψ *are arbitrary smooth functions of* t, $T_t>0$.

Corollary 1. 1. $G^{\mathrm{equiv}}\simeq\mathrm{Aut}\,A^{\cup}$. 2. *Let* A^1 *and* A^2 *be the maximal Lie invariance algebras of equations from class* (1) *for some potentials, and* $\mathcal{V}^i=\{V\,|\,A^{\max}(V)=A^i\}$, $i=1,2$. *Then* $\mathcal{V}^1\sim\mathcal{V}^2 \bmod G^{\mathrm{equiv}}$ *iff* $A^1\sim A^2 \bmod \mathrm{Aut}\,A^{\cup}$.

Lemma 1. *A complete list of* $\mathrm{Aut}\,A^{\cup}$*-inequivalent one-dimensional subalgebras of* $A^{\cup}$ *is exhausted by the algebras* $\langle\partial_t\rangle$, $\langle\partial_x\rangle$, $\langle tM\rangle$, $\langle M\rangle$.

Proof. Consider any operator $Q\in A^{\cup}$, i.e. $Q=D(\xi)+G(\chi)+\lambda M$. Depending on the values of ξ, χ and λ it is equivalent under $\mathrm{Aut}\,A^{\cup}$ and multiplication by a number to one from the following operators: $D(1)$ if $\xi\neq0$; $G(1)$ if $\xi=0$ and $\chi\neq0$; tM if $\xi=\chi=0$, $\lambda_t\neq0$; M if $\xi=\chi=\lambda_t=0$. ■

Corollary 2. *If* $A^{\max}(V)\neq A^{\ker}$ *then* $V_tV_x=0 \bmod G^{\mathrm{equiv}}$.

Proof. Under the corollary assumption there exists an operator $Q = D(\xi) + G(\chi) + \lambda M \in A^{\max}(V)$ which do not belong to $\langle M\rangle$. Condition (3) implies $(\xi,\chi) \neq (0,0)$. Therefore, in force of Lemma 1 $\langle Q\rangle \sim \langle\partial_t\rangle$ or $\langle\partial_x\rangle$ mod $\operatorname{Aut} A^{\cup}$, i.e. $V_tV_x = 0$ mod G^{equiv}. ■

Theorem 5. *A complete set of inequivalent cases of V admitting extensions of the maximal Lie invariance algebra of equations* (1) *is exhausted by the potentials given in Table 1.*

Table 1. Results of classification. Here $W(t), \nu, \alpha, \beta \in \mathbb{R}$, $(\alpha,\beta) \neq (0,0)$.

N	V	Conditions mod G^{equiv}	Basis of $A^{\max}$
0	$V(t,x)$		M
1	$iW(t)$		$M,\ \partial_x,\ G(t)$
2	$\frac{i}{2}\frac{t+\nu}{t^2+1}$	$\nu \geq 0$	$M,\ \partial_x,\ G(t),\ D(t^2+1)$
3	$i\nu t^{-1},\ \nu \neq 0, \frac{1}{2}$	$\nu \geq \frac{1}{4}$	$M,\ \partial_x,\ G(t),\ D(t)$
4	i		$M,\ \partial_x,\ G(t),\ \partial_t$
5	0		$M,\ \partial_x,\ G(t),\ \partial_t,\ D(t)$
6	$V(x)$		$M,\ \partial_t$
7	$(\alpha+i\beta)x^{-2}$	$\beta \geq 0$	$M,\ \partial_t,\ D(t)$

If we use Corollary 2, then to prove Theorem 5 it is sufficient to study two cases: $V_x = 0$ and $V_t = 0$. In fact, below we obtain the complete results of group classifications for both special cases and then unite them for the general case under consideration.

Lemma 2. *Let $V_x = 0$, i.e. $V = V(t)$.*

1. $A^{\ker}_{V_x=0} = \langle M, G(1), G(t)\rangle$. $A^{\text{equiv}}_{V_x=0} = \langle M'(\lambda)\ \forall\lambda = \lambda(t), G'(1), G'(t), D'(1), D'(t), D'(t^2)\rangle$. $G^{\text{equiv}}_{V_x=0}$ *is generated by I_t, I_x and the transformations of form* (4) *where $X = c_1t + c_0$, $T = (a_1t+a_0)/(b_1t+b_0)$, Ψ is an arbitrary smooth function of t. a_i, b_i and c_i are arbitrary constants such that $a_1b_0 - b_1a_0 > 0$.*

2. $V \sim iW$ mod $G^{\text{equiv}}_{V_x=0}$ *where* $W = \operatorname{Im} V$. $A^{\max}(iW) \subset A^{\cup}_{\{iW\}} = A^{\ker}_{V_x=0} \ni \langle D(1), D(t), D(t^2)\rangle$. $A^{\ker}_{\{iW\}} = A^{\ker}_{V_x=0}$. $A^{\cup}_{\{iW\}} = \langle \bigcup_W A^{\max}(iW)\,\rangle$. $A^{\text{equiv}}_{\{iW\}} = \langle M, G'(1), G'(t), D'(1), D'(t), D'(t^2)\rangle$. $G^{\text{equiv}}_{\{iW\}} = G^{\text{equiv}}_{V_x=0}\big|_{\Psi=\text{const}}$. $A^{\cup}_{\{iW\}} \simeq A^{\text{equiv}}_{\{iW\}} = \operatorname{pr}_{(V,V^*)} A^{\cup}_{\{iW\}}$.

3. $S = \langle D(1), D(t), D(t^2)\rangle \simeq sl(2,\mathbb{R})$. *The complete list of* $\operatorname{Aut} A^{\cup}_{\{iW\}}$*-inequivalent proper subalgebras of S is exhausted by* $\langle D(1)\rangle$, $\langle D(t)\rangle$, $\langle D(t^2+1)\rangle$, $\langle D(1), D(t)\rangle$.

4. *Let A^1 and A^2 be the maximal Lie invariance algebras of equations from class (1) for some potentials from $\{iW(t)\}$, and $\mathcal{W}^i = \{\, W(t) \,|\, A^{\max}(iW) = A^i\}$, $i = 1,2$. Then $\mathcal{W}^1 \sim \mathcal{W}^2$* mod $G^{\text{equiv}}_{\{iW\}}$ *iff $A^1 \bigcap S \sim A^2 \bigcap S$* mod $\operatorname{Aut} S$.

5. *If $A^{\max}_{\{iW\}} \neq A^{\ker}_{V_x=0}$ the potential $iW(t)$ is $G^{\text{equiv}}_{\{iW\}}$-equivalent to one from Cases 2–5 of Table 1.*

Note 2. For any W $A^{\max}(iW) \not\supset S$ (otherwise, condition (3) would imply an incompatible system for W). If $W = \text{const}$ $W \in \{0,1\}$ mod $G^{\text{equiv}}_{\{iW\}}$ (Cases 5 and 4 correspondingly). Cases 2_ν and $2_{\tilde\nu}$ (3_ν and $3_{\tilde\nu}$ where $\nu, \tilde\nu \geq \frac{1}{4}$) are G^{equiv}-inequivalent if $\nu \neq \tilde\nu$. Since $D(t^2+1)$ cannot be contained in any two-dimensional subalgebra of S it is not possible to extend $A^{\max}$ in Case 2. There are two possibilities for extension of $A^{\max}(i\nu t^{-1})$, namely with either $D(1)$ (for $\nu = 0$, Case 5) or $D(t^2)$ (for $\nu = \frac{1}{2}$ that is equivalent to Case 5 with respect to $G^{\text{equiv}}_{\{iW\}}$).

Lemma 3. *Let $V_t = 0$, i.e. $V = V(x)$.*

1. *$A^{\ker}_{V_t=0} = \langle M, D(1)\rangle$. $A^{\rm equiv}_{V_t=0} = \langle M'(1), M'(t), G'(1), D'(1), D'(t)\rangle$. $G^{\rm equiv}_{V_t=0}$ consists of I_t, I_x and the transformations of form (4) where $T_{tt} = X_t = \Psi_{tt} = 0$.*
2. *If $A^{\max}(V) \neq A^{\ker}_{V_t=0}$ the potential $V(x)$ is $G^{\rm equiv}_{V_t=0}$-equivalent to one from cases of Table 2.*

Table 2. Results of classification for the subclass $\{V = V(x)\}$. Here $\nu, \alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq (0, 0)$.

N	N_1	V	mod $G^{\rm equiv}$	Basis of $A^{\max}$
0	6	$V(x)$		$M,\ \partial_t$
1	7	$(\alpha+i\beta)x^{-2}$	$\beta \geq 0$	$M,\ \partial_t,\ D(t)$
2	7	$x^2 + i + (\alpha+i\beta)x^{-2}$		$M,\ \partial_t,\ D(e^{4t})$
3	4	i		$M,\ \partial_t,\ \partial_x,\ G(t)$
4	4	$x + i\nu$	$\nu > 0$	$M,\ \partial_t,\ G(1)+tM,\ G(2t)+t^2M$
5	2	$-x^2 + i\nu$	$\nu \geq 0$	$M,\ \partial_t,\ G(\sin 2t),\ G(\cos 2t)$
6	3	$x^2 + i\nu,\ \nu \neq \pm 1$	$\nu \geq 0$	$M,\ \partial_t,\ G(e^{2t}),\ G(e^{-2t})$
7	5	0		$M,\ \partial_t,\ \partial_x,\ G(t),\ D(t)$
8	5	x		$M,\ \partial_t,\ G(1)+tM,\ G(2t)+t^2M,\ D(2t)+G(3t^2)+t^3M$
9	5	$x^2 + i$		$M,\ \partial_t,\ G(e^{2t}),\ G(e^{-2t}),\ D(e^{4t})$

Proof. 2. Let $V = V(x)$ and $A^{\max}(V) \neq A^{\ker}_{V_t=0}$. Consider an arbitrary operator $Q = D(\xi) + G(\chi) + \lambda M \in A^{\max}(V)$. Under Lemma's assumption, the condition (3) implies a set of equations on V of the general form

$$(ax+b)V_x + 2aV = c_2x^2 + c_1x + \tilde{c}_0 + ic_0, \qquad \text{where} \quad a, b, c_2, c_1, \tilde{c}_0, c_0 = \text{const} \in \mathbb{R}.$$

The exact number k of such equations with the linear independent sets of coefficients can be equal to either 1 or 2. (The value $k = 0$ corresponds to the general case $V_t = 0$ without any extensions of $A^{\max}$.)

For $k = 1$ $(a, b) \neq (0, 0)$ and there exist two possibilities $a = 0$ and $a \neq 0$. If $a = 0$ without loss of generality we can put $b = 1$. Condition (3) results in $\xi_t = 0$, $c_2 = c_0 = 0$, i.e. $V_x = c_1x + \tilde{c}_0$, and then $k = 2$ that is impossible.

Therefore, $a \neq 0$ and we can put $a = 1$. $\tilde{c}_0, b = 0 \bmod G^{\rm equiv}_{V_t=0}$. Condition (3) results in $\chi = 0$ (then $c_1 = 0$), $\lambda_t = 0$, $\xi_{tt} = 2c_0\xi_t$ and $c_2 = c_0^2$. Depending on $c_0 = 0$ and $c_0 \neq 0$, we obtain Cases 1 and 2 (of Table 2) correspondingly.

The condition $k = 2$ involves $V = d_2x^2 + d_1x + \tilde{d}_0 + id_0$. $\tilde{d}_0 = 0 \bmod G^{\rm equiv}_{V_t=0}$. Considering different possibilities for values of the constants d_2, d_1 and d_0, we obtain Cases 3–9:

$$d_2 = d_1 = d_0 = 0 \ \to\ 7; \qquad d_2 = d_1 = 0,\ d_0 \neq 0 \ \to\ 3;$$
$$d_2 = d_0 = 0,\ d_1 \neq 0 \ \to\ 8; \qquad d_2 = 0,\ d_0, d_1 \neq 0 \ \to\ 4;$$
$$d_2 < 0 \ \to\ 5; \qquad d_2 > 0,\ d_2 \neq d_0^2 \ \to\ 6; \qquad d_2 > 0,\ d_2 = d_0^2 \ \to\ 9.$$

Note 3. To prove Theorem 5, it is sufficient to consider only the case $k = 1$, $a \neq 0$ in Lemma 3 since the other cases of extensions of $A^{\max}(V)$ with $V = V(x)$ admit operators of the form $G(\chi) + \lambda M$ $(\chi \neq 0)$ and, therefore (by Corollary 1), are equivalent to Cases 1–5 of Table 1.

Note 4. The number N_1 for each line of Table 2 is equal to the number of the same or equivalent case in Table 1. The corresponding equivalence transformations have the form (4) where the functions T, X and Ψ are as follows:

$$2 \to 7,\ 9 \to 5: \quad T = -e^{-4t},\ X = \Psi = 0;$$
$$6 \to 3(\tilde{\nu} = \tfrac{1-\nu}{4}): \quad T = -e^{-4t},\ X = \Psi = 0; \qquad 5 \to 2(\tilde{\nu} = \nu): \quad T = \tan 2t,\ X = \Psi = 0;$$
$$8 \to 5: \quad T = t,\ X = -t^2,\ \Psi = \frac{t^3}{3}; \qquad 4 \to 4: \quad T = |\nu|t,\ X = -\sqrt{|\nu|}\,t^2,\ \Psi = \frac{t^3}{3}.$$

The results of the group classification obtained in this paper can be used to construct both invariant and partially invariant exact solutions of equations having the form (1). Moreover, we plan to study conditional symmetries of (1) to find non-Lie exact solutions.

Another direction for our future research to develop the above results is investigation of a more general class of $(1+n)$-dimensional NSchEs with potentials

$$i\psi_t + \Delta\psi + F(\psi,\psi^*) + V(t,\vec{x})\psi = 0, \tag{5}$$

where the $F = F(\psi,\psi^*)$ is an arbitrary complex-valued smooth function of the variables ψ and ψ^*. We have already described all possible inequivalent forms of the parameter-function F (without any restriction on the dimension n) for which an equation of the form (5) with a some potential V has an extension of the maximal Lie invariance algebra. We believe that the classification method suggested in this paper can be effectively applied to complete the group classification in (5) for the small values of n.

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Universal Structure of Jet Space

Maido RAHULA

Institute of Pure Mathematics, University of Tartu, 2 Liivi, 50409 Tartu, Estonia
E-mail: *Maido.Rahula@math.ut.ee*

Operators of total differentiation D, Cartan forms ω and infinitesimal symmetries P constitute the structure of infinite jet space $J_{n,m}$. We describe these notions compactly for the space $J_{1,1}$ though reserve the possibility to pass with the help of multi-indices to general case $J_{n,m}$. Our aim is to show the universality of this structure. Every time when we differentiate a function f with respect to the vector field X on a manifold M we can determine a map $\varphi : M \to J_{1,1}$ and connect the triple (X, s, F) with the triple (D, t, U) in $J_{1,1}$, where F is the set of derivatives $f^{(k)} = X^k f$, $k = 0, 1, 2, \ldots$; s is canonical parameter of X, U is the set of fiber coordinates $u^{(k)} = D^k u$, $k = 0, 1, 2, \ldots$, and t is canonical parameter of D. Then all the invariants and symmetries of D as well as all the covariant tensors including Cartan forms can be transformed from $J_{1,1}$ onto the manifold M. The structure is universal as final object in the category of triples (X, s, F).

Let $f : V_n \to V_m$ be a smooth mapping. The infinite jet of the map f is determined by the coordinates t^i, u^α of the points $t \in V_n$ and $u = f(t) \in V_m$, and by the values of partial derivatives at t:

$$u^\alpha_i = \frac{\partial f^\alpha}{\partial u^i}(t), \quad u_{ij} = \frac{\partial^2 f^\alpha}{\partial u^i \partial u^j}(t), \quad \ldots,$$
$$i, j = 1, 2, \ldots, \quad n = \dim V_n, \quad \alpha = 1, 2, \ldots, \quad m = \dim V_m.$$

The set of the jets of f is called *jet space* $J_{m,n}$ where the quantities

$$t^i, \quad u^\alpha, \quad u^\alpha_i, \quad u^\alpha_{ij}, \quad \ldots \tag{1}$$

are *jet coordinates*.

In the space $J_{1,1}$ we have the coordinates

$$t, \quad u, \quad u', \quad u'', \quad \ldots \tag{2}$$

or briefly (t, U) where U is the column of elements $u, u', u'', \ldots$.

In $J_{1,1}$ one has the *natural basis* $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial U}; dt, dU\right)$ associated with the coordinates (2). Here $\frac{\partial}{\partial U}$ is the row of elements $\frac{\partial}{\partial u}, \frac{\partial}{\partial u'}, \ldots$ and dU is the column of elements $du, du', du'', \ldots$.

Let us introduce the infinite-dimensional unit matrix E and the shift matrix C as follows:

$$E = \begin{pmatrix} 1 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \cdot \\ 0 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 & \cdot \\ 0 & 0 & 1 & \cdot \\ 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and define in $J_{1,1}$ the *total differentiation operator* by formula:

$$D = \frac{\partial}{\partial t} + \frac{\partial}{\partial U} U', \qquad \text{where} \qquad U' = CU. \tag{3}$$

Proposition 1. *The operator D is a linear vector field in the jet space $J_{1,1}$ and its flow is determined by exponential law (see [5]),*

$$U' = CU \quad \Rightarrow \quad U_t = e^{Ct} U. \tag{4}$$

The curves (t, U_t) are the trajectories of D.

Proposition 2. *If the operator $\frac{\partial}{\partial t}$ in the frame $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial U}\right)$ is replaced by D then the differentials dU in the coframe (dt, dU) have to be replaced by Cartan forms*

$$\omega = dU - U'dt. \tag{5}$$

The new basis in the matrix form

$$\begin{pmatrix} D & \frac{\partial}{\partial U} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial U} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ U' & E \end{pmatrix}, \qquad \begin{pmatrix} dt \\ \omega \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -U' & E \end{pmatrix} \cdot \begin{pmatrix} dt \\ dU \end{pmatrix}.$$

is called *adapted basis* in $J_{1,1}$. The term "adapted basis" proceeds from the theory of connections (see [4, p. 23]).

Proposition 3. *The derivation formulae valid for the adapted basis (vertical part):*

$$\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U}C, \qquad \omega' = C\omega. \tag{6}$$

The stroke means Lie derivative with respect to D. The frame $\frac{\partial}{\partial U}$ and the coframe ω are transported by the flow of D according to the law (4):

$$\left(\frac{\partial}{\partial U}\right)' = -\frac{\partial}{\partial U}C \qquad \Rightarrow \qquad \left(\frac{\partial}{\partial U}\right)_t = \frac{\partial}{\partial U}e^{-Ct},$$
$$\omega' = C\omega \qquad \Rightarrow \qquad \omega_t = e^{Ct}\omega.$$

Proposition 4. *The quantities*

$$I = e^{-Ct}U \tag{7}$$

are the invariants of D because $I' = e^{-Ct}(U' - CU) = 0$. Replacing U by I in the fibers of $J_{1,1}$ we have the invariant basis:

$$\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial I}; dt, dI\right).$$

The exponential e^{-Ct} is *integrating matrix* for Cartan forms ω and the operators $\frac{\partial}{\partial I}$ are *infinitesimal symmetries* of D (*infinitesimals* after [1]) in the following sense:

$$dI = e^{-Ct}\omega, \qquad \frac{\partial}{\partial I} = \frac{\partial}{\partial U}e^{Ct}. \tag{8}$$

Infinitesimal symmetries of D are called *Lie vector fields* in $J_{1,1}$.

Proposition 5. *A vector field P written in three frames of $J_{1,1}$, natural, adapted and invariant as follows*

$$P = \frac{\partial}{\partial t}\xi + \frac{\partial}{\partial U}\lambda = D\xi + \frac{\partial}{\partial U}\mu = \frac{\partial}{\partial t}\xi + \frac{\partial}{\partial I}\nu \tag{9}$$

has for components the entities

$$\xi = Pt, \qquad \lambda = PU, \qquad \mu = \omega(P), \qquad \nu = PI, \tag{10}$$

with relations

$$\nu = e^{-Ct}\mu, \qquad \mu = \lambda - U'\xi. \tag{11}$$

The field P is a Lie vector field if and only if one of equivalent conditions is satisfied:

$$\nu' = 0, \qquad \mu' = C\mu, \qquad \lambda' = C\lambda + U'\xi'. \tag{12}$$

It is obvious from the Lie derivatives:

$$P' = D\xi' + \frac{\partial}{\partial U}(\lambda' - C\lambda - U'\xi') = D\xi' + \frac{\partial}{\partial U}(\mu' - C\mu) = D\xi' + \frac{\partial}{\partial I}\nu',$$
$$L_p\omega = (\lambda' - C\lambda - U'\xi')dt + \left(\frac{\partial\lambda}{\partial U} - U'\frac{\partial\xi}{\partial U}\right)\omega = (\mu' - C\mu)dt + \frac{\partial\mu}{\partial U}\omega = e^{Ct}\nu' dt + \frac{\partial\nu}{\partial I}\omega.$$

The most simple condition $\nu' = 0$ says that the components ν in invariant frame are invariants of D.

The condition $\mu' = C\mu$ means that each entry of column μ is the derivative of preceding entry. Thus all entries of column μ in adapted frame are generated by the first entry $\mu_0 = f$ (*generating function*, see [1, p. 454]) by means of differentiation:

$$\mu_k = f^{(k)} = D^k f, \qquad k = 0, 1, 2, \ldots.$$

There is an obvious analogy between two equations $I = e^{-Ct}U$ and $v = e^{-Ct}\mu$.

The most complicated condition $\lambda' = C\lambda + U'\xi'$ is principal for the calculation of symmetries in natural basis (see [1, p. 244], [2, p. 110], [3, p. 55]).

Remark 1. In $J_{1,1}$ the invariants $I = e^{-Ct}U$ are described as follows:

$$I_k = \sum_{\ell=0}^{\infty} u^{(k+\ell)}\frac{(-t)^\ell}{\ell!}, \qquad k = 0, 1, 2, \ldots.$$

The operators $\frac{\partial}{\partial I}$ are basic Lie vector fields with generating functions $1, t, \frac{t^2}{2}, \ldots$ respectively, that is

$$\frac{\partial}{\partial I_0} = \frac{\partial}{\partial u},$$
$$\frac{\partial}{\partial I_1} = t\frac{\partial}{\partial u} + \frac{\partial}{\partial u'},$$
$$\frac{\partial}{\partial I_2} = \frac{t^2}{2}\frac{\partial}{\partial u} + t\frac{\partial}{\partial u'} + \frac{\partial}{\partial u''}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Remark 2. In $J_{n,m}$ instead of D we have a system of n operators D_i, $i = 1, 2, \ldots, n$, and instead of 1-dimensional trajectories we have n-dimensional orbits of the additive group $\mathbb{R}^n$. For example, in the space $J_{2,1}$ there are the 2-dimensional time $t = (t_1, t_2)$ and two operators

$$D_1 = \frac{\partial}{\partial t_1} + u_1\frac{\partial}{\partial u} + u_{11}\frac{\partial}{\partial u_1} + u_{12}\frac{\partial}{\partial u_2} + \cdots,$$
$$D_2 = \frac{\partial}{\partial t_2} + u_2\frac{\partial}{\partial u} + u_{12}\frac{\partial}{\partial u_1} + u_{22}\frac{\partial}{\partial u_2} + \cdots.$$

Herewith 2-dimensional orbits of $\mathbb{R}^2$ are determined by the series

$$u_t = u + u_1t_1 + u_2t_2 + \frac{1}{2}\left[u_{11}(t_1)^2 + 2u_{12}t_1t_2 + u_{22}(t_2)^2\right] + \cdots$$

and its partial derivatives of all orders with respect to t_1 and t_2.

Remark 3. In $J_{2,1}$ the Lie field P with the generating function f can be represented in adapted and natural basis as follows:

$$P = \xi^1 D_1 + \xi^2 D_2 + f\frac{\partial}{\partial u} + f_1\frac{\partial}{\partial u_1} + f_2\frac{\partial}{\partial u_2} + \cdots$$

$$= \xi^1 \frac{\partial}{\partial t_1} + \xi^2 \frac{\partial}{\partial t_2} + \left(f + u_1\xi^1 + u_2\xi^2\right) \frac{\partial}{\partial u} + \left(f_1 + u_{11}\xi^1 + u_{12}\xi^2\right) \frac{\partial}{\partial u_1}$$
$$+ \left(f_2 + u_{12}\xi^1 + u_{22}\xi^2\right) \frac{\partial}{\partial u_2} + \cdots ,$$

where $f_i = D_i f$, $i = 1, 2$. The components $\lambda_k = f_k + u_{ki}\xi^i$, $k = 0, 1, 2, \ldots$ are consistent with the relation $\lambda = \mu + U'\xi$.

Theorem 1. *Any smooth vector field X without singularities on a manifold M can be connected with the total differentiation operator D in the jet space $J_{1,1}$, i.e. there exists a smooth map $\varphi : M \longrightarrow J_{1,1}$ such that the vector field X is φ-connected with the operator D.*

Proof. Let s be the canonical parameter of the vector field X, herewith $Xs = 1$. Take a smooth function f and calculate its derivatives with respect to X, $f^{(k)} = X^k f$, $k = 1, 2, \ldots$. Let F be the infinite column of elements $f, f', f'', \ldots$ and let us define the mapping φ by the relations

$$t \circ \varphi = s, \qquad U \circ \varphi = F. \tag{13}$$

At some step n we get the conditions

$$\Theta = df \wedge df' \wedge df'' \wedge \cdots \wedge df^{(n-1)} \neq 0 \qquad \text{and} \qquad \Theta \wedge df^{(n)} = 0. \tag{14}$$

There are two possible cases: a) $n = \dim M$, or b) $n < \dim M$.

Case a) $n = \dim M$. Let the functions $f, f', f'', \ldots, f^{(n-1)}$ be the coordinates on M and let us represent the field X as follows:

$$X = f' \frac{\partial}{\partial f} + f'' \frac{\partial}{\partial f'} + \cdots + f^{(n)} \frac{\partial}{\partial f^{(n-1)}}.$$

The Jacobian matrix of φ relate the components of X to the components of D (the subscripts mean the partial derivatives):

$$\begin{pmatrix} s_1 & \cdots & s_n \\ 1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 1 \\ f_1^{(n)} & \cdots & f_n^{(n)} \\ \cdots & \cdots & \cdots \end{pmatrix} \cdot \begin{pmatrix} f' \\ \cdots \\ f^{(n)} \end{pmatrix} = \begin{pmatrix} s' \\ f' \\ \cdots \\ f^{(n)} \\ f^{(n+1)} \\ \cdots \end{pmatrix} = \begin{pmatrix} 1 \\ u' \\ \cdots \\ u^{(n)} \\ u^{(n+1)} \\ \cdots \end{pmatrix} \circ \varphi.$$

The rank of the Jacobian matrix is equal to n and φ is an immersion of M into $J_{1,1}$. The triple (X, s, F) on the manifold M is φ-connected with the triple (D, t, U) in the jet space $J_{1,1}$.

Case b) $n < N = \dim M$. It follows from $\Theta \wedge df^{(n)} = 0$ that $df^{(n)}$ is a linear combination of $df, df', df'', \ldots, df^{(n-1)}$,

$$df^{(n)} = \sum_{i=1}^{n} \alpha_i df^{(n-i)} \qquad \text{and} \qquad L_X\Theta = \alpha_1\Theta.$$

The functions $f, f', f'', \ldots, f^{(n-1)}$ determine a submersion $\pi : M \longrightarrow W$. The vector field X transports the fibers of π into the fibers of the same bundle and because of this the field X can be projected on the n-dimensional manifold W. In the coordinates $v^{(i)}$,

$$v^{(i)} \circ \pi = f^{(i)}, \qquad i = 0, 1, 2, \ldots, n-1,$$

the projection of X is a vector field

$$T\pi X = v'\frac{\partial}{\partial v} + v''\frac{\partial}{\partial v'} + \cdots + v^{(n-1)}\frac{\partial}{\partial v^{(n-2)}} + f^{(n)}\frac{\partial}{\partial v^{(n-1)}}$$

which can be connected by a map $\tilde{\varphi} : W \longrightarrow J_{1,1}$ with the operator D. Then the vector field X is φ-connected with D, where $\varphi = \tilde{\varphi} \circ \pi$.

General case. How to make the correspondence between a system of n vector fields Y_i on a manifold M with the operators of total differentiation D_i in the jet space $J_{n,m}$? Let u^α be the coordinates on M, u^i the canonical parameters of Y_i, $Y_i u^j = \delta_i^j$, and y_i^α the natural components of the fields Y_i. The operators

$$X_i = \frac{\partial}{\partial u^i} + Y_i = \frac{\partial}{\partial u^i} + y_i^\alpha \frac{\partial}{\partial u^\alpha}$$

determine a n-dimensional distribution in the "space-time" $R^n \times M$ with the coordinates (u^i, u^α), $i = 1, 2, \ldots, n$; $\alpha = n+1, \ldots, n+m$; $m = \dim M$. This is a particular case of connection in the fiber space, see [4], where the operators

$$X_i = \frac{\partial}{\partial u^i} + \Gamma_i^\alpha \frac{\partial}{\partial u^\alpha} \tag{15}$$

form in the coordinates (u^i, u^α) an adapted frame of the horizontal distribution Δ_h, with the components

$$\Gamma_i^\alpha = \Gamma_i^\alpha(u^j, u^\beta).$$

In our case we have $\Gamma_i^\alpha = y_i^\alpha(u^j)$. Let us immerse the operators X_i in the space $J_{n,m}$ with the help of the map $\varphi : M \longrightarrow J_{n,m}$ supposing

$$\begin{aligned} &t^i \circ \varphi = u^i, \quad u^\alpha \circ \varphi = u^\alpha, \quad u_i^\alpha \circ \varphi = \Gamma_i^\alpha, \\ &u_{ij}^\alpha \circ \varphi = X_{(i}\Gamma_{j)}^\alpha, \quad u_{ijk}^\alpha \circ \varphi = X_{(i}X_j\Gamma_{k)}^\alpha, \quad \ldots. \end{aligned} \tag{16}$$

The operators X_i and the vector fields Y_i are φ-connected with the operators D_i. ■

As corollaries we have the next Propositions.

Proposition 6. *If the vector field X is φ-connected with the operator D then for any function I in $J_{1,1}$ the derivatives $X(I \circ \varphi)$ and DI are φ-connected, i.e. $X(I \circ \varphi) = (DI) \circ \varphi$. From this it follows that $DI = 0 \Longrightarrow X(I \circ \varphi) = 0$ and all the invariants of D can be transported on the manifold M in the invariants of the vector field X. In particular the invariants $I = e^{-Ct}U$ are transported from $J_{1,1}$ on M in the invariants $I \circ \varphi = e^{-Cs}F$.*

Proposition 7. *If the vector field X is φ-connected with the operator D then all the covariant tensors can be transported from $J_{1,1}$ on the manifold M. For example, the Cartan forms $\omega = dU - U'dt$ can be transported in the forms $\omega \circ T\varphi = dF - F'dt$, where $F' = XF$. The sequence of Lie derivatives with respect to D (Cartan forms)*

$$\omega_0 = du - u'dt, \quad \omega_0' = du' - u''dt, \quad \omega_0'' = du'' - u''dt, \quad \ldots$$

induces the sequence of Lie derivatives with respect to X:

$$\omega_0 \circ T\varphi = df - f'ds, \quad \omega_0' \circ T\varphi = df' - f''ds, \quad \omega_0'' \circ T\varphi = df'' - f'''ds, \quad \ldots.$$

Proposition 8. *In the general case (16) the Cartan forms in $J_{n,m}$*

$$\omega^\alpha = du^\alpha - u^\alpha dt^i, \quad \omega_i^\alpha = du_i^\alpha - u_{ij}^\alpha dt^j, \quad \dots$$

induce on the manifold $\mathbb{R}^n \times M$ the sequence of 1-forms

$$\theta^\alpha = \omega^\alpha \circ T\varphi = du^\alpha - \Gamma_i^\alpha du^i, \quad \theta_i^\alpha = \omega_i^\alpha \circ T\varphi = d\Gamma_i^\alpha - X_{(i}\Gamma_{j)}^\alpha du^j, \quad \dots.$$

The horizontal distribution Δ_h is the annulator of the forms θ^α, i.e. $\theta^\alpha(X_i) = 0$. The forms θ_i^α imply the appearance of two important objects:

$$K_{ij}^\alpha = X_{[i}\Gamma_{j]}^\alpha, \quad \text{object of curvature,}$$
$$\Gamma_{i\beta}^\alpha = -\partial_\beta \Gamma_i^\alpha, \quad \text{object of connection.}$$

Namely, because $d\Gamma_i^\alpha = X_j\Gamma_i^\alpha du^j + \partial_\beta \Gamma_i^\alpha \theta^\beta$ and $X_j\Gamma_i^\alpha = X_{(i}\Gamma_{j)}^\alpha - X_{[i}\Gamma_{j]}^\alpha$ we have

$$\theta_i^\alpha = -K_{ij}^\alpha du^j - \Gamma_{i\beta}^\alpha \theta^\beta.$$

For the linear connection the quantities Γ_i^α are linear functions on the fibers: $\Gamma_i^\alpha = -\Gamma_{i\beta}^\alpha u^\beta$, and we have $K_{ij}^\alpha = -K_{ij\beta}^\alpha u^\beta$, where $K_{ij\beta}^\alpha = \partial_{[i}\Gamma_{j]\beta}^\alpha + \Gamma_{[i|\gamma|}^\alpha \Gamma_{j]\beta}^\gamma$, (see [4, p. 26]).

Extending the linear connection onto the tangent bundle $TM \longrightarrow M$ we get the affine connection on the manifold M in the classical sense.

Proposition 9. *The vertical distribution Δ_v is integrable because $\Delta_v = \operatorname{Ker} T\pi$ and the vector fields* (15) *are infinitesimals of Δ_v. For any coframe θ^i of Δ_v there exists an integrating matrix B_j^i such that $B_j^i\theta^j = du^i$. Then $B_k^i\theta^k(Xj) = \delta_j^i$ is unit matrix and B_j^i is inverse to the matrix $\theta^i(X_j)$.*

Let us mention that from (8) we have the same situation $e^{-Ct}\omega(\frac{\partial}{\partial I}) = E$. This generalizes the known property of integrating factor for $n = 1$ (see [1, p. 60]).

Proposition 10. *The vector field P represented in natural and adapted frames as follows (see [5, p. 286])*

$$P = \xi^i \frac{\partial}{\partial u^i} + \lambda^\alpha \frac{\partial}{\partial u^\alpha} = \xi^i X_i + \mu^\alpha \frac{\partial}{\partial u^\alpha}, \qquad \mu^\alpha = \lambda^\alpha - \Gamma_i^\alpha \xi^i,$$

is an infinitesimal symmetry of horizontal distribution Δ_h if and only if either

$$X_i\lambda^\alpha - P\Gamma_i^\alpha - \Gamma_j^\alpha X_i\xi^j = 0 \tag{17}$$

or

$$X_i\mu^\alpha + \Gamma_{i\beta}^\alpha \mu^\beta + 2K_{ij}^\alpha \xi^j = 0. \tag{18}$$

For the case

$$\Gamma_i^\alpha = -\Gamma_{i\beta}^\alpha u^\beta, \qquad \mu^\alpha = \mu_\beta^\alpha u^\beta, \qquad \lambda^\alpha = \lambda_\beta^\alpha u^\beta, \qquad \mu_\beta^\alpha = \lambda_\beta^\alpha + \Gamma_{i\beta}^\alpha \xi^i$$

the conditions (17) and (18) are equivalent to

$$\partial_i\lambda_\beta^\alpha - \lambda_\gamma^\alpha\Gamma_{i\beta}^\gamma + \Gamma_{i\gamma}^\alpha\lambda_\beta^\gamma + \partial_i\Gamma_{j\beta}^\alpha\xi^j + \Gamma_{j\beta}^\alpha X_i\xi^j = 0, \tag{19}$$
$$\partial_i\mu_\beta^\alpha - \mu_\gamma^\alpha\Gamma_{i\beta}^\gamma + \Gamma_{i\gamma}^\alpha\mu_\beta^\gamma - 2K_{ij\beta}^\alpha\xi^j = 0. \tag{20}$$

On the tangent bundle $TM \longrightarrow M$ we have the correspondence

$$(u^i, u^\alpha) \sim (u^i, du^i), \qquad (\xi^i, \lambda^\alpha) \sim (\xi^i, d\xi^i), \qquad \lambda_\beta^\alpha \sim \frac{\partial \xi^i}{\partial u^j}, \qquad \mu^\alpha \sim \frac{\partial \xi^i}{\partial u^j} + \Gamma_{kj}^i \xi^k$$

and the conditions (19) and (20) define P as an *affine collineation* (infinitesimal movement in the space of affine connection or *Killing's vector field* in Riemannian geometry), see [6, p. 37, formulae (2.30) and (2.31)].

Remark 4. For ODE $y' + p(x)y + q(x) = 0$ we have $\omega = (py + q)dx + dy$ and the condition (18) gives $\mu = e^{-\int p dx}$. The form

$$\frac{\omega}{\mu} = d\left(\frac{y}{\mu}\right) + \frac{q}{\mu}dx$$

is exact and determines the first integral (see [1, p. 160])

$$\frac{y}{\mu} + \int \frac{q}{\mu}dx.$$

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2-Dimensional Plasticity: Boundary Problems and Conservation Laws, Reproduction of Solutions

Sergey I. SENASHOV † *and Alexander YAKHNO* ‡

† *Krasnoyarsk State Trade-Economic Institute, 2 Prushinskoi Str., 660075 Krasnoyarsk, Russia*
E-mail: *senashov@kgtei.kts.ru*

‡ *University of Guadalajara, C.U.C.E.I., Guadalajara, Mexico*
E-mail: *ay223627@cucei.udg.mx*

For the system of two-dimensional plasticity equations with the help of conservation laws basic boundary problems are solved. We have considered solutions of some concrete boundary problems. Using the symmetry transformations, from the well known Prandtle's solution we deduce some new solutions, describing the pressing of a layer between two rugged slabs. Also, some new properties of characteristics of the plane plasticity are shown.

1 Introduction

Among hyperbolic systems of nonlinear equations with partial derivatives the systems of the quasilinear equations of two independent variables are most investigated. These systems describe, in particular, unsteady one-dimensional and supersonic two-dimensional stationary flows of compressible gases and liquids, two-dimensional deformed plastic state of a continuous medium and so on. A lot of them are reduced to a hyperbolic system of homogeneous quasilinear equations

$$\begin{aligned} &u_x + A(u,v)u_y = 0,\\ &v_x + B(u,v)v_y = 0, \end{aligned} \tag{1}$$

where $u = u(x,y)$, $v = v(x,y)$, an indices below mean derivation with respect to the corresponding variables.

In the Section 2 we describe a method of the analytical solving of boundary-value problems for the system (1). This method is based on the conservations laws that allow us to consider linearization of problems without nonsingular transformations and to obtain exact solutions of boundary-value problems in explicit form. In the Section 3 we consider some concrete systems.

Other way to solve some boundary-value problems is using of symmetries of equations. The theory of symmetries is widely used in investigation and in resolution of differential equations. A group of symmetries allows to find exact solutions of differential equations and to classify PDE's by admitting groups.

Symmetries admitted by the system of PDE's have a great property: under its action any solutions of the system are transformed to the solutions of this system. This property allows to construct new solutions without integrating of the given system, only by means of group transformations of known solutions. In such a way some of interesting results were obtained [1]. Note that this way is effective only if we have a sufficiently rich group of point transformations. In the Section 4 Prandtl's solution was transformed. As a result we obtained a lot of exact solutions. We selected only restricted ones along Oy-axis ones and these solutions can be used for the solving of boundary-value problems for compression of a plastic layer by rigid parallel slabs. We can hope that with the help of these solutions it will be possible to describe not only compression of the thin layers, but the thick ones too.

2 Conservation laws

Let us set Cauchy's problem: in some neighborhood of an arc $a \le \tau \le b$ of a smooth curve

$$L = \{(x, y) : x = x(\tau), y = y(\tau), \tau \in [a, b]\}$$

in the plane xOy it is required to find a solution of the system (1) that takes given values on L

$$u(x, y)|_L = u(x(\tau), y(\tau)) = u^0(\tau), \qquad v(x, y)|_L = v(x(\tau), y(\tau)) = v^0(\tau).$$

The characteristics equations of the system (1) look like

$$\frac{dx}{dy} = A, \qquad \frac{dx}{dy} = B, \tag{2}$$

with the relations on the characteristics $u = u^0$, $v = v^0$ respectively.

We search a conservation law of the set of equations (1) in the form of a relation

$$C_x + D_y = 0, \tag{3}$$

where $C = C(u, v)$, $D = D(u, v)$, which should vanish only on the solutions of the system (1):

$$C_u u_x + C_v v_x + D_u u_y + D_v v_y = -AC_u u_y - BC_v v_y + D_u u_y + D_v v_y = 0,$$

hence

$$D_u - AC_u = 0, \qquad D_v - BC_v = 0. \tag{4}$$

The equation (3), if conditions of Green's theorem are satisfied, is equivalent to a relation

$$\oint_\Gamma -Cdy + Ddx = 0,$$

where Γ is an arbitrary closed contour.

In the plane xOy we will consider the closed path MNK, where $M(x_m(a), y_m(a))$, $N(x_n(b), y_n(b)) \in L$, $K(x_k, y_k)$ is the point of an intersection of characteristics $v = v^0$, $u = u^0$, drawn through the points M, N respectively.

Then,

$$\oint_{MNK} -Cdy + Ddx = \int_{MN} Ddx - Cdy + \int_{NK} Ddx - Cdy + \int_{KM} Ddx - Cdy = 0. \tag{5}$$

Taking into account the expressions (2)

$$\int_{NK} Ddx - Cdy = \int_{NK} (D - AC)dx = x(D - AC)\Big|_{x_n}^{x_k} - \int_{NK} x\partial_v(D - AC)dv.$$

Similarly,

$$\int_{KM} Ddx - Cdy = \int_{KM} (D - BC)dx = x(D - BC)\Big|_{x_k}^{x_m} - \int_{KM} x\partial_u(D - BC)du.$$

Let us assume

$$\partial_v \left[(D - AC)|_{u=u^0(x(b),y(b))}\right] = 0, \qquad \partial_u \left[(D - BC)|_{v=v^0(x(a),y(a))}\right] = 0. \tag{6}$$

Let us denote

$$\phi(u,v) = D - AC, \qquad \psi(u,v) = D - BC.$$

Then

$$D = (A\psi - B\phi)/(A-B), \qquad C = (\psi - \phi)/(A-B),$$

where $A \neq B$, because the considered system (1) is the hyperbolic one and has two different families of characteristics.

In the new variables, the system (4) has the form

$$\begin{aligned} &\phi_u + K_1(\psi - \phi) = 0, \\ &\psi_v + K_2(\psi - \phi) = 0, \end{aligned} \tag{7}$$

where $K_1 = A_u/(A-B)$, $K_2 = B_v/(A-B)$.

We will write the conditions (6) as follows

$$\phi|_{u=u^0} = \text{const}_1 = 1, \qquad \psi|_{v=v^0} = \text{const}_2 = 0. \tag{8}$$

Coming back to (5) and taking into account (8), we obtain

$$\begin{aligned} \int_{MN} Ddx - Cdy &= -(x(D-AC)|_{x_n}^{x_k} + x(D-BC)|_{x_k}^{x_m}) \\ &= -(x_k\phi|_{u=u^0,v=v^0} - x_n\phi|_{u=u^0} + x_m\psi|_{v=v^0} - x_k\psi|_{u=u^0,v=v^0}) = x_n - x_k. \end{aligned} \tag{9}$$

If we find a solution of the linear system (7) satisfying to the boundary conditions (8), then we will determine the coordinate x_k from the equation (9).

On the other hand, for y-coordinate

$$\int_{NK} Ddx - Cdy = \int_{NK} \left(\frac{D}{A} - C\right) dy = y\,\frac{\phi}{A}\Big|_{y_n}^{y_k} - \int_{NK} y\partial_v\left(\frac{\phi}{A}\right) dv.$$

Similarly,

$$\int_{KM} Ddx - Cdy = \int_{KM} \left(\frac{D}{B} - C\right) dy = y\,\frac{\psi}{B}\Big|_{y_k}^{y_m} - \int_{KM} y\partial_u\left(\frac{\psi}{B}\right) du.$$

Let us assume

$$\partial_v\left[\frac{\phi}{A}\Big|_{u=u^0(x(b),y(b))}\right] = 0, \qquad \partial_u\left[\frac{\psi}{B}\Big|_{v=v^0(x(a),y(a))}\right] = 0. \tag{10}$$

We can write the conditions (10) as follows

$$\phi|_{u=u^0} = A(u^0, v), \qquad \psi|_{v=v^0} = 0. \tag{11}$$

From (5), taking into account (10), we have

$$\begin{aligned} \int_{MN} Ddx - Cdy &= -y_k\,\frac{\phi}{A}\Big|_{u=u^0,v=v^0} + y_n\,\frac{\phi}{A}\Big|_{u=u^0} \\ &- y_m\,\frac{\psi}{B}\Big|_{v=v^0} + y_k\,\frac{\psi}{B}\Big|_{u=u^0,v=v^0} = y_n - y_k. \end{aligned} \tag{12}$$

The solution of the problem (7), (11) makes possible to find the coordinate y_k from the equation (12). Thus, we will determine the coordinates of the point K, where the values of the functions u, v can be reconstructed.

3 Examples of application of the method

1. In the work [2] the solution was found that takes given values on L

$$\sigma(x,y)|_L = \sigma_0, \qquad \theta(x,y)|_L = \theta_0$$

for the initial problem for the system of 2-dimensional plasticity under von Mises condition

$$\begin{aligned} &\sigma_x - 2k\,(\theta_x \cos 2\theta + \theta_y \sin 2\theta) = 0,\\ &\sigma_y - 2k\,(\theta_x \sin 2\theta - \theta_y \cos 2\theta) = 0, \end{aligned} \tag{13}$$

where σ is a hydrostatic pressure, θ is an angle between the first main direction of a stress tensor and the ox-axis, k is a constant of plasticity. This system in the form (1) is

$$\xi_x + \xi_y \tan\theta = 0, \qquad \eta_x - \eta_y \cot\theta = 0,$$

where $2\theta = \eta - \xi$, $\sigma = k(\eta + \xi)$. Solution of problem (7), (8) has the form: $\phi = \rho/\cos\theta$, $\psi = 2\rho_\xi/\sin\theta$, where

$$\rho(\xi,\eta) = R(\xi,\xi_0,\eta,\eta_0)\cos\left(\frac{\eta_0-\xi_0}{2}\right) - \frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\sin\left(\frac{\tau-\xi_0}{2}\right)\,d\tau.$$

Accordingly, the solution of the problem (7), (11) is

$$\rho(\xi,\eta) = R(\xi,\xi_0,\eta,\eta_0)\sin\left(\frac{\eta_0-\xi_0}{2}\right) + \frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\cos\left(\frac{\tau-\xi_0}{2}\right)\,d\tau,$$

where $R(\xi,\eta,\xi_0,\eta_0) = I_0\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}\right)$ is Bessel function of a zero order of imaginary argument, $I_0(0) = 1$, $I_0'(0) = 0$.

2. The generalization of (13) is a system of ideal plasticity equations under Coulomb's plasticity condition that has the form:

$$\begin{aligned} &\sigma_x(1+\cos 2\alpha\cos 2\Theta) + \sigma_y \cos 2\alpha \sin 2\Theta = 2(\sigma\cos 2\alpha + k\sin 2\alpha)(\Theta_x \sin 2\Theta - \Theta_y \cos 2\Theta),\\ &\sigma_x \cos 2\alpha \sin 2\Theta + \sigma_y(1-\cos 2\alpha\cos 2\Theta) = -2(\sigma\cos 2\alpha + k\sin 2\alpha)(\Theta_x\cos 2\Theta + \Theta_y \sin 2\Theta), \end{aligned}$$

where $\pi/2 - 2\alpha$ is a constant angle of internal friction, $\Theta = \theta + \pi/4$. If $\alpha = \pi/4$, then we have the system (13). This system in the form (1) is:

$$\xi_x + \xi_y \tan(\Theta - \alpha) = 0, \qquad \eta_x - \eta_y \tan(\Theta+\alpha) = 0,$$

where $\xi = \frac{1}{2}\tan 2\alpha \ln(\sigma\cot 2\alpha + k) - \Theta$, $\eta = \frac{1}{2}\tan 2\alpha \ln(\sigma\cot 2\alpha + k) + \Theta$. Solution of problem (7), (8) has a form [3]: $\phi = \gamma(-\xi,-\eta)V(\xi,\eta)/\cos(\Theta-\alpha)$, $\gamma(\xi,\eta) = \exp(-(\xi+\eta)/2\cot 2\alpha)$, where:

$$\begin{aligned} V = {}& \gamma(\xi_0,\eta_0)R(\xi,\xi_0,\eta,\eta_0)\cos\left((\eta_0-\xi_0)/2-\alpha\right)\\ &- \frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\gamma(\xi_0,\tau)\left[\sin\left((\tau-\xi_0)/2-\alpha\right) - \cot 2\alpha\cos\left((\tau-\xi_0)/2-\alpha\right)\right]\,d\tau. \end{aligned} \tag{14}$$

The solution of the problem (7), (11) is

$$\begin{aligned} V = {}& \gamma(\xi_0,\eta_0)R(\xi,\xi_0,\eta,\eta_0)\sin\left((\eta_0-\xi_0)/2-\alpha\right)\\ &- \frac{1}{2}\int_{\eta_0}^{\eta} R(\xi,\xi_0,\eta,\tau)\gamma(\xi_0,\tau)\left[\cot 2\alpha\sin\left((\tau-\xi_0)/2-\alpha\right) - \cos\left((\tau-\xi_0)/2-\alpha\right)\right]\,d\tau, \end{aligned} \tag{15}$$

$R(\xi,\eta,\xi_0,\eta_0) = I_0\left(\sqrt{(\xi-\xi_0)(\eta-\eta_0)}/\sin 2\alpha\right)$. From the equation (9) taking into account (14) we will find a coordinate x_k. Using (15), from the equations (12) we will obtain a coordinate y_k. Thus, we have determined a point K, in which the values of functions ξ, η are reconstructed.

4 Reproduction of Prandle's solution

Let us consider the system (13) that has two families of characteristics (in the theory of plasticity they are named slide lines) with the following equations and relations along characteristics:

$$\frac{dy}{dx} = \tan\theta, \qquad \frac{\sigma}{2k} - \theta = c_1,$$
$$\frac{dy}{dx} = -\cot\theta, \qquad \frac{\sigma}{2k} + \theta = c_2.$$

This system admits the infinite group of symmetries [4]. Its point subgroup is generated by following generators:

$$X_1 = \partial_x, \qquad X_2 = \partial_y, \qquad X_3 = x\partial_x + y\partial_y, \qquad X_4 = -x\partial_y + y\partial_x + \partial_\theta,$$
$$X_5 = \partial_\sigma, \qquad X_6 = \xi_1\partial_x + \xi_2\partial_y + 4k\theta\partial_\sigma - \frac{\sigma}{k}\partial_\theta, \qquad X = \xi\partial_x + \eta\partial_y, \tag{16}$$

where

$$\xi_1 = -x\cos 2\theta - y\sin 2\theta - y\frac{\sigma}{k}, \qquad \xi_2 = y\cos 2\theta - x\sin 2\theta + x\frac{\sigma}{k},$$

and (ξ, η) is an arbitrary solution of the following lineal system of equations:

$$\xi_\theta - 2k\left(\xi_\sigma\cos 2\theta + \eta_\sigma\sin 2\theta\right) = 0,$$
$$\eta_\theta - 2k\left(\xi_\sigma\sin 2\theta - \eta_\sigma\cos 2\theta\right) = 0, \tag{17}$$

Transformations, which correspond to any generator of (16), convert the system (13) to itself. Thus, for the generator X we have transformations of independent variables:

$$x' = x + a\xi, \qquad y' = y + a\eta, \tag{18}$$

where a is an arbitrary parameter, (ξ, η) is an arbitrary solution of (17). As X is an admitted generator, the transformations (18) do not change the form of the system (13). Note that we have shown the variables that are changed under this transformation, other variables are omitted.

Under the action of the symmetry group, any solution of the system (13) is transformed to the solution of this system. Furthermore, it is easy to demonstrate

Proposition 1. *The characteristics of the system* (13) *are transformed to the characteristics of this system under the action of symmetry group* (16).

The system (13) is being investigated for more than 100 years, but there are a few of its exact solutions. Certainly, one of the well-known solutions was constructed by Prandtl [5]. Let us write this solution in the form

$$\sigma = -kx + k\left(1 - y^2\right)^{\frac{1}{2}}, \qquad 2\theta = \arccos y. \tag{19}$$

From proposition, in particular, it is follows that for the construction of slide lines we should substitute the characteristic relations to the solution. Let us substitute the relation along the first characteristic $\sigma/(2k) - \theta = c_1$ to the solution (19). We obtain the equation of the first family of slide lines:

$$x = -2\theta + \sin 2\theta + K_1, \qquad y = \cos 2\theta.$$

By analogy, we obtain the second family of slide lines with relation $\sigma/(2k) + \theta = c_2$:

$$x = 2\theta + \sin 2\theta + K_2, \; y = \cos 2\theta.$$

Under the transformations (18), Prandtl's solution will transform to the following one:

$$\sigma = -kx + ak\xi + k\sin 2\theta, \qquad y = \cos 2\theta + a\eta, \tag{20}$$

and the slide lines will have the form:

$$x = \mp 2\theta + a\xi + \sin 2\theta + K_i, \qquad y = \cos 2\theta + a\eta, \qquad i = 1, 2.$$

Therefore, in order to obtain new solutions, we have to find some solutions of the system (17). This is the system of PDE's of the first order with the variable coefficients, and it is possible to find solutions in the form of series:

$$\xi = \sum_{i=1}^{\infty} \left[f_i(\sigma)\cos(i\theta) + g_i(\sigma)\sin(i\theta) \right], \qquad \eta = \sum_{i=1}^{\infty} \left[F_i(\sigma)\cos(i\theta) + G_i(\sigma)\sin(i\theta) \right],$$

where $f_i(\sigma)$, $g_i(\sigma)$, $F_i(\sigma)$, $G_i(\sigma)$ are the functions, obtained as solutions of a system of ODE's. This way can be successful, but it is difficult to give an analysis of results and use them for practical goals. Therefore, we can find some solutions of (17) in a simpler way, which can be used in practice. Solutions of the system (17) we can find in the forms:

$$\xi = \alpha\sigma + F(\theta), \qquad \eta = \beta\sigma + G(\theta), \tag{21}$$

$$\xi = f(\theta)\exp\left(\frac{\sigma}{2k}\right), \qquad \eta = g(\theta)\exp\left(\frac{\sigma}{2k}\right), \tag{22}$$

$$\xi = \alpha(\theta)\sin\frac{\sigma}{2k} + \beta(\theta)\cos\frac{\sigma}{2k}, \qquad \eta = F(\theta)\sin\frac{\sigma}{2k} + G(\theta)\cos\frac{\sigma}{2k}, \tag{23}$$

$$\xi = \alpha(\theta)\sinh\frac{\sigma}{2k} + \beta(\theta)\cosh\frac{\sigma}{2k}, \qquad \eta = F(\theta)\sinh\frac{\sigma}{2k} + G(\theta)\cosh\frac{\sigma}{2k}, \tag{24}$$

where functions f, g, α, β, F, G are determined as a solution of corresponding systems of ODEs.

It is easy to see that under the transformation (21) the Prandtl's solutions is transformed into the solution of the same form (there is no new solution).

Under the transformations (22), (24) the solution will be transformed to the non-restricted ones along the *oy*-axis, hence we do not consider solutions of this form in this article.

Let us consider the action of ultimate transformation (23) to the slide lines. Substituting (23) to the system (17) we have the system of equations for a determination of functions α, β, F, G

$$\begin{aligned} &\alpha' + \beta\cos 2\theta + G\sin 2\theta = 0, \qquad \beta' - \alpha\cos 2\theta - F\sin 2\theta = 0, \\ &F' + \beta\sin 2\theta - G\cos 2\theta = 0, \qquad G' - \alpha\sin 2\theta + F\cos 2\theta = 0. \end{aligned} \tag{25}$$

Let us derive each of the equations (25) with respect to θ. Then, according to (25), we obtain the equations:

$$\begin{aligned} &\alpha'' + \alpha + 2F' = 0, \qquad F'' + F - 2\alpha' = 0, \\ &\beta'' + \beta + 2G' = 0, \qquad G'' + G - 2\beta' = 0, \end{aligned} \tag{26}$$

Let us find the solution of the first system of (26) in the form

$$\alpha = Ce^{\lambda\theta}, \qquad F = De^{\lambda\theta}, \qquad C, D = \text{const}.$$

Therefore, for the determination of λ, C, D we have the algebraic equations

$$C\lambda^2 + C + 2D\lambda = 0, \qquad D\lambda^2 + D - 2C\lambda = 0.$$

Hence $C = \pm iD$, $\lambda_{1,2} = i(1 \pm \sqrt{2})$, $i^2 = -1$. Then, a real solution for α, F has the form

$$\alpha = C_1 \cos\left(1 \pm \sqrt{2}\right)\theta - C_2 \sin\left(1 \pm \sqrt{2}\right)\theta,$$
$$F = C_1 \sin\left(1 \pm \sqrt{2}\right)\theta + C_2 \cos\left(1 \pm \sqrt{2}\right)\theta,$$

where C_1, C_2 are arbitrary constants.

Then, solution for β, G of the second system (26), according to (25) will take the form:

$$\beta = \left(1 \pm \sqrt{2}\right)\left(C_2 \cos\left(1 \mp \sqrt{2}\right)\theta - C_1 \sin\left(1 \mp \sqrt{2}\right)\theta\right),$$
$$G = \left(1 \pm \sqrt{2}\right)\left(C_2 \sin\left(1 \mp \sqrt{2}\right)\theta + C_1 \cos\left(1 \mp \sqrt{2}\right)\theta\right)$$

For a simplicity let us $C_2 = 0$, and let us take the lower sign, then (23) we can write down in the form:

$$\xi = \cos\left(1 - \sqrt{2}\right)\theta \sin\frac{\sigma}{2k} - \left(1 - \sqrt{2}\right)\sin\left(1 + \sqrt{2}\right)\theta \cos\frac{\sigma}{2k},$$
$$\eta = \sin\left(1 - \sqrt{2}\right)\theta \sin\frac{\sigma}{2k} + \left(1 - \sqrt{2}\right)\cos\left(1 + \sqrt{2}\right)\theta \cos\frac{\sigma}{2k}.$$

The transformations of the slide lines can be written as:

$$x = \mp 2\theta + a\left[\cos\left(1 - \sqrt{2}\right)\theta \sin(\pm\theta + c_i) - \left(1 - \sqrt{2}\right)\sin\left(1 + \sqrt{2}\right)\theta \cos(\pm\theta + c_i)\right]$$
$$+ \sin 2\theta + K_i, \qquad i = 1, 2,$$
$$y = \cos 2\theta + a\left[\sin\left(1 - \sqrt{2}\right)\theta \sin(\pm\theta + c_i) + \left(1 - \sqrt{2}\right)\cos\left(1 + \sqrt{2}\right)\theta \cos(\pm\theta + c_i)\right]. \quad (27)$$

From relations (27) it follows, that new solution has restricted slide lines, and we can use these solutions for description of plastic flows between two rugged slabs.

With the increment of parameter a the thickness of layer increases and becomes approximately equal to $2(h + a)$, where h is a thickness of the initial layer. The new solution that we named S-solution, i.e. obtained from the Prandtl's solution by means of symmetry transformations (20), has the form:

$$\sigma = -kx + a\left[\cos\left(1 - \sqrt{2}\right)\theta \sin\frac{\sigma}{2k} - \left(1 - \sqrt{2}\right)\sin\left(1 + \sqrt{2}\right)\theta \cos\frac{\sigma}{2k}\right] + k\left(1 - y^2\right)^{\frac{1}{2}},$$
$$y = \cos 2\theta + a\left[\sin\left(1 - \sqrt{2}\right)\theta \sin\frac{\sigma}{2k} + \left(1 - \sqrt{2}\right)\cos\left(1 + \sqrt{2}\right)\theta \cos\frac{\sigma}{2k}\right],$$

where a is an arbitrary parameter. For large values of parameter a we have S-solution far different from Prandle's solution. It can be used for analysis of not thin plastic flows.

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The Structure of Cosymmetries and a Simple Proof of Locality for Hierarchies of Symmetries of Odd Order Evolution Systems

Artur SERGYEYEV [1]

The Jacob Blaustein Institute for Desert Research,
Ben-Gurion University of the Negev, Sede Boqer Campus, 84990 Israel
E-mail: *Artur.Sergyeyev@math.slu.cz*

We derive a formula for the leading terms of formal conservation laws (and hence for the leading terms of higher order cosymmetries as well) for a large class of odd order evolution systems in (1+1)-dimensions. This result yields a simple proof of locality for hierarchies of symmetries generated using master symmetries and recursion operators of such systems.

Introduction

It is well known that members of integrable hierarchies of systems of partial differential equations (PDEs) usually have an infinite number of common conservation laws, see e.g. [1–4]. The latter are typically extracted from the Lax pair or zero curvature representation for the system in question, cf. e.g. [4].

On the other hand, one often has to deal with the inverse problem, that is, to check whether a given conserved density or, more broadly, a cosymmetry [1] (in other terminology, a conserved covariant [5]) is indeed shared by the evolution systems associated with the symmetries of the original system. For instance, we may need to verify whether a given physical quantity (charge, momentum, etc.) is invariant under the higher flows of the hierarchy. This is also required when we wish to verify locality of the hierarchy of symmetries produced by the repeated application of a recursion operator or by the repeated commutation with a master symmetry, cf. [2, 3, 6–11].

Checks of this kind can be quite a difficult task, especially when no Lax pair or zero curvature representation for the system in question is yet found. A possible workaround is to use scaling-based arguments, see e.g. [12], but then one has to know the weights of all (homogeneous) cosymmetries for the system in question. Finding these weights is a fairly nontrivial problem, and in the present paper we solve it for the odd order evolution systems in (1+1) dimensions.

First of all, we derive, under some minor technical assumptions, an explicit formula (8) for the leading terms of formal conservation laws for such systems, see Lemma 1 for details. As the directional derivative of a cosymmetry is a formal conservation law, this yields the formula (10) for the weight of any homogeneous cosymmetry of sufficiently high order.

With this in mind, we can find when a homogeneous cosymmetry is shared by the flow associated with a given (homogeneous) symmetry, i.e., when the Lie derivative of this cosymmetry along the symmetry in question vanishes, see Lemma 2 below for details. Using this result, we obtain new easily verifiable sufficient conditions ensuring the locality of symmetries generated with usage of recursion operator or master symmetry, see Propositions 1 and 3 and Corollaries 1 and 2.

For the hierarchies generated using hereditary recursion operators one can [7, 8, 10] prove locality without finding weights of all cosymmetries. Unfortunately, the check of hereditariness can be quite difficult by itself, especially for the multicomponent systems. Proposition 1 and

[1] On leave from Silesian University in Opava, Mathematical Institute, Na Rybníčku 1, 746 01 Opava, Czech Republic.

Corollary 1 of the present work enable one to avoid this check altogether and prove the locality of corresponding hierarchy in a simple and straightforward manner.

The absence of immediate analog of hereditariness for master symmetries makes the technique of [7,8] useless for proving locality of the associated hierarchies. If the only nonlocalities in master symmetries are linear combinations of potentials of the so-called canonical conservation laws, we can prove locality of generated symmetries using the results from [11]. However, if master symmetries involve other nonlocalities as well, one needs other methods for proving locality of symmetries, and the one presented below in Proposition 3 and Corollary 2 provides a convenient and easy-to-use alternative.

1 Basic definitions and structures

Consider an evolution system with constraints (cf. [13–15])

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{F}(x,t,\boldsymbol{u},\boldsymbol{u}_1,\dots,\boldsymbol{u}_{n'},\vec{\omega}), \tag{1}$$

$$\partial\omega_\kappa/\partial x = X_\kappa(x,t,\boldsymbol{u},\boldsymbol{u}_1,\dots,\boldsymbol{u}_h,\vec{\omega}), \tag{2}$$

$$\partial\omega_\kappa/\partial t = T_\kappa(x,t,\boldsymbol{u},\boldsymbol{u}_1,\dots,\boldsymbol{u}_h,\vec{\omega}). \tag{3}$$

Here $\boldsymbol{u} = (u^1,\dots,u^s)^T$, $\boldsymbol{u}_0 \equiv \boldsymbol{u}$, $\boldsymbol{u}_l = \partial^l\boldsymbol{u}/\partial x^l$, $l = 1,2,\dots$, $\vec{\omega} = (\omega_1,\dots,\omega_c)^T$, $\kappa = 1,\dots,c$. The superscript T here and below stands for the matrix transposition.

Let $\mathcal{A}_j$ be an algebra (with respect to the standard multiplication) of locally analytic functions of $x,t,\boldsymbol{u},\dots,\boldsymbol{u}_j,\omega_1,\dots,\omega_c$, cf. [13]. Set $\mathcal{A} = \bigcup_{j=1}^\infty \mathcal{A}_j$. In what follows we assume that X_κ, $T_\kappa \in \mathcal{A}$ for all $\kappa = 1,\dots,c$.

Let $D_x \equiv D = \partial/\partial x + \sum\limits_{i=0}^{\infty} \boldsymbol{u}_{i+1}\partial/\partial\boldsymbol{u}_i + \sum\limits_{\kappa=1}^{c} X_\kappa\partial/\partial\omega_\kappa$ and $D_t = \partial/\partial t + \sum\limits_{i=0}^{\infty} D^i(\boldsymbol{F})\partial/\partial\boldsymbol{u}_i + \sum\limits_{\kappa=1}^{c} T_\kappa\partial/\partial\omega_\kappa$ be the operators of total derivatives on $\mathcal{A}$, cf. [13–15].

As in [13–15], we require that $[D,D_t] = 0$ or, equivalently, $D_t(X_\kappa) = D(T_\kappa)$ for all $\kappa = 1,\dots,c$. In what follows we tacitly assume that the kernel of D in $\mathcal{A}$ is exhausted by the functions of t alone. We shall employ the notation $\operatorname{Im} D$ for the image of D in $\mathcal{A}$.

Consider the set $\operatorname{Mat}_p(\mathcal{A})[\![D^{-1}]\!]$ of *formal series* in powers of D of the form $\mathfrak{H} = \sum\limits_{j=-\infty}^{q} h_j D^j$, where h_j are $p\times p$ matrices with entries from $\mathcal{A}$. The greatest $m\in\mathbb{Z}$ such that $h_m \neq 0$ is called the *degree* of $\mathfrak{H}$ ($\deg\mathfrak{H} \stackrel{\text{def}}{=} m$). If $\det h_m \neq 0$, then $\mathfrak{H}$ is called *nondegenerate*. We assume that $\deg 0 = -\infty$, cf. e.g. [2].

For $\mathfrak{A} = \sum\limits_{i=-\infty}^{k} a_i D^i$ define [16,2] its formal adjoint $\mathfrak{A}^\dagger \stackrel{\text{def}}{=} \sum\limits_{i=-\infty}^{k} (-D)^i \circ a_i^T$. Here $\circ$ stands for the multiplication generalizing the well-known Leibniz rule of calculus. For monomials it reads

$$aD^i \circ bD^j = a\sum_{q=0}^{\infty} \frac{i(i-1)\cdots(i-q+1)}{q!} D^q(b) D^{i+j-q}.$$

It is extended by linearity to the whole $\operatorname{Mat}_p(\mathcal{A})[\![D^{-1}]\!]$ and makes it into an algebra [2,16]. The commutator $[\mathfrak{A},\mathfrak{B}] = \mathfrak{A}\circ\mathfrak{B} - \mathfrak{B}\circ\mathfrak{A}$ further makes $\operatorname{Mat}_p(\mathcal{A})[\![D^{-1}]\!]$ into a Lie algebra. In the sequel we shall omit $\circ$ unless this leads to confusion.

The directional derivative $\vec{f}'$ of $\vec{f}\in\mathcal{A}^q$ is defined as (cf. [13])

$$\vec{f}' = \sum_{\beta,\kappa=1}^{c} \frac{\partial\vec{f}}{\partial\omega_\beta}\big((D-W)^{-1}\big)_{\beta\kappa} \circ \sum_{j=0}^{h} \partial X_\kappa/\partial\boldsymbol{u}_j \circ D^j + \sum_{i=0}^{\infty} \frac{\partial\vec{f}}{\partial\boldsymbol{u}_i} D^i.$$

Here W is a $c\times c$ matrix with the entries $\partial X_\alpha/\partial\omega_\beta$, and $(D-W)^{-1}=D^{-1}\circ\left(\mathbb{I}-W\circ D^{-1}\right)^{-1}=\mathbb{I}D^{-1}+D^{-1}\circ W\circ D^{-1}+\cdots\equiv\mathbb{I}D^{-1}+\sum\limits_{j=-\infty}^{-2}W_jD^j$, where W_j are $c\times c$ matrices with entries from $\mathcal{A}$ and $\mathbb{I}$ denotes a $c\times c$ unit matrix. In other words, for the entries of $(D-W)^{-1}$ we have

$$((D-W)^{-1})_{\alpha\beta}\equiv\delta_{\alpha\beta}D^{-1}+\sum_{j=-\infty}^{-2}(W_j)_{\alpha\beta}D^j,$$

where $(W_j)_{\alpha\beta}$ is the (α,β)-th entry of the matrix W_j, and $\delta_{\alpha\beta}$ is Kronecker delta.

Clearly, for $\boldsymbol{K}\in\mathcal{A}^s$ we have $\boldsymbol{K}'\in\mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$, so we can define the *formal order* of $\boldsymbol{G}\in\mathcal{A}^s$ as $\mathrm{ford}\,\boldsymbol{G}\stackrel{\mathrm{def}}{=}\deg\boldsymbol{G}'$.

Next, define [1,2] the *Lie bracket* for $\boldsymbol{K},\boldsymbol{H}\in\mathcal{A}^s$ as $[\boldsymbol{K},\boldsymbol{H}]=\boldsymbol{H}'[\boldsymbol{K}]-\boldsymbol{K}'[\boldsymbol{H}]$. It is skew-symmetric, but in general for $\boldsymbol{K},\boldsymbol{H}\in\mathcal{A}^s$ the Lie bracket $[\boldsymbol{K},\boldsymbol{H}]$ is not obliged to belong to $\mathcal{A}^s$ unless $\mathcal{A}=\mathcal{A}_{\mathrm{loc}}$, where $\mathcal{A}_{\mathrm{loc}}$ is a subalgebra of *local* (that is, independent of $\vec{\omega}$) functions in $\mathcal{A}$. Note that the restriction of the Lie bracket $[\cdot,\cdot]$ to $\mathcal{A}^s_{\mathrm{loc}}$ satisfies the Jacobi identity, see e.g. [2].

Following [1,2], we say that $\boldsymbol{G}\in\mathcal{A}^s$ is a *symmetry* for (1)–(3), if

$$\partial\boldsymbol{G}/\partial t+[\boldsymbol{F},\boldsymbol{G}]=0,\tag{4}$$

or equivalently [14]

$$D_t(\boldsymbol{G})-\boldsymbol{F}'[\boldsymbol{G}]=0.$$

Let $S_F^{(k)}(\mathcal{A})=\{\boldsymbol{G}\in\mathcal{A}^s|\frac{\partial\boldsymbol{G}}{\partial t}+[\boldsymbol{F},\boldsymbol{G}]=0\text{ and }\mathrm{ford}\boldsymbol{G}\le k\}$, $S_F(\mathcal{A})=\bigcup_{j=0}^\infty S_F^{(j)}(\mathcal{A})$, $\mathrm{Ann}_F(\mathcal{A})=\{\boldsymbol{G}\in S_F(\mathcal{A})|\partial\boldsymbol{G}/\partial t=0\}$, and $\boldsymbol{F}'\equiv\sum\limits_{i=-\infty}^{n}\phi_iD^i$, where $n=\mathrm{ford}\,\boldsymbol{F}$. Set

$$n_0=\begin{cases}1-j,\text{ if }\phi_i'=0\text{ for }i=n-j,\dots,n,\\ 2\text{ otherwise.}\end{cases}$$

A *formal conservation law* of rank m for (1)–(3) is [16,2] a formal series $\mathfrak{L}\in\mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$ such that

$$\deg(D_t(\mathfrak{L})+\mathfrak{L}\circ\boldsymbol{F}'+(\boldsymbol{F}')^\dagger\circ\mathfrak{L})\le\deg\boldsymbol{F}'+\deg\mathfrak{L}-m.\tag{5}$$

Likewise, a *formal symmetry* of rank m for (1)–(3) is [16,2] a formal series $\mathfrak{L}\in\mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$ such that

$$\deg(D_t(\mathfrak{L})-[\boldsymbol{F}',\mathfrak{L}])\le\deg\boldsymbol{F}'+\deg\mathfrak{L}-m.\tag{6}$$

Recall [5,1] that $\boldsymbol{\gamma}\in\mathcal{A}^s$ is called a *cosymmetry* (or a *conserved covariant*) for (1)–(3) if

$$D_t(\boldsymbol{\gamma})+(\boldsymbol{F}')^\dagger(\boldsymbol{\gamma})=0.\tag{7}$$

Set $CS_F(\mathcal{A})=\{\boldsymbol{\gamma}\in\mathcal{A}^s|D_t(\boldsymbol{\gamma})+(\boldsymbol{F}')^\dagger(\boldsymbol{\gamma})=0\}$ and $SCS_F(\mathcal{A})=\{\boldsymbol{\gamma}\in CS_F(\mathcal{A})|\partial\boldsymbol{\gamma}/\partial t=0\}$.

If $\mathrm{ford}\boldsymbol{\gamma}\equiv\deg\boldsymbol{\gamma}'=r\ge n_0$, then $\boldsymbol{\gamma}'$ is easily seen to be a formal conservation law of rank $r-n_0+2$. Likewise, if $\boldsymbol{G}$ is a symmetry of (1)–(3) of formal order $k\ge n_0$, then $\boldsymbol{G}'$ is a formal symmetry of rank $k-n_0+2$, cf. [16,17].

Define (see e.g. [2,3]) the operator of variational derivative

$$\delta/\delta\boldsymbol{u}=\sum_{j=0}^\infty(-D)^j\frac{\partial}{\partial\boldsymbol{u}_j}.$$

If $\rho\in\mathcal{A}_{\mathrm{loc}}$ is a conserved density for (1)–(3), that is, $D_t(\rho)=D(\sigma)$ for some $\sigma\in\mathcal{A}_{\mathrm{loc}}$, then $\boldsymbol{\gamma}=\delta\rho/\delta\boldsymbol{u}$ is a cosymmetry for (1)–(3), cf. e.g. [5,16].

2 NWD systems

Let Φ denote the leading coefficient of $\boldsymbol{F}'$ (so $\boldsymbol{F}' \equiv \Phi D^n + \cdots$). Suppose that $n = \deg \boldsymbol{F}' \geq 2$ and Φ is diagonalizable, i.e., there exists a matrix Γ such that $\Lambda = \Gamma\Phi\Gamma^{-1}$ is diagonal, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_s)$. Further assume that $\det \Phi \neq 0$ and $\lambda_i \neq \lambda_j$ if $i \neq j$. Following [17], we shall call the systems (1)–(3) that enjoy these properties *nondegenerate weakly diagonalizable (NWD)*, because for such systems there exists [16, 17] a unique formal series

$$\mathfrak{T} = \Gamma + \sum_{j=-1}^{-\infty} \Gamma_j \Gamma D^j \in \operatorname{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$$

with the property $\operatorname{diag} \Gamma_j = 0$, $j = -1, -2, \ldots$, such that all coefficients of the formal series $\mathfrak{V} = \mathfrak{T}\boldsymbol{F}'\mathfrak{T}^{-1} + D_t(\mathfrak{T})\mathfrak{T}^{-1}$ are diagonal matrices. Notice that any system (1)–(3) with $s = 1$ is (trivially) NWD with $\mathfrak{T} = 1$.

We have $\mathfrak{V} \equiv \Lambda D^n + \Psi D^{n-1} + \cdots$, where Ψ is a diagonal matrix by construction, so $\Psi \equiv \operatorname{diag}(\psi_1, \ldots, \psi_s)$, $\psi_i \in \mathcal{A}$. For i such that $\psi_i/\lambda_i \in \operatorname{Im} D$ let $z_i \in \mathcal{A}$ be solutions of the equations $D(z_i) = \psi_i/\lambda_i$.

Lemma 1. *Let (1)–(3) be an NWD system with odd $n \equiv \deg \boldsymbol{F}' \geq 3$. Then any its formal conservation law $\mathfrak{L}$ of degree k and of rank $m \geq 2$ can be written in the form*

$$\mathfrak{L} = \mathfrak{T}^\dagger \circ (b_k \Lambda^{k/n} D^k + d_k D^{k-1}) \circ \mathfrak{T} + \mathfrak{N}, \qquad \deg \mathfrak{N} < k - 1. \tag{8}$$

Here $\Lambda^{k/n} = \operatorname{diag}(\lambda_1^{k/n}, \ldots, \lambda_s^{k/n})$, $b_k = \operatorname{diag}(c_1(t)\exp(2z_1/n), \ldots, c_s(t)\exp(2z_s/n))$, d_k is a diagonal $s \times s$ matrix; $c_i(t) = 0$ if $\psi_i/\lambda_i \notin \operatorname{Im} D$.

Proof. Set $\tilde{\mathfrak{L}} = (\mathfrak{T}^\dagger)^{-1} \circ \mathfrak{L} \circ \mathfrak{T}^{-1}$. Then equation (5) is [16] equivalent to the following:

$$\deg(D_t(\tilde{\mathfrak{L}}) + \tilde{\mathfrak{L}} \circ \mathfrak{V} + \mathfrak{V}^\dagger \circ \tilde{\mathfrak{L}}) \leq \deg \boldsymbol{F}' + \deg \tilde{\mathfrak{L}} - m. \tag{9}$$

Let $\tilde{\mathfrak{L}} \equiv L_k D^k + L_{k-1} D^{k-1} + \cdots$. Then equating to zero the coefficient at D^{n+k} in (9) yields $[L_k, \Lambda] = 0$, hence L_k is diagonal: $L_k = \operatorname{diag}(h_1, \ldots, h_s)$.

Diagonal entries of the matrix $[L_{k-1}, \Lambda]$ are zeros, so equating to zero the diagonal part of the coefficient at D^{m+k-1} in (9) yields

$$kh_i D(\lambda_i) - n\lambda_i D(h_i) + 2h_i\psi_i = 0, \qquad i = 1, \ldots, s.$$

Solving these equations yields $h_i = c_i(t)\lambda_i^{k/n}\exp(2z_i/n)$, where c_i are arbitrary functions of t; if $\psi_i/\lambda_i \notin \operatorname{Im} D$ for some i, then we have $c_i = 0$ for these values of i.

Finally, equating to zero the antidiagonal part of the coefficient at D^{m+k-1} in (9) yields $[L_{k-1}, \Lambda] = 0$, so L_{k-1} is a diagonal matrix. ■

Notice that using (8) for $\mathfrak{L} = \boldsymbol{\gamma}'$ one can readily find the form of leading terms for cosymmetries $\boldsymbol{\gamma}$ of formal order $q \geq n_0$. This complements known results (cf. e.g. [16, 18, 19]) and can be very useful e.g. while searching for cosymmetries and conserved densities for (1)–(3).

Remark 1. If $\psi_i/\lambda_i \notin \operatorname{Im} D$ for all $i = 1, \ldots, s$, then the system (1)–(3) has no formal conservation laws from $\operatorname{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$ of rank $m \geq 2$ [16]. However, if (1)–(3) has a nondegenerate formal symmetry of rank not lower than $n + 2$, then [16] $\chi_i \equiv \psi_i/\lambda_i$ satisfy $D_t(\chi_i) = D(\xi_i)$, where $\xi_i \in \mathcal{A}$, for all $i = 1, \ldots, c$, and we can proceed as follows.

Let I_0 be the set of all $i \in \{1, \ldots, c\}$ such that $\chi_i \notin \operatorname{Im} D$. Next, let I be a maximal subset of I_0 such that χ_i for $i \in I$ are linearly independent. Introduce additional nonlocal variables $\tilde{\omega}_i$, $i \in I$, such that $\partial\tilde{\omega}_i/\partial t = \xi_i$ and $\partial\tilde{\omega}_i/\partial x = \chi_i$. Then $\chi_i \in \operatorname{Im} D$, as $\chi_i = D(\tilde{\omega}_i)$, so we can set $z_i = \tilde{\omega}_i$, and moreover [20], $\ker D$ is still exhausted by functions of t, so Lemma 1 remains valid in this extended setting.

3 Homogeneous cosymmetries of NWD systems

Following the literature (cf. e.g. [5,1,7]), define the Lie derivatives along $\boldsymbol{G} \in \mathcal{A}^s$ by the following formulae: a) $L_{\boldsymbol{G}}(f) = f'[\boldsymbol{G}]$ for $f \in \mathcal{A}$, b) $L_{\boldsymbol{G}}(\boldsymbol{H}) = [\boldsymbol{G}, \boldsymbol{H}]$ for a symmetry $\boldsymbol{H}$, c) $L_{\boldsymbol{G}}(\boldsymbol{\gamma}) = \boldsymbol{\gamma}'[\boldsymbol{G}] + (\boldsymbol{G}')^\dagger(\boldsymbol{\gamma})$ for a cosymmetry $\boldsymbol{\gamma}$.

Let (1)–(3) be an NWD system such that $\partial X_\kappa/\partial t = \partial T_\kappa/\partial t = 0$ for all $\kappa = 1, \dots, c$. Consider a scaling vector field $\boldsymbol{S} = x\boldsymbol{u}_1 + \beta\boldsymbol{u}$, where β is a constant. A function $f \in \mathcal{A}$ such that $\partial f/\partial t = 0$ (resp. some other quantity $\mathcal{X}$ (a symmetry, a cosymmetry etc.) such that $\partial\mathcal{X}/\partial t = 0$) is called *$\boldsymbol{S}$-homogeneous of weight μ* if $L_{\boldsymbol{S}}(f) - xD(f) = \mu f$ (resp. $L_{\boldsymbol{S}}(\mathcal{X}) = \mu\mathcal{X}$) for some constant μ. We write this as $\operatorname{wt} f = \mu$ (resp. $\operatorname{wt}\mathcal{X} = \mu$), cf. e.g. [7,8].

Motivated by the prototypic example of the Korteweg–de Vries equation $u_t = u_3 + uu_1$, let us introduce the following

Definition 1. Let (1)–(3) be an NWD system such that a) $\partial X_\kappa/\partial t = \partial T_\kappa/\partial t = 0$ for all $\kappa = 1, \dots, c$; b) $n \equiv \deg \boldsymbol{F}'$ is odd and $n \geq 3$; c) $\partial\Phi/\partial t = 0$ (recall that Φ is the leading coefficient of $\boldsymbol{F}'$). Further assume that there exists a scaling $\boldsymbol{S} = x\boldsymbol{u}_1 + \beta\boldsymbol{u}$, where β is a constant, such that a) all entries of Φ are $\boldsymbol{S}$-homogeneous, all of them have the same weight $\tilde{\alpha}$, and $\tilde{\alpha} \neq -n$; b) it is possible to choose z_i, introduced in the previous section, in such a way that $\partial z_i/\partial t = 0$, $\exp(2z_i/n)$ are $\boldsymbol{S}$-homogeneous, and $\operatorname{wt}(\exp(2z_i/n)) = \operatorname{wt}(\exp(2z_j/n))$ for all $i, j = 1, \dots, s$. Then we shall call the system (1)–(3) *weakly KdV-like* with the scaling $\boldsymbol{S}$.

Remark 2. Let $\boldsymbol{\zeta} \in SCS_F(\mathcal{A})$ be an $\boldsymbol{S}$-homogeneous time-independent cosymmetry of a weakly KdV-like system (1)–(3), and $k \equiv \deg \boldsymbol{\zeta}' \geq n_0$. Then it is immediate from (8) that

$$\operatorname{wt}(\boldsymbol{\zeta}) = \operatorname{wt}(\exp(2z_1/n)) + ((\tilde{\alpha} + n)/n)k + 2\beta - 1. \tag{10}$$

Note that this formula is very useful on its own, for instance, for finding nonlocal parts of recursion operators and master symmetries for weakly KdV-like systems, cf. [7,8].

Lemma 2. *Assume that* (1)–(3) *is weakly KdV-like with a scaling $\boldsymbol{S}$. Let $\mathcal{L}$ be a subspace of $SCS_F(\mathcal{A})$, and $\mathcal{M}_{\mathcal{L}} = \{\boldsymbol{\zeta} \in \mathcal{L} | \deg \boldsymbol{\zeta}' < n_0$ and $\boldsymbol{\zeta}$ is $\boldsymbol{S}$-homogeneous$\}$.*

Then $L_{\boldsymbol{P}}(\boldsymbol{\gamma}) = 0$ for any $\boldsymbol{S}$-homogeneous $\boldsymbol{P} \in \operatorname{Ann}_F(\mathcal{A})$ and any $\boldsymbol{S}$-homogeneous $\boldsymbol{\gamma} \in SCS_F(\mathcal{A})$ such that $L_{\boldsymbol{P}}(\boldsymbol{\gamma}) \in \mathcal{L}$, $p \equiv \operatorname{ford} \boldsymbol{P} \geq n_0$, $\operatorname{wt}\boldsymbol{\zeta} \neq p(\tilde{\alpha} + n)/n + \operatorname{wt}\boldsymbol{\gamma}$ for all $\boldsymbol{\zeta} \in \mathcal{M}_{\mathcal{L}}$, $\boldsymbol{\zeta} \neq 0$, and either p is odd and $\deg\boldsymbol{\gamma}' \geq n_0$ or there exists no integer j such that $j = p + n(\operatorname{wt}\boldsymbol{\gamma} - 2\beta + 1 - \operatorname{wt}(\exp(2z_1/n)))/(\tilde{\alpha} + n)$ and $n_0 \leq j \leq \max(p + n_0 - 2, p + \deg\boldsymbol{\gamma}')$.

Proof. Consider first the case when p is odd and $\deg\boldsymbol{\gamma}' \geq n_0$. Using (10) and the formula $\operatorname{wt}(\boldsymbol{P}) = p(\tilde{\alpha} + n)/n$ from [17], we obtain $\operatorname{wt} L_{\boldsymbol{P}}(\boldsymbol{\gamma}) = \operatorname{wt}(\boldsymbol{P}) + \operatorname{wt}(\boldsymbol{\gamma}) = \operatorname{wt}(\exp(2z_1/n)) + 2\beta - 1 + (p + \deg\boldsymbol{\gamma}')(\tilde{\alpha} + n)/n$. On the other hand, as p is odd and $\deg\boldsymbol{\gamma}' \geq n_0$, a straightforward computation with usage of (8) shows that $\deg(L_{\boldsymbol{P}}(\boldsymbol{\gamma}))' \leq p + \deg\boldsymbol{\gamma}' - 1$. Along with our assumption that $L_{\boldsymbol{P}}(\boldsymbol{\gamma}) \in \mathcal{L}$, this implies that $\deg(L_{\boldsymbol{P}}(\boldsymbol{\gamma}))' < n_0$, and thus $L_{\boldsymbol{P}}(\boldsymbol{\gamma}) \in \mathcal{M}_{\mathcal{L}}$. Indeed, otherwise (10) yields $\operatorname{wt} L_{\boldsymbol{P}}(\boldsymbol{\gamma}) = \operatorname{wt}(\exp(2z_1/n)) + 2\beta - 1 + \deg(L_{\boldsymbol{P}}(\boldsymbol{\gamma}))'(\tilde{\alpha} + n)/n \neq \operatorname{wt}(\exp(2z_1/n)) + 2\beta - 1 + (p + \deg\boldsymbol{\gamma}')(\tilde{\alpha} + n)/n$. Thus, $L_{\boldsymbol{P}}(\boldsymbol{\gamma}) \in \mathcal{M}_{\mathcal{L}}$, but $\operatorname{wt}\boldsymbol{\zeta} \neq \operatorname{wt} L_{\boldsymbol{P}}(\boldsymbol{\gamma}) = \operatorname{wt}(\exp(2z_1/n)) + 2\beta - 1 + (p + \deg\boldsymbol{\gamma}')(\tilde{\alpha} + n)/n$ for all $\boldsymbol{\zeta} \in \mathcal{M}_{\mathcal{L}}$, $\boldsymbol{\zeta} \neq 0$, and hence $L_{\boldsymbol{P}}(\boldsymbol{\gamma}) = 0$.

Likewise, if there is no integer j such that $j = p + n(\operatorname{wt}\boldsymbol{\gamma} - 2\beta + 1 - \operatorname{wt}(\exp(2z_1/n)))/(\tilde{\alpha} + n)$ and $n_0 \leq j \leq \max(p + n_0 - 2, p + \deg\boldsymbol{\gamma}')$, then there exists no $\boldsymbol{S}$-homogeneous cosymmetry $\boldsymbol{\zeta} \in \mathcal{L}$ with $\deg\boldsymbol{\zeta}' \geq n_0$ such that $\operatorname{wt} L_{\boldsymbol{P}}(\boldsymbol{\gamma}) = \operatorname{wt}\boldsymbol{\zeta}$, so $L_{\boldsymbol{P}}(\boldsymbol{\gamma}) \in \mathcal{M}_{\mathcal{L}}$, and proceeding as above we conclude that $L_{\boldsymbol{P}}(\boldsymbol{\gamma}) = 0$. ■

4 Applications to recursion operators and master symmetries

Recall [5, 2, 1] that $\mathfrak{R} \in \operatorname{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$ is called a (*adjoint*) *recursion operator* for (1)–(3), if $D_t(\mathfrak{R}) - [\boldsymbol{F}', \mathfrak{R}] = 0$ (resp. $D_t(\mathfrak{R}) + [(\boldsymbol{F}')^\dagger, \mathfrak{R}] = 0$), and a (*inverse*) *Noether operator*, if $D_t(\mathfrak{R}) - \mathfrak{R} \circ (\boldsymbol{F}')^\dagger - \boldsymbol{F}' \circ \mathfrak{R} = 0$ (resp. $D_t(\mathfrak{R}) + \mathfrak{R} \circ \boldsymbol{F}' + (\boldsymbol{F}')^\dagger \circ \mathfrak{R} = 0$).

Consider an operator of the form

$$\mathfrak{R} = \sum_{i=0}^{r} a_i D^i + \sum_{\kappa=1}^{k} \boldsymbol{G}_\kappa \otimes D^{-1} \circ \boldsymbol{\gamma}_\kappa, \tag{11}$$

where $r \geq 0$, a_i are $s \times s$ matrices with local entries, $\boldsymbol{G}_\kappa \in \mathcal{A}^s_{\mathrm{loc}}$, $\boldsymbol{\gamma}_\kappa = \delta\rho_\kappa/\delta\boldsymbol{u}$, and $\rho_\kappa \in \mathcal{A}_{\mathrm{loc}}$ are local conserved densities for (1)–(3), i.e., $D_t(\rho_\kappa) = D(\sigma_\kappa)$ for some $\sigma_\kappa \in \mathcal{A}_{\mathrm{loc}}$.

Proposition 1. *Suppose that a weakly KdV-like system* (1)–(3) *with a scaling* $\boldsymbol{S}$ *has a recursion (resp. inverse Noether) operator of the form* (11) *with* $\boldsymbol{\gamma}_\kappa \in SCS_F(\mathcal{A}_{\mathrm{loc}})$*, and the requirements of Lemma 2 are met for a symmetry* $\boldsymbol{P} \in \mathrm{Ann}_F(\mathcal{A}_{\mathrm{loc}})$ *and for all cosymmetries* $\boldsymbol{\gamma}_\kappa$*,* $\kappa = 1, \ldots, k$*.*

Then $\mathfrak{R}(\boldsymbol{P}) \in S_F(\mathcal{A}_{\mathrm{loc}})$ *(resp.* $\mathfrak{R}(\boldsymbol{P}) \in CS_F(\mathcal{A}_{\mathrm{loc}})$*), i.e.,* $\mathfrak{R}(\boldsymbol{P})$ *is again a* local *symmetry (resp. cosymmetry) for* (1)–(3)*.*

Proof. We have to prove that $\boldsymbol{\gamma}_\kappa \cdot \boldsymbol{P} = D(\eta_\kappa)$ for some local η_κ, or equivalently [2] that $\delta(\boldsymbol{\gamma}_\kappa \cdot \boldsymbol{P})/\delta\boldsymbol{u} = (\boldsymbol{\gamma}'_\kappa)^\dagger(\boldsymbol{P}) + (\boldsymbol{P}')^\dagger(\boldsymbol{\gamma}_\kappa) = 0$. But by Lemma 2 $L_{\boldsymbol{P}}(\boldsymbol{\gamma}_\kappa) = 0$, so $(\boldsymbol{P}')^\dagger(\boldsymbol{\gamma}_\kappa) = -\boldsymbol{\gamma}'_\kappa[\boldsymbol{P}]$, whence $\delta(\boldsymbol{\gamma}_\kappa \cdot \boldsymbol{P})/\delta\boldsymbol{u} = ((\boldsymbol{\gamma}'_\kappa)^\dagger - \boldsymbol{\gamma}'_\kappa)(\boldsymbol{P})$. As $\boldsymbol{\gamma}_\kappa = \delta\rho_\kappa/\delta\boldsymbol{u}$ implies (see e.g. [5]) $(\boldsymbol{\gamma}'_\kappa)^\dagger = \boldsymbol{\gamma}'_\kappa$, we have $\delta(\boldsymbol{\gamma}_\kappa \cdot \boldsymbol{P})/\delta\boldsymbol{u} = 0$, as desired. ■

Consider now an operator of the form

$$\mathfrak{N} = \sum_{i=0}^{r} b_i D^i + \sum_{\kappa=1}^{k} \boldsymbol{\gamma}_\kappa \otimes D^{-1} \circ \boldsymbol{G}_\kappa, \tag{12}$$

where $r \geq 0$, b_i are $s \times s$ matrices with local entries, $\boldsymbol{\gamma}_\kappa \in \mathcal{A}^s_{\mathrm{loc}}$, $\boldsymbol{G}_\kappa \in \mathrm{Ann}_F(\mathcal{A}_{\mathrm{loc}})$ are time-independent local symmetries for (1)–(3). In complete analogy with Proposition 1 we obtain the following result.

Proposition 2. *Let a weakly KdV-like system* (1)–(3) *with a scaling* $\boldsymbol{S}$ *have an adjoint recursion (resp. Noether) operator* $\mathfrak{N}$ *of the form* (12)*. Assume that the requirements of Lemma 2 are met for a* local *time-independent cosymmetry* $\boldsymbol{\zeta} \in SCS_F(\mathcal{A}_{\mathrm{loc}})$ *and for all symmetries* $\boldsymbol{G}_\kappa$*,* $\kappa = 1, \ldots, k$*, and let* $\boldsymbol{\zeta} = \delta\chi/\delta\boldsymbol{u}$ *for some local function* χ*.*

Then $\mathfrak{N}(\boldsymbol{\zeta})$ *is again a* local *cosymmetry (resp. symmetry) for* (1)–(3)*.*

Now let us turn to the case when (1)–(3) has a master symmetry of degree one or equivalently [6] a symmetry linear in time t.

Proposition 3. *Let a weakly KdV-like system* (1)–(3) *with a scaling* $\boldsymbol{S}$ *be such that* $\partial\boldsymbol{F}/\partial t = 0$*,* $\partial\boldsymbol{F}/\partial\omega_\kappa = 0$ *and* $X_\kappa, T_\kappa \in \mathcal{A}_{\mathrm{loc}}$ *for all* $\kappa = 1, \ldots, c$*. Further suppose that* (1)–(3) *has a symmetry* $\boldsymbol{K} = \boldsymbol{\tau} + t[\boldsymbol{\tau}, \boldsymbol{F}] \in S_F(\mathcal{A})$*, where* $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \sum_{\kappa=1}^{c} \boldsymbol{W}_\kappa\omega_\kappa$*,* $\boldsymbol{\tau}_0 \in \mathcal{A}^s_{\mathrm{loc}}$*,* $\partial\boldsymbol{\tau}_0/\partial t = 0$*, and* $\boldsymbol{W}_\kappa \in \mathrm{Ann}_F(\mathcal{A}_{\mathrm{loc}})$ *for all* $\kappa = 1, \ldots, c$*. Let the requirements of Lemma 2 be met for a* local *time-independent symmetry* $\boldsymbol{P} \in \mathrm{Ann}_F(\mathcal{A}_{\mathrm{loc}})$ *and for all cosymmetries* $\boldsymbol{\gamma}_\kappa = \delta X_\kappa/\delta\boldsymbol{u} \in SCS_F(\mathcal{A}_{\mathrm{loc}})$ *with* $\kappa \in \{1, \ldots, c\}$ *such that* $\boldsymbol{W}_\kappa \neq 0$*. Finally, suppose that* $[\boldsymbol{P}, \boldsymbol{W}_\kappa] = 0$ *for all* $\kappa = 1, \ldots, c$*.*

Then $[\boldsymbol{K}, \boldsymbol{P}]$ *is again a* local *symmetry for* (1)–(3)*.*

Proof. By virtue of the assumptions made $[\boldsymbol{\tau}, \boldsymbol{F}]$ is local [11], and hence so is $[[\boldsymbol{\tau}, \boldsymbol{F}], \boldsymbol{P}]$. Since $[\boldsymbol{P}, \boldsymbol{W}_\kappa] = 0$, it remains to prove that $\omega'_\kappa[\boldsymbol{P}]$ are local or, equivalently, that $\delta(\boldsymbol{\gamma}_\kappa \cdot \boldsymbol{P})/\delta\boldsymbol{u} = 0$ for all $\kappa = 1, \ldots, c$ such that $\boldsymbol{W}_\kappa \neq 0$. This is done as in the proof of Proposition 1 above. ■

Let $\mathrm{ad}_{\boldsymbol{H}}(\boldsymbol{G}) \equiv [\boldsymbol{H}, \boldsymbol{G}]$. Set $\boldsymbol{Q}_j = \mathrm{ad}^j_{\boldsymbol{K}}(\boldsymbol{P})$. Clearly, under minor technical assumptions the repeated application of Proposition 1 (resp. Proposition 3) enables one to prove locality for all symmetries $\mathfrak{R}^j(\boldsymbol{P})$ (resp. $\boldsymbol{Q}_j$), $j = 1, 2, \ldots$, where $\mathfrak{R}$ is a recursion operator for (1)–(3). In particular, the following assertions hold.

Corollary 1. *Under the assumptions of Proposition 1 suppose that $\mathfrak{R}$ is a recursion operator for (1)–(3), r is even, the coefficients of $\mathfrak{R}$ are time-independent, and $\mathfrak{R}$ is $\boldsymbol{S}$-homogeneous of some weight μ, i.e., $L_{\boldsymbol{S}}(\mathfrak{R}) \equiv \mathfrak{R}'[\boldsymbol{S}] - [\boldsymbol{S}', \mathfrak{R}] = \mu\mathfrak{R}$. Further assume that $p \equiv \operatorname{ford} \boldsymbol{P}$ is odd, and for any $j \in \mathbb{N}$ and all $\kappa = 1, \dots, k$ $L_{\mathfrak{R}^j(\boldsymbol{P})}(\boldsymbol{\gamma}_\kappa) \in \mathcal{L}$, $\deg \boldsymbol{\gamma}_\kappa \geq n_0$, and $\operatorname{wt} \boldsymbol{\zeta} \neq (p + rj + \deg \boldsymbol{\gamma}'_\kappa)(\tilde{\alpha} + n)/n + \operatorname{wt}(\exp(2z_1/n)) + 2\beta - 1$ for all $\boldsymbol{\zeta} \in \mathcal{M}_{\mathcal{L}}$, $\boldsymbol{\zeta} \neq 0$.*

Then the symmetries $\mathfrak{R}^j(\boldsymbol{P})$ are local for all $j = 1, 2, 3, \dots$.

Corollary 2. *Under the assumptions of Proposition 3 suppose that $q - n$ is even, $\boldsymbol{\tau}$ is $\boldsymbol{S}$-homogeneous, i.e., $[\boldsymbol{S}, \boldsymbol{\tau}] = \mu\boldsymbol{\tau}$ for some constant μ, $q = \operatorname{ford}[\boldsymbol{\tau}, \boldsymbol{F}] > \max(\operatorname{ford} \boldsymbol{\tau}, n)$, $p \equiv \operatorname{ford} \boldsymbol{P} > n$, p is odd, $[\boldsymbol{Q}_j, \boldsymbol{W}_\kappa] = 0$ and $[\boldsymbol{Q}_j, [\boldsymbol{\tau}, \boldsymbol{F}]] = 0$ for all $\kappa = 1, \dots, c$ and all $j = 0, 1, \dots, b-1$. Further assume that for all $j = 0, \dots, b-1$ and all $\kappa = 1, \dots, c$ such that $\boldsymbol{W}_\kappa \neq 0$ we have $L_{\boldsymbol{Q}_j}(\boldsymbol{\gamma}_\kappa) \in \mathcal{L}$, $\deg \boldsymbol{\gamma}_\kappa \geq n_0$, and $\operatorname{wt} \boldsymbol{\zeta} \neq (p + (q-n)j + \deg \boldsymbol{\gamma}'_\kappa)(\tilde{\alpha} + n)/n + \operatorname{wt}(\exp(2z_1/n)) + 2\beta - 1$ for all $\boldsymbol{\zeta} \in \mathcal{M}_{\mathcal{L}}$, $\boldsymbol{\zeta} \neq 0$.*

Then $\boldsymbol{Q}_j = \operatorname{ad}^j_{\boldsymbol{K}}(\boldsymbol{P})$ are local for all $j = 1, 2, \dots, b$.

Thus, proving locality for the hierarchies generated using master symmetries is slightly more complicated, as we must proceed inductively and make sure that the conditions of Corollary 2 are satisfied for all $b = 1, 2, 3, \dots$. However, this considerably simplifies verification of the relations $[\boldsymbol{Q}_j, \boldsymbol{W}_\kappa] = 0$ and $[\boldsymbol{Q}_j, [\boldsymbol{\tau}, \boldsymbol{F}]] = 0$, because we can use the information on locality obtained at the previous step. There are various ways of proving these relations, see e.g. Proposition 5 of [17] for a scaling-based approach. Note that the conditions $q = \operatorname{ford}[\boldsymbol{\tau}, \boldsymbol{F}] > \max(\operatorname{ford} \boldsymbol{\tau}, n)$ and $p > n$ in Corollary 2 ensure [17] that $\operatorname{ford} \boldsymbol{Q}_{j+1} > \operatorname{ford} \boldsymbol{Q}_j$.

5 Examples

Example 1. Consider the Hirota–Satsuma system (see [21, 7] and references therein)

$$u_t = (1/2)u_3 + 3uu_1 - 6vv_1, \qquad v_t = -v_3 - 3uv_1.$$

and its recursion operator

$$\mathfrak{R} = \begin{pmatrix} D^3/2 + D \circ u + uD & D \circ v + vD \\ D \circ v + vD & D^3/2 + D \circ u + uD \end{pmatrix} \circ \begin{pmatrix} D/2 + D^{-1} \circ u + uD^{-1} & -2D^{-1} \circ v \\ -2vD^{-1} & -2D \end{pmatrix}.$$

The nonlocal part of this operator can be written [7] as $\mathfrak{R}_- = \boldsymbol{G}_1 \otimes D^{-1} \circ \boldsymbol{\gamma}_1 + \boldsymbol{G}_2 \otimes D^{-1} \circ \boldsymbol{\gamma}_2$, where $\boldsymbol{G}_1 = ((1/2)u_3 + 3uu_1 - 6vv_1, -v_3 - 3uv_1)^T$, $\boldsymbol{G}_2 = (u_1, v_1)^T$, $\boldsymbol{\gamma}_1 = (1, 0) = \delta\rho_1/\delta\boldsymbol{u}$ for $\rho_1 = u$, $\boldsymbol{\gamma}_2 = (u, -2v) = \delta\rho_2/\delta\boldsymbol{u}$ for $\rho_2 = u^2/2 - v^2$. It is immediate that the system in question is weakly KdV-like with the scaling $\boldsymbol{S} = x\boldsymbol{u}_1 + 2\boldsymbol{u}$, and the requirements of Proposition 1 are met for any x, t-independent local $\boldsymbol{S}$-homogeneous generalized symmetry $\boldsymbol{P}$ of odd formal order not lower than one, if we set $\mathcal{L} = \{\boldsymbol{\zeta} \in SCS_F(\mathcal{A}_{\mathrm{loc}}) | \partial\boldsymbol{\zeta}/\partial x = 0\}$, and thus the symmetries $\mathfrak{R}^j(\boldsymbol{P})$ are local for all $j = 1, 2, \dots$. In particular, $\mathfrak{R}^j(\boldsymbol{G}_\kappa)$ are local for all $j \in \mathbb{N}$ and $\kappa = 1, 2$.

Example 2. Consider an NWD system with constraints made up from the Schwarzian Korteweg–de Vries equation $u_t = u_3 - (3/2)u_2^2/u_1$, see e.g. [3, 7] and references therein, and the equations defining the potential ω of the conserved density $\rho = (1/2)u_2^2/u_1^2$:

$$u_t = u_3 - (3/2)u_2^2/u_1,$$
$$\partial\omega/\partial t = u_2u_4/u_1^2 - (1/2)u_3^2/u_1^2 - 2u_2^2u_3/u_1^3 + (9/8)u_2^4/u_1^4, \qquad \partial\omega/\partial x = (1/2)u_2^2/u_1^2.$$

It is weakly KdV-like with the scaling $\boldsymbol{S} = xu_1$ and satisfies the requirements of Proposition 3 with $\boldsymbol{P}$ being any x, t, u-independent local $\boldsymbol{S}$-homogeneous symmetry of odd formal order not lower than three, $\mathcal{L} = \{\boldsymbol{\zeta} \in SCS_F(\mathcal{A}_{\mathrm{loc}}) | \partial\boldsymbol{\zeta}/\partial x = 0 \text{ and } \partial\boldsymbol{\zeta}/\partial u = 0\}$, and master symmetry $\boldsymbol{\tau} = u_2 + x(u_3 - (3/2)u_2^2/u_1) - u_1\omega$. Hence by Proposition 3 and Corollary 2 the symmetries $\operatorname{ad}^j_{\boldsymbol{K}}(u_3 - (3/2)u_2^2/u_1) = \operatorname{ad}^j_{\boldsymbol{\tau}}(u_3 - (3/2)u_2^2/u_1)$ are local for all $j = 1, 2, \dots$.

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Application of the Orbits Method to Integration of Linear Differential Equations with Non-Commutative Symmetries

Igor V. SHIROKOV

Physics Department, Omsk State University, 55-a Mira Ave., 644077 Omsk, Russia
E-mail: *shirokov@univer.omsk.su*

Linear PDE's are considered in the article. The equation to be solved is reduced to the equation on Lagrange submanifold to the coadjoint orbit. As a result we obtained necessary and sufficient conditions of integrability for invariant linear differential equations.

1 Introduction

The most widespread integration method the linear partial differential equations

$$H(z,\partial_z)\varphi(z) = E\varphi(z), \qquad z \in \mathbb{Z} \subset \mathbb{R}^N, \quad \varphi(z) \in C^\infty(\mathbb{Z}) \tag{1}$$

is separation of variables [1] that uses commutative algebra of symmetries of equation (1). The main goal in separation of variables is to solve eigenvalue problem

$$Y_\mu\psi(z) = \lambda_\mu\psi(z), \qquad Y_1 = H, \qquad \lambda_1 = E, \qquad [Y_\mu, Y_\nu] = 0. \tag{2}$$

Commutative algebra of symmetries is not a sufficient condition for the separation of variables. It is necessary for the solvability of system (2) that operators Y_μ must meet additional requirements [2].

Let linear differential equation (1) allow a non-Abelian symmetry group G. We shall now investigate the most efficient way how the non-Abelian symmetry group can be used for the integration of that equation.

2 Main theorem

Let $M = G/H$ be the regular orbit of group action G on Z; x are local coordinates on homogeneous space M, $X_A(x,\partial_x) \equiv X_A^a(x)\partial_{x^a}$ are generators of the transformation group which form Lie algebra $\mathcal{G}$: $[X_A, X_B] = C_{AB}^C X_C$, y are the invariants of the transformation group ($X_A y = 0$), $z \leftrightarrow (x,y)$.

Theorem 1. *Equation* (1) *with N variables that allows a Lie symmetry group G can be reduced to linear differential equations with N' variables:*

$$N' = N - \dim M + d(M), \tag{3}$$

where

$$d(M) = \frac{1}{2}\dim \mathcal{G}/\mathcal{G}^\lambda - \dim \mathcal{H}/\mathcal{H}^\lambda. \tag{4}$$

Here $\mathcal{H}$ is Lie algebra of isotropy subgroup H, λ is regular element of the space $\mathcal{H}^\perp = \{f \in \mathcal{G}^* \mid \langle f, \mathcal{H}\rangle = 0\} \subset \mathcal{G}^*$, $\mathcal{G}^\lambda$ is annihilator of the covector $\lambda \in \mathcal{H}^\perp$, $\mathcal{H}^\lambda = \mathcal{G}^\lambda \cap \mathcal{H}$.

Non-negative integer $d(M)$, which is called *defect* of homogeneous space M can be easily calculated using structural constants of Lie algebra $\mathcal{G}$ and its subalgebra $\mathcal{H}$ [3]:

$$d(M) = \frac{1}{2}\operatorname{rank}\langle\lambda, [\mathcal{G}, \mathcal{G}]\rangle - \operatorname{rank}\langle\lambda, [\mathcal{G}, \mathcal{H}]\rangle.$$

In the case when N' from (3) equals 1 we obtain ordinary differential equation and we call initial equation *integrable*.

Homogeneous spaces for which $d(M) = 0$ are called *commutative*. The class of commutative spaces includes, in particular, all of the symmetric and weakly symmetries spaces [4].

Let us show the main constructions, sufficient for the proof of the Theorem 1 [5].

Let us consider associative algebra D, generated by the finite set of elements $\{E_\mu\}$. Let $\mathcal{F}$ be a linear space with the basis $\{E_\mu\}$, $S(\mathcal{F})$ is a symmetric algebra, then $S(\mathcal{F}) \simeq D$.

We define a skew-symmetric bilinear form (commutator)

$$[A, B] \in S(\mathcal{F}), \qquad \forall\, A, B \in \mathcal{F},$$

such that Jacobi identities are satisfied

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B] = 0, \qquad \forall\, A, B, C \in \mathcal{F}.$$

(We assume here that the form acts on $S(\mathcal{F})$ according to the Leibnitz rule).

For basis elements we have:

$$[E_\mu, E_\nu] = \Omega_{\mu\nu}(E) \in S(\mathcal{F}), \qquad \mu, \nu = 1, \ldots, \dim \mathcal{F}.$$

Algebra $S(\mathcal{F})$ is called *functional algebra* ($\mathcal{F}$-algebra). In particular, if $\Omega_{\mu\nu}$ in formula (4) appears to be a quadratic polynomial, then we call it a quadratic algebra, and so on.

The space of smooth functions on $\mathcal{F}^*$ is Poisson algebra with the Poisson bracket:

$$\{\varphi, \psi\}^{\mathcal{F}}(g) = \Omega_{\mu\nu}(g)\frac{\partial\varphi(g)}{\partial g_\mu}\frac{\partial\psi(g)}{\partial g_\nu}, \qquad g = g_\mu E^\mu \in \mathcal{F}^*, \qquad \varphi, \psi \in C^\infty(\mathcal{F}^*).$$

The Poisson–Lie bracket is defined on the space of functions on $\mathcal{G}^*$:

$$\{\varphi, \psi\}(f) = C^C_{AB} f_C \frac{\partial\varphi(f)}{\partial f_A}\frac{\partial\psi(f)}{\partial f_B}, \qquad f = f_A e^A \in \mathcal{G}^*, \qquad \varphi, \psi \in C^\infty(\mathcal{G}^*). \tag{5}$$

Every Lie group G, acting on homogeneous space M, corresponds to $\mathcal{F}$-algebra of invariant linear operators on $C^\infty(M)$. Let us note as $L_\mu(x, \partial_x)$ independent operators, forming the $\mathcal{F}$-algebra

$$[L_\mu(x, \partial_x), X_A(x, \partial_x)] = 0, \qquad \mu = 1, \ldots, \dim \mathcal{F}, \qquad A = 1, \ldots, \dim \mathcal{G},$$
$$[L_\mu, L_\nu] = \Omega_{\mu\nu}(L) \in S(\mathcal{F}). \tag{6}$$

The dimension of the $\mathcal{F}$-algebra invariant operators is given by the following formula [3]

$$\dim \mathcal{F} = \dim \mathcal{G} + \dim \mathcal{H}^\lambda - 2\dim \mathcal{H}, \qquad \lambda \in \mathcal{H}^\perp. \tag{7}$$

We introduce symbols of operators $X_A(x, p) \equiv X^a_A(x)p_a$, $L_\mu(x, p) \in C^\infty(T^*M)$:

$$\{X_A(x, p), X_B(x, p)\} = C^C_{AB} X_C(x, p), \qquad \{L_\mu(x, p), L_\nu(x, p)\} = \Omega_{\mu\nu}(L), \tag{8}$$
$$\{X_A(x, p), L_\mu(x, p)\} = 0. \tag{9}$$

Momentum mappings

$$\mu:\ T^*X \to \mathcal{G}^*, \quad X(x,p) = f \in \mathcal{G}^*, \qquad \tilde{\mu}:\ T^*X \to \mathcal{F}^*, \quad L(x,p) = g \in \mathcal{F}^*$$

are Poisson mappings on Poisson algebra of functions on T^*M in Poisson algebra on $\mathcal{G}^*$ and $\mathcal{F}^*$. Also symplectic sheets $\Omega \subset \mathcal{G}^*$ and $\tilde{\Omega} \subset \mathcal{F}^*$ mutually correspond [6]:

$$\Omega = \mu\big(\tilde{\mu}^{-1}(\tilde{\Omega})\big), \qquad \tilde{\Omega} = \tilde{\mu}\big(\mu^{-1}(\Omega)\big), \qquad \operatorname{codim}\Omega = \operatorname{codim}\tilde{\Omega}.$$

Let Q be a Lagrange submanifold on the symplectic sheet Ω (coadjoint orbit) in $\mathcal{G}^*$. Let us represent the algebra $\mathcal{G}$ by differential operators $l(q,\partial_q,J)$, which form an irreducible representation of algebra $\mathcal{G}$ in the space of functions on Q [7]:

$$[l_A(q,\partial_q;J), l_B(q,\partial_q;J)] = C^C_{AB} l_C(q,\partial_q;J). \tag{10}$$

Here q are local coordinates on Q, J are parameters, "numerating" orbits in $\mathcal{G}^*$ and satisfying the condition for orbits to be integer.

We note as U a Lagrange submanifold to symplectic sheet $\tilde{\Omega}$. The defect $d(M)$ of homogeneous space M is defined as dimension of Lagrange submanifold to symplectic sheet on the coalgebra of invariant operators [3]:

$$d(M) = \dim U = \frac{1}{2}\dim\tilde{\Omega}. \tag{11}$$

We will note *index* of $\mathcal{F}$-algebra ($\operatorname{ind}\mathcal{F}$) the number of functionally independent elements that generate center of the enveloping field of algebra $\mathcal{F}$.

Obviously the equality

$$\dim\mathcal{F} - \operatorname{ind}\mathcal{F} = \dim\tilde{\Omega} = 2d(M) \tag{12}$$

takes place.

We now construct irreducible representation of algebra $\mathcal{F}$ on Lagrange submanifold U of symplectic sheet $\tilde{\Omega}$ in $\mathcal{F}^*$ with differential operators $\zeta(u,\partial_u;J)$:

$$[\zeta_\mu(u,\partial_u;J), \zeta_\nu(u,\partial_u;J)] = -\Omega_{\mu\nu}(\zeta(u,\partial_u;J)). \tag{13}$$

Set of generalized functions $D^J_{qu}(x)$, is defined from the equations

$$(X_A(x,\partial_x) + l_A(q,\partial_q;J))D^J_{qu}(x) = 0, \qquad (L_\mu(x,\partial_x) - \zeta_\mu(u,\partial_u;J))D^J_{qu}(x) = 0. \tag{14}$$

It is full and orthogonal on $C^\infty(M)$ [5]:

$$\int_M D^{\tilde{J}}_{\tilde{q}\tilde{u}}(x)\overline{D^J_{qu}(x)}\,d\mu(x) = \delta(J,\tilde{J})\delta(q,\tilde{q})\delta(u,\tilde{u}), \tag{15}$$

$$\int D^J_{qu}(x)\overline{D^J_{qu}(\tilde{x})}\,d\mu(J)d\mu(q)d\mu(u) = \delta(x,\tilde{x}). \tag{16}$$

Since operator $H(y,\partial_y,x,\partial_x)$ of equation (1) is G-invariant, so it may be expressed as follows:

$$H(y,\partial_y,x,\partial_x) = F(y,\partial_y,L(x,\partial_x)).$$

Because of that solution basis of equation (1), marked by the parameters (J,q) may be constructed following way:

$$\varphi^J_q(y,x) = \int_U \tilde{\varphi}^J(y,u)D^J_{qu}(x)\,d\mu(u), \tag{17}$$

where function $\tilde{\varphi}^J(y,u)$ is a solution of the equation

$$\tilde{H}(y,\partial_y,u,\partial_u;J)\tilde{\varphi}^J(y,u)=0. \tag{18}$$

Here $H(y,\partial_y,u,\partial_u;J)=F(y,\partial_y,\zeta^+(u,\partial_u;J))$.

Since the expression is correct (11), the number of independent variables N' in equation (18) is defined by formula (3).

Finally from (11) we have the number of independent variables N' in equation (18) to be given by formula (3).

3 Example

Let us illustrate the Theorem presented in this article by a non-trivial example. We consider an equation of type (1), where H is the Laplace operator

$$\Delta\varphi(x)\equiv\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}g^{ij}(x)\sqrt{g}\frac{\partial}{\partial x^j}\varphi(x)=E\varphi(x),\qquad g\equiv\det g_{ij} \tag{19}$$

on the Riemannian manifold with metric (below $c_i=\text{const}$)

$$g^{ij}(x)=\begin{pmatrix} c_1e^{2x_4}x_3^2 & c_1e^{2x_4}x_3 & -c_4x_3e^{-x_4}/2 & c_3x_3e^{x_4}/2+c_4e^{-x_4}/2\\ c_1e^{2x_4}x_3 & c_1e^{2x_4} & -c_4e^{-x_4}/2 & c_3e^{x_4}/2\\ -c_4x_3e^{-x_4}/2 & -c_4e^{-x_4}/2 & 0 & 0\\ c_3x_3e^{x_4}/2+c_4e^{-x_4}/2 & c_3e^{x_4}/2 & 0 & c_2 \end{pmatrix}. \tag{20}$$

That metric is not Stakkel which means that Hamilton–Jacobi equation as well as Klein–Gordon and Dirac equations cannot be solved using separation of variables. From the point of existing methods the corresponding equation is not integrable.

Metric (20) allows five-dimensional motion group formed by operators

$$\begin{aligned} &X_1=\partial_1,\qquad X_2=\partial_2,\qquad X_3=x_2\partial_1+\partial_3,\\ &X_4=-x_1\partial_1+x_2\partial_2-2x_3\partial_3+\partial_4,\qquad X_5=x_1\partial_2-x_3^2\partial_3+x_3\partial_4, \end{aligned} \tag{21}$$

that form a basis of the Lie algebra $\mathcal{G}=\{e_A\}$ with the following nonzero commutation rules:

$$\begin{aligned} &[e_1,e_4]=-e_1,\qquad [e_1,e_5]=e_2,\qquad [e_2,e_3]=e_1,\qquad [e_2,e_4]=e_2,\\ &[e_3,e_4]=-2e_3,\qquad [e_3,e_5]=e_4,\qquad [e_4,e_5]=-2e_5. \end{aligned}$$

Here this group acts on a four-dimensional homogeneous Riemannian space with one dimensional isotropy subgroup. Substituting $x=0$ in operators X_A (21), we find isotropy subalgebra $\mathcal{H}=\{e_5\}$. Algebra $\mathcal{G}$ is of an odd index 1, i.e. it has only one Casimir function $K(f)=f_1f_2f_4+f_1^2f_5-f_2^2f_3$, generating center of the Poisson algebra with the Poisson–Lie bracket defined by (5). Regular element of space $\mathcal{H}^\perp$ is non-degenerate and looks like: $\lambda=(\lambda_1,\lambda_2,\lambda_3,\lambda_4,0)$. Annihilator of that covector λ is one-dimensional and, because it is degenerate, is the gradient of Casimir function:

$$\mathcal{G}^\lambda=\big\{\nabla K(f)|_{f=\lambda}=\lambda_2\lambda_4e_1+(\lambda_1\lambda_4-2\lambda_2\lambda_3)e_2-\lambda_2^2e_3+\lambda_1\lambda_2e_4+\lambda_1^2e_5\big\}.$$

So we get: $\mathcal{H}^\lambda=\{0\}$. According to (4) our Riemannian space is non-commutative and its defect is $d(M)=1$. Making a substitution in (3) $N=\dim M=4$, $d(M)=1$, we get $N'=1$, which means according to the Theorem that equation (19) may be reduced to a linear first-order differential equation. Also formulae (7), (12) give $\dim\mathcal{F}=3$, $\operatorname{ind}\mathcal{F}=1$, i.e. the algebra

of invariant operators is three-dimensional and center of enveloping field is generated by one operator.

Let us find the algebra of invariant operators for this case. Symmetry operators X_A are projections of left-invariant vector fields ξ_A: $X_A = \pi_* \xi_A$, where mapping $\pi : G \to M = G/H$ is a projection of group G on the right coset space. Here we note as η_A the algebra of right-invariant vector fields:

$$[\xi_A, \xi_B] = C_{AB}^C \xi_C, \qquad [\eta_A, \eta_B] = C_{AB}^C \eta_C, \qquad [\xi_A, \eta_B] = 0.$$

In canonical coordinates of type II: $g = g_5 g_4 g_3 g_2 g_1$ on the Lie group, where $g_k = \exp(x_k e_k)$, left- and right-invariant vector fields are:

$$\begin{aligned}
&\xi_1 = \partial_1, \qquad \xi_2 = \partial_2, \qquad \xi_3 = x_2\partial_1 + \partial_3,\\
&\xi_4 = -x_1\partial_1 + x_2\partial_2 - 2x_3\partial_3 + \partial_4, \qquad \xi_5 = x_1\partial_2 - x_3^2\partial_3 + x_3\partial_4 + e^{-2x_4}\partial_5,\\
&\eta_1 = -\left(e^{-x_4} + x_3x_5e^{x_4}\right)\partial_1 - e^{x_4}x_5\partial_2, \qquad \eta_2 = -e^{x_4}x_3\partial_1 - e^{x_4}\partial_2,\\
&\eta_3 = -e^{-2x_4}\partial_3 - x_5\partial_4 + x_5^2\partial_5, \qquad \eta_4 = -\partial_4 + 2x_5\partial_5, \quad \eta_5 = -\partial_5.
\end{aligned}$$

(It is obvious that symmetry operators X_A are left-invariant vector fields restricted on the space of functions, which are independent of coordinate x_5). Invariant operators $L(x, \partial_x)$ are operator functions of right-invariant field restricted on the class of functions which are constant on every right coset (in our case functions, which are independent of coordinate x_5) and commuting with every field η_α, which form isotropy algebra $\mathcal{H} = \{\eta_\alpha\}$ (in our case, commuting with η_5). Invariant functions $L(x, p)$, which are symbols of invariant operators are easily obtained:

$$L_\mu(x, p) = a_\mu(f)|_{f=\eta^{cl}}, \qquad \eta_A^{cl} \equiv \eta_A^i(x)p_i,$$

where functions $a_\mu(f) \in C^\infty(\mathcal{G}^*)$ are solutions of the equations

$$\left(C_{\alpha A}^B f_B \frac{\partial a(f)}{\partial f_A}\right)\Bigg|_{f=\lambda} = 0, \qquad \lambda \in \mathcal{H}^\perp, \qquad \alpha = 1, \ldots, \dim \mathcal{H}. \tag{22}$$

In our case equation (22) transforms in

$$f_2\frac{\partial a(f)}{\partial f_1} + f_4\frac{\partial a(f)}{\partial f_3} = 0,$$

As we solve that equation, we get: $a_1 = f_2$, $a_2 = f_4$, $a_3 = f_1f_4 - f_2f_3$. So, invariant functions are:

$$\begin{aligned}
&L_1(x, p) = \eta_2^{cl}(x, p)|_{p_5=0} = -e^{x_4}(x_3p_1 + p_2), \qquad L_2(x, p) = \eta_4^{cl}(x, p)|_{p_5=0} = -p_4,\\
&L_3(x, p) = (\eta_1^{cl}(x, p)\eta_4^{cl}(x, p) - \eta_2^{cl}(x, p)\eta_3^{cl}(x, p))|_{p_5=0} = e^{-x_4}(p_1p_4 - p_2p_3 - x_3p_1p_3).
\end{aligned}$$

Invariant operators look as follows $L(x, \partial_x)$

$$L_1 = ie^{x_4}(x_3\partial_1 + \partial_2), \qquad L_2 = i(\partial_4 + 1), \qquad L_3 = -e^{-x_4}(\partial_{14} - x_3\partial_{13} - \partial_{23}),$$

are self-adjoint in respect to Riemannian measure $\sqrt{g}\,dx = C\exp(2x_4)\,dx$ and form solvable three-dimensional Lie algebra: $i[L_1, L_2] = L_1$, $[L_1, L_3] = 0$, $i[L_2, L_3] = L_3$. Center of the enveloping $\mathcal{F}$-algebra is generated by one element $Z = L_1L_3$, which coincides with Casimir operator $K(-iX)$ for algebra $\mathcal{G}$: $Z(x, \partial_x) = L_1(x, \partial_x)L_3(x, \partial_x) = K(-iX(x, \partial_x))$:

$$L_1L_3 = -i\{X_1, X_2, X_4\} - i\{X_1, X_1, X_5\} + i\{X_2, X_2, X_3\}.$$

Here and below figure brackets note the symmetrized product of operators. The Laplace operator belongs to the enveloping field of the $\mathcal{F}$-algebra since it is invariant operator:

$$-\Delta = c_1 L_1^2(x,\partial_x) + c_2 L_2^2(x,\partial_x) + c_3\{L_1(x,\partial_x), L_2(x,\partial_x)\} + c_4 L_3(x,\partial_x) + c_2. \tag{23}$$

Let us construct an irreducible representation of algebra $\mathcal{G}$ in the space of function on the Lagrange submanifold Q of coadjoint orbit Ω (symplectic sheet) (10):

$$l_1 = iq_1, \qquad l_2 = -iq_2, \qquad l_3 = q_1\partial_{q_2}, \qquad l_4 = q_1\partial_{q_1} - q_2\partial_{q_2}, \qquad l_5 = q_2\partial_{q_1} + iJ/q_1^2, \tag{24}$$

here $(q_1, q_2) \in Q = \mathbb{R}^2$. Because of irreducibility $K(-il(q,\partial_q;J)) = J$. Let us note that operators l_A (24) are skew-Hermitian in respect to measure $d\mu(q) = dq_1 dq_2$ on Q. Parameter J that numerates coadjoint orbits is not quantized and may be of any real value.

Let us construct in the same way irreducible representation of $\mathcal{F}$-algebra in the space of function on Lagrange submanifold U of the symplectic sheet $\tilde{\Omega}$ (13) corresponding with representation (24):

$$\zeta_1 = u, \qquad \zeta_2 = -iu\partial_u, \qquad \zeta_3 = J/u, \qquad \left(u \in U = \mathbb{R}^1\backslash\{0\}\right). \tag{25}$$

These operator are self-adjoint in respect to measure $d\mu(u) = du/u$ on U.

We find set of functions $D^J_{qu}(x)$, numerated by variables (J,q,u) from solution of overdetermined system (14):

$$D^J_{qu}(x) = \exp\left(i(q_2x_2 - q_1x_1) - iJe^{x_4}/q_1u\right)\ \delta\left(e^{x_4}(x_3q_1 - q_2) - u\right). \tag{26}$$

That set of functions satisfies ortogonality and completeness conditions on M (15), (16) (in the right sides of these formulae delta-functions are presented with respect to corresponding measures $d\mu(J) = dJ/(2\pi)^3$).

Let us present solution $\varphi^J_q(x)$ of equation (19) as (17)

$$\varphi^J_q(x) = \int_U \tilde{\varphi}^J(u) D^J_{qu}(x)\, d\mu(u), \tag{27}$$

then, using expressions (14), (23), (25) for function $\tilde{\varphi}^J(u)$ we obtain ordinary differential equation

$$\left(c_2u^2\partial_u^2 + u(c_2 - ic_3u)\partial_u + ic_3u/2 - c_1u^2 - c_4J/u - c_2\right)\tilde{\varphi}^J(u) = E\tilde{\varphi}^J(u).$$

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Group Explanation for the Conditional Similarity Reductions of the (2+1)-Dimensional KdV Equation

Xiao-Yan TANG and Sen-Yue LOU

Physics Department of Shanghai Jiao Tong University, Shanghai, 200030, P.R. China
E-mail: *xytang@sjtu.ed.cn, sylou@sjtu.edu.cn*

The conventional Lie group approach is extended successfully to give out the group explanation to the new *conditional* similarity reductions obtained by modifying the Clarkson and Kruskal's (CK's) direct method for the (2+1)-dimensional Korteweg–de Vries (KdV) equation.

1 Introduction

As is well-known that the classical Lie group approach, the nonclassical Lie group approach, the Clarkson and Kruskal's (CK's) direct method are three powerful methods in finding similarity reductions for a given nonlinear partial differential equation (NPDE) [1–6]. In many cases, the similarity reductions obtained by the classical Lie group approach can also be yielded by the CK's direct method; and those obtained by the CK's direct method while not by the classical Lie group approach can be reobtained by the nonclassical Lie group approach. There have been several modifications of these three methods in the literature [7–12].

Three years ago, we for the first time proposed the modified CK's direct method to construct the so-called *conditional* similarity reductions of the (2+1)-dimensional KdV equation in the integrable case [9]. We call a reduction conditional similarity reduction since one reduction field need to satisfy more than one reduction equation. Since then, similar work has been carried out on several other NPDEs including the nonintegrable (2+1)-dimensional KdV equation [10], the Jimbo–Miwa (JM) equation [11] and the Boussinesq equation [12]. It is noticed that the conditional similarity reductions obtained by means of the modified CK's direct method cannot be recovered by utilizing the classical or even the nonclassical Lie group approach in their present forms. The very reason lies in the fact that the constrained equation introduced in the present nonclassical Lie group approach does not offer an additional conditional reduction equation for the reduction field. Consequently, in order to reobtain the conditional similarity reductions by using the classical Lie group approach or the nonclassical Lie group approach, a conditional equation which will lead to the additional reduction equation must be introduced. In Ref. [13], all the conditional similarity reductions of the JM equation resulting from the modified CK's direct method were retrieved by introducing a conditional equation, the KP equation, to form an equation system and then applying the classical Lie group approach to the system. Thus, the whole group theoretical explanation is given of the conditional similarity reductions for the JM equation. In fact, how to introduce a conditional equation so that the conditional similarity reductions got by the modified CK's direct method can also be yielded by the classical Lie group approach and/or the nonclassical Lie group approach has not yet been precisely known.

The aim of this paper is to report the recent progress on the Lie group approach which gives out the group explanation of the conditional similarity reduction solutions obtained by the modified CK's direct method. The modified CK's direct method and the conditional similarity reduction solutions for the (2+1)-dimensional KdV equation will be reviewed in the next section. In Section 3, the conventional Lie group approach is developed further to give out the full group explanation for the conditional similarity reduction solutions given in Section 2. The last section is a short summary and discussion.

2 Conditional similarity reduction solutions

In the traditional CK's direct method, in order to find the similarity solutions of a general Nth order n-dimensional nonlinear system,

$$E_0\left(x_1,x_2,\ldots,x_n,u,u_{x_i},u_{x_ix_j},\ldots,u_{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}}\right)\equiv E_0=0,\qquad \sum_{j=1}^{n} i_j=N, \tag{1}$$

one seeks solutions of the kind $u=U(x_1,x_2,\ldots,x_n,P(\xi_1,\xi_2,\ldots,\xi_{n-1}))$, where ξ_j, $j=1,2,\ldots,n-1$ are all functions with respect to $\{x_1,x_2,\ldots,x_n\}$. For some types of models, the solution u may commonly be simplified to the linear ansatz

$$u=\alpha+\beta P(\xi_1,\xi_2,\ldots,\xi_{n-1}), \tag{2}$$

where α and β are both functions of $\{x_1,x_2,\ldots,x_n\}$. In general, it is known that the ansatz (2) may not be valid for other types of models, say, the Harry–Dym equation [3]. Substituting (2) into equation (1) gives out

$$\sum_{l=1}^{L} r_l(\alpha,\alpha_{x_i},\beta,\beta_{x_i},\xi_{x_i},\ldots)F_l(\xi_j,P,P_{\xi_j},\ldots)\equiv\sum_{l=1}^{L} r_lF_l=0. \tag{3}$$

Since the similarity reduction function P satisfies only one reduction equation, equation (3) becomes an $(n-1)$-dimensional PDE $G(\xi_j,P,P_{\xi_j},\ldots)\equiv G(P)=0$, only for all the ratios of r_l being functions of $\{\xi_1,\xi_2,\ldots,\xi_{n-1}\}$. Namely, $r_l=r_k\Gamma_l$ should be satisfied for some fixed nonzero r_k, and Γ_l are functions of $\{\xi_1,\xi_2,\ldots,\xi_{n-1}\}$.

In order to find the conditional similarity reductions of (1), we relax the condition that *the similarity reduction function P satisfies only one reduction equation* as that *P is allowed to satisfy more than one reduction equation at the same time*. Based on this idea, we make use of the same reduction ansatz (2) and then separate the resulting equation (3) into m parts

$$\sum_{k=1}^{m}\sum_{l=1}^{M} A_{lk}F_l=0,\qquad M\geq L, \tag{4}$$

with the condition $\sum\limits_{k=1}^{m} A_{lk}=r_l$, $(l\leq L)$, $\sum\limits_{k=1}^{m} A_{lk}=0$, $(l>L)$, where F_l for $l>L$ may be some suitable functions of P and its partial differential derivatives with respect to $\{\xi_1,\xi_2,\ldots,\xi_{n-1}\}$. Then we can see that the reduction function P may satisfy m reduction equations $\sum\limits_{l=1}^{M} A_{lk}F_l=0$, $(k=1,2,\ldots,m)$. When applying this modified direct method, one must make sure that all the ratios of A_{lk} are functions of $\{\xi_j\}$ for the *same* k, while cannot be functions of $\{\xi_1,\xi_2,\ldots,\xi_{n-1}\}$ for the *different* k. The reason is that if the ratio of any two A_{lk} is a function of $\{\xi_1,\xi_2,\ldots,\xi_{n-1}\}$, then they can be put into the same part of (4). The arbitrariness of m and the functions F_l for $l>L$ makes it a hard job to cover all the cases of the conditional similarity reductions. Therefore, up to now, we have just considered the case $m=2$ and $F_l\equiv 0$ for $l>L$. Obviously, many more meaningful conditional similarity reductions may be found for $m>2$ and/or for $F_l\neq 0$ when $l>L$.

Some types of the conditional similarity reductions of

$$u_{xt}-u_{xxxy}-4u_xu_{xy}-4u_{xx}u_y=0, \tag{5}$$

which is the potential form ($v=u_x$) of the (2+1)-dimensional KdV equation

$$v_t-v_{xxy}-4vv_y-4v_x\partial_x^{-1}v_y=0.$$

have been discussed in detail via the modified CK's direct method in [9]. Here, we will list the general known conditional similarity solution of (5) which reads

$$u=\frac{\eta_t}{4\eta_y}x+\frac{1}{4}\int\left(\frac{\sigma_t\eta_y-\sigma_y\eta_t}{\theta\eta_y}+\theta(\omega_{14}\sigma+B\sigma_y)+\gamma_{14}\sigma\eta_y\right)\mathrm{dy}+\theta P,\qquad \xi=\theta x+\sigma,$$

where the function $P=P(\xi,\eta)$ satisfies the conditional similarity reduction equations (i.e. (116) and (117) of Ref. [9])

$$P_{\xi\xi\eta}+4P_\xi P_\eta+\gamma_{14}(\xi P_\xi-P)+\gamma_{29}P+(\gamma_{21}\eta+\gamma_{20})\xi+f_1=0,$$
$$P_{\xi\xi\xi}+4P_\xi^2+\omega_{14}(\xi P_\xi-P)+BP_\xi+\omega_{29}P+(\omega_{21}\eta+\omega_{20})\xi+f_0=0,$$

with $B=B(\eta)$, $f_1=f_1(\eta)$ and $f_0=f_0(\eta)$ being arbitrary functions of η. For the other functions ($\theta=\theta(t)$, $\sigma=\sigma(y,t)$, $\eta=\eta(y,t)$) and constants (γ_{14}, γ_{29}, γ_{21}, γ_{20}, ω_{14}, ω_{29}, ω_{21}, ω_{20}), seven possible selections were given in [9] which will be explicitly written down again in Table 1 later.

3 Group explanation of the conditional similarity reduction solutions

In order to give out the whole group theoretical explanation of the conditional similarity reductions obtained via the modified CK's direct method described in the last section, we have to extend the present classical Lie group approach and the nonclassical Lie group approach.

Simply speaking, the extended classical and nonclassical conditional Lie group approaches are realized if we introduce some constrained equations when solving the model equation and then applying the standard group approach to the formed equation system. As a first attempt, we worked on the JM equation [13] where the conditional similarity reductions yielded by the modified CK's direct method have been recovered totally. However, the constrained equation in this case comes quite specifically so that it cannot be a good candidate for all the NPDEs in observation. Consequently, a more systemic way should be established to find out a common equation which can then be considered for a class of NPDEs. Starting from this standpoint, a more general conditional equation for the integrable (2+1)-dimensional KdV equation is imported when we try to give out the group explanation of the results [9] obtained via the modified CK's direct method, which read

$$u_{xxx}+A_1u_{xxy}+A_2u_xu_y+A_3u_x+A_4u_y+A_5u_x^2+A_6u+A_7=0, \tag{6}$$

with the coefficients A_i, $(i=1,2,\ldots,7)$ being suitable functions with respect to the space-time variables $\{x,y,t\}$ to be determined later. Writing down the constrained equation (6) is based on the fact that the orders of the differentiations and the nonlinearity of the conditional reductions obtained by the modified CK's direct method in [9] are not higher than those of the original system (5). Like the conventional nonclassical Lie group approach, when using the extended nonclassical Lie group approach, we have to utilize the following constraint condition

$$Xu_x+Yu_y+Tu_t-U=0, \tag{7}$$

where X, Y, T, U of $\{x,y,t,u\}$ are called the infinitesimals of the transformation functions related to the infinitesimal transformations $\{x,y,t,u\}\to\{x,y,t,u\}+\epsilon\{X,Y,T,U\}$ with ϵ being the group parameter, under which the model equation and the constrained equations should be form invariant.

Ignoring the concrete calculations, we directly give out the whole group explanation in Table 1 where the first column is the selections corresponding to the known conditional similarity solutions in Ref. [9], the second column is the solutions for the coefficients of the constrained equation (6) and the last column is the concrete form of the related infinitesimal transformations.

Table 1. Group explanation for the results obtained by the modified CK's direct method.

	Selections	Parameters	Infinitesimal Transformations
1	$\gamma_{14}=c_1,\gamma_{29}=c_1-1$ $\gamma_{20}=c_0,\ \gamma_{21}=-\frac{1}{2}(c_1+1)$ $\omega_{14}=\omega_{29}=\omega_{21}=\omega_{20}=0$ $\theta=c_2t^{-1/(3+c_1)}$ $\eta=-c_2^{-2}\frac{1}{3+c_1}t^{-(1+c_1)/(3+c_1)}y$ $+c_5t^{-(1+c_1)/(3+c_1)}+\frac{2c_0}{1+c_1}$ $+c_6t^{2/(3+c_1)}$ $\sigma\equiv\sigma(y,t)$ arbitrary	$A_1=A_2=A_4=A_5=A_7=0$ $A_3=-\theta^2(2C_1\eta-B-4c_0)$ $+2c_2^2c_6C_2$ $A_6=-\frac{1}{4}c_6c_2^2\theta^2C_2(2C_1\eta-4c_0$ $-B)+\frac{1}{4}\theta^4[(C_1\eta-2c_0)(C_1\eta$ $-2c_0-B)-4f_0]+\frac{1}{4}c_2^4c_6^2C_2^2$ where $C_1\equiv1+c_1,\ C_2\equiv3+c_1$	$X=-\sigma_t+\frac{1}{C_2t}[c_2t^{-1/C_2}x-\sigma_y(C_1y$ $-c_2^2C_2(c_5C_1-2c_6t))]$ $Y=t^{-1/C_2}[-\frac{c_2C_1(c_5c_2^2C_2-y)}{C_2t}+2c_6c_2^3]$ $T=c_2t^{-1/C_2}$ $U=\frac{\sigma y}{4C_2^2t^2}[c_2^2C_2Bt^{\frac{C_1}{C_2}}(2c_2^2c_6C_2t-c_5c_2^2$ $C_1C_2+C_1y)+2C_1^2y^2-4c_2^2C_1C_2(c_5C_1$ $-2c_6t)y-8c_2^4c_6tC_2^2(c_5C_1-c_6t)+2c_2^4$ $c_5^2C_1^2C_2^2]+\frac{c_2}{4C_2t^2}t^{-1/C_2}[C_1(x-1)$ $(c_5c_2^2C_2-y)+2t(2\alpha-c_6c_2^2C_2x-2u)]$ $+c_2\alpha_tt^{-1/C_2}+\frac{1}{4C_2^2t^2}(c_1\sigma-2C_2\sigma_tt)$ $[c_2^2C_2(c_5C_1-2c_6t)-C_1y]$ $\alpha=-\frac{1}{4c_2C_2}t^{1/C_2}[(1+2c_1)t^{-1}\int\sigma\mathrm{d}y$ $-C_2\int\sigma_t\mathrm{d}y-c_2^2C_2t^{-2/C_2}\int\sigma_yB\mathrm{d}y$ $+\sigma(C_1(c_5c_2^2C_2-y)t^{-1}-2c_2^2c_6C_2)]$
2	$\gamma_{14}=-3,\gamma_{29}=-4$ $\gamma_{20}=c_0,\gamma_{21}=1$ $\omega_{14}=\omega_{29}=\omega_{21}=\omega_{20}=0$ $\theta=\exp(c_2t)$ $\eta=(c_2y+c_3t+c_4)e^{-2c_2t}-c_1$ $\sigma\equiv\sigma(y,t)$ arbitrary	$A_1=A_2=A_4=A_5=A_7=0$ $A_3=\theta^2(4\eta+4c_1+B)-\frac{2c_3}{c_2}$ $A_6=-\frac{1}{4c_2^2}[-2c_2^2\theta^4((\eta+c_1)$ $(2\eta+2c_1+B)-2f_0)$ $+c_2c_3\theta^2(4\eta+4c_1+B)-c_3^2]$	$X=-\sigma_t-2I\sigma_y-c_2x\exp(c_2t)$ $Y=2c_2ye^{c_2t}+(2c_4+2c_3t-\frac{c_3}{c_2})e^{c_2t}$ $T=\exp(c_2t)$ $U=\frac{1}{2}I[2\sigma_t-3c_2\sigma+\sigma_y(B\exp(2c_2t)+4I)]$ $-(\frac{1}{2}c_3-c_2u+c_2\alpha-\alpha_t)\exp(c_2t)$ $\alpha=-\frac{\exp(-c_2t)}{4}[\int(5c_2\sigma-\sigma_t)\mathrm{d}y+2I\sigma]$ $+\frac{\exp(c_2t)}{4}\int\sigma_yB\mathrm{d}y$ where $I\equiv c_2y+c_3t+c_4-\frac{c_3}{2c_2}$
3	$\gamma_{14}=c_1,\gamma_{29}=c_1-1$ $\gamma_{20}=\gamma_{21}=\omega_{14}=\omega_{29}=\omega_{21}=0$ $\theta=c_2t^{-1/(3+c_1)}$ $\eta=-\frac{c_2^{-2}y}{3+c_1}t^{(-1-c_1)/(3+c_1)}+\eta_0$ $\sigma=\xi_0+\frac{y}{4c_2^2c_3}t^{2/(3+c_1)}$ $[(3+c_1)t\eta_{0tt}+2(1+c_1)\eta_{0t}]$ $-\frac{(1+c_1)y^2}{4c_2^4c_3(3+c_1)^2}t^{-2(1+c_1)/(3+c_1)}$ $\xi_0\equiv\xi_0(t),\ \eta\equiv\eta_0(t)$ arbitrary	$A_1=A_2=A_4=A_5=A_7=0$ $A_3=-\theta^2[2C_1\eta'-B]$ $+2c_2^{C_2}C_2\eta_{0t}\theta^{-C_1}$ $A_6=\frac{\theta^4}{4}[C_1\eta'(C_1\eta'-B)+4c_3\xi$ $-4f_0]+\frac{1}{4}c_2^{2C_2}C_2^2\eta_{0t}^2\theta^{-2C_1}$ $-\frac{1}{4}c_2^{C_2}C_2\eta_{0t}\theta^{1-c_1}[2C_1\eta'-B]$ $\eta'\equiv\eta-\eta_0,\ C_1\equiv1+c_1$ $C_2\equiv3+c_1$	$X=\frac{c_2Ax}{(3+c_1)t}-\frac{1}{2c_3}(3+c_1)(1+c_1)t\eta_{0t}^2$ $-\frac{1}{4c_3}(3+c_1)^2t^2\eta_{0tt}\eta_{0t}-\xi_{0t}$ $-\frac{1}{A^2}[\frac{(2+c_1)(1+c_1)y\eta_{0t}}{2c_2^2c_3(3+c_1)t}+\frac{(2+c_1)y\eta_{0tt}}{c_2^2c_3}$ $+\frac{(3+c_1)ty\eta_{0ttt}}{4c_2^2c_3}]$ $Y=\frac{c_2AC_1y}{C_2t}+c_2^3A^3C_2\eta_{0t}t$ $T=c_2A$ $U\equiv U_{\mathrm{III}}$ $\alpha\equiv\alpha_{\mathrm{III}}$ where $A\equiv t^{-\frac{1}{3+c_1}}$
4	$\gamma_{14}=-3,\gamma_{29}=-4,\omega_{20}=c_3$ $\gamma_{20}=\gamma_{21}=\omega_{14}=\omega_{29}=\omega_{21}=0$ $\theta=\exp(c_2t)$ $\eta=c_2y\exp(-2c_2t)+\eta_0$ $\sigma=-\frac{y\exp(-2c_2t)}{4c_3c_2}(4c_2\eta_{0t}+\eta_{0tt})$ $+\frac{c_2^2y^2}{2c_3}\exp(-4c_2t)-\xi_0$ $\xi_0\equiv\xi_0(t),\ \eta\equiv\eta_0(t)$ arbitrary	$A_1=A_2=A_4=A_5=A_7=0$ $A_3=\theta^2(4\eta-4\eta_0-2c_2^{-1}\eta_{0t}+B)$ $A_6=\frac{\theta^4}{4c_2^2}[-c_2\eta_{0t}(4\eta-4\eta_0+B)$ $+2c_2^2(\eta-\eta_0)B+4c_2^2\eta(\eta-2\eta_0)$ $+4c_2^2(\eta_0^2+c_3\xi-f_0)+\eta_{0t}^2]$	$X=\frac{y\exp(-2c_2t)}{4c_2c_3}(4c_2^2\eta_{0t}+4c_2\eta_{0tt}$ $+\eta_{0ttt})+\xi_{0t}-c_2x\exp(c_2t)$ $-\frac{\eta_{0t}\eta_{0tt}+4c_2\eta_{0t}^2}{4c_3c_2^2}$ $Y=2c_2y\exp(c_2t)-\frac{1}{c_2}\eta_{0t}\exp(3c_2t)$ $T=\exp(c_2t)$ $U\equiv U_{\mathrm{IV}}$ $\alpha\equiv\alpha_{\mathrm{IV}}$
5	$\gamma_{14}=1,\gamma_{29}=0$ $\gamma_{21}=c_2,\ \gamma_{20}=c_3$ $\omega_{14}=\omega_{29}=c_1$ $\omega_{21}=c_4,\omega_{20}=c_5$ $\eta=\theta^{-3}\theta_ty+\eta_0$ $\sigma=(\frac{\theta_{tt}}{\theta^2\theta_tc_1}-\frac{5\theta_t}{c_1\theta^3})+\xi_0$ $\xi_0\equiv\xi_0(t)$ arbitrary while $\eta_0\equiv\eta_0(t)$ and $\theta\equiv\theta(t)$ are given by equations (8) and (9).	$A_1=A_2=A_4=A_5=A_7=0$ $A_3=\theta^2(6\eta'+c_1\xi+B)-2\theta^3\theta_t^{-1}$ $\eta_{0t}-2\theta^3\theta_{tt}\theta_t^{-2}\eta'$ $A_6=\frac{\theta^4}{4}[9\eta'^2-4f_0+3(c_1\xi+B)\eta'$ $+4(c_4\eta+c_5)\xi]+\frac{1}{4}\theta^6\theta_t^{-4}\theta_{tt}\eta'$ $(\theta_{tt}\eta'+2\theta_t\eta_{0t})+\frac{1}{4}\theta^5\theta_t^{-2}[\theta\eta_{0t}^2$ $-(6\eta'+c_1\xi+B)\theta_{tt}\eta']$ $-\frac{1}{4}\theta^5\theta_t^{-1}\eta_{0t}[6\eta'+c_1\xi+B]$ where $\eta'\equiv\eta-\eta_0$	$X=-\theta_tx+\frac{(2\theta_{tt}^2\theta-\theta_t^2\theta_{ttt}-\theta\theta_t\theta_{ttt})y}{c_1\theta^3\theta_t^2}$ $+\frac{\theta_{tt}\eta_{0t}\theta-5\theta_t^2\eta_{0t}-c_1\xi_{0t}\theta_t^2}{c_1\theta_t^2}$ $Y=-\theta_t^{-1}\left[(\theta_{tt}\theta-3\theta_t^2)y+\eta_{0t}\theta^4\right]$ $T=\theta$ $U\equiv U_{\mathrm{V}}$ $\alpha=\frac{\theta\theta_{tt}-5\theta_t^2}{4c_1\theta^2\theta_t}\int B\mathrm{d}y+\frac{y^2}{8c_1\theta^5\theta_t^2}[4\theta_t^2(5\theta_t^2$ $-2\theta_{tt}\theta)+\theta^2(\theta_{ttt}\theta_t-\theta_{tt}^2)]$ $-\frac{y}{4c_1\theta^2\theta_t^2}[\theta\theta_{tt}(\theta\eta_{0t}-c_1\theta_t\xi_0)$ $-\theta_t^2(c_1\theta\xi_{0t}-4c_1\theta_t\xi_0+5\theta\eta_{0t})]$
6	$\gamma_{14}=c_3,\gamma_{29}=c_2,\gamma_{21}=c_5$ $\omega_{14}=c_1+c_2,\gamma_{20}=c_6$ $\omega_{29}=c_1,\omega_{20}=c_4$ $\omega_{21}=c_7=\frac{c_5(c_1-c_2)}{c_2-c_3}$	$A_2=4A_1,A_5=4,A_6=0$ $A_1=-\frac{c_2\theta^3t}{b_2}$ $A_3=\frac{t}{c_1^2b_2\theta^2\eta_2^2}\{-c_1\eta_{1t}\eta_2\theta(3c_2\theta_t$ $+2c_1b_8\eta_2\theta^3)-c_2(c_2-2c_1)\eta'\eta_2\theta_t^2$	$X=-b_1b_2t^{b_2-1}x-\xi_{0t}-\frac{b_2c_2(2b_2b_4t+b_3b_6)}{c_1b_5t^{2b_2+2}}$ $Y=\frac{b_1t^{b_2-1}}{b_5}[b_5b_6y-2c_2b_2b_4b_1^2t-c_2b_3b_6b_1^2]$ $T=b_1t^{b_2}$ $U\equiv U_{\mathrm{VI}}$

	$-\frac{c_1(c_2^2+2c_1c_2-3c_1c_3)}{2(c_2-c_1)}$ $\theta = b_1 t^{b_2}, \quad \sigma = \frac{\theta_t}{c_1\theta^3}y + \xi_0$ $\eta = \eta_2 y + \eta_1 \equiv \frac{3-5b_6}{2c_2 b_1^2 t^{b6}}y$ $+\{\frac{2b_5 b_7}{c_1 c_2^2 b_2 b_6} + \frac{b_3 t + b_4}{t^{b_6}}\}$ $b_2 = \frac{c_1(c_3-c_2)}{4c_1c_2-5c_1c_3+c_2^2}$ $b_5 = 5b_2+1, \ b_6 = 2b_2+1$ $b_7 = c_2c_8b_2 - c_6c_1b_5$ $\xi_0 \equiv \xi_0(t)$ arbitrary	$+c_1^2c_2\theta^4\xi_{0t}\eta_2^2 + c_1c_2\theta_t\theta(\eta' 4\eta_{2t}$ $+\eta_2^2\theta^2(c_2\xi_0 + c_1\xi_0 - c_3\eta' + B))$ $-c_1^2c_2\eta_2\theta^4\eta_{2t}(\xi_0 - \xi) + c_1\eta'\eta_2\theta$ $(2c_1\theta^3\eta_{2t}b_8 - c_2\theta_{tt}) + c_1^2\eta_2^3\theta^6$ $(b_8(c_1\xi + B) + c_2(c_3\xi_0 - c_2\xi))\}$ $A_4 = \frac{c_2\theta^3 t(\eta_{1t}\eta_2 - \eta_{2t}\eta')}{b_2\eta_2^2}$ $A_7 \equiv A_{\rm VI7}$ where $\eta' \equiv \eta_1 - \eta, \ b_8 \equiv c_3 - c_2$	$\alpha = -\frac{c_2 y}{4c_1c_2b_1b_5t^{b_2+1}}[c_1b_5((3b_2+1)\xi_0 - \xi_{0t}t)$ $+c_2b_2t^{-b_6}(2b_2b_4t + b_3b_6)] - \frac{b_2y^2(3b_2+1)}{8c_1b_1^3t^{3b_2+2}}$ $+\frac{b_2}{4c_1b_1t^{b_2+1}}\int B\mathrm{d}y$
7	$\gamma_{14} = \gamma_{29} = \omega_{29} = c_1$ $\omega_{14} = 2c_1, \omega_{20} = c_4, \omega_{21} = c_7$ $\gamma_{21} = c_5, \gamma_{20} = c_6$ $\eta = \eta_2 y + \eta_1$ $\sigma = \frac{\theta_t}{c_1\theta^3}y + \xi_0$ $\xi_0 \equiv \xi_0(t)$ arbitrary, $\eta_1 \equiv \eta_1(t)$ $\eta_2 \equiv \eta_2(t)$ and $\theta \equiv \theta(t)$ are determined by Eqs. (10)–(12)	$A_2 = 4A_1, A_5 = 4, A_6 = 0$ $A_1 = -\frac{c_1\theta^4}{\theta_t}$ $A_3 = \frac{\eta'}{3c_1\theta\theta_t\eta_2}(27c_1\theta_t^2 + 4\theta^3\eta_2$ $(2c_1^2\theta_t + c_7\theta_t + c_1c_5\theta^3\eta_2))$ $-\frac{3\eta_{1t}}{\eta_2} + \theta^2(B - c_1\xi_0 + 3c_1\xi)$ $+\frac{c_1\theta^3\xi_{0t}}{\theta_t}$ $A_4 = -\frac{c_1\theta^3\eta'(c_1\eta_2\theta^3 + 3\theta_t)}{\eta_2\theta_t}$ $+\frac{c_1\theta^4\eta_{1t}}{\theta_t\eta_2}$ $A_7 \equiv A_{\rm VII7}$, where $\eta' \equiv \eta_1 - \eta$	$X = -\theta_t x - \xi_{0t} + \frac{2y}{3c_1^2\theta^4}(2c_1c_5\theta^6\eta_2^2 + 3c_1\theta_t^2$ $+\theta^3\theta_t\eta_2(c_1^2 + 2c_7)) + \frac{\theta_t\eta_{1t}}{c_1\eta_2\theta^3}$ $Y = -\frac{\theta(\eta_{2t}y + \eta_{1t})}{\eta_2}, \quad T = \theta, \quad U \equiv U_{\rm VII}$ $\alpha = \frac{y^2}{24\ c_1^2\theta^5}[\theta_t\theta^3\eta_2(3c_1c_3 - 2c_1^2 - 4c_7)$ $+3(c_2 - c_1)\theta_t^2 - 4c_1c_5\theta^6\eta_2^2] + \frac{y}{4c_1\theta^4\eta_2}$ $[-\eta_{1t}\theta_t + c_1\eta_2\theta^3(c_3\xi_0\theta^2\eta_2 + \xi_{0t})$ $+(c_1 + c_2)\theta_t\theta^2\xi_0\eta_2] + \frac{\theta_t}{4c_1\theta^2}\int B\mathrm{d}y.$

Note. The concrete forms of $U_{\rm III}$, $\alpha_{\rm III}$, $U_{\rm IV}$, $\alpha_{\rm IV}$, $U_{\rm V}$, $U_{\rm VI}$, $A_{\rm VI7}$, $U_{\rm VII}$, $A_{\rm VII7}$ and equations (8)–(12) are all given out in the Appendix.

Hence, the full group theoretical explanation is given out for all the conditional similarity reductions of the integrable (2+1)-dimensional KdV equation obtained by the modified CK's direct method in Ref. [9]. It is worth pointing out that the coefficients of the conditional equation used in giving out the group explanation of the results resulting from the modified CK's direct method are permitted to be functions with respect to the space-time arguments. In the case that all the coefficients in this added equation are fixed to be constants, we are still able to find many conditional symmetry reductions by utilizing the extended classical and nonclassical Lie group approaches [14] which are definitely recoverable by the modified direct method though the related work has not yet been carried out.

4 Summary and discussions

By extending both the CK's direct method and the Lie group approach, we can obtain the conditional similarity reductions. The crucial point for this kind of reduction is that the one reduction field of a given NPDE needs to satisfy more than one reduction equation. From the detailed description in the last two sections, one can notice much freedom or arbitrariness in both the modified CK's direct method and the extended Lie group approach. For the modified CK's direct method, we need to separate the resulting equation (3). Obviously, the division is quite arbitrary. Moreover, the values of F_l for $l > L$ can be set arbitrarily. While for the extended Lie group approach, the choice of the constrained equation is rather arbitrary. Many different useful results will be generated with different selections of the conditional equations. Anyhow, we do believe that the conditional similarity reductions obtained from the modified CK's direct method can definitely be reobtained from the extended Lie group approach and vice versa through ascertaining or balancing the arbitrariness between these two methods.

Furthermore, on the other hand, one can also see that the constrained equations introduced in the extended Lie group approach have little relation with the model equation under investigation. However, as for the nonintegrable JM equation, the conditional equation forms an integrable system with the JM equation. Therefore, probably we can decrease the arbitrariness by considering a constrained equation which has some possible relation with the studied equation especially for the nonintegrable models. All in all, the search of a method to introduce

a constrained equation for the model equation into the extended classical and nonclassical Lie group approaches is still in progress.

Appendix

$$-\frac{1}{2}c_1\theta^2\theta_{tt}^2+\left(\frac{3}{4}c_1+c_4\right)\theta\theta_t^2\theta_{tt}+\frac{1}{4}\theta_t c_1\theta^2\theta_{ttt}+\left(c_1c_1-\frac{3}{2}c_1-5c_4\right)\theta^4=0, \tag{8}$$

$$(4c_4\theta_t\theta\theta_{tt}-20c_4\theta^3+4c_1c_2\theta_t^3)\eta_0+\theta^2\theta_t c_1\eta_{0tt}$$
$$+(6c_1\theta\theta_t^2-2c_1\theta^2\theta_{tt})\eta_{0t}-20c_5\theta_t^3+4c_1c_3\theta_t^3+4c_5\theta\theta_t\theta_{tt}=0. \tag{9}$$

$$\theta\eta_{2t}-c_1\theta^3\eta_2^2-3\eta_2\theta_t=0, \tag{10}$$

$$c_1\theta\eta_{1tt}-2c_1\eta_{1t}(c_1\eta_2\theta^3+3\theta_t)+4\theta^2\eta_2\theta_t(c_8+c_7\eta_1)+4c_1\theta^5\eta_2^2(c_6+c_5\eta_1)=0, \tag{11}$$

$$3c_1\theta\theta_{tt}-12c_1\theta_t^2-(c_1^2-4c_7)\eta_2\theta^3\theta_t+4c_1c_5\eta_2^2\theta^6=0. \tag{12}$$

$$U_{\rm III}=c_2A\alpha_t+\frac{c_1C_1^2y^3}{16c_3(c_2C_2tA)^4}-\frac{c_1C_1\xi_{0t}y}{4C_2^2t^2}+\frac{1}{4c_3}c_2^2C_1C_2^2A^2t^2\eta_{0t}^3+\frac{C_1y^2\eta_{0ttt}}{8c_3c_2^2A^2}+\frac{C_1\xi_{0t}y}{2C_2t}$$
$$+\frac{\eta_{0t}^2}{8c_3}[c_2^2C_2^3A^2t^3\eta_{0tt}+c_2^2C_1C_2A^2Bt+3C_1(2+c_1)y]+\frac{c_2A}{4C_2t^2}[C_1y(1-x)$$
$$-4(u-\alpha)t]+\eta_{0t}\left[\frac{C_2t}{16c_3}(\eta_{0tt}(c_2^2C_2A^2Bt+9(2+c_1)y)+2C_2\eta_{0ttt}yt)+\frac{1}{2}c_2^2C_2A^2\xi_{0t}t\right.$$
$$\left.+\frac{c_1C_1By}{8c_3C_2t}-\frac{1}{4}c_2^3(C_2x+C_1)A^3-\frac{1}{4}c_2^2c_1A^2\xi_0+\frac{C_1(2c_1^2+11c_1+8)y^2}{16c_3(c_2C_2At)^2}\right]$$
$$-\frac{C_1^2By^2}{8c_3c_2^2C_2^3A^2t^3}+\frac{\eta_{0tt}}{16c_3}\left[C_1By-4c_2^3c_3C_2A^3t+\frac{C_1(16+7c_1)y^2}{C_2c_2^2A^2t}\right],$$

$$\alpha_{\rm III}=\frac{C_1C_2\eta_{0t}^2yt}{8c_2c_3A}+\frac{\eta_{0t}}{16c_2c_3A}\left[C_2^2\eta_{0tt}yt^2+2C_1\int B{\rm d}y+\frac{2C_1y^2}{c_2^2C_2A^2t}\right]+\frac{C_2\eta_{0ttt}y^2t}{32c_2^3c_3A^3}-\frac{c_1\xi_{0t}y}{4c_2C_2At}$$
$$+\frac{\eta_{0tt}}{32c_2c_3A}\left[2C_2t\int B{\rm d}y+\frac{y^2(3c_1+8)}{c_2^2A^2}\right]-\frac{C_1\int yB{\rm d}y}{8c_3c_2^3C_2^2A^3t^2}+\frac{c_1C_1y^3}{48c_3c_2^5C_2^3A^5t^3}+\frac{\xi_{0t}y}{4c_2A},$$

$$\alpha_{\rm IV}=-\frac{(4c_2\eta_{0t}+\eta_{0tt})e^{-c_2t}}{16c_2c_3}\int B{\rm d}y+\frac{c_2^2e^{-3c_2t}}{4c_3}\int yB{\rm d}y-\frac{c_2^3y^3e^{-5c_2t}}{8c_3}+\frac{y^2e^{-3c_2t}}{32c_2c_3}$$
$$\times(-\eta_{0ttt}+8c_2^2\eta_{0t}-c_2\eta_{0tt})+\frac{ye^{-c_2t}}{16c_3c_2^2}(-4c_3c_2^2\xi_{0t}+12c_2^3c_3\xi_0+4c_2\eta_{0t}^2+\eta_{0t}\eta_{0tt}),$$

$$U_{\rm IV}=(c_2u-c_2\alpha+\alpha_t)e^{c_2t}+\frac{y^2e^{-2c_2t}}{8c_3}(4c_2^3B+7c_2^2\eta_{0t}-5c_2\eta_{0tt}-2\eta_{0ttt})$$
$$+\frac{e^{3c_2t}}{4c_2}(2c_2\eta_{0t}+\eta_{0tt})+\frac{e^{2c_2t}}{16c_2^3c_3}[Bc_2\eta_{0t}(4c_2\eta_{0t}+\eta_{0tt})-2\eta_{0t}(4c_2\eta_{0t}^2+\eta_{0t}\eta_{0tt}$$
$$+6c_2^3c_3\xi_0-4c_3c_2^2\xi_{0t})]-\frac{3c_2^4y^3e^{-4c_2t}}{4c_3}+\frac{y}{16c_3c_2^2}[8c_3c_2^3(3c_2\xi_0-2\xi_{0t})$$
$$-2c_2^2B(6c_2\eta_{0t}+\eta_{0tt})+\eta_{0t}(12c_2^2\eta_{0t}+9c_2\eta_{0tt}+2\eta_{0ttt})],$$

$$U_{\rm V}=\frac{y^2(\theta_{tt}\theta-3\theta_t^2)}{4c_1\theta_t^3\theta^5}(3\theta_{tt}^2\theta^2-2\theta_t\theta_{ttt}\theta^2+7\theta_t^2\theta_{tt}\theta-20\theta_t^4)-\frac{x(\theta_t^2-\theta_{tt}\theta)}{4\theta\theta_t^2}(3\theta_t^2y-\theta\theta_{tt}y$$
$$-\eta_{0t}\theta^4)-\frac{y}{4c_1\theta^2\theta_t^3}[\theta_t^5(12c_1\xi_0-3c_1\theta+15B)-\theta\eta_{0t}(10\theta_t^4-9\theta\theta_t^2\theta_{tt}+5\theta^2\theta_{tt}^2$$
$$-2\theta^2\theta_t\theta_{ttt})+\theta^2\theta_t\theta_{tt}^2(c_1\theta+B+c_1\xi_0)-\theta\theta_t^2\theta_{tt}(-3c_1\theta\theta_t-2c_1\theta\xi_{0t}+7c_1\theta_t\xi_0+8\theta_tB)$$
$$-c_1\theta\theta_t^2(6\theta_t^2\xi_{0t}+\theta^2\theta_{ttt})]+\frac{1}{4c_1\theta_t^3}[\theta^2\theta_t\eta_{0t}(-\theta\theta_{tt}(c_1\theta+B+c_1\xi_0)-2c_1\theta\theta_t\xi_{0t}+(3c_1\theta$$
$$+5B+4c_1\xi_0)\theta_t^2)+4c_1\theta\theta_t^3\alpha_t+2\theta^3\eta_{0t}^2(\theta\theta_{tt}-5\theta_t^2)+c_1\theta_t^2(\theta^4\eta_{0tt}-4\theta_t^2\alpha)]+\theta_tu,$$

$$A_{\rm VI7}=-\frac{t}{4b_2c_1^2\theta^2\eta_2^4}\{-c_1\eta_{1t}^2\eta_2^2\theta(2c_2\theta_t+c_1b_8\eta_2\theta^3)-c_2\eta_2\eta_{2t}\eta'(c_1^2\eta_2\theta^4\xi_{0t}-(c_2-2c_1)\eta'\theta_t^2)$$
$$+\eta_2\eta_{1t}[c_2(2c_1-c_2)\theta_t^2\eta_2\eta'+c_1^2\theta^6\eta_2^3(b_8(B+c_1\xi)+c_2(c_3\xi_0-c_2\xi))$$
$$+c_1^2\theta^4\eta_2(2b_8\eta_{2t}\eta'+c_2\eta_{2t}(\xi-\xi_0)+c_2\xi_{0t}\eta_2)+c_1c_2\theta^3\theta_t\eta_2^2(B+(c_2+c_1)\xi_0-c_3\eta')$$
$$+c_1c_2\theta\eta'(5\theta_t\eta_{2t}-\theta_{tt}\eta_2)]-c_1c_2\theta\theta_t\eta'\eta_{2t}[3\eta'\eta_{2t}+\eta_2^2\theta^2((c_2+c_1)\xi_0-c_3\eta'+B)]$$
$$+c_1\eta_{2t}\eta'\eta_2\theta[c_2\eta'\theta_{tt}-c_1\eta_2^2\theta^5(b_8(B+c_1\xi)+c_2(c_3\xi_0-c_2\xi))]-4c_1^2\eta_2^5\theta^8(b_8f_0-c_2f_1$$
$$+(c_8b_8-c_2c_6)\xi+(c_7b_8-c_2c_5)\xi\eta)+c_1^2\eta_2\theta^4\eta_{2t}^2\eta'(-b_8\eta'+c_2(\xi_0-\xi))\},$$

$$U_{\rm VI}=-\frac{b_1xt^{b_2-2}}{4b_5}[b_6b_5y-c_2b_1^2(b_3b_6+2b_2b_4t)]+\frac{y}{4c_1b_5t^2}[c_1b_5b_6(2\xi_{0t}t-(3b_2+1)\xi_0+b_1t^{b_2})$$
$$+c_2b_2b_5b_6B-c_2b_2(b_2+1)t^{-2b_2-1}(b_3b_6+2b_2b_4t)]+b_1t^{b_2-1}(b_2u+b_1\alpha_tt)$$
$$-\frac{c_2b_1^2}{4c_1b_5t^2}(2b_2b_4t+b_3b_6)[2c_1\xi_{0t}t+b_2B-c_1\xi_0(3b_2+1)]-\frac{b_2b_6(3b_2+1)y^2}{4c_1b_1^2t^{2b_2+3}}$$
$$+\frac{b_1}{4c_1b_5^2}[2c_2^2b_1b_2b_3b_6t^{-2b_2-3}(b_3b_6+4b_2b_4t)+8c_2^2b_1b_2^3b_4^2t^{-b_6}$$
$$-c_1b_5t^{b_2-2}(c_2b_1^2b_3b_6+4b_2b_5\alpha t)],$$

$$A_{\rm VII7}=\frac{c_1\theta^2\xi_{0t}}{4\theta_t\eta_2}[\eta'(3\theta_t+c_1\eta_2\theta^3)-\theta\eta_{1t}]+\frac{\theta_t\eta'}{4c_1\eta_2^2\theta}[3c_1(B\theta^2\eta_2+4\eta_{1t})+\theta^2\eta_2\eta'(4c_7+11c_1^2)$$
$$-3c_1^2\theta^2\eta_2(\xi_0-3\xi)]-\frac{\theta^2\eta_{1t}}{12c_1\eta_2}[(4c_7+11c_1^2)\eta'+3c_1(B-c_1\xi_0+3c_1\xi)]$$
$$+\frac{\theta^4\eta'}{12}[(5c_1^2+4c_7+12c_5)\eta'+3c_1(B-c_1\xi_0+3c_1\xi)]-\frac{\theta^5}{3\theta_t}\big[c_5\eta'(\eta_{1t}-c_1\theta^2\eta_2\eta')$$
$$+3c_1\theta^2\eta_2(c_6\xi+c_5\xi\eta+f_1)\big]+\frac{9(\eta_1-\eta)^2\theta_t^2+\theta^2\eta_{1t}^2}{2\eta_2^2\theta^2},$$

$$U_{\rm VII}=-\frac{x}{4\theta\eta_2}(5\theta_t+c_1\eta_2\theta^3)(3\eta_2\theta_ty+c_1\eta_2^2\theta^3y+\theta\eta_{1t})-\eta_2\theta^5(c_5\eta_1+c_6)+\theta\alpha_t+\theta_t(u-\alpha)$$
$$+\frac{\theta^2}{4c_1}[c_1\eta_{1t}(c_1\theta-c_3\xi_0)-4\theta_t(c_8-c_7\eta_1)]-\frac{\eta_{1t}}{4c_1\eta_2^2\theta^3}[c_1\theta^3\eta_2(2\xi_{0t}-3\theta_t)-2\theta_t\eta_{1t}$$
$$+\theta_t\theta^2\eta_2(B+(c_1+c_2)\xi_0)]+\frac{(c_1\eta_2\theta^3+3\theta_t)y^2}{12c_1^2\theta^5}[4(c_1^2+2c_7)\theta^3\theta_t\eta_2$$
$$-c_1\theta^3\eta_2(3c_3\theta_t-8c_5\theta^3\eta_2)-3(c_2-3c_1)\theta_t^2]-\frac{y}{12c_1^2\eta_2\theta^4}\{3c_1^2\theta^9\eta_2^3(4c_5-c_1^2)$$
$$+3c_1^3c_3\theta^8\eta_2^3\xi_0+3\eta_{1t}\theta_t^2(c_2-9c_1)-2c_1\theta^6\eta_2^2(9c_1^2\theta_t-3c_1^2\xi_{0t}-6c_7\theta_t+4c_5\eta_{1t})$$
$$+3c_1^2\theta_t\theta^5\eta_2^2((c_1+c_2)\xi_0+3c_3\xi_0+B)+9c_1\theta_t^2\theta^2\eta_2((c_1+c_2)\xi_0+B)$$
$$+\theta_t\theta^3\eta_2[c_1^2(18\xi_{0t}-27\theta_t-10\eta_{1t})+\eta_{1t}(3c_1c_3-8c_7)]\}.$$

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Dynamical Symmetries of Autonomous Differential Second Order Equations

Mikhail I. TIMOSHIN

Samara State Pedagogical University, Russia
E-mail: *midvolga@mail.ru*

In the report the possibility of dynamic symmetries to an analytical integration of autonomous ODE of second order is presented. The correlation with known results is pointed out. The concept of a defining system of dynamic symmetries is given. The example indicating to the effectiveness of an offered method is produced.

1 Introduction

The concept of dynamic symmetry is given, for example, in [1]. The differential equation, in this case, is replaced by ODE system of the first order:

$$y'' = f(x, y, y') \;\Leftrightarrow\; \frac{dy}{dx} = z, \qquad \frac{dz}{dx} = f(x, y, z). \tag{1}$$

Then it is possible to consider the question of an infinitesimal transformation

$$X = \xi(x, y, z)\frac{\partial}{\partial x} + \eta(x, y, z)\frac{\partial}{\partial y} + \mu(x, y, z)\frac{\partial}{\partial z}, \tag{2}$$

translating a solution of system (1) again into a solution. That is operator (2) should satisfy the condition

$$[X, A] = \lambda(x, y, z)A, \tag{3}$$

where

$$A = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z}.$$

As against the contact symmetries, set by the characteristic function $\Omega(x, y, z)$

$$X = \Omega_z\frac{\partial}{\partial x} + (z\,\Omega_z - \Omega)\frac{\partial}{\partial y} - (\Omega_x + z\,\Omega_y)\frac{\partial}{\partial z},$$

of the operator of dynamic symmetry (2), the components ξ, η, μ are defined only by condition (3).

2 Dynamical symmetries of autonomous differential second order equations

Let us consider the autonomous differential equation of the second order

$$y'' = f(y, y'). \tag{4}$$

The equation (4) admits point symmetry $X_1 = \frac{\partial}{\partial x}$ and dynamic symmetry X_2. Following the scheme of S. Lie, we shall write out the most general transformations retaining autonomy (4). For this purpose it is necessary that the transformed system should suppose symmetry $\bar{X}_1 = \frac{\partial}{\partial t}$. Thus two cases are possible:

I uncrossed transformation **II crossed transformation**

$$X_1 \Leftrightarrow \bar{X}_1 \qquad\qquad X_1 \quad \bar{X}_1$$

$$\qquad\qquad\qquad\qquad \times$$

$$X_2 \Leftrightarrow \bar{X}_2 \qquad\qquad X_2 \quad \bar{X}_2$$

In case of **I** transformations have the form

$$x = t + \alpha(u, w), \qquad y = \beta(u, w), \qquad z = \gamma(u, w). \tag{5}$$

In case of **II** transformations will be defined by a choice of the form of the transformed dynamic system. By taking dynamic system

$$\frac{du}{dt} = R(w)u + P(w), \qquad \frac{dw}{dt} = Q(w), \tag{6}$$

it is possible to speak about linearizing of the equation (4) according to the circuit **I** or **II**. In case of **I** the class of linearizing equations is most clearly defined

$$y'' = \begin{vmatrix} \dfrac{d^2\beta}{dt^2} & \dfrac{d\beta}{dt} \\ \dfrac{d^2(t+\alpha)}{dt^2} & \dfrac{d(t+\alpha)}{dt} \end{vmatrix} \Bigg/ \left(\frac{d\,(t+\alpha)}{dt}\right)^3. \tag{7}$$

The general solution of the equation (7) has the form:

$$\begin{aligned} x &= \int \frac{dw}{Q(w)} + \alpha(u, w) + C_1, \qquad y = \beta(u, w), \\ u &= e^{\int \frac{R(w)}{Q(w)} dw} \left(\int \frac{P(w)}{Q(w)} e^{-\int \frac{R(w)}{Q(w)} dw} dw + C_2 \right). \end{aligned} \tag{8}$$

Before consideration of type **II** transformation we shall present additional information concerning dynamic symmetries. In the book [1] it is pointed out that if the symmetry X of dynamic system of the equations determined by the operator A is known, then

$$\hat{X} = X + \rho(x, y, z)A \tag{9}$$

with any function $\rho(x, y, z)$, is also a dynamic symmetry. That is explained by the fact that the following

$$[\hat{X}, A] = \lambda A - (A\rho)A = \hat{\lambda} A$$

takes place. Using ratio (9) it is always possible to proceed from the operator X to the operator $\widehat{X}$ at which the coefficient at $\frac{\partial}{\partial x}$ will be equal to zero. In the case of point symmetry

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \cdots \tag{10}$$

and a differential equation of the second order $y'' = f(x, y, y')$ we shall obtain

$$\hat{X} = (\eta - y'\xi)\frac{\partial}{\partial y} + \cdots . \tag{11}$$

In the book [2] the proof by Boyer (1967) that the operators (10), (11) are equivalent is presented. By repeating Boyer's reasonings it is easy to show that in case of dynamic system

$$\frac{dy}{dx}=\varphi(x,y,z),\qquad \frac{dz}{dx}=f(x,y,z),\qquad A=\frac{\partial}{\partial x}+\varphi(x,y,z)\frac{\partial}{\partial y}+f(x,y,z)\frac{\partial}{\partial z},$$

generators

$$X=\xi(x,y,z)\frac{\partial}{\partial x}+\eta(x,y,z)\frac{\partial}{\partial y}+\mu(x,y,z)\frac{\partial}{\partial z}\qquad\text{and}\qquad \hat{X}=(\eta-\varphi\xi)\frac{\partial}{\partial y}+(\mu-f\xi)\frac{\partial}{\partial z}$$

are equivalent. Let us consider now linearization of the second type. By taking as new variables the first integrals of the system (6)

$$s=ue^{-\int\frac{R(w)}{Q(w)}dw}-\int\frac{P(w)}{Q(w)}e^{-\int\frac{R(w)}{Q(w)}dw}dw,\qquad v=\int\frac{dw}{Q(w)}-t,$$

we shall obtain the following system

$$\frac{ds}{dt}=0,\qquad \frac{dv}{dt}=0. \tag{12}$$

The system (12) has the dynamic symmetry

$$X=\xi(t,s,v)\frac{\partial}{\partial t}+\eta(s,v)\frac{\partial}{\partial s}+\mu(s,v)\frac{\partial}{\partial v}$$

with any functions $\xi(t,s,v)$, $\eta(s,v)$, $\mu(s,v)$. Going back to variables t, u, w we shall obtain the dynamic symmetry of system (6)

$$X=(\xi+\mu)\frac{\partial}{\partial t}+\left[(\xi+\mu)(R(w)u+P(w))+\eta e^{\int\frac{R(w)}{Q(w)}dw}\right]\frac{\partial}{\partial u}+(\xi+\mu)Q(w)\frac{\partial}{\partial w}.$$

Going from the found symmetry to the equivalent operator, we obtain dynamic symmetry in the following form

$$\hat{X}=\eta e^{\int\frac{R(w)}{Q(w)}dw}\frac{\partial}{\partial u}. \tag{13}$$

The most general transformation reducing the generator (13) to $\frac{\partial}{\partial x}$ has the form

$$x=\alpha(s,v)+\Phi(t,w),\qquad y=\Psi(t,w),\qquad z=\Omega(t,w), \tag{14}$$

where $\alpha_s=\frac{1}{\eta(s,v)}$. The transformation (14) allows to proceed from system (6) to system

$$\frac{dy}{dx}=\left(\frac{d}{dt}\Psi(t,w)\right)\Big/\left(\frac{d}{dt}\Phi(t,w)\right),\qquad \frac{dz}{dx}=\left(\frac{d}{dt}\Omega(t,w)\right)\Big/\left(\frac{d}{dt}\Phi(t,w)\right). \tag{15}$$

To bring system (15) into correspondence with equation (4) it is necessary to put

$$z=\Omega(t,w)=\left(\frac{d\Psi}{dt}\right)\Big/\left(\frac{d\Phi}{dt}\right). \tag{16}$$

Thus we obtain the second class of linearizing autonomous ODE of the second order as

$$y''=\begin{vmatrix}\frac{d^2\Psi}{dt^2} & \frac{d\Psi}{dt}\\ \frac{d^2\Phi}{dt^2} & \frac{d\Phi}{dt}\end{vmatrix}\Big/\left(\frac{d\Phi}{dt}\right)^3. \tag{17}$$

The general solution of the equation (17) looks as follows:

$$x = C_1 + \Phi(t, w), \qquad y = \Psi(t, w), \qquad t = C_2 + \int \frac{dw}{Q(w)}. \tag{18}$$

Comparing the general solutions (8), (18) it is easy to notice that they will coincide if in (8) functions α, β are considered as depending on arguments s, w: $\alpha = \alpha(s, w)$, $\beta = \beta(s, w)$, and in (18) functions Φ, Ψ depending on arguments v, w: $\Phi = \Phi(v, w)$, $\Psi = \Psi(v, w)$. It makes possible to speak not about two classes of autonomous ODE of the second order but about one. Of three functions Ω, Ψ, Φ only two are independent, the third function is determined from a condition (16). By taking as arguments variables t, v it is easy to obtain well known results. With the help of functions $\Omega(t, v)$, $\Psi(t, v)$ the autonomous differential equation of the second order is determined as

$$\Omega \frac{d\Omega}{dt} = \frac{d\Psi}{dt} f(\Psi, \Omega). \tag{19}$$

In spite of the fact that at transition from the equation (4) to the equation (19) there was a reduction of the order, it does not always facilitate a task of integration of the differential equations.

The group approach with the help of criterion (3) allows to receive additional equations of the second order with respect to required functions Ω, Ψ, Φ. In the coordinate form the criterion (3) is equivalent to the system of two equations of the form:

$$\begin{aligned} &\mu - \eta_x - z\eta_y - f\eta_z + z\xi_x + z^2\xi_y + zf\xi_z = 0, \\ &\eta f_y + \mu f_z - \mu_x - z\mu_y - f\mu_z + f\xi_x + zf\xi_y + f^2\xi_z = 0. \end{aligned} \tag{20}$$

The transformation (14) allows to write out the type of dynamic symmetries in variables x, y, z:

$$X = (\Phi_t - \alpha_v)\frac{\partial}{\partial x} + \Psi_t \frac{\partial}{\partial y} + \Omega_t \frac{\partial}{\partial z}.$$

Replacing in the adduced out system variables y, z with the help of the formulas (14), and adding the equations

$$\Psi_t + \Psi_w Q - (\Phi_t + \Phi_w Q)\Omega = 0, \qquad \Omega_t + \Omega_w Q - (\Phi_t + \Phi_w Q) f(\Psi, \Omega) = 0, \tag{21}$$

we can receive a system of four equations with three unknown functions. As against classical system of the determining equations used while finding point symmetries, the constructed system of the equations is not linear. However, it is easy to notice, that with the help of the equations (21) equations with partial derivatives can be reduced to the ordinary differential equations of the second order, in which one of variables plays a role of parameter. It makes possible to apply the well elaborated algorithms of point symmetries to investigation of the constructed system of the determining equations of dynamic symmetries.

3 Example

As a testing example underlining efficiency of the offered technique, we shall consider the equation of the second order having the form

$$y''(A + By') + y'^3 C + y'^2 D + y' E = 0 \tag{22}$$

where

$$A = r_3 l^2 k - r_1 k^3 - r_3 h^2 k^3 - r_2 k^3 h - k^2 g_1 l,$$

$$B = 2r_3kl - r_2k^2 - 2r_3hk^2 - k^2g_1,$$
$$C = r_2kk_y - r_3k_yl - r_3kl_y + r_3h_yk^2 + 2r_3hkk_y + g_1kk_y,$$
$$D = r_1k^2k_y + 2r_3h_yk^2l + g_1kk_yl + r_2kk_yl + 2r_3hkk_yl + r_2hk^2k_y + r_3h^2k^2k_y - g_1k^3h_y - r_2k^2l_y - 2r_3hk^2l_y - r_3l^2k_y,$$
$$E = r_1k^2k_yl + r_2k^2k_yhl + r_3l^2k^2h_y - g_1k^3lh_y + r_3h^2k^2k_yl - r_3h^2k^3l_y - r_1k^3l_y - r_2k^3hl_y,$$

in which k, l, h are arbitrary functions of argument y, and r_1, r_2, r_3, g_1 are arbitrary constants.

Using the standard method of reduction to form (19), from the equation (22) we obtain the Abel equation of the second type

$$\frac{d\Omega}{dy}(A + B\Omega) + \Omega^2C + \Omega D + E = 0. \tag{23}$$

The equation (22) is linearized with the help of transformation

$$u = \frac{k}{z+l} - \frac{r_3+1}{g_1}, \qquad w = \frac{z}{k} + h. \tag{24}$$

Thus we obtain equation

$$\left(r_1 + r_2w + r_3w^2\right)\frac{du}{dw} = g_1u + 1,$$

whose general solution looks as follows:

$$(u + 1/g_1) - C_2e^{\int \frac{g_1dw}{r_1+r_2w+r_3w^2}} = 0. \tag{25}$$

For representation of the general solution of the equation (22) as (8), it is necessary to define the form of function $\alpha(u, w)$. Using ratio (5), and considering the function α on arguments (y, C_2) we obtain expression

$$\alpha(y, C_2) = \int\left(\frac{1}{z} + \frac{kl_y - k_yl - h_yk^2}{(kr_2 - 2r_3l + 2r_3hk + kg_1)z - l^2r_3 + k^2r_1 + lkg_1 + r_3h^2k^2 + k^2hr_2}\right)dy.$$

In this case the integral is calculated with the assumption that variable z is function from y determined by ratios (24), (25). Thus the general solution of equation (22) has the form

$$x = \alpha(y, C_2) + \int \frac{dw}{r_1 + r_2w + r_3w^2} + C_1,$$
$$\frac{k}{z+l} - \frac{r_3}{g_1} + C_2e^{\int \frac{g_1dw}{r_1+r_2w+r_3w^2}} = 0, \qquad w = \frac{z}{k} + h.$$

Let us also note that last two equations give the general solution of Abel equation (23). It is interesting to notice that in variable y, $w(y)$ equation (23) takes the form

$$r_1 + r_2w + r_3w^2 + \frac{dw}{dy}\left\{\frac{w\left(2r_3k(l-kh) - k^2(g_1+r_2)\right) - r_1k^2 + r_3(l-kh)^2 + kg_1(kh-l)}{k_yl - kl_y + h_yk^2}\right\} = 0,$$

and the standard transformation is brought to the canonic form:

$$\frac{d\vartheta}{d\tau}\vartheta + \vartheta + \frac{3\tau}{16} - \frac{3^{\frac{2}{3}}\left(2g_1^2\lambda^{\frac{4}{3}} - r_3\lambda^{\frac{1}{3}}\right)}{24\tau^{\frac{1}{3}}r_3^2\lambda^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}}{144\tau^{\frac{5}{3}}r_3^2\lambda^{\frac{2}{3}}} = 0, \tag{26}$$

where

$$\lambda = \frac{r_3}{g_1^2 + 4r_3r_1 - r_2^2}.$$

In directory [3] the transformation of equation of type (26) into another Abel equation and then to Riccati equation is presented. The obtained equation was integrated by Koyalovich [4]. The author of the directory asserts that the form of the obtained solution is very bulky and consequently it was not presented. In [3] solutions of special cases of the equation (26) through Bessel's functions are also given.

4 Concluding remark

In the conclusion we shall note that the proposed approach to investigation of integrated cases of the autonomous differential equations of the second order can be considered as some analogue of the Galois theory for polynomials. In investigating the solubility of the equation

$$P_n = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

the various ratios between roots of x_i equation are considered, i.e. the equation is replaced with a system of the equations for roots $F_1(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) = 0$, …, $F_m(x_{i_1}, x_{i_2}, \ldots, x_{i_r}) = 0$, which can be solved easier than the initial equation.

In the proposed method the question of integration of differential equations is actually reduced to investigation of the system of differential equations

$$F_i(x, y, z, x_t, y_t, z_t, x_w, y_w, z_w, x_{tw}, y_{tw}, z_{tw}, x_{tt}, y_{tt}, z_{tt}, x_{ww}, y_{ww}, z_{ww}) = 0,$$

where $i = 1, 2, 3, 4$. And while investigating the constructed system, the standard algorithms of the group analysis can be used.

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Symmetry Reduction of Nonlinear Differential Equations

Ivan TSYFRA

Institute of Mathematics, University of Bialystok, 2 Akademicka Str., Bialystok, Poland;
Institute of Geophysics of NAS of Ukraine, 32 Palladin Ave., Kyiv, Ukraine
E-mail: *tsyfra@math.uwb.edu.pl*

We propose the symmetry reduction method of partial differential equations to the system of differential equations with fewer number of independent variables. We also obtain generalized sufficient conditions for the solution found by conditional symmetry method to be an invariant one in classical sense.

1 Introduction

In recent years the symmetry method is often used for reduction of partial differential equations to the equations with fewer number of independent variables and thus for construction of exact solutions for different mathematical physics problems. To construct a corresponding ansatz generators of classical Lie point transformations are used as well as operators of conditional symmetry. In this connection the application of combination of conditional and generalized symmetry is fruitful as was shown in [1,2] on the examples of evolution equation in two-dimensional case (see also [3]). In [4] Svirshchevskii proposed the symmetry reduction method based on the invariance of linear ordinary differential equations (see also [5]). It is the symmetry explanation of "nonlinear" separation of variables [6] for the evolution-type equations.

Here we propose an approach applicable for symmetry reduction of partial differential equations which are not restricted to evolution type ones. It can be used in multi-dimensional case. This approach is the generalization of the method introduced in [4].

2 Generalized symmetry and reduction of partial differential equations

Let us consider partial differential equation

$$U\big(x,u,\underset{1}{u},\underset{2}{u},\ldots,\underset{k}{u}\big)=0, \tag{1}$$

where $x=(x_1,x_2,\ldots,x_n)$, $u=u(x)\in C^k\big(\mathbb{R}^n,\mathbb{R}^1\big)$, and $\underset{k}{u}$ denotes all partial derivatives of k-th order. Replacing u by $u+\epsilon w$ and then equating coefficients at ϵ in Taylor series expansion we obtain linearized equation (1)

$$L(x,u,w)=0. \tag{2}$$

It has been proved that the following property is fulfilled in this case. If equation (1) admits Lie–Bäcklund vector field $Q=\eta\big(x,u,\underset{1}{u},\underset{2}{u},\ldots,\underset{r}{u}\big)\partial_u$ and $u=f(x)$ is a solution of equation (1) then

$$w=Qu\big|_{u=f(x)} \tag{3}$$

is a solution of equation (2).

This property is illustrated by the connection between the solutions of Liouville and Moutard equations. It is well known that the Liouville equation

$$u_{xy} = 2\exp u \tag{4}$$

is invariant with respect to the Lie group of transformations with the generator

$$Q_1 = f(x)\partial_x + g(y)\partial_y - (f' + g')\partial_u,$$

where $f(x)$ and $g(y)$ are arbitrary smooth functions. Then from the solution

$$u = \ln \frac{X'Y'}{(X+Y)^2}$$

of Liouville equation we easily obtain the solution

$$u = \frac{X_1'}{X'} + \frac{Y_1'}{Y'} - 2\frac{X_1 + Y_1}{X+Y}$$

of the Moutard equation

$$w_{xy} = 2\frac{X'Y'}{(X+Y)^2}w$$

with potential $V = 2\frac{X'Y'}{(X+Y)^2}$, where $X(x)$, $Y(y)$ are arbitrary smooth functions of their arguments, $fX' = X_1$, $gY' = Y_1$, by using (3).

This property can also be used for reduction of partial differential equations to system of equations with smaller number of independent variables. For simplicity consider ordinary differential equation

$$H\big(x, u, \dots, \underset{m'}{u}\big) = 0, \tag{5}$$

where $\underset{m'}{u}$ denotes the derivative of u with respect to one variable x_1 of m'-th order, and H are a smooth function of its arguments. Suppose that Q is the operator of Lie–Bäcklund symmetry of equation (5). Let

$$u = F(x, C_1, \dots, C_{m'}), \tag{6}$$

where F is a smooth function of variables x, $C_1, \dots, C_{m'}$, $C_1, \dots, C_{m'}$ are arbitrary functions of parametric variables $x_2, x_3, \dots, x_n$, be a general solution of equation (5). Then operator Q transforms solution (6) to the solution of linearized version of equation (5), i.e. the linear homogeneous ordinary differential equation. It means that Q maps the set of solutions (6) into a m'-dimensional vector space M. Moreover, we proved that partial derivatives $\frac{\partial F}{\partial C_i}$, $i = \overline{1, m'}$ form the basis of M provided that H, U and F are sufficiently smooth functions of their arguments. In this connection the following statement holds.

Theorem 1. *Let equation* (5) *be invariant with respect to the Lie–Bäcklund operator* Q*. Then the ansatz*

$$u = F(x, \phi_1, \phi_2, \dots, \phi_{m'}), \tag{7}$$

where $\phi_1, \phi_2, \dots, \phi_{m'}$ *depend on* $n-1$ *variables* $x_2, x_3, \dots, x_n$ *reduces partial differential equation*

$$\eta\big(x, u, \underset{1}{u}, \underset{2}{u}, \dots, \underset{r}{u}\big) = 0 \tag{8}$$

to the system of k_1 *equations for unknown functions* $\phi_1, \phi_2, \dots, \phi_m$ *with* $n-1$ *independent variables and* $k_1 \le m'$*.*

Taking into account the above-mentioned arguments we proved the theorem. It can be easily generalized for $\phi_1, \phi_2, \ldots, \phi_{m'}$ depending on $\omega_l(x)$, $l = \overline{1, n-1}$, where $\omega_l(x)$ are some functions of variables x. Note that the case when equation (5) is linear ordinary differential equation and η is the function of special type (evolution type) considered in [4].

We consider several examples illustrating the application of the theorem. Firstly consider equation

$$u_t = f(u_x)u_{xx}, \tag{9}$$

where $f(u_x)$ is a smooth function of u_x. Let us consider the equation (5) in the following form

$$u_{xx} = u_x^3. \tag{10}$$

In accordance with our approach we study the invariance of the equation (10) with respect to Lie–Bäcklund operator

$$K_1 = (u_t - f(u_x)u_{xx})\partial_u. \tag{11}$$

Function $f(u_x)$ is determined from the condition of invariance of the equation (10) with respect to the operator (11). It has the form

$$f = \frac{A}{u_x^3} + \frac{B}{u_x^2},$$

where A, B are arbitrary real constants. Then we proved that the equation (10) admits the operators

$$K_2 = uu_x\partial_u, \qquad K_3 = h\left(u + u_x^{-1}\right)\partial_u,$$

where h is arbitrary smooth function. Therefore the equation (10) is invariant with respect to the group of Lie–Bäcklund transformations with infinitesimal operator $\alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3$, where α_1, α_2, α_3 are arbitrary real constants. From this it follows that the theorem can be used for the equation

$$u_t = \left(\frac{A}{u_x^3} + \frac{B}{u_x^2}\right)u_{xx} + \lambda u u_x + \lambda_1 h\left(u + u_x^{-1}\right), \tag{12}$$

where λ, λ_1 are arbitrary real constants. The ansatz corresponding to the equation (10) has the form

$$u = \phi_2(t) - \sqrt{\phi_1(t) - 2x}, \tag{13}$$

where $\phi_1(t)$ and $\phi_2(t)$ are unknown functions. The ansatz (13) reduces the equation (12) to the system of ordinary differential equations

$$\phi_2' = A - \lambda + \lambda_1 h(\phi_2), \qquad -\phi_1' = 2(B + \lambda\phi_2). \tag{14}$$

For different $h(\phi_2)$ we receive different solutions. Let $h(\phi_2) = \phi_2$. In this case the solution of the system (14) is

$$\phi_2 = Ce^{\lambda_1 t} - (A - \lambda)\lambda_1^{-1}, \qquad -\phi_1 = 2\left(B - (A - \lambda)\lambda\lambda_1^{-1}\right)t + 2\lambda_1^{-1}\lambda C e^{\lambda_1 t} - C_1. \tag{15}$$

Substituting (15) into (13) we obtain the solution

$$u = Ce^{\lambda_1 t} - (A - \lambda)\lambda_1^{-1} - \sqrt{2\left((A - \lambda)\lambda\lambda_1^{-1} - B\right)t - 2\lambda_1^{-1}\lambda C e^{\lambda_1 t} + C_1 - 2x}$$

of equation (12). This solution cannot be obtained by using classical Lie method of point symmetry.

Note that the ansatz (13) can be used also for reduction of equations obtained by commuting of operators K_1, K_2, K_3.

Next we show the application of the method to an equation associated with inverse scattering problem. As we know the group-theoretical background of the inverse scattering problem method was given for the first time in [7].

Namely we study the symmetry of the linear ordinary differential equation

$$u_{xx} = f(t,x)u. \tag{16}$$

where variable t play the role of parameter in this equation, with respect to the Lie–Bäcklund operator

$$Q_1 = \left(u_t + u_{xxx} - 3\frac{u_{xx}u_x}{u} + \alpha(t)u\right)\partial_u, \tag{17}$$

where $\alpha(t)$ is a function of t.

We have proved that the equation (16) admits the operator (17) if and only if f satisfies the Korteweg–de Vries equation in the form

$$f_t + f_{xxx} - 6ff_x = 0. \tag{18}$$

This statement is valid for arbitrary smooth $\alpha(t)$. Thus if one can construct the general solution of equation (16) for some solution $f(t,x)$ of (18) then the corresponding ansatz will reduce the partial differential equation

$$u_t + u_{xxx} - 3\frac{u_{xx}u_x}{u} + \alpha(t)u = 0 \tag{19}$$

to the system of two ordinary differential equations with independent variable t. We stress that a solution $f(t,x)$ should not necessarily vanish at the infinity.

To solve the Cauchy problem

$$u|_{t=t_0} = g(x) \tag{20}$$

for equation (19) one should construct the solution $f = p(t,x)$ of equation (18) satisfying the condition $p(t_0,x) = \frac{g_{xx}}{g}$ and then to integrate the ordinary differential equation

$$u_{xx} = p(t,x)u.$$

If we can construct the ansatz in this way then we reduce the Cauchy problem (20) for the equation (19) to the Cauchy problem for the system of two ordinary differential equations.

In addition note that Theorem 1 helped us to generalize the theorem proved in [8] concerning the sufficient conditions for the solution obtained by using conditional symmetry operators to be an invariant solution in the classical sense. Namely consider involutive family of operators

$$Q_a = \sum_{j=1}^{n} \xi_{aj}(x,u)\partial_{x_j} + \eta_a(x,u)\partial_u, \qquad a = \overline{1,m}. \tag{21}$$

Suppose that the equation (1) is conditionally invariant with respect to involutive family of operators (21) and corresponding ansatz reduces this equation to an ordinary differential equation of k_1-th order. Then the following statement holds.

Theorem 2. *Let equation* (1) *be invariant with respect to s-dimensional Lie algebra* AG_s *and conditionally invariant with respect to involutive family of operators* $\{Q_i\}$. *If the system*

$$\sum_{l=1}^{n} \xi_i^l \frac{\partial u}{\partial x_l} = \eta_i(x,u)$$

is also invariant under the algebra AG_s *and* $s \geq k_1 + 1$, *then conditionally invariant solution of equation* (1) *with respect to involutive family of operators* $\{Q_i\}$ *is an invariant solution in the classical Lie sense.*

It is necessary to note that this theorem can be generalized to the case when the algebra AG_m contains Lie–Bäcklund operators too.

3 Conclusion

We showed that the symmetry of ordinary differential equations can be used for reduction of partial differential equations to a system with fewer number of independent variables. It follows from Theorem 1 that the linearity is not a necessary condition for this reduction. Thus the restriction obtained in [4] on the dimension of vector space W_k: $k \leq 2p+1$, where p is the order of studied evolution equation, is not valid in this case. It is obvious that the suggested method is applicable in a multi-dimensional case.

In addition note that this approach can be used in solving the problem of integrability of a partial differential equations. It gives the possibility to reduce this problem to the problem of integrability for an ordinary differential equation. We consider this method to be important in studying quasi-exactly solvable systems.

We also obtained the generalized sufficient condition for the solution constructed by using conditional symmetry to be an invariant one in classical sense.

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On Applications of Non-Point and Discrete Symmetries for Reduction of the Evolution-Type Equations

Ivan TSYFRA [†], *Antonino MESSINA* [‡], *Anna NAPOLI* [‡] *and Violeta TRETYNYK* [§]

[†] *University of Bialystok, 2 Academicka Str., Bialystok, Poland;*
Intitute of Geophysics of NAS of Ukraine, 32 Palladina Ave., 03142 Kyiv-142, Ukraine
E-mail: *tsyfra@math.uwb.edu.pl*

[‡] *University of Palermo, 36 Archirafi Str., 90123 Palermo, Italy*
E-mail: *messina@fisica.unipa.it, napoli@fisica.unipa.it*

[§] *Internat. Science and Technology University, 3 Magnitohorsky Prov., 02094 Kyiv-94, Ukraine*
E-mail: *vtretynyk@yahoo.com*

Combination of non-point and conditional symmetries of the evolution-type equations for construction of its exact solutions are presented. Also we analyzed discrete symmetries of Maxwell equations in vacuum and decoupled ones to the four independent equations that can be solved independently.

1 Introduction

Generalization of Lie classical approach to non-Lie approach that take into consideration conditional symmetries, non-point symmetries, supersymmetries, parasupersymmetries, discrete symmetries opens us additional ways of investigation of partial differential equations (PDEs). For instance, it is possible to seek invariance algebra of DE in a class of high-order differential operators or even of integro-differential operators. In this paper we consider two possibilities for expansion of applications of symmetries to (PDEs) by using non-point symmetries and discrete symmetries. First we consider a problem of construction of ansatzes not for the dependent variable $u(t,x)$ (as it usually used by non-classical method, by method of nonlinear separation and by method of generalized symmetry) but for its *derivatives*:

$$\frac{\partial u}{\partial t} = R_1(t,x,u,\varphi_1(\omega),\varphi_2(\omega)),$$
$$\frac{\partial u}{\partial x} = R_2(t,x,u,\varphi_1(\omega),\varphi_2(\omega)), \qquad \omega = \omega(t,x),$$

that reduce our evolution equation (F is an arbitrary smooth function)

$$u_t = F(u_{xx}) \tag{1}$$

to a system of two DE for function $\varphi_1(\omega)$ and $\varphi_2(\omega)$. The main idea we exploit here is to find symmetries of the system of equations

$$v_1^2 = v_2^1,$$
$$v_2^2 = \Phi\big(v^1\big), \qquad \Phi = F^{-1}, \qquad v^1 \equiv \frac{\partial u}{\partial t}, \qquad v^2 \equiv \frac{\partial u}{\partial x}, \tag{2}$$

(lower indices designate the respect derivatives, i.e., $v_2^1 = \frac{\partial v^1}{\partial x_2}$, $v_1^2 = \frac{\partial v^2}{\partial x_1}$ and $x_1 \equiv t$, $x_2 \equiv x$) corresponding to (1) instead of symmetries of initial equation (1). Taking into account the above

designations, it is evident that second equation of system (2) corresponds to (1), and the first one is the compatibility condition. More detailed explanation that (1) yields system (2) see in [1–3].

We note that by investigating the symmetry properties of system (2) we can find symmetry operators that would not correspond to any infinitesimal operators of group of point transformations of (1) and thus we can extend the class of symmetry operators of (1).

Another goal of our paper is to show an effective way of simplifying a given problem using discrete symmetries of DE. A successful using of discrete symmetries for investigation of PDE can be found in [5–9].

2 Symmetry classification of evolution equation

Here we investigate the system of (2) and find the maximal invariance algebra of the system. We prove that

Theorem 1. *System* (2) *is invariant with respect to a local Lie algebra generated by operator* ($\xi^1_{v_1}$, $\xi^1_{v_2}$, $\xi^2_{v_1}$, $\xi^2_{v_2}$ *are not equal to zero simultaneously):*

$$X = \xi^1(x_1, x_2, v_1, v_2)\frac{\partial}{\partial x_1} + \xi^2(x_1, x_2, v_1, v_2)\frac{\partial}{\partial x_2} + \eta^1(x_1, x_2, v_1, v_2)\frac{\partial}{\partial v^1} + \eta^2(x_1, x_2, v_1, v_2)\frac{\partial}{\partial v^2},$$

if function $\Phi(v^1)$ *is a solution of equation:*

$$a(\Phi_{v^1}v^1)_{v^1} = b\Phi\Phi_{v^1} + c\Phi_{v^1v^1} + d\Phi_{v^1}, \tag{3}$$

where a, b, c, d *are arbitrary real constants and* $\Phi_{v^1} \equiv \frac{d\Phi}{dv^1}$, $\Phi_{v^1v^1} \equiv \frac{d^2\Phi}{dv_1^2}$.

We notice that Theorem 1 defines us a class of equations (1) admitting Lie symmetries. Fixing values of a, b, c, d in (3) we find various explicit form of $\Phi\big(v^1\big)$. We consider only such forms of $\Phi\big(v^1\big)$ that cannot be transformed to each other by point transformations of equivalence of (1).

Theorem 2. *The maximal invariance algebra of system* (2) *is a 5-dimensional Lie algebra:*

1. $\langle P_1, P_2, P_3, D_1, Q_1\rangle$ *if* $\Phi = \dfrac{1}{\alpha \ln v^1}$, $\alpha = \text{const}$;
2. $\langle P_1, P_2, P_3, D_1, Q_2\rangle$ *if* $\Phi = \dfrac{1}{e^{v^1} - c}$, $c = \text{const}$;
3. $\langle P_1, P_2, P_3, D_1, Q_3\rangle$ *if* $\Phi = \dfrac{1}{1 - \big(v^1\big)^r}$, $r \in \mathbb{R},\ r \neq 0, \pm 1$;
4. $\langle P_1, P_2, P_3, D_1, Q_4\rangle$ *if* $\Phi = \tan v^1$;
5. $\langle P_1, P_2, P_3, D_1, Q_5\rangle$ *if* $\Phi = \tan\big(\alpha \ln v^1\big)$, $\alpha = \text{const}$.

Theorem 3. *The maximal invariance algebra of system* (2) *is a 7-dimensional Lie algebra:*

$$\langle P_1, P_2, P_4, P_5, D_2, Q_7, Q_8\rangle \quad \textit{if} \quad \Phi = \frac{1}{1 - (v^1)^{-3}}$$

(it is a particular case of 3 from Theorem 2 when $r = -3$*).*

Theorem 4. *The maximal invariance algebra of system* (2) *is an infinite-dimensional Lie algebra:*

$$\langle P_1, P_2, P_3, D_1, Q_6\rangle \quad \textit{if} \quad \Phi = \frac{1}{v^1}.$$

The basis elements of the Lie algebras have the following form:

$$P_1=\frac{\partial}{\partial x_1},\quad P_2=\frac{\partial}{\partial x_2},\quad P_3=\frac{\partial}{\partial v^2},\quad P_4=x_2\frac{\partial}{\partial x_3}+\frac{\partial}{\partial v^2},\quad P_5=\frac{\partial}{\partial x_3},$$
$$D_1=2x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+v^2\frac{\partial}{\partial v^2},\quad D_2=2x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+2x_3\frac{\partial}{\partial x_3}+v^2\frac{\partial}{\partial v^2},$$
$$Q_1=-x_1\frac{\partial}{\partial x_1}+v^2\frac{\partial}{\partial x_2}+v^1\frac{\partial}{\partial v^1},\quad Q_2=\big(x_2+2cv^2\big)\frac{\partial}{\partial x_2}+2\frac{\partial}{\partial v^1}-v^2\frac{\partial}{\partial v^2},$$
$$Q_3=-(r+1)x_1\frac{\partial}{\partial x_1}-rv^2\frac{\partial}{\partial x_2}+v^1\frac{\partial}{\partial v^1}-rv^2\frac{\partial}{\partial v^2},$$
$$Q_4=-2v^2\frac{\partial}{\partial x_2}+2\frac{\partial}{\partial v^1}+2x_2\frac{\partial}{\partial v^2},\quad Q_5=-x_1\frac{\partial}{\partial x_1}-\alpha v^2\frac{\partial}{\partial x_2}+v^1\frac{\partial}{\partial v^1}+\alpha x_2\frac{\partial}{\partial v^2},$$
$$Q_6=-B_{v^2}\frac{\partial}{\partial x_2}+B_{x_1}\frac{\partial}{\partial v^1},$$
$$Q_7=2x_1\frac{\partial}{\partial x_1}+3v^2\frac{\partial}{\partial x_2}+\left[\frac{3}{2}\big(v^2\big)^2+3x_3\right]\frac{\partial}{\partial x_3}+v^1\frac{\partial}{\partial v^1}+3v^2\frac{\partial}{\partial v^2},$$
$$Q_8=\big[3\big(v^2\big)^2-2x_3\big]\frac{\partial}{\partial x_2}+2\big(v^2\big)^3\frac{\partial}{\partial x_3}++2v^1v^2\frac{\partial}{\partial v^1}+2(v^2)^2\frac{\partial}{\partial v^2},$$

where $B\big(x_1,v^2\big)$ is an arbitrary solution of equation:

$$B_{v^2v^2}+B_{x_1}=0$$

(we designate $B_{x_1}\equiv\frac{\partial B}{\partial x_1}$, $B_{v^2v^2}\equiv\frac{\partial^2 B}{\partial v^2\partial v^2}$, ...).

The Theorems are proved by means of the Lie algorithm [11, 12]. It is worth noting that operators Q_a ($a=\overline{1,8}$) correspond to non-point symmetry operators of (1). The cases 4 and 5 of Theorem 2 have been obtained under classical symmetry investigation in [13] and under studying of potential symmetries in [14].

3 Ansätze and exact solutions

Let us use operators of non-point symmetries Q_a, $a=\overline{1,8}$ for reducing of (2) to a system of two ordinary differential equations. It is obvious that we can use also the first prolongation operators of point symmetry of (1). In order to construct an ansatz reducing given equation to system of two ordinary DEs we should use two-dimensional subalgebra of invariance algebra of equation. At the same time according to [15] the obtained solution will be an invariant solution in the Lie sense and can be obtained by classical Lie method. It follows that we should use operators of non-point symmetries to obtain new results.

We note that (1) connected with nonlinear heat equation:

$$\chi_t+(C(\chi)\chi_x)_x=0, \tag{4}$$

Indeed, by double differentiation of (1) with respect to x we obtain equation (4), where $\chi=u_{xx}$, $C(\chi)=F'(\chi)$.

Thus, it is possible to find exact solutions for function χ instead of function u. The ansatz

$$v^1=\frac{\varphi_1\big(v^2\big)}{x_1},\qquad v^2=\frac{x_2}{\varphi_2\big(v^2\big)-\ln x_1}$$

constructed by the operator Q_1 reduces system (2) to the following ordinary differential equations ($\alpha=1$):

$$\frac{d\varphi_1}{dv^2}=v^2,\qquad \ln\varphi_1-\varphi_2=v^2\frac{d\varphi_2}{dv^2}.$$

Integrating the latter system we find exact solution of (4) (when $C(\chi) = -\frac{1}{\chi^2}\exp\left(\frac{1}{\chi}\right)$) in the form:

$$\exp\left(\frac{1}{\chi}\right) = \frac{\theta^2+c}{2t}, \qquad \theta = \frac{x}{\ln\frac{\theta^2+c}{2te^2} + \frac{2\sqrt{c}}{\theta}\arctan\frac{\theta}{\sqrt{c}} + \frac{c_1}{\theta}}, \quad \text{if} \quad c>0;$$

$$\exp\left(\frac{1}{\chi}\right) = \frac{\theta^2+c}{2t}, \qquad \theta = \frac{x}{\ln\frac{\theta^2+c}{2te^2} + \frac{\sqrt{-c}}{\theta}\ln\left|\frac{\theta-\sqrt{-c}}{\theta+\sqrt{-c}}\right| + \frac{c_1}{\theta}}, \quad \text{if} \quad c<0,$$

c, c_1 are constants, $\theta \equiv v^2$. In an analogous way we can find exact solutions for the other cases listed in Theorem 2. Here we represent operators, corresponding ansatzes and reduced systems of ODEs:

Operator	Ansätze	Reduced system
Q_1+P_3	$v^1 = \frac{\varphi_1(\omega)}{x_1}$ $v^2 = \varphi_2(\omega) - \ln x_1$ $\omega = x_2 - \frac{(v^2)^2}{2}$	$\frac{d\varphi_1}{d\omega} = -1$ $(\ln\varphi_1 - \varphi_2)\frac{d\varphi_2}{d\omega} = 1$
Q_1+D_1	$v^1 = x_1\varphi_1(\omega)$ $v^2 = x_1\varphi_2(\omega)$ $\omega = \frac{x_2}{v^2} - \ln v^2$	$\frac{d\varphi_1}{d\omega} = {\varphi_2}^2$ $\varphi_2 + \left(1+\omega+\ln\frac{\varphi_2}{\varphi_1}\right)\frac{d\varphi_2}{d\omega} = 0$
Q_2-D_1	$v^1 = \varphi_1(\omega) - \ln(x_1\varphi_2(\omega))$ $v^2 = x_1\varphi_2(\omega)$ $\omega = x_2 + c(v^2)^2$	$\exp(\varphi_1)\frac{d\varphi_1}{d\omega} = \varphi_2$ $\varphi_2\frac{d\varphi_1}{d\omega} - \frac{d\varphi_2}{d\omega} = {\varphi_2}^2$
Q_3	$v^1 = x_1^{\frac{-1}{(r+1)}}\varphi_1(\omega)$ $v^2 = x_1^{\frac{r}{r+1}}\varphi_2(\omega)$ $\omega = x_2 - v^2$	$\frac{r+1}{r}\varphi_2 = \frac{d\varphi_1}{d\omega}$ $\varphi_1^r\frac{d\varphi_2}{d\omega} = -1$
Q_3+rD_1	$v^1 = x_1^{\frac{1}{(r-1)}}\varphi_1(v^2)$ $v^2 = x_2 - x_1^{\frac{r}{r-1}}\varphi_2(v^2)$	$\frac{r}{1-r}\varphi_2 = \frac{d\varphi_1}{dv^2}$ $\varphi_1^{-r}\frac{d\varphi_2}{dv^2} = -1$
Q_4	$v^1 = \varphi_2(x_1) - \arcsin\frac{x_2}{\sqrt{\varphi_1(x_1)}}$ $v^2 = \sqrt{\varphi_1(x_1) - x_2^2}$	$\frac{d\varphi_1}{dx_1} = -2$ $\tan\varphi_2 = 0$
Q_5	$v^1 = \frac{\varphi_1(\omega)}{x_1}$ $v^2 = \sqrt{\varphi_2(\omega) - x_2^2}$ $\omega = \ln x_1 - \arcsin\frac{x_2}{\sqrt{\varphi_2}}$	$\frac{d\varphi_1}{d\omega} = \frac{1}{2}\frac{d\varphi_2}{d\omega}$ $\frac{d\varphi_2}{d\omega} = \frac{2\tan\{\ln(\varphi_1-\omega)\}\varphi_2}{1-\tan\{\ln(\varphi_1-\omega)\}}$

We stress that the obtained ansatzes cannot be found by classical point symmetries.

4 Discrete symmetries of Maxwell equations

Let us consider the Maxwell equations for vacuum in Dirac form [4]:

$$L\Psi = (p_0 - \vec{\alpha}\vec{p})\Psi = 0 \tag{5}$$

where Ψ is a 4-component function:

$$\Psi = \begin{pmatrix} 0 \\ E_1 - iH_1 \\ E_2 - iH_2 \\ E_3 - iH_3 \\ E_4 - iH_4 \end{pmatrix}, \qquad \alpha_a\alpha_b + \alpha_b\alpha_a = 2\delta_{ab}$$

Equation 5 admits the following operators of discrete symmetry (ODS):

$$Q_1 = \alpha_1 R_{23}, \qquad Q_2 = \alpha_2 R_{13}, \qquad Q_3 = \alpha_3 R_{12}, \qquad Q_4 = CT$$

(where R_{ab}: $x_a \to -x_a$, $x_b \to -x_b$, $a \neq b$, $a, b = 1, 2, 3$, C is operator of complex conjugation, T is operator of time inversion) as much as the invariance condition $[L, Q]\,\Psi = 0$ holds.

It is easy to verify that these ODS satisfy the following commutation relations ($a = 1, 2, 3$):

$$[Q_1, Q_2] = 2iQ_3, \qquad [Q_1, Q_3] = 2iQ_2, \qquad [Q_2, Q_3] = 2iQ_1, \qquad [Q_a, Q_4] = 0$$

and form $AO(3) \oplus Q_4$ algebra. Finding the diagonalizing operators for Q_1 and Q_2 we reduce the equation (5) to four independent one-component equations. Diagonalizing operator of Q_1 has the form:

$$U = U_1 U_2 U_3, \qquad U_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad U_1^{-1} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$U_2 = \frac{(1 + i\sigma_1)}{\sqrt{2}} \otimes I_2, \qquad U_2^{-1} = \frac{(1 - i\sigma_1)}{\sqrt{2}} \otimes I_2,$$

$$U_3 = \left(R_+^{23} + i\alpha_2 R_-^{23}\right), \qquad U_3^{-1} = \left(R_+^{23} - i\alpha_2 R_-^{23}\right), \qquad R_\pm^{23} = \frac{1 \pm R_{23}}{2}, \qquad \Psi' = U\Psi.$$

Thus the corresponding reducing equation has the form:

$$L'\Psi' = U_1 U_2 U_3 L U_3^{-1} U_2^{-1} U_1^{-1} \Psi' = \{p_0 - ip_2 R_{23} \mp p_1 R_{23} \pm p_3\}\Psi' = 0.$$

In a similar way we find that the diagonalizing operator of Q_2

$$U = U_1 U_2, \quad U_1 = \frac{1}{2}\begin{pmatrix} i & i & 1 & 1 \\ -1 & 1 & i & -i \\ -1 & -1 & -i & -i \\ i & -i & -1 & 1 \end{pmatrix}, \qquad U_1^{-1} = \frac{1}{2}\begin{pmatrix} -i & -1 & -1 & -i \\ -i & 1 & -1 & i \\ 1 & -i & i & -1 \\ 1 & i & i & 1 \end{pmatrix},$$

$$U_2 = (R_+ + i\alpha_1 R_-), \qquad U_2^{-1} = (R_+ - i\alpha_1 R_-)$$

reduces the corresponding equation (5) to the following uncoupled equations:

$$L'\Psi' = U_1 U_2 L U_2^{-1} U_1^{-1} \Psi' = \{p_0 - ip_1 R_{13} \mp p_2 R_{13} \mp p_3\}\Psi' = 0.$$

5 Conclusion

We have shown how we can use non-Lie symmetries (discrete, conditional and non-point ones) for reducing a given problem to a simpler one and to construct its exact solutions. It enabled us to find new classes of nonlinear heat equations for which the reduction method can be successfully applied. With our symmetries equation (1) can be reduced to system of ODEs. We are then able to derive the exact solutions of the heat equation (4) with nonlinearities:

$$C_1(\chi) = -\frac{2}{\chi^5}\exp\left(\frac{1}{\chi}\right), \qquad C_2(\chi) = \frac{1}{\chi(C\chi + 1)}, \quad C = \text{const},$$

$$C_3(\chi) = \frac{1}{r}\left\{\frac{(\chi - 1)^{(1-r)/r}}{\chi^{\frac{2+r}{r}}}\right\}, \qquad C_4(\chi) = \frac{1}{1 + \chi^2}, \qquad C_5(\chi) = \frac{\arctan\chi \exp(\arctan\chi)}{1 + \chi^2}.$$

As far as we know, this is the first application of the non-point and conditional symmetries of an evolution equation for construction its exact solutions. In our opinion using of generalized conditional symmetry in the proposed approach is very promising. It would be interesting to develop this technique for searching exact solutions of DEs with supersymmetries and extended supersymmetries.

Thus extending of symmetry ideas (non-point symmetry, conditional symmetry, supersymmetry) enables us to obtain new results which are important from both mathematical and physical point of view.

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Nonlocal Symmetry and Generating Solutions for the Inhomogeneous Burgers Equation

Valentyn TYCHYNIN † *and Olexandr RASIN* ‡

† *Prydniprovsk State Academy of Civil Engineering and Architecture, 24a Chernyshevsky Str., 49000 Dnipropetrovsk, Ukraine*
E-mail: *tychynin@pgasa.dp.ua, tychynin@ukr.net*

‡ *Dnipropetrovsk National University, 13 Naukovy Per., 49050 Dnipropetrovsk, Ukraine*
E-mail: *sasha_rasin@rambler.ru*

In the present paper we consider a class of inhomogeneous Burgers equations. Nonlocal transformations of a dependent variable that establish relations between various equations of this class were constructed. We identified the subclass of the equations, invariant under the appropriate substitution. The formula of non-local superposition for the inhomogeneous Burgers equation was constructed. We also present examples of generation of solutions.

1 Non-local invariance of the inhomogeneous Burgers equation

Let us consider an inhomogeneous Burgers equation:

$$u_t + uu_x - u_{xx} = c, \qquad u = u(x,t), \tag{1}$$

where $c = c(x,t)$ is an arbitrary smooth function, and $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

Let us make the first order non-local substitution of the dependent variable:

$$u = f(t,x,v,v_x), \tag{2}$$

where $v = v(x,t)$ is the new dependent variable.

We seek the transformation of the equation (1) into another inhomogeneous Burgers equation:

$$v_t + vv_x - v_{xx} = g, \tag{3}$$

with arbitrary smooth function $g = g(x,t)$. To obtain this transformation we substitute (2) and its differential prolongations into equation (1). For differential prolongations of the equation (3) we obtain the determining relation:

$$\begin{aligned} &f_t - f_v v_x v + f_v g - f_{v_x} v_{xx} v - f_{v_x} {v_x}^2 + f_{v_x} g_x + f f_x + f f_v v_x + f f_{v_x} v_{xx} - f_{xx} \\ &\quad - 2f_{vx} v_x - 2f_{v_x x} v_{xx} - f_{vv} {v_x}^2 - 2v_x f_{vv_x} v_{xx} - f_{v_x v_x} {v_{xx}}^2 - c = 0. \end{aligned}$$

Having split these expressions with respect to the derivative v_{xx}, we obtain the system of equations:

$$\begin{aligned} &f_{v_x v_x} = 0, \\ &-f_{v_x} v + f f_{v_x} - 2f_{vv_x} v_x - 2f_{v_x x} = 0, \\ &f_t - f_v v_x v + f_v g + f f_v v_x - f_{v_x} {v_x}^2 + f_{v_x} g_x + f f_x - c - f_{vv} {v_x}^2 - f_{xx} - 2f_{vx} v_x = 0. \end{aligned} \tag{4}$$

The nontrivial solution of the system (1) is

$$f = \frac{-2v_x + 4F_x + v^2 - 2vF}{v - 2F}, \tag{5}$$

$$c = 2F_{xx} + 2F_t + 4F_xF, \tag{6}$$
$$g = -2F_{xx} + 2F_t + 4F_xF. \tag{7}$$

Here $F = F(x,t)$ is an arbitrary smooth function.

Let us consider a non-local invariance transformation, namely $c = g$. The equations (6), (7) give us the condition $F_{xx} = 0$. Integration yields $F = A(t)x + B(t)$, where $A(t)$ and $B(t)$ are arbitrary smooth functions. So we have found the non-local invariant transformation for the inhomogeneous Burgers equation in a general form:

$$f = \frac{2v_x - 4A - v^2 + 2vAx + 2vB}{-v + 2Ax + 2B}, \tag{8}$$
$$g = c = 2A_tx + 2B_t + 4A^2x + 4AB. \tag{9}$$

Substituting $c = g = 0$ into expressions (6), (7), we find the system:

$$2F_{xx} + 2F_t + 4F_xF = 0,$$
$$-2F_{xx} + 2F_t + 4F_xF = 0.$$

The general and trivial solutions of this system are:

$$F = \frac{x + c_1}{2t + 2c_2}, \qquad F = 0. \tag{10}$$

Here c_1, c_2 are an arbitrary constants. Having substituted (10) into (5), we get the following non-local invariant transformations of the homogeneous Burgers equation for the general and trivial solutions respectively:

$$u = \frac{2v_xt + 2v_xc_2 - 2 - v^2t - v^2c_2 + vx + vc_1}{-vt - vc_2 + x + c_1}, \qquad u = \frac{-2v_x + v^2}{v}.$$

Example 1. Using (5), (6), (7) we transform the inhomogeneous Burgers equation:

$$u_t + uu_x - u_{xx} = 8\frac{\sin x}{\cos^3 x}, \tag{11}$$

into homogeneous one. We obtain the system of the equations for F by substituting $c = 8\frac{\sin x}{\cos^3 x}$ and $g = 0$ in (6), (7):

$$F_t + 2F_xF - F_{xx} = 0,$$
$$F_t + 2F_xF + F_{xx} = 4\frac{\sin x}{\cos^3 x}.$$

There is the general solution of the system $F = \mathrm{tg}x$. It gives us the following substitution:

$$u = \frac{2v_x\cos^2 x - 4 - v^2\cos^2 x + 2v\sin x\cos x}{\cos x(-v\cos x + 2\sin x)}. \tag{12}$$

This expression can be applied to generation of solutions of the equation (11). Thus, the partial solution of the homogeneous Burgers equation:

$$v = \frac{-4x}{2t + x^2}, \tag{13}$$

generates the following solution of the equation (11):

$$u = -2\frac{2\cos^2 x + 2t + x^2 + 2x\sin x\cos x}{\cos x(2x\cos x + 2t\sin x + x^2\sin x)}. \tag{14}$$

Example 2. It is an example of application of the non-local transformation (8) for a partial case of equation (1): $A = 1, B = t$. Then we have the equation:

$$u_t + uu_x - u_{xx} = 4x + 4t + 2, \tag{15}$$

and corresponding invariant transformation:

$$\overset{(2)}{u} = \frac{2\overset{(1)}{u}_x - 4 - \overset{(1)}{u}^2 + 2\overset{(1)}{u}x + 2\overset{(1)}{u}t}{-\overset{(1)}{u} + 2x + 2t}. \tag{16}$$

One of the similarity solutions of the equation (15) (see Appendix 1) is

$$u = -2 - 2x - 2t + e^{2t}\mathrm{tg}\left(\frac{1}{4}(2x + 1 + 2t)\, e^{2t}\right).$$

Having substituted it into (16) we find a new solution of the equation (15):

$$\overset{(2)}{u} = -\frac{(6t + 6x + 4)\,\mathrm{tg}\left(\frac{e^{2t}}{4}(2x + 1 + 2t)\right) e^{2t} - 12 - 8x^2 - 12x - 12t - 8t^2 - 16xt}{-2 - 4x - 4t + e^{2t}\mathrm{tg}\left(\frac{e^{2t}}{4}(2x + 1 + 2t)\right)}.$$

Using this algorithm we get a chain of solutions:

$$\begin{aligned} -2x - 2t - 2 &\to -2\frac{3 + 2x^2 + 4xt + 3x + 2t^2 + 3t}{2x + 2t + 1} \\ &\to -2\frac{4x^3 + 8x^2 + 12x^2t + 12xt^2 + 16xt + 15x + 15t + 4t^3 + 7 + 8t^2}{3 + 4x^2 + 8xt + 4x + 4t^2 + 4t} \to \cdots. \end{aligned}$$

2 Linearization of the inhomogeneous Burgers equation

We are looking for a non-local transformation

$$u = f(v, v_x), \tag{17}$$

of equation (1), where $v = v(x, t)$ is a smooth function, which yields equation:

$$v_t - v_{xx} + \varphi = 0. \tag{18}$$

Here $\varphi = \varphi(x, t, v)$ is an arbitrary smooth function. To obtain this transformation we substitute (17) and its differential prolongations into equation (1). For differential prolongations of the equation (18) we obtain the determining correlation:

$$f_v\varphi + f_{v_x}\varphi_x + f_{v_x}\varphi_v v_x + ff_v v_x + ff_{v_x}v_{xx} - f_{vv}{v_x}^2 - 2v_x f_{vv_x}v_{xx} - f_{v_xv_x}{v_{xx}}^2 + c = 0.$$

Having split these expressions with respect to the derivative v_{xx}, we obtain the system of equations:

$$\begin{aligned} &f_{v_xv_x} = 0, \\ &2f_{vv_x}v_x - ff_{v_x} = 0, \\ &f_v\varphi + f_{v_x}\varphi_x + f_{v_x}\varphi_v v_x + ff_v v_x + c - f_{vv}{v_x}^2 = 0. \end{aligned}$$

The nontrivial solution is

$$f = -2\frac{v_x}{v + c_1}, \tag{19}$$

$$c = -2F_x, \tag{20}$$
$$\varphi = -F(v + c_1). \tag{21}$$

Here c_1 is an arbitrary constant and $F = F(x,t)$ is an arbitrary smooth function. Thus we obtain the Cole–Hopf substitution with the parameter c_1.

$$u = -2\frac{v_x}{v + c_1}. \tag{22}$$

So the inhomogeneous Burgers equation may be transformed into the linear equation with variable coefficients [2, 4]:

$$v_t - F(v + c_1) - v_{xx} = 0.$$

Here the function F is obtained from the equation (20). In the case $c_1 = 0$, we obtain the transformation into the homogeneous equation:

$$v_t - Fv - v_{xx} = 0. \tag{23}$$

Theorem 1. *Formula of a nonlinear superposition for inhomogeneous Burgers equation can be written in the following way:*

$$\begin{aligned} &\overset{(3)}{u} = -2\partial_x \ln\left(\overset{(1)}{\tau} + \overset{(2)}{\tau}\right), \\ &-2\partial_x \ln \overset{(k)}{\tau} = \overset{(k)}{u}, \qquad k = 1, 2, \\ &-2\partial_t \ln \overset{(k)}{\tau} = \overset{(k)}{u}_x - \frac{1}{2}\overset{(k)}{u}^2 + \psi, \qquad \psi_x = c(x,t) \end{aligned} \tag{24}$$

Here $\overset{(1)}{u}$, $\overset{(2)}{u}$ *are known solutions, and* $\overset{(3)}{u}$ *is the new one.*

Proof. Let $\overset{(1)}{\tau}$, $\overset{(2)}{\tau}$ be solutions of equation (23). Then $\overset{(3)}{\tau} = \overset{(1)}{\tau} + \overset{(2)}{\tau}$ is a new solution of equation (23). By using the substitution $u = -2\partial_x \ln(\tau)$ we can find $\overset{(3)}{u}$:

$$\overset{(3)}{u} = -2\partial_x \ln\left(\overset{(3)}{\tau}\right) = -2\partial_x \ln\left(\overset{(1)}{\tau} + \overset{(2)}{\tau}\right).$$

On the other side $\overset{(k)}{\tau}$, $k = 1, 2$ are connected with $\overset{(k)}{u}$, $k = 1, 2$ in the following way:

$$-2\partial_x \ln \overset{(k)}{\tau} = \overset{(k)}{u}, \qquad -2\partial_t \ln \overset{(k)}{\tau} = \overset{(k)}{u}_x - \frac{1}{2}\overset{(k)}{u}^2 + \psi, \qquad \psi_x = c(x,t), \qquad k = 1, 2.$$

So superposition formula (24) is obtained. ■

Example 3. We can use superposition formula for equation (15). There are two solutions of equation (15):

$$\begin{aligned} &\overset{(1)}{u} = -2 - 2x - 2t + e^{2t}\mathrm{tg}\left(\frac{1}{4}(2x + 1 + 2t)e^{2t}\right), \\ &\overset{(2)}{u} = -2x - 2t - 2, \end{aligned}$$

The formula (24) gives us a third one:

$$\overset{(3)}{u} = -2 - 2x - 2t + \frac{\mathrm{tg}\left(\left(\frac{x}{2} + \frac{1}{4} + \frac{t}{2}\right)e^{2t}\right)e^{2t}}{1 + e^{\frac{e^{4t}}{16}}\sec^2\left(\left(\frac{x}{2} + \frac{1}{4} + \frac{t}{2}\right)e^{2t}\right)}.$$

3 Lie symmetries for the inhomogeneous Burgers equation

To apply the classical method [3] to (1) we require the infinitesimal operator to be of this form:

$$X = \xi_0(x,t,u)\partial_t + \xi_1(x,t,u)\partial_x + \eta(x,t,u)\partial_u.$$

The invariance condition for equation (1) yields an overdetermined system of differential equations for the coordinates of X. Having solved the system of equations we obtain the following expressions for infinitesimals and function $c(x,t)$:

$$\begin{aligned}
&\xi_0 = F_1, \qquad \xi_1 = \frac{1}{2}F_1' x + F_2, \qquad \eta = -\frac{1}{2}uF_1' + \frac{1}{2}F_1'' x + F_2',\\
&c = \left(\frac{1}{2}\int F_1''' F_1 \int F_2 F_1^{-3/2} dt dt - \frac{1}{2}\int F_1''' F_1 dt \int F_2 F_1^{-3/2} dt\right.\\
&\qquad \left. + \int F_1^{1/2} F_2'' dt + F_3(\omega)\right) F_1^{-3/2} + \frac{x}{2}\int F_1''' F_1 dt F_1^{-2},\\
&\omega = \left(x - F_1^{1/2}\int F_2 F_1^{-3/2} dt\right) F_1^{-1/2}.
\end{aligned}$$

Here $F_1 = F_1(t)$, $F_2 = F_2(t)$, $F_3 = F_3(\omega)$ are arbitrary smooth functions. If $F_1 = 0$ we have:

$$c = \frac{F_2'' x}{F_2} + F_3,$$

Obviously, the connection between the right hand side of the equation (1) and its Lie symmetries exists.

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Symmetry Analysis of (2 + 1)-Dimensional Doebner–Goldin Equations

Jörg VOLKMANN and Gerd BAUMANN

Department of Mathematical Physics, University of Ulm,
Albert–Einstein–Allee 11, D–89069 Ulm/Donau, Germany
E-mail: *volk@physik.uni-ulm.de, gerd.baumann@physik.uni-ulm.de*

The paper discusses the application of *MathLie* in connection with Lie group analysis. The examined example is the (2 + 1)-dimensional case of the Doebner–Goldin equations after Madelung representation. The related Lie algebras are calculated and classified. Furthermore we discuss the determination of an optimal system for the 7-dimensional case of one Lie algebra.

1 Introduction

The application of Lie's theory to examine systems of partial differential equations is one of the most efficient methods to calculate solutions for equations of motions. Furthermore Lie's theory allows the classification of solutions and related algebras. One can use Lie's transformation theory as a microscope to get information about the properties of a physical model [1, 2]. Lie's method is discussed in literature in great detail [3–6]. With the algorithms at hand one can generate computer programs such as *MathLie* to automatically carry out the calculations. Today there are a large number of symbolic computing programs for the algebraic manipulation of equations.

In this discussion we show investigations to get information about the structure of solutions of the Doebner–Goldin–Madelung equations. The symmetry investigations of these equations are carried out by using the Mathematica program *MathLie*. Section 3 is concerned with the algebra investigation of a 7-dimensional algebra using the *Mathematica* program *MathLieAlg* by R. Schmid [7]. Section 4 deals with an algorithm of calculating optimal systems. The example discussed in Section 4 is a 7-dimensional algebra generated by one of the Doebner–Goldin–Madelung equation.

2 Derivation of the Doebner–Goldin equations

The investigation of Borel quantization for S^1 leads to a non-linear Schrödinger equation of the form (here with $m = 1$, $\hbar = 1$):

$$\mathrm{i}\partial_t\psi = -\frac{1}{2}\Delta\psi + V(\vec{x},t)\psi + \frac{\mathrm{i}}{2}KR_2(\psi)\psi + \sum_{j=1}^{5} D_jR_j(\psi)\psi, \tag{1}$$

derived by Dobrev, Doebner and Twarock [8] and nowadays called Doebner–Goldin equations. Here, the $R_j(\psi)$ with $j \in \{1,\ldots,5\}$ are real valued functionals of the real valued density $\varrho = \overline{\varrho} = \psi\overline{\psi}$ and the real valued current $\vec{j} = \overline{\vec{j}} = \frac{\mathrm{i}\hbar}{2m}\left(\psi\nabla\overline{\psi} - \overline{\psi}\nabla\psi\right)$ and can be found in [8, 9].

In the following we consider the (2+1)-dimensional case of equation (1) without potential V. Applying the Madelung transformation

$$\psi \to \sqrt{\varrho(\vec{x},t)}\exp\left(\mathrm{i}S(\vec{x},t)\right), \qquad \overline{\psi} \to \sqrt{\varrho(\vec{x},t)}\exp\left(-\mathrm{i}S(\vec{x},t)\right)$$

to equation (1) and dividing the resulting equations into real and imaginary parts we find the following system:

$$\varrho_t + S_x\varrho_x + S_y\varrho_y + \varrho S_{xx} + \varrho S_{yy} - \delta\varrho_{xx} - \delta\varrho_{yy} = 0, \tag{2}$$

$$(1+8D_5)\big(\varrho_x^2+\varrho_y^2\big) + 4\varrho^2\big(2S_t + S_x^2 + 2D_3S_x^2 + (1+2D_3)S_y^2 + 2D_1S_{xx} + 2D_1S_{yy}\big) \tag{3}$$
$$+ 2\varrho(4D_1S_x\varrho_x + 4D_4S_x\varrho_x + 4(D_1+D_4)S_y\varrho_y - \varrho_{xx} + 4D_2\varrho_{xx} + (-1+4D_2)\varrho_{yy}) = 0,$$

where $m = 1$ and $\hbar = 1$. Here, D_1, D_2, D_3, D_4, and δ are real valued parameters. By permutating these parameters we receive 63 different model equations (see table in [9]) of nonlinear Schrödinger type. This set of equations is called the set of Doebner–Goldin–Madelung equations. The whole set of equations are examined with the Mathematica program *MathLie*.

3 Symmetry analysis of the (2 + 1)-dimensional Doebner–Goldin–Madelung equations

In order to find the symmetry group of equations (2), (3), we apply the algorithms described in text books such as [3, 1]. We look for an algebra of vector fields of the form

$$V = \xi[1]\partial_x + \xi[2]\partial_t + \phi[1]\partial_\varrho + \phi[2]\partial_S,$$

where the infinitesimals $\xi[1]$, $\xi[2]$ and $\phi[1]$, $\phi[2]$ depend on x, t, ϱ, and S in general.

These coefficients are determined from the requirement that the second prolongation of V should annihilate the equation on the solution set of the equation. This was done by using the Mathematica program *MathLie* [3] for all 63 model equations of the Doebner–Goldin–Madelung equations.

The next step of our discussion is related to the investigation of the 7-dimensional algebra of the equation with the parameters D_3, D_5 (see [9]). We use a *Mathematica* program *MathLieAlg* by R. Schmid [7]. The generators of this equation are:

$$V[1] = \partial_t, \qquad V[2] = \varrho\partial_\varrho, \qquad V[3] = -4t\partial_t - 2x\partial_x - 2y\partial_y,$$
$$V[4] = \partial_x, \qquad V[5] = \partial_S, \qquad V[6] = -y\partial_x + x\partial_y, \qquad V[7] = \partial_y. \tag{4}$$

The only non-zero commutators of the vector fields are following:

$$[V[1],V[3]] = -4V[1], \qquad [V[3],V[4]] = 2V[4], \qquad [V[3],V[7]] = 2V[7],$$
$$[V[4],V[6]] = V[7], \qquad [V[6],V[7]] = V[4].$$

Nontrivial algebra elements are $\{V[1],V[2],V[3],V[5],V[6]\}$. Algebras can be generated by the following sets:

$$\{V[1],V[2],V[3],V[4],V[5],V[6]\}, \qquad \{V[1],V[2],V[3],V[5],V[6],V[7]\},$$
$$\{V[1],V[2],V[3],V[4],V[5],V[6],V[7]\}.$$

The Cartan metric of this algebra reads:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 24 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The algebraic properties are following: not semisimple, solvable, not nilpotent.

From this algebra the subalgebras listed in Table 1 can be derived:

Table 1. Table of subalgebras[1].

Dimension	Number of subalgebras	Subalgebra elements
0	1	$\{\}$
1	7	$\{V[1]\},\{V[2]\},\{V[3]\},\{V[4]\},\{V[5]\},\{V[6]\},\{V[7]\}$
2	19	$\{V[1],V[2]\},\{V[1],V[3]\},\{V[1],V[4]\},\{V[1],V[5]\},\{V[1],V[6]\},$ $\{V[1],V[7]\},\{V[2],V[3]\},\{V[2],V[4]\},\{V[2],V[5]\},\{V[2],V[6]\},$ $\{V[2],V[7]\},\{V[3],V[4]\},\{V[3],V[5]\},\{V[3],V[6]\},\{V[3],V[7]\},$ $\{V[4],V[5]\},\{V[4],V[7]\},\{V[5],V[6]\},\{V[5],V[7]\}$
3	27	$\{V[1],V[2],V[3]\},\{V[1],V[2],V[4]\},\{V[1],V[2],V[5]\},\{V[1],V[2],V[6]\},$ $\{V[1],V[2],V[7]\},\{V[1],V[3],V[4]\},\{V[1],V[3],V[5]\},\{V[1],V[3],V[6]\},$ $\{V[1],V[3],V[7]\},\{V[1],V[4],V[5]\},\{V[1],V[4],V[7]\},\{V[1],V[5],V[6]\},$ $\{V[1],V[5],V[7]\},\{V[2],V[3],V[4]\},\{V[2],V[3],V[5]\},\{V[2],V[3],V[6]\},$ $\{V[2],V[3],V[7]\},\{V[2],V[4],V[5]\},\{V[2],V[4],V[7]\},\{V[2],V[5],V[6]\},$ $\{V[2],V[5],V[7]\},\{V[3],V[4],V[5]\},\{V[3],V[4],V[7]\},\{V[3],V[5],V[6]\},$ $\{V[3],V[5],V[7]\},\{V[4],V[5],V[7]\},\{V[4],V[6],V[7]\}$
4	23	$\{V[1],V[2],V[3],V[4]\},\{V[1],V[2],V[3],V[5]\},\{V[1],V[2],V[3],V[6]\},$ $\{V[1],V[2],V[3],V[7]\},\{V[1],V[2],V[4],V[5]\},\{V[1],V[2],V[4],V[7]\},$ $\{V[1],V[2],V[5],V[6]\},\{V[1],V[2],V[5],V[7]\},\{V[1],V[3],V[4],V[5]\},$ $\{V[1],V[3],V[4],V[7]\},\{V[1],V[3],V[5],V[6]\},\{V[1],V[3],V[5],V[7]\},$ $\{V[1],V[4],V[5],V[7]\},\{V[1],V[4],V[6],V[7]\},\{V[2],V[3],V[4],V[5]\},$ $\{V[2],V[3],V[4],V[7]\},\{V[2],V[3],V[5],V[6]\},\{V[2],V[3],V[5],V[7]\},$ $\{V[2],V[4],V[5],V[7]\},\{V[2],V[4],V[6],V[7]\},\{V[3],V[4],V[5],V[7]\},$ $\{V[3],V[4],V[6],V[7]\},\{V[4],V[5],V[6],V[7]\}$
5	13	$\{V[1],V[2],V[3],V[4],V[5]\},\{V[1],V[2],V[3],V[4],V[7]\},\{V[1],V[2],V[3],V[5],V[6]\},$ $\{V[1],V[2],V[3],V[5],V[7]\},\{V[1],V[2],V[4],V[5],V[7]\},\{V[1],V[2],V[4],V[6],V[7]\},$ $\{V[1],V[2],V[3],V[4],V[5]\},\{V[1],V[3],V[4],V[5],V[7]\},\{V[1],V[3],V[4],V[6],V[7]\},$ $\{V[1],V[4],V[5],V[6],V[7]\},\{V[2],V[3],V[4],V[5],V[7]\},\{V[2],V[3],V[4],V[6],V[7]\},$ $\{V[2],V[4],V[5],V[6],V[7]\},\{V[3],V[4],V[5],V[6],V[7]\}$
6	5	$\{V[1],V[2],V[3],V[4],V[5],V[7]\},\{V[1],V[2],V[3],V[4],V[6],V[7]\},$ $\{V[1],V[2],V[4],V[5],V[6],V[7]\},\{V[1],V[3],V[4],V[5],V[6],V[7]\},$ $\{V[2],V[3],V[4],V[5],V[6],V[7]\}$
7	1	$\{V[1],V[2],V[3],V[4],V[5],V[6],V[7]\}$

Examining the subalgebras we can find the ideals listed in [9]. The normalizer of all ideals is the algebra $\{V[1],V[2],V[3],V[4],V[5],V[6],V7]\}$ and the radical reads $\{V[1],V[2],V[3],V[4],V[5],V[6],V7]\}$. For the center we find $\{V[2],V[5]\}$. The adjoint representation of our 7-dimensional algebra can be found in [9].

4 The optimal system of the seven-dimensional algebra

Let us first consider the general system of differential equations

$$F\big(x,u,u^{(n)}\big)=0 \tag{5}$$

and the related symmetry group G. For every s-parametric subgroup H_s one can calculate similarity solutions under the assumption[2], that $s<\min(r,n')$, where n' is the number of independent variables and r is the order of the system of differential equations [4]. In this set of similarity solutions there are such solutions, which can be calculated by a transformation of the symmetry group from other similarity solutions. Our aim is to derive a minimal set of similarity solutions from which one can gain all the other solutions by a transformation. Such a list is called optimal system of similarity solutions with elements which are essentially different types of similarity solutions. The application of the conjugation (see e.g. [12]) and a theorem

[1] $\{\}$ is empty set.

[2] Following [12] the number of parameters s of a subgroup H of a symmetry group G will be written under the symbol of the subgroup.

by Olver [6] allows us to transform the problem of classifying solutions to that of classifying subgroups. The adjoint representation maps this problem to the classification of subalgebras with respect to inner automorphisms. The result is the optimal system of subalgebras. More details can be found in [4, 6].

The literature presents several methods of classifying subalgebras. A detailed discussion of the procedures are given in [10, 4, 5, 11]. A common property of these methods is that they all start with algebras of very low dimension.

Due to Ovsyannikov [13–15] we organize our calculations by the following definition and theorem.

Definition 1 ([15]). An optimal system ΘL is said to be normalized if $\mathrm{Nor}\, K \in \Theta L$ whenever a subalgebra K is in ΘL.

The existence of such optimal system follows from

Theorem 1 ([15]). *For any finite-dimensional Lie algebra there exists a normalized optimal system ΘL of subalgebras.*

The following discussion demonstrates the application of our algorithm [2] to the 7-dimensional algebra 4. We start our calculation with the series of ideals

$$\{\} \subset \{V[1], V[4]\} \subset \{V[1], V[4], V[7]\} \subset \{V[1], V[3], V[4], V[5], V[6], V[7]\}$$
$$\subset \{V[1], V[2], V[3], V[4], V[5], V[6], V[7]\} = L_7$$

and take the maximal Abelian ideal $I_{\max} = \{V[1], V[4], V[7]\}$. The related factor algebra $L_7 \backslash I_{\max}$ is $\{V[2], V[3], V[5], V[6]\}$. Now we are doing the same step with this factor algebra. The series of ideals is

$$0 \subset \{V[2], V[5]\} \subset \{V[2], V[3], V[5], V[6]\}$$

with the maximal Abelian subalgebra $\{V[2], V[5]\}$. The related factor algebra reads $\{V[3], V[6]\}$. By starting with the smallest factoralgebra we have to express a general vector Y by a linear combination of the vectors $V[3]$, $V[6]$:

$$Y_1 = x_3 V[3] + x_6 V[6], \qquad Y_2 = y_3 V[3] + y_6 V[6]. \tag{6}$$

The coefficient matrix of (6) reads $\begin{pmatrix} x_3 & x_6 \\ y_3 & y_6 \end{pmatrix}$ which allows manipulation by row and application of the adjoint representation of the ranks 2, 1, 0. The classification of the above matrix delivers the result in Table 2.

Table 2. Optimal system of the two-dimensional subalgebra $\{V[3], V[6]\}$.

Dimension	Subalgebras
2	$\{V[3], V[6]\}$
1	$\{V[3] + x_6 V[6]\}$, $x_6 \neq 0$, $\{V[3]\}$, $\{V[6]\}$
0	$\{\}$

Now we have to consider the first extension of the algebra. An equation similar to (6) leads to the matrix

$$\begin{pmatrix} u_2 & u_5 & u_3 & u_6 \\ v_2 & v_5 & v_3 & v_6 \\ x_2 & x_5 & x_3 & x_6 \\ y_2 & y_5 & y_3 & y_6 \end{pmatrix} \tag{7}$$

which has to be classified. It allows the ranks 4, 3, 2, 1, 0. In addition there exists a block structure. The 2×2 matrix in the lower right corner with the indices $(3, 6)$ and the matrix in the upper left corner with the indices $(2, 5)$.

We start our calculation with the matrix in the upper left corner related to the indices $(2, 5)$, which allows the ranks 2, 1, 0. The results are

Table 3. Optimal system of the subalgebra $\{V[2], V[5]\}$.

Dimension	Subalgebras
2	$\{V[2], V[5]\}$
1	$\{V[2] + u_5V[5]\}$, $u_5 \neq 0$, $\{V[2]\}$, $\{V[5]\}$

For the rank 0 the upper left matrix $(2, 5)$ only contains 0. So we have to consider the matrix containing the last two lines of (7). The block matrix with $(3, 6)$ index runs through all subalgebras of Table 2. The final result of this calculation step is

Table 4. Optimal system of the subalgebra $\{V[2], V[5], V[3], V[6]\}$.

Dimension	Subalgebras
4	$\{V[3], V[6], V[2], V[5]\}$
3	$\{V[3], V[6], V[2] + x5V[5]\}, \{V[3], V[6], V[2]\}, \{V[3], V[6], V[5]\}$ $\{V[3] + x6V[6], V[2], V[5]\}, \{V[3], V[2], V[5]\}, \{V[6], V[2], V[5]\}$
2	$\{V[3] + x_6V[6], V[2] + x_5V[5]\}, \{V[3] + x_6V[6], V[2]\}, \{V[3] + x_6V[6], V[5]\},$ $\{V[3], V[2] + x_5V[5]\}, \{V[3], V[2]\}, \{V[3], V[5]\}, \{V[6], V[2] + x_5V[5]\}, \{V[6], V[2]\},$ $\{V[6], V[5]\}, \{V[2], V[5]\}, \{x_2V[2] + x_5V[5] + V[3], y_2V[2] + y_5V[5] + V[6]\},$ $\{x_2V[2] + x_5V[5] + V[3] + x_6V[6], y_2V[2] + y_5V[5]\}, \{x_2V[2] + x_5V[5] + V[3], y_2V[2] + y_5V[5]\},$ $\{x_2V[2] + x_5V[5] + V[6], y_2V[2] + y_5V[5]\}, \{V[3], V[6]\}$
1	$\{V[2] + x_5V[5]\}, \{V[2]\}, \{V[5]\}, \{V[3] + x_6V[6]\}, \{V[3]\}, \{V[6]\}$

In the next step of our algorithm [2], we have to consider the properties of the matrix

$$\begin{pmatrix} a_1 & a_4 & a_7 & a_5 & a_3 & a_6 \\ b_1 & b_4 & b_7 & b_5 & b_3 & b_6 \\ c_1 & c_4 & c_7 & c_5 & c_3 & c_6 \\ u_1 & u_4 & u_7 & u_5 & u_3 & u_6 \\ v_1 & v_4 & v_7 & v_5 & v_3 & v_6 \\ x_1 & x_4 & x_7 & x_5 & x_3 & x_6 \\ y_1 & y_4 & y_7 & y_5 & y_3 & y_6 \end{pmatrix}. \tag{8}$$

It desintegrates into a block structure where we have to consider the matrix in the upper left corner (first three lines) with the indices $(1, 3, 7)$. It allows the ranks 3, 2, 1, 0. The result of this classification is

Table 5. Optimal system of the subalgebra $\{V[1], V[4], V[7]\}$.

Dimension	Subalgebras
3	$\{V[1], V[4], V[7]\}$
2	$\{V[1] + a_7V[7], V[4] + b_7V[7]\}, \{V[1] + a_4V[4], V[7]\}, \{V[4], V[7]\}$
1	$\{V[1] + a_4V[4] + a_7V[7]\}, \{V[1] + a_7V[7]\}, \{V[1]\}, \{V[4] + a_7V[7]\}, \{V[7]\}$

By taking into account the block structure we have the matrix in the upper left corner with the indices $(1, 4, 7)$ (first three lines) and the matrix in the right left corner with the indices $(2, 5, 3, 6)$ (last four lines). To classify the whole matrix (8) every subalgebra of the upper left corner from Table 5 has to combine with every subalgebra of the lower right corner from Table 4.

For the rank 0 we have to investigate the matrix built by the last four lines of (8). The block matrix with the indices $(2, 5, 3, 6)$ runs through all subalgebras of Table 4. The final result of classifying this matrix is given in Table 6.

Table 6. Optimal system of the subalgebra $\{V[1], V[4], V[7], V[2], V[5], V[3], V[6]\}$.

Dimension	Optimal system
4	$\{V[3], v_1V[1]+V[6], x_1V[1]+V[2]+x_5V[5], y_1V[1]\}$, $\{V[3], v_1V[1]+V[6], V[5], y_1V[1]\}$, $\{V[2]+u_7V[7], V[5]+v_7V[7], V[3], y_7V[7]\}$, $\{V[2]+u_4V[4]+u_7V[7], V[5], V[3], y_4V[4]+y_7V[7]\}$, $\{V[2]+u_4V[4], V[3]+v_6V[6], V[4], V[7]\}$, $\{V[2]+u_7V[7], V[3]+v_6V[6], V[4], V[7]\}$, $\{V[5]+u_7V[7], V[3]+v_6V[6], V[4], V[7]\}$, $\{u_4V[4]+V[5], V[3]+v_6V[6], V[4], V[7]\}$, $\{V[2]+u_5V[5]+u_7V[7], V[3], V[4], V[7]\}$, $\{V[2]+u_5V[5], V[3], V[1], V[4]+y_7V[7]\}$, $\{u_1V[1]+V[2]+u_5V[5], V[3], V[1], V[7]\}$, $\{V[2]+u_4V[4]+u_5V[5], V[3], V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_7V[7], V[3], V[1], V[7]\}$, $\{u_1V[1]+V[2]+u_7V[7], V[3], V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4], V[3], V[1], V[4]\}$, $\{u_1V[1]+V[2], V[3], V[1], V[7]\}$, $\{V[2]+u_4V[4], V[3], V[4], V[7]\}$, $\{V[5]+u7V[7], V[3], V[4], V[7]\}$, $\{u_1V[1]+V[5], V[3], V[1], V[4]+y_7V[7]\}$, $\{u_1V[1]+V[5], V[3], V[1], V[7]\}$, $\{u_4V[4]+V[5], V[3], V[4], V[7]\}$, $\{u_1V[1]+V[6], v_1V[1]+V[2]+v_5V[5]+v_7V[7], V[4], V[7]\}$, $\{u_1V[1]+V[6], v_1V[1]+V[2]+v_4V[4]+v_5V[5], V[4], V[7]\}$, $\{u_1V[1]+V[6], v_1V[1]+V[2]+v_7V[7], V[4], V[7]\}$, $\{u_1V[1]+V[6], v_1V[1]+V[2]+v_4V[4], V[4], V[7]\}$, $\{u_1V[1]+V[6], v_1V[1]+V[5]+v_7V[7], V[4], V[7]\}$ $\{u_1V[1]+V[6], v_1V[1]+v_4V[4]+V[5], V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+x_7V[7], V[1]+x_7V[7], V[4]+y_7V[7]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]+x_4V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]+x_7V[7], V[4]+y_7V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]+x_4V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[4], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_4V[4]+V[6]+v_7V[7], V[4], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_4V[4]+V[6], V[4], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5]+u_6V[6], v_2V[2]+v_5V[5]+v_7V[7], V[4], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5]+u_6V[6], v_2V[2]+v_4V[4]+v_5V[5], V[4], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_1V[1]+v_2V[2]+v_5V[5]+v_7V[7], V[4], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_1V[1]+v_2V[2]+v_5V[5], V[1], V[4]+y_7V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_1V[1]+v_2V[2]+v_5V[5], V[1], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_4V[4]+v_5V[5], V[4], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]+V[6]+v_7V[7], V[4], V[7]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_1V[1]+v_2V[2]+v_4V[4]+v_5V[5]+V[6], V[1], V[4]\}$, $\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_4V[4]+v_5V[5]+V[6], V[4], V[7]\}$, $\{u_1V[1]+u_2V[2]+u_5V[5]+V[6], v_1V[1]+v_2V[2]+v_5V[5]+v_7V[7], V[4], V[7]\}$, $\{u_1V[1]+u_2V[2]+u_5V[5]+V[6], v_1V[1]+V[2]+v_4V[4]+v_5V[5], V[4], V[7]\}$, $\{V[3], V[6]+v_7V[7], V[4], V[7]\}$, $\{V[3], v_4V[4]+V[6], V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_5V[5]+V[6], V[1], V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4]+v_5V[5], V[1], V[4], V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4]+v_5V[5], V[1], V[4], V[7]\}$, $\{u_1V[1]+V[2]+v_7V[7], V[1], V[4], V[7]\}$, $\{u_1V[1]+V[2]+v_4V[4], V[1], V[4], V[7]\}$, $\{u_1V[1]+V[5]+u_7V[7], V[1], V[4], V[7]\}$, $\{u_1V[1]+u_4V[4]+V[5], V[1], V[4], V[7]\}$, $\{V[3]+u_6V[6], V[1], V[4], V[7]\}$, $\{V[3], V[1], V[4], V[7]\}$, $\{u_1V[1]+V[6], V[1], V[4], V[7]\}$
3	$\{V[3], V[6], V[2]+x_5V[5]\}$, $\{V[3], V[6], V[2]\}$, $\{V[3], V[6], V[5]\}$, $\{V[2], V[5], V[3]+x_6V[6]\}$, $\{V[2], V[3], V[5]\}$,$\{u_1V[1]+V[2], v_1V[1]+V[5], x_1V[1]+V[6]\}$, $\{u_1V[1]+V[2], V[3]+v_6V[6], V[1]\}$, $\{u_1V[1]+V[5], V[3]+v_6V[6], V[1]\}$, $\{u_1V[1]+V[2]+u_5V[5], V[3], V[1]\}$, $\{V[2]+u_5V[5], V[3], V[4]+x_7V[7]\}$, $\{V[2]+u_4V[4]+u_5V[5], V[3], V[4]\}$, $\{V[2]+u_5V[5], V[3], V[7]\}$, $\{u_1V[1]+V[2], V[3], V[1]\}$, $\{V[2], V[3], V[4]+x_7V[7]\}$, $\{V[2]+u_4V[4], V[3], V[4]\}$, $\{V[2], V[3], V[7]\}$, $\{V[5]+u_7V[7], V[3], V[7]\}$, $\{u_1V[1]+V[5], V[3], V[1]\}$, $\{V[5], V[3], V[4]+x_7V[7]\}$, $\{u_4V[4]+V[5], V[3], V[4]\}$, $\{V[5], V[3], V[7]\}$, $\{u_1V[1]+V[6], v1V[1]+V[2]+v_5V[5], V[1]\}$, $\{u_1V[1]+V[6], v1V[1]+V[2], V[1]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]+x_4V[4]+x_7V[7]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]+x_7V[7]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[4]+x_7V[7]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[4]\}$, $\{u_1V[1]+V[2]+u_7V[7], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]+x_4V[4]+x_7V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]+x_7V[7]\}$, $\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[1]\}$,

Dimension	Optimal system
3	$\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[4]\}$,
	$\{u_1V[1]+V[2]+u_4V[4], v_1V[1]+v_4V[4]+V[5]+v_7V[7], V[7]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_1V[1]+v_2V[2]+v_5V[5]+V[6], V[1]\}$,
	$\{u_2V[2]+V[3]+u_5V[5]+u_6V[6], v_1V[1]+v_2V[2]+v_5V[5], V[1]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_1V[1]+v_2V[2]+v_5V[5], V[1]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5], V[4]+x_7V[7]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_4V[4]+v_5V[5], V[4]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5], V[7]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]+V[6], V[1]\}$,
	$\{V[3], v_1V[1]+V[6], V[1]\}, \{u1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], v_4V[4], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[1]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[4], V[7]\}, \{u_1V[1]+V[2]+u_4V[4], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4], V[1]+v_4V[4], V[7]\}, \{u_1V[1]+V[2]+u_4V[4], V[4], V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[1]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[4], V[7]\}, \{u_1V[1]+u_4V[4]+V[5], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+u_4V[4]+V[5], V[1]+v_4V[4], V[7]\}, \{u_1V[1]+u_4V[4]+V[5], V[4], V[7]\}$,
	$\{V[3]+u_6V[6], V[4], V[7]\}, \{V[3], V[1]+v_7V[7], V[4]+x_7V[7]\}, \{V[3], V[1]+v_4V[4], V[7]\}$,
	$\{V[3], V[4], V[7]\}, \{u_1V[1]+V[6], V[4], V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_7V[7], V[4]+x_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5], +u_7V[7], V[1]+v_4V[4]+V[7]\}$
2	$\{V[2]+u_5V[5], V[3]+v_6V[6]\}, \{V[2], V[3]+v_6V[6]\}, \{u_1V[1]+V[5], V[3]+v_6V[6]\}$,
	$\{V[2]+u_5V[5], V[3]\}, \{V[2], V[3]\}, \{V[5], V[3]\}, \{u_1V[1]+V[6], v_1V[1]+V[2]+v_5V[5]\}$,
	$\{u_1V[1]+V[6], v_1V[1]+V[2]\}, \{u_1V[1]+V[6], v_1V[1]+V[5]\}$,
	$\{u_1V[1]+u_7V[7]+V[2], v_1V[1]+v_4V[4]+v_7V[7]+V[5]\}$,
	$\{u_1V[1]+u_4V[4]+V[2], v_1V[1]+v_4V[4]+v_7V[7]+V[5]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]\}, \{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]+V[6]\}$,
	$\{u_2V[2]+V[3]+u_5V[5], v_2V[2]+v_5V[5]+V[6]\}$,
	$\{u_1V[1]+u_2V[2]+u_5V[5]+V[6], v_1V[1]+v_2V[2]+v_5V[5]\}$,
	$\{V[3], V[6]\}, \{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_4V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]+v_7V[7]\}, \{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[1]\}$,
	$\{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[4]+v_7V[7]\}, \{u_1V[1]+V[2]+u_5V[5]+u_7V[7], V[4]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]+v_4V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]+v_7V[7]\}, \{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[1]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[4]+v_7V[7]\}, \{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[4]\}$,
	$\{u_1V[1]+V[2]+u_4V[4]+u_5V[5], V[7]\}, \{u_1V[1]+V[2]+u_7V[7], V[1]+v_4V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[1]+v_7V[7]\}, \{u_1V[1]+V[2]+u_7V[7], V[1]\}$,
	$\{u_1V[1]+V[2]+u_7V[7], V[4]+v_7V[7]\}, \{u_1V[1]+V[2]+u_7V[7], V[4]\}$,
	$\{u_1V[1]+V[2]+v_4V[4], V[1]+v_4V[4]+v_7V[7]\}, \{u_1V[1]+V[2]+v_4V[4], V[1]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+v_4V[4], V[1]\}, \{u_1V[1]+V[2]+v_4V[4], V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[2]+v_4V[4], V[4]\}, \{u_1V[1]+V[2]+v_4V[4], V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[1]+v_4V[4]+v_7V[7]\}, \{u_1V[1]+V[5]+u_7V[7], V[1]+v_7V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[1]\}, \{u_1V[1]+V[5]+u_7V[7], V[4]+v_7V[7]\}$,
	$\{u_1V[1]+V[5]+u_7V[7], V[4]\}, \{u_1V[1]+V[5]+u_7V[7], V[7]\}$,
	$\{u_1V[1]+u_4V[4]+V[5], V[1]+v_4V[4]+v_7V[7]\}, \{u_1V[1]+u_4V[4]+V[5], V[1]+v_7V[7]\}$,
	$\{u_1V[1]+u_4V[4]+V[5], V[1]\}, \{u_1V[1]+u_4V[4]+V[5], V[4]+v_7V[7]\}$,
	$\{u_1V[1]+u_4V[4]+V[5], V[4]\}, \{u_1V[1]+u_4V[4]+V[5], V[7]\}, \{V[3]+u_6V[6], V[1]\}$,
	$\{V[3], V[1]+v_4V[4]+v_7V[7]\}, \{V[3], V[1]+v_7V[7]\}, \{V[3], V[1]\}, \{V[3], V[4]+v_7V[7]\}$,
	$\{V[3], V[4]\}, \{V[3], V[7]\}, \{u_1V[1]+V[6], V[1]\}$
1	$\{u_1V[1]+u_7V[7]+V[2]+u_5V[5]\}, \{u_1V[1]+u_4V[4]+V[2]+u_5V[5]\}$,
	$\{u_1V[1]+u_7V[7]+V[2]\}, \{u_1V[1]+u_4V[4]+V[2]\}$,
	$\{u_1V[1]+u_7V[7]+V[5]\}, \{u_1V[1]+u_4V[4]+V[5]\}, \{V[3]\}$

Therefore the subalgebras of L_7 are classified. We can find one 7-dimensional, 9 six-dimensional, 39 five-dimensional, 145 four-dimensional, 168 three-dimensional, 88 two-dimensional, 7 one-dimensional and one zero-dimensional subalgebras.

5 Conclusion

In our examination we have calculated the infinitesimals for each equation of the set of Doebner–Goldin–Madelung models. We found 10-, 8- and 7-dimensional algebras which were investigated. For the 7-dimensional algebra we have determined the optimal systems.

By application of the statements at the beginning of Section 4 we can calculate for every one and two-dimensional optimal-system the related reduction. In the case of the one-dimensional optimal system the result is a system of equations with two new independent variables whether for the 2-dimensional optimal system the reduced system will be an ordinary differential equation system which can be solved. More details can be found in [2]. In this situation we want to emphasize that it is necessary to take isomorphic investigations into account. A closer look at the commutator table shows that the algebras of each dimension look very similar.

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Group Classification with Respect to Hidden Symmetry

Irina YEHORCHENKO

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *iyegorch@imath.kiev.ua*

We present approaches to systematic description of PDE that display hidden symmetry, with reduced equations having additional symmetry operators compared to the initial equations.

1 Introduction

The concept of conditional symmetry was extensively developed recently following the papers [1] and allowed finding many new solutions of nonlinear differential equations. Here we will consider another facet of the conditional symmetry – additional symmetry of the reduced equations.

"Hidden symmetry" in various contexts is usually a symmetry not obtainable by some standard and straightforward procedure applicable to the models in this context. This term shares the usage of other related terms like "conditional symmetry", "approximate symmetry", and "symmetry" or "invariance" when the same words may denote rather different concepts. Here we will consider hidden symmetry of partial differential equations similarly to Type II hidden symmetry of ordinary differential equations according to [2]. With respect to ODE such symmetry arises as symmetry of equations with reduced order that is not a symmetry of the original equations. In the same way, for a PDE it is symmetry of the reduced equation (with reduced number of independent variables) not present in the original equation.

Definition 1. A differential equation is said to have *hidden symmetry* under an operator X if after the process of reduction of the number of independent variables the resulting reduced equation is invariant under the operator X_1 (being the projection of the operator X in new variables) while the original equation is not invariant under the operator X.

Such symmetry is "classical" in the sense that full hidden symmetry of either ODE or PDE may be found by consecutive reduction of the original equation and investigation of Lie symmetries of the reduced equations, as provided by L. Ovsyannikov's *Submodels* programme [3]. "Symmetry" or "Lie symmetry" is determined in accordance to the procedures that may be found e.g. in [4, 5].

The definition of hidden symmetry can be reformulated in terms of the conditional symmetry according to the book [6]. Here we also will take into account the result proved in the paper [7] that conditional invariance of a differential equation under an involutive family of first-order differential operators Q_a is equivalent to possibility of reduction of this equation by means of the ansatz corresponding to this family of operators. Thus, the additional symmetry under the operator X_1 of the reduced equation (hidden symmetry under the operator X for the original equation) turns out to be a conditional symmetry of the original equation under conditions $Q_a[u] = 0$ ($Q_a[u]$ designates characteristics of the vector field Q_a) with all appropriate differential consequences. Note that X_1 is a Lie symmetry of the reduced equation, and we do not add the condition $X[u] = 0$ to the set of conditions, so such operator will not present a purely Q-conditional symmetry in the sense of [6].

Q-conditional symmetry can also be hidden – that is being a new Q-conditional symmetry of the reduced equation. For examples of such symmetries see [8].

In this paper we give an outlook of description of partial differential equations possessing certain hidden symmetries.

Group classification for classes of differential equations is aimed at identification of equations having wider symmetries than the equations of the class in general. For an overview of the group classification problems and the extensive list of related references see [9]. Usually two types of such problems are considered – finding the equations within a general class that are invariant under specific symmetry group, and description of all symmetries (up to appropriately chosen equivalence) of equations that belong to the specific class. On the basis of the known algorithms for group classification of differential equations in the Lie's sense we develop approaches for a systematic description of classes of nonlinear PDEs that display hidden symmetry.

Definition 2. A class of equations can be regarded as *general* if any local transformations of dependent and independent variables transform an equation from this class into another equation within the same class.

An example is the class of all kth-order PDEs $F = F(x, u, \underset{1}{u}, \dots, \underset{k}{u}) = 0$ with x, u being respectively n- and m-dimensional independent and dependent variables, $\underset{r}{u}$ being the set of all rth-order partial derivatives of the function $u = \big(u^1, u^2, \dots, u^m\big)$. Group classification even with respect to the Lie symmetry for the general classes is usually an overwhelming task, and, to our knowledge, such problem was completely solved only for single ordinary differential equations by S. Lie [10]. A restricted, but practically important problem for the general classes of equations would be description of all equations within the class invariant under some specified symmetry group that can be done by describing all differential invariants for such group. Similarly, description of equations having specified Lie symmetry and specified hidden symmetry may be done by means of conditional differential invariants, as shown in Section 2.

For a more specific class it may be possible to make full group classification of a system consisting of the original equation together with the reduction conditions of the type $Q_a[u] = 0$ (with appropriate prolongations of the conditions), as shown in the Section 3.

2 Description of all equations in a general class possessing particular hidden symmetry

The first example is the general first-order partial differential equation with one dependent and three independent variables of the form

$$F = F(x, y, z, u, u_x, u_y, u_z) = 0. \tag{1}$$

We describe all such equations having Lie symmetry with respect to the operator ∂_x and hidden symmetry with respect to the operator ∂_y after reduction by means of the operator ∂_x. The condition of such Lie and hidden symmetry, according to Definition 1, is invariance of the equation (1) under the operator ∂_y on condition that $u_x = 0$:

$$\partial_x F\big|_{F=0} = 0, \qquad \partial_y F\big|_{F=0,\ u_x=0} = 0. \tag{2}$$

The general solution of the condition (2) will be a function of all invariants of the operators ∂_x and ∂_y, that is z, u, u_x, u_y, u_z, and of the conditional invariant $r = u_x R(y, z, u, u_x, u_y, u_z)$ (being an absolute invariant of ∂_x), where R is an arbitrary function that is reasonably determined on the manifold $u_x = 0$:

$$F(r, z, u, u_x, u_y, u_z) = 0, \tag{3}$$

F has to be a function of the invariants of the hidden symmetry operator on the manifold determined by the reduction condition, and have arbitrary form elsewhere. Note that the function r in (3) is not arbitrary: we cannot take e.g.

$$r = R_1(y, z, u, u_x, u_y, u_z) = u_x \frac{R_1}{u_x}$$

as such $R = \frac{R_1}{u_x}$ will be in the general case undetermined on the manifold $u_x = 0$.

Definition 3 ([12]). A function $F(x, u, \underset{1}{u}, \dots, \underset{k}{u})$ is a *conditional differential invariant* of the operator Q, if under the conditions $G(x, u, \underset{1}{u}, \dots, \underset{r}{u}) = 0$ the relations $Q[F] = 0$, $Q[G] = 0$ are satisfied. We take prolongations of the operators of the order $\max(k, r)$.

A set of invariants of the order $r \le k$ of the operator Q with the conditions $G = 0$ is called a *generating set* of the kth-order conditional differential invariants of Q if all other invariants can be represented as functions of invariants from this set.

Invariants in such generating set may be both absolute invariants of Q and G-conditional of the form $G^{(l)} \times R_{(l)}$, where $G^{(l)}$ are derivatives of G of the order $l \le k - r$ and $R_{(l)}$ are arbitrary functions determined on the manifold $G^{(k)} = 0$ for all values of k.

Number of functionally independent Q-absolute invariants in the generating set of conditional differential invariants can be calculated similarly to the number of invariants in a functional basis of absolute differential invariants, as $m - 1$, where m is the number of variables in the set $x, u, \underset{1}{u}, \dots, \underset{k}{u}$. Number of linearly independent purely G-conditional invariants is equal to the number of independent conditions of the type $G^{(l)} = 0$.

In some cases we would be able to construct a functional basis of conditional invariants, i.e. the maximal set of functionally independent conditional invariants. That is possible in the case when we put a requirement that our conditional invariants should be also absolute invariants of some Lie algebra L, and additional conditions $G = 0$ in the Definition 3 and their relevant differential consequences are not invariant under L.

Equation (3) is reduced by means of the operator ∂_x to the equation $F(0, z, u, 0, u_y, u_z) = 0$ that is invariant with respect to ∂_y. If $R_y \neq 0$ in the expression for r in (3) then this equation is not invariant with respect to the operator ∂_y, so such symmetry is purely hidden.

This example is easily generalised for larger order of equations or number of reduction operators.

The second example is hidden projective symmetry presented by the operator ($m \neq 0$)

$$A = (x_0 - x_2)^2(\partial_0 - \partial_2) + (x_0 - x_2)x_1\partial_1 + \frac{imx_1^2}{2}(u\partial_u - u^*\partial_{u^*}) - (x_0 - x_2)(u\partial_u + u^*\partial_{u^*}) \tag{4}$$

for the d'Alembert equation

$$\Box u = \lambda u|u|^4, \tag{5}$$

here $u = u(x_0, x_1, x_2)$ is a complex-valued function, $|u| = (uu^*)^{1/2}$.

The maximal invariance algebra of the equation (5) is the Poincaré algebra extended by the dilation and charge operators (see e.g. [6] and references therein), with basis operators

$$p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu,$$
$$D = x_\mu p_\mu - \frac{i}{4}(u\partial_u + u^*\partial_{u^*}), \quad J = i(u\partial_u - u^*\partial_{u^*}), \tag{6}$$

where μ, ν take the values 0, 1, 2; the summation is implied over the repeated indices (small Greek letters) in the following way: $x_\nu x_\nu \equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - x_2^2$, $g_{\mu\nu} = \operatorname{diag}(1, -1, -1)$.

This equation can be reduced by means of the ansatz ($m \neq 0$)

$$u = \exp\left(\frac{-im(x_0 + x_2)}{2}\right)\Phi(x_0 - x_2, x_1) \tag{7}$$

to the Schrödinger equation

$$2im\Phi_t + \Phi_{xx} = \Phi|\Phi|^4, \qquad t = x_0 - x_2, \qquad x = x_1. \tag{8}$$

The ansatz (7) corresponds to the following additional condition with its conjugate:

$$u_0 + u_2 + imu = 0. \tag{9}$$

Such reduction allowed construction of numerous new solutions for the nonlinear wave equation by means of the solutions of a nonlinear Schrödinger equation [11]. Here reduction gives additional symmetry properties for equation (5), related to the symmetry properties of the equation (8). This equation is invariant, beside the Galilei algebra and the dilation operator

$$\begin{aligned} &\partial_t, \quad \partial_x, \quad J_1 = i(\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*}), \quad G = t\partial_x + xJ_1, \\ &\hat{D}_1 = 2t\partial_t + x\partial_x - I \quad (\text{here } I = i(\Phi\partial_\Phi + \Phi^*\partial_{\Phi^*})) \end{aligned} \tag{10}$$

(see e.g. [6]), under the projective operator equal to the operator A in the new variables t, x, Φ

$$A_1 = t^2\partial_t + tx\partial_x + \frac{im}{2}x^2J_1 - tI. \tag{11}$$

This is one of the immediate examples of hidden symmetry: when presence of some conformal- or projective-type symmetry in a nonlinear equation depends on both the degree of the polynomial nonlinearity and the number of independent variables, such symmetry will exist as hidden in equations with the degree of the polynomial nonlinearity corresponding to the smaller number of independent variables. Note that the operator $\hat{D}_1$ in (10) is not a projection of D in (6), but leads to another hidden symmetry operator $\hat{D} = x_\mu p_\mu - i(u\partial_u + u^*\partial_{u^*})$.

Here we will present full description of Poincaré-invariant equations possessing such hidden symmetry with respect to the operator (4). This is done by means of conditional differential invariants introduced in [12]. Note that in our example we do not require Poincaré invariance of the additional condition itself, so we are able to construct a functional basis of conditional differential invariants.

We adduce functional bases of differential invariants that may be utilised for construction of classes of Poncaré-invariant equations with hidden symmetry (4).

First we present according to [13] a functional basis of absolute differential invariants of the second order for the Poincaré algebra $\langle p_\mu, J_{\mu\nu}\rangle$ (6) and the complex-valued scalar function $u = u(x_0, x_1, x_2)$. It consists of 17 invariants

$$\begin{aligned} &u,\ u^*,\ u_\mu u_\mu,\ u^*_\mu u^*_\mu,\ \Box u,\ \Box u^*,\ u_{\mu\nu}u_{\mu\nu},\ u_{\mu\nu}u^*_{\mu\nu},\ u^*_{\mu\nu}u^*_{\mu\nu},\ u_\mu u_{\mu\nu}u_\nu,\ u^*_\mu u_{\mu\nu}u^*_\nu, \\ &u_\mu u_{\mu\nu}u_{\nu\lambda}u_\lambda,\ \ u^*_\mu u_{\mu\nu}u_{\nu\lambda}u^*_\lambda,\ \ u_{\mu\nu}u_{\nu\lambda}u_{\lambda\mu},\ \ u_{\mu\nu}u_{\nu\lambda}u^*_{\lambda\mu},\ \ u_{\mu\nu}u^*_{\nu\lambda}u^*_{\lambda\mu},\ \ u^*_{\mu\nu}u^*_{\nu\lambda}u^*_{\lambda\mu}. \end{aligned} \tag{12}$$

A functional basis of differential invariants for the Galilei algebra (10), mass $m \neq 0$, of the second order for the complex-valued scalar function $\Phi = \Phi(t, x)$ consists of 10 invariants that can be checked by direct calculation.

For simplification of the expressions for differential invariants we introduced the following notations: $\phi = \ln\Phi$, and to ensure that ϕ is single-valued we choose it so as $\operatorname{Im}\Phi = \arctan\frac{\operatorname{Re}\phi}{\operatorname{Im}\phi}$.

The elements of the functional basis may be chosen as follows:

$$\phi + \phi^*, \quad \phi_x + \phi^*_x, \quad M_1 = 2im\phi_t + \phi^2_x, \quad M^*_1, \quad \phi_{xx}, \quad \phi^*_{xx},$$

$$\theta = im\phi_{tx} + \phi_x\phi_{xx}, \quad \theta^*, \quad M_2 = -m^2\phi_{tt} + 2im\phi_x\phi_{tx} + \phi_x^2\phi_{xx}, \quad M_2^*. \tag{13}$$

A functional basis of differential invariants for the Galilei algebra extended by the dilation operator (10) and the projective operator (11) may be chosen as follows:

$$N_1e^{-4(\phi+\phi^*)}, \quad \frac{N_1}{N_1^*}, \quad \frac{N_2}{N_1^2}, \quad \frac{N_2^*}{(N_1^*)^2}, \quad \frac{\phi_{xx}+\phi_{xx}^*}{N_1}, \quad \frac{\theta}{N_1^{\frac{3}{2}}}, \quad \frac{\theta^*}{N_1^{\frac{3}{2}}}, \quad \frac{(\phi_x+\phi_x^*)^2}{N_1},$$

where

$$N_1 = M_1 + \phi_{xx} = 2im\phi_t + \phi_{xx} + \phi_x^2, \qquad N_2 = \phi_{xx}N_1 + \frac{\phi_{xx}^2}{2} + M_2. \tag{14}$$

An algorithm for construction of conditional differential invariants may be derived directly from the Definition 3. We can construct differential invariants of the Poincaré algebra $\langle p_\mu, J_{\mu\nu}\rangle$ (6) being conditional differential invariants of the projective operator (4) solving the system

$$\underset{2}{A}_1F(\mathrm{Inv}_P) = 0, \qquad u_0 + u_2 + imu = 0, \qquad \frac{\partial}{\partial x_\mu}(u_0 + u_2 + imu) = 0, \quad \mu = 0,1,2,$$

where Inv_P are all differential invariants (12) of the Poincaré algebra $\langle p_\mu, J_{\mu\nu}\rangle$ (6). Using the fact that the ansatz (7) is the general solution of the additional condition (9), we can directly substitute this ansatz into differential invariants (12). The expression $\Box u$ transforms into uN_1, where N_1 is an expression entering into expression for differential invariants (14). Substituting the ansatz (7) into all elements of the fundamental basis (12) of second-order differential invariants of the Poincaré algebra, we can obtain a reduced basis of differential invariants that may be used for construction of all equations reducible by means of this ansatz to equations possessing the projective symmetry. We can give the following representation of the Poincaré invariants using expressions M_k (13) and N_k (14):

$$u_\mu u_\mu = u^2M_1, \ u_\mu u_\mu^* = \frac{uu^*}{2}\big(M_1 + M_1^* - (\phi_x+\phi_x^*)^2\big), \ u_{\mu\nu}u_{\mu\nu} = u^2\big(2M_2 + M_1^2 + \phi_{xx}^2\big),$$
$$u_{\mu\nu}u_{\mu\nu}^* = uu^*\Big(M_2 + M_2^* + \frac{1}{4}\big(M_1 + M_1^* - (\phi_x+\phi_x^*)^2\big)^2 - 2(\theta+\theta^*)(\phi_x+\phi_x^*)$$
$$+ \big(\phi_{xx}+\phi_{xx}^*\big)(\phi_x+\phi_x^*)^2 + \phi_{xx}\phi_{xx}^*\Big), \quad u_\mu u_\nu u_{\mu\nu} = u^3\big(M_2 + M_1^2\big),$$
$$u_\mu^* u_\nu^* u_{\mu\nu} = uu^{*2}\Big(M_2 - 2\theta(\phi_x+\phi_x^*) + \phi_{xx}(\phi_x+\phi_x^*)^2 + \frac{1}{4}\big(M_1 + M_1^* - (\phi_x+\phi_x^*)^2\big)^2\Big),$$

with similar, however more cumbersome expressions for other invariants. Note that it is convenient to use for expression of the projective conditional invariants not only invariants from the fundamental basis (12) but also other invariants (that are anyway functions of (12)).

We construct Poincaré-invariant conditional differential invariants of the projective operator (4) under the condition (9) using differential invariants (13)

$$I_1 = N_1e^{-4(\phi+\phi^*)} = \frac{\Box u}{u(uu^*)^2}, \quad I_2 = \frac{N_1}{N_1^*} = \frac{u^*\Box u}{u\Box u^*},$$
$$I_3 = \frac{N_2}{N_1^2} = \left(uu_\mu u_\nu u_{\mu\nu} + \frac{3}{2}u^2(\Box u)^2 + \frac{1}{2}(u_\mu u_\mu)^2 - 2u\Box u(u_\mu u_\mu)\right)\big(u^2(\Box u)^2\big)^{-1}, \quad I_3^*,$$
$$I_4 = \frac{(\phi_x+\phi_x^*)^2}{N_1} = \frac{u_\mu u_\mu u^{*2} + u_\mu^* u_\mu^* u^2 - 2u_\mu u_\mu^* uu^*}{uu^{*2}\Box u},$$
$$I_5 = \frac{(\phi_{xx}+\phi_{xx}^*)}{N_1} = \frac{u^*\Box u + u\Box u^* - 2u_\mu u_\mu^*}{u^*\Box u}, \quad I_6 = \frac{\theta}{2N_1} = \big(- u^3u_\mu^*u_\nu^*u_{\mu\nu}$$

$$+ u^{*2}\big(uu_\mu u_\nu u_{\mu\nu} - (u_\mu u_\mu)^2\big) + (u\Box u - u_\mu u_\mu)\big(u_\mu u_\mu u^{*2} + u^*_\mu u^*_\mu u^2 - 2u_\mu u^*_\mu u u^*\big)$$
$$+ (uu_\mu u^*_\mu)^2\big)\big(u^2 u^* \Box u (u_\mu u_\mu u^{*2} + u^*_\mu u^*_\mu u^2 - 2u_\mu u^*_\mu u u^*)^{1/2}\big)^{-1}, \quad I^*_6. \tag{15}$$

The expressions (15) form a functional basis of invariants that are absolute differential invariants for the operators of the extended Poincaré algebra (6) and conditional invariants of the operator A (4) with the additional condition (9). Functional independence of the expressions (15) can be checked directly, most easily by application of the additional conditions, that is by substitution of the ansatz (7) into (15), with the "reduced basis" obtained as a result. The reduced invariants in the basis will depend on new functions without conditions and will be absolute invariants of the operators $\langle G, J, \hat{D}_1, A_1\rangle$.

Calculation of the number of invariants in such basis may be also considered as follows:

$$\begin{aligned}\text{Number of invariants} = {}& \text{Number of (variables + derivatives)}\\ & - \big(\text{Rank of the set of operators}\\ & + \text{Number of independent conditions and their derivatives}\big).\end{aligned}$$

We have 20 dependent variables and derivatives, 8 conditions ((9) with the conjugate and derivatives), and the rank of the set of operators $\langle J_{\mu\nu}, \hat{D}, J, A\rangle$ under the conditions (9) with the conjugate and derivatives is equal to 6 with all conditions taken into account.

However, the number of conditional differential invariants in the considered example is 8 and not $6 = 20 - 8 - 6$ as it may seem to be. However, if we apply the additional condition (9) and its first derivatives with respective conjugates (8 conditions) to the fundamental basis of invariants of the Poincaré algebra (12) (17 invariants), we will obtain 11 independent reduced invariants, as may be checked by direct calculation. So two of the conditions appear to be dependent on the manifold of invariants, and with other 3 conditions of invariance with respect to operators $\langle \hat{D}, J, A\rangle$ we obtain 8 invariants in the functional basis of conditional differential operators. Thus, we arrived to the following statement.

Proposition 1. *All equations of the form*

$$F(I_1, I_2, I_3, I^*_3, I_4, I_5, I_6, I^*_6) = 0 \tag{16}$$

are conditionally invariant with respect to the operator A (4) *with the additional condition* (9). *All equations that are invariant with respect to the algebra* (6) *and conditionally invariant with respect to the operator* A (4) *with the additional condition* (9) *have the form* (16).

Note that in this example we required the equations to be invariant in the Lie sense under the algebra (6), and the condition (9) is not invariant with respect to this algebra. So we do not have conditional invariants with arbitrary functions of the type $(u_0 + u_2 + imu) \times$ (arbitrary function). In general, if we require some hidden or conditional invariance producing conditional invariants with arbitrary functions together with some Lie invariance, we may require Lie invariance of an equation depending on such conditional invariants. Moreover, the operators $\langle J_{\mu\nu}, \hat{D}, J, A\rangle$ do not form an algebra, and we would not be able to use absolute differential invariants in the Lie sense to describe invariant equations similarly to the example of the equation (1) and its reduction under the operator ∂_y.

3 Finding hidden symmetries

"Simple" hidden symmetries (Lie symmetries of reduced equations) for a particular class of equations can be found by means of consecutive Lie reductions and consecutive finding Lie symmetries of the reduced equations. Group classification of such class with respect to hidden

symmetries of reduced equations will involve description of all possible reductions and group classification of the respective classes of reduced equations. Here we will consider a rather simple example of the nonlinear wave equation in two spatial dimensions

$$\Box u = f(x_0, x_1, x_2, u) \tag{17}$$

that will allow us to illustrate the proposed algorithm. We present group classification of this equation with respect to hidden symmetries for reduction by means of the operator ∂_{x_2}.

Such reduction leads to the two-dimensional wave equation $u_{00} - u_{11} = f(x_0, x_1, u)$. The next step is the usual group classification of the reduced equation. It will be more convenient to use the substitution $\tau = x_0 + x_1$, $\omega = x_0 - x_1$ and to go to the equation

$$u_{\tau\omega} = F(\tau, \omega, u). \tag{18}$$

The equivalence group of equation (18) (see [14]) consists of the continuous transformations (for the definitions and calculation procedures of equivalence transformation groups see [4, 15])

$$\tau' = T(\tau), \quad \omega' = W(\omega), \quad u' = e^{\lambda}u + U(\tau, \omega), \quad F' = \frac{1}{\dot{T}(\tau)\dot{W}(\omega)}(e^{\lambda}F + U_{\tau\omega}), \tag{19}$$

where T, W, U are arbitrary (T, W do not take constant values simultaneously) functions of their arguments and dots over the letters mean derivatives, and discrete transformations $\tau' = \epsilon\epsilon_1\omega + (1-\epsilon)\epsilon_2\tau$, $\omega' = \epsilon\epsilon_3\tau + (1-\epsilon)\epsilon_4\omega$, $u' = \epsilon_5 u$, $F' = \epsilon_5(\epsilon\epsilon_1\epsilon_3 + (1-\epsilon)\epsilon_2\epsilon_4)\, F(\tau', \omega', u')$ ($\epsilon = 0, 1$, $\epsilon_k = \pm 1$, $k = 1, 2, \ldots, 5$), together with products of continuous and discrete transformations.

Group classification of the equation (18) is given in [4] ($F_{uu} = 0$) and in [14] ($F_{uu} \neq 0$). Such group classification was performed up to transformations from the equivalence group of the equation. However, if we multiply the resulting inequivalent invariant equations by means of transformations from the equivalence group (19) that do not belong to the equivalence group of equation (17) and then add u_{22} ("reverse reduction"), we will obtain description of equations with hidden symmetry.

For example, when $F = F(\rho_1(\tau) - \rho_2(\omega), u)$, equation (18) is invariant with respect to the operator $Q = \rho_1'\partial_\tau + \rho_2'\partial_\omega$ (primes here designate derivatives). So, the equations from the class (17) $\Box u = f(\rho_1(\tau) - \rho_2(\omega), x_2, u)$ will have hidden invariance with respect to the operator Q with appropriate change of variables in Q. It quite obvious that such transformed Q is not a Lie symmetry operator for the above equation except if $\rho_1 = a\tau + b$, $\rho_2 = a\omega + c$, a, b, c are constants.

Similarly, for $f = \exp(\rho_1(\tau) + \rho_2(\omega))\, \phi(u \exp(-\rho_1(\tau) - \rho_2(\omega)), x_2)$, there is hidden invariance of (17) with respect to the operators $Q_1 = \rho_1'\partial_\tau + u\partial_u$, $Q_2 = \rho_2'\partial_\omega + u\partial_u$. Here ρ_1, ρ_2 are arbitrary but fixed functions.

We present an algorithm for group classification with respect to hidden symmetry:

Step 1. Obtain reduced equations of for the class of initial PDE.

Step 2. Perform group classification of the reduced equations.

Step 3. Multiply inequivalent invariant reduced equations by means of transformations from the equivalence group of this class.

Step 4. Reverse of the equation: find equations from the initial class of PDE corresponding to the multiplied reduced equations.

Step 5. Find all inequivalent equations with respect to transformations from the equivalence group of the initial class of PDE.

4 Discussion

We see that a nontrivial hidden symmetry for partial differential equations stems from the reduced equations having wider equivalence group than the original equations. So group classification of the classes of equations with respect to hidden symmetry involves study of equivalence

groups of such classes, in the similar way as it is done for classification with respect to the Lie symmetry. Description of hidden symmetry of physically interesting classes of equations would allow to identify wider classes of reducible equations than it is possible by means of group classification in the Lie sense. The special interest in such classification lies in that reduced equations often have infinite-dimensional equivalence groups that allows to obtain classes of equations with hidden symmetry that have large degree of arbitrariness.

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Sets of Conditional Symmetry Operators and Exact Solutions for Wave and Generalised Heat Equations

Irina YEHORCHENKO † *and Alla VOROBYOVA* ‡

† *Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *iyegorch@imath.kiev.ua*

‡ *Mykolayiv Petro Mogyla State Humanities University, Mykolaiv, Ukraine*

Reduction of some wave and heat equations by means of ansatzes in special coordinate systems results in PDEs with additional symmetry operators compared to original ones. Such operators will allow constructing sets of conditional symmetry operators that represent hidden symmetry of the equations. Classes of generalised wave and heat equations possessing these symmetry properties are described, and families of respective nonclassical solutions are listed.

1 Example 1: Linear wave equation

Let us consider the linear wave equation

$$\Box u = ku \tag{1}$$

for the real-valued function $u = u(x_0, x_1, x_2, x_3)$, $x_0 = t$ is the time variable, x_1, x_2, x_3 are space variables. $\Box$ is the d'Alembert operator

$$\Box u = -\frac{\partial^2 u}{\partial x_0^2} + \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}.$$

The maximal invariance algebra (without trivial operators $u\partial_u$, $f(x)\partial_u$ ($\Box f = 0$) taken into account) of the equation (1), $k \neq 0$, is the Poincaré algebra $AP(1,3)$ with the basis operators

$$p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu,$$

where μ, ν take the values 0, 1, 2, 3; $g_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$.

Similarity solutions for the equation (1) can be found by means of symmetry reduction by non-equivalent subalgebras of the algebra $AP(1,3)$ [1–4]. In such a way it is possible to obtain solutions in the form $u = \phi(\omega)$, where ω is an invariant of some subalgebra of the Poincaré algebra $AP(1,3)$.

Here we will look for solutions of the equation (1) in the form $u = f(x)\phi(\omega)$, with f and ω being some functions on x. Ansatzes of such form are usually obtained in the process of reduction of wave equations having scale invariance under subalgebras containing scale transformations $D = x^\nu p_\nu + \lambda i u\partial_u$. However, we can obtain a solution with such ansatzes for the equation (1) that is not invariant scale transformations.

We can reduce the equation (1) by means of the ansatz

$$u = f(r)\phi(\omega), \qquad r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \qquad \omega = r + vx_0.$$

The general form of the reduced equation will be

$$\left(v^2 - 1\right) f\phi'' - 2\left(f' + \frac{1}{r}f\right)\phi' - \left(f'' + \frac{2}{r}f' + kf\right)\phi = 0, \tag{2}$$

and the reduction conditions (when $f \neq 0$ and $\phi \neq 0$) are as follows: when v is an arbitrary parameter, $v^2 - 1 \neq 0$, then

$$f' + \frac{1}{r} f = \lambda f, \qquad f'' + \frac{2}{r} f' = \gamma f,$$

with λ, γ being arbitrary constants. This system is compatible if and only if $\gamma = \lambda^2$, and the ansatz takes the form

$$u = \frac{c \exp \lambda r}{r} \phi(\omega), \tag{3}$$

where $\omega = r + vx_0$; c, λ, $v \neq \pm 1$, $v \neq 0$ are arbitrary constants. The reduced equation has the form $\left(v^2 - 1\right) \phi'' - 2\lambda\phi' - \left(\lambda^2 + k\right) \phi = 0$.

When $v = \pm 1$, there may be several types of the reduction conditions.

1. With

$$f' + \frac{1}{r} f \neq 0, \qquad f'' + \frac{2}{r} f' + kf \neq 0, \qquad \alpha \neq 0,$$

the reduction condition will have the form

$$\alpha \left(f' + \frac{1}{r} f \right) = f'' + \frac{2}{r} f' + kf,$$

and the corresponding ansatzes are

$$u = \frac{1}{r} \exp \frac{\alpha r}{2} [c_1 \cosh \beta r + c_2 \sinh \beta r] \phi(\omega), \qquad 0 > k - \frac{\alpha^2}{4} = -\beta^2; \tag{4}$$

$$u = \frac{1}{r} \exp \frac{\alpha r}{2} [c_1 \cos \beta r + c_2 \sin \beta r] \phi(\omega), \qquad 0 < k - \frac{\alpha^2}{4} = \beta^2; \tag{5}$$

$$u = \frac{1}{r} \exp \frac{\alpha r}{2} [c_1 r + c_2] \phi(\omega), \qquad 0 = k - \frac{\alpha^2}{4}; \tag{6}$$

the reduced equation for ϕ will be $\phi' + \alpha\phi = 0$ and its solution $\phi = c \exp\left(-\frac{\alpha\omega}{2}\right)$ gives exact solutions for the equation (1).

2. With

$$f' + \frac{1}{r} f = 0, \qquad f'' + \frac{2}{r} f' + kf = 0,$$

ϕ may be an arbitrary function. It is evident that $f = 0$ if $k \neq 0$. If $k = 0$, $f = \frac{c}{r}$, and we obtain the well-known Euler solution for the wave equation $\Box u = 0$:

$$u = \frac{c}{r} \phi(r \pm t),$$

where ϕ is an arbitrary function. This solution is obtained from the degenerate case of the reduced equation (2) with all coefficients at and functions vanishing, so the solution of the resulting reduced equation is an arbitrary function. This solution corresponds to the hidden symmetry of the wave equation, represented by Lie infinite-dimensional symmetry algebra of the reduced equation $\phi_{tt} - \phi_{rr} = 0$, $u = \frac{c}{r} \phi(r, t)$.

3. With

$$f' + \frac{1}{r} f \neq 0, \qquad f'' + \frac{2}{r} f' + kf = 0,$$

$\phi' = 0$, $\phi = \text{const}$. The resulting solutions have the form $u = u(r)$ and are similarity solutions that may be obtained from the Lie symmetry of the equation (1).

It is quite obvious that ansatzes of the form (3)–(6) cannot be obtained by reduction of the equation (1) by subalgebras of the Poincaré algebra $AP(1,3)$

The ansatz (3) corresponds to the set of operators Q_1, J_{ab}, where $a,b=1,2,3$ and Q_1 has the form

$$Q_1 = x_a\partial_a - \frac{r}{v}\partial_0 + (\lambda r - 1)u\partial_u. \tag{7}$$

This set of operators determines conditional invariance of the equation (1), and the ansatz (3) allows to obtain new solutions compared to those that may be obtained by means of Lie symmetry operators. Note that these operators do not form an algebra.

We present other such operators $Q_{(1,i)}$ determining conditional invariance of the equation (1):

$$Q_{(1,1)} = x_a\partial_a + r\partial_0 + k^{1/2}ru\partial_u;$$
$$Q_{(1,2)} = x_a\partial_a + r\partial_0 + \left(\frac{\alpha}{2} - \frac{1}{r} + \left(k - \frac{\alpha^2}{4}\right)^{1/2}\operatorname{tg} r\right)u\partial_u;$$
$$Q_{(1,3)} = x_a\partial_a + r\partial_0 + \left(\frac{\alpha}{2} - \frac{1}{r} + \left(k - \frac{\alpha^2}{4}\right)^{1/2}\operatorname{ctg} r\right)u\partial_u.$$

These operators generate scale-type transformations of the space variables.

We adduce here some of the families of exact solutions for the equation (1):

$$u = \frac{c\exp\lambda r}{r}[c_1\exp(\lambda+\alpha)l\omega + c_2\exp(\lambda-\alpha)l\omega], \qquad 0 < v^2(k+\lambda^2) - k = \alpha^2;$$
$$u = \frac{c\exp(\lambda r+\lambda l\omega)}{r}[c_1\cos l\alpha\omega + c_2\sin l\alpha\omega], \qquad 0 > v^2(k+\lambda^2) - k = -\alpha^2;$$
$$u = \frac{c\exp(\lambda r+\lambda l\omega)}{r}[c_1\omega + c_2], \qquad v^2(k+\lambda^2) - k = 0,$$

$l = 1/(v^2-1)$; c_1, c_2 are arbitrary constants.

2 Example 2: Generalised heat equation

We consider a nonlinear generalised heat equation in the form

$$u_0 + u_{11} + u_{22} + u_{33} = F(x_0, x_1, x_2, x_3, u), \tag{8}$$

where $u_\mu = \frac{\partial u}{\partial x_\mu}$, $u_{\mu\mu} = \frac{\partial^2 u}{\partial x_\mu^2}$; $\mu = 0,1,2,3$.

Classical Lie symmetries with two and three spatial variables with F not depending on x_μ were described in [5].

We consider reductions of the equation (8) to an ODE by means of an ansatz of the form

$$u = \frac{c}{r}\phi(\omega), \qquad r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \qquad \omega = r + vx_0, \tag{9}$$

where v is an arbitrary parameter, when $F = \frac{1}{r}\Phi(\omega, ur)$. The reduced equation will then have the form $\phi'' + v\phi' = \Phi(\omega,\phi)$.

The maximal invariance algebra of the equation (8) for the general form of $F = \frac{1}{r}\Phi(\omega, ur)$ is the three-dimensional algebra of rotation operators with a basis J_{ab}, $a,b=1,2,3$, and the ansatz (9) corresponds to the set of operators $\{Q_2, J_{ab}\}$, where $a,b=1,2,3$ and Q_2 has the form

$$Q_2 = x_a\partial_a - \frac{r}{v}\partial_0 + u\partial_u. \tag{10}$$

This set of operators determines conditional invariance of the equation (8), and the ansatz (9) allows to obtain new solutions compared to those that may be obtained by means of Lie symmetry operators. We will discuss the reduced equation for (8) by means of the ansatz (9) with $F = \lambda r^{k-1} u^k$ that has the form

$$\phi'' + v\phi' = \lambda\phi^k.$$

This equation may be transformed to the second-type Abel equation

$$p'p + vp = \lambda y^k \tag{11}$$

with substitution $\phi' = p$, $\phi = y$, $p = p(y)$.

1. If $k = \frac{1}{2}$ then the equation (11) by means of the transformation $\frac{s'}{s} = y^{1/2}$, $-\frac{1}{v}p = -\frac{z}{2A}$, $A = \frac{\lambda}{v^2}$ reduces to the Bessel equation

$$s'' - \frac{z}{2A}s = 0.$$

2. If $k = -1$ then the equation (11) is the Liouville equation $p'p + vp = \lambda y^{-1}$. By means of the substitution $-\frac{1}{v}p = y + z$, $y = \frac{1}{u}$ it is reduced to a linear one with respect to the function $u = u(z)$

$$\frac{\lambda}{v^2}u_z + zu + 1 = 0.$$

3. If $k = -2$ then the equation (11) by means of the substitution $-\frac{1}{v}p = y + z$, $\frac{1}{y} = u - \frac{z^2}{2}$ may be transformed to the Riccati equation for the function $z = z(u)$

$$\frac{\lambda}{v^2}z_u + u - \frac{z^2}{2} = 0,$$

whose solution may be expressed via the Bessel function.

3 Definitions and discussion

The concept of non-classical, or conditional symmetry, originated in its various facets in the papers [6–11], and later it was developed by numerous authors into the theory and a number of algorithms for studying symmetry properties of equations of mathematical physics and for construction of their exact solutions (see e.g. [12]). Here we will use the following definition of the conditional symmetry:

Definition 1. The equation $\Phi(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = 0$, where $\underset{k}{u}$ is the set of all kth-order partial derivatives of the function $u = (u^1, u^2, \ldots, u^m)$, is called conditionally invariant [3] under the operator

$$Q = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r}$$

if there is an additional condition

$$G(x, u, \underset{1}{u}, \ldots, \underset{l_1}{u}) = 0, \tag{12}$$

such that the system of two equations $\Phi = 0$, $G = O$ is invariant (according to the Lie definition, see e.g. [13]) under the operator Q.

If (12) has the form $G = Q[u]$, $Q[u]$ designates $Q[u] = \eta^r(x,u) - \xi^i(x,u)u_{x_i}$, then the equation $\Phi = 0$ is called Q-conditionally invariant under the operator Q [3]. These definitions of the conditional invariance of some equation are based on what is in reality Lie symmetry (see e.g. the classical texts [13–15]) of the same equation with a certain additional condition.

In this paper we develop results previously obtained by the authors in [16–18]. Our examples may be used to illustrate a generalisation of this definition to conditional invariance with a set of conditions $G_m = 0$. We may speak about $\langle Q_m \rangle$-conditional invariance where additional conditions have the form

$$G_m(x, u, \underset{1}{u}, \ldots, \underset{l_1}{u}) = Q_m[u] = 0.$$

However, we see that these examples contain features of two different types of non-classical invariance – hidden symmetry (see [30]) and Q-conditional symmetry.

Definition 2. An equation is said to have hidden conditional invariance if a reduced equation is conditionally invariant under some additional condition.

The equation (1) possesses hidden Q-conditional symmetry with respect to the operators Q_1, $Q_{1,i}$, and the equation (8) possesses hidden Q-conditional symmetry with respect to the operator Q_2, under conditions $J_{ab}u = 0$, $a, b = 1, 2, 3$.

The conditional symmetry of the Klein–Gordon equation was studied in [19, 20]. The conditional symmetry of the nonlinear heat equations with one space variable was extensively studied by numerous authors [6, 21–27, 29]. In [28] all conditional symmetries for an n-dimensional homogeneous linear heat equation were described.

The operators of the hidden conditional symmetry presented in this paper do not belong to sets of operators described in the above-mentioned papers, and they are not obtainable from conditional symmetry of equations with one space dimension. However, the operators we found are linked to the conditional symmetry of the respective radial wave and heat equations

$$u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F_1(u),$$
$$u_t + u_{rr} + \frac{n-1}{r}u_r = F_2(u)$$

(n is the number of spatial variables, here $n = 3$) that present considerable interest by themselves. See e.g. [31, 32], where exact solutions for a n-dimensional radial wave equation with a general power nonlinearity are derived and studied, and [33] for discussion of the linear radial heat equation.

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Symmetries of Radial Maxwell–Vlasov Equations

Zygmunt Jacek ZAWISTOWSKI

Institute of Fundamental Technological Research, Polish Academy of Sciences,
Swietokrzyska 21, 00-049 Warsaw, Poland
E-mail: *zzawist@ippt.gov.pl*

Lie group of point symmetries is determined for Vlasov–Maxwell equations of spherically symmetric plasmas. The case of multi-component plasma is considered. Classification of possible invariant solutions is presented. Comparison with the case of one-dimensional rectilinear motion of plasma particles is made.

1 Introduction

The case of purely radial motion of charges is singled out by the general properties of Maxwell equations. Similarly as in one-dimensional rectilinear motion of particles of plasma there is no generation of time dependent magnetic field in this case [1]. Thus, there is no radiation and the self-consistent solutions of the Vlasov–Maxwell equations (distribution functions, electric field and no magnetic field) exist.

The case of one-dimensional rectilinear motion is widely employed in plasma physics [2–5]. It is used as a starting point for analysis of more complicated systems with very significant applications. The same is expected in the case of radial motion, however, this case is not popular and not widely employed. Thus, research into that topic is important and promising.

In this paper we find Lie symmetry groups of point transformations of the Vlasov–Maxwell equations for collision-less multi-component plasma without magnetic field in the case of purely radial motion of particles. We find the optimal system of one-parameter subgroups and classify corresponding invariant solutions.

We use the direct method for determining symmetry groups of integro-differential equations (IDE's) presented in papers [6–8]. For the IDE's of the type

$$F(x_1,\dots,x_n,y,\underset{1}{y},\dots,\underset{m}{y})+\int_X dx_1\cdots dx_l f(x_1,\dots,x_n,y,\underset{1}{y},\dots,\underset{k}{y})=0, \tag{1}$$

where the symbol $\underset{m}{y}$ denotes the set of all partial derivatives of m-order, it consists in the following infinitesimal criterion of invariance

$$G^{(m)}F+\int_X dx_1\cdots dx_l\left[G^{(k)}f+f\sum_{i=1}^{l}\partial_i\xi_i\right]=0 \qquad \text{on solutions of (1)}, \tag{2}$$

where $G^{(m)}$ is the extended to m-th order generator of the point transformation

$$\begin{aligned}&\widetilde{x}^i=e^{\epsilon G}x^i=x^i+\epsilon\xi^i(x,y)+\mathcal{O}\big(\epsilon^2\big),\\ &\widetilde{y}=e^{\epsilon G}y=y+\epsilon\eta(x,y)+\mathcal{O}\big(\epsilon^2\big),\end{aligned} \tag{3}$$

with the generator

$$G=\xi^i(x,y)\partial_{x^i}+\eta(x,y)\partial_y. \tag{4}$$

The method is a natural generalization of the well known Ovsiannikov's method for partial differential equations which can be found in monographs [9–12]. The short review with relevant references of other methods of finding symmetries of integro-differential equations is given in [8].

2 Symmetries of radial multi-component plasma

In the case of purely radial motion the Vlasov–Maxwell system of equations for collision-less, multi-component, plasmas without magnetic field has the following form

$$\begin{aligned}
&\partial_t f_\alpha + u\partial_r f_\alpha + \frac{q_\alpha}{m_\alpha} E \partial_u f_\alpha = 0,\\
&\partial_t E + \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int_0^\infty du\, u^3 f_\alpha = 0, \qquad \partial_r E + \frac{2}{r}\,E - \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int_0^\infty du\, u^2 f_\alpha = 0,
\end{aligned} \tag{5}$$

where $E = E(t,r)$ is the radial component of electric field vector, u is the radial component of vector velocity, $f_\alpha = f_\alpha(t,r,u)$ is the radial distribution function of α-plasma component (integrated over the solid angle in velocity space with 4π included), q_α, m_α are charge and mass of α-particles respectively, and ϵ_0 is electric permittivity of free space.

In this case, the generators (4) of point transformations (3) take the form

$$G = \tau\partial_t + \xi\partial_r + \rho\partial_u + \sum_\alpha \eta_\alpha \partial_{f_\alpha} + \zeta\partial_E. \tag{6}$$

Using the criterion (2) we obtain

$$0 = \partial_{f_\alpha}\tau = \partial_{f_\alpha}\xi = \partial_{f_\alpha}\rho = \partial_E\tau = \partial_E\xi = \partial_E\rho,$$

and the following determining equations (limits 0 and ∞ of integrals are dropped):

$$\begin{aligned}
&0 = \partial_E\eta_\alpha \qquad 0 = \partial_{f_\alpha}\zeta, \qquad 0 = u\partial_u\tau - \partial_u\xi, \qquad 0 = \left(\frac{q_\alpha}{m_\alpha} - \frac{q_\beta}{m_\beta}\right)\partial_{f_\beta}\eta_\alpha \quad \text{for } \alpha \neq \beta,\\
&0 = \partial_t\eta_\alpha + u\partial_r\eta_\alpha + \frac{q_\alpha}{m_\alpha}\,E\partial_u\eta_\alpha, \qquad 0 = u\partial_t\tau - \partial_t\xi + \rho + u^2\partial_r\tau - u\partial_r\xi,\\
&0 = \partial_t\zeta + \frac{2}{r}\,E\partial_t\xi, \qquad 0 = \partial_r\zeta - \frac{2}{r^2}\,E\xi + \frac{2}{r}\,\zeta - \frac{2}{r}\,E\partial_E\zeta + \frac{2}{r}\,E\partial_r\xi,\\
&0 = \frac{q_\alpha}{m_\alpha}E\left(\partial_t\tau + u\partial_r\tau + \frac{q_\alpha}{m_\alpha}E\partial_u\tau - \partial_u\rho\right) + \frac{q_\alpha}{m_\alpha}\,\zeta - \partial_t\rho - u\partial_r\rho,\\
&0 = (\partial_t\tau - \partial_E\zeta)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u^3 f_\beta - (\partial_t\xi)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u^2 f_\beta\\
&\qquad + \sum_\beta \frac{q_\beta}{\epsilon_0}\int du(3u^2\rho f_\beta + u^3\eta_\beta + u^3 f_\beta\partial_u\rho),\\
&0 = (\partial_E\zeta - \partial_x\xi)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u^2 f_\beta + (\partial_r\tau)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u^3 f_\beta\\
&\qquad - \sum_\beta \frac{q_\beta}{\epsilon_0}\int du\big(2u\rho f_\beta + u^2\eta_\beta + u^2 f_\beta\partial_u\rho\big).
\end{aligned}$$

For different species $\alpha \neq \beta$ also $q_\alpha/m_\alpha \neq q_\beta/m_\beta$ for real plasmas (except the special case of deuterium and helium nuclei). This leads to a symmetry group which is completely different from the case of one-component plasma when there is only one value of q/m. This is the reason for the oscillatory like generators for one-component plasma in the case of rectilinear motion of particles to appear, as found by V. Taranov [14]. We easily find that

$$0 = \partial_u\tau = \partial_u\xi = \partial_t\eta_\alpha = \partial_r\eta_\alpha = \partial_u\eta_\alpha = \partial_E\eta_\alpha, \qquad 0 = \partial_{f_\beta}\eta_\alpha \quad \text{for } \alpha \neq \beta, \qquad \zeta = \lambda_1 E.$$

Then, the last two integro-differential equations lead to

$$0 = \int du \left[f_\alpha (u^3 \partial_u \tau - \lambda_1 u^3 - u^2 \partial_t \xi + 3u^2 \rho + u^3 \partial_u \rho) + u^3 \eta_\alpha \right],$$
$$0 = \int du \left[f_\alpha (\lambda_1 u^2 - u^2 \partial_r \xi + u^3 \partial_r \tau - 2u\rho - u^2 \partial_u \rho) + u^2 \eta_\alpha \right].$$

We assume that the point transformations (3) are analytic functions of the point $(t, r, u, f_\alpha, \zeta)$. In general, analyticity with respect to the parameter ϵ and infinite differentiability with respect to the point is assumed for Lie groups. However, the latter dependence is in fact also analytic due to a physical interpretation. Expanding $\eta_\alpha(f_\alpha)$ in the Taylor series, using the generalized mean value theorem and well known special stationary solutions of the Vlasov–Maxwell equations (5), which depend only on velocity, we find that the coefficients η_α can depend on f_α only linearly $\eta_\alpha = \lambda_2 f_\alpha$. Thus, we can apply the Lagrange lemma of calculus of variations [13] and obtain *differential* equations for integrands.

Solutions of the determining equations lead to the following three generators

$$G_1 = \partial_t, \qquad G_2 = -t\partial_t - 2r\partial_r - u\partial_u + 5\sum_\alpha f_\alpha \partial_{f_\alpha},$$
$$G_3 = -3t\partial_t - r\partial_r + 2u\partial_u + 5E\partial_E, \tag{7}$$

which span the Lie algebra of the group of point symmetry transformations of the Vlasov–Maxwell equations (5). Non-vanishing commutators between these generators are given by

$$[G_1, G_2] = -G_1, \qquad [G_1, G_3] = -3G_1, \qquad [G_2, G_3] = 0.$$

The algebra is solvable.

Summing up the Lie series we obtain one-parameter subgroups of the symmetry group of transformations corresponding to the generators (7). For G_1 we have translations in time:

$$\widetilde{t} = t + \epsilon, \qquad \widetilde{r} = r, \qquad \widetilde{u} = u, \qquad \widetilde{f_\alpha} = f_\alpha, \qquad \widetilde{E} = E.$$

This symmetry follows from the fact that coefficients of equations (5) do not depend on variable t. The G_2 and G_3 generate the following scaling transformations:

$$\widetilde{t} = t\exp(-\epsilon), \qquad \widetilde{r} = r\exp(-2\epsilon), \qquad \widetilde{u} = u\exp(-\epsilon), \qquad \widetilde{f_\alpha} = f_\alpha \exp(5\epsilon), \qquad \widetilde{E} = E,$$

and

$$\widetilde{t} = t\exp(-3\epsilon), \qquad \widetilde{r} = r\exp(-\epsilon), \qquad \widetilde{u} = u\exp(2\epsilon), \qquad \widetilde{f_\alpha} = f_\alpha, \qquad \widetilde{E} = E\exp(5\epsilon).$$

The lack of Galilean symmetry and spatial translational symmetry, contrary to the case of one-dimensional rectilinear motion [6–8], follows from the fact that for radial motion there exists the distinguished frame of reference connected with the point $r = 0$.

3 Classification of invariant solutions

Using the adjoint representation we find the optimal system of one-dimensional subalgebras (see [9, 10]). They are generated by the following operators:

$$G_1, \quad G_2, \quad G_3, \quad \pm G_1 - 3G_2 + G_3, \quad a_2 G_2 + a_3 G_3 \ \text{ for arbitrary } a_2, a_3 \neq 0.$$

Since G_2 commutes with G_3, we can express the finite transformations corresponding to the generator $a_2 G_2 + a_3 G_3$ as compositions of one-dimensional subgroups generated by operators G_2

and G_3. Similarly, because $[\pm G_1, -3G_2 + G_3] = 0$ the finite transformation corresponding to $\pm G_1 - 3G_2 + G_3$ is a composition of subgroups generated by G_1, G_2 and G_3. Using these transformations we find invariants built from independent and dependent variables. From these invariants follow the forms of invariant solutions corresponding to the optimal system. For example,

$$e^{\epsilon(\pm G_1 - 3G_2 + G_3)} = e^{\epsilon \pm G_1} e^{\epsilon(G_3 - 3G_2)} = e^{\pm \epsilon G_1} e^{-3\epsilon G_2} e^{\epsilon G_3}$$

leads to the following transformation

$$\widetilde{t'} = t' e^{\pm\epsilon}, \qquad \widetilde{r} = r e^{5\epsilon}, \qquad \widetilde{u} = u e^{5\epsilon}, \qquad \widetilde{f_\alpha} = f_\alpha e^{-15\epsilon}, \qquad \widetilde{E} = E e^{5\epsilon},$$

where $t' = e^t$. Invariants have the forms

$$r e^{\mp 5t}, \quad u e^{\mp 5t}, \quad r u^{-1}, \quad f_\alpha e^{\pm 15t}, \quad r^3 f_\alpha, \quad u^3 f_\alpha, \quad E e^{\mp 5t}, \quad r^{-1} E, \quad u^{-1} E, \quad \ldots.$$

We choose independent invariants (two of them built from independent variables – see [11])

$$y = r e^{\mp 5t}, \qquad z = u e^{\mp 5t}, \qquad f_{\alpha 0} = f_\alpha e^{\pm 15t}, \qquad E_0 = E e^{\mp 5t}$$

and look for solutions in the form

$$f_\alpha = e^{\mp 15t} f_{\alpha 0}\big(r e^{\mp 5t}, u e^{\mp 5t}\big), \qquad E = e^{\pm 5t} E_0\big(r e^{\mp 5t}\big),$$

because electric field does not depend on velocity u.

The results of this classification of essentially independent invariant solutions are gathered in the following table

No	Subgroup	Form of the solution
1	G_1	$f_\alpha(r,u)\,,\ E(r)$
2	G_2	$t^{-5} f_\alpha\big(t^2 r^{-1}, r^{-1}u^2\big), \quad E\big(t^2 r^{-1}\big)$
3	G_3	$f_\alpha\big(t r^{-3}, r^{-1}u^2\big), \quad t^{-1} r^{-2} E\big(t r^{-3}\big)$
4	$\pm G_1 - 3G_2 + G_3$	$e^{\mp 15t} f_\alpha\big(r e^{\mp 5t}, u e^{\mp 5t}\big), \quad e^{\pm 5t} E\big(r e^{\mp 5t}\big)$
5	$a_2 G_2 + a_3 G_3$	$r^{-2} u f_\alpha\big(t^{(2a_2+a_3)} r^{-(a_2+3a_3)}, t r^{-1} u\big),$ $t^{-2} r E\big(t^{(2a_2+a_3)} r^{-(a_2+3a_3)}\big)$

For example, stationary solutions $f_\alpha(r,u)$, $E(r)$ corresponding to G_1 are generalizations of nonlinear electrostatic BGK-waves (Bernstein, Greene, Kruskal [15]). The BGK-solutions are found by solving an integro-differential equation for electrostatic potential. This equation, which is obtained [2] by substituting the general solution of the Vlasov equation into the Poisson equation, is now more complicated due to the term $2Er^{-1}$. In the case of vanishing field $E = 0$ we obtain the equilibrium solutions of the same form as in the rectilinear case: $f_\alpha(u)$, where f_α is an arbitrary function.

4 Conclusions

The results obtained in this paper provide a new example of effectiveness of the method presented in [6–8]. Namely, we have found symmetry groups for Vlasov–Maxwell equations for spherically symmetric systems of multi-component collisionless plasma. The optimal system

of one-parameter subgroups is found and classification of corresponding invariant solutions is carried out.

Spherical symmetry of our problem leads to determining equations, which are different from the case of one-dimensional rectilinear motion due to higher powers of velocity in the Vlasov–Maxwell equations.

The symmetry group is narrower, only three-dimensional, than the symmetry group for one-dimensional rectilinear case [6,7,14]. The lack of the Galilean symmetry and spatial translational symmetry for the radial case is obvious since for radial motion there exists the distinguished frame of reference connected with the point $r = 0$. Other differences are of minor importance.

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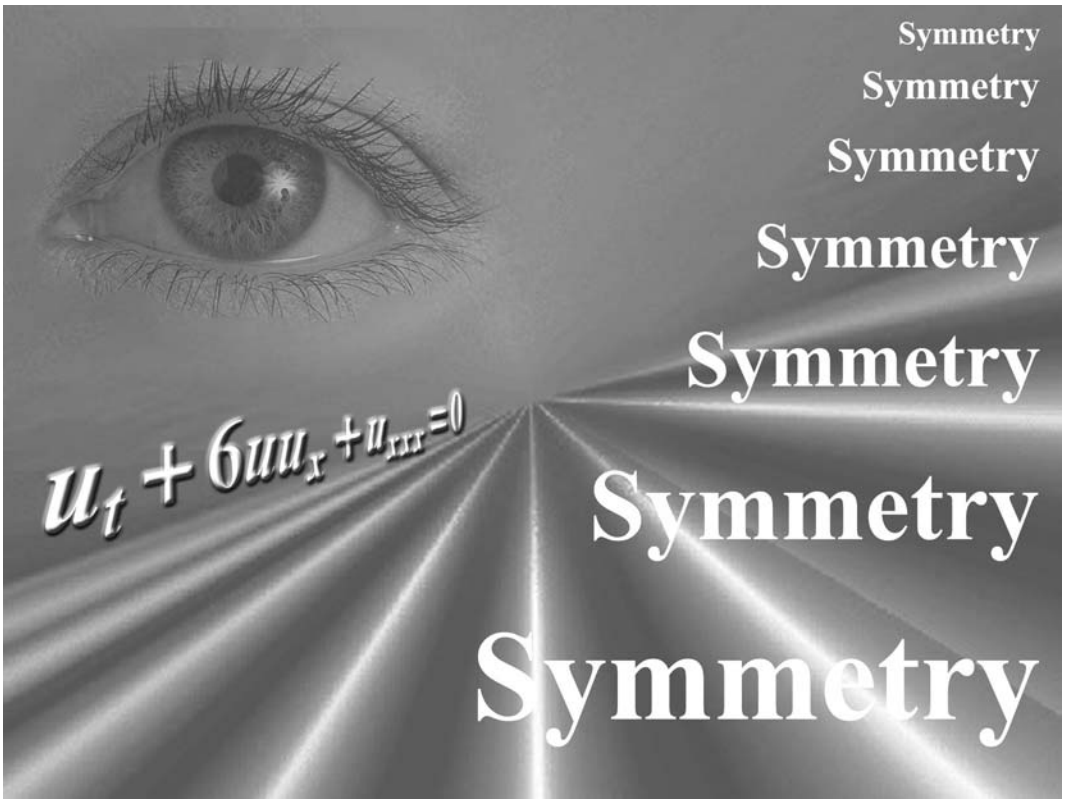
Solitons and Integrability



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Conformal Maps and Integrable Hamiltonian Structures

Eugene BELOKOLOS

Dept. Theor. Physics, Institute of Magnetism, 36-b Vernadskii Ave., 03142 Kyiv, Ukraine
E-mail: *bel@imag.kiev.ua*

A discussion of integrable structures of complex analysis is presented. It is shown that a conformal map $z(w)$ of the exterior of the unit disc to the exterior of a simply-connected domain and a map $\overline{z}\big(w^{-1}\big)$ are conjugate variables with respect to some Poisson structure. Deformations of the functions $z(w)$, $\overline{z}\big(w^{-1}\big)$ with respect to the harmonic moments of the domain are described by the Hamiltonian equations. These equations are known as the Lax–Sato equations.

1 Introduction

Basic notions of the complex analysis and the physics have deep inter-connections. It is well known that an analytic function of complex variable may be considered as a complex potential of plane vector field and this fact leads to construction of the conformal field theory. But it is less known that a conformal map may be considered as an integrable Hamiltonian system, although this connection between complex analysis and mechanics was established some years ago. In our report we present a pedagogical introduction to this problem.

2 Moments of a curve (domain). Generating function of moments

We designate a curve on complex plane z as a closed analytic curve if it is an analytic image of circle, i.e. if it is parametrized by a function $z(w)$ that is analytic in a ring $a < |w| < b$ containing a circle $|w| = 1$.

Let us consider on a complex plane z a closed analytic curve Γ and designate by D_+ and D_- internal and external domains with respect to this curve. We shall assume that the point $z = 0$ belongs to the domain D_+.

For the curve Γ (or domains D_+ and D_-) we introduce two sets of the harmonic moments, namely, the external moments $\{t_0, t_1, \ldots\}$,

$$t_0 = \frac{1}{\pi}\int_{D_-} \mathrm{d}^2 z, \qquad t_k = -\frac{1}{\pi k}\int_{D_-} z^{-k}\mathrm{d}^2 z, \qquad k \geq 1,$$

and the internal moments $\{v_0, v_1, \ldots\}$,

$$v_0 = \frac{2}{\pi}\int_{D_+} \log|z|\mathrm{d}^2 z, \qquad v_k = \frac{1}{\pi}\int_{D_+} z^k \mathrm{d}^2 z, \qquad k \geq 1.$$

Moments t_0, v_0 are real numbers, other moments are complex numbers.

Let us introduce a generating function of moments F and τ-function

$$F = \log\tau = -\frac{1}{\pi}\int_{D_+}\mathrm{d}^2 z \int_{D_+}\mathrm{d}^2 z' \log\left|\frac{1}{z} - \frac{1}{z'}\right|.$$

The following theorem is valid.

Theorem 1. *A differential of the generating function of moments F (or the logarithm of τ-function) is of a form*

$$\mathrm{d}F = \mathrm{d}\log\tau = v_0\mathrm{d}t_0 + \sum_{k=1}^{\infty}\left(v_k\mathrm{d}t_k + \overline{v}_k\mathrm{d}\overline{t}_k\right),$$

and it allows to express the internal moments in terms of external ones,

$$v_0 = \frac{\partial F}{\partial t_0} = \frac{\partial\log\tau}{\partial t_0}, \qquad v_k = \frac{\partial F}{\partial t_k} = \frac{\partial\log\tau}{\partial t_k}, \qquad \overline{v}_k = \frac{\partial F}{\partial \overline{t}_k} = \frac{\partial\log\tau}{\partial \overline{t}_k}.$$

Thus, the sets of external and internal moments are not independent, there is connection between them. Any of these sets of moments defines uniquely the curve Γ (or domains D_+ and D_-) [1–3].

3 The Schwarz function $S(z)$ and its potential $\Omega(z)$

Let us present the equation $g(x,y)=0$ of an analytic curve Γ in complex coordinates, $g(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}) = 0$, and resolve the latter one with respect to the variable z. We obtain the equation of the curve Γ in the following form

$$\overline{z} = S(z).$$

The function $S(z)$ is called the Schwarz function of the curve Γ. This function is analytic in a strip that contains the curve [4].

The Schwarz function maps a point z into the conjugate one $S(z)$ with respect to the curve Γ. On the curve itself the Schwarz function coincides obviously with the complex conjugation.

According to definition the Schwarz function cannot be an arbitrary one, it must satisfy the so called unitary condition

$$z = \overline{S}(S(z)),$$

i.e. the inverse function must coincide with with the complex conjugated one. We remark here that if a function $f(z)$ is defined with the Laurent series $f(z) = \sum_j f_j z^j$, then $\overline{f}(z) = \sum_j \overline{f}_j z^j$.

Theorem 2. *The Laurent series for the Schwarz function $S(z)$ of a curve Γ is of the form*

$$S(z) = \sum_{k=-\infty}^{\infty} s_k z^{k-1} = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1},$$

i.e. the Laurent series coefficients of the Schwarz function are expressed in terms of the moments of the curve.

Proof. Let us assume that a function $f(z)$ is analytic in the domain $D_\pm$. Then for the function $g(z) = f(z)\overline{z} = f(z)S(z)$ the Green–Ostrogradskii formula

$$\int_{D_\pm}\frac{\partial g(z)}{\partial\overline{z}}\mathrm{d}^2 z = \pm\frac{1}{2i}\int_\Gamma g(z)\mathrm{d}z,$$

leads to the relation

$$\int_{D_\pm} f(z)\mathrm{d}^2 z = \pm\frac{1}{2i}\int_\Gamma f(z)S(z)\mathrm{d}z.$$

Using this formula in the case $f(z) = z^k$, $k \in \mathbb{Z}$, we can express the moments of curve Γ in terms of the Schwarz function $S(z)$,

$$t_k = \frac{1}{2\pi i k} \int_\Gamma z^{-k} S(z) \mathrm{d}z, \qquad v_k = \frac{1}{2\pi i} \int_\Gamma z^k S(z) \mathrm{d}z.$$

Substituting in integrals the Schwarz function in the form of its Laurent series,

$$S(z) = \sum_{k=-\infty}^{\infty} s_k z^{k-1}$$

and calculating integrals by means of the orthogonality relation

$$\frac{1}{2\pi i} \int_\Gamma z^j \mathrm{d}z = \delta_{j,-1},$$

we obtain

$$s_0 = t_0, \qquad s_k = k t_k, \qquad s_{-k} = v_k, \qquad k \geq 1.$$

Thus the moments of the curve Γ are the coefficients of the Laurent series for the Schwarz function $S(z)$,

$$S(z) = \sum_{k=-\infty}^{\infty} s_k z^{k-1} = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1}. \qquad \blacksquare$$

Now we introduce the function $\Omega(z)$ that is a potential for the Schwarz function,

$$S(z) = \partial_z \Omega(z).$$

According to definition we can present the function $\Omega(z)$ in a form of the Laurent series,

$$\Omega(z) = \sum_{k=1}^{\infty} t_k z^k - \frac{1}{2} v_0 + t_0 \log z - \sum_{k=1}^{\infty} \frac{v_k}{k} z^{-k},$$

and hence as a sum of the following functions,

$$\Omega(z) = \Omega^{(+)}(z) + \Omega^{(-)}(z) - \frac{1}{2} v_0,$$

where the functions $\Omega^{(\pm)}(z)$ are analytic in domains $D_\pm$ appropriately. Here we have used the notations

$$\Omega^{(+)}(z) = \sum_{k=1}^{\infty} t_k z^k = \frac{1}{\pi} \int_{D_-} \log\left(1 - \frac{z}{z'}\right) \mathrm{d}^2 z',$$

$$\Omega^{(-)}(z) = t_0 \log z - \sum_{k=1}^{\infty} \frac{v_k}{k} z^{-k} = \frac{1}{\pi} \int_{D_+} \log\left(z - z'\right) \mathrm{d}^2 z'.$$

At the curve Γ we can write down the potential $\Omega(z)$ in the form

$$\Omega(z) = \frac{1}{2}|z|^2 + i 2A(z), \qquad z \in \Gamma,$$

where $|z|$ is the distance from the origin of coordinates to the point z, and $A(z)$ is the area of the domain D_+, bounded with a ray $\phi = \arg z$ and a real line.

The following theorem is valid.

Theorem 3. *The differential of the function $\Omega(z)$ is of the form*

$$\mathrm{d}\Omega = S\mathrm{d}z + \log w\,\mathrm{d}t_0 + \sum_{k=1}^{\infty}\left(H_k\,\mathrm{d}t_k - \overline{H}_k\,\mathrm{d}\overline{t}_k\right),$$

where

$$H_k(z) = \partial_{t_k}\Omega(z) = \left(z^k(w)\right)_+ + \frac{1}{2}\left(z^k(w)\right)_0,$$

$$\overline{H}_k(z) = -\partial_{\overline{t}_k}\Omega(z) = -\left(\overline{z}^k(w)\right)_- - \frac{1}{2}\left(\overline{z}^k(w)\right)_0.$$

Here the symbol $(f(w))_\pm$ designates the part of the Laurent series that contains summands only with positive (negative) powers w, and $(f(w))_0$ designates the summand with w^0.

4 Conformal maps as Hamiltonian systems with respect to moments

Let us consider the disc of unit radius $U(0,1)$ on a plane w, and the domain D on a plane z. Let us consider the univalent analytic function $z(w)$, that maps conformally a complement of the unit disc to a complement of the domain D. In order to emphasize a dependence of this function on the domain D it would be reasonable to write down the function in the form $z(w,D)$ or $z(w,t_0,t_1,\ldots)$ since moments $(t_0,t_1,\ldots)$ of the domain D define it uniquely. We shall use the latter designation.

For arbitrary functions $f(w,t_0)$, $g(w,t_0)$ let us define the Poisson brackets as follows

$$\{f,g\} = \frac{\partial f}{\partial \log w}\frac{\partial g}{\partial t_0} - \frac{\partial g}{\partial \log w}\frac{\partial f}{\partial t_0} = w\left(\frac{\partial f}{\partial w}\frac{\partial g}{\partial t_0} - \frac{\partial g}{\partial w}\frac{\partial f}{\partial t_0}\right).$$

It is easy to verify that according to definition these Poisson brackets have the following properties:

1)Antisymmetry

$$\{f,g\} = -\{g,f\},$$

2) Linearity

$$\{f_1+f_2,g\} = \{f_1,g\} + \{f_2,g\},$$

3) The Jacobi identity

$$\{f,\{g,h\}\} = \{g,\{h,f\}\} = \{h,\{f,g\}\}.$$

We can present the functions

$$z(w) = \overline{S}\left(\overline{z}\left(w^{-1}\right)\right), \qquad \overline{z}\left(w^{-1}\right) = S(z(w)),$$

by means of series

$$z(w) = rw + \sum_{j=0}^{\infty} u_j w^{-j}, \qquad \overline{z}\left(w^{-1}\right) = rw^{-1} + \sum_{j=0}^{\infty}\overline{u}_j w^j,$$

where r is the so called conformal radius.

Using the latter expressions we can prove the following theorem.

Theorem 4 ([5]). *A pair of variables* $(\log w, t_0)$ *and* $\big(z(w,t_0), \overline{z}\big(w^{-1},t_0\big)\big)$ *are canonical variables with respect to the Poisson brackets defined above,*

$$\big\{z(w,t_0), \overline{z}\left(w^{-1},t_0\right)\big\} = 1.$$

Proof. According to definition of the Poisson bracket

$$\big\{z(w,t_0), \overline{z}\left(w^{-1},t_0\right)\big\} = w\left(\frac{\partial z(w,t_0)}{\partial w}\frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial t_0} - \frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial w}\frac{\partial z(w,t_0)}{\partial t_0}\right).$$

Now using the formula $\overline{z}\big(w^{-1},t_0\big) = S(z(w,t_0))$ we get

$$\frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial t_0} = \frac{\partial S(z,t_0)}{\partial t_0} + \frac{\partial S(z,t_0)}{\partial z}\frac{\partial z(w,t_0)}{\partial t_0}, \qquad \frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial w} = \frac{\partial S(z,t_0)}{\partial z}\frac{\partial z(w,t_0)}{\partial w},$$

and substituting these expressions in the Poisson bracket we obtain the equality

$$\big\{z(w,t_0), \overline{z}\left(w^{-1},t_0\right)\big\} = w\frac{\partial S(z,t_0)}{\partial t_0}\frac{\partial z(w,t_0)}{\partial w}.$$

The r.h.s. of this equality due to

$$S(z) = \sum_{k=1}^{\infty} k t_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} v_k z^{-k-1}, \qquad \text{and} \qquad z(w) = rw + \sum_{j=0}^{\infty} u_j w^{-j},$$

is 1 plus positive powers in w.

Similarly using the formula $z(w) = \overline{S}\big(\overline{z}\big(w^{-1}\big)\big)$, we prove that

$$\big\{z(w,t_0), \overline{z}\left(w^{-1},t_0\right)\big\} = -w\frac{\partial \overline{S}(\overline{z},t_0)}{\partial t_0}\frac{\partial \overline{z}\left(w^{-1},t_0\right)}{\partial w}.$$

The r.h.s. of this equality due to

$$\overline{S}(z) = \sum_{k=1}^{\infty} k \overline{t}_k z^{k-1} + \frac{t_0}{z} + \sum_{k=1}^{\infty} \overline{v}_k z^{-k-1}, \qquad \text{and} \qquad \overline{z}\left(w^{-1}\right) = rw^{-1} + \sum_{j=0}^{\infty} \overline{u}_j w^{j},$$

is 1 plus negative powers of w.

Comparing two expressions for the Poisson bracket we prove the theorem. ■

Thus we established the symplectic structure of the conformal map.

Theorem 5 ([5]). *The deformation of the conformal map* $z(w)$ *as a result of variation of moments* $\{t_k\}_{k=0}^{\infty}$ *is described with the so called Lax–Sato equations*

$$\frac{\partial z(w)}{\partial t_k} = \{H_k, z(w)\}, \qquad \frac{\partial \overline{z}\left(w^{-1}\right)}{\partial t_k} = \left\{H_k, \overline{z}\left(w^{-1}\right)\right\}.$$

Self-consistency conditions of the Lax–Sato equations are

$$\partial_{t_k} H_l - \partial_{t_l} H_k + \{H_k, H_l\} = 0.$$

5 An example: ellipse

Let us consider an example when the curve Γ is ellipse and is characterized with three external moments t_0, t_1, t_2, all other moments are equal to zero, $t_3 = t_4 = \cdots = 0$.

An ellipse with center at a point $z = 0$ and half-axes a, b along coordinate axes x, y is described by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1.$$

Three non-zero external moments of the ellipse in terms of half-axes a, b looks as follows,

$$t_0 = ab, \qquad t_1 = 0, \qquad t_2 = \frac{1}{2}\frac{a-b}{a+b}.$$

All internal moments of the ellipse are non-zero and the first two of them are of the following form,

$$v_1 = 0, \qquad v_2 = \frac{2t_0^2 t_2}{1-4t_2^2}.$$

The Schwarz function of ellipse is

$$S(z) = \frac{a^2+b^2}{a^2-b^2}z - \frac{2ab}{a^2-b^2}\sqrt{z^2-(a^2-b^2)},$$

and its Laurent series looks as follows,

$$S(z) = 2t_2 z + \frac{t_0}{z} + \frac{v_2}{z^3} + \cdots .$$

A logarithm of τ-function is

$$\log\tau = \frac{1}{2}t_0^2\left(\log\frac{t_0}{1-t_2^2} - \frac{3}{2}\right).$$

The conformal map of a complement of unit disc to a complement of ellipse is the function

$$z(w) = rw + u_1 w^{-1}$$

with coefficients

$$r^2 = \frac{t_0}{1-4t_2^2} = \frac{1}{4}(a+b)^2, \qquad u_1^2 = \frac{4t_0 t_2^2}{1-4t_2^2} = \frac{1}{4}(a-b)^2.$$

The first two Hamiltonians are

$$H_1 = rw, \qquad H_2 = r^2 w^2 + u_1 r,$$

and the appropriate Hamiltonian flows do not change the form of disc. The Lax–Sato equations with the Hamiltonians $H_3, H_4, \ldots$ describe a deformation of the disc into the ellipse.

6 The Dirichlet boundary problem

The Dirichlet boundary problem

$$\Delta u(z) = 4\partial_z \partial_{\overline{z}} u(z) = f(z), \quad z \in D; \qquad u(z)|_{\partial D} = g(z),$$

has a solution

$$u(z) = \frac{1}{2\pi} \iint_D f(\zeta) G(z,\zeta) \mathrm{d}\zeta \mathrm{d}\overline{\zeta} + \frac{1}{2\pi} \int_{\partial D} g(\zeta) \frac{\partial}{\partial n} G(z,\zeta) |\mathrm{d}\zeta|,$$

where $G(z,\zeta)$ is the Green function which is uniquely defined by the following properties:

1) $G(z,\zeta) - \log|z-\zeta|$ is a symmetric, bounded and harmonic function in both arguments everywhere in the domain D,

2) $G(z,\zeta) = 0$ if any one of their arguments is at the boundary of the domain D.

We can present the Green function by the expression

$$G(z,\zeta) = |w(z,\zeta)|,$$

where $w(z,\zeta)$ is the analytic function which describe a conformal map of the domain D to the unit disc U.

According to J. Hadamard [6, 7] variations of the Green function under small deformations of the domain looks as follows

$$\delta G(z,\zeta) = \frac{1}{2\pi} \int_{\partial D} \partial_n G(z,\eta) \partial_n G(\zeta,\eta) \delta h(\eta) |\mathrm{d}\eta|,$$

where $\delta h(\eta)$ is a shift from the curve ∂D to the deformed curve along the normal n at the point $\eta \in \partial D$.

It is possible to write down the deformations $\delta h(\eta)$ of the domain boundary in terms of harmonic moments of the domain and as a result of that to express the variations of the Green function in these terms.

7 Quantization

Now let us quantize the above Hamiltonian structure and introduce respectively basic quantities of the quantized theory.

The Lax operators are

$$L = r(t_0) \exp\left(\hbar \frac{\partial}{\partial t_0}\right) + \sum_{k=0}^{\infty} u_k(t_0) \exp\left(-k\hbar \frac{\partial}{\partial t_0}\right),$$

$$\overline{L} = \exp\left(-\hbar \frac{\partial}{\partial t_0}\right) r(t_0) + \sum_{k=0}^{\infty} \exp\left(k\hbar \frac{\partial}{\partial t_0}\right) \overline{u}_k(t_0).$$

The Hamiltonians are

$$H_k = \left(L^k\right)_+ + \frac{1}{2}\left(L^k\right)_0, \qquad \overline{H}_k = \left(\overline{L}^k\right)_- + \frac{1}{2}\left(\overline{L}^k\right)_0,$$

where the symbol $\left(L^k\right)_{\pm}$ means positive (negative) parts of series for the shift operator $\exp\left(\hbar \frac{\partial}{\partial t_0}\right)$.

The Lax–Sato equations attain now the following form

$$\hbar \frac{\partial L}{\partial t_k} = [H_k, L], \qquad \hbar \frac{\partial L}{\partial \overline{t}_k} = [\overline{H}_k, L].$$

The spectrum of Lax operator is defined by the equation

$$L\Psi = z\Psi,$$

and the appropriate wave function can be presented as follows,

$$\Psi(z,t_0,t_1,\ldots) = z^{\hbar^{-1}t_0}\tau_\hbar^{-1}(t_0,t_1,\ldots)\exp\left(\frac{1}{\hbar}\sum_{k>0}t_kz^k\right)\exp\left(\hbar\sum_{k>0}\frac{z^{-k}}{k}\frac{\partial}{\partial t_k}\right)\tau_\hbar(t_0,t_1,\ldots).$$

By means of the last expression we introduce the function

$$\tau_\hbar(t_0,t_1,\ldots).$$

The commutator

$$[L,\overline{L}] = \hbar$$

is called the selection rule or the string equation

In the semiclassical limit $\hbar \to 0$ we get the following correspondence with the notions considered above:

$$W = \exp\left(\hbar\frac{\partial}{\partial t_0}\right) \to w, \qquad L \to z(w),$$
$$\Psi \to \exp\left(\frac{\Omega}{\hbar}\right), \qquad \tau_\hbar \to \exp\left(\log\frac{\tau}{\hbar^2}\right),$$
$$\frac{1}{\hbar}[L,\overline{L}] = 1 \to \left\{z(w),\overline{z}\left(w^{-1}\right)\right\} = 1.$$

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Integration of Bi-Hamiltonian Systems by Using the Dressing Method

Yuriy BERKELA

Carpathian Biosphere Reserve, Rakhiv, Ukraine
E-mail: *yuri@rakhiv.ukrtel.net*

The integration by the dressing method of integrable by Lax systems from the $\mathcal{D}$HcmKP hierarchy is considered. The applicability of this method to construction of exact solutions of the nonlinear bi-Hamiltonian systems, also with nonstandart (recursive) Lax representations, is shown.

1 Introduction

The paper [1] introduces the so-called scalar $\mathcal{D}$-Hermitian constrained modified Kadomtsev–Petviashvili ($\mathcal{D}$HcmKP) hierarchy. The integrable systems of this hierarchy contain already known nonlinear models of the soliton theory and their new modifications and vector (multi-component) generalizations. The unified form of the Lax operator (4) allows to construct a general method of the Lax flows investigation describing a group of transformation operators (group of $\mathcal{D}$-unital Volterra operators) which corresponds to the Lie algebra of the integro-differential symbols of $\mathcal{D}$-skew-Hermitian operators.

This paper continues integration of integrable systems from $\mathcal{D}$HcmKP by using the method of dressing transformations. For $n = 2$, under additional reduction, Lax operator (4) becomes the generating operator for the modified Korteweg–de Vries equation (mKdV) in a real case. The vector generalization of the mKdV equation can be reduced to common Korteweg–de Vries equation (KdV).

In Section 2 we submit basic definitions in the $\mathcal{D}$HcmKP hierarchy and reductions to well-known dynamical systems.

In Sections 3 we propose the method of construction of exact solutions for the KdV equation and the mKdV equation.

2 $\mathcal{D}$HcmKP hierarchy and its reductions

Let us consider the algebra ζ of the micro-differential operators [2],

$$\zeta := \left\{ L = \sum_{i=-\infty}^{n(L)} a_i \mathcal{D}^i : \ a_i = a_i(x, y, t_m); \ i, n(L) \in \mathbb{Z} \right\}.$$

The coefficients a_i are, in general, smooth $(N \times N)$-matrix-valued functions of $x \in \mathbb{R}$ and of finite quantity of the evolution parameters $t_m \in \mathbb{R}$, $t_2 := y$, $t_3 := t$. The micro-differential operator $L \in \zeta$ satisfies additional constraints. The Hermitian-conjugated operator $L^* := \sum\limits_{i=-\infty}^{n(L)} (-1)^i \mathcal{D}^i a_i^*$, where $a_i^* = \bar{a}^\top$, $(\alpha \partial_y)^* := -\bar{\alpha}\partial_y$, $(\beta \partial_{t_m})^* := -\bar{\beta}\partial_{t_m}$.

Definition 1. We say that an operator $L \in \zeta$ is $\mathcal{D}$-Hermitian ($\mathcal{D}$-skew-Hermitian) if $L^* = \mathcal{D}L\mathcal{D}^{-1}$ ($L^* = -\mathcal{D}L\mathcal{D}^{-1}$).

Definition 2. We say that an integral operator $W \in \zeta_{<1} := \left\{ L_{<1} := \sum\limits_{i=-\infty}^{0} u_i \mathcal{D}^i \right\}$ is $\mathcal{D}$-unital if $W^{-1} = \mathcal{D}^{-1} W^* \mathcal{D}$.

Lemma 1 ([1]). *Let L be a $\mathcal{D}$-Hermitian ($\mathcal{D}$-skew-Hermitian) operator and W is a $\mathcal{D}$-unital operator. Then $\hat{L} := WLW^{-1}$ is a $\mathcal{D}$-Hermitian ($\mathcal{D}$-skew-Hermitian) operator.*

Let $\varphi = \varphi(x, y, t_m)$ be a matrix $N \times K$ function, $\Omega := C + \int_{-\infty}^{x} \varphi^* \varphi_x dx$ be a non-degenerate $K \times K$ function such that the improper integral $\int_{-\infty}^{x} \varphi^* \varphi_x dx$ converges absolutely, C be a constant complex $K \times K$-matrix.

Theorem 1 ([1]). 1. *Let $C^* = -C = \text{const} \in \text{Mat}_{K\times K}(\mathbb{C})$ and $w_0 := I_N - \varphi \Omega^{-1} \varphi^*$ (where I_N is a unitary $(N \times N)$-matrix. Then $w_0^{-1} = w_0^* = I_N - \varphi \Omega^{*-1} \varphi^*$, where $\Omega^* = \varphi^* \varphi - \Omega$.*

2. *Operator $W := w_0 + \varphi \Omega^{-1} \mathcal{D}^{-1} \varphi_x^*$ is a $\mathcal{D}$-unital and the inverse operator is defined by the formula $W^{-1} := w_0^{-1} + \varphi \mathcal{D}^{-1} (\Omega^{*-1} \varphi^*)_x$.*

Lemma 2 ([3]). *The following property holds true:*

$$\det w_0 = (-1)^K \frac{\det \Omega^*}{\det \Omega}.$$

Remark 1. Let $\varphi \in \text{Mat}_{N\times K}(\mathbb{R})$ and $C^\top = -C$. Then $\det w_0 = \det(I_N - \varphi \Omega^{-1} \varphi^\top) = (-1)^K$.

Let us consider the modified Korteweg–de Vries equation (mKdV)

$$u_t = u_{xxx} + a u^2 u_x,$$

where $u = u(x, t) \in C^{(\infty)}(\mathbb{R}^2, \mathbb{R})$.

$$u_t = \mathcal{L}(\nabla H_1) = \mathcal{M}(\nabla H_2),$$

where $\mathcal{L} = -\mathcal{D}$, $\mathcal{M} = \mathcal{D}^3 + \frac{2}{3} a u \mathcal{D} u \mathcal{D}^{-1} u \mathcal{D}$ is a Hamiltonian pair and the functionals

$$H_1 = \int_{\mathbb{R}} \left(\frac{1}{2} u_x^2 - \frac{1}{12} a u^4 \right) dx, \qquad H_2 = \int_{\mathbb{R}} \frac{1}{2} u^2 dx$$

are first integrals of the mKdV equation.

The generation operators $\Lambda_{\text{mKdV}} = \mathcal{L}^{-1} \mathcal{M}$ and $\Lambda^\tau_{\text{mKdV}} = \mathcal{M} \mathcal{L}^{-1}$ have the form

$$\Lambda = -\mathcal{D}^2 - \frac{2}{3} a u \mathcal{D}^{-1} u \mathcal{D}, \qquad \Lambda^\tau = -\mathcal{D}^2 - \frac{2}{3} a \mathcal{D} u \mathcal{D}^{-1} u \tag{1}$$

and satisfy the equation of Lax type $\Lambda_{t_m} = [\Lambda, K'^\tau]$, where $K'^\tau = -\mathcal{D}^3 - a u \mathcal{D}$.

Let us consider also the KdV equation

$$u_t = u_{xxx} + a u u_x = K[u]. \tag{2}$$

Similarly to the previous equation, the operators will have the form

$$\begin{aligned} &\Lambda_{\text{KdV}} = \mathcal{D}^2 + \frac{2}{3} a u - \frac{1}{3} a \mathcal{D}^{-1} u_x, \qquad \Lambda^\tau_{\text{KdV}} = \mathcal{D}^2 + \frac{2}{3} a u + \frac{1}{3} a u_x \mathcal{D}^{-1}, \\ &K' = \mathcal{D}^3 + a u \mathcal{D} + a u_x, \qquad K'^\tau = -\mathcal{D}^3 - a u \mathcal{D}. \end{aligned} \tag{3}$$

The operators Λ, K'^τ are $\mathcal{D}$-Hermitian and $\mathcal{D}$-skew-Hermitian respectively. These operators are partial cases of the $\mathcal{D}$-Hermitian constrained modified Kadomtsev–Petviashvili ($\mathcal{D}$HcmKP) hierarchy introduced in the paper [1] and determined in the following form: let

$$\zeta \ni L_{\mathcal{D}\text{HcmKP}} := L_n = \mathcal{D}^n + u_{n-1} \mathcal{D}^{n-1} + \cdots + u_1 \mathcal{D} - \boldsymbol{q} \mathcal{M} \mathcal{D}^{-1} \boldsymbol{q}^* \mathcal{D}, \tag{4}$$

$\mathcal{M}^* = (-1)^n\mathcal{M}$ is a complex constant $(l \times l)$-matrix, $\boldsymbol{q}(x, t_m) = (q_1, \dots, q_l)$, $k, l \in \mathbb{N}$, and an additional reduction for operator L_n: $L_n^* = \mu\mathcal{D}L_n\mathcal{D}^{-1}$, $\mu = \pm 1$.

We consider the evolution equations

$$\alpha_m L_{n_{t_m}} = [B_m, L_n], \tag{5}$$

where $L_n := L_{\mathcal{D}\mathrm{HcmKP}}$, and B_m are fractional powers m/n of L_n; $n, m \in \mathbb{N}$.

Let $n = 2$. For $L_2 = \mathcal{D}^2 + iu\mathcal{D} - \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*\mathcal{D}$ we obtain that $B_2 = (L_2)_{>0} = \mathcal{D}^2 + iu\mathcal{D}$, $B_3 = \mathcal{D}^3 + \frac{3}{2}iu\mathcal{D}^2 - \left(\frac{3}{8}u^2 - \frac{3i}{4}u_x + \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*\right)\mathcal{D}$, $\mathcal{M}^* = \mathcal{M}$. For $\alpha_2 = i$, $\alpha_3 = 1$ the following systems of equations are consequences of (5)

$$i\boldsymbol{q}_{t_2} = \boldsymbol{q}_{xx} + iu\boldsymbol{q}_x, \qquad u_{t_2} = 2(\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*)_x \tag{6}$$

and

$$\begin{aligned}
\boldsymbol{q}_{t_3} &= \boldsymbol{q}_{xxx} + \frac{3}{2}iu\boldsymbol{q}_{xx} - \left(\frac{3}{8}u^2 + \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^* - \frac{3}{4}iu_x\right)\boldsymbol{q}_x,\\
u_{t_3} &= \frac{1}{4}u_{xxx} + \frac{3}{8}u^2u_x - \frac{3}{2}(\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*u)_x + \frac{3}{2}i(\boldsymbol{q}_x\mathcal{M}\boldsymbol{q}^* - \boldsymbol{q}\mathcal{M}\boldsymbol{q}_x^*)_x.
\end{aligned} \tag{7}$$

System (6) is a multi-component modification of the integrable Yajima–Oikawa model [4], which describes the interaction of the Laengmur wave packets in the physics of plasma. System (2) is a modification of a higher Yajima–Oikawa model. Note that equation (2) admits some interesting reductions on invariant sub-manifolds, where evolution is introduced by known dynamical systems. So, for $\boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*\mathcal{D} \equiv 0$ we have the scalar mKdV equation $u_{t_3} = \frac{1}{4}u_{xxx} + \frac{3}{8}u^2u_x$. For $u \equiv 0$ the complex multi-component mKdV equation obtained

$$\boldsymbol{q}_{t_3} = \boldsymbol{q}_{xxx} - \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*\boldsymbol{q}_x \tag{8}$$

with the differential condition

$$(\boldsymbol{q}\mathcal{M}\boldsymbol{q}_x^* - \boldsymbol{q}_x\mathcal{M}\boldsymbol{q}^*)_x = 0, \tag{9}$$

and for $\bar{\boldsymbol{q}} \equiv \boldsymbol{q}$, $\mathcal{M} \in \mathrm{Mat}_{l\times l}(\mathbb{R})$ its real version is

$$\boldsymbol{q}_{t_3} = \boldsymbol{q}_{xxx} - \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^\top\boldsymbol{q}_x. \tag{10}$$

In this case, the differential constraint is satisfied.

Remark 2. The vector generalization of the complex mKdV equation can be obtained from system (8)–(9), if we satisfy condition (9) in such a way: let us $\boldsymbol{q} = (\vec{\boldsymbol{q}}, \bar{\vec{\boldsymbol{q}}})$, $\vec{\boldsymbol{q}} = (q_1, \dots, q_k)$ be a k-component vector-function ($l = 2k$) and the matrix $\mathcal{M}$ have a block form

$$\mathcal{M} = \begin{pmatrix} B & A \\ \bar{A} & \bar{B} \end{pmatrix}, \qquad A^\top = A, \qquad B^* = B. \tag{11}$$

Condition (9) is satisfied similarly.

3 Integration of some bi-Hamiltonian systems

Theorem 2. *Let:* 1) $\varphi(x, t_3)$ *be a real* K*-component solution of the system of equations*

$$\varphi_{t_3} = \varphi_{xxx}, \qquad \varphi_{xx} = \varphi\Lambda. \tag{12}$$

2) $C^\top = -C$ *is a real skew-symmetric matrix.*

3) *Matrix function* $\Omega := C + \int_{-\infty}^{x}\varphi^\top\varphi_x dx$ *is non-degenerate on the left semi-axis.*

Then the function $\boldsymbol{q}(x, t_3) := \varphi\Omega^{-1}$ *satisfies mKdV equation* (10)*, where* $\mathcal{M} = C\Lambda - \Lambda^\top C$.

Proof. Let $L_0 := \mathcal{D}^2$, then using Theorem 1

$$L := W\mathcal{D}^2W^{-1} = \mathcal{D}^2 - 2w_{0x}w_0^{-1}\mathcal{D} - \left[\varphi_{xx} - \varphi\Omega^{-1}\int_{-\infty}^{x}\varphi^*\varphi_{xxx}dx\right]\mathcal{D}^{-1}\Omega^{*-1}\varphi^*\mathcal{D}$$
$$- \varphi\Omega^{-1}\mathcal{D}^{-1}\left[\varphi^*_{xx} - \int_{-\infty}^{x}\varphi^*_{xxx}\varphi dx\Omega^{*-1}\varphi^*\right]\mathcal{D}.$$

Under conditions a)–b) we obtain that

$$L = \mathcal{D}^2 - 2w_{0x}w_0^{-1}\mathcal{D} - \varphi\Omega^{-1}(C\Lambda - \Lambda^*C)\mathcal{D}^{-1}\Omega^{*-1}\varphi^*\mathcal{D}.$$

Let $M_0 := \partial_{t_3} - \mathcal{D}^3$, similarly we proceed to the operator

$$M := WM_0W^{-1} = \partial_{t_3} - \mathcal{D}^3 + 3w_{0x}w_0^{-1}\mathcal{D}^2$$
$$- \left[w_0(w_0^{-1})_{xx} - 2w_0\varphi_{xx}\Omega^{*-1}\varphi^* - w_0\varphi_x(\Omega^{*-1}\varphi^*)_x\right.$$
$$\left.+ \varphi\Omega^{-1}\varphi^*_x(w_0^{-1})_x - \varphi\Omega^{-1}\varphi^*_{xx}w_0^{-1} - \varphi\Omega^{-1}\varphi^*_x\varphi_x\Omega^{*-1}\varphi^*\right]\mathcal{D}$$
$$- \left[\varphi_{t_3} - \varphi_{xxx} - \varphi\Omega^{-1}\int_{-\infty}^{x}\varphi^*(\varphi_{t_3} - \varphi_{xxx})_x dx\right]\mathcal{D}^{-1}\Omega^{*-1}\varphi^*\mathcal{D}$$
$$+ \varphi\Omega^{-1}\mathcal{D}^{-1}\left[\varphi^*_{t_3} - \varphi^*_{xxx} - \int_{-\infty}^{x}(\varphi^*_{t_3} - \varphi^*_{xxx})_x\varphi dx\Omega^{*-1}\varphi^*\right]\mathcal{D}.$$

Under condition a) the integral parts are equal to zero. With Remark 1 the formulas for the "dressed" operators L and M are:

$$L = \mathcal{D}^2 - \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^\top\mathcal{D}, \qquad M := \partial_{t_3} - \mathcal{D}^3 + \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^\top\mathcal{D}. \tag{13}$$

For $l = 1$ (that is $\boldsymbol{q} = q$ is a scalar function) the operators L and M (13) constitute the recursive Lax pair for the mKdV equation (1). ■

Let us consider Lax operators (13) for the K-component real mKdV equation (10),

$$[L, M] = 0 \;\Leftrightarrow\; \boldsymbol{q}_{t_3} = \boldsymbol{q}_{xxx} - \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^\top\boldsymbol{q}_x = K[\boldsymbol{q}]. \tag{14}$$

The operator L is a K-generalization [5] of the generating operator Λ_{mKdV} (1) for the scalar mKdV equation and M^τ is its linearization ($M^\tau = \partial_{t_3} - K'[\boldsymbol{q}]$).

Let $K = 2k$, $\boldsymbol{q} = (\vec{\alpha}, \vec{\boldsymbol{q}})$, $\vec{\alpha} \in \mathbb{R}^k$, $\vec{\boldsymbol{q}} = \vec{\boldsymbol{q}}(x, t_3)$,

$$\mathcal{M} = \begin{pmatrix} 0 & -\Lambda^\top \\ -\Lambda & 0 \end{pmatrix}, \tag{15}$$

where Λ is a constant real $(k \times k)$-matrix. In this case, operators (13) reduce to the form

$$L = \mathcal{D}^2 - (\vec{\alpha}, \vec{\boldsymbol{q}})\mathcal{M}(\vec{\alpha}, \vec{\boldsymbol{q}})^\top + (\vec{\alpha}, \vec{\boldsymbol{q}})\mathcal{M}\mathcal{D}^{-1}(\vec{\alpha}, \vec{\boldsymbol{q}})^\top_x$$
$$= \mathcal{D}^2 + 2\vec{\alpha}\Lambda^\top\vec{\boldsymbol{q}}^\top - \mathcal{D}^{-1}(\vec{\alpha}\Lambda^\top\vec{\boldsymbol{q}}^\top)_x = \mathcal{D}^2 + 2u - \mathcal{D}^{-1}u_x =: \Lambda_{KdV}, \tag{16}$$
$$M = \partial_{t_3} - \mathcal{D}^3 - 3u\mathcal{D}, \tag{17}$$

where $u = \vec{\alpha}\Lambda^\top\vec{\boldsymbol{q}}^\top$, and constitute a recursive Lax pair for KdV equation (see (2)–(3) for $a = 3$)

$$[\partial_{t_3} + K'^\tau[u], \Lambda_{KdV}] = 0 \;\Leftrightarrow\; \frac{\partial}{\partial t_3}\Lambda_{KdV} = [\Lambda_{KdV}, K'^\tau]$$
$$\Leftrightarrow u_{t_3} = K[u] = u_{xxx} + 3uu_x. \tag{18}$$

Described process of reduction of the real version of the vector mKdV equation (14) to scalar KdV equation (18) allows to formulate the following statement.

Theorem 3. *Let:* 1) $\varphi(x,t_3)=(\vec{\varphi},\vec{\alpha})$, *where* $\vec{\varphi}=\vec{\varphi}(x,t_3)$ *is a* k*-component real field,* $\mathbb{R}^k\ni\vec{\alpha}$ *is a* k*-component real vector.*

2) $\vec{\varphi}$ *is a solution of the linear system*

$$\vec{\varphi}_{t_3}=\vec{\varphi}_{xxx},\qquad \vec{\varphi}_{xx}=\vec{\varphi}\Lambda,\qquad \Lambda\in \mathrm{Mat}_{k\times k}(\mathbb{R}).$$

3) $\Omega:=C+\int_{-\infty}^{x}\varphi^{\top}\varphi_x dx$ *is a non-degenerate on the left semi-axis* $(2k)\times(2k)$*-matrix function, where* $C=\begin{pmatrix}0 & I_k\\ -I_k & 0\end{pmatrix}$.

Then the function

$$u(x,t_3):=\vec{\alpha}\Lambda^{\top}(\vec{\varphi}^{\top}\vec{\alpha}-I_k)^{-1}\left(\vec{\varphi}^{\top}+\int_{-\infty}^{x}\vec{\varphi}_x^{\top}\vec{\varphi}\,dx\vec{\alpha}^{\top}\right)$$

is a solution of KdV equation (18).

Proof. We consider

$$\Omega:=C+\int_{-\infty}^{x}\varphi^{\top}\varphi_x\,dx=\begin{pmatrix}\displaystyle\int_{-\infty}^{x}\vec{\varphi}^{\top}\vec{\varphi}_x\,dx & I_k\\ \vec{\alpha}^{\top}\vec{\varphi}-I_k & 0\end{pmatrix}.$$

In order to prove the Theorem we use the known formula for the matrix A^{-1} that is inverse the matrix for the block $(2k)\times(2k)$-matrix A

$$A=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\ \to\ A^{-1}=\begin{pmatrix}A_{11}^{-1}(I+A_{12}T^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}T^{-1}\\ -T^{-1}A_{21}A_{11}^{-1} & T^{-1}\end{pmatrix},$$

where $T=A_{22}-A_{21}A_{11}^{-1}A_{12}$. In our case $A_{12}=I_k$, $A_{22}=0:=0_k$, $T=-A_{21}A_{11}^{-1}\Rightarrow T^{-1}=-A_{11}A_{21}^{-1}$, whence we derive the simple formula for the matrix Ω^{-1}

$$\Omega^{-1}=\begin{pmatrix}0 & A_{21}^{-1}\\ I_k & -A_{11}A_{21}^{-1}\end{pmatrix}=\begin{pmatrix}0 & (\vec{\alpha}^{\top}\vec{\varphi}-I_k)^{-1}\\ I_k & -\displaystyle\int_{-\infty}^{x}\vec{\varphi}^{\top}\vec{\varphi}_x\,dx\,(\vec{\alpha}^{\top}\vec{\varphi}-I_k)^{-1}\end{pmatrix}.$$

Thus $\boldsymbol{q}:=\varphi\Omega^{-1}=(\vec{\alpha},\vec{\boldsymbol{q}})$, where

$$\vec{\boldsymbol{q}}=\left(\vec{\varphi}+\vec{\alpha}\int_{-\infty}^{x}\vec{\varphi}^{\top}\vec{\varphi}_x\,dx\right)(\vec{\alpha}^{\top}\vec{\varphi}-I_k)^{-1}. \tag{19}$$

The matrix $\mathcal{M}=C\hat{\Lambda}-\hat{\Lambda}^{\top}C$ is of the form $\mathcal{M}=\begin{pmatrix}0 & -\Lambda^{\top}\\ -\Lambda & 0\end{pmatrix}$, as $\hat{\Lambda}=\begin{pmatrix}\Lambda & 0\\ 0 & 0\end{pmatrix}$. For the completion of the proof it is sufficient to refer to formulas (15)–(19) and to the corresponding results of Theorem 3 for the mKdV equation (14). ■

The possibility of application of other reductions in $\mathcal{D}$HcmKP hierarchy requires further research.

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New Classes of Nonlinear Evolutionary Equations

Lev M. BERKOVICH

Samara State University, 1 Academician Pavlov Str., 443011 Samara, Russia
E-mail: *berk@ssu.samara.ru*

We consider new classes of nonlinear evolutionary equations (NEEs) of the n-th order depending on two arbitrary functions such that the solutions to these equations of traveling wave type satisfy the corresponding nonlinear superposition principles (NSPs). The construction is based on the methods of the factorization and exact linearization of ordinary differential equations (ODEs)

1 Introduction

In the papers [1] and [2] some higher analogs of the nonlinear evolutionary equation of Korteweg–de Vries are considered. Some NEEs of the higher order are represented in handbook [3]. In the present paper we have constructed new types of NEEs of higher order. For their construction the methods of factorization and exact linearization were used (see [4–8]).

In Section 2 the summary of results on the method of a exact linearization is given. In Section 3 the NEEs of n-th order depending on one arbitrary function and reducible to linear evolutionary equations (LEEs) by a nonlinear substitution of dependent variable are constructed. In Section 4 the NEEs of n-th order depending on two arbitrary functions are constructed. Their solutions of traveling wave type satisfying to nonlinear ODEs, linearized by transformation dependent and independent variables. In Section 5 the NEEs of n-th order depending on two arbitrary functions also are are constructed. Their stationary solutions satisfy to the nonlinear ODEs reduced to the semilinear equations.

2 On the method of exact linearization for ordinary differential equations

Lemma 1. *An autonomous second-order ODE*

$$F(y, y', y'') = 0, \qquad ()' = \frac{d}{dx} \tag{1}$$

can be reduced to the linear form

$$z''(s) + 2b_1 z'(s) + b_2 z(s) = 0, \qquad b_1, b_2 = \text{const}, \tag{2}$$

by a nonlinear transformation of the dependent and independent variables

$$y = v(y)z, \qquad ds = ydx \tag{3}$$

if and only if (1) *can be factored into noncommutative nonlinear differential operators as*

$$\left(D - \left(\frac{v^*}{v} + \frac{u^*}{u}\right)y' - r_2 u\right)\left(D - \frac{v^*}{v}y' - r_1 u\right)y = 0, \qquad D = \frac{d}{dx}, \qquad ()^* = \frac{d}{dy} \tag{4}$$

or into commutative operators as

$$\left(\frac{1}{u}D - \frac{v^*}{vu}y' - r_2\right)\left(\frac{1}{u}D - \frac{v^*}{vu}y' - r_1\right)y = 0,$$

where r_1 and r_2 are the roots of the characteristic equation

$$r^2 + 2b_1 r + b_2 = 0. \tag{5}$$

Lemma 2. *Equation* (1) *can be linearized by transformation* (3)*, if and only if it can be represented in the form*

$$y'' + fy'^2 + 2b_1\varphi y' + b_2\varphi \exp\left(-\int f dy\right)\int \varphi \exp\left(\int f dy\right) dy = 0, \tag{6}$$

where $f = f(y)$, $\varphi = \varphi(y)$, that reduces to (2) by the transformation

$$z = \beta \int \varphi \exp\left(\int f dy\right) dy, \qquad ds = \varphi(y)dx, \tag{7}$$

where $\beta = \text{const} \neq 0$ is a normalizing factor.

Lemma 3. *An autonomous n-th order ODE*

$$F\left(y, y', \ldots, y^{(n)}\right) = 0 \tag{8}$$

can be reduced to a linear autonomous form

$$M_n(z) \equiv z^{(n)}(s) + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(s) = 0, \qquad b_k = \text{const}$$

by transformation (3) *if and only if* (8) *can be represented in the form*

$$\prod_{k=n}^{1} \left[D - \left(\frac{1}{y} - \left(\ln \int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy\right)^* + (k-1)\frac{\varphi^*}{\varphi}\right) y' - r_k\varphi\right] y = 0, \tag{9}$$

where r_k are the roots of the characteristic equation

$$M_n(r) \equiv r^n + \sum_{k=1}^{n} \binom{n}{k} b_k r^{n-k} = 0; \tag{10}$$

the linearizing transformation (2) then has the form

$$z = \beta \int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy, \qquad ds = \varphi dx, \tag{11}$$

where $\beta = \text{const} \neq 0$ is a normalizing factor.

Note 1. The structure of linearizable equations. Linearizable equations depend on two arbitrary functions and n parameters serving as coefficients of the linear equations. They are algebraic with respect to the derivatives of the dependent variable which they include. The higher-order equations are constructed on the basis of recursive relations. The order of the nonlinear term is determined as the sum of the products of the orders of derivatives by their exponents. Each equation belonging to the class under examination can be represented as an algebraic sum of terms (with coefficients expressed through the dependent variable) each of which consists of nonlinear terms of the same order. The order of the term not depending on the coefficients of the transformed linear equation equals the order of the equation. All the other terms have smaller orders and contain coefficients (parameters) of the linear equation. Then, a linearizable equation can be represented in the form

$$\sum_{k_1+2k_2+\cdots+nk_n=n} \Psi^{12\ldots n}_{k_1 k_2 \ldots k_n} y^{(1)k_1} y^{(2)k_2} \ldots y^{(n)k_n}$$

$$+\sum_{m=1}^{n-1}\binom{n}{m}b_m\varphi^m\left(\sum_{l_1+2l_2+\cdots+(n-m)l_{n-m}=n-m}\Psi^{12\ldots n-m}_{l_1l_2\ldots l_{n-m}}y^{(1)l_1}y^{(2)l_2}\ldots y^{(n-m)l_{n-m}}\right)$$
$$+b_n\exp\left(-\int f dy\right)\int\varphi^{\frac{n^2+n-2}{2n}}\exp\left(\int f dy\right)dy=0, \tag{12}$$

where the coefficients Ψ depend on $f(y)$ and $\varphi(y)$ and

$$\Psi^{12\ldots n}_{00\ldots1}=1,\qquad \Psi^{12\ldots n-m}_{00\ldots1}=1,\qquad \Psi^{12\ldots n-1n}_{10\ldots10}=nf(y).$$

3 Nonlinear evolutionary equations reducible to linear evolutionary equations

Proposition 1. *An NEE*

$$\frac{\partial y}{\partial t}=F\left(y,\frac{\partial y}{\partial x},\ldots,\frac{\partial^n y}{\partial x^n}\right),\qquad y=y(t,x) \tag{13}$$

reduces to a linear evolutionary equation (LEE)

$$\frac{\partial z}{\partial t}=\sum_{k=0}^{n}\binom{n}{k}b_k\frac{\partial^{n-k}z}{\partial x^{n-k}}, \tag{14}$$

by substitution of the form (see (3)*)*

$$y=v(y)z, \tag{15}$$

if and only if (13) *can be factored as*

$$\left(1-\frac{v^*}{v}y\right)\frac{\partial y}{\partial t}=\prod_{k=1}^{n}\left(\frac{\partial}{\partial x}-\frac{v^*}{v}\frac{\partial y}{\partial x}-r_k\right)y, \tag{16}$$

where r_k are the roots of characteristic equation (10)*.*

Theorem 1. *NEE* (13) *reduces to LEE* (14) *by substitution* (15) *if and only if it can be represented in the form*

$$\frac{\exp\left(\int f dy\right)y}{\int\exp(\int f dy)dy}\frac{\partial y}{\partial t}=\prod_{k=1}^{n}\left[\frac{\partial}{\partial x}-\left(\frac{1}{y}-\frac{\exp\left(\int f dy\right)}{\int\exp(\int f dy)dy}\right)\frac{\partial y}{\partial x}-r_k\right]y, \tag{17}$$

linearizing substitution (15) *then has an explicit form*

$$z=\beta\int\exp\left(\int f dy\right)dy,\qquad \beta=\text{const}\neq0. \tag{18}$$

Proof. We apply Lemma 3, equation (16), and the formula

$$1-\frac{v^*}{v}y=\frac{\exp\left(\int f dy\right)y}{\int\exp\left(\int f dy\right)dy}.$$ ■

Equation (13) in form (16) satisfies the nonlinear superposition principle (NSP)

$$z=\int\exp\left(\int f dy\right)dy=\sum_{k=1}^{n}c_kz_k(t,x),$$

where c_k are arbitrary constants and $z_k(t,x)$ are linearly independent partial solutions to LEE (14).

Example 1. The equation[1]

$$y_t = \frac{h}{2}y_{xx} - \frac{1}{2}y_x^2, \qquad h = \text{const} \tag{19}$$

belongs to the class (17) and it is reduced to the LEE

$$z_t = -hz_{xx} \tag{20}$$

by the transformation $z = \exp(-y/h)$. Let $z_1(t,x)$ and $z_2(t,x)$ be its linearly independent solutions. Then the complete integral $z = c_1z_1 + c_2z_2$ of LEE (20), where c_1 and c_2 are arbitrary constants, is a linear superposition principle, and the formula

$$y = -h\ln\left[\exp\left(-\frac{1}{h}y_1 - \frac{\lambda_1}{h}\right) + \exp\left(-\frac{1}{h}y_2 - \frac{\lambda_2}{h}\right)\right]$$

is a NSP for (19), where λ_1 and λ_2 are arbitrary constants, $y_1(x,t)$ and $y_2(x,t)$ are particular solutions, and $y(x,t)$ is a complete integral of equation (19).

Note that equation (19) can be linearized into the equation $z_t = z_{ss}$ by the transformation $z = \exp(-y/h)$, $ds = \sqrt{2/h}dx$.

Note 2. In idempotent analysis, the correspondence principle (in the sense of Maslov) is an NSP (see for example [9]).

Example 2. The equation

$$\begin{aligned} y_t = {} & y^{\text{iv}} + 4fy'y''' + 3fy''^2 + 6(f^2 + f^*)y'^2y'' + (f^3 + 3ff^* + f^{**})y'^4 \\ & + 4b_1(y''' + 3fy'y'' + (f^2 + f^*)y'^3) \\ & + 6b_2(y'' + fy'^2) + 4b_3y' + b_4\exp\left(-\int f dy\right)\int \exp\left(\int f dy\right)dy = 0 \end{aligned}$$

can be reduced by substitution (18) to the corresponding fourth-order LEE.

A nonlinear ODE linearizable by transformation (18) was constructed in [10].

4 Nonlinear evolutionary equations with linearizable right-hand sides

Proposition 2. *NEE* (13) *has a solution of the traveling wave type*

$$y(\tau) = y(x - at) \tag{21}$$

and can be reduced to a linear ODE by the substitution

$$y = v(y)z, \qquad ds = u(y)d\tau, \tag{22}$$

if and only if

$$u^{n-1}\left(1 - \frac{v^*}{v}y\right)\frac{\partial y}{\partial t} = \prod_{k=n}^{1}\left[\frac{\partial}{\partial x} - \left(\frac{v^*}{v} + (k-1)\frac{u^*}{u}\right)\frac{\partial u}{\partial x} - r_ku\right]y, \tag{23}$$

where r_k are the roots of characteristic equation (10).

[1]This example was adduced in works of V.P. Maslov and his coauthors.

Proof. Sufficiency. First, note that the right-hand side of equation (23) generalizes formula (4). We seek a traveling wave type solution (21) for (23):

$$-au^{n-1}\left(1-\frac{v^*}{v}y\right)y_\tau=\prod_{k=n}^{1}\left[D_\tau-\left(\frac{v^*}{v}+(k-1)\frac{u^*}{u}\right)y_\tau-r_ku\right]y,\qquad D_\tau=\frac{d}{d\tau}.\tag{24}$$

Let us rewrite the left-hand side of (24) in the form

$$-au^{n-1}\left(D_\tau-\frac{v^*}{v}y_\tau\right)y$$

and apply substitution (22). We obtain, successively,

$$-au^{n-1}vD_\tau z=v\prod_{k=n}^{1}\left[D_\tau-(k-1)\frac{u^*}{u}y_\tau-r_ku\right]z,$$

$$-au^{n-1}vuD_sz=vu^n\prod_{k=n}^{1}(D_s-r_k)z,\qquad D_s=uD_\tau,\tag{25}$$

$$\sum_{k=0}^{n}\binom{n}{k}b_kz^{(n-k)}(s)+az'(s)=0,\qquad b_0=1,\qquad ()'=\frac{d}{ds}.\tag{26}$$

The **necessity** of conditions (23) is proved on the basis of equation (25). ■

Theorem 2. *NEE* (13) *with condition* (23) *has a traveling wave type solution* (21) *if and only if* (23) *can be represented in the form*

$$\frac{\varphi^{\frac{3n^2-3n+2}{2n}}\exp\left(\int f dy\right)y}{\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy}\frac{\partial y}{\partial t}$$
$$=\prod_{k=n}^{1}\left[\frac{\partial}{\partial x}-\left(\frac{1}{y}-\frac{\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)}{\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy}+(k-1)\frac{\varphi^*}{\varphi}\right)\frac{\partial y}{\partial x}-r_k\varphi\right]y,\tag{27}$$

where r_k *with* $k=1,2,\dots,n$ *are the roots of characteristic equation* (10)*, or in lexicographic form*

$$\varphi^{n-1}y_t=\sum_{k_1+2k_2+\cdots+nk_n=n}\Psi^{12\dots n}_{k_1k_2\dots k_n}y^{(1)k_1}y^{(2)k_2}\dots y^{(n)k_n}$$
$$+\sum_{m=1}^{n-1}\binom{n}{m}b_m\varphi^m\left(\sum_{l_1+2l_2+\cdots+(n-m)l_{n-m}=n-m}\Psi^{12\dots n-m}_{l_1l_2\dots l_{n-m}}y^{(1)l_1}y^{(2)l_2}\dots y^{(n-m)l_{n-m}}\right)$$
$$+b_n\exp\left(-\int f dy\right)\int\varphi^{\frac{n^2+n-2}{2n}}\exp\left(\int f dy\right)dy=0,\tag{28}$$

where the coefficients Ψ *depend on* $f(y)$ *and* $\varphi(y)$ *and*

$$\Psi^{12\dots n}_{00\dots1}=1,\qquad \Psi^{12\dots n-m}_{00\dots1}=1,\qquad \Psi^{12\dots n-1n}_{10\dots10}=nf(y).$$

Then substitution (21) *which gives linear ODE* (25) *has the form*

$$z=\beta\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy,\qquad ds=\varphi d\tau.$$

Proof. We apply Lemma 3, Proposition 2, formulas (9), (11), (12), (14), and the formula

$$1-\frac{v^*}{v}y=\frac{\varphi^{\frac{n^2-3n+2}{2n}}\exp\left(\int f dy\right)y}{\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy}.$$

■

Theorem 3 (see [5]). *Let $z_1(s), z_2(s), \dots, z_n(s)$ be linearly independent particular solutions to equation* (25)*. Then the general integral of* (25) *is*

$$z=c_1z_1(s)+c_2z_2(s)+\cdots+c_nz_n(s),$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants, and this formula is an LSP. The NSP for (27), (28) *is obtained by the formulas*

$$z=\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)=\sum_{k=1}^{n}c_kz_k(s),\qquad ds=\varphi d\tau,\qquad \tau=x-at.$$

The general form of a second-order NEE belonging to class (28) *(see also* (5) *and* (6)*) is*

$$\varphi y_t=y_{xx}+fy_x^2+2b_1\varphi y_x+b_2\varphi\exp\left(-\int f dy\right)\int\varphi\exp\left(\int f dy\right)dy.$$

Next, we have

$$-a\varphi y_\tau=y_{\tau\tau}+fy_\tau^2+2b_1\varphi y_\tau+b_2\varphi\exp\left(-\int f dy\right)\int\varphi\exp\left(\int f dy\right)dy.$$

The substitution (see formula (7)) $z=\beta\int\varphi\exp\left(\int f dy\right)dy$, $ds=\varphi d\tau$ yields a linear equation

$$z''(s)+2b_1z'(s)+b_2z(s)+az'(s)=0.$$

Example 3. Consider the equation

$$yy_t=y_{xx}+3yy_x+y^3. \tag{29}$$

Seeking a solution of type (21) for equation (29) leads to the ODE $-ayy_\tau=y_{\tau\tau}+3yy_\tau+y^3$. Using the substitution $y^2=z$, $ds=yd\tau$, we obtain a linear equation

$$z''(s)+3z'(s)+2z(s)+az'(s)=0.$$

The general form of an NEE of order $n=3$ belonging to a class (28), is

$$\begin{aligned}\varphi^2y_t=y_{xxx}+3fy_xy_{xx}+\left(\frac{1}{3}\frac{\varphi_{yy}}{\varphi}-\frac{5}{9}\frac{\varphi_y^2}{\varphi^2}-\frac{1}{3}f\frac{\varphi_y}{\varphi}+f^2+f_y\right)y_x^3\\+3b_1\varphi\left[y_{xx}+\left(f+\frac{1}{3}\frac{\varphi_y}{\varphi}\right)y_x^2\right]+3b_2\varphi^2y_x\\+b_3\varphi^{5/3}\exp\left(-\int f dy\right)\int\varphi^{4/3}\exp\left(\int f dy\right)dy.\end{aligned} \tag{30}$$

Example 4. Consider the Harry–Dym equation

$$y_t=y^3y_{xxx}. \tag{31}$$

It belongs to class (30), namely

$$\varphi^2 y_t = y_{xxx} + \left(\frac{1}{3}\frac{\varphi_{yy}}{\varphi} - \frac{5}{9}\frac{\varphi_y^2}{\varphi^2}\right) y_x^3 + b_3\varphi^{3/5}\int \varphi^{4/5} dy$$

with $f = 0$, $b_3 = 0$, $\varphi = y^{-3/2}$. It admits the representation

$$y^{-3} y_t = \left(\partial_x + \frac{y_x}{y}\right)\left(\partial_x - \frac{1}{2}\frac{y_x}{y}\right) y_x.$$

The ordinary differential equation corresponding to a solution of (31) of type (21) has the form

$$y'''(\tau) + a y^{-3} y'(\tau) = 0. \tag{32}$$

Equation (32) is reduced to the linear form $z'''(s) + az'(s) = 0$ by the substitution $z = 1/y$, $ds = y^{-3/2} d\tau$.

5 Nonlinear evolutionary equations with factorable right-hand sides

The new classes of the nonlinear evolutionary equations (NEE) of n-th order, depending on two arbitrary functions and $n-1$ parameters, are constructed (see also [11]):

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial x}\left[\frac{v}{v - v^* y}\prod_{k=n-1}^{1}\left(\frac{\partial}{\partial x} - \left(\frac{v^*}{v} + (k-1)\frac{u^*}{u}\right)\frac{\partial y}{\partial x} - r_k u\right)\right] y, \tag{33}$$

where $r_k = \text{const}$, $k = 1, \ldots, n-1$, $v = v(y)$, $u = u(y)$, $()^* = d/dy$.

Besides solutions such as a traveling wave, stationary solutions of the equation (33) are also of interest. The corresponding ODE have the form

$$\prod_{k=n-1}^{1}\left[\frac{\partial}{\partial x} - \left(\frac{v^*}{v} + (k-1)\frac{u^*}{u}\right)\frac{dy}{dx} - r_k u\right] y = C\left(1 - \frac{v^*}{v} y\right), \qquad D = \frac{d}{dx}. \tag{34}$$

By substitution $y = v(y)z$, $ds = u(y)dx$ the equation (34) is reduced to a semilinear equation

$$\prod_{k=n-1}^{1}(D_s - r_k) z = C\left(1 - \frac{v^*}{v} y\right) v^{-1} u^{1-n}, \tag{35}$$

where right-hand side is a function of z.

Example 5. The Korteweg–de Vries equation (KdV)

$$y_t + y_{xxx} - 6 y y_x = 0$$

belongs to the class (33) and admits a representation of the form

$$y_t + \partial_x(y_{xx} - 3y^2) = 0$$

and also a factorization

$$y_t + \frac{2}{3}\partial_x\left(\partial_x - r_2 y^{1/2}\right)\left(\partial_x + \frac{1}{2}\frac{y_x}{y} - r_1 y^{1/2}\right) y = 0,$$
$$r_{1,2} = \mp\frac{3}{\sqrt{2}}, \qquad u = y^{1/2}, \qquad v = y^{-1/2}.$$

The semilinear equation corresponding to (35) is the equation

$$z''(s) - \frac{9}{2}z = Cz^{-1/3}.$$

Example 6. Consider modified KdV-equation (MKdV)

$$y_t + y_{xxx} - 6y^2y_x = 0.$$

It admits a factorization $y_t + \partial_x(y_{xx} - 2y^3) = 0$, and the following

$$y_t + \frac{1}{2}\partial_x(\partial_x - 2y)\left(\partial_x + \frac{y_x}{y} + 2y\right)y = 0,$$

here $u = y$, $v = y^{-1}$, $r_{1,2} = \pm 2$.

Example 7. Generalized MKdV-equation

$$y_t + y_{xxx} - ay^k y_x = 0$$

admits first a representation

$$y_t + \partial_x\left(y_{xx} + \frac{a}{k+1}y^{k+1}\right) = 0,$$

and then a factorization

$$y_t + \frac{2}{k+2}\partial_x\left(\partial_x - r_2y^{k/2}\right)\left(\partial_x + \frac{k}{2}\frac{y_x}{y} - r_1y^{k/2}\right)y = 0,$$

where $u = y^{k/2}$, $v = y^{-k/2}$, $r_{1,2} = \pm i\sqrt{\frac{a(k+2)}{2(k+1)}}$.

Corresponding semilinear equation has the form

$$\ddot{z} + \frac{a(k+2)}{2(k+1)}z = Cz^{-\frac{k}{k+2}}.$$

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Systematic Construction of Separable Systems with Quadratic in Momenta First Integrals

Maciej BŁASZAK

Institute of Physics, A. Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland
E-mail: *blaszakm@amu.edu.pl*

Liouville integrable separable systems with quadratic in momenta first integrals are considered. Particular attention is paid to the systems generated by so-called special conformal Killing tensors, i.e. Benenti systems. Then infinitely many new classes of separable systems are constructed by appropriate deformations of Benenti class systems.

1 Introduction

The separation of variables for solving by quadratures of the Hamilton–Jacobi (HJ) equations of related Liouville integrable dynamic systems with quadratic in momenta first integrals has a long history as a part of analytical mechanics. There are some milestones of that theory. First, in 1891 Stäckel initiated a program of classification of separable systems presenting conditions for separability of the HJ equations in orthogonal coordinates [1–3]. Then, in 1904 Levi-Civita found a test for the separability of a Hamiltonian dynamics in a given system of canonical coordinates [4]. The next was Eisenhart [5–7], who in 1934 inserted a separability theory in the context of Riemannian geometry, making it coordinate free and introducing the crucial objects of the theory, i.e. Killing tensors. This approach was then developed by Woodhouse [8], Klanins [9, 10] and others. Finally, in 1992, Benenti [11–13] constructed a particular, but very important subclass of separable systems, based on the so called special conformal Killing tensors.

The first constructive theory of separated coordinates for dynamical systems was made by Sklyanin [15]. He adopted the method of soliton theory, i.e. the Lax representation, to systematic derivation of separated coordinates. In that approach involutive functions appear as coefficients of characteristic equation (spectral curve) of Lax matrix. The method was successfully aplied to separation of variables for many integrable systems [15–20].

Recently, a modern geometric theory of separability on bi-Poisson manifolds was constructed [21–26], related to the so-called Gel'fand–Zakharevich (GZ) bi-Hamiltonian systems [27, 28]. Obviously, it contains as a special case Liouville integrable systems with all constants of motion being quadratic in momenta functions. Indeed, Ibort et.al. [29] proved that the Benenti class of systems can be lifted to the GZ bi-Hamiltonian form.

In the following paper we construct in a systematic way all Liouville integrable systems on Riemannian spaces, which are of the GZ type, including as a special case the Benenti class of systems. What is important, infinitely many classes of separable systems are constructed from appropriate deformations of the Benenti class of systems. In that sense we demonstrate the crucial role of this particular class of systems in the separability theory of dynamic systems with quadratic in momenta first integrals.

2 Separable dynamics on Riemannian spaces

Let (Q, g) be a Riemann (pseudo-Riemann) manifold with covariant metric tensor g and local coordinates $q^1, \ldots, q^n$. Moreover, let $G := g^{-1}$ be a contravariant metric tensor satisfying

$\sum\limits_{j=1}^{n} g_{ij}G^{jk} = \delta_i^k$. The equations

$$q_{tt}^i + \Gamma_{jk}^i q_t^j q_t^k = G^{ik}\partial_k V(q), \quad i = 1, \ldots, n, \qquad q_t \equiv \frac{dq}{dt} \tag{1}$$

describe the motion of a particle in the Riemannian space with the metric g, where Γ_{jk}^i are the Levi-Civita connection components. Equations (1) can be obtained by varying the Lagrangian

$$\mathcal{L}(q, q_t) = \frac{1}{2}\sum_{i,j} g_{ij} q_t^i q_t^j - V(q) \tag{2}$$

and are called Euler–Lagrange equations. Obviously, for $G = I$ equations (1) reduce to Newton equations of motion.

One can pass in a standard way to the Hamiltonian description of dynamics, where the Hamiltonian function takes the form

$$H(q,p) = \sum_{i=1}^{n} q_t^i \frac{\partial \mathcal{L}}{\partial q_t^i} - \mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} G^{ij} p_i p_j + V(q), \qquad p_i := \frac{\partial \mathcal{L}}{\partial q_t^i} = \sum_j g_{ij} q_t^j$$

and equations of motion are

$$\begin{pmatrix} q \\ p \end{pmatrix}_t = \theta_0 dH = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = X_H \iff q_t^i = \frac{\partial H}{\partial p_i}, \qquad p_{it} = -\frac{\partial H}{\partial q^i}.$$

X_H denotes the Hamiltonian vector field with respect to a canonical Poisson tensor θ_0 and the whole dynamics takes place on the phase space $M = T^*Q$ in local coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$.

Of special importance is the geodesic motion $V(q) \equiv 0$, with Euler–Lagrange equations and Hamiltonian representation in the form

$$q_{tt}^i + \Gamma_{jk}^i q_t^j q_t^k = 0, \quad i = 1, \ldots, n \iff \begin{pmatrix} q \\ p \end{pmatrix}_t = \theta_0 dE = X_E, \quad E = \frac{1}{2}\sum_{i,j=1}^{n} G^{ij} p_i p_j.$$

Functionally independent Hamiltonian functions H_i, $i = 1, \ldots, n$ are said to be separable in the canonical coordinates (λ, μ) if there are n relations, called the separation conditions (Sklyanin [15]), of the form

$$\varphi_i(\lambda^i, \mu_i; H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n, \qquad \det\left[\frac{\partial \varphi_i}{\partial H_j}\right] \neq 0, \tag{3}$$

which guarantee the solvability of the appropriate Hamilton–Jacobi equations and involutivity of H_i. A special case, when all separation relations (3) are affine in H_i, is given by the set of equations

$$\sum_{k=1}^{n} \phi_i^k(\lambda_i, \mu_i) H_k = \psi_i(\lambda_i, \mu_i), \quad i = 1, \ldots, n, \tag{4}$$

where ϕ and ψ are arbitrary smooth functions of their arguments, is called the *Stäckel separation conditions* and the related dynamic systems are called Stäckel separable.

We are going to present a subclass of one-particle dynamics, containing Liouville integrable and separable systems with n quadratic in momenta constants of motion. Stäckel [1–3] was the first who gave the characterization of equations of motion integrable by separation of variables.

He proved that if in a system of orthogonal coordinates (λ,μ) there exists a non-singular matrix $\varphi=(\varphi_k^l(\lambda_k))$, called a *Stäckel matrix* such that the Hamiltonians H_r are of the form

$$H_r=\frac{1}{2}\sum_{i=1}^{n}(\varphi^{-1})_r^i(\mu_i^2+\sigma_i(\lambda_i)), \tag{5}$$

then H_r are functionally independent, pairwise commute with respect to the canonical Poisson bracket and the Hamilton–Jacobi equation associated to H_1 is separable. Indeed, for quadratic in momenta constants of motion, the Stäckel separation conditions (4) take the general form

$$\sum_{k=1}^{n}\phi_i^k(\lambda^i)H_k=\frac{1}{2}f_i(\lambda^i)\mu_i^2+\gamma_i(\lambda^i),\qquad i=1,\dots,n, \tag{6}$$

where f_i, γ_i, ϕ_i^k are arbitrary smooth functions of its argument and the normalization $\phi_i^n=1$, $i=1,\dots,n$ is assumed. To get the explicit form of $H_k=H_k(\lambda,\mu)$ one has to solve the system of linear equations (6). The results are the following

$$\varphi=\begin{pmatrix}\frac{\phi_1^1(\lambda^1)}{f_1(\lambda^1)} & \frac{\phi_1^2(\lambda^1)}{f_1(\lambda^1)} & \dots & \frac{\phi_1^{n-1}(\lambda^1)}{f_1(\lambda^1)} & \frac{1}{f_1(\lambda^1)}\\ \vdots & \vdots & \dots & \vdots & \vdots\\ \frac{\phi_n^1(\lambda^n)}{f_n(\lambda^n)} & \frac{\phi_n^2(\lambda^n)}{f_n(\lambda^n)} & \dots & \frac{\phi_n^{n-1}(\lambda^n)}{f_n(\lambda^n)} & \frac{1}{f_n(\lambda^n)}\end{pmatrix},\qquad \sigma_i(\lambda_i)=\gamma_i(\lambda^i)/f_i(\lambda^i).$$

Eisenhart considered Stäckel separable systems in the frame of one-particle dynamics on Riemannian (pseudo-Riemannian) space (Q,g). He gave a coordinate-free representation for Stäckel geodesic motion introducing a special family of *Killing tensors* [5–7]. As known, a $(1,1)$-type tensor $K=(K_j^i)$ (or a $(2,0)$-type tensor $KG=A=(A^{ij})$) is called a Killing tensor with respect to g if

$$\Big\{\sum A^{ij}p_ip_j,\sum G^{ij}p_ip_j\Big\}_{\theta_0}=0,$$

where $\{\cdot,\cdot\}_{\theta_0}$ means a canonical Poisson bracket. He proved [5–7] that the geodesic Hamiltonians can be transformed into the Stäckel form (5) if the contravariant metric tensor $G=g^{-1}$ has $(n-1)$ commuting independent contravariant Killing tensors A_r of a second order such that

$$E_r=\frac{1}{2}\sum_{i,j}A_r^{ij}p_ip_j,$$

admitting a common system of closed eigenforms α_i

$$(A_r^*-v_r^iG)\alpha_i=0,\qquad d\alpha_i=0,\qquad i=1,\dots,n,$$

where v_r^i are eigenvalues of $(1,1)$ Killing tensor $K_r=A_rg$ $(K_r^*=gA_r^*)$.

For n degrees of freedom, let us consider n Stäckel Hamiltonian functions in separated coordinates in the following form

$$H_r=\frac{1}{2}\sum_{i=1}^{n}v_r^iG^{ii}\mu_i^2+V_r(\lambda)=\frac{1}{2}\mu^TK_rG\mu+V_r(\lambda),\quad r=1,\dots,n, \tag{7}$$

where $\mu=(\mu_1,\dots,\mu_n)^T$ and $V_r(\lambda)$ are appropriate potentials separable in (λ,μ) coordinates. For the integrable system (7) calculated from (6)

$$G^{ii}=(-1)^{i+1}\frac{f_i(\lambda^i)\det W^{i1}}{\det W},\qquad v_r^i=(-1)^{r+1}\frac{\det W^{ir}}{\det W^{i1}},\qquad V_r=\sum_{i=1}^{n}(-1)^{i+r}\gamma_i(\lambda^i)\frac{\det W^{ir}}{\det W},$$

where

$$W = \begin{pmatrix} \phi_1^1(\lambda^1) & \phi_1^2(\lambda^1) & \cdots & \phi_1^{n-1}(\lambda^1) & 1 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \phi_n^1(\lambda^n) & \phi_n^2(\lambda^n) & \cdots & \phi_n^{n-1}(\lambda^n) & 1 \end{pmatrix}$$

and W^{ik} is the $(n-1)\times(n-1)$ matrix obtained from W after we cancel its ith row and kth column.

In our further considerations we restrict to the so-called GZ case, when $\phi_i^k(\lambda_i)$ are monomials, $f_i = f$ and $\gamma_i = \gamma$. Then, the Stäckel separation conditions are n copies of a so-called separation curve

$$H_1\xi^{m_1} + \cdots + H_n\xi^{m_n} = \frac{1}{2}f(\xi)\mu^2 + \gamma(\xi), \qquad m_n = 0 < m_{n-1} < \cdots < m_1 \in \mathbb{N}, \tag{8}$$

with $(\xi,\mu) = (\lambda^i, \mu_i)$, $i = 1,\dots,n$.

In (λ,μ) coordinates n related Hamilton–Jacobi (HJ) equations

$$H_r\left(\lambda, \frac{\partial W}{\partial \lambda}\right) = a_r, \qquad r = 1,\dots,n,$$

for a generation function $W(\lambda,a) = \sum\limits_{i=1}^{n} W_i(\lambda_i,a)$, decouple into n ordinary differential equations

$$\frac{1}{2}f(\lambda^i)\left(\frac{dW_i}{d\lambda^i}\right)^2 + \gamma(\lambda^i) = a_1(\lambda^i)^{m_1} + a_2(\lambda^i)^{m_2} + \cdots + a_n \equiv a(\lambda^i)$$

and hence, the implicit solution of dynamical system with Hamiltonian function H_r is given by

$$\sum_{k=1}^{n}\int^{\lambda^k} \frac{\xi^{m_i}}{\sqrt{\psi(\xi)}}d\xi = t_r\delta_{ri} + \text{const}_i, \qquad i = 1,\dots,n,$$

where $2f(\xi)[a(\xi) - \gamma(\xi)] \equiv \psi(\xi)$, called the *inverse Jacobi problem*.

In this context, a question about classification and construction in natural coordinates of all separable systems on Riemannian spaces, with n quadratic in momenta constants of motion, arises. The classification can be made with respect to the admissible forms of Stäckel separability conditions. The right hand side of the conditions (8) is always the same for the class of systems considered

$$\text{r.h.s.} = \frac{1}{2}f(\lambda^i)\mu_i^2 + \gamma(\lambda^i) = \psi(\lambda^i,\mu_i), \tag{9}$$

so different classes of separable systems are described by different forms of the l.h.s. of Stäckel conditions, while systems from the same class are described by different f and γ in (9).

3 Separable systems in natural coordinates

Among all Stäckel systems a particularly important subclass consists of these considered by Benenti [11–13] and constructed with the help of the so-called *conformal Killing tensor*. Let $L = (L_j^i)$ be a second order mixed type tensor on Q and let $\overline{L} : M \to \mathbb{R}$ be a function on M defined as $\overline{L} = \frac{1}{2}\sum\limits_{i,j=1}^{n}(LG)^{ij}p_ip_j$. If

$$\{\overline{L}, E\}_{\theta_0} = \kappa E, \qquad \text{where} \quad \kappa = \{\varepsilon, E\}_{\theta_0}, \quad \varepsilon = \text{Tr}\,(L),$$

then L is called a conformal Killing tensor with the associated potential $\varepsilon = \mathrm{Tr}\,(L)$. If we assume additionally that L has simple eigenvalues and its Nijenhuis torsion vanishes, then L is called a special conformal Killing tensor [14].

For the Riemannian space (Q, g, L), geodesic flow has n constants of motion of the form

$$E_r = \frac{1}{2}\sum_{i,j=1}^{n} A_r^{ij} p_i p_j = \frac{1}{2}\sum_{i,j=1}^{n} (K_r G)^{ij} p_i p_j, \qquad r = 1, \ldots, n, \tag{10}$$

where A_r and K_r are Killing tensors of type $(2,0)$ and $(1,1)$, respectively. Moreover, as was shown by Benenti [11, 12], all the Killing tensors K_r with a common set of eigenvectors, are constructed from L by

$$K_{r+1} = \sum_{k=0}^{r} \rho_k L^{r-k},$$

where ρ_r are coefficients of the characteristic polynomial of L

$$\det(\xi I - L) = \xi^n + \rho_1 \xi^{n-1} + \cdots + \rho_n, \qquad \rho_0 = 1, \tag{11}$$

or equivalently by the following 'cofactor' formula [30–32]

$$\mathrm{cof}\,(\xi I - L) = \sum_{i=0}^{n-1} K_{n-i} \xi^i,$$

where $\mathrm{cof}\,(A)$ stands for the matrix of cofactors, so that $\mathrm{cof}\,(A)A = (\det A) I$. So, for a given metric tensor g, the existence of a special conformal Killing tensor L is a sufficient condition for the geodesic flow on Q to be a Liouville integrable Hamiltonian system with all constants of motion quadratic in momenta. Moreover, the basic separable potentials $V_r^{(m)}$ are given by the following recursion relation [30–32]

$$V_r^{(m+1)} = \rho_r V_1^{(m)} - V_{r+1}^{(m)}, \tag{12}$$

where the first nontrivial potentials are $V_r^{(0)} = -\rho_r$.

It turns out that we can (generically) associate with the tensor L a coordinate system on Q in which the geodesic flows associated with all the functions E_r separate. Namely, let $(\lambda^1(q), \ldots, \lambda^n(q))$ be n distinct, functionally independent eigenvalues of L, i.e. solutions of the characteristic equation $\det(\xi I - L) = 0$. Solving these relations with respect to q we get the transformation $\lambda \to q: \ q_i = \alpha_i(\lambda)$. The remaining part of the transformation to the separation coordinates can be reconstructed from the generating function $W(p, \lambda) = \sum\limits_i p_i \alpha_i(\lambda)$. In the (λ, μ) coordinates the Stäckel separation conditions (8) for Benenti Hamiltonian functions $H_r = E_r + V_r^{(j)}$ are given by the separation curve of the form

$$H_1 \xi^{n-1} + H_2 \xi^{n-2} + \cdots + H_n = \frac{1}{2} f(\xi)\mu^2 + \xi^{n+j}, \qquad j = 0, 1, 2, \ldots. \tag{13}$$

It is important to notice that all Liouville integrable systems of classical mechanics, with quadratic in momenta first integrals, that was separated in XIX and XX centuries, belong to the Benenti class. The reason is that only the Benenti class contains dynamic systems on flat and constant curvature Riemannian spaces. Another important feature of Benenti systems is that all of them can be lifted to one Casimir bi-Hamiltonian form [29].

Now we present the way to construct all remaining classes of separable systems by an appropriate deformations of the Benenti class. Let us start from the separability condition (8) for n Hamiltonian functions in the following form

$$\widetilde{H}_1\xi^{(n+k)-1} + \widetilde{H}_2\xi^{(n+k)-2} + \cdots + \widetilde{H}_{n+k} = \frac{1}{2}f(\xi)\mu^2 + \xi^{n+j}, \qquad k \in \mathbb{N}, \qquad j \geq k, \tag{14}$$

where $\widetilde{H}_{n_1} = \widetilde{H}_{n_2} = \cdots = \widetilde{H}_{n_k} = 0$, $1 < n_1 < \cdots < n_k < n+k$, and the separability condition for Benenti systems with the same separation coordinates (λ, μ)

$$H_1\xi^{n-1} + H_2\xi^{n-2} + \cdots + H_n = \frac{1}{2}f(\xi)\mu^2 + \xi^{n+j}. \tag{15}$$

Theorem 1. *Deformation of the Benenti system with separation curve* (15) *to the new system with separation curve* (14) *is given by a following determinant form*

$$\widetilde{H}_r = \begin{vmatrix} H_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ H_{n_1-k} & \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ H_{n_k-k} & \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix} \Bigg/ \begin{vmatrix} \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}, \tag{16}$$

where ρ_i, $i = 0, \ldots, n$ are coefficients of the characteristic polynomial of the conformal Killing tensor L (11) *related to the Benenti system.*

The formula (16) applies separately to the geodesic and the potential parts. The geodesic part $\widetilde{E}_r$ can be presented in the following form

$$\widetilde{E}_r = \frac{1}{2}p^T\widetilde{K}_r\widetilde{G}p, \qquad r = 1, \ldots, n+k, \tag{17}$$

where metric tensor $\widetilde{G}$ and Killing tensors $\widetilde{K}_r$ are

$$\widetilde{G} = (-1)^k\varphi^{-1}D_0G, \qquad \widetilde{K}_r = K_r - K_{r-1}D_1D_0^{-1} + \cdots + (-1)^kK_{r-k}D_kD_0^{-1},$$

$$\varphi = \begin{vmatrix} \rho_{n_1-1} & \cdots & \rho_{n_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{n_k-1} & \cdots & \rho_{n_k-k} \end{vmatrix}, \qquad D_0 = \begin{vmatrix} K_{n_1-1} & \cdots & K_{n_1-k} \\ \cdots & \cdots & \cdots \\ K_{n_k-1} & \cdots & K_{n_k-k} \end{vmatrix},$$

$$D_i = \begin{vmatrix} K_{n_1} & \cdots & K_{n_1-i+1} & K_{n_1-i-1} & \cdots & K_{n_1-k+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{n_k} & \cdots & K_{n_k-i+1} & K_{n_k-i-1} & \cdots & K_{n_k-k+1} \end{vmatrix}, \qquad i = 1, \ldots, k$$

and K_m in determinant calculations are treated as symbols, not as matrices. The proof of the theorem as well as other details can be found in [32].

Systems constructed in such way although obtained through the deformation procedure on the level of Hamiltonian functions, are far from being trivial generalizations of Benenti systems. There is no obvious relations between solutions of a given Benenti system and all its deformations. In each case we have a different inverse Jacobi problem to solve. Notice that the common feature of appropriate deformed systems is the same set of separated coordinates determined by the related Benenti system. Moreover, all of them can be lifted to a multi-Casimir bi-Hamiltonian form [32].

4 Example

Consider the case $n = 2$. Let Q be a two dimensional flat space parametrized by canonical coordinates $q = (q^1, q^2)$ with the contravariant metric tensor G and related special conformal Killing tensor L of the form

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} q^1 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & 0 \end{pmatrix} \implies K_2 = \begin{pmatrix} 0 & \frac{1}{2}q^2 \\ \frac{1}{2}q^2 & -q^1 \end{pmatrix}.$$

Two geodesic Hamiltonians E_1 and E_2, according to (10) are

$$E_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2, \qquad E_2 = \frac{1}{2}q^2 p_1 p_2 - \frac{1}{2}q^1 p_2^2.$$

Let us choose the potential $V^{(2)}$. As $\rho_1 = -q^1$ and $\rho_2 = -\frac{1}{4}(q^2)^2$ hence

$$V_1^{(2)} = (q^1)^3 + \frac{1}{2}q^1(q^2)^2, \qquad V_2^{(2)} = \frac{1}{16}(q^2)^4 + \frac{1}{4}(q^1)^2(q^2)^2.$$

It is one of the integrable cases of the Henon–Heiles system with Hamiltonian function $H_1 = E_1 + V_1^{(3)}$, second constant of motion $H_2 = E_2 + V_2^{(3)}$ and Newton equations

$$(q^1)_{tt} = -3(q^1)^2 - \frac{1}{2}(q^2)^2, \qquad (q^2)_{tt} = -q^1 q^2.$$

The transformation to separated coordinates (λ, μ) takes the form

$$q^1 = \lambda^1 + \lambda^2, \qquad q^2 = 2\sqrt{-\lambda^1\lambda^2},$$
$$p_1 = \frac{\lambda^1\mu_1}{\lambda^1 - \lambda^2} + \frac{\lambda^2\mu_2}{\lambda^2 - \lambda^1}, \qquad p_2 = \sqrt{-\lambda^1\lambda^2}\left(\frac{\mu_1}{\lambda^1 - \lambda^2} + \frac{\mu_2}{\lambda^2 - \lambda^1}\right),$$

and the separation curve is

$$H_1\xi + H_2 = \frac{1}{2}\xi\mu^2 + \xi^4.$$

Now, let us consider the simplest deformation given by $k = 1$, $n_1 = 2$. Then,

$$\widetilde{G} = -\frac{1}{\rho_1}G = \begin{pmatrix} \frac{1}{q^1} & 0 \\ 0 & \frac{1}{q^1} \end{pmatrix}, \quad \widetilde{K}_2 = 0, \quad \widetilde{K}_3 = -K_2^2 = \begin{pmatrix} -\frac{1}{4}(q^2)^2 & \frac{1}{2}q^1q^2 \\ \frac{1}{2}q^1q^2 & -\frac{1}{4}(q^2)^2 - (q^1)^2 \end{pmatrix},$$
$$\widetilde{V}_1 = -\frac{1}{\rho_1}V_1, \quad \widetilde{V}_2 = 0, \quad \widetilde{V}_3 = V_2 - \frac{\rho_2}{\rho_1}V_1,$$

hence

$$\widetilde{H}_1 = \frac{1}{2}\frac{1}{q^1}p_1^2 + \frac{1}{2}\frac{1}{q^1}p_2^2 + (q^1)^2 + \frac{1}{2}(q^2)^2,$$
$$\widetilde{H}_3 = -\frac{1}{8}\frac{q_2^2}{q^1}p_1^2 + \frac{1}{2}q^2p_1p_2 - \frac{1}{8}\frac{q_2^2}{q^1}p_2^2 - \frac{1}{16}(q^2)^4.$$

$\widetilde{H}_1$ and $\widetilde{H}_2$ are in involution and are separated in the same coordinates (λ, μ) as the Henon–Heiles system. The appropriate separation curve takes the form

$$\widetilde{H}_1\xi^2 + \widetilde{H}_3 = \frac{1}{2}\xi\mu^2 + \xi^4.$$

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Asymptotic Irrelevance of the KdV Hierarchy

Luis J. BOYA

Departamento de Física Teórica, Universidad de Zaragoza, E-50009 Zaragoza, Spain
E-mail: *luisjo@unizar.es*

All the equations of the KdV hierarchy share many common features, like single and multiple soliton solutions, singular algebraic potentials, etc.; in particular the phase shifts for soliton-soliton scattering are all the same in the chain KdV_1, KdV_2, etc., depending only on the momenta. We exhibit the reason for this behaviour: as these equations are originated as isopectral deformations of the Schrödinger equation in one dimension, the above features are really properties of the Schrödinger operator, and the different KdV equations just prescribe different "speeds", given by the dispersion law, for reaching the asymptotic regime. In particular, we show how the "interaction" is sum of two-body forces, what is the key for the solubility of these equations in the first place, and then calculate the phase shifts from inverse scattering (as a double-Darboux factorization), obtaining the known result, independent of the hierarchy, without ever using any of the KdV equations. We also obtain the Hirota formula for N solitons as a Wronskian, by creating them one by one from the $u(x) = 0$ potential.

1 Isospectral problem for finite matrices

Consider a Hermitian $n \times n$ matrix M and an automorphism $M \to M^{\#}$; as in any mathematical category, we are interested in the orbit $M/\{\mathrm{Aut}\}$. Now in a complete matrix algebra any automorphism is *inner*, that is

$$M \to M^{\#} = UMU^{-1} \tag{1}$$

with U unitary to preserve Hermiticity. The spectrum of M, $\mathrm{Spec}\, M$, is preserved, and we call it a hard datum. The orbit of a prefixed M is

$$\{M\} = U(n)/(\text{isotropy subgroup for } M) \equiv U(n)/H(M); \tag{2}$$

e.g. if $\mathrm{Spec}\, M$ is nondegenerate, that is for a generic M,

$$H(M) = U(1) \times \cdots \times U(1) \qquad \text{and} \qquad \{M\} = \text{Flag manifold} = U(n)/(U(1)^n). \tag{3}$$

So a generic Hermitian matrix is characterised by n different real numbers, the Spectrum, the hard datum, which is deformation invariant and $n^2 - n$ real numbers, variable under deformation, which specify the Orientation of M, and constitute the soft datum. The isospectral deformation changes the orientation but maintains the spectrum; it translates the matrix along the flag manifold.

At any moment the spectrum is retrieved from the powers of traces of the matrix by

$$I_k = (1/k)\,\mathrm{Tr}\, M^k, \tag{4}$$

i.e. $c_1 := \sum \lambda_i = I_1$, $c_2 := \sum_{i<j} \lambda_i \lambda_j = I_1^2/2 - I_2$, etc., where c_i are the elementary symmetric polynomials on the eigenvalues. Alternatively the spectrum is given by the poles of the resolvent

$$\mathrm{Res}\, M(\lambda) = [M - \lambda]^{-1}. \tag{5}$$

In the following we apply this analysis to the Schrödinger operator as a (infinite dimensional local) Hermitian operator.

2 Hard and soft data for the Schrödinger operator

Consider the Schrödinger equation on the real line $\mathbb{R}$; the operator is

$$H = -D^2 + u(x), \tag{6}$$

where $x \in \mathbb{R}$, $D \equiv d/dx$, and $u(x)$ is a real function decaying "fast enough" at $x \to \pm\infty$. There are *three* differences with the above finite case: infinite dimension, presence of continuous spectrum, and differential operator (the most general operator would be an *integral* operator). The data for the Discrete Spectrum are the bound states

$$H\Psi_j(x) = E_j\Psi_j(x), \qquad E_j < 0, \qquad j : 1, 2, \ldots, N, \qquad N+1 \in \mathbb{N}. \tag{7}$$

For $x \to +\infty$

$$\Psi_j(x) \to c_j \exp(-\kappa_j x), \qquad \|\Psi\| = 1, \qquad \kappa_j^2 = -E_j. \tag{8}$$

The *eigenvalues* E_i are the (discrete) hard data; the norming constants c_j localise the bound states; there are position (soft data), not spectral data. Note we take *normalised* discrete eigenfunctions.

Continuous Spectrum. It is contained in the *S-matrix*, which in our $1D$ case is a 2×2 matrix: we have forward/backward amplitudes for the direct and the *zurdo* scattering (zurdo or $\hat{}$: incoming particle form the right):

$$S(k) = \begin{pmatrix} t(k) & b(k) \\ \hat{b}(k) & \hat{t}(k) \end{pmatrix}, \qquad k \in [0, \infty). \tag{9}$$

Our potential is real so time reversal holds; hence $t(k) = \hat{t}(k)$; also by causality $t(k)$ is related to the *discrete* spectrum [1]; hence the forward amplitude is a hard datum, as part of the spectral density. There are also restrictions coming from the *unitarity* of S:

$$\begin{aligned} &S^\dagger S = 1 = SS^\dagger \Rightarrow |t|^2 + |b|^2 = |\hat{t}|^2 + |\hat{b}|^2 = 1, \\ &\operatorname{Arg} t + \operatorname{Arg} \hat{t} (= 2\operatorname{Arg} t) - \operatorname{Arg} b - \operatorname{Arg} \hat{b} = \pi. \end{aligned} \tag{10}$$

Hence the *argument* of either reflected amplitude (say, in direct scattering) is the soft datum for the continuous spectrum. Hence the complete data for the Schrödinger operator are

$\kappa_j (j : 1, \ldots, N)$ and $|b(k)|$ are hard data, fixed by $\operatorname{Spec} H$,
c_j and $\operatorname{Arg} b(k)$ are soft data, fixed by the Orientation of H.

In particular κ_j and $|b(k)|$ fix the *spectral density*. It is to be remarked that the natural "observables": energies E_j and scattering *coefficients* $\sigma_+ = |t(k) - 1|^2$ and $\sigma_- = |b(k)|^2$ are *hard data*.

3 Inverse problem

We just recall the usual procedure to recover the potential from the scattering data, cf. e.g. [2]. There are three steps:

1) Fourier transform of the total (soft + hard) data:

$$B(x) \equiv \frac{1}{2\pi} \int b(k)\, dk \exp(ikx) + \sum_j c_j^2 \exp(-\kappa_j x). \tag{11}$$

2) Solve for the kernel $K(x,y)$ in the Marchenko integral equation

$$K(x,z) + B(x+z) + \int_x^\infty K(x,y)B(y+z)dy = 0. \tag{12}$$

3) Retrieve the potential by

$$u(x) = -2K_x(x,x), \qquad \text{where} \quad K_x \equiv dK/dx. \tag{13}$$

Notice the solution is complete, i.e. we recover the full potential, and hence both the hard and the soft data are used. For example, for the *solitonic potentials* (defined as reflectionless potentials by $b(k) \equiv 0$), it is

$$u(x) = -2\big(d^2/dx^2\big) \log \det A, \tag{14}$$

where

$$A_{mn} = \delta_{mn} + c_n^2[\exp -(\kappa_m + \kappa_n)x]/(\kappa_m + \kappa_n), \tag{15}$$

where the discrete spectrum data κ_j, c_j are given arbitrarily, subject only to all eigenvalues κ_j different. For example for $N = 1$ $\{\kappa, c\}$ we recover the one soliton potential $u(x) = -2\kappa^2 \operatorname{sech}^2\kappa(x - x_0)$, where $E = -\kappa^2$ and $c = \sqrt{2\kappa}\exp(\kappa x_0)$.

The exponential part in (15) is related to the soliton scattering phase shift for any of the KdV hierarchy equations (see later); we have approached it here from inverse scattering theory, with no reference to the hierarchy at all! (15) was first obtained by Kay and Moses [3] from inverse scattering and later by Hirota [4] from his bilinear method. We shall obtain it below also by the Darboux method.

4 General isospectral problem

We know that the KdV equations are isospectral evolution equations; hence, their solutions $u(x) = u(x, t = 0) \to u(x,t)$ are a particular case of the general isospectral problem. It is this: *given a potential $u(x)$, find the most general potential $v(x)$ with the same hard data.* The formal solution of the inverse problem sketched above allows us to set and solve, in principle, the general isospectral problem: There are again three steps:

1) Solve the direct problem for $u(x)$: find

$$\kappa_j, \quad c_j, \quad |b(k)| \quad \text{and} \quad \operatorname{Arg} b(k). \tag{16}$$

2) Change arbitrarily the soft data c_j and $\operatorname{Arg} b(k)$.

3) Solve the inverse problem from the new c and $b(k)$.

Then the new potential $v(x)$ is the most general potential strictly isospectral to $u(x)$.

Some restrictions apply: for example, the new $b(k)$ has to be "nice", i.e. to admit Fourier transform guaranteeing that the new potential $v(x)$ falls off at infinity fast enough. Also c_j have to be positive (by an initial convention).

From the above it is clear that the KdV equations realize part of this program; before seeing it in detail, let us consider the simplest case: to obtain the most general potential isospectral to the standard one-soliton potential

$$u(x) = -2\operatorname{sech}^2(x); \tag{17}$$

it has a single bound state with $\kappa = 1$, $c = \sqrt{2}$ and it is reflectionless, $b(k) = 0$. To change the soft data means just to change $\sqrt{2}$ to an arbitrary c; then the inverse method produces inmediately

$$v(x) = -2\operatorname{sech}^2(x - x_1), \qquad \text{with} \quad x_1 = \log\big(c/\sqrt{2}\big). \tag{18}$$

The only effect on the potential is just a traslation of the "well", where the soliton is located.

We shall see now that two methods: the Double Darboux method (DD) and the KdV hierarchy are equivalent to finding deformed isospectral potentials in somewhat complementary ways from a given potential. We first dispose of the DD method [5].

In the simplest application of the DD method one kills the ground state and reinserts it in another "position"; therefore, it changes only the first norming constant c_1. The procedure is constructive, i.e. the new ground state wavefunction is calculated completely from the original one (see e.g. [6]):

$$\Psi(x,\lambda) = \Psi(x)/[1+\lambda\int_{-\infty}^{x}\Psi^2(x)\,dx], \tag{19}$$

and therefore the new norming constant is (as $\|\Psi\| = 1$)

$$c_1' = c_1/(1+\lambda). \tag{20}$$

The iterated DD method repeats the process changing simultaneously the positions of two or more arbitrary bound states (by killing and reinstating them somewhere else), and leaving the phases of the reflection amplitudes unchanged. The procedure can be always implemented (and indeed, not only for fast-decaying potentials), and the solutions are also algebraic, but the procedure is cumbersome and tedious!

The original method was devised by Darboux (1882) [5] to relate linear differential operators, not necessarily of second order. The general application to solutions of the Schrödinger equation was found by Abraham and Moses (1980) [6], Mielnick (1984) [7] and others.

5 The KdV hierarchy

The Korteweg–de Vries equation ($\mathrm{KdV} = \mathrm{KdV}_1$) was discovered in 1895 as an equation exhibiting soliton behaviour, and the hierarchy KdV_n was first presented by P. Lax (1968) [8]. The time evolution of the solutions of the equations of the KdV hierarchy, $u(x) \to u(x,t)$ represents the isospectral change of the Hamiltonian of the Schrödinger equation by an uniparametric group of automorphims whose generator is polynomial in the derivatives.

In the beautiful exposition of Lax (1970) [8], we have

$$H \to H(t) = U(t)HU^{-1}(t). \tag{21}$$

Define $B(t)$ by $B(t) = (dU(t)/dt)U^{-1}(t)$. Then

$$H_t = [B,H], \qquad u_t = K[u], \qquad \text{and} \qquad \Psi_t = B\Psi \tag{22}$$

with $a_t = \partial a/\partial t$ throughout. Then the first equation defines the problem (i.e., to find B). The second equation is the Non-Linear evolution equation (KdV hierarchy). The third equation is important for the evolution of the scattering data.

The KdV hierarchy obtains solving for B order by order in derivatives:

$B = D$: The equation is $u_t = u_x$; the chiral wave equation.

$B = D^3 + \{u, D\}$. The equation is, of course

$$u_t = 6uu_x - u_{xxx}, \qquad (\text{Korteweg–de Vries}, 1895) \tag{23}$$

with a cubic dispersion law. There are at least one equation of each odd order of dispersion; for systematics we just write the (natural) quintic equation [9]

$$u_t = -30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} - u_{xxxxx}. \tag{24}$$

The key point why these hierarchical equations can be solved *inspite of being partial and nonlinear* is that *the scattering data of the potential change with the linear part of the operator B only*, which has the highest derivative: the rest is function of the potential u, and goes to zero faster, as u itself decays fast.

Let us interpret the evolution in the original KdV equation, with cubic dispersion law, as an one-parameter isospectral deformation; at time t (=deformation parameter), the hard data are of course invariant:

$$\kappa_j(t) = \kappa_j(0), \qquad |b(k,t)| = |b(k,0)|, \tag{25}$$

but the soft data evolve in a simple manner:

$$c_j(t) = \exp\left(4\kappa_j^3 t\right) c_j(0), \qquad b(k,t) = \exp\left(8ik^3 t\right) b(k,0), \tag{26}$$

i.e. the "bumps" of the bound state wavefunctions move all harmonically, and only the phase of the reflected amplitudes changes. As it is expected, the other equations in the KdV hierarchy just trade, as far as soft data are concerned, the cubic "3" of KdV_1 for any greater odd number.

It is interesting the comparison of the Double Darboux method and the KdV solutions for new potentials:

– In the DD method, we change one position of the discrete spectrum at a time arbitrarily.

– With the KdV equations, we change all the soft data with a particular dispersion law.

6 Irrelevance of the KdV hierarchy

It is common lore that the "nonlinear" character of the KdV equations is the agent responsible for the phase shift in soliton-soliton scattering; the shift is obtained from the asymptotic behaviour, of course; now the motion of the solitons through space means the centers x_i (or the norming constants c_i) move with a particular dispersion law, but of course they are irrelevant at the asymptotic limit: initial $x_i \to -\infty$, final $x_i \to +\infty$. We conclude, therefore, that the phase shift is a hard datum; hence, we should be able to get it directly from the Schrödinger equation; this we show next, limiting ourselves to the case of two solitons.

To create two arbitrary solitons we use the direct Darboux method twice starting from the $u(x) = 0$ solution, the vacuum [10]. The procedure is well known, so we just quote some formulas and results:

Take the $E_1 = -\kappa_1^2$, unphysical solution $\phi_1(x) = \cosh \kappa_1(x - x_1)$. Factorize $H - E_1 = A^\dagger A$, where $A = D + W' = \phi_1 \cdot D \cdot \phi_1^{-1}$ with $\phi_1(x) = \exp(-W)$. The new, first step potential is

$$v_1(x) = -2\kappa_1^2 \operatorname{sech}^2(x - x_1). \tag{27}$$

To create the second solution, with $E_2 < E_1$, take now the free solution $\phi_2(x) = \sinh \kappa_2(x - x_2)$, which becomes a zeroless unphysical solution of the first step potential by

$$\tilde{\phi}_2(x) = A\phi_2(x) = (D - \tanh \kappa_1(x - x_1))\phi_2(x) = \phi_1 \cdot D \cdot \phi_1^{-1}\phi_2(x). \tag{28}$$

The new potential is now

$$v_2(x) = -2D^2 \log \phi_1(x) - 2D^2 \log \tilde{\phi}_2(x) = -2D^2 \log W(\phi_1(x), \phi_2(x)), \tag{29}$$

where W is the *Wronskian* $W = \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x)$.

This is an *exact* solution of the two soliton potential; to compute the phase shift we let now $x_2 \to -\infty$ and compare the new center of the remaining (original) soliton x_1' with the original x_1; the phaseshift is twice this value times κ_1 (as Φ is dimensionless): as the second soliton is created

at $x = -\infty$, the first gets repelled and moves a distance $x_1' - x_1$: when the bigger soliton moves to $+\infty$, the old one recedes to $-x_1' - x_1$.The messy algebra leads to a well known expression (recall $\kappa_2 > \kappa_1$)

$$\Phi = \log(\kappa_2 + \kappa_1)/(\kappa_2 - \kappa_1) \tag{30}$$

checked numerically (it is enough to take $\kappa_1 = 1$, and $x_1 = 0$). The second soliton repels the first by a sort of exclusion principle: the bound eigenvalues of the Schrödinger equations are nondegenerate, hence $\kappa_1 = \kappa_2$ would lead to an infinite phase shift, and "by continuity" we get (30); writing it in the form

$$\Phi = 2\,\mathrm{Arg}\,\tanh(\kappa_1/\kappa_2) \tag{31}$$

looks like a very common formula is scattering theory (with arc tan instead; recall κ is like an imaginary wavenumber; our shift is not really a phase shift, as it is not periodic; it is more a space (vs time) delay). For instance, scattering for a point impurity of strength g produces an even phase shift [11] $\mathrm{tg}\,\delta_+ = g/k$.

Formula (30) is of course obtained from the KdV_n equations: the generalization for n solitons leads, from KdV, to the Hirota formula [4]; in our method we get directly the Wronskian for n independent $E < 0$ free solutions; the identity of the Kay–Moses & Hirota formula with the Wronskian we leave to the reader to check; for this is essential to use the affine invariance of the equation (14), namely $\det A = \det BA$, where B is a column matrix with entries $\exp(a_i x + b_i)$, up to terms which die in calculating the potential.

Notice also the full n-soliton interaction is sum of independent two-body "forces", which accounts for the completely solubility of the system.

7 Singular solutions

We can see also how many special solutions, common to all KdV hierarchy equations, are obtained directly by the Darboux method starting again from the $u(x) = 0$ potential. For example, the first step of the $u(x) = 0$, $E = 0$ solution $\phi(x) = ax + b$ produces the centrifugal potential

$$u(x) = +2/(x - x_1)^2 \tag{32}$$

which is also a rational solution of KdV. Iteration with the $\phi_2(x) = x^2$ solution produces e.g.

$$u(x) = 6x\big(x^3 - 24x_1\big)/\big(x^3 + 12x_1\big)^2 \tag{33}$$

which again is a rational solution of KdV (with $t = x_1$) (cf. [9]).

8 Final remarks

Soliton properties, including scattering, are really properties of the Schrödinger operator. In particular, the one-soliton potential is just the partner potential to the vacuum potential $u(x) = 0$ doubled by the (unphysical, negative energy) solution $\phi(x) = \cosh(x)$. The iteration, always with negative energy, will produce the most general multi-soliton solution at any x, hence should reproduce the phase shifts, as we checked for the two soliton case; solitons "forces" are exclusively the effect of non-degeneracy of the normalizable eigenfunctions.

As the solitons move in one direction only, they behave as chiral fields. In the Lagrangian formulation, these chiral fields are rather peculiar, because the space of potentials $\{u\}$ (or rather

of the integrals $\varphi = \int u$, which is what enters in the local Lagrangian) is already the phase space, not the configuration space. This implies, in particular, that in the quantum version the solitons should behave like fermions (Jackiw proved this directly for the chiral, KdV_0 case [12]). This fermi character is a translation, we believe, of the purely classical effect: namely two solitons of close wavenumber cannot superpose (P. Lax), reflecting the case that for equal wavenumber they should be at infinite distance, because the Schrödinger equation does not allow degeneracy in the bound states in one dimension.

We believe also that upon quantization these solitons should behave like nonrelativistic Majorana–Weyl fermions, reflecting the "chiral" behaviour of the classical counterpart. But we have not looked carefully yet to the quantization problem.

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On the Use of the Lie–Bäcklund Groups in the Context of Asymptotic Integrability

Georgy I. BURDE

The Jacob Blaustein Institute for Desert Research,
Ben-Gurion University, Sede Boker Campus 84990, Israel
E-mail: *georg@bgumail.bgu.ac.il*

A new approach to the problem of asymptotic integrability of physical systems is developed and applied to the KdV equation with higher-order corrections. A central object of the approach is an integrable reference equation, which is constructed by defining a proper Lie–Bäcklund group of transformations and applying it to the leading order equation. It is shown that the solitary wave solutions of the asymptotically integrable equations, derived using asymptotic transformations, fail to approximate solutions of those equations in some (rather wide) range of the soliton and system parameters.

1 Introduction

For many important physical systems, the leading order term in an asymptotic perturbation expansion is given by an integrable nonlinear equation. This implies that those physical systems are integrable at the nontrivial leading orders in an asymptotic sense. The meaning of the term "asymptotic integrability" is that the equation with higher-order corrections is integrable up to a certain order in the asymptotic sense. Asymptotic integrability at certain order is commonly ascertained by developing an asymptotic transformation relating this higher-order equation to an integrable equation. If such a transformation is impossible, the nonintegrable effects (the "obstacles" to integrability [1]) are defined as additional terms appearing in the target equation.

The most extensively studied example is that of weakly nonlinear long waves, when the leading order equation is the Korteweg–de Vries (KdV) equation. The KdV equation first arose as an approximate equation governing the unidirectional small amplitude long waves in inviscid incompressible fluid [2] but later it was introduced in many different physical contexts. If effects of higher-order are of interest, then extension of terms up to the next orders in the (small) wave amplitude, with the weakly nonlinear and weakly dispersive effects being in balance, leads to the extended KdV equation including higher-order corrections. The KdV equation with the first-order corrections is integrable in the asymptotic sense as it has been shown by Kodama [3] (see also [4]) who has found a transformation which maps the perturbed KdV equation to the integrable equation (normal form) obtained by combining the KdV equation with its first commuting flow. In [4], the normal form was introduced up to the second-order corrections. It was found that, in general, the asymptotic integrability cannot be extended to the second-order, and the asymptotic transformation from the second order KdV equation to the equation representing a normal form plus one obstacle was defined.

It is commonly accepted that the asymptotic transformations from the high-order KdV equations to the corresponding integrable equation (if there are no obstacles), or to the form given by the symmetries plus obstacles, can be used for investigating the properties of the high-order KdV solitons (see, e.g. [5,6]).

In the present paper, we develop an approach allowing us to study the approximation properties of the asymptotic solutions of the higher-order KdV equations that are obtained by Kodama's asymptotic transformation from the normal form solitary waves. Our approach, as

applied in the context of asymptotic integrability, is aimed at constructing some asymptotically integrable higher-order KdV equations, for which the asymptotic solutions, alternative to those derived via Kodama's transformation, may be defined. The central object of the approach is some reference integrable equation, which depends on a small parameter of the physical system in such a way that its asymptotic perturbation expansion, to a certain order, has a form of the corresponding order KdV equation. The reference integrable equation is constructed by applying the properly defined Lie–Bäcklund group of transformations to the leading order equation.

2 The reference equation

We will consider a physical system that can be described by an asymptotic perturbation expansion with the leading order term given by an integrable nonlinear equation. We will assume the following expansion form with a small parameter ϵ and a leading order term representing a scalar differential equation:

$$F^{(0)}(x,u,u_{(1)},u_{(2)},\dots,u_{(k)})+\epsilon F^{(1)}(x,u,u_{(1)},u_{(2)},\dots,u_{(k+r)}) \\ +\epsilon^2F^{(2)}(x,u,u_{(1)},u_{(2)},\dots)+\dots+\epsilon^NF^{(N)}(x,u,u_{(1)},u_{(2)},\dots)=O\big(\epsilon^{N+1}\big), \tag{1}$$

where $x=(x^1,x^2,\dots,x^n)$ are n independent variables, u is the dependent variable, and $u_{(j)}$ denotes the set of all jth-order partial derivatives of u with respect to x.

Following the ideas of the approach, developed in [7] for the point transformations, we will consider the one-parameter (a) group of the Lie–Bäcklund transformations

$$\bar u=\phi(x,u,u_{(1)},u_{(2)},\dots;a) \tag{2}$$

with (canonical) Lie–Bäcklund infinitesimal generator

$$U=\eta(x,u,u_{(1)},u_{(2)},\dots)\frac{\partial}{\partial u}, \tag{3}$$

where the order of derivatives appearing in the generator is, in general, not restricted.

The group (2) is defined from the requirement that the leading order equation transform, under an infinitesimal action of the group (for small values of the group parameter $a=\epsilon$), into the first-order perturbed equation obtained by retaining only the leading order and first-order terms in (1), as follows

$$F^{(0)}(x,u,u_{(1)},u_{(2)},\dots,u_{(k)})+\epsilon F^{(1)}(x,u,u_{(1)},u_{(2)},\dots,u_{(k+r)})=0. \tag{4}$$

To define the group the following steps are made:

(i) The group (2) is applied to the *unperturbed equation* $F^{(0)}=0$ written in the variables $\bar u$, $\bar u_{(1)}$ and so on, as

$$F^{(0)}(x,\bar u,\bar u_{(1)},\bar u_{(2)},\dots,\bar u_{(k)})=0. \tag{5}$$

As the result, equation (5) is transformed to

$$H(x,u,u_{(1)},u_{(2)},\dots,u_{(s)};a)=0 \tag{6}$$

or infinitesimally

$$F^{(0)}(x,\bar u,\bar u_{(1)},\bar u_{(2)},\dots,\bar u_{(k)})=F^{(0)}(x,u,u_{(1)},u_{(2)},\dots,u_{(k)}) \\ +aU^{(k)}F^{(0)}(x,u,u_{(1)},u_{(2)},\dots,u_{(k)})\Big|_{[F^{(0)}]}+O\left(a^2\right)\qquad(a\ll1), \tag{7}$$

where $U^{(k)}$ is the kth extended infinitesimal generator of (3) defined by

$$U^{(k)} = \eta\frac{\partial}{\partial u} + D_j(\eta)\frac{\partial}{\partial u_j} + D_iD_j(\eta)\frac{\partial}{\partial u_{ij}} + \cdots \tag{8}$$

and $[F^{(0)}]$ denotes the set of equations including the equation $F^{(0)} = 0$ and its differential consequences:

$$D_iF^{(0)} = 0, \qquad D_iD_jF^{(0)} = 0, \qquad \ldots. \tag{9}$$

(ii) It is required that

$$U^{(k)}F^{(0)}(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)})\Big|_{[F^{(0)}]} = F^{(1)}(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k+r)}). \tag{10}$$

It yields determining equations for the Lie–Bäcklund group generator $\eta(x, u, u_{(1)}, u_{(2)}, \ldots)$.

(iii) With the group generator defined, the finite transformations (2) can be determined as a solution of the Lie–Bäcklund equation

$$\frac{d\bar{u}}{da} = \eta(x, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, \ldots), \qquad \bar{u}|_{a=0} = u(x). \tag{11}$$

It is evident from (i)–(iii) that the group (2) defined in such a way is neither a symmetry group of equation (1) nor a symmetry group of the unperturbed equation.

In some cases, for specific relations among the coefficients of differential monomials in the first-order term of (1), the Lie–Bäcklund equation (11) can be solved in a closed form so that the finite transformation (2) is represented by an analytical function. Using this closed form transformation one can construct the equation (6) in an explicit form (not as a series). This new equation is dependent on a parameter a (for the sake of brevity we will name it the "a-equation") and possesses the following properties:

a) The a-equation is integrable, since there exists an exact transformation (inverse to (2)) that converts equation (6) into the leading order integrable equation (5). Therefore any exact solution of equation (5) yields the exact solution of the a-equation.

b) When $a \ll 1$, the a-equation coincides with the initial perturbed equation (1) up to first-order in $a = \epsilon$.

c) In the case when (1) is an evolution equation and the x-derivative terms at each order represent differential polynomials of specific weights, the corresponding order terms in an expansion of the a-equation in series with respect to a have the same differential structure as those in the original equation but with the monomial coefficients specified in a certain way.

In view of the property (c), the a-equation can be considered as a model for the original physical system in a sense, since it contains impacts of all orders of the asymptotic perturbation expansion on the solution. This model may correspond to a real physical system if there are enough freedoms to satisfy the required conditions on the coefficients.

From another point of view, an expansion of the a-equation up to a certain order produces equations that can be treated as asymptotically integrable at this order since the equation expanded is exactly integrable and the terms dropped may be considered as having an asymptotically negligible influence upon the solution.

3 Applications to the higher-order KdV equations

It can be shown with the use of an appropriate perturbation method (e.g., see [2]) that the higher-order correction to the KdV equation has the following expansion form with a small parameter ϵ:

$$u_t + K^{(0)}[u] + \epsilon K^{(1)}[u] + \epsilon^2 K^{(2)}[u] + \cdots + \epsilon^N K^{(N)}[u] = O\big(\epsilon^{N+1}\big), \tag{12}$$

where $K^{(0)}[u]$ gives the KdV flow in its standard form

$$K^{(0)}[u] = u_3 + 6uu_1 \tag{13}$$

and the next two terms in (12) are given by

$$K^{(1)}[u] = a_1^{(1)}u_5 + a_2^{(1)}u_3u + a_3^{(1)}u_2u_1 + a_4^{(1)}u_1u^2, \tag{14}$$

$$K^{(2)}[u] = a_1^{(2)}u_7 + a_2^{(2)}u_5u + a_3^{(2)}u_4u_1 + a_4^{(2)}u_3u^2 + a_5^{(2)}u_3u_2 + a_6^{(2)}u_2u_1u + a_7^{(2)}u_1u^3 + a_8^{(2)}u_1^3. \tag{15}$$

The subscript t denotes derivatives with respect to time variable and other subscripts denote derivatives of the corresponding order with respect to the space variable x. Each polynomial $K^{(n)}[u]$ has the scaling property with the homogeneous weight $2n+5$ if one assigns the weight 2 to u and 1 to $\partial/\partial x$.

Following the approach described in the previous section, we consider the one-parameter (a) Lie–Bäcklund group of transformations

$$\bar{u} = \phi(x, t, u, p, u_1, u_2, \ldots; a), \qquad U = \eta(x, t, u, p, u_1, u_2, \ldots)\frac{\partial}{\partial u}, \tag{16}$$

where the nonlocal terms containing

$$p = D^{-1}(u), \qquad D^{-1}(\cdot) = \int_{-\infty}^{x} (\cdot)dx' \tag{17}$$

may appear. The transformation (16) is defined by the requirement that it converts the leading order equation $u_t + K^{(0)}[u] = 0$ written in the variables $\bar{u}$, $\bar{u}_1$ and so on, as

$$F^{(0)}(\bar{u}) = \bar{u}_t + \bar{u}_3 + 6\bar{u}\bar{u}_1 = 0 \tag{18}$$

into a new equation $H(u; a) = 0$ which possesses the property that, for small values of the group parameter a, it coincides with the perturbed equation (12) up to first-order in $a = \epsilon$:

$$H(u; a) = u_t + K^{(0)}[u] + aK^{(1)}[u] + O\left(a^2\right) \qquad (a \ll 1). \tag{19}$$

Applying this requirement (expressed in terms of the group generator by (10)) yields determining equations for the Lie–Bäcklund group generator $\eta(x, t, u, p, u_1, u_2, \ldots)$.

It is evident that the generator $\eta(x, t, u, p, u_1, u_2, \ldots)$ defined in such a way is not unique since, in addition to the terms depending on the coefficients $a_1^{(1)}, \ldots, a_4^{(1)}$ of the differential polynomial $K^{(1)}[u]$, it may include the terms representing symmetries of the leading order equation $F^{(0)}(u) = 0$ with arbitrary coefficients. In what follows, we will deal with a minimal form of the group generator excluding symmetries, since only it yields the transformation generating the a-equation possessing the property (c): the terms of its expansion with respect to a, at each order, represent differential polynomials of the homogeneous weight like (14), (15) and so on. This minimal form is uniquely defined from the determining equations yielded by (10) as

$$\eta = \lambda_1 u^2 + \lambda_2 u_2 + \lambda_3 u_1 p + \lambda_4 x(u_3 + 6uu_1) \tag{20}$$

or equivalently as

$$\eta = \lambda_1 u^2 + \lambda_2 u_2 + \lambda_3 p u_1 - \lambda_4 x u_t, \tag{21}$$

where the parameters $\lambda_1, \ldots, \lambda_4$ are expressed through the coefficients $a_1^{(1)}, \ldots, a_4^{(1)}$ of $K^{(1)}[u]$ by the relations

$$\lambda_1 = \frac{1}{6}\left(a_4^{(1)} - a_2^{(1)} - 4a_1^{(1)}\right), \qquad \lambda_2 = \frac{1}{12}\left(a_4^{(1)} - a_3^{(1)} + 6a_1^{(1)}\right),$$
$$\lambda_3 = \frac{1}{3}\left(a_2^{(1)} - 8a_1^{(1)}\right), \qquad \lambda_4 = \frac{1}{3}a_1^{(1)}. \tag{22}$$

It is seen that the group generator depends explicitly on x but it does not contain t.

With the group generator defined, the finite transformations (16) are determined as a solution of the Lie–Bäcklund equation

$$\frac{d\bar{u}}{da} = \lambda_1\bar{u}^2 + \lambda_2\bar{u}_2 + \lambda_3\bar{u}_1\bar{p} + \lambda_4 x(\bar{u}_3 + 6\bar{u}\bar{u}_1), \qquad \bar{u}|_{a=0} = u(x,t), \tag{23}$$

where, alternatively, the expression for η on the right-hand side, may be defined as in (21). We will concentrate on the cases where the Lie–Bäcklund equation (23) can be solved in a closed form, which correspond to the specific relations between the coefficients $a_1^{(1)}, \ldots, a_4^{(1)}$ of the differential polynomial $K^{(1)}[u]$.

In the present paper, we will consider only the following case

$$a_2^{(1)} = 8a_1^{(1)}, \qquad a_3^{(1)} = a_4^{(1)} + 6a_1^{(1)} \tag{24}$$

which, according to (22), corresponds to the generator (21) with the coefficients λ_2 and λ_3 vanishing. Then the Lie–Bäcklund equations (23) can be solved to give

$$\bar{u}(x,t;a) = \frac{u(\xi,\tau)}{1 - a\lambda_1 u(x,\tau)}, \qquad \xi = x, \qquad \tau = t - a\lambda_4 x, \tag{25}$$

where

$$\lambda_1 = \frac{a_4^{(1)}}{6} - 2a_1^{(1)}, \qquad \lambda_4 = \frac{a_1^{(1)}}{3}. \tag{26}$$

In (25), $\bar{u}$ is a solution of the KdV equation (18) and u is a solution of the reference equation $H(u;a) = 0$ which possesses the property (19). The equation for $u(\xi,\tau)$ obtained by substituting (25) into (18) is

$$H(u;a) = \frac{1}{a^4\lambda_1^4 v^4}\Big\{6a^2\lambda_1^2\left[L(v)\right]^3 - 6vL(v)\left[1 + a^2\lambda_1^2 L^2(v)\right]$$
$$+ a\lambda_1 v^2\left[a\lambda_1 v_\tau - 6L(v) + a\lambda_1 L^3(v)\right]\Big\} = 0, \tag{27}$$

where

$$v = u - \frac{1}{a\lambda_1}, \qquad L = \frac{\partial}{\partial\xi} - a\lambda_4\frac{\partial}{\partial\tau}. \tag{28}$$

The a-equation (27) treated as an equation for $u(x,t)$ is dependent on a – not only explicitly but also through the independent variable τ as is defined in (25). It possesses the properties (a)–(c) formulated in the previous section. Solutions of the a-equation (27) can be obtained from solution of the integrable KdV equation (18) by the transformation (inverse to (25):

$$u(x,t;a) = \frac{\bar{u}(x,z)}{1 + a\lambda_1\bar{u}(x,z)}, \qquad z = t + a\lambda_4 x, \tag{29}$$

where $\bar{u}(x,z)$ satisfies $\bar{u}_z + \bar{u}_3 + 6\bar{u}\bar{u}_1 = 0$.

Despite the fact that the maximal order of derivatives in equation (27) is three – the same as in the leading order equation, its expansion in to series with respect to a includes the terms with higher-order derivatives (which appear due to the a-dependence of the independent variable τ), and those high-order terms have the same differential structure as the corresponding order terms in the original equation (12). However, the monomial coefficients in the higher-order terms are now specified in a certain way.

In the expansion of equation (27), the coefficients in the first-order term are specified according to (24). We will also write out the relations defining the monomial coefficients in the second-order term

$$\begin{aligned}
&a_1^{(2)} = \frac{4}{3}a_1^{(1)^2}, \qquad a_2^{(2)} = 14a_1^{(1)^2}, \qquad a_3^{(2)} = \frac{1}{3}a_1^{(1)}\left(84a_1^{(1)} + 5a_4^{(1)}\right),\\
&a_4^{(2)} = \frac{4}{3}a_1^{(1)}\left(18a_1^{(1)} + a_4^{(1)}\right), \qquad a_5^{(2)} = \frac{2}{3}a_1^{(1)}\left(66a_1^{(1)} + 5a_4^{(1)}\right),\\
&a_6^{(2)} = 36a_1^{(1)^2} + 10a_1^{(1)}a_4^{(1)} + \frac{1}{6}a_4^{(1)^2}, \qquad a_7^{(2)} = \frac{1}{6}a_4^{(1)^2}, \qquad a_8^{(2)} = \frac{1}{6}a_4^{(1)}\left(12a_1^{(1)} + a_4^{(1)}\right).
\end{aligned} \tag{30}$$

4 Asymptotic integrability

The analysis made in [4] is based on a near identity transformation $T_\epsilon : v \to u$

$$u = T_\epsilon(v) = v + \epsilon\Phi^{(1)}(v) + \cdots \tag{31}$$

such that the perturbed KdV equation (12) is transformed to

$$v_t + K^{(0)}[v] + \epsilon G^{(1)}[v] + \epsilon^2 G^{(2)}[v] + \cdots + \epsilon^N G^{(N)}[v] = O\big(\epsilon^{N+1}\big), \tag{32}$$

where $G^{(n)}(v)$ are differential polynomials of the same structure as the corresponding polynomials in (12). In particular, if the calculations are aimed at defining obstacles to the asymptotic integrability, then $G^{(n)}(v)$ are represented as

$$G^{(n)}(v) = a_1^{(n)}K_0^{(n)}(v) + R^{(n)}(v), \qquad R^{(n)}(v) = \sum_{i=1}^{\Delta(n)} \mu_i^{(n)} R_i^{(n)}(v). \tag{33}$$

Here $K_0^{(n)}(v)$ are symmetries of the leading order KdV equation. The number $\Delta(n)$ in (33) is the total number of the conditions $\mu_i^{(n)} = 0$ (number of obstacles) needed to be satisfied for existence of a transformation to the normal form containing only symmetries.

At order ϵ, the transform T_ϵ is given by the first-order term $\Phi^{(1)}$ of the generating function. There is no obstacles to asymptotic integrability at the first-order. The transform T_ϵ at the next orders is defined via an expansion of the generating function Φ in the power series of ϵ [4]. At order ϵ^2, there is one obstacle to asymptotic integrability, which means that one condition on the coefficients $a_1^{(1)}, \ldots, a_4^{(1)}, a_1^{(2)}, \ldots, a_8^{(2)}$ needs to be satisfied for existence of a transformation to the normal form.

We will study the approximate solitary wave solutions for the KdV equation (12) of a specific form, namely, one with the first- and second-order corrections given by the terms of an expansion of the a-equation (27) in series with respect to a. There is no obstacle to asymptotic integrability of this equation via Kodama's transformation to the normal form – the relations (24) and (30), defining the expansion of the a-equation (27) satisfy the condition for vanishing the obstacle.

The one-soliton solution of the leading order KdV equation and the corresponding solution of the second-order normal form are

$$u = 2k^2\mathrm{sech}^2\left[k\left(x - 4k^2t\right)\right], \tag{34}$$

$$v = 2k^2\mathrm{sech}^2\left\{k\left[x - \left(4k^2 + 16\epsilon a_1^{(1)}k^4 + 64\epsilon^2 a_1^{(2)}k^6\right)t\right]\right\}, \tag{35}$$

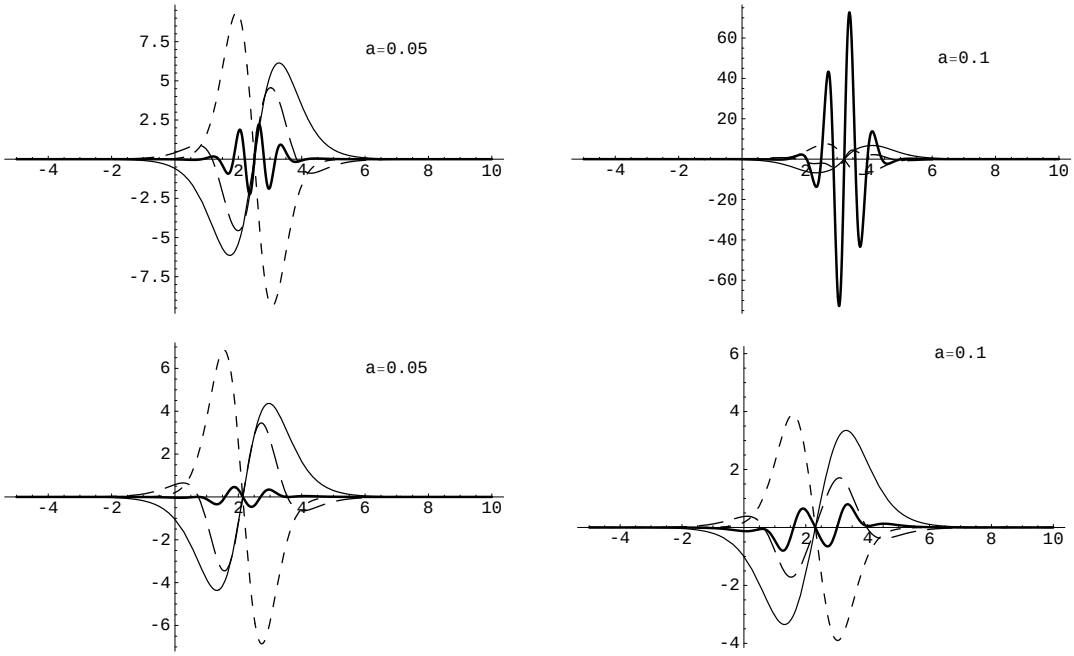

Figure 1. A residual error for the solutions of the second-order KdV equation defined by (24) and (30) in comparison with the terms of the leading order KdV equation (thick solid for the error, thin solid for u_t, short-dashed for $6uu_x$ and long-dashed for u_{xxx}): for the solution (36) obtained by Kodama's transformation ($\epsilon = a$) from the normal form solitary wave (top) and for the solution (37) of the a-equation (bottom). For both solutions $a_1^{(1)} = 1$, $a_4^{(1)} = 30$, $k = 1$.

where a phase constant has been omitted for the sake of simplicity. Applying Kodama's transformation to the solution (35) yields

$$\begin{aligned} u(x,t) = {} & 2k^2\mathrm{sech}^2 Z + \frac{2}{3}\epsilon k^4\left(16a_1^{(1)} - a_4^{(1)} + 4a_1^{(1)}\cosh 2Z\right)\mathrm{sech}^4 Z \\ & + \frac{2}{9}\epsilon^2 k^6\left(282{a_1^{(1)}}^2 - 32a_1^{(1)}a_4^{(1)} + {a_4^{(1)}}^2 + 8a_1^{(1)}\left(19a_1^{(1)} - a_4^{(1)}\right)\cosh 2Z\right. \\ & \left. + 14{a_1^{(1)}}^2\cosh 4Z\right)\mathrm{sech}^6 Z, \qquad Z = k\left[x - \left(4k^2 + 16\epsilon a_1^{(1)}k^4 + \frac{256}{3}\epsilon^2{a_1^{(1)}}^2k^6\right)t\right]. \end{aligned} \tag{36}$$

It is readily checked that this solution satisfies the second-order KdV equation with the coefficients defined by (24) and (30) if the terms of the order ϵ^3 are dropped.

Applying the transformation (29) to the solution (34) yields the corresponding one-soliton solution of the a-equation (27), as follows:

$$u = \frac{12k^2}{3 + 2a\left(a_4^{(1)} - 12a_1^{(1)}\right)k^2 + 3\cosh\left\{2k\left[x - 4k^2\left(t + \frac{aa_1^{(1)}x}{3}\right)\right]\right\}}. \tag{37}$$

The results presented in Fig. 1 and Fig. 2 demonstrate that the solitary wave solutions, constructed from the normal form solitons using a near identity transformation, although being formally asymptotic solutions of the higher-order KdV equations, have very bad approximation properties for the values of the soliton wave-number k larger than one. Even for $k = 1$ an acceptable approximation is achieved only for values of the expansion parameter $\epsilon \sim 10^{-2}$. For $k = 2$ (the second soliton in Fig. 2) the solution is invalid already for $\epsilon < 10^{-2}$.

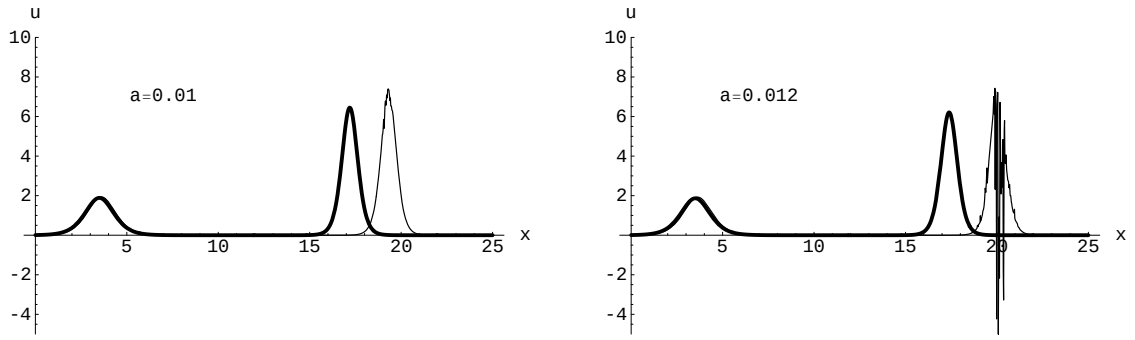

Figure 2. Comparison of different two-soliton ($k_1 = 1$; $k_2 = 2$) solutions of the KdV equation defined by (24) and (30) at $t = 1$: thick solid – the solution of the a-equation; thin solid – the solution obtained by Kodama's transformation ($\epsilon = a$) from the second-order normal form two-soliton solution. For both solutions $a_1^{(1)} = 1$, $a_4^{(1)} = 30$.

It should be noted in this connection that the solutions constructed via Kodama's transformation are ones of an excessive accuracy. For example, in the one-soliton solution (36), one should have taken $Z = k\left[x - \left(4k^2 + 16\epsilon a_1^{(1)} k^4\right) t\right]$ in the terms multiplied by ϵ, and $Z = k\left(x - 4k^2 t\right)$ in the terms multiplied by ϵ^2. However, the approximation properties for such a solution were found to be even worse than those for (36).

We also conclude, based on some features of the solitary wave solutions of the a-equation (not discussed here), that the solutions constructed via Kodama's transformation cannot (because of the k-dependence of their approximation properties) reflect some solution features intrinsic for the original physical system.

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Aspects of Quantum Groups and Integrable Systems

Robert CARROLL

Department of Mathematics, University of Illinois, 1409 W. Green Str., Urbana, USA
E-mail: *rcarroll@math.uiuc.edu*

Some algebro-geometric origins of q-differential equations are examined.

1 Introduction

Many q-differential operators arise in the study of q-special functions, Casimir operators, quantum groups, and representation theory. There are also natural origins via quantum integrable systems and the quantization of classical integrable systems. The latter is often expressed via a q-hierarchy picture akin to the standard Hirota–Lax–Sato formulation and this has many canonical aspects. On the other hand one can produce a great number of q-differential operators by more or less ad hoc manipulation of noncommutative differential calculi or by variations of classical Lie group methods applied to quantum groups. We examine first the hierarchy picture briefly and notice that although the standard methods generalize quite readily the resulting KP or KdV equations for example seem to have an infinite number of terms whereas many "ad hoc" derivations from differential calculi or e.g. Maurer–Cartan arguments have only a finite number of terms and there is no clear way to determine if in fact such equations have any intrinsic meaning. We show that the standard derivation of KdV via vector fields on the unit circle and the smooth dual space of Lax operators can be extended to a q-situation using a q-Virasoro algebra and we produce a corresponding qKdV equation with an infinite number of terms; this seems to be a fairly canonical derivation and we suggest that it could be equivalent to the hierarchy qKdV equation (not yet proved). This approach to KdV via the unit circle with its attendant UrKdV–mKdV equations, Schwarzian derivatives, projective geometry, etc. has another connection to quantum mechanics (QM) via the beautiful equivalence principle of Faraggi–Matone and we use this as a motivational background (cf. [1]). In this connection we remark that in fact KP for example can already be considered as a Moyal quantization of dKP (dispersionless KP) and it is not clear just what role the qKP or qKdV equations can play in quantum mechanics; however for completeness we also indicate a few approaches to Moyal type integrable equations.

2 The equivalence principle

The equivalence principle (EP) of Faraggi–Matone (cf. refs. [1, 2]) is based on the idea that all physical systems can be connected by a coordinate transformation to the free situation with vanishing energy (i.e. all potentials are equivalent under coordinate transformations). This automatically leads to the quantum stationary Hamilton–Jacobi equation (QSHJE) which is a third-order nonlinear differential equation providing a trajectory representation of quantum mechanics (QM). The theory transcends in several respects the Bohm theory and in particular utilizes a Floydian time (cf. ref. [3]) leading to **(A1)** $\dot{q} = p/m_Q \neq p/m$ where **(A2)** $m_Q = m(1 - \partial_E Q)$ is the "quantum mass" and Q the "quantum potential". Thus the EP is reminscient of the Einstein equivalence of relativity theory. This latter served as a midwife to the birth of relativity but was somewhat inaccurate in its original form. It is better put as saying that all laws of physics

should be invariant under general coordinate transformations (cf. ref. [4]). This demands that not only the form but also the content of the equations be unchanged. More precisely the equations should be covariant and all absolute constants in the equations are to be left unchanged (e.g. c, $\hbar$, e, m and $\eta_{\mu\nu}$ = Minkowski tensor). Now for the EP, the classical picture with $S^{cl}(q, Q^0, t)$ the Hamilton principal function ($p = \partial S^{cl}/\partial q$) and P^0, Q^0 playing the role of initial conditions involves the classical HJ equation (CHJE) **(A3)** $H(q,p) = (\partial S^{cl}/\partial q), t) + (\partial S^{cl}/\partial t) = 0$. For time independent V one writes $S^{cl} = S_0^{cl}(q, Q^0) - Et$ and arrives at the classical stationary HJ equation (CSHJE) **(A4)** $(1/2m)(\partial S_0^{cl}/\partial q)^2 + \mathfrak{W} = 0$ where $\mathfrak{W} = V(q) - E$. In the Bohm theory one looked at Schrödinger equations **(A5)** $i\hbar\psi_t = -(\hbar^2/2m)\psi'' + V\psi$ with $\psi = \psi(q)\exp(-iEt/\hbar)$ and **(A6)** $\psi(q) = R(q\exp(i\hat{W}/\hbar)$ leading to

$$\left(\frac{1}{2m}\right)(\hat{W}')^2 + V - E - \frac{\hbar^2 R''}{2mR} = 0, \qquad (R^2\hat{W}')' = 0, \tag{1}$$

where **(A7)** $\hat{Q} = -\hbar^2 R''/2mR$ was called the quantum potential; this can be written in the Schwartzian form **(A8)** $\hat{Q} = (\hbar^2/4m)\{\hat{W}; q\}$ (via $R^2\hat{W}' = c$). Here **(A9)** $\{f; q\} = (f'''/f') - (3/2)(f''/f')^2$. Writing $\mathfrak{W} = V(q) - E$ as in **(A4)** we have the quantum stationary HJ equation (QSHJE) **(A10)** $(1/2m)(\partial\hat{W}'/\partial q)^2 + \mathfrak{W}(q) + \hat{Q}(q) = 0$ ($\equiv \mathfrak{W} = -(\hbar^2/4m)\{\exp(2iS_0/\hbar); q\}$). This was worked out in the Bohm school (without the Schwarzian connections) but **(A6)** is not appropriate for all situations; the Bohm theory is incomplete and can lead to incorrect predictions. The technique of Faraggi–Matone (FM) is completely general and with only the EP as guide one exploits the relations between Schwarzians, Legendre duality, and the geometry of a second-order differential operator $D_x^2 + V(x)$ (Möbius transformations play an important role here) to arrive at the QSHJE in the form

$$\frac{1}{2m}\left(\frac{\partial S_0^v(q^v)}{\partial q^v}\right)^2 + \mathfrak{W}(q^v) + \mathfrak{Q}^v(q^v) = 0, \tag{2}$$

where $v : q \to q^v$ represents an arbitrary locally invertible coordinate transformation. Note in this direction for example that the Schwarzian derivative of the the ratio of two linearly independent elements in $ker\,(D_x^2 + V(x))$ is twice $V(x)$. In particular given an arbitrary system with coordinate q and reduced action $S_0(q)$ the system with coordinate q^0 corresponding to $V - E = 0$ involves **(A11)** $\mathfrak{W}(q) = (q^0; q)$ where (q^0, q) is a cocycle term which has the form **(A12)** $(q^a; q^b) = -(\hbar^2/4m)\{q^a; q^b\}$. In fact it can be said that the essence of the EP is the cocycle condition **(A13)** $(q^a; q^c) = (\partial_{q^c} q^b)^2[(q^a; q^b) - (q^c; q^b)]$.

In addition FM developed a theory of (x, ψ) duality (cf. ref. [1]) which related the space coordinate and the wave function via a prepotential (free energy) in the form $\mathfrak{F} = (1/2)\psi\bar{\psi} + iX/\epsilon$ for example. A number of interesting philosophical points arise (e.g. the emergence of space from the wave function) and we connected this to various features of dispersionless KdV in refs. [5, 6] in a sort of extended WKB spirit. One should note here that although a form **(A6)** is not generally appropriate it is correct when one is dealing with two independent solutions of the Schrödinger equation ψ and $\bar{\psi}$ which are not proportional. In this context we utilized some interplay between various geometric properties of KdV which involve the Lax operator $L^2 = D_x^2 + V(x)$ and of course this is all related to Schwartzians, Virasoro algebras, and vector fields on S^1 (see e.g. refs. [7–10]. Thus the simple presence of the Schrödinger equation (SE) in QM automatically incorporates a host of geometrical properties of $D_x = d/dx$ and the circle S^1. In fact since the FM theory exhibits the fundamental nature of the SE via its geometrical properties connected to the QSHJE one could speculate about trivializing QM to a study of S^1 and ∂_x!

In any event KdV in its geometrical glory is important and we want to look at qKdV and qKP. A main theme here is to understand the relation of the qKdV hierarchy picture in terms of a geometrical origin of qKdV type equations. The hierarchy picture is described below. We also

give various derivations of qKdv and qKP type equations from various geometrical or quantum group points of view (most of which are not equivalent to the hierarchy pictures). However we do find a qKdV equation from a q-Virasoro context which could agree with the hierarchy picture (see Section 6). Even so it is not entirely obvious that the hierarchy picture is the only legitimate object of study – although the analytic approach of ref. [11] and the algebraic (oper) approach of ref. [12] do point in this direction.

3 KP and KDV hierarchies

For classical KP and KdV one has (cf. refs. [5, 6, 13, 14]

$$L = \partial + \sum_1^\infty u_{n+1}(t)\partial^{-n}, \qquad W = 1 + \sum_1^\infty w_n\partial^{-n}, \qquad L = W\partial W^{-1} \tag{3}$$

with **(A14)** $\psi = We^\xi = W(\lambda)e^\xi$; $\xi = \sum_1^\infty t_n\lambda^n$; $L\psi = -\lambda\psi$; $\partial_m\psi = B_m\psi = L_+^m\psi$ and **(A15)** $\partial_m L = [B_m, L]$; $\partial_n B_m - \partial_m B_n = [B_n, B_m]$; $\partial_n W = -(L_-^n)W$. Recall **(A16)** $\psi(t,\lambda) = \frac{X(\lambda)\tau}{\tau} = \frac{e^\xi\tau\left(t_j - \frac{1}{j\lambda^j}\right)}{\tau(t)} = \frac{e^\xi\tau_-}{\tau}$ where τ is the famous tau function with **(A17)** $u = \partial_x^2\log(\tau)$ with $u \sim u_2$ in 3. Typical forms for KP and KdV are **(A18)** $u_t - u_{xxx} + 6uu_x - 3\partial^{-1}u_{yy} = 0$ and $u_t - u_{xxx} + 6uu_x = 0$. For KdV one has $L_+^2 = L^2$ and the hierarchy picture is much simplified; we discuss this below in the context of qKdV.

4 *Q*-hierarchies

For qKP we go to refs. [11, 12, 14–23]; the hierarchy picture is straightforward but the resulting equations are much more complicated. Thus one writes e.g. **(A19)** $\partial_q f(x) = [f(qx) - f(x)]/(q-1)x$ and $Df(x) = f(qx)$. Then in the hierarchy constructions x and $t = t_1$ are both used in the first variable position with

$$\tau_q(x,t) = \tau(t_i + c(x)_i), \qquad c(x) = \left(\frac{(1-q)x}{1-q}, \ldots, \frac{(1-q)^n x^n}{n(1-q^n)}, \ldots\right), \tag{4}$$

where τ is an ordinary tau function for KP. One takes then ($\partial_1 \sim \partial/\partial t_1, \ldots, \partial_n \sim \partial/\partial t_n$)

$$L_q = \partial_q + \sum_0^\infty a_n(t)\partial_q^{-n}, \qquad \partial_n L_q = [(L_q^n)_+, L_q] = [B_n^q, L_q] \tag{5}$$

(note $a_0 \neq 0$ appears here and $u \sim a_1$). Further ($\xi = \sum_1^\infty t_k\lambda^k$)

$$\psi_q = \frac{\tau_q(x, t - [\lambda^{-1}])}{\tau_q(x,t)} e_q(x\lambda)\exp(\xi), \qquad (q;q)_k = (1-q)\cdots(1-q^k), \tag{6}$$

$$\exp_q(x) = \sum_1^\infty \frac{(1-q)^k x^k}{(q;q)_k} = \exp\left(\sum_1^\infty \frac{1-q)^k x^k}{k(1-q^k)}\right), \qquad \partial_q e_q(x\lambda) = \lambda e_q(x),$$

where $[\lambda^{-1}] = (1/n\lambda^n)$. Note also **(A20)** $\psi_q = W_q e_q(x\lambda)e^\xi$; $L_q\psi_q = \lambda\psi_q$ with

$$L_q = W_q\partial_q W_q^{-1}, \qquad W_q = 1 + \sum_1^\infty \tilde{w}_j\partial_q^{-j}, \qquad \partial_j W_q = -(L_q^j)_- W_q. \tag{7}$$

Next note $e_q(x\lambda)^{-1} = e_{1/q}(-x\lambda)$ and the adjoint wave function ψ_q^* is determined via

$$\psi_q^* = (W^*)^{-1}_{x/q} \exp_{1/q}(-x\lambda)e^{\xi}, \qquad L^*_{x/q}\psi_q^* = \lambda\psi_q^*,$$
$$\psi_q^* = \frac{\tau_q(x, t + [\lambda^{-1}])}{\tau_q(x,t)} \exp_{1/q}(-x\lambda)e^{-\xi}, \tag{8}$$

where $V = \sum v_i\partial_q^i \sim V^* = \sum(\partial_q^*)^i v_i$ with $\partial_q^* = -q^{-1}\partial_{1/q}$ and $V_{x/q} \sim \sum v_i(x/q)q^i\partial_q^i$. Recall also the Schur polynomials

$$\sum p_n(t)\lambda^n = \exp(\xi(t,\lambda)), \qquad \sum \tilde{p}_n(x,t)\lambda^n = \exp_q(x\lambda)e^{\xi(t,\lambda)},$$
$$\tilde{p}_k(x,t) = p_k(t + c(x)). \tag{9}$$

Next the classical Hirota bilinear identity is

$$\oint \psi^*(t,\lambda)\psi(t',\lambda)d\lambda = 0 \quad \Rightarrow \quad \oint \tau(t + y + [\lambda^{-1}])\tau(t - y - [\lambda^{-1}])e^{-2\sum y_i\lambda^i} = 0 \tag{10}$$

and this leads to classical Hirota equations **(A21)** $\partial_1\partial_n\tau\cdot\tau = 2p_{n+1}(\tilde{\partial})\tau\cdot\tau$ (Hirota notation – also $\tilde{\partial} = (\partial_1, (1/2)\partial_2, \ldots)$). One can produce analogous formulas in q-theory but their usefulness is limited and not even clear (cf. ref. [14] for an extensive discussion). The problem lies in the fact that $u = \partial^2 \log\tau$ does not generalize but instead

$$u = -(q-1)x\partial_q\left(\frac{p_2(\tilde{\partial})\tau_q}{\tau_q}\right) + \frac{\partial_q\tau_q}{\tau_q}(q-1)x\partial_q\left(\frac{\partial_1\tau_q}{\tau_1}\right) + \partial_q\left(\frac{\partial_q\tau_q}{\tau_q}\right). \tag{11}$$

The classical Hirota equations are compatible with and lead to formulas involving $\log\tau$ (and hence u) but in the q-situation (11) makes this impossible. In fact I have never seen a qKP equation analogous to **(A18)** for example.

The problems are more easily seen with qKdV and there one has (recall $Df(x) = f(qx)$)

$$L_q^2 = \partial_q^2 + (q-1)xu\partial_q + u, \qquad u = \partial_q\partial_1\log[\tau_q(x,t)D\tau_q(x,t)],$$
$$L_q = \partial_q + s_0 + s_1\partial_q^{-1} + \cdots. \tag{12}$$

Considerable calculation leads to ($u_1 = (q-1)xu$)

$$s_0 = (q-1)x\partial_1\partial_q\log\tau_q, \qquad \partial_t u = (\partial_q^3 u) + w_2(\partial_q^2 u) + w_1(\partial_q u) - [(\partial_q^2 w_0) + u_1(\partial_q w_0)], \tag{13}$$
$$w_2 = D^2 s_0 + u_1 = D^2 s_0 + Ds_0 + s_0, \tag{14}$$
$$w_1 = (q+1)(D\partial_q s_0) + \tau^2 s_1 + [(Ds_0) + s_0](Ds_0) + u,$$
$$w_0 = \partial_q^2 s_0 + (q+1)(D\partial_q s_1) + u_1\partial_q s_0 + u_1(Ds_1) + us_0 + D^2 s_2$$

and the determination of the s_i requires infinite series calculations based on formulas like

$$s_1 + Ds_1 = u - \partial_q s_0 - s_0^2 = f \quad \Rightarrow \quad s_1(x) = \sum_0^\infty (-1)^n f(q^n x). \tag{15}$$

Similarly **(A22)** $Ds_2 + s_2 = -\partial_q s_1 - s_0 s_1 - s_1 D^{-1} s_0$, etc. This seems to suggest that an explicit form of qKdV (as in **(A18)**) arising in the hierarchy picture with coefficients specified in terms of u (or s_0) will involve an infinite number of terms.

5 Origins of Q-equations

5.1 Linear equations

In refs. [14–16] we sketched a number of sources for linear q-differential equations in various contexts (see the references there for details and cf. also refs. [24, 25]). This includes many relations involving q-special functions, Casimir operators, quantum groups, and representation theory with linear q-differential equations arising naturally. A good source of general information with Lie theoretic background is refs. [26, 27]. There is also a development involving special functions and q-special functions where tau functions arise as generating functions for matrix elements in general group representations (cf. ref. [15] and references there).

Remark 1. The development in ref. [26] for example emphasizes the ideas of intertwining and group invariant differential operators in a general Lie theoretic context with representations involving Verma modules (cf. also ref. [27] and references there). One arrives e.g. at a Maxwell hierarchy and its quantum group version in the context of a particular q-Minkowski space-time. The spirit is that of QG symmetries described via invariant q-differential operators and here the equations are also q-conformally invariant. Other work involving Dirac equations is described in ref. [15] for example and the differential operators involved seem to be primarily linear. However the treatment in ref. [27] provides interesting nonlinear differential operators in the context of multilinear intertwining differential operators. Bilinear and trilinear formulas for example are written out for $\mathfrak{sl}(2)$ and there is an interesting connection to KdV. Thus let G be a Lie group with representations T, T' in spaces C, C'. An intertwining operator $\mathfrak{I}$ is a continuous linear map **(A23)** $\mathfrak{I} : C \to C' : f \to j$ such that **(A24)** $\mathfrak{I} \circ T(g) = T'(g) \circ \mathfrak{I}$. The equation **(A25)** $\mathfrak{I}f = j$ is then a G-invariant equation. Omitting details one defines multilinear intertwining differential operators via

$$ {}_k\mathfrak{I} : f \otimes \cdots \otimes f \to j;\ {}_k\mathfrak{I} \circ T(g) \otimes \cdots \otimes T(g) = T'(g) \circ {}_k\mathfrak{I} \tag{16} $$

and arrives at generalized Schwarzians **(A26)** $\widetilde{Sch}_n(\phi) = [1/(\phi')^2]_2\mathfrak{I}^0_{n\alpha}(\phi)$. Recalling that the KdV equation can be written in the Krichever–Novikov form **(A27)** $\partial_t f + \widetilde{Sch}_4(f)f' = 0$ (note $\widetilde{Sch}_4(f) = S(f)$ is the Schwarzian derivative) one can pass to the standard KdV form **(A28)** $\partial_t u + u''' - 6uu' = 0$ where the substitution $u = -(1/2)\widetilde{Sch}_4(f)$ is used. It is conjectured in ref. [27] that the equations **(A29)** $\partial_t f + \widetilde{Sch}_n(f)f' = 0$ ($n \in 2\mathbb{N}+2$) are integrable, and if so they should coincide with the KdV hierarchy. One expects that a similar analysis in the q-context would be of great interest in understanding qKdV.

5.2 Differential calculi

When we come to nonlinear equations such as KP or KdV there are a number of classical derivations of intrinsic geometrical or algebraic interest which should morally have a q-version. Moreover one would expect the q-versions to bear some natural or canonical relation to the hierarchy version described earlier. In refs. [14–16] we examined some such derivations but without being able to establish a clear connection to the hierarchy picture. The constructions however seem interesting enough and well modeled on meaningful classical situations so some further study is indicated. First in a somewhat experimental manner consider a differential calculus approach following refs. [14–16, 28]. Thus

Example 1. Consider a calculus based on **(A)** $dt^2 = dx^2 = dxdt + dtdx = 0$ **(B)** $[dt, t] = [dx, t] = [dt, x] = 0$ and $[dx, x] = \eta dt$. Assuming the Leibnitz rule $d(fg) = (df)g + f(dg)$ for functions and $d^2 = 0$ one obtains **(A30)** $df = f_x dx + (f_t + (1/2)\eta f_{xx})dt$. For a connection $A = wdt + udx$ the zero curvature condition $F = dA + A^2 = 0$ leads to **(A31)** $(u_t - w_x + (\eta/2)u_{xx} + \eta u u_x = 0$ which for $w_x = 0$ is a form of Burger's equation.

Example 2. Next consider **(A)** $[dt,t]=[dx,t]=[dt,x]=[dy,t]=[dt,y]=[dy,y]=0$ with **(B)** $[dx,x]=2bdy$ and $[dx,y]=[dy,x]=3adt$. Further **(C)** $dt^2=dy^2=dtdx+dxdt=dydt+dtdy=dydx+dxdy=0$. Then **(A32)** $df=f_xdx+(f_y+bf_{xx})dy+(f_t+3af_{xy}+abf_{xxx}dt)$. For $A=vdx+wdt+udy$ one finds that $dA+A^2=F=0$ implies

$$u_x=v_y+bv_{xx}+2bvv_x,\qquad w_x=3av_{xy}+abv_{xxx}+3auv_x+3av(v_y+bv_{xx}),\tag{17}$$
$$w_x+bw_{xx}=u_t+3au_{xy}+abu_{xxx}+3auu_x-v[2bw_x-3a(u_y+bu_{xx})].$$

Taking e.g. $w_x=(3a/2b)u_y+(3a/2)u_{xx}$ in the last equation to decouple one arrives at **(A33)** $\partial_x(u_t-(ab/2)u_{xxx}+3auu_x)=(3a/2b)u_{yy}$; for suitable a, b this is KP. If the equation is independent of y we obtain a version of KdV.

It is surprisingly difficult to convert these examples into meaningful q-calculus equations and in that spirit for guidance we were motivated to develop many formulas concerning qKP, qKdV, etc. One has e.g. a first-order differential calculus (FODC) Γ_+ from ref. [25] on a quantum plane or Manin plane (cf. also ref. [15]); this is based on $xp=qpx$ with standard formulas as in ref. [25]. In this FODC the partial derivatives ∂_i of Γ_+ act on $\mathfrak{A}=$ formal power series with x, p ordering, via ($\partial_p x^n=q^nx^n\partial_p$ and $\partial_x p^n=q^np^n\partial_x$)

$$\partial_x(f(x)h(p))=(D^x_{q^2}f(x))h(p),\qquad \partial_p(f(x)h(p))=(T_qf(x))(D^p_{q^2}h(p)),\tag{18}$$

$$\partial_x(x^n)=D_{q^2}x^n=[[n]]_{q^2}x^{n-1},\qquad [[n]]_{q^2}=\frac{q^{2n}-1}{q^2-1},\qquad \partial_p p^n=[[n]]_{q^2}p^{n-1}.\tag{19}$$

Example 3. The q-plane itself doesn't seem immediately fruitful here so consider the generalized q-plane with an algebra generated by x, y, x^{-1}, y^{-1} where $xy=qyx$, $xdx=qdxx$, $ydx=q^{-1}dxy$, $xdy=qdyx$, and $ydy=q^{-1}dyy$. Also from $qdyx=dxy$ we have $qdydx=dxdy$ and a little calculation yields **(A34)** $dx^n=[(1-q^{-n})/(1-q^{-1})]x^{n-1}dx$ with $dy^m=[(1-q^m)/(1-q)]y^{m-1}dy$. Working from $f=\sum a_{nm}x^ny^m$ one obtains then (note $dxy^m=q^my^mdx$)

$$df=D_yD^x_{q^{-1}}fdx+D^y_qfdy.\tag{20}$$

Set then $A=wdy+udx$ with $dA=D_yD^x_{q^{-1}}wdxdy+D^y_qudydx$ and, noting that $dyx^n=q^{-n}x^ndy$, $dyy^m=q^my^mdy$, and $dxy^m=q^my^mdx$ with $dxx^n=q^{-n}x^ndx$ one gets $dyw=D_x^{-1}D_ywdy$ and $dxu=D_x^{-1}D_yudx$ leading to $A^2=wD_x^{-1}D_yudydx+uD_x^{-1}D_ywdxdy$; and

$$dA+A^2=0=qD_yD^x_{q^{-1}}w+D^y_qu+wD_x^{-1}D_yu+quD_x^{-1}D_yw.\tag{21}$$

Setting then e.g. $qw=D_y^{-1}D^x_{q^{-1}}u$ one gets

$$D^y_qu+(D^x_{q^{-1}})^2u+q^{-1}(D_x^{-1}D_yu)(D_y^{-1}D^x_{q^{-1}}u)+uD_x^{-1}D^x_{q^{-1}}u=0.\tag{22}$$

For $q\to 1$ we have **(A35)** $u_y+u_{xx}+2uu_x=0$ so this appears to be an exact q-form of Burger's equation.

Example 4. We will try now a somewhat different approach. First we take q-derivatives only in x, as in the case of qKP for example and we know from Example 2 that **(A32)** leads to interesting consequences so begin with an assumption ($f_y=\partial f/\partial y$, etc.)

$$df=D^x_qfdx+\left(f_y+b(D^x_q)^2f\right)dy+\left(f_t+3a\partial_yD^x_qf+ab(D^x_q)^3f\right)dt.\tag{23}$$

Then we can determine what elementary commutation relations between the variables are consistent with (23). This is rather ad hoc but we stipulate x, y, t ordering and then there are relations

$$dxx=qxdx+b[2]_qdy,\qquad dyx=q^2xdy+a[3]_qdt,$$
$$dtx=q^3xdt,\qquad dxy=ydx+3adt\tag{24}$$

along with

$$[dt,y]=[dy,y]=[dy,t]=[dx,t]=[dt,t]=[dx,y]=0 \tag{25}$$

which are determined by (23). The underlying structure for x,y,t is not visible from (23) but (24)–(25) do lead to (23) and whatever zero curvature equations subsequently arise (see ref. [16] for details). Returning now to (23) it remains to check now the zero curvature equation arising in the spirit of Example 2 (some extra factors and terms will arise via noncommutativity). Thus assume first $dx^2=dy^2=dt^2=0$ and take $A=vdx+wdt+udy$; after some computation modeled on Example 2 one arrives at a rather clumsy q-version of KP. For y independent u we obtain

$$\begin{aligned} &u_t+ab(D_q^x)^3u-\frac{[3]_qab}{[2]_q}D_q^xD_x(D_q^x)^2u+[3]_qauD_x^2D_q^xu\\ &\quad=\frac{[3]_qa}{[2]_qb}\left(D_xD_q^xuD_x^3u-uD_x^3D_q^xu\right) \end{aligned} \tag{26}$$

which is a quantum KdV type equation.

5.3 Geometry

We indicate two "geometrical" contexts in a classical vein and subsequently give "quantum" versions of these.

Example 5. One can devise a procedure directly from ref. [29]. Thus look at $SL(2,\mathbb{R})$ with matrices **(A36)** $X=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $ad-bc=1$. The right invariant Maurer–Cartan (MC) form is **(A37)** $\omega=dXX^{-1}=(w_j^i)$ $(i,j=1,2)$ where $\omega_1^1+\omega_2^2=0$. The structure equation of $SL(2,\mathbb{R})$ or MC equation is **(A38)** $d\omega=\omega\wedge\omega$ or explicitly **(A39)** $d\omega_1^1=\omega_1^2\wedge\omega_2^1$; $d\omega_1^2=2\omega_1^1\wedge\omega_1^2$; $d\omega_2^1=2\omega_2^1\wedge\omega_1^1$. Now let U be a neighborhood in the (x,t) plane and consider a smooth map $f:U\to SL(2,\mathbb{R})$. The pullback of the MC form can be written as **(A40)** $\omega_1^1\sim\eta dx+Adt$; $\omega_1^2\sim Qdx+Bdt$; $\omega_2^1\sim rdx+Cdt$ with coefficient functions of x, t. The equations **(A39)** become **(A41)** (i) $-\eta_t+A_x-QC+rB=0$, (ii) $-Q_t+B_x-2\eta B+2QA=0$, and (iii) $-r_t+C_x-2rA+2\eta C=0$. Take $r=1$ with η independent of (x,t) and set $Q=u(x,t)$. Then from (i) and (iii) one gets **(A42)** $A=\eta C+\frac{1}{2}C_x$; $B=uC-\eta C_x-\frac{1}{2}C_{xx}$. Putting this in the (ii) above yields $u_t=K(u)$ where **(A43)** $K(u)=u_xC+2uC_x+2\eta^2C_x-\frac{1}{2}C_{xxx}$. In the special case $C=\eta^2-(1/2)u$ one gets the KdV equation **(A44)** $u_t=\frac{1}{4}u_{xxx}-\frac{3}{2}uu_x$.

Example 6. Following ref. [30], let $\mathrm{Vec}\,(S^1)$ denote the Lie algebra of smooth vector fields on S^1 and then the Virasoro algebra is $\mathrm{Vir}=\mathrm{Vec}(S^1)\oplus\mathbb{R}=\mathfrak{W}\oplus\mathbb{R}$ with (note the minus sign convention involving $f'g-fg'$)

$$[(f(x)\partial_x,a),(g(x)\partial_x,b)]=\left((f'g-fg')\partial_x,\int_{S^1}f'g''dx\right) \tag{27}$$

($\mathfrak{W}\sim$ Witt algebra). Here $\int_{S^1}f'g''dx$ is called the Gelfand–Fuks cocycle, where a cocycle on a Lie algebra $\mathfrak{g}$ is a bilinear skew symmetric form $c(\cdot,\cdot)$ satisfying **(A45)** $\sum c([f,g],h)=0$ over cyclic permutations of f, g, h. This means that $\hat{\mathfrak{g}}=\mathfrak{g}\oplus\mathbb{R}$ (central extension) with commutator $[(f,a),(g,b)]=([f,g],c(f,g))$ satisfies the Jacobi identity of a Lie algebra. Now the Euler equation corresponding to geodesic flow is a 1-parameter family of KdV equations. To see how this arises consider **(A46)** $\mathrm{Vir}^*=\{(u(x)dx^2,c);\ u \text{ smooth on } S^1 \text{ and } c\in\mathbb{R}\}$. Then **(A47)** $\langle(v(x)\partial_x,a),(u(x)dx^2,c)\rangle=\int_{S^1}v(x)u(x)dx+ac$. The coadjoint action of $(f\partial_x,a)\in\mathrm{Vir}$ on $(udx^2,c)\in\mathrm{Vir}^*$ is $(\mathrm{ad}_v^*:\mathfrak{g}^*\to\mathfrak{g}^*,\ \mathrm{ad}_v^*w(u)=w(ad_vu))$ **(A48)** $\mathrm{ad}^*_{(f\partial_x,a)}(udx^2,c)=$

$(2f'u + fu' + cf''')dx^2, 0)$ which arises from the identity **(A49)** $\langle[(f\partial_x, a), (g\partial_x, b)], (udx^2, c)\rangle = \langle(g\partial_x, b), \mathrm{ad}^*_{(f\partial_x,a)}(udx^2, c)\rangle$. Now for S^1 there are no boundary terms in integration (for single valued functions) so, integrating by parts,

$$\int_{S^1} g(2f'u + fu' + cf''')dx = \int_{S^1} [u(gf' - fg') + cf'g'']dx. \tag{28}$$

Now a function H on $\mathfrak{g} = \mathrm{Vir}$ determines a tautological inertia operator $A : \mathrm{Vir} \to \mathrm{Vir}^* : (u\partial_x, c) \to (udx^2, c)$ and hence a quadratic Hamiltonian on Vir^* via

$$H(udx^2, c) = \frac{1}{2}\langle(u\partial_x, c), (udx^2, c)\rangle = \frac{1}{2}\langle(u\partial_x, c), A(u\partial_x, c)\rangle. \tag{29}$$

Following ref. [30] the corresponding Euler equation is $\dot{m} = -\mathrm{ad}^*_{A^{-1}m}m$ $(m \in \hat{\mathfrak{g}})$ which here takes the form **(A50)** $\partial_t(udx^2, c) = -\mathrm{ad}^*_{A^{-1}(udx^2,c)}(udx^2, c)$. Using **(A51)** $(f\partial_x, a) = A^{-1}(udx^2, c) = (u\partial_x, c)$ this becomes an equation **(A52)** $\partial_t u = -2u'u - uu' - cu''' = -3uu' - cu'''$ where c is independent of time.

6 Q-versions

6.1 Q-Virasoro

We recall first from refs. [16, 31] the following information regarding q-Virasoro constructions. Thus work on S^1 with $(q \neq 0, \pm 1)$

$$\partial_q z = \frac{q^m z^m - q^{-m} z^m}{(q - q^{-1})z} = z^{m-1}[m], \qquad [m] = \frac{q^m - q^{-m}}{q - q^{-1}}. \tag{30}$$

We adapt the formalism of ref. [31] as follows. Let $D_n = -z^{n+1}\partial$ with $\partial : z^m \to q^m[m]z^{m-1}$ so $\partial \sim \partial_q\tau$ where $\tau f(z) = f(qz)$. Generally we will think of $z = e^{i\theta} \in S^1$ so $(1/2\pi i)\int_{S^1} z^n dz = (1/2\pi)\int z^{n+1}d\theta = \delta_{(-1,0)}$ which will be written as **(A52)** $\int z^n = \delta_{(-1,0)}$. Write also **(A53)** $\ell_n \sim D_n = -z^{n+1}\partial_q\tau$ and it is known that q-brackets are needed now where **(A54)** $[\ell_m, \ell_n]_q = q^{m-n}\ell_m\ell_n - q^{n-m}\ell_n\ell_m = [m-n]\ell_{m+n}$. For a central term in a putative Vir_q one wants (cf. refs. [16, 31]) a formula **(A55)** $c[m+1][m][m-1]\delta_{m+m,0}$ (see below for an optimal term). First we want to formulate the q-bracket in terms of vector fields as follows (the central term will be added later). This can be done as a direct calculation using the basic definition of ∂ above (cf. also (37) below). Thus

$$[z^n\partial, z^m\partial]_q \sim q^{n-m}z^n\partial(z^m\partial) - q^{m-n}z^m\partial(z^n\partial) = [n-m](-z^{m+n-1}\partial). \tag{31}$$

Let now $v \sim \sum a_n z^n$ and $w \sim \sum b_m z^m$; then we define a bracket in $Vec(S^1)$ via **(A56)** $[v\partial, w\partial]_q = -\sum a_n b_m [n-m]z^{m+n-1}\partial$. We defined a bracket of vector fields in ref. [16] so that from **(A56)** there resulted a correspondence **(A57)** $v'w - vw' \sim -[v\partial_x, w\partial_x] \sim -[v\partial, w\partial]_q = -\{(\tau v)(\partial_q w) - (\tau w)(\partial_q v)\}\tau$. This dangling τ creates some complications and is removed below.

Remark 2. In ref. [31] one defines the q-analogue of the enveloping algebra of the Witt algebra $\mathfrak{W}$ as the associative algebra $\mathfrak{U}_q(\mathfrak{W})$ having generators ℓ_m $(m \in \mathbb{Z})$ and relations **(A54)**. The q-deformed Virasoro algebra is defined as the associative algebra $\mathfrak{U}_q(\mathrm{Vir})$ having generators ℓ_m $(m \in \mathbb{Z})$ and relations $(q \neq$ root of unity)

$$q^{m-n}\ell_m\ell_n - q^{n-m}\ell_n\ell_m = [m-n]\ell_{m+n} + \delta_{m+n,0}\frac{[m+1][m][m-1]}{[2][3]\langle m\rangle}\hat{c}, \tag{32}$$

where $\langle m\rangle = q^m + q^{-m}$ and $\hat{c}\ell_m = q^{2m}\ell_m\hat{c}$ (thus $\hat{c}$ is an operator which we examine below and we refer to refs. [16, 31] for the central term). Then $\mathfrak{U}_q(\mathrm{Vir}) \sim \mathrm{Vir}_q$ is a $\mathbb{Z}$ graded algebra

with $\deg(\ell_m) = m$ and $\deg(\hat{c}) = 0$. One also introduces in ref. [31] a larger algebra $\mathfrak{U}(V_q) =$ associative algebra generated by $J^{\pm 1}$, $\hat{c}$, d_m $(m \in \mathbb{Z})$ with relations

$$JJ^{-1} = J^{-1}J = 1, \qquad Jd_mJ^{-1} = q^m d_m, \qquad \hat{c}J = J\hat{c}, \qquad \hat{c}d_m = q^m d_m\hat{c}, \tag{33}$$
$$q^m d_m d_n J - q^n d_n d_m J = [m-n]d_{m+n} + \delta_{m+n,0}\frac{[m+1][m][m-1]}{[2][3]\langle m\rangle}\hat{c}.$$

The subalgebra of $\mathfrak{U}(V_q)$ generated by $\ell'_m = d_m J$ and $\hat{c}' = \hat{c}J$ $(m \in \mathbb{Z})$ is the same as $\mathfrak{U}_q(\mathrm{Vir})$. It is stated in ref. [31] that V_q is the universal quantum central extension of $\mathfrak{W}_q$ and thus (33) is better adapted for optimal algebraic and geometric meaning; it is this aspect which we emphasize below.

Now we mimic the framework of Example 6 and it is interesting to note that an ordinary integral $\int_{S^1}$ will suffice. One does not need a Jackson type integral in order to deal with integration by parts. Thus we observe that **(A58)** $\int_{S^1} f = \int \sum f_n z^n = f_{-1}$; $\int \partial_q f = (q - q^{-1})^{-1}(f_0 - f_0) = 0$. Since $\partial_q(fh) = (\tau f)(\partial_q h) + (\partial_q f)(\tau^{-1}h)$ we have an integration by parts formula

$$\int (\tau f)(\partial_q h) = -\int (\partial_q f)(\tau^{-1}h) \Rightarrow \int f\partial_q(\tau h) = -\int \partial_q(\tau^{-1}f)h. \tag{34}$$

This can be written as (recall $\partial \sim \partial_q\tau$) **(A59)** $\int f\partial h = -\int h\hat{\partial}f$ for $\hat{\partial} = \partial_q\tau^{-1}$. Now we think of $\mathfrak{U}_q(\mathrm{Vir})$ with elements $(f\partial, a)$ satisfying **(A60)** $[(f\partial, a), (g\partial, b)] = (-[f\partial, g\partial]_q\partial, \psi(f\partial, g\partial))$. The central term could be defined tentatively via e.g. **(A61)** $\int(\tau\partial^3 f)(\tau g)\hat{c} = \psi(f\partial, g\partial)$ (where one notes that **(A62)** $\psi(f\partial, g\partial) = q^{-1}\int g\partial^3 f\hat{c}$). We will want to put the central operator $\hat{c}$ into the integral, acting on f, and will see below that $\hat{c} \sim \tau^2$ for example and $\tau^2 F(z) = F(q^2 z)\tau^2$ so it eventually automatically passes to the right in our qKdV type equations. Hence for the moment think of $\hat{c} = \tau^2$ put into **(A61)** via e.g. $\tau\partial^3\tau^{-2}\hat{c}f \equiv \tau\partial^3 f$ and ignored at the end except when exhibiting formulas like (32) on generators (see also remarks below).

We recall now from ref. [31] (second paper)

$$[d_m, d_n] = Jd_mJ^{-1}d_nJ - Jd_nJ^{-1}d_mJ = q^{-n}Jd_md_n - q^{-m}Jd_nd_m. \tag{35}$$

Further $\hat{c}d_m = q^m d_m\hat{c}$ suggests $\hat{c} = \tau$ here and d_m is being used as $\ell_m\tau^{-1}$. Dropping the minus sign momentarily, from $\ell_m = z^{m+1}\partial_q\tau$ we get then **(A63)** $d_m = z^{m+1}\partial_q$. Then, using **(A64)** $\partial_q\tau = q\tau\partial_q$, one obtains $\tau d_m = \tau(z^{m+1}\partial_q) = q^m z^{m+1}\partial_q\tau$ and $\tau^{-1}d_m = q^{-m}d_m\tau^{-1}$. In addition $Jd_mJ^{-1} = q^m d_m$ corresponds to $Jd_m = q^m d_m J$ so we identify $J = \tau$. Writing $\ell_m = d_m J = d_m\tau$ we can also easily see that the brackets $[d_m, d_n]$ above are exactly the q-brackets **(A65)** $[\ell_m, \ell_n]_q = q^m\ell_m\ell_n - q^n\ell_n\ell_m$.

Now in ref. [31] (second paper) a Jacobi type identity is used involving an operator $\sigma(x) = (1/2)(\tau + \tau^{-1})(x)$ for $x \in \oplus\mathbb{C}d_n$. This seems to be better phrased in terms of an operator **(A66)** $\Gamma(d_p) = \langle p\rangle d_p$ which avoids the need to carry τ around to other terms. Then we can check that the rule in (33) rewritten as **(A67)** $[d_m, d_n] = [m-n]d_{m+n} + \gamma_m\delta_{m+n,0}\hat{c}$ will yield

$$[[d_m, d_n], \Gamma(d_p)] + [[d_n, d_p], \Gamma(d_m)] + [[d_p, d_m], \Gamma(d_n)] = \Xi_{m,n,p} = 0. \tag{36}$$

This is based on two identities; one is **(A68)** $[m-n][m+n-p]\langle p\rangle + [n-p][n+p-m]\langle m\rangle + [p-m][p+m-n]\langle n\rangle = 0$ and the second is

$$[p+1][p][p-1][m-n] + [m+1][m][m-1][n-p] + [n+1][n][n-1][p-m] = 0 \qquad (m+n+p=0). \tag{37}$$

The proofs are essentially straightforward (cf. ref. [32] for details). Thus V_q will be a genuine central extension of $\mathfrak{W}_q$, with a reasonable Jacobi identity (36).

Remark 3. Now in 31–**(A57)** we recall $\ell_m \sim -z^{m+1}\partial = -z^{m+1}\partial_q\tau$ and a d_m formulation would drop the τ. Thus work with ∂_q instead of $\partial = \partial_q\tau$ with **(A69)** $\partial_q z^{p+1} = q^{p+1}z^{p+1}\partial_q + [p+1]z^p\tau^{-1}$ based on $\partial_q f = (\tau f)\partial_q + (\partial_q f)\tau^{-1}$. Then

$$\begin{aligned}[d_m, d_n] &= [z^{m+1}\partial_q, z^{n+1}\partial_q] \\ &= q^m d_m d_n\tau - q^n d_n d_m\tau = [n-m]z^{n+m+1}\partial_q = [m-n]d_{m+n}. \end{aligned} \tag{38}$$

In this context the $\delta_{m+n,0}$ term does not arise. Note here $\tau^{-1}\partial_q = q\partial_q\tau^{-1}$ and **(A70)** $q^{m+1}[n+1] - q^{n+1}[m+1] = [n-m]$. Let now $v \sim \sum v_{n+1}z^{n+1}$ and $w = \sum w_{m+1}z^{m+1}$; then

$$[v\partial_q, w\partial_q] = [\sum v_{n+1}d_n, \sum w_{m+1}d_m] = \sum v_{n+1}w_{m+1}[n-m]z^{n+m+1}\partial_q. \tag{39}$$

Going back to (38) this corresponds then to **(A71)** $[v\partial_q, w\partial_q] = [(\tau v)(\partial_q w) - (\tau w)(\partial_q v)]\partial_q$ and this replaces **(A57)**.

Now we build in a cocycle term by using (33) in (38); a term **(A72)** $[z^{m+1}\partial_q, z^{n+1}\partial_q] = [d_m, d_n] = [m-n]d_{m+n} + \gamma_m\delta_{m+n,0}\tau$ arises. For $[v\partial_q, w\partial_q]$ we get then an additional term

$$\sum v_{n+1}w_{m+1}\gamma_n\delta_{m+n,0}\tau = \sum v_{n+1}w_{-n+1}\frac{[n+1][n][n-1]\tau}{\langle n\rangle[2][3]}. \tag{40}$$

Then for $a = 1/[2][3]$ and $(\tau + \tau^{-1})z^n = (q^n + q^{-n})z^n = \langle n\rangle z^n$ define

$$a\int w(\partial_q^2(\tau+\tau^{-1})^{-1}(\partial_q v)) = \sum v_{n+1}w_{-n+1}\gamma_n = \phi(v\partial_q, w\partial_q). \tag{41}$$

One checks that ϕ is antisymmetric and satisfies the q-cocycle condition

$$\phi([v\partial_q, w\partial_q], \Gamma(u\partial_q)) + \phi([w\partial_q, u\partial_q], \Gamma(v\partial_q)) + \phi([u\partial_q, v\partial_q], \Gamma(w\partial_q)) = 0. \tag{42}$$

Consequently (cf. ref. [32] for details).

Theorem 1. *The term $\phi(v\partial_a, w\partial_q)$ in (41) is a q-cocycle and following the constructions in ref.* [16] *one has a possibly canonical qKdV equation in the form* ($a = 1/[2][3]$)

$$u_t + c'\partial_q^2(\tau+\tau^{-1})^{-1}\partial_q u + \partial_q(u\tau u) + \tau^{-1}u\partial_q\tau^{-1}u. \tag{43}$$

Proof. We modify slightly the constructions in ref. [16] and write

$$\begin{aligned} &q\langle[f\partial_q, a), (g\partial_q, b)], (u, c)\rangle \\ &\quad = -q\int[(\tau f)(\partial_q g) - (\tau f)(\partial_q f)]u + caq\int g\partial_q^2(\tau+\tau^{-1})^{-1}\partial_q^2 f. \end{aligned} \tag{44}$$

We note from ref. [16] that **(A73)** $q\int fg = \int\tau^{-1}f\tau^{-1}g$ and $\int\partial_q f = 0$ while via **(A74)** $\partial_q(gfu) = \partial_q f\tau^{-1}(fu) + (\tau g)\partial_q(fu)$ (via $\partial_q(ab) = (\tau a)\partial_q b + (\partial_a)\tau^{-1}b$). Then **(A75)** $\int g[\tau^{-1}\times \partial_q(\tau u\tau^2 f) + \tau^{-1}(u\partial_q f) + aqc\partial_q^2(\tau+\tau^{-1})^{-1}\partial_q f]$ follows from (44). Putting $f = u$ we obtain the Euler equation as in ref. [16], namely **(A76)** $qu_t = -qca\partial_q^2(\tau+\tau^{-1})^{-1}\partial_q u - \tau^{-1}\partial_q(\tau u\tau^2 u) - \tau^{-1}(u\partial_q u)$. Using also $\tau^{-1}\partial_q = q\partial_q\tau^{-1}$ we obtain (43) with $c' = ac$. ■

Remark 4. In view of the expression **(A77)** $(\tau+\tau^{-1})^{-1} \sim \tau\sum(-1)^n\tau^{2n}$ the equation (43) involves an infinite number of terms (much as are indicated for qKdV in the hierarchy picture in ref. [16]. Since we now have a derivation with all of the classical algebraic and geometrical structure duplicated it seems that (43) could be a good candidate for a canonical form.

Remark 5. The Maurer–Cartan (MC) formulas were indicated in Example 5 and a version of this in a q-plane context appears in ref. [25]. A version of this can also be developed directly from the discussion of duality (cf. ref. [16] for details). A more interesting version comes from a quantum line $\mathbb{R}^1_q$ coupled with a time variable (e.g. $A_q = C(\mathbb{R}) \otimes \mathbb{R}^1_q$) with e.g. $x\Lambda = q\Lambda x$; $xdx = qdxx$; $dx\Lambda = q\Lambda dx$; $xd\Lambda = qd\Lambda x$; $e_1 x = q\Lambda x$ and $e_1\Lambda = 0$; $df(e_1) = e_1 f$; $e_2\Lambda = q\Lambda x$; $e_2 x = 0$; $df(e_2) = e_2 f$ (Λ is introduced for technical reasons (cf. ref. [16])). The MC equations are **(A78)** $d\omega_0 = \mathfrak{Q}(\omega_0 \wedge \omega_1)$; $d\omega_1 = -\omega_0 \wedge \omega_2$; $d\omega_2 = \mathfrak{Q}(\omega_1 \wedge \omega_2)$ ($\mathfrak{Q} = q^2 + q^4$) and we must find expressions df and $d\omega$ for functions and 1-forms. After considerable calculation based on zero curvature ideas (and some assumptions on Λ, $d\Lambda$) one can say that if $u = u(x,t)$ does not depend on Λ a qKdV type equation based on the quantum line arises in the form **(A79)** $u_t = \frac{1}{\mathfrak{Q}^2}\partial^x_{q^{-1}}(\partial^x_q)^2 u + \frac{\eta}{\mathfrak{Q}}\left((\partial^x_{q^{-1}}D_x)^2 - D_x^{-1}(\partial^x_q)^2\right)u - \frac{1}{\mathfrak{Q}}[q(\partial^x_q)D_x u + (1+q)u\partial^x_q u] + \eta u(1 - D_x)u$.

7 Moyal approach

One knows that Moyal quantization plays an important role in mathematical physics (for discussion cf. ref. [5, 15]). In fact KP can be identified with a Moyal quantization of dKP (cf. ref. [15] for details). Generally speaking one writes

$$f * g = f \exp\left[\kappa(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right] g = f\left(x + \kappa\overrightarrow{\partial}_p, p - \kappa\overrightarrow{\partial}_x\right) g(x,p) \tag{45}$$

and $\{f,g\}_M = (f*g - g*f)/2\kappa$ is the Moyal bracket. This leads to **(A80)** $\{f,g\}_M = \frac{1}{\kappa}\{f\sin[\kappa(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x)]g\}$. One can also write (using x, t variables and $\kappa \sim \theta/2$) **(A81)** $f * g = m \circ \exp(\theta P/2)(f \otimes g)$; $m(f\otimes g) = fg$; $P = \partial_t \otimes \partial_x - \partial_x \otimes \partial_t$. Now go to ref. [28] as developed in ref. [15] where one deals with bicomplexes (BC) $M = \oplus_{r\geq 0} M^r$ with linear maps $d, \delta : M^r \to M^r$ satisfying **(A82)** $d^2 = \delta^2 = d\delta + \delta d = 0$ (note δ is not the standard metric adjoint). In the spirit of Examples 1, 2 one develops zero curvature equations to produce integrable equations such as KdV and KP.

Example 7. Let $M = C^\infty(\mathbb{R}^3)\otimes\Lambda_2$ ($\Lambda_2 \sim \oplus_0^2\Lambda^j$) with **(A83)** $df = (f_t - f_{xxx})\tau + (1/2)(f_y - f_{xx})\xi$ and $\delta f = (3/2)(f_y + f_{xx})\tau + f_x\xi$ where $\tau, \xi \in \Lambda^1$. Deform now (or "dress") d to $Df = df + \delta(vf) - v\delta(f)$ so

$$Df = [f_t - f_{xxx} + (3/2)(v_y + v_{xx})f + 3v_x f_x]\tau + (1/2)(f_y - f_{xx} + 2v_x f)\xi. \tag{46}$$

Then the required BC condition $D^2 = 0$ becomes **(A84)** $v_{xt} - (1/4)v_{xxxx} + 3v_x v_{xx} - (3/4)v_{yy} = 0$ which is equivalent to KP for $u \sim -v_x$. There are also other forms of dressing of one BC to another while preserving the BC conditions. The underlying ideas here are zero curvature, cohomology (to get hierarchies), and gauge transformations (involving Seiberg–Witten (SW) maps – cf. ref. [5, 15]). In particular SW maps preserve zero curvature and e.g. solutions of the KdV equation determine solutions of a noncommutative KdV (NCKdV) equation in a manner similar to what happens with the SW map between commutative and noncommutative gauge theories (cf. ref. [15, 28] for details). Thus take $M = C^\infty(\mathbb{R}^2)\otimes\Lambda_2$ with $\tau, \xi \in \Lambda^1$ satisfying $\tau^2 = \xi^2 = \tau\xi + \xi\tau = 0$. One can then define d, δ on $M^0 = C^\infty(\mathbb{R}^2)$ and extend by linearity via $d(f\tau + h\xi) = (df)\tau + (dh)\xi$ for example. Start with **(A85)** $df = -f_{xx}\xi + (f_t + 4f_{xxx})\tau$ and $\delta f = f_x\xi - 3f_{xx}\tau$ (similar to Example 1). Apply a dressing to d in the form ($u = \phi_x$)

$$Df = df + \delta(\phi * f) - \phi * \delta f = -(f_{xx} + u * f)\xi + (f_t + 4f_{xxx} - 6u * f_x - 3u_x * f)\tau \tag{47}$$

(note ∂_x and ∂_t are derivations for $*$ which is a product as in (45) based on x, t). The only nontrivial BC equation is now $D^2 = 0$ and this is equivalent to the NCKdV equation **(A86)** $u_t + u_{xxx} - 3(u * u_x + u_x * u) = 0$. There are many more developments but we stop here.

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Bi-Hamiltonian Formulation of Generalized Toda Chains

Pantelis A. DAMIANOU

Department of Mathematics and Statistic, University of Cyprus, 1678, Nicosia, Cyprus
E-mail: *damianou@ucy.ac.cy*

In this paper we prove that the Bogoyavlensky–Toda lattices corresponding to simple, classical Lie groups are bi-Hamiltonian. We illustrate in detail the case of B_N Toda systems.

1 Introduction

In this paper we establish for the first time the bi-Hamiltonian nature of the Toda lattices corresponding to classical simple Lie groups. These are systems that generalize the usual finite, non-periodic Toda lattice (which corresponds to a root system of type A_n). This generalization is due to Bogoyavlensky [2]. These systems were studied extensively in [10] where the solution of the system was connected intimately with the representation theory of simple Lie groups. There are also studies by Adler, van Moerbeke [1] and Olshanetsky, Perelomov [15]. We call such systems the Bogoyavlensky–Toda lattices. We make the following more general definition which involves systems with exponential interaction: Consider a Hamiltonian of the form

$$H = \frac{1}{2}(\boldsymbol{p}, \boldsymbol{p}) + \sum_{i=1}^{m} e^{(\boldsymbol{v}_i, \boldsymbol{q})}, \tag{1}$$

where $\boldsymbol{q} = (q_1, \ldots, q_N)$, $\boldsymbol{p} = (p_1, \ldots, p_N)$, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are vectors in $\mathbb{R}^N$ and $(\ ,\ )$ is the standard inner product in $\mathbb{R}^N$. The set of vectors $\Delta = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is called the spectrum of the system. In this paper we limit our attention to the case where the spectrum is a system of simple roots for a simple Lie algebra $\mathcal{G}$. In this case $m = l = \operatorname{rank} \mathcal{G}$. It is worth mentioning that the case where m, N are arbitrary is an open and unexplored area of research. The main exception is the work of Kozlov and Treshchev [11] where a classification of system (1) is performed under the assumption that the system possesses N polynomial (in the momenta) integrals. Such systems are called Birkhoff integrable.

Example. The usual Toda lattice corresponds to a Lie algebra of type A_{N-1}. In other words, $l = N - 1$ and we choose Δ to be the set:

$$\boldsymbol{v}_1 = (1, -1, 0, \ldots, 0), \quad \ldots\ldots\ldots, \quad \boldsymbol{v}_{N-1} = (0, 0, \ldots, 0, 1, -1).$$

The Hamiltonian becomes:

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}, \tag{2}$$

which is the well-known classical Toda lattice.

The results of this paper were announced in [5] where it was pointed out that the higher Toda flows of the B_N, C_N and D_N systems are bi-Hamiltonian, but there was no similar result for the initial flow. Usually, it is more convenient to work instead in the space of the natural (q, p) variables, with the Flaschka variables (a, b). We end-up with a new set of polynomial equations

in the variables (a, b). One can write the equations in Lax pair form. The Lax pair $(L(t), B(t))$ in $\mathcal{G}$ can be described in terms of the root system. The symplectic bracket is transformed in (a, b) coordinates into a Lie–Poisson bracket denoted by π_1. The construction of the bi-Hamiltonian pair may be summarized as follows: Define a recursion operator $\mathcal{R}$ in (a, b) space by finding a second bracket, π_3, and inverting the initial Poisson bracket π_1. Define the negative recursion operator $\mathcal{N}$ by inverting the second Poisson bracket π_3. This recursion operator is the inverse of the operator $\mathcal{R}$. Finally, define a new rational bracket π_{-1} by $\pi_{-1} = \mathcal{N}\pi_1 = \pi_1\pi_3^{-1}\pi_1$. We obtain a bi-Hamiltonian formulation of the system:

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4,$$

where $H_i = \frac{1}{i} \operatorname{tr} L^i$. The brackets π_1 and π_{-1} are compatible and Poisson. In this paper we treat in detail the B_N case. For complete results and full proofs for all classical simple Lie groups, see [6]. We begin with a review of the classical Toda lattice.

2 A_N Toda lattice

Equation (2) is the classical, finite, nonperiodic Toda lattice. This system was investigated in [7–9, 13, 14] and in numerous of other papers that are impossible to list here. To determine the constants of motion, one uses Flaschka's transformation:

$$a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \qquad b_i = -\frac{1}{2} p_i.$$

Then Hamilton's equations become

$$\dot{a}_i = a_i(b_{i+1} - b_i),$$
$$\dot{b}_i = 2(a_i^2 - a_{i-1}^2).$$

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix},$$

and B is the skew-symmetric part of L (In the decomposition, lower Borel plus skew-symmetric). This is an example of an isospectral deformation; the entries of L vary over time but the eigenvalues remain constant. It follows that the functions $H_i = \frac{1}{i} \operatorname{tr} L^i$ are constants of motion.

Consider $\mathbb{R}^{2N}$ with coordinates $(q_1, \ldots, q_N, p_1, \ldots, p_N)$, the standard symplectic bracket

$$\{f, g\}_s = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right),$$

and the mapping $F : \mathbb{R}^{2N} \to \mathbb{R}^{2N-1}$ defined by

$$F : (q_1, \ldots, q_N, p_1, \ldots, p_N) \to (a_1, \ldots, a_{N-1}, b_1, \ldots, b_N).$$

There exists a bracket on $\mathbb{R}^{2N-1}$, which satisfies

$$\{f,g\}\circ F=\{f\circ F, g\circ F\}_s\ .$$

This bracket (up to a constant multiple) is given by

$$\{a_i,b_i\}=-a_i,\qquad \{a_i,b_{i+1}\}=a_i, \tag{3}$$

all other brackets are zero. $H_1=b_1+b_2+\cdots+b_N$ is the only Casimir. The Hamiltonian in this bracket is $H_2=\frac{1}{2}\operatorname{tr}L^2$. The Lie algebraic interpretation of this bracket can be found in [10]. We denote this bracket by π_1. The invariants H_i are in involution with respect to this Lie Poisson bracket π_1.

The quadratic Toda bracket appears in conjunction with isospectral deformations of Jacobi matrices. First, let λ be an eigenvalue of L with normalized eigenvector v. Standard perturbation theory shows that

$$\nabla\lambda=(2v_1v_2,\dots,2v_{N-1}v_N,v_1^2,\dots,v_N^2)^T:=U^\lambda\ ,$$

where $\nabla\lambda$ denotes $\left(\frac{\partial\lambda}{\partial a_1},\dots,\frac{\partial\lambda}{\partial b_N}\right)$. Some manipulations show that U^λ satisfies

$$\pi_2U^\lambda=\lambda\pi_1U^\lambda,$$

where π_1 and π_2 are skew-symmetric matrices. It turns out that π_1 is the matrix of coefficients of the Poisson tensor (3), and π_2, whose coefficients are quadratic functions of the a's and b's, that can be used to define a new Poisson tensor. It is a Poisson bracket in which the Hamiltonian vector field generated by H_1 is the same as the Hamiltonian vector field generated by H_2 with respect to the π_1 bracket. The defining relations are:

$$\{a_i,a_{i+1}\}=\frac{1}{2}a_ia_{i+1},\qquad \{a_i,b_i\}=-a_ib_i,\qquad \{a_i,b_{i+1}\}=a_ib_{i+1},\qquad \{b_i,b_{i+1}\}=2a_i^2,$$

all other brackets are zero. This bracket, due to Adler, has $\det L$ as Casimir and $H_1=\operatorname{tr}L$ is the Hamiltonian. The eigenvalues of L (and therefore the H_i as well) are still in involution. Furthermore, π_2 is compatible with π_1. We also have

$$\pi_2\nabla H_l=\pi_1\nabla H_{l+1},\qquad l=1,2,\dots. \tag{4}$$

These relations are similar to the Lenard relations for the KdV equation; they are generally called the Lenard relations. Taking $l=1$ in (4), we conclude that the Toda lattice is bi-Hamiltonian. In fact, using results from [5], we can prove that the Toda lattice is multi-Hamiltonian:

$$\pi_2\,\nabla H_1=\pi_1\nabla H_2=\pi_0\nabla H_3=\pi_{-1}\nabla H_4=\cdots\ .$$

The notion of bi-Hamiltonian system is due to Magri [12].

The sequence of Poisson tensors can be extended to form an infinite hierarchy. In order to produce the hierarchy of Poisson tensors one uses master symmetries. The first two Poisson brackets are precisely the linear and quadratic brackets we mentioned above. If a system is bi-Hamiltonian and one of the brackets is symplectic, one can find a recursion operator by inverting the symplectic tensor. The recursion operator is then applied to the initial symplectic bracket to produce an infinite sequence. However, in the case of Toda lattice (in Flaschka variables (a,b)) both operators are non-invertible and therefore this method fails. Recursion operators were introduced by Olver [16].

In the case of Toda equations, the master symmetries map invariant functions to other invariant functions. Hamiltonian vector fields are also preserved. New Poisson brackets are generated by using Lie derivatives in the direction of these vector fields and they satisfy interesting deformation relations. We quote the results from refs. [3, 4].

Theorem 1. *There exists a sequence of vector fields X_i, for $i \geq -1$, and a sequence of contravariant 2-tensors π_j, $j \geq 1$, satisfying:*

i) *π_j are all Poisson;*

ii) *the functions H_i, $i \geq 1$, are in involution with respect to all of the π_j;*

iii) *$X_i(H_j) = (i+j)H_{i+j}$, $i \geq -1$, $j \geq 1$;*

iv) *$L_{X_i}\pi_j = (j-i-2)\pi_{i+j}$, $i \geq -1$, $j \geq 1$;*

v) *$[X_i, X_j] = (j-i)X_{i+j}$, $i \geq 0$, $j \geq 0$;*

vi) *$\pi_j \nabla H_i = \pi_{j-1}\nabla H_{i+1}$, where π_j denotes the Poisson matrix of the tensor π_j.*

Remark 1. Theorem 1 was extended for all integer values of the index in [5].

Remark 2. To define the vector fields X_n one considers expressions of the form

$$\dot{L} = [B, L] + L^{n+1}.$$

This equation is similar to a Lax equation, but in this case the eigenvalues satisfy $\dot{\lambda} = \lambda^{n+1}$ instead of $\dot{\lambda} = 0$.

3 B_N Toda systems

In this section, we show that higher polynomial brackets exist also in the case of B_n Toda systems. We will prove that these systems possess a recursion operator and we will construct an infinite sequence of compatible Poisson brackets in which the constants of motion are in involution. Finally, we show that these systems are bi-Hamiltonian. The Hamiltonian for B_n is

$$H = \frac{1}{2}\sum_{1}^{n} p_j^2 + e^{q_1 - q_2} + \cdots + e^{q_{n-1}-q_n} + e^{q_n}.$$

We make a Flaschka-type transformation

$$a_i = \frac{1}{2}e^{\frac{1}{2}(q_i - q_{i+1})}, \qquad a_n = \frac{1}{2}e^{\frac{1}{2}q_n}, \qquad b_i = -\frac{1}{2}p_i.$$

Then

$$\dot{a}_i = a_i(b_{i+1} - b_i), \qquad \dot{b}_i = 2(a_i^2 - a_{i-1}^2), \qquad i = 1, \ldots, n,$$

with the convention that $a_0 = b_{n+1} = 0$.

These equations can be written as a Lax pair $\dot{L} = [B, L]$, where L is the symmetric matrix

$$L = \begin{pmatrix} b_1 & a_1 & & & & & & \\ a_1 & \ddots & \ddots & & & & & \\ & \ddots & \ddots & a_{n-1} & & & & \\ & & a_{n-1} & b_n & a_n & & & \\ & & & a_n & 0 & -a_n & & \\ & & & & -a_n & -b_n & \ddots & \\ & & & & & \ddots & \ddots & -a_1 \\ & & & & & & -a_1 & -b_1 \end{pmatrix},$$

and B is the skew-symmetric part of L (In the decomposition, lower Borel plus skew-symmetric). We note that the determinant of L is zero.

In the new variables a_i, b_i, the canonical bracket on $\mathbb{R}^{2N}$ is transformed into a bracket π_1 which is given by

$$\{a_i, b_i\} = -a_i, \qquad \{a_i, b_{i+1}\} = a_i.$$

It is easy to show by induction that

$$\det \pi_1 = a_1^2 a_2^2 \cdots a_n^2.$$

The invariant polynomials for B_n, which we denote by

$$H_2, \quad H_4, \quad \ldots, \quad H_{2n}$$

are defined by $H_{2i} = \frac{1}{2i} \operatorname{Tr} L^{2i}$.

We look for a bracket π_3 which satisfies

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4 \ .$$

We define the following homogeneous cubic bracket π_3:

$$\begin{aligned}
&\{a_i, a_{i+1}\} = a_i a_{i+1} b_{i+1}, \qquad \{a_i, b_i\} = -a_i b_i^2 - a_i^3, \qquad \{a_n, b_n\} = -a_n b_n^2 - 2a_n^3,\\
&\{a_i, b_{i+2}\} = a_i a_{i+1}^2, \qquad \{a_i, b_{i+1}\} = a_i b_{i+1}^2 + a_i^3, \qquad \{a_i, b_{i-1}\} = -a_{i-1}^2 a_i,\\
&\{b_i, b_{i+1}\} = 2a_i^2(b_i + b_{i+1}), \qquad i = 1, 2, \ldots, n-1
\end{aligned}$$

We summarize the properties of this new bracket in the following:

Theorem 2. *The bracket π_3 satisfies the following:*

1. *π_3 is Poisson;*
2. *π_3 is compatible with π_1;*
3. *H_{2i} are in involution.*

Define $\mathcal{R} = \pi_3 \pi_1^{-1}$. Then $\mathcal{R}$ is a recursion operator. We obtain a hierarchy

$$\pi_1, \ \pi_3, \ \pi_5, \ \ldots$$

consisting of compatible Poisson brackets of odd degree in which the constants of motion are in involution.

4. *$\pi_{j+2} \operatorname{grad} H_{2i} = \pi_j \operatorname{grad} H_{2i+2} \ \forall \ i, \ j$.*

The proof of this result is in [4].

Following the procedure outlined in the introduction we obtain a bi-Hamiltonian formulation of the system. In other words, we define $\pi_{-1} = \mathcal{N}\pi_1 = \pi_1 \pi_3^{-1} \pi_1$ and use it to obtain the desired formulation. We illustrate with the B_2 Toda system. In this case $\det \pi_1 = a_1^2 a_2^2$ and $\det \pi_3 = a_1^2 a_2^2 \Delta^2 = \det \pi_1 \Delta^2$, where

$$\Delta = a_1^4 + 2a_2^2 a_1^2 + 2a_2^2 b_1^2 + b_1^2 b_2^2 - 2a_1^2 b_1 b_2.$$

The explicit formula for π_{-1} is

$$\pi_{-1} = \frac{1}{\Delta} A,$$

where

$$A=\begin{pmatrix} 0 & -a_1a_2b_2 & -a_1(b_2^2+a_1^2+2a_2^2) & a_1(b_1^2+a_1^2+2a_2^2)\\ a_1a_2b_2 & 0 & a_1^2a_2 & -a_2(b_1^2+2a_1^2)\\ a_1(b_2^2+a_1^2+2a_2^2) & -a_1^2a_2 & 0 & -2a_1^2(b_1+b_2)\\ -a_1(b_1^2+a_1^2+2a_2^2) & a_2(b_1^2+2a_1^2) & 2a_1^2(b_1+b_2) & 0 \end{pmatrix}.$$

This bracket is Poisson by construction. It is also compatible with π_1. We note that $\Delta=\sqrt{\det\mathcal{R}}$ and it is also equal to the product of the non-zero eigenvalues of L. Using the rational bracket π_{-1} we establish the bi-Hamiltonian nature of the system, i.e.

$$\pi_1\nabla H_2=\pi_{-1}\nabla H_4.$$

In fact the system is multi-Hamiltonian:

$$\pi_1\nabla H_2=\pi_{-1}\nabla H_4=\pi_{-3}\nabla H_6=\cdots,$$

where

$$\pi_{-(2i-1)}=\mathcal{N}^i\pi_1.$$

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On Construction of Zero-Curvature Representations for Some Chiral-Type Three-Field Systems

Dmitry K. DEMSKOI

Oryol State University, 95 Komsomolskaya Str., 302015 Oryol, Russia
E-mail: *demskoy@iname.com*

The problem of construction of matrix zero-curvature representations for some chiral-type three field systems is considered. The systems belong to the class described by the Lagrangian $L = \frac{1}{2} g_{ij}(u) u^i_x u^j_t + f(u)$, where g_{ij} is the metric of three-dimensional reducible Riemannian space. The investigation is based on the analysis of evolutionary system $u_t = S(u)$, where S is a higher symmetry.

1 Introduction

We consider here the systems belonging to the following class of two-dimensional fields

$$u^i_{tx} + \Gamma^i_{jk}(u) u^j_x u^k_t = f^i(u), \qquad f^i = g^{ij} \frac{\partial f}{\partial u^j}, \tag{1}$$

where g^{ij} is the metric tensor and Γ^i_{jk} are the Christoffel symbols of the configurational space with the coordinates u^i, the low indices x and t denote the partial derivatives, also we assume summation over repeating indices. The system (1) possesses the following Lagrangian

$$L = \frac{1}{2} g_{ij}(u) u^i_x u^j_t + f(u).$$

In the article [2] the systems with the Lagrangian

$$L = \frac{1}{2} \left(u_t u_x + \psi(v, w)(v_t w_x + v_x w_t) \right) + f(u, v, w), \tag{2}$$

were studied. There we found the systems possessing the nontrivial higher polynomial (with respect to derivatives $u^i_n = \partial u^i / \partial x^n$) symmetries of the 2nd, 3rd, 4th or 5th order. It was proved that the non-degenerate symmetries exist if and only if $\psi = (vw + c)^{-1}$, $c = \text{const}$, and f takes one of the following forms

$$f = ave^{\sqrt{2}u} + bwe^{-\sqrt{2}u}, \tag{3a}$$
$$f = av^2 e^{2u} + bw^2 e^{-2u}, \tag{3b}$$
$$f = av^2 e^{2u} + bwe^{-u}, \tag{3c}$$
$$f = ave^{u} + bwe^{-u}, \tag{3d}$$
$$f = (vw + c/2) \left[ae^{\sqrt{2}u} + be^{-\sqrt{2}u} \right], \tag{3e}$$
$$f = a\,(vw + c/2)\, e^{\sqrt{2}u} + be^{-\sqrt{2}u}, \tag{3f}$$
$$f = a\,(vw + c/2)\, e^{\sqrt{2}u} + be^{-2\sqrt{2}u}, \tag{3g}$$
$$f = \left(v^2 w + 2/3\, vc \right) e^{\sqrt{2}u}, \tag{3h}$$
$$f = ve^{\sqrt{2/3}u}, \tag{3i}$$

where a and b are arbitrary constants. We assume that $c \neq 0$, so the connection Γ^i_{jk} is nontrivial.

Evidently the simplest case for investigation is when function f is linear with respect to v and w. Among the two-exponential functions (3a)–(3g) there are only two functions of that kind, they are given by (3a), (3d). Hyperbolic systems (1) corresponding to these functions have the following forms

$$u_{tx} = \sqrt{2}\big[ave^{\sqrt{2}u} - bwe^{-\sqrt{2}u}\big], \qquad v_{tx} = b\psi^{-1}e^{-\sqrt{2}u} + \psi w v_x v_t,$$
$$w_{tx} = a\psi^{-1}e^{\sqrt{2}u} + \psi v w_x w_t, \tag{4}$$

and

$$u_{tx} = ave^{u} - bwe^{-u}, \qquad v_{tx} = b\psi^{-1}e^{-u} + \psi w v_t v_x, \qquad w_{tx} = a\psi^{-1}e^{u} + \psi v w_t w_x. \tag{5}$$

It was shown in [3] that each system posessing Lagrangian (2) can be represented in an explicit Hamiltonian form

$$u_t = D_x^{-1}\frac{\delta H}{\delta u}, \qquad v_t = e^{\varphi}D_x^{-1}e^{-\varphi}\psi^{-1}\frac{\delta H}{\delta w}, \qquad w_t = \psi^{-1}e^{-\varphi}D_x^{-1}e^{\varphi}\frac{\delta H}{\delta v},$$

which is possibly nonlocal. Existence of the Hamiltonian form gives us a hope that both systems (4) and (5) are the first nonlocal members of the corresponding sequences of integrable Hamiltonian evolution systems

$$\boldsymbol{u}_{t_n} = S_n(\boldsymbol{u}) = J^{-1}\frac{\delta H_n}{\delta \boldsymbol{u}}, \tag{6}$$

where J^{-1} is Hamiltonian operator, S_0 is a corresponding nonlocal vector field, $S_1(\boldsymbol{u}) = \boldsymbol{u}_x$ and S_n, $n > 1$ are higher order vector fields; H_n are the corresponding Hamiltonians.

2 Zero-curvature representations

If a nonlinear system can be represented as the compatibility condition of a linear system $\Psi_x = U\Psi$, $\Psi_t = V\Psi$, i.e. as

$$U_t - V_x + [U, V] = 0, \tag{7}$$

then the system is said to possess zero-curvature representation. In the equation (7) $[U, V]$ is the commutator of matrices. The matrices U, V depend on field functions, and finite set of theirs derivatives, and on a parameter λ, usually called the spectral parameter.

The zero-curvature representation can be constructed starting directly from equation (7). For simplicity, one can choose the matrices U, V in the form

$$U = U_i(u)u_x^i + \overline{U}(u), \qquad V = V_i(u)u_t^i + \overline{V}(u) \tag{8}$$

(this choice corresponds to the solutions given below). Substituting (8) to (7), and requiring the system (1) to be obtained, one can obtain a matrix system for U_i, V_i, $\overline{U}$, and $\overline{V}$ involving covariant derivatives:

$$\nabla_j U_i - \nabla_i V_j + [U_i, V_j] = 0, \qquad f^i(U_i - V_i) + [\overline{U}, \overline{V}] = 0,$$
$$\nabla_i \overline{U} + [\overline{U}, V_i] = 0, \qquad \nabla_i \overline{V} + [\overline{V}, U_i] = 0. \tag{9}$$

The direct computation of the matrices U and V from the equations (9) is a fundamentally difficult problem, and we therefore use a different approach. Existence of the explicit Hamiltonian

form allows expecting that the systems (4) and (5) are nonlocal terms of hierarchies of evolutionary systems $u_t = S_n(u)$. Adopting this assumption we can try to find the zero curvature representation for any system (6) in the following form

$$U_{,t} - V_{n,x} + [U, V_n] = 0. \tag{10}$$

The matrix U must be one and the same for all equations of the hierarchy, but the matrices V_n are different. For the evolution systems such problems are usually solvable by the prolongation method [4, 5]. We assume that the matrix U for systems (4) and (5) has form (8). Replacing in (7) u_t, v_t, and w_t according to (6), we obtain matrices U, V in the form

$$U = \sum_i p_i(u) u_x^i A_i + \sum_j q_j(u) A_j, \qquad V_n = \sum_i g_i(u, u_1, \dots, u_k) A_i, \tag{11}$$

where k is the order of a higher symmetry $S_n(u)$, A_i are some constant matrices which satisfy commutation relations

$$[A_i, A_j] = C_{ij}^k A_k. \tag{12}$$

Note, that table of commutators (12) is not closed. There are some ways to solve the systems like (12). For example, one can choose one of the matrices in the Jordan normal form and try to solve the equations directly. But this way leads to an excessive branching if the matrix size is large. Therefore we applied a modification of the prolongation method by H.D. Wahlquist and F.B. Estabrook [4, 5] to close the table of commutators (12). There are two possibilities for any unknown commutator $[A_i, A_j]$:

(i) $[A_i, A_j]$ is a linear combination of the known elements of the Lie algebra or (ii) $[A_i, A_j]$ is a new element linearly independent of the previous elements. In the first case we write $[A_i, A_j]$ as a linear combination of all elements $A_1, \dots, A_n$ which we have for the current step and try to find the coefficients with the help of the Jacobi identity. In the second case we introduce the new element $A_{n+1} = [A_i, A_j]$ of the algebra. Then using the Jacobi identity we try to find the new commutational relations for A_{n+1} and so on. After a number of such steps we obtain a closed table of commutators. The obtained algebra can possess a centre. To obtain an algebra with the lowest dimension we construct a factor algebra, setting the elements of the centre as zeros. As a result, the matrices U and V were always embedded into simple classical algebras. To construct a representation of resulting algebra we use the standard algorithm: find a Cartan subalgebra, construct a Cartan–Weyl basis, and build all the matrices explicitly.

Zero-curvature representation for system (4). The simplest higher symmetry of this system is

$$\begin{aligned} &u_{t_1} = \sqrt{2}\psi v_x w_x, \qquad v_{t_1} = v_{xx} - 2v\psi v_x w_x + \sqrt{2} u_x v_x, \\ &w_{t_1} = -w_{xx} + 2w\psi v_x w_x + \sqrt{2} u_x w_x. \end{aligned} \tag{13}$$

Using prolongation technique of Whalquist–Estabrook we obtain the following matrices of the zero-curvature representation for system (13)

$$U = \begin{pmatrix} 0 & -v^{-1}(3vw + c)be^{-\sqrt{2}u} & \frac{2c^2}{3} v^{-1} v_x \psi \\ 0 & \frac{c}{3} v^{-1} v_x \psi & \lambda v a e^{\sqrt{2}u} \\ \frac{1}{3} v^{-1} v_x \psi & v^{-1} b e^{-\sqrt{2}u} & -\frac{c}{3} v^{-1} v_x \psi \end{pmatrix}, \tag{14}$$

$$V_1 = \begin{pmatrix} -\lambda ab & 3be^{-\sqrt{2}u} w_x & 2c^2 g - \lambda ab(3vw + c) \\ \frac{\lambda}{3} a e^{\sqrt{2}u} v_x \psi & cg & \left(1 - \frac{2}{3} c\psi\right) \lambda a e^{\sqrt{2}u} v_x \\ g & 0 & -cg \end{pmatrix}, \tag{15}$$

here

$$g = v^{-1}\psi(v_{xx} + \sqrt{2}v_x u_x)/3 - \psi^2 v_x w_x/3.$$

Now we can obtain the matrix V_0 for original hyperbolic system (4). Substituting matrix (14) to equation (10), one can easily find that the matrix V_0 has the following form

$$V_0 = u_t B_1 + v^{-1}v_t B_2 + w_t v\psi B_3 + B_4.$$

The constant matrices B_i are of the form

$$B_1 = \sqrt{2}/3(\boldsymbol{e}_{22} - \boldsymbol{e}_{11} - \boldsymbol{e}_{33}), \qquad B_4 = \lambda^{-1}(2c\boldsymbol{e}_{12} + \boldsymbol{e}_{32}) - (\boldsymbol{e}_{21} + c\boldsymbol{e}_{23})/3,$$
$$B_2 = (\boldsymbol{e}_{11} + \boldsymbol{e}_{22} - 2\boldsymbol{e}_{33})/3 + c\boldsymbol{e}_{13}, \qquad B_3 = (2\boldsymbol{e}_{11} - \boldsymbol{e}_{22} - \boldsymbol{e}_{33})/3 - 2c\boldsymbol{e}_{13},$$

here $\boldsymbol{e}_{ij}$ are the Weyl matrices. Thus the matrix V is given by

$$V_0 = \begin{pmatrix} h_1 & 2c\lambda^{-1} & cv^{-1}v_t - 2cw_t v\psi \\ -1/3 & h_2 - h_1 & -c/3 \\ 0 & \lambda^{-1} & -h_2 \end{pmatrix}, \tag{16}$$

where $h_1 = \frac{1}{3}(v^{-1}v_t + 2w_t v\psi - \sqrt{2}u_t)$, $h_2 = \frac{1}{3}(2v^{-1}v_t + w_t v\psi + \sqrt{2}u_t)$.

Zero-curvature representation for system (5). The simplest higher symmetry of this system is

$$\begin{aligned}
u_t &= -\frac{1}{2}u_{xxx} + \frac{3}{2}\psi(v_{xx}w_x - v_x w_{xx}) + \frac{1}{4}u_x^3 + \frac{9}{2}\psi u_x v_x w_x + \frac{3}{2}\psi^2 v_x w_x(v_x w - v w_x), \\
v_t &= v_{xxx} + \frac{3}{2}u_{xx}v_x + 3v_{xx}(u_x - \psi v w_x) + \frac{9}{4}u_x^2 v_x - 6\psi u_x v v_x w_x + 3\psi v_x w_x\left(\psi v^2 w_x - \frac{1}{2}v_x\right), \\
w_t &= w_{xxx} - \frac{3}{2}u_{xx}w_x - 3w_{xx}(u_x + \psi v_x w) + \frac{9}{4}u_x^2 w_x + 6\psi u_x v_x w w_x \\
&\quad + 3\psi v_x w_x\left(\psi w^2 v_x - \frac{1}{2}w_x\right).
\end{aligned} \tag{17}$$

The matrices U and V_1 of the zero-curvature representation for the system (17) is embedded into $sl(4,\mathbb{C})$. The matrix U is given by

$$U = \begin{pmatrix} -\frac{1}{2}w\psi v_1 & -\lambda e^u & 0 & -we^{-u} \\ -\psi^{-1}e^{-u} & \frac{1}{2}w\psi v_1 & we^{-u} & 0 \\ 0 & -\lambda v e^u & \frac{1}{2}w\psi v_1 & -\psi^{-1}e^{-u} \\ -\lambda v e^u & 0 & \lambda e^u & -\frac{1}{2}w\psi v_1 \end{pmatrix}. \tag{18}$$

The matrix V_1 is too cumbersome, and we omit it. Let us consider now the zero-curvature representation for the hyperbolic system (5). We take the matrix U in the form (18), and moreover, we assume that the matrix V_0 is linear with respect to u_t, v_t and w_t

$$V_0 = f_1 u_t + f_2 v_t + f_3 w_t + f_4.$$

If one substitutes V_0 into the equation (10) and substitutes therein u_{tx}, v_{tx} and w_{tx} from (5), then this equation must be an identity. This implies, in particular

$$f_1 = C_1, \qquad f_2 = C_2, \qquad f_4 = C_4, \qquad f_3 = f_3(v,w),$$

where C_i are constant matrices. The matrix f_3 satisfies the system

$$\frac{\partial f_3}{\partial v} = cA_1\psi^2 + w\psi[A_1, f_3], \qquad \frac{\partial f_3}{\partial w} = -v\psi f_3,$$

which has the following general solution $f_3 = \psi(C_3 + vA_1)$, and we obtain

$$V_0 = C_1u_t + C_2v_t + \psi(C_3 + vA_1)w_t + C_4.$$

Now the equation (10) yields a considerable number of commutational relations for the constant matrices C_i and A_j. We solved these relations using (18) and found

$$C_1 = \boldsymbol{e}_{44} - \boldsymbol{e}_{22}, \qquad C_2 = \boldsymbol{e}_{31}, \qquad C_3 = \boldsymbol{e}_{13}, \qquad C_4 = \frac{b}{2}(\boldsymbol{e}_{32} - \boldsymbol{e}_{41}) - \frac{a}{2\lambda}(\boldsymbol{e}_{14} + \boldsymbol{e}_{23}),$$

and

$$V_0 = \begin{pmatrix} -\frac{1}{2}\psi w_t v & 0 & \psi w_t & -a/(2\lambda) \\ 0 & -u_t + \frac{1}{2}\psi w_t v & -a/(2\lambda) & 0 \\ v_t & b/2 & \frac{1}{2}\psi w_t v & 0 \\ -b/2 & 0 & 0 & u_t - \frac{1}{2}\psi w_t v \end{pmatrix}. \tag{19}$$

3 Conserved currents

To construct the conserved currents of the systems (4) and (5) we use the algorithm presented in [1]. Let the vector function Ψ satisfy the system

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi. \tag{20}$$

Introducing the function $\Phi = \Psi/(C, \Psi)$ where (C, Ψ) is the Euclidean scalar product and C is a constant vector, one can easily check that

$$\Phi_x = U\Phi - \Phi(C, U\Phi), \qquad \Phi_t = V\Phi - \Phi(C, V\Phi), \qquad (C, \Phi) = 1. \tag{21}$$

Multiplying (20) by vector $C/(C, \Psi)$ we obtain $(\log(C, \Psi))_x = (C, U\Phi)$, $(\log(C, \Psi))_t = (C, V\Phi)$. This implies the following conservation law

$$D_t(C, U\Phi) = D_x(C, V\Phi). \tag{22}$$

Expanding now Φ in a power series with respect to λ, one can obtain an infinite sequence of local conserved currents. Hence we may call the functions

$$R = (C, U\Phi) \qquad \text{and} \qquad T = (C, V_0\Phi) \tag{23}$$

the generating functions for the conserved densities and the fluxes accordingly.

Conserved currents for system (4). Let us consider equations (21) with matrices (14) and (16), where we set $a = 1$ and assume that $b \neq 0$. In this case, we can choose $c = (0, 1, 0)$. Then identity $(c, \varphi) = 1$ implies that $\varphi = (\varphi_1, 1, \varphi_3)$, and the first system in (21) takes the following form

$$\Phi_x = -c\frac{v_x}{v}\psi\Phi - 3b\frac{e^{-\sqrt{2}u}}{v\psi} - \lambda v e^{\sqrt{2}u}\Phi\varphi_3, \qquad \varphi_{3,x} = \frac{v_x}{3v}\psi\Phi + b\frac{e^{-\sqrt{2}u}}{v} - \lambda v e^{\sqrt{2}u}\varphi_3^2, \tag{24}$$

where $\Phi = \varphi_1 - 2c\varphi_3$. Generating functions (23) are then written as

$$\rho = \lambda v e^{\sqrt{2}u}\varphi_3 + \frac{cv_x}{3v}\psi, \qquad \theta = \frac{1}{3}\left(2\sqrt{2}u_t + \frac{v_t}{v} - vw_t\psi\right) - c\varphi_3 - \frac{1}{3}\Phi. \tag{25}$$

To obtain WKB-expansion for system (24), we set

$$\lambda = k^2/b, \qquad \Phi = gk^{-1}v^{-1}b\exp(-\sqrt{2}u), \qquad \varphi_3 = hk^{-1}v^{-1}b\exp(-\sqrt{2}u),$$

then equations (24) take simpler form

$$h_x = \left(\frac{v_x}{v} + \sqrt{2}u_x\right)h + \frac{v_x}{3v}\psi g + k(1-h^2), \quad g_x = (v_x w\psi + \sqrt{2}u_x)g - k(3\psi^{-1} + gh). \tag{26}$$

It is now clear that the expansions for g, and h must be given by

$$h = 1 + \sum_{i=1}^{\infty} h_i k^{-i}, \qquad g = -3\psi^{-1} + \sum_{i=1}^{\infty} g_i k^{-i}. \tag{27}$$

Substituting this expansions in (26), we get the recursion relations

$$\begin{aligned}
&h_{i+1} = \frac{1}{2}\left(\sqrt{2}u_x + \frac{v_x}{v}\right)h_i - \frac{1}{2}D_x h_i + \frac{v_x}{6v}\psi g_i - \frac{1}{2}\sum_{j=1}^{i} h_j h_{i-j+1},\\
&g_{i+1} = 3\psi^{-1}h_{i+1} + (\sqrt{2}u_x + v_x w\psi)g_i - D_x g_i - \sum_{j=1}^{i} h_j g_{i-j+1},\\
&h_1 = u_x/\sqrt{2}, \qquad g_1 = 3vw_x - 3/\sqrt{2}\,u_x\psi^{-1},
\end{aligned} \tag{28}$$

where $i \geqslant 1$. Applying all the substitutions to (25), we obtain the series

$$\rho = k + \sum_{i=0}^{\infty} \rho_i k^{-i}, \qquad \theta = \sum_{i=0}^{\infty} \theta_i k^{-i},$$

which determine canonical conserved currents (ρ_i, θ_i) of the system (4):

$$\begin{aligned}
&\rho_0 = \frac{\sqrt{2}}{2}u_x + \frac{cv_x}{3v}\psi, \qquad \rho_i = h_{i+1}, \qquad i \geq 1,\\
&\theta_0 = \frac{1}{3}\left(2\sqrt{2}u_t + \frac{v_t}{v} - vw_t\psi\right), \qquad \theta_1 = wb\exp(-\sqrt{2}u),\\
&\theta_{i+1} = -v^{-1}b\exp(-\sqrt{2}u)\,(ch_i + g_i/3), \qquad i \geq 1.
\end{aligned} \tag{29}$$

With the help of relations (28), and (29) one can easily obtain any number of conserved currents. For example

$$\begin{aligned}
&\rho_1 = -\frac{\sqrt{2}}{4}\omega_1, \qquad \rho_2 = \frac{\sqrt{2}}{8}D_x(\omega_1) - \frac{1}{2}\omega_3,\\
&\rho_3 = \frac{3}{4}D_x(\omega_3) - \frac{\sqrt{2}}{16}D_x^2(\omega_1) - \frac{1}{16}\omega_1^2 - \frac{1}{2}\omega_2\omega_3, \qquad \theta_2 = -b\exp(-\sqrt{2}u)\left(w_x - \frac{\sqrt{2}}{2}wu_x\right),\\
&\theta_3 = bw\exp(-\sqrt{2}u)\left(-\frac{\sqrt{2}}{4}\omega_1 + (v_x w\psi)^{-1}\omega_3\right),
\end{aligned}$$

where the functions ω_i are given by

$$\begin{aligned}
&\omega_1 = (\sqrt{2}u_2 - u_1^2 - 2v_1 w_1\psi)/6, \qquad \omega_2 = v_2 v_1^{-1} + u_1/\sqrt{2} - \psi w_1 v,\\
&\omega_3 = \psi v_1(w_2 - v_1 w_1\psi w - \sqrt{2}u_1 w_1).
\end{aligned} \tag{30}$$

Conserved currents for system (5). To construct the conserved currents for the system (5), let us simplify the matrix U with the help of the gauge transformation $\widetilde{U} = S^{-1}(US - S_x)$, $\widetilde{V}_0 = S^{-1}(V_0 S - S_t)$. We choose the matrix S in the following form

$$S = \exp(-\varphi/2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-u} & 0 & 0 \\ v & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{u} \end{pmatrix},$$

where $\varphi = D_x^{-1}\psi w v_1$ is the quasi-local variable: $D_t\varphi = \psi v w_t - u_t$. Then the transformed matrices are

$$\widetilde{U} = \begin{pmatrix} 0 & -\lambda & 0 & -w \\ -c & \psi w v_1 + u_1 & w & 0 \\ -c\psi v_1 & 0 & \psi w v_1 & -c \\ 0 & 0 & \lambda & -u_1 \end{pmatrix}, \tag{31}$$

$$\widetilde{V}_0 = \begin{pmatrix} \psi v w_t - \frac{1}{2}u_t & 0 & \psi w_t & -\frac{a}{2\lambda}e^{u} \\ -\frac{av}{2\lambda}e^{u} & \psi v w_t - \frac{1}{2}u_t & -\frac{a}{2\lambda}e^{u} & 0 \\ 0 & \frac{b}{2}e^{-u} & -\frac{1}{2}u_t & \frac{av}{2\lambda}e^{u} \\ -\frac{b}{2}e^{-u} & 0 & 0 & -\frac{1}{2}u_t \end{pmatrix}. \tag{32}$$

Let us set $C = (0, 0, 0, 1)$, then the first of the systems (21) takes the following form

$$\begin{aligned} &\Phi_{1,x} = u_1\Phi_1 - \lambda\Phi_2 - w - \lambda\Phi_1\Phi_3, \qquad \Phi_{2,x} = -c\Phi_1 + (2u_1 + \psi w v_1)\Phi_2 + w\Phi_3 - \lambda\Phi_2\Phi_3, \\ &\Phi_{3,x} = -c\psi v_1\Phi_1 + (u_1 + \psi w v_1)\Phi_3 - c - \lambda\Phi_3^2. \end{aligned} \tag{33}$$

The generating functions (23) take now the simplest form

$$R = \lambda\Phi_3 - u_x, \qquad T = -(be^{-u}\Phi_1 + u_t)/2. \tag{34}$$

To obtain the local conserved densities we set $\lambda = -k^2/c$ and adopted the following formal series for Φ_i:

$$\Phi_1 = \frac{1}{k}\left(w + \sum_{i=1}^{\infty} h_i k^{-i}\right), \quad \Phi_2 = \frac{1}{k^3}\left(\frac{cw_1}{2} + \sum_{i=1}^{\infty} g_i k^{-i}\right), \quad \Phi_3 = \frac{c}{k}\left(1 + \sum_{i=1}^{\infty} \rho_i k^{-i}\right). \tag{35}$$

Substituting these series into the equations (33) we found that $\rho_1 = -u_x/2$ and the first conservation law is trivial $D_t u_x = D_x u_t$. The next conservation laws are given by

$$D_t\rho_i = D_x\frac{b}{2}e^{-u}h_{i-2}, \qquad i \geq 2, \tag{36}$$

where ρ_i and h_i satisfy the following relations

$$\rho_{i+1} = \frac{1}{2}\psi v_1 h_i - \frac{1}{2}(u_1 + \psi w v_1)\rho_i + \frac{1}{2}D_x\rho_i - \frac{1}{2}\sum_{j=0}^{i-1}\rho_{j+1}\rho_{i-j}, \qquad i \geq 1,$$

$$h_{i+1} = D_x h_i - w\rho_{i+1} - u_1 h_i - \frac{1}{c}g_i - \sum_{j=0}^{i-1}\rho_{j+1}h_{i-j}, \qquad i \geq 1,$$

$$g_{i+1} = \frac{c}{4}\delta_{i0}(w_2 - 2u_1 w_1 - \psi v_1 w w_1) + \frac{1}{2}D_x g_i - cw\rho_{i+2} - \left(u_1 + \frac{1}{2}\psi w v_1\right)g_i - \frac{c}{4}w_1\rho_{i+1}$$

$$-\frac{1}{2}\sum_{j=0}^{i-1}\rho_{j+1}g_{i-j}+\frac{c}{2}D_x h_{i+1}-\frac{c}{2}u_1h_{i+1}-\frac{c}{2}\sum_{j=0}^{i}\rho_{j+1}h_{i+1-j},\qquad i\geq 0,$$

$$\rho_1=-u_1/2,\qquad g_0=0,\qquad h_0\equiv w,\qquad h_1=(w_1-u_1w)/2. \tag{37}$$

Using relations (37) we found in particular

$$\begin{aligned}
&\rho_2=\omega_1/4,\qquad \rho_3=D_x(\rho_2)/2,\qquad \rho_4=(4\omega_2-\omega_1^2+2D_x^2(\omega_1))/32,\\
&\omega_1=u_1^2/2+\psi v_1w_1-u_2,\\
&\omega_2=-\psi v_1w_3+\psi u_2v_1w_1+\psi^2v_1v_2ww_1+3\psi u_1v_1w_2+2\psi^2v_1^2ww_2\\
&\qquad-3\psi^2u_1v_1^2ww_1-2\psi u_1^2v_1w_1+c\psi^3v_1^2w_1^2+\psi^2v_1^2w_1^2/2-2\psi^3v_1^3w^2w_1.
\end{aligned} \tag{38}$$

It can be seen that $\omega_2/8-\omega_1^2/32$ is not a total derivative, hence the densities ρ_1 and ρ_3 are trivial, and ρ_2, ρ_4 are non-trivial.

It is easy to see from (29), and (36) that if we set $b=0$, then all conserved densities of the systems (4), and (5) become pseudo-constants and the systems become Liouvillian ones. To prove this statement we must present three independent pseudo-constants along each characteristic (see for instance [6]).

Obviously, functions (30) constitute complete set of pseudo-constants for system (4) along the characteristic $D_t\omega=0$. Lorentz invariance $x\longleftrightarrow t$ of the system allows us to obtain the pseudo-constants along the characteristic $D_x\omega=0$ simply by the substitutions $u_x\to u_t$, $u_{xx}\to u_{tt}$, ... from ω_1, ω_2 and ω_3.

The two obvious independent pseudo-constants of the system (5) are ω_1 and ω_2 given by (38). To find the third pseudo-constant let us rewrite the second equation of (5) under the condition $b=0$ in the following form

$$D_t\log v_x=\psi v_tw.$$

This implies $D_x(D_t\log v_x+u_t)=D_x(\psi v_tw+u_t)$. Using the conservation law $D_t\psi vw_x=D_x(\psi v_tw+u_t)$ we find $D_t(D_x\log v_x+u_x-\psi vw_x)=0$. Thus, the function

$$\omega_3=D_x\log v_x+u_x-\psi vw_x \tag{39}$$

is the third independent pseudo-constant along the characteristic $D_t\omega=0$ for system (5).

In the case when $a=0$ (and $b\neq 0$), the complete set of the pseudo-constants can be obtained with the help of the discrete symmetry of the system (5) $v\longleftrightarrow w$, $u\longrightarrow -u$.

4 Conclusion

We believe that the modification of the Wahlquist and Estabrook method applied here may be used for construction of the zero curvature representations for other hyperbolic systems too.

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New Lumps of Veselov–Novikov Integrable Nonlinear Equation and New Exact Rational Potentials of Two-Dimensional Stationary Schrödinger Equation via $\overline{\partial}$-Dressing Method

V.G. DUBROVSKY and I.B. FORMUSATIK

Novosibirsk State Technical University, 20 K. Marx Ave., 630092 Novosibirsk, Russia
E-mail: *dubrovsky@online.nsk.su*

Recently calculated via $\overline{\partial}$-dressing method new lumps of 2+1-dimensional integrable nonlinear Veselov–Novikov (VN) evolution equation and exact rational potentials of two-dimensional stationary Schrödinger (2DSchr) equation with multiple pole of order two wave functions are reviewed. Among the constructed rational solutions and rational potentials are both nonsingular and singular.

1 Introduction

Exact solutions of differential equations of physics are very important for the understanding of various physical phenomena. The generation and application of new methods of calculation of exact solutions was and is actual task in all times for human scientific civilization. In the last two decades the Inverse Spectral Transform (IST) method has been generalized and successfully applied to various two-dimensional nonlinear evolution equations such as Kadomtsev–Petvashvili, Davey–Stewartson, Nizhnik–Veselov–Novikov, Zakharov–Manakov system, Ishimori, two-dimensional integrable sine-Gordon and others (see books [1–4] and references therein). The nonlocal Riemann–Hilbert problem [5], $\overline{\partial}$-problem [6] and more general $\overline{\partial}$-dressing method of Zakharov and Manakov [7–10] are now basic tools for solving two-dimensional integrable nonlinear evolution equations. Great task for mathematicians and physicists now is the generalization of IST method to multidimensional differential equations of mathematical physics.

In the present short note the results of recent calculations [19] of new exact rational so called multiple pole solutions of the famous two-dimensional integrable Veselov–Novikov (VN) nonlinear equation [11]

$$u_t + \kappa u_{zzz} + \overline{\kappa} u_{\overline{z}\overline{z}\overline{z}} + 3\kappa \left(u\partial_z^{-1} u_{\overline{z}}\right)_{\overline{z}} + 3\overline{\kappa} \left(u\partial_{\overline{z}}^{-1} u_z\right)_z = 0 \tag{1}$$

via $\overline{\partial}$-dressing method of Zakharov and Manakov are summarized. Here and in what follows $z = x + iy$, $\partial_z \equiv \frac{\partial}{\partial z} = \frac{1}{\partial 2}(\partial_x - i\partial_y)$, $\partial_{\overline{z}} \equiv \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$; $u_z \equiv \frac{\partial u}{\partial z}$, $\partial_z^{-1}\partial_z = \partial_{\overline{z}}^{-1}\partial_{\overline{z}} = 1$; κ is arbitrary complex constant; $u(z, \overline{z}, t)$ is scalar function of space-time variables. It is well known that VN equation can be represented as compatibility condition for two linear auxiliary problems

$$\begin{aligned} &L_1\psi = \left(\partial_{z\overline{z}}^2 + u(z, \overline{z}, t)\right)\psi = 0, \\ &L_2\psi = \left(\partial_t + \kappa\partial_z^3 + \overline{\kappa}\partial_{\overline{z}}^3 + 3\kappa\left(\partial_z^{-1} u_{\overline{z}}\right)\partial_z + 3\overline{\kappa}\left(\partial_{\overline{z}}^{-1} u_z\right)\partial_{\overline{z}}\right)\psi = 0 \end{aligned} \tag{2}$$

in the form of Manakov's triad representation

$$[L_1, L_2] = BL_1, \qquad B = 3\left(\kappa\partial_z^{-1} u_{\overline{z}\overline{z}} + \overline{\kappa}\partial_{\overline{z}}^{-1} u_{zz}\right). \tag{3}$$

The $\overline{\partial}$-dressing method of Zakharov and Manakov [7–10] allows

- To construct integrable nonlinear equations together with corresponding linear auxiliary problems.
- Using the solution of linear auxiliary problems via reconstruction formula to calculate broad classes of exact solutions and to solve Cauchy problem for integrable nonlinear evolution equations.
- To construct explicitly broad classes of exactly solvable variable coefficients-fields and corresponding wave functions of linear auxiliary problems.

In conclusion of this introduction let us mention that exact integration of VN equation has remarkable history, more detailed information about all known cases of exact integration of (1) one can found in [11–15] and in doctorate dissertation of Grinevich [16], see also the review [17] and books [1–4]. The calculated in the paper [19] and reviewed in the present note multiple pole of order two rational solutions of VN equation (1) are completely new.

2 General formulas for calculating of exact multiple pole rational solutions

The basic equation of $\overline{\partial}$-dressing method in our case is the following scalar non-local $\overline{\partial}$-problem [7–9]:

$$\frac{\partial\chi}{\partial\overline{\lambda}} = (\chi * R)(\lambda,\overline{\lambda}) = \iint d\mu\wedge d\overline{\mu}\;\chi(\mu,\overline{\mu})\,R(\mu,\overline{\mu};\lambda,\overline{\lambda}) \tag{4}$$

stated here in the auxiliary space of spectral variables μ, λ. Via "long" derivatives $D_1=\partial_\xi+i\lambda$, $D_2=\partial_\eta-i\frac{i\epsilon}{\lambda}$, $D_3=\partial_t+i\left(\kappa_1\lambda^3-\kappa_2\frac{\epsilon^3}{\lambda^3}\right)$ containing the "spectral" variable λ one can construct the auxiliary linear problems for VN equation (1):

$$\begin{aligned} &L_1\chi=(D_1D_2+V_1D_1+V_2D_2+u)\chi=0,\\ &L_2\chi=\left(D_3+\kappa D_1^3+\overline{\kappa}D_2^3+W_1D_1^2+W_2D_2^2+W_3D_1+W_4D_2+W\right)\chi=0 \end{aligned} \tag{5}$$

explicitly containing the "spectral" parameter λ. In terms of the wave function

$$\psi=\chi e^{F(\lambda)},\qquad F(\lambda):=i\left(\lambda z-\frac{\epsilon}{\lambda}\overline{z}\right)-i\left(k\lambda^3-\overline{k}\frac{\epsilon^3}{\lambda^3}\right)t \tag{6}$$

the linear auxiliary problems (5) in potential case for the operator L_1 (when $V_1=V_2=0$) coincide with the problems (2). In the $\overline{\partial}$-dressing method the non-analytic wave functions χ are explored. The detailed derivation of reconstruction formulas for the variable coefficients V_1, V_2, u and W_1, W_2, W_3, W_4, W of auxiliary linear problems (5) and of the conditions of reality and potentiality in the framework of $\overline{\partial}$-dressing method in the paper [19] are presented (see also [20]).

The multiple pole solutions u of VN equation (1) correspond to special analytic structure of wave function χ (or ψ): the solution u is multiple pole solution of order m if the maximum order of pole terms in Loran expansion of χ near some points λ_k is equal to m. The solution of $\overline{\partial}$-problem (4) in the case of canonical normalization, $\chi\to\infty$ as $\lambda\to1$, is equivalent to the solution of singular integral equation:

$$\chi(\lambda)=1+\iint_C\frac{d\lambda'\wedge d\overline{\lambda'}}{2\pi i(\lambda'-\lambda)})\iint_C d\mu\wedge d\overline{\mu}\;\chi(\mu,\overline{\mu})\,R_0(\mu,\overline{\mu};\lambda',\overline{\lambda'})e^{F(\mu)-F(\lambda')}, \tag{7}$$

where $F(\lambda)$ is given by the formula (6). The kernel R_0 of $\overline{\partial}$-problem (4) in the following form corresponds to the multiple pole rational solutions u of VN equation (1):

$$R_0(\mu,\bar{\mu};\lambda,\bar{\lambda}) = \frac{\pi}{2}\sum_{p}\sum_{k,m}^{N_p} r_k^{(p)}(\mu) l_m^{(p)}(\lambda)\delta^{(k)}(\mu-\lambda_p)\delta^{(m)}(\lambda-\lambda_p). \tag{8}$$

The insertion of (8) into (7) gives the Loran expansion of wave function χ near the points $\lambda=\lambda_p$ and this leads to multiple pole solutions u of VN equation. The scheme for calculating of 2+1-dimensional integrable nonlinear evolution equations via $\overline{\partial}$-dressing method of multiple pole solutions was developed at first in the paper [18].

Let us review here some recent results [19] of calculation of exact order two multiple pole rational solutions of VN equation (1) with constant asymptotic values $-\epsilon$ at infinity:

$$u = \widetilde{u}(x,y,t) - \epsilon, \qquad \widetilde{u}(x,y,t) \underset{|z|\to\infty}{\longrightarrow} 0. \tag{9}$$

In this case the first linear auxiliary problem (5) (expressed in terms of wave function ψ (6)) in potential case ($V_1 = V_2 = 0$) coincides with problem (2a) or with two-dimensional stationary Schrödinger equation

$$(\partial^2_{z\bar{z}} + \widetilde{u})\psi = \epsilon\psi. \tag{10}$$

So the construction of exact rational multiple pole solutions u with constant asymptotic values at infinity $u \underset{|z|\to\infty}{\longrightarrow} -\epsilon$ of VN equation (1) means simultaneously construction of exact multiple pole rational potentials $\widetilde{u} := u + \epsilon$ of two-dimensional stationary Schrödinger equation (10). Recently the calculations of exact rational multiple pole of order two solutions of VN equations for the kernel R_0 of the form

$$R_0 = \frac{\pi}{2}\sum_{k=1}^{N}\left[A_k\delta_\mu(\mu-\lambda_k)\delta_\lambda(\lambda-\lambda_k) + B_k\delta_\mu(\mu+\lambda_k)\delta_\lambda(\lambda+\lambda_k)\right. \tag{11}$$
$$\left. + \frac{\epsilon^3}{|\mu|^2|\lambda|^2\bar{\mu}\bar{\lambda}}\left(\overline{A}_k\delta_{\frac{\epsilon}{\mu}}\left(\frac{\epsilon}{\mu}+\bar{\lambda}_k\right)\delta_{\frac{\epsilon}{\bar{\lambda}}}\left(\frac{\epsilon}{\lambda}+\bar{\lambda}_k\right) + \overline{B}_k\delta_{\frac{\epsilon}{\mu}}\left(\frac{\epsilon}{\mu}-\bar{\lambda}_k\right)\delta_{\frac{\epsilon}{\bar{\lambda}}}\left(\frac{\epsilon}{\lambda}-\bar{\lambda}_k\right)\right)\right]$$

containing the terms with first derivatives of delta-functions and satisfying to the reality condition $u = \overline{u}$ for the solutions have been performed by us (D.V.G. and F.I.B.) [19]. The obtained results for nonsingular and singular rational multiple pole of order two solutions u in the following two sections are stated.

3 Rational nonsingular solutions of VN equation corresponding to double pole wave functions

Let us consider at first the case of negative energies $E = -\epsilon = -|\lambda_k|^2 < 0$ in the standard representation $(-\frac{1}{4}\Delta - \widetilde{u})\psi = E\psi)$ of stationary Schrödinger equation. Some difficulties (really this is the main problem in calculations of exact solutions of VN equation) presents the fulfillment of the potentiality condition $V_2 = 0$ for the operators L_1 of the first linear auxiliary problem (5). Long calculations [19] lead in the considered case $\epsilon = |\lambda_k|^2 > 0$ to the following restrictions from potentiality on the constants A_k, B_k of the kernel R_0 (11):

$$\frac{\lambda_k^3}{B_k} - \frac{\lambda_k^3}{A_k} = \frac{i}{2}, \qquad \frac{\bar{\lambda}_k^3}{\bar{B}_k} = -\frac{\lambda_k^3}{B_k}, \qquad \frac{\bar{\lambda}_k^3}{\bar{A}_k} = -\frac{\lambda_k^3}{A_k}, \qquad k = 1,\ldots,N. \tag{12}$$

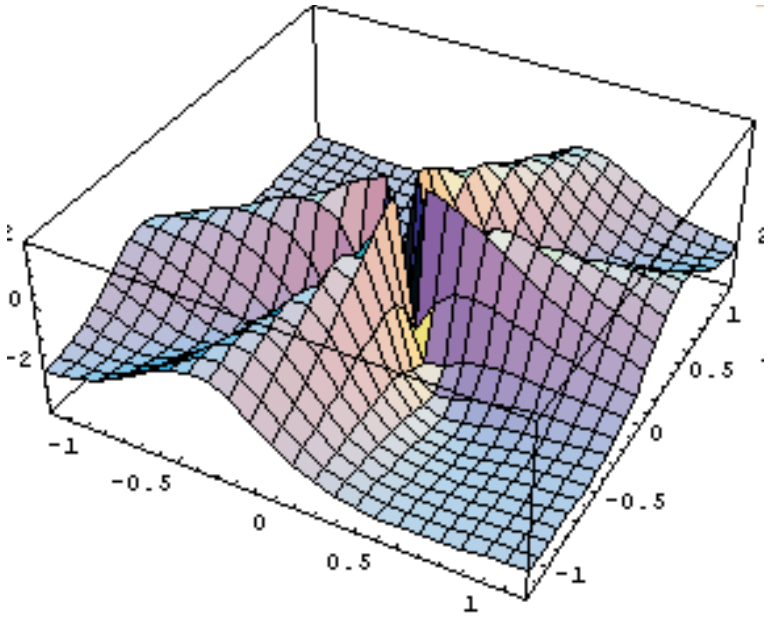

Figure 1. Nonsingular multiple order two pole rational (lump) solution of VN equation.

For the case $N = 1$ in the sum (11) under the additional condition on κ and λ_1 of the form $\kappa\lambda_1^3 = -\overline{\kappa}\overline{\lambda}_1^3$ one obtains the following exact rational nonsingular order two multiple pole solution [19] with constant asymptotic value $-\epsilon$ at infinity of VN equation (1):

$$u = -\epsilon - 32\epsilon\frac{(\lambda_R\hat{x} - \lambda_I\hat{y})^2[4(\lambda_R\hat{x} - \lambda_I\hat{y})^4 - 9(\lambda_I\hat{x} + \lambda_R\hat{y})^2]}{[4(\lambda_R\hat{x} - \lambda_I\hat{y})^4 + 3\,|\lambda_1|^2\,(\hat{x}^2 + \hat{y}^2)]^2}. \tag{13}$$

Here $\lambda_1 = \lambda_R + i\lambda_I$ and the wave variables $\hat{x}$ and $\hat{y}$ are introduced: $\hat{x} = x - V_1 t - x_0$, $\hat{y} = y - V_2 t - y_0$ with real $V_1 = 18i\frac{\lambda_I}{|\lambda_1|^2}k\lambda_1^3$, $V_2 = 18i\frac{\lambda_R}{|\lambda_1|^2}k\lambda_1^3$. This solution represents a new rational nonsingular lump, with behavior at infinity as $u \underset{|z|\to\infty}{\longrightarrow} -\epsilon + O\left(\frac{1}{|z|^2}\right)$, of VN equation propagating on the plane x, y with the velocity $\overrightarrow{V} = (V_1, V_2)$. One can see an illustration for this new lump solution of VN equation on the Fig. 1[1].

Corresponding new exact rational nonsingular potential of two-dimensional stationary Schrödinger equation (10) with multiple pole of order two wave function ψ (6) gives the formula $\widetilde{u} = u + \epsilon$ with u from (13).

4 Rational singular solutions of VN equation corresponding to double pole wave functions

Let us consider now the case of positive energies $E = -\epsilon = |\lambda_k|^2 > 0$ in the standard representation $\left(-\frac{1}{4}\Delta - \widetilde{u}\right)\psi = E\psi$ of stationary Schrödinger equation. Some difficulties (as in the previous case $\epsilon > 0$ of Section 3) presents fulfillment of the potentiality condition $V_2 = 0$ for the operators L_1 of the first linear auxiliary problem (5). Long calculations [19] in the considered case $\epsilon = -|\lambda_k|^2 < 0$ lead to the following restriction from potentiality on the constants A_k, B_k of the kernel R_0 (11):

$$\frac{\lambda_k^3}{B_k} - \frac{\lambda_k^3}{A_k} = \frac{i}{2}, \qquad \frac{\bar{\lambda}_k^3}{\bar{A}_k} = -\frac{\lambda_k^3}{B_k}, \qquad k = 1, \ldots, N. \tag{14}$$

[1]Figures in colour will be available only in electronic version.

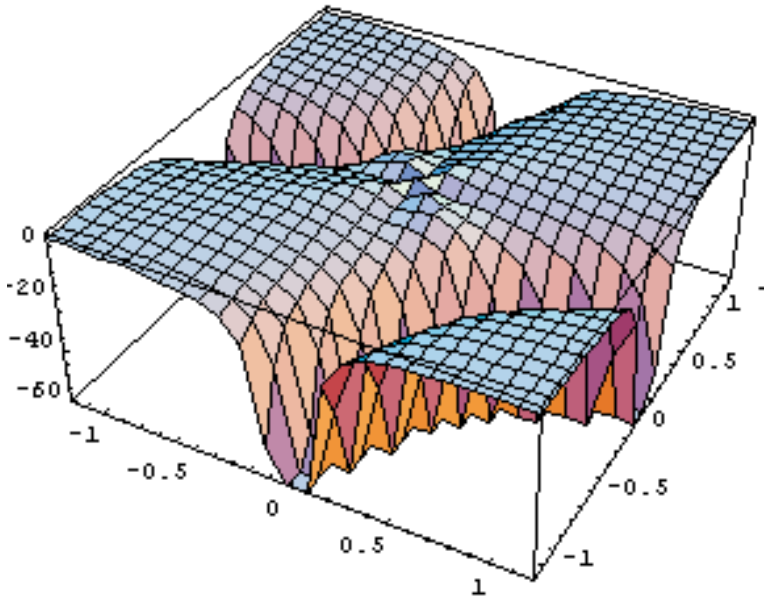

Figure 2. Singular multiple order two pole rational solution of VN equation.

For the case $N = 1$ in the sum (11) under the additional condition on κ and λ_1 of the form $\kappa\lambda_1^3 = \overline{\kappa}\overline{\lambda}_1^3$ one obtains the following exact rational singular multiple pole of order two solution with constant asymptotic value $-\epsilon$ at infinity of VN equation (1):

$$u = -\epsilon - 32\epsilon\frac{(\lambda_I\hat{x} + \lambda_R\hat{y})^2[4(\lambda_I\hat{x} + \lambda_R\hat{y})^4 + 9(\lambda_R\hat{x} - \lambda_I\hat{y})^2]}{[4(\lambda_I\hat{x} + \lambda_R\hat{y})^4 - 3\,|\lambda_1|^2\,(\hat{x}^2 + \hat{y}^2)]^2}. \tag{15}$$

Here $\lambda_1 = \lambda_R + i\lambda_I$ and the wave variables $\hat{x} = x - V_1t - x_0$, $\hat{y} = y - V_2t - y_0$ with real $V_1 = 9\frac{\lambda_R}{|\lambda_1|^2}k\lambda_1^3$, $V_2 = -9\frac{\lambda_I}{|\lambda_1|^2}k\lambda_1^3$ are introduced. The singular rational solution (15) propagates on the plane x, y with the velocity $\overrightarrow{V} = (V_1, V_2)$ One can see an illustration for this new singular rational solution of VN equation on the Fig. 2.

The corresponding new exact rational singular potential of two-dimensional stationary Schrödinger equation (10) gives the formula $\widetilde{u} = u + \epsilon$ with u from (15).

In conclusion of this note let us mention that for VN nonlinear integrable equation (1) the first auxiliary linear problem (2), i.e. two-dimensional stationary Schrödinger equation (5), is self-adjoint, but multiple pole solutions exist as the present calculations show [19]. For the KP or mKP equations the first auxiliary linear problems are non-self-adjoint, and with the non-self-adjoint character of these problems the existence of multiple pole solutions has been connected [18]. By VN equation we have the first explicit example which shows that multiple pole solutions may exist even for cases with self-adjoint first auxiliary linear problems. Our work on multiple pole rational solutions of VN equation and multiple pole rational potentials of 2DSchr equation for more general choices of kernel R_0 (8) and its parameters in general position is in progress and the results will be published elsewhere.

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On N-Wave and NLS Type Systems: Generating Operators and the Gauge Group Action: the $so(5)$ Case

Vladimir S. GERDJIKOV † *and Georgi G. GRAHOVSKI* †‡

† *Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Science, 72 Tzarigradsko chaussee, 1784 Sofia, Bulgaria*

‡ *Dipartimento di Matematica "Federigo Enriques", Università degli studi di Milano, Via C. Saldini 50, 20133 Milan, Italy*

We first review some recent developments [2] of the N-wave equations and their gauge equivalent ones related to the $so(5)$ algebra. These include the form of the equations and the relevant recursion operators, the properties of the scattering data etc. Next we generalize these results to the multicomponent nonlinear Schrödinger equations and their gauge equivalent ones known as the Heisenberg ferromagnet type equations. The explicit form of the $so(5)$ HF equations and the relevant recursion operators are derived.

1 Introduction

We start by briefly reviewing our previous results on N-wave equations and their gauge equivalent ones [1, 2] and extend them for the multicomponent NLS-type equation and their gauge equivalent.

Both classes of the above-mentioned nonlinear evolution equations (NLEE) are solvable by applying the inverse scattering method [3, 4] to the generalized Zakharov–Shabat system [5]:

$$L(\lambda)\psi \equiv \left(i\frac{d}{dx} + q(x,t) - \lambda J\right)\psi(x,t,\lambda) = 0, \tag{1}$$

related to the simple Lie algebra $\mathfrak{g}$ of rank $r > 1$. The gauge in (1) is fixed up by choosing J to be a constant element of the Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$. Then the potential $q(x,t)$ can be cast in the form $q(x,t) = [J, Q(x,t)]$, where $Q(x,t)$ is a generic element of $\mathfrak{g}$. We also assume that $q(x,t)$ tends to zero fast enough for $x \to \pm\infty$.

For the N-wave type equations J is chosen to be a regular element of $\mathfrak{h}$; as a consequence the subalgebra $\mathfrak{g}_J \subset \mathfrak{g}$ of elements commuting with J coincides with $\mathfrak{h}$. For the multi-component NLS (MNLS) type equations J as a rule is **not** a regular element of $\mathfrak{h}$ and the corresponding subalgebra $\mathfrak{g}_J$ is a non-commutative one. This makes more difficult the derivations of: i) the fundamental analytic solutions (FAS) of (1); ii) the construction of the related recursion operators and iii) the application of the gauge transformation, i.e. the transition to the pole gauge in which:

$$\tilde{L}\tilde{\psi}(x,t,\lambda) \equiv \left(i\frac{d}{dx} - \lambda\mathcal{S}(x,t)\right)\tilde{\psi}(x,t,\lambda) = 0, \tag{2}$$

$$\tilde{\psi}(x,t,\lambda) = g^{-1}(x,t)\psi(x,t,\lambda), \qquad \mathcal{S}(x,t) \equiv \mathrm{Ad}_g \cdot J = g^{-1}(x,t)Jg(x,t),$$

$$g(x,t) = \psi(x,t,\lambda=0), \qquad \text{i.e.,} \qquad \left(i\frac{d}{dx} + q(x,t)\right)g(x,t) = 0. \tag{3}$$

To be more specific let us write down the explicit form of the above mentioned NLEE along with their Lax representations and dispersion laws. The N-wave equations and their gauge

equivalent read:

$$i[J,Q_t]-i[I,Q_x]+[[I,Q],[J,Q]]=0, \tag{4}$$

$$\mathcal{S}_t-\frac{d}{dx}f(\mathcal{S})=0, \tag{5}$$

where $Q(x,t)$ is related to the potential $q(x,t)$ in (1) by $q(x,t)=[J,Q(x,t)]$. The constant regular element $I\in\mathfrak{h}$ can be expressed as a function (polynomial) of J: $I=f(J)$, see [2]. The corresponding M-operators of (4) and (5) equal:

$$M(\lambda)\psi\equiv\left(i\frac{d}{dt}+[I,Q(x,t)]-\lambda I\right)\psi(x,t,\lambda)=0, \tag{6}$$

$$\tilde{M}\tilde{\psi}(x,t,\lambda)\equiv\left(i\frac{d}{dt}-\lambda f(\mathcal{S})\right)\tilde{\psi}(x,t,\lambda)=0, \tag{7}$$

and the dispersion law for both equations is linear in λ: $f_{N-w}(\lambda)=\lambda I$. An important point here is that I and J are both regular and *linearly independent;* otherwise the corresponding equations (4) and (5) become linear.

As we mentioned above the Lax operator for the MNLS equations formally has the form (1) but now J is no more regular element of $\mathfrak{h}$. This means that the subalgebra $\mathfrak{g}_J\subset\mathfrak{g}$ of elements commuting with J (i.e., the kernel of the operator ad_J) is a non-commutative one. The dispersion law of the MNLS equation is quadratic in λ: $f_{\mathrm{MNLS}}=2\lambda^2 J$. The MNLS equation, its gauge equivalent MHF equation and their M-operators have the form:

$$i\frac{dq}{dt}+2\,\mathrm{ad}_J^{-1}\frac{d^2q}{dx^2}+[q,\pi_0[q,\mathrm{ad}_J^{-1}q]]-2i(\mathbb{1}-\pi_0)[q,\mathrm{ad}_J^{-1}q_x]=0, \tag{8}$$

$$i\frac{d\mathcal{S}}{dt}+2\frac{d}{dx}\left(\mathrm{ad}_{\mathcal{S}}^{-1}\frac{d\mathcal{S}}{dx}\right)=0, \tag{9}$$

$$M(\lambda)\psi\equiv\left(i\frac{d}{dt}-V_0^{\mathrm{d}}+2i\,\mathrm{ad}_J^{-1}q_x(x,t)+2\lambda q(x,t)-2\lambda^2 J\right)\psi(x,t,\lambda)=0, \tag{10}$$

$$\tilde{M}\tilde{\psi}(x,t,\lambda)\equiv\left(i\frac{d}{dt}-2i\lambda\,\mathrm{ad}_{\mathcal{S}}^{-1}\mathcal{S}_x-2\lambda^2\mathcal{S}\right)\tilde{\psi}(x,t,\lambda)=0, \tag{11}$$

where $V_0^d=\pi_0\left([q,\mathrm{ad}_J^{-1}q_x]\right)$ and π_0 is the projector onto $\mathfrak{g}_J$; (see also Section 3 below).

The interpretation of the inverse scattering method (ISM) as a generalized Fourier transform and the expansions over the so called "squared solutions" (see e.g. [5] for regular J and [6] for non-regular J) allows one to study all the fundamental properties of the relevant NLEE's. These include: (i) the description of the whole class NLEE related to the Lax operator $L(\lambda)$ (1) solvable by the ISM; (ii) derivation of the infinite family of integrals of motion and (iii) the Hamiltonian formulation of the NLEE's.

The "squared solutions" that appeared first in [7,8] were later generalized in [5,6] to Lax operators of the type (1); they can also be viewed as natural generalizations of the usual exponentials and are introduced by:

$$e_\alpha^{\pm}(x,t,\lambda)=(\mathbb{1}-\pi_0)\left(\chi^{\pm}(x,t,\lambda)E_\alpha\hat{\chi}^{\pm}(x,t,\lambda)\right), \tag{12}$$

where $\chi^{\pm}(x,t,\lambda)$ is the fundamental analytic solution of the Lax operator L (1). In fact their completeness relations [5,6] provide us the spectral decompositions of the operators:

$$\Lambda_+e_{\pm\alpha}^{\pm}=\lambda e_{\pm\alpha}^{\pm},\qquad \Lambda_-e_{\mp\alpha}^{\pm}=\lambda e_{\pm\alpha}^{\pm},\qquad \alpha\in\Delta_+, \tag{13}$$

known as the generating (or recursion) operators; they play crucial role in deriving the properties of the NLEE. The recursion operator appeared first in the AKNS-approach [7] as a tool to

generate the class of all M-operators as well as the NLEE related to the given Lax operator. Next I.M. Gel'fand and L.A. Dickii [9] discovered that the class of these M-operators is contained in the diagonal of the resolvent of L which could be explicitly defined in terms of the FAS $\chi^{\pm}(x,\lambda)$.

In the present paper we outline the construction of the recursion operators $\Lambda_{\pm}$ related to L (1) both for regular and non-regular elements J. In order to avoid a number of technicalities we consider only the Lax operators (1) related to the algebra $so(5)$. Our ideas, however, can be generalized to any simple Lie algebra thus allowing one to treat the corresponding class of NLEE. Another important point in our considerations is the explicit gauge covariant approach, which allows one to reformulate everything also for the pole gauge, thus extending the results in [10,6].

In Section 2 we briefly summarize our results on N-wave type equations related to $so(5)$ and their gauge equivalent, see [2, 11]. In Section 3 we resolve the technical problems in constructing the recursion operators $\Lambda_{\pm}$ for the $so(5)$ MNLS type equations by using the spectral decomposition of the operator ad_J. As a result we obtain explicit expressions for the recursion operators $\widetilde{\Lambda}_{\pm}$, gauge equivalent to $\Lambda_{\pm}$. In Section 4 we briefly show how these recursion operators generate the hierarchies of Hamiltonian structures of the MNLS equations and their gauge equivalent ones.

2 N-wave type systems and their gauge-equivalent: $so(5)$-case

The $so(5)$ algebra has two simple roots α_1, α_2 and four positive roots $\Delta_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$, $\alpha_1 + \alpha_2 = e_1$, $\alpha_1 + 2\alpha_2 = e_1 + e_2$. Here we use the following parametrization for J and $q(x)$

$$q(x,t) \equiv \sum_{\alpha\in\Delta_+} (q_\alpha E_\alpha + p_\alpha E_{-\alpha}) = \begin{pmatrix} 0 & q_{10} & q_{11} & q_{12} & 0 \\ p_{10} & 0 & q_1 & 0 & q_{12} \\ p_{11} & p_1 & 0 & q_1 & -q_{11} \\ p_{12} & 0 & p_1 & 0 & q_{10} \\ 0 & p_{12} & -p_{11} & p_{10} & 0 \end{pmatrix}, \quad J = \begin{pmatrix} J_1 & 0 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -J_2 & 0 \\ 0 & 0 & 0 & 0 & -J_1 \end{pmatrix}, \quad (14)$$

where q_{jk} and p_{jk} are related to the roots $j\alpha_1 + k\alpha_2$ and $-j\alpha_1 - k\alpha_2$ respectively. Since in this section J is regular we can assume that $J_1 > J_2 > 0$; then the positive roots satisfy $\alpha(J) > 0$. The corresponding 4-wave system has the form [11]:

$$\begin{aligned} &i(J_1 - J_2)q_{10,t} - i(I_1 - I_2)q_{10,x} + 2\kappa q_{11}q_{01}^* = 0, \\ &iJ_2 q_{01,t} - iI_2 q_{01,x} + \kappa(q_{11}^* q_{12} + q_{11}q_{10}^*) = 0, \\ &iJ_1 q_{11,t} - iI_1 q_{11,x} + \kappa(q_{12}q_{01}^* - q_{10}q_{01}) = 0, \\ &i(J_1 + J_2)q_{12,t} - i(I_1 + I_2)q_{12,x} - 2\kappa q_{11}q_{01} = 0, \end{aligned} \quad (15)$$

where we put $p_{jk} = q_{jk}^*$, I_k are the matrix elements of $I = \mathrm{diag}\,(I_1, I_2, 0, -I_2, -I_1)$ and $\kappa = J_1 I_2 - J_2 I_1$. The system (15) can be written down as: $i\,\mathrm{ad}_J^{-1}\frac{dq}{dt} - \Lambda[I, \mathrm{ad}_J^{-1} q(x,t)] = 0$, where the recursion operators $\Lambda_{\pm}$ satisfying (13) and $\Lambda = (\Lambda_+ + \Lambda_-)/2$ are defined by:

$$\Lambda_{\pm} Z(x) = \mathrm{ad}_J^{-1}(\mathbb{1} - \pi_0)\left(i\frac{dZ}{dx} + [q(x), Z(x)] + i\left[q(x), \pi_0 \int_{\pm\infty}^{x} dy\,[q(y), Z(y)]\right]\right). \quad (16)$$

The equation (15) find an applications in nonlinear optics [3, 11], and their gauge equivalent – in differential geometry [13]. The equations, gauge equivalent to (15) are:

$$\frac{d\mathcal{S}}{dt} - f_1\frac{d\mathcal{S}}{dx} - f_3\frac{d\mathcal{S}^3}{dx} = 0, \qquad f_1 = \frac{I_2 J_1^3 - I_1 J_2^3}{J_1 J_2\left(J_1^2 - J_2^2\right)}, \qquad f_3 = \frac{I_1 J_2 - I_2 J_1}{J_1 J_2\left(J_1^2 - J_2^2\right)}, \quad (17)$$

where the matrix $\mathcal{S}(x,t) \in so(5)$ is constrained by:

$$\mathrm{tr}\,\mathcal{S}^2 = 2\left(J_1^2 + J_2^2\right), \qquad \mathrm{tr}\,\mathcal{S}^4 = 2\left(J_1^4 + J_2^4\right), \qquad \mathcal{S}\left(\mathcal{S}^2 - J_1^2\right)\left(\mathcal{S}^2 - J_2^2\right) = 0. \quad (18)$$

We are using explicitly gauge covariant approach which allows one to reformulate all fundamental properties of the NLEE to be easily reformulated from one gauge to another. In particular, the gauge-equivalent "squared solutions" and recursion operators are given by:

$$\widetilde{e}^{\pm}_{\alpha}(x,t,\lambda) = g^{-1}(x,t)e^{\pm}_{\alpha}(x,t,\lambda)g(x,t), \qquad \widetilde{\Lambda}_{\pm}\widetilde{Z} = (g^{-1}(x,t)\Lambda g(x,t))\widetilde{Z}(x,t), \tag{19}$$

where

$$\widetilde{\Lambda}_{\pm}\widetilde{Z} = i\,\mathrm{ad}^{-1}_{S(x)}(\mathbb{1}-\widetilde{\pi}_0(x))\left\{\frac{d\widetilde{Z}}{dx}+\sum_{k=1}^{2}[\widetilde{h}_k(x),\mathrm{ad}^{-1}_{\mathcal{S}(x)}]\int_{\pm\infty}^{x}dy\left\langle[\widetilde{h}_k(y),\mathrm{ad}^{-1}_{\mathcal{S}(y)}\mathcal{S}_y],\widetilde{Z}(y)\right\rangle\right\}, \tag{20}$$

where $\widetilde{h}_k(x,t) = g^{-1}(x,t)H_k g(x,t)$, $\langle H_k, H_j\rangle = \langle \widetilde{h}_k(x,t), \widetilde{h}_j(x,t)\rangle = \delta_{jk}$. The main difficulty in deriving the explicit form of the gauge equivalent equations is in the need to express all relevant quantities, such as $g^{-1}q(x,t)g(x,t)$, $g^{-1}Ig(x,t)$, etc. in terms of $\mathcal{S}$ only. The starting point is that both I and J have a common set of eigensubspaces and also the fact that if $J\in\mathfrak{h}$ then also $J^3\in\mathfrak{h}$. In what follows we express the Cartan generators $H_1 = \mathrm{diag}\,(1,0,0,0,-1)$ and $H_2 = \mathrm{diag}\,(0,1,0,-1,0)$ as polynomials of J, see [11],

$$h_1 = \frac{J}{J_1}\frac{J^2-J_2^2}{J_1^2-J_2^2}, \qquad h_2 = \frac{J}{J_2}\frac{J^2-J_1^2}{J_2^2-J_1^2}, \qquad g^{-1}(x,t)q(x,t)g(x,t) = -i\,\mathrm{ad}^{-1}_{\mathcal{S}(x)}\mathcal{S}_x, \tag{21}$$

$$\widetilde{h}_1(x,t) = \frac{\mathcal{S}}{J_1}\frac{\mathcal{S}^2-J_2^2}{J_1^2-J_2^2}, \qquad \widetilde{h}_2(x,t) = \frac{\mathcal{S}}{J_2}\frac{\mathcal{S}^2-J_1^2}{J_2^2-J_1^2}. \tag{22}$$

Thus we have $I = I_1h_1 + I_2h_2$. Applying the gauge transformation to this formula allows us to find the desired expression for $g^{-1}Ig(x,t)$ in terms of $\mathcal{S}$:

$$f(\mathcal{S}) = I_1\widetilde{h}_1(x,t) + I_2\widetilde{h}_2(x,t) = f_1\mathcal{S} + f_3\mathcal{S}^3,$$

compare with equation (17). The set of constraints on $\mathcal{S}$ come from the fact that it has constant eigenvalues equal to $\pm J_1$, $\pm J_2$ and 0. These formulas together with the set of nonlinear constraints mentioned above completely determine the NLEE's gauge equivalent to the N-wave ones. Note that the number of the independent coefficients in $\mathcal{S}$ is equal to the number of the roots of the algebra, equal to 4 in the case of $so(5)$.

One can also derive explicitly the operator inverse to $\widetilde{\Lambda}$ generalizing the result in [12]:

$$\widetilde{\Lambda}^{-1}_{\pm}\widetilde{Z} = \frac{1}{i}(\mathbb{1}-\widetilde{\pi}_0)\int_{\pm\infty}^{x}dy\,[S(y),\widetilde{Z}(y)]. \tag{23}$$

The direct scattering problem for the Lax operator (1) is based on the Jost solutions and the scattering matrix $T(\lambda)$:

$$\lim_{x\to\infty}\psi(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \qquad \lim_{x\to-\infty}\phi(x,\lambda)e^{i\lambda Jx} = \mathbb{1}, \quad T(\lambda) = (\psi(x,\lambda))^{-1}\phi(x,\lambda). \tag{24}$$

The FAS $\chi^{\pm}(x,\lambda)$ of $L(\lambda)$ are related to the Jost solutions by [3, 5, 14]

$$\chi^{\pm}(x,\lambda) = \phi(x,\lambda)S^{\pm}(\lambda) = \psi(x,\lambda)T^{\mp}(\lambda)D^{\pm}(\lambda), \tag{25}$$

where $T^{\pm}(\lambda)$ and $S^{\pm}(\lambda)$, $D^{\pm}(\lambda)$ are elements of the group $SO(5)$ and are factors in the Gauss decomposition of the scattering matrix: $T(\lambda) = T^{-}(\lambda)D^{+}(\lambda)\hat{S}^{+}(\lambda) = T^{+}(\lambda)D^{-}(\lambda)\hat{S}^{-}(\lambda)$. Here $\hat{S}\equiv S^{-1}$, $S^{\pm}(\lambda,t) = \exp\left(\sum_{\alpha\in\Delta_+}s^{\pm}_{\alpha}(\lambda,t)E_{\pm\alpha}\right)$, and similar expressions for $T^{\pm}(\lambda,t)$. In particular $D^{\pm}(\lambda) = \exp(\pm d_1^{\pm}(\lambda)H_1 \pm d_2^{\pm}(\lambda)H_2)$ are analytic functions of λ for $\mathrm{Im}\,\lambda > 0$ and $\mathrm{Im}\,\lambda < 0$ respectively.

On the real axis $\chi^+(x,\lambda)$ and $\chi^-(x,\lambda)$ are related by $\chi^+(x,\lambda) = \chi^-(x,\lambda)G_0(\lambda)$, $G_0(\lambda) = \hat{S}^-(\lambda)S^+(\lambda)$, and the function $G_0(\lambda)$ can be considered as a minimal set of scattering data in the case of absence of discrete eigenvalues of (1) [3, 14, 5].

If the potential $q(x,t)$ of $L(\lambda)$ (1) satisfies the N-wave equation (15) then $f_{N-w}(\lambda) = \lambda I$ and

$$i\frac{dS^\pm}{dt} - [f_{N-w}(\lambda), S^\pm(\lambda,t)] = 0, \qquad i\frac{dT^\pm}{dt} - [f_{N-w}(\lambda), T^\pm(\lambda,t)] = 0, \qquad \frac{dD^\pm}{dt} = 0, \tag{26}$$

i.e., $S^\pm(\lambda)$ and $T^\pm(\lambda)$ satisfy linear ODE while the coefficients $d_j^\pm(\lambda)$ are generating functional of the local integrals of motion of the NLEE.

The scattering data for the gauge equivalent equations $\tilde{S}^\pm(\lambda)$, $\tilde{T}^\pm(\lambda)$ and $\tilde{D}^\pm(\lambda)$ are related to the scattering data of the "canonical" systems as follows:

$$\tilde{S}^\pm(\lambda) = T(0)S^\pm(\lambda)\hat{T}(0), \qquad \tilde{T}^\pm(\lambda) = T^\pm(\lambda), \qquad \tilde{D}^\pm(\lambda) = D^\pm(\lambda)\hat{T}(0). \tag{27}$$

$T(0)$ is an element of the Cartan subgroup of $SO(5)$. On the real λ axis again $\tilde{G}_0(\lambda) = \hat{\tilde{S}}^-(\lambda)\tilde{S}^+(\lambda)$ can be considered as a *minimal set of scattering data* for the gauge equivalent systems.

In order to obtain the soliton solutions for the gauge equivalent N-wave systems one needs to apply the Zakharov–Shabat dressing method to a regular FAS $\tilde{\chi}^\pm_{(0)}(x,\lambda)$ of $\tilde{L}$ with potential $\mathcal{S}_{(0)}$. Thus one gets a new singular solution $\tilde{\chi}^\pm_{(1)}(x,\lambda)$ of the Riemann–Hilbert problem with singularities located at prescribed positions $\lambda_1^\pm$. It is related to the regular one by the dressing Zakharov–Shabat factors [3] $\tilde{u}(x,\lambda)$:

$$\tilde{\chi}^\pm_{(1)}(x,\lambda) = \tilde{u}(x,\lambda)\tilde{\chi}^\pm_{(0)}(x,\lambda)\tilde{u}_-^{-1}(\lambda), \qquad \tilde{u}_-(\lambda) = \lim_{x\to-\infty}\tilde{u}(x,\lambda), \tag{28}$$

and the dressing factors for the gauge equivalent equations $\tilde{u}(x,\lambda)$ are related to $u(x,\lambda)$ by

$$\tilde{u}(x,\lambda) = g_{(0)}^{-1}(x,t)u^{-1}(x,\lambda=0)u(x,\lambda)g_{(0)}. \tag{29}$$

Then the new solutions $\tilde{\chi}^\pm_{(1)}(x,\lambda)$ will correspond to a potential $\mathcal{S}_{(1)}$ of $\tilde{L}$ with two discrete eigenvalues $\lambda_1^\pm$. The explicit construction of the dressing factors is discussed in [11] for the "canonical" N-wave systems and in [2] for their gauge equivalent. Note only that getting the explicit form of the dressing factors one gets the 1-soliton solution as well. The n-soliton solutions could be obtained by multiple applying the dressing method (n-times).

3 Generating (recursion) operators for MNLS type models

The MNLS type system for has a dispersion law $f(\lambda) = 2\lambda^2 J$. Our choice for J is a degenerate one, because $\alpha_1(J) = 0$; the set of roots $\Delta_1^+ = \{\alpha_2, \alpha_3, \alpha_4\}$ of $so(5)$, for which $\alpha(J) \neq 0$ labels the coefficients of the potential $q(x,t)$:

$$q(x,t) \equiv \sum_{\alpha\in\Delta_1^+}(q_\alpha E_\alpha + p_\alpha E_{-\alpha}) = \begin{pmatrix} 0 & 0 & q_{11} & q_{12} & 0 \\ 0 & 0 & q_1 & 0 & q_{12} \\ p_{11} & p_1 & 0 & q_1 & -q_{11} \\ p_{12} & 0 & p_1 & 0 & 0 \\ 0 & p_{12} & -p_{11} & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & -a \end{pmatrix}, \tag{30}$$

where we assumed $p_{jk} = q_{jk}^*$. The corresponding MNLS type system is of the form:

$$i\frac{dq_{12}}{dt} + \frac{1}{2a}\frac{d^2q_{12}}{dx^2} - \frac{1}{a}q_{12}(|q_1|^2 + |q_{11}|^2 + |q_{12}|^2) + \frac{i}{a}q_1q_{11,x} - \frac{i}{a}q_{11}q_{1,x} = 0,$$

$$i\frac{dq_{11}}{dt}+\frac{1}{a}\frac{d^2q_{11}}{dx^2}-\frac{1}{a}q_{11}\left(|q_1|^2+|q_{11}|^2+\frac{1}{2}|q_{12}|^2\right)+\frac{i}{a}q_{12}q^*_{1,x}+\frac{i}{2a}q_{12,x}q^*_1=0, \tag{31}$$
$$i\frac{dq_1}{dt}+\frac{1}{a}\frac{d^2q_1}{dx^2}-\frac{1}{a}q_1\left(|q_1|^2+|q_{11}|^2+\frac{1}{2}|q_{12}|^2\right)-\frac{i}{a}q_{12}q^*_{11,x}-\frac{i}{2a}q_{12,x}q^*_{11}=0.$$

Our choice of J means that $K=\mathrm{ad}_J$ has five different eigenvalues: $-2a$, $-a$, 0, a and $2a$. Then the minimal characteristic polynomial for K is:

$$K\left(K^2-a^2\right)\left(K^2-4a^2\right)=0. \tag{32}$$

Let us also introduce the projectors onto the eigensubspaces of K as follows:

$$\pi_{\pm2}=\frac{K\left(K^2-a^2\right)(K\pm2a)}{24a^4},\qquad \pi_{\pm1}=-\frac{K\left(K^2-4a^2\right)(K\pm a)}{6a^4},$$
$$\pi_0=\frac{\left(K^2-a^2\right)\left(K^2-4a^2\right)}{4a^4}.$$

Using the characteristic equation (32) it is easy to check that π_j are orthogonal projectors; i.e. they satisfy: $\pi_j\pi_k=\delta_{jk}\pi_j$ for all $j,k=\pm2$, ±1, 0 and that $K\pi_{\pm2}=\pm2a\pi_{\pm2}$, $K\pi_{\pm1}=\pm a\pi_{\pm1}$, $K\pi_0=0$. Thus any function $f(K)$ can be expressed in terms of these projectors: $f(K)=f(2a)\pi_2+f(a)\pi_1+f(0)\pi_0+f(-a)\pi_{-1}f(-2a)\pi_{-2}$ provided $f(\lambda)$ is regular for $\lambda=\pm2a$, $\pm a$ and 0.

The phase space $\mathcal{M}_J$ of the MNLS equations is the co-adjoint orbit of the $so(5)$ determined by J; its elements are matrices $q(x,t)$ satisfying $\pi_0q(x,t)=0$. Note also that $\mathrm{ad}_J=K$ introduces a grading on $\mathfrak{g}=\overset{2}{\underset{j=-2}{\oplus}}\mathfrak{g}_j$ and the projectors π_j project precisely onto $\mathfrak{g}_j$. Obviously $\mathfrak{g}_j=\pi_j\mathfrak{g}$, $\mathfrak{g}_0\equiv\mathfrak{g}_J$ and $\mathcal{M}_J\simeq\mathfrak{g}\backslash\mathfrak{g}_J$. The "squared solutions" again have the form (12), but now the corresponding FAS and the projectors π_0 are different due to the choice of J:

$$\chi^{\pm}(x,t,\lambda)=\phi(x,t,\lambda)S^{\pm}_J(t,\lambda)=\psi(x,t,\lambda)T^{\mp}_J(t,\lambda)D^{\pm}_J(\lambda). \tag{33}$$

Here $S^{\pm}_J$, $T^{\pm}_J$ and $D^{\pm}_J$ are the factors in the generalized Gauss decompositions of $T(t,\lambda)$ [6]:

$$T(t,\lambda)=T^-_J(t,\lambda)D^+_J(\lambda)\hat{S}^+_J(t,\lambda)=T^+_J(t,\lambda)D^-_J(\lambda)\hat{S}^-_J(t,\lambda), \tag{34}$$
$$S^{\pm}_J(t,\lambda)=\exp\left(\sum_{\alpha\in\Delta^+_1}s^{\pm}_{J,\alpha}(t,\lambda)E_{\pm\alpha}\right),\qquad T^{\pm}_J(t,\lambda)=\exp\left(\sum_{\alpha\in\Delta^+_1}t^{\pm}_{J,\alpha}(t,\lambda)E_{\pm\alpha}\right), \tag{35}$$
$$D^{\pm}_J(\lambda)=\exp\left(\pm d^{\pm}_1(\lambda)H_1\pm2d^{\pm}_2(\lambda)H_2+d^{\pm}_{\alpha_1}(\lambda)E_{\alpha_1}+d^{\pm}_{-\alpha_1}(\lambda)E_{-\alpha_1}\right). \tag{36}$$

If $q(x,t)$ evolves according to the MNLS (3) then

$$i\frac{dS^{\pm}_J}{dt}-2\lambda^2[J,S^{\pm}_J(t,\lambda)]=0,\qquad i\frac{dT^{\pm}_J}{dt}-2\lambda^2[J,T^{\pm}_J(t,\lambda)]=0,\qquad \frac{dD^{\pm}_J}{dt}=0. \tag{37}$$

This means that the MNLS equation (3) has four series of integrals of motion as compared to the two series of integrals for the $so(5)$ 4-wave system. This is due to the special (degenerate) choice of the dispersion law $f_{\mathrm{MNLS}}=2\lambda^2J$. We have to remember, however, that only two of these four series are in involution, which in turn is related to the non-commutativity of the subalgebra $\mathfrak{g}_J$. As a result we get the following expression for the recursion operator:

$$\Lambda_{\pm}Z=K^{-1}(\mathbb{1}-\pi_0)\left(i\frac{dZ}{dx}+[q(x),Z(x)]+i\left[q(x),\int_{\pm\infty}^{x}dy\;\pi_0[q(y),Z(y)]\right]\right), \tag{38}$$

where we assume that $Z\in\mathcal{M}_J$, i.e. $\pi_0Z(x)=0$.

In order to evaluate the gauge equivalent recursion operator we again make use of the gauge covariant approach. First it allows one easily to recalculate the projectors on the eigensubspaces of $\mathrm{ad}_{S(x)} \equiv \widetilde{K}(x) = g_0^{-1} K g_0(x,t)$:

$$\widetilde{\pi}_{\pm 2} = \frac{\widetilde{K}(\widetilde{K}^2 - a^2)(\widetilde{K} \pm 2a)}{24a^4}, \qquad \widetilde{\pi}_{\pm 1} = -\frac{\widetilde{K}(\widetilde{K}^2 - 4a^2)(\widetilde{K} \pm a)}{6a^4},$$
$$\widetilde{\pi}_0 = \frac{(\widetilde{K}^2 - a^2)(\widetilde{K}^2 - 4a^2)}{4a^4}.$$

In particular, these formulae allow us to cast the MHF system (9) in the form:

$$iS_t - \frac{5}{4a^2}[S, S_{xx}] + \frac{1}{4a^4}\left((\mathrm{ad}_S)^3 S_x\right)_x = 0, \tag{39}$$

where $\mathcal{S}$ is constrained by $\mathcal{S}\left(\mathcal{S}^2 - a^2\right)^2 = 0$. In addition the operator $\widetilde{K}(x,t)$ satisfy the equation (32). One way to derive the explicit form of the recursion operators $\widetilde{\Lambda}_\pm$ is to apply the gauge transformation to $\Lambda_\pm$ (38). For degenerate choice of J this is more difficult due to the fact that $\mathfrak{g}_J$ is non-Abelian. Here we only provide the explicit form of the operators inverse to $\widetilde{\Lambda}_\pm$

$$\tilde{\Lambda}_\pm^{-1}\tilde{Z} = \frac{1}{i}(\mathbb{1} - \widetilde{\pi}_0)\int_{\pm\infty}^{x} dy\,[S(y), \widetilde{Z}(y)], \qquad \widetilde{\pi}_0\widetilde{Z}(x) = 0. \tag{40}$$

The scattering data for the gauge equivalent MHF equations $\tilde{S}_J^\pm(\lambda)$, $\tilde{T}_J^\pm(\lambda)$ and $\tilde{D}_J^\pm(\lambda)$ are related to the scattering data of the "canonical" systems as follows: $\tilde{S}_J^\pm(\lambda) = T_J(0) S_J^\pm(\lambda)\hat{T}_J(0)$, $\tilde{T}_J^\pm(\lambda) = T_J^\pm(\lambda)$, and $\tilde{D}_J^\pm(\lambda) = D_J^\pm(\lambda)\hat{T}_J(0)$ and $\tilde{T}_J(0)$ is an element of the subgroup $\mathcal{G}_J$ of $SO(5)$. On the real λ axis again $\tilde{G}_{J,0}(\lambda) = \hat{\tilde{S}}_J^-(\lambda)\tilde{S}_J^+(\lambda)$ can be considered as a *minimal set of scattering data* for the gauge equivalent systems.

In order to obtain the soliton solutions for the gauge equivalent MHF systems one needs to apply the Zakharov–Shabat dressing method to a regular FAS $\tilde{\chi}_{(0)}^\pm(x,\lambda)$ of $\tilde{L}$ with potential $\mathcal{S}_{(0)}$. Thus one gets a new singular solution $\tilde{\chi}_{(1)}^\pm(x,\lambda)$ of the Riemann–Hilbert problem with singularities located at prescribed positions $\lambda_1^\pm$. It is related to the regular one by the dressing factors $\tilde{u}(x,\lambda)$ whose structure is analogous to the ones presented in Section 2 above.

4 Hierarchy of Hamiltonian structures for MNLS type models

It is well known that both classes of NLEE possess hierarchies of Hamiltonian structures. The phase spaces $\mathcal{M}_J$ and $\widetilde{\mathcal{M}}_\mathcal{S}$ corresponding to the standard and the pole gauge

$$\mathcal{M}_J \equiv \{q(x,t),\ \pi_0 q(x,t) = 0\}, \qquad \widetilde{\mathcal{M}}_\mathcal{S} \equiv \{S(x,t),\ S(x,t) = g^{-1}Jg(x,t)\}, \tag{41}$$

where in addition we assume that $q(x,t)$ and $S(x,t) - J$ are smooth functions tending to zero fast enough for $|x| \to \infty$.

The hierarchies of symplectic structures defined on $\mathcal{M}_J$ and $\widetilde{\mathcal{M}_\mathcal{S}}$ are generated by the corresponding recursion operators and are given by the following families of compatible two-forms:

$$\Omega_{\mathrm{q}}^{(k)} = i\int_{-\infty}^{\infty} dx \left\langle \delta q \underset{,}{\wedge} \Lambda^k \mathrm{ad}_J^{-1}\delta q(x,t)\right\rangle, \qquad \tilde{\Omega}_\mathcal{S}^{(k)} = i\int_{-\infty}^{\infty} dx \left\langle \delta\mathcal{S} \underset{,}{\wedge} \tilde{\Lambda}^k \mathrm{ad}_\mathcal{S}^{-1}\delta\mathcal{S}(x,t)\right\rangle.$$

Note also that the gauge transformation relates nontrivialy the symplectic structures, i.e. $\Omega_{\mathrm{q}}^{(k)} \simeq \tilde{\Omega}_\mathcal{S}^{(k+2)}$ (for NLS–HFE equivalence, see [16, 10]). These two hierarchies are dynamically equivalent.

5 Discussion

Here we derived new results about the construction of the recursion operators for MNLS models and their gauge equivalent MHF systems by using the gauge covariant approach. These results can be naturally extended in several directions. First, one can apply the reduction group method [17] in order to investigate the $\mathbb{Z}_2$ and other finite order reduction of the MNLS systems thus extending the results of [11, 1] for the N-wave systems. Second, it is natural to extend these reductions also to the gauge equivalent systems (5) and (9). Such results may display additional properties of the class isoparametric hypersurfaces [13] described by the equations (17) in the framework of the $so(5)$ algebra. A third open problem is to study reductions of the gauge equivalent systems and the spectral decompositions for the relevant recursion operators.

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Nonlocal Brackets and Integrable String Models

Victor D. GERSHUN

Kharkiv Institute of Physics and Technology, 1 Akademicheskaya Str., 61108 Kharkiv, Ukraine
E-mail: *gershun@kipt.kharkov.ua*

The closed string model in the background gravity field is considered as a bi-Hamiltonian system in assumption that string model is the integrable model for particular kind of the background fields. The dual nonlocal Poisson brackets (PB), depending of the background fields and of their derivatives, are obtained. The integrability condition is formulated as compatibility of the bi-Hamiltonity condition and the Jacobi identity of the dual Poisson bracket. It is shown that the dual brackets and dual Hamiltonians can be obtained from the canonical PB and from the initial Hamiltonian by imposing the second kind constraints on the initial dynamical system, on the closed string model in the constant background fields, as example. Two types of the nonlocal brackets are introduced.Constant curvature and time-dependent metrics are considered, as examples. It is shown, that the Jacobi identities for the nonlocal brackets have particular solution for the space-time coordinates, as matrix representation of the simple Lie group.

1 Introduction

The bi-Hamiltonian approach [1–3] to the integrable systems was initiated by Magri [4] for investigation of the integrability of the KdV equation.

Definition 1. A finite dimensional dynamical system with $2N$ degrees of freedom x^a, $a = 1, \ldots, 2N$ is integrable, if it is described by the set of the n integrals of motion $F_1, \ldots, F_n$ in involution under some Poisson bracket (PB)

$$\{F_i, F_k\}_{\mathrm{PB}} = 0.$$

The dynamical system is completely solvable, if $n = N$. Any integral of motion (or any linear combination of them) can be considered as the Hamiltonian $H_k = F_k$.

Definition 2. The bi-Hamiltonity condition has following form:

$$\dot{x}^a = \frac{dx^a}{dt} = \{x^a, H_1\}_1 = \cdots = \{x^a, H_N\}_N. \tag{1}$$

The hierarchy of new PB arises in this connection:

$$\{,\}_1,\ \{,\}_2,\ \ldots,\ \{,\}_N.$$

The hierarchy of new dynamical systems arises under the new time coordinates t_k.

$$\frac{dx^a}{dt_{n+k}} = \{x^a, H_n\}_{k+1} = \{x^a, H_k\}_{n+1}. \tag{2}$$

The new equations of motion describe the new dynamical systems, which are dual to the original system, with the dual set of the integrals of motion. The dual set of the integrals of motion can be obtained from the original one by the mirror transformations and by contraction of the integrals of motion algebra. Contraction of the integrals of motion algebra means that

the dynamical system belongs to the orbits of corresponding generators and describes the invariant subspace. The set of the commuting integrals of motion belongs to Cartan subalgebra of this algebra. KdV equation is one of the most interesting examples of the infinite-dimensional integrable mechanical systems with soliton solutions. We consider the dynamical systems with constraints. In this case, first-kind constraints are generators of the gauge transformations and they are integrals of motion. First-kind constraints $F_k(x^a) \approx 0$, $k = 1, 2, \ldots$ form the algebra of constraints under some PB

$$\{F_i, F_k\}_{\mathrm{PB}} = C_{ik}^l F_l \approx 0.$$

The structure functions C_{ik}^l may be functions of the phase space coordinates in general case. The second kind constraints $f_k(x^a) \approx 0$ are the representations of the first kind constraints algebra. The second kind constraints is defined by the condition

$$\{f_i, f_k\} = C_{ik} \neq 0.$$

The reversible matrix C_{ik} is not a constraint, and also it is a function of phase space coordinates. The second-kind constraints take part in deformation of the $\{,\}_{\mathrm{PB}}$ to the Dirac bracket $\{,\}_D$. As a rule, such deformation leads to nonlinear and to nonlocal brackets. The bi-Hamiltonity condition leads to the dual PB that are nonlinear and nonlocal brackets as a rule. We suppose that the dual brackets can be obtained from the initial canonical bracket under the imposition of the second kind constraints. We have applied [5–9] bi-Hamiltonity approach to the investigation of the integrability of the closed string model in the arbitrary background gravity field and antisymmetric B-field. The bi-Hamiltonity condition and the Jacobi identities for the dual brackets were considered as the integrability condition for a closed string model. They led to some restrictions on the background fields. B.A. Dubrovin and S.P. Novikov have considered the local dual PB of the similar type [10] in the application to the Hamiltonian hydrodynamical models. The PB of the hydrodynamical type for the phase coordinate functions $u^i(x,t)$ is defined by the formula

$$\{u^i(x), u^k(y)\} = g^{ik}(u(x))\frac{\partial}{\partial x}\delta(x-y) - g^{ij}\Gamma_{jl}^k(u(x))u_x^l\delta(x-y).$$

There $g_{ik}(u)$, $\Gamma_{jl}^k(u)$ are the arbitrary functions of the phase space coordinates and $u_x = \partial_x u$. The Jacobi identity is satisfied if g_{ik} is the Riemannian metric without torsion,the curvature tensor is equal to zero. The metric tensor is constant, locally. O.I. Mokhov and E.V. Ferapontov introduced the nonlocal PB [11, 12]. The Ferapontov's nonlocal PB is:

$$\begin{aligned}\{u^i(x), u^k(y)\} = {} & g^{ik}(u)\frac{\partial}{\partial x}\delta(x-y) - g^{ij}\Gamma_{jl}^k u_x^l\delta(x-y) \\ & + \omega_k^{(s)i}(u(x))u_x^j\nu(x-y)\omega_d^{(s)k}(u(y))u_y^d,\end{aligned}$$

where $\nu(x-y) = \mathrm{sgn}\,(x-y) = (\frac{d}{dx})^{-1}\delta(x-y)$. This PB was used for description of the Hamiltonian system of the hydrodynamical type. There are systems with functionals of the hydrodynamical type. The density of this functionals does not depend on the derivatives u_x^k, u_{xx}^k, ..., and Hamiltonian is also a functional of the hydrodynamical type. On the contrary to these models, the functionals of the closed string model is depended of the derivatives of the string coordinates. The plan of the paper is the following. In the second section we considered closed string model in the arbitrary background gravity field. We suppose that this model is an integrable model for some configurations of the background fields. The bi-Hamiltonity condition and the Jacobi identities for the dual PB must be a result of the integrability condition, which is restricted the possible configurations of the background fields.

2 Closed string in the background fields

The string model in the background gravity field is described by the system of equations:

$$\ddot{x}^a - x''^a + \Gamma^a_{bc}(x)(\dot{x}^b\dot{x}^c - x'^b x'^c) = 0,$$
$$g_{ab}(x)(\dot{x}^a\dot{x}^b + x'^a x'^b) = 0, \qquad g_{ab}(x)\dot{x}^a x'^b = 0,$$

where $\dot{x}^a = \frac{dx^a}{d\tau}$, $x'^a = \frac{dx^a}{d\sigma}$. We will consider the Hamiltonian formalism. The closed string in the background gravity field is described by first-kind constraints in the Hamiltonian formalism:

$$h_1 = \frac{1}{2}g^{ab}(x)p_a p_b + \frac{1}{2}g_{ab}(x)x'^a x'^b \approx 0, \qquad h_2 = p_a x'^a \approx 0, \tag{3}$$

where $a, b = 0, 1, \ldots, D-1$, $x^a(\tau,\sigma)$, $p_a(\tau,\sigma)$ are the periodical functions on σ with the period on π. The original PB are the symplectic PB

$$\{x^a(\sigma), p_b(\sigma')\}_1 = \delta^a_b\delta(\sigma-\sigma'), \{x^a(\sigma), x^b(\sigma')\}_1 = \{p_a(\sigma), p_b(\sigma')\}_1 = 0.$$

The Hamiltonian equations of motion of the closed string, in the arbitrary background gravity field under the Hamiltonian $H_1 = \int_0^\pi h_1 d\sigma$ and PB $\{\ ,\ \}_1$, are

$$\dot{x}^a = g^{ab}p_b, \qquad \dot{p}_a = g_{ab}x''^b - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c - \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b}.$$

The dual PB are obtained from the bi-Hamiltonity condition

$$\dot{x}^a = \left\{x^a, \int_0^\pi h_1(\sigma')d\sigma'\right\}_1 = \left\{x^a, \int_0^\pi h_2(\sigma')d\sigma'\right\}_2,$$
$$\dot{p}_a = \left\{p_a, \int_0^\pi h_1(\sigma')d\sigma'\right\}_1 = \left\{p_a, \int_0^\pi h_2(\sigma')d\sigma'\right\}_2. \tag{4}$$

They have the following form:

Proposition 1.

$$\begin{aligned}\{A(\sigma), B(\sigma')\}_2 &= \frac{\partial A}{\partial x^a}\frac{\partial B}{\partial x^b}[[\omega^{ab}(\sigma)+\omega^{ab}(\sigma')]\nu(\sigma'-\sigma) + [\Phi^{ab}(\sigma)+\Phi^{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma)\\
&+ [\Omega^{ab}(\sigma)+\Omega^{ab}(\sigma')]\delta(\sigma'-\sigma)] + \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial p_b}[[\omega_{ab}(\sigma)+\omega_{ab}(\sigma')]\nu(\sigma'-\sigma)\\
&+ [\Phi_{ab}(\sigma)+\Phi_{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma) + [\Omega_{ab}(\sigma)+\Omega_{ab}(\sigma')]\delta(\sigma'-\sigma)]\\
&+ \left[\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_b} + \frac{\partial A}{\partial p_b}\frac{\partial B}{\partial x^a}\right][[\omega^a_b(\sigma)+\omega^a_b(\sigma')]\nu(\sigma'-\sigma) + [\Phi^a_b(\sigma)+\Phi^a_b(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma)]\\
&+ \left[\frac{\partial A}{\partial x^a}\frac{\partial B}{\partial p_b} - \frac{\partial A}{\partial p_b}\frac{\partial B}{\partial x^a}\right][\Omega^a_b(\sigma)+\Omega^a_b(\sigma')]\delta(\sigma'-\sigma).\end{aligned}$$

The arbitrary functions A, B, ω, Φ, Ω are the functions of the $x^a(\sigma)$, $p_a(\sigma)$. The functions ω^{ab}, ω_{ab}, Φ^{ab}, Φ_{ab} are the symmetric functions on a, b and Ω^{ab}, Ω_{ab} are the antisymmetric functions to satisfy the condition $\{A,B\}_2 = -\{B,A\}_2$.

The equations of motion under the Hamiltonian $H_2 = \int_0^\pi h_2(\sigma')d\sigma'$ and PB $\{\ ,\ \}_2$ are

$$\begin{aligned}\dot{x}^a &= -\omega^a_b x^b + 2\omega^{ab}p_b + 2\Phi^{ab}p'_b - 2\Phi^a_b x''^b + 2\Omega^a_b x'^b - 2\Omega^{ab}p'_b\\
&+ \int_0^\pi d\sigma'\left[\omega^a_b x'^a + \frac{d\omega^{ab}}{d\sigma'}p_b\right]\nu(\sigma'-\sigma) + \frac{d\Phi^{ab}}{d\sigma}p'_b - \frac{d\Phi^a_b}{d\sigma}x'^b,\end{aligned}$$

$$\dot{p}_a = -\omega_{ab}x^b - 2\Phi_{ab}x''^b + 2\Omega_{ab}x'^b + 2\omega_a^b p_b + 2\Phi_a^b p_b'' + 2\Omega_a^b p_b' + \int_0^\pi d\sigma' \left[\omega_{ab}x'^b + \frac{d\omega_a^b}{d\sigma'}p_b\right]\nu(\sigma'-\sigma) - \frac{d\Phi_{ab}}{d\sigma}x'^b + \frac{d\Phi_a^b}{d\sigma}p_b'.$$

The bi-Hamiltonity condition (4) is led to the two constraints

$$-\omega_b^a x^b + 2\omega^{ab}p_b + 2\Phi^{ab}p_b'' - -2\Phi_b^a x''^b + 2\Omega_b^a x'^b - 2\Omega^{ab}p_b' + \int_0^\pi d\sigma' \left[\omega_b^a x'^a + \frac{d\omega^{ab}}{d\sigma'}p_b\right]\nu(\sigma'-\sigma) + \frac{d\Phi^{ab}}{d\sigma}p_b' - \frac{d\Phi_b^a}{d\sigma}x'^b = g^{ab}p_b,$$

$$-\omega_{ab}x^b - 2\Phi_{ab}x''^b + 2\Omega_{ab}x'^b + 2\omega_a^b p_b + 2\Phi_a^b p_b'' + 2\Omega_a^b p_b' + \int_0^\pi d\sigma' \left[\omega_{ab}x'^b + \frac{d\omega_a^b}{d\sigma'}p_b\right]\nu(\sigma'-\sigma) - \frac{d\Phi_{ab}}{d\sigma}x'^b + \frac{d\Phi_a^b}{d\sigma}p_b' = g_{ab}x''^b - \frac{1}{2}\frac{\partial g^{bc}}{\partial x^a}p_b p_c - \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a}x'^b x'^c + \frac{\partial g_{ac}}{\partial x^b}x'^b x'^c.$$

In reality, there is the list of the constraints depending on the possible choice of the unknown functions ω, Ω, Φ. In the general case, there are both the first-kind constraints and the second-kind constraints too. Also it is possible to solve the constraints equations as the equations for the definition of the functions ω, Φ, Ω. We considered the latter possibility and we obtained the following consistent solution of the bi-Hamiltonity condition:

$$\Phi^{ab} = 0, \qquad \Omega^{ab} = 0, \qquad \Phi_b^a = 0, \qquad \Omega_b^a = 0, \qquad \frac{\partial\omega^{ab}}{\partial x^c}x^c + 2\omega^{ab} = g^{ab},$$

$$\omega_{ab} = \frac{1}{2}\frac{\partial^2\omega^{cd}}{\partial x^a\partial x^b}p_c p_d, \qquad \omega_b^a = -\frac{\partial\omega^{ac}}{\partial x^b}p_c,$$

$$\Phi_{ab} = -\frac{1}{2}g_{ab}, \qquad \Omega_{ab} = \frac{1}{2}\left(\frac{\partial\Phi_{bc}}{\partial x^a} - \frac{\partial\Phi_{ac}}{\partial x^b}\right)x'^c, \qquad \frac{\partial\omega^{ab}}{\partial p_c} = 0.$$

Remark 1. In distinct from the PB of the hydrodynamical type, we need to introduce the separate PB for the coordinates of the Minkowski space and for the momenta because, the gravity field is not depend of the momenta. Although, this difference is vanished under the such constraint as $f(x^a, p_a) \approx 0$.

Consequently, the dual PB for the phase space coordinates are

$$\{x^a(\sigma), x^b(\sigma')\}_2 = [\omega^{ab}(\sigma) + \omega^{ab}(\sigma')]\nu(\sigma'-\sigma),$$

$$\{p_a(\sigma), p_b(\sigma')\}_2 = \left[\frac{\partial^2\omega_{cd}(\sigma)}{\partial x^a\partial x^b}p_c p_d + \frac{\partial^2\omega_{cd}(\sigma')}{\partial x^a\partial x^b}p_c p_d\right]\nu(\sigma'-\sigma) - \frac{1}{2}[g_{ab}(\sigma) + g_{ab}(\sigma')]\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma) + \left[\frac{\partial g_{ac}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^a}\right]x'^c(\sigma)\delta(\sigma'-\sigma),$$

$$\{x^a(\sigma), p_b(\sigma')\}_2 = -\left[\frac{\partial\omega^{ac}(\sigma)}{\partial x^b}p_c + \frac{\partial\omega^{ac}(\sigma')}{\partial x^b}p_c\right]\nu(\sigma'-\sigma),$$

$$\{p_a(\sigma), x^b(\sigma')\}_2 = -\left[\frac{\partial\omega^{bc}(\sigma)}{\partial x^a}p_c + \frac{\partial\omega^{bc}(\sigma')}{\partial x^c}p_c\right]\nu(\sigma'-\sigma). \tag{5}$$

The function $\omega^{ab}(x)$ is satisfied on the equation:

$$\frac{\partial\omega^{ab}}{\partial x^c}x^c + 2\omega^{ab} = g^{ab}. \tag{6}$$

The Jacobi identities for the PB $\{\,,\,\}_2$ are led to the nonlocal consistence conditions on the unknown function $\omega^{ab}(\sigma)$. We can calculate unknown metric tensor $g^{ab}(\sigma)$ by substitution of the solution of the consistence condition for ω^{ab} to the equation (6). The Jacobi identity

$$\{x^a(\sigma), x^b(\sigma')\}x^c(\sigma'')\}_J \\ \equiv \{x^a(\sigma), x^b(\sigma')\}x^c(\sigma'')\} + \{x^c(\sigma''), x^a(\sigma)\}x^b(\sigma')\} + \{x^b(\sigma'), x^c(\sigma'')\}x^a(\sigma)\} = 0 \quad (7)$$

is led to the following nonlocal analogy of the WDVV consistence condition:

$$\left[\frac{\partial\omega^{ab}(\sigma)}{\partial x^d}[\omega^{dc}(\sigma)+\omega^{dc}(\sigma'')] - \frac{\partial\omega^{ac}(\sigma)}{\partial x^d}[\omega^{db}(\sigma)+\omega^{db}(\sigma')]\right]\nu(\sigma'-\sigma)\nu(\sigma''-\sigma) \quad (8)$$
$$+\left[\frac{\partial\omega^{cb}(\sigma')}{\partial x^d}[\omega^{da}(\sigma')+\omega^{da}(\sigma)] - \frac{\partial\omega^{ab}(\sigma')}{\partial x^d}[\omega^{dc}(\sigma')+\omega^{dc}(\sigma'')]\right]\nu(\sigma-\sigma')\nu(\sigma''-\sigma')$$
$$+\left[\frac{\partial\omega^{ac}(\sigma'')}{\partial x^d}[\omega^{db}(\sigma'')+\omega^{db}(\sigma')] - \frac{\partial\omega^{cb}(\sigma'')}{\partial x^d}[\omega^{da}(\sigma'')+\omega^{da}(\sigma)]\right]\nu(\sigma-\sigma'')\nu(\sigma'-\sigma'')=0.$$

This equation has the particular solution of the following form:

$$\frac{\partial\omega^{ab}(\sigma)}{\partial x^d}[\omega^{dc}(\sigma)+\omega^{dc}(\sigma'')] - \frac{\partial\omega^{ac}(\sigma)}{\partial x^d}[\omega^{db}(\sigma)+\omega^{db}(\sigma')] \\ = [T^b,T^c]T^a]f(\sigma,\sigma',\sigma'')\nu(\sigma''-\sigma)\nu(\sigma'-\sigma). \quad (9)$$

T^a, $a=0,1,\ldots,D-1$, is the matrix representation of the simple Lie algebra and $f(\sigma,\sigma',\sigma'')$ is arbitrary function. The Jacobi identity is satisfied on the Jacobi identity of the simple Lie algebra in this case:

$$([T^a,T^b]T^c] + [T^c,T^a]T^b] + [T^b,T^c]T^a])f(\sigma,\sigma',\sigma'') = 0$$

and we used the relation $\nu^2(\sigma-\sigma')=1$. The local solution of the Jacobi identities is led to the constant metric tensor. The rest Jacobi identities are cumbrous and we do not reduce this expressions here. The symmetric factor of σ, σ' of the antisymmetric functions $\nu(\sigma'-\sigma)$, $\frac{\partial}{\partial\sigma}\delta(\sigma-\sigma')$ in the right side of the PB can be both sum of the functions of σ and σ', and production of them. Last possibility can be used in the vielbein formalism.

Proposition 2. *The bi-Hamiltonity condition can be solved in the terms PB $\{\,,\,\}_2$, which have the following form:*

$$\{x^a(\sigma),x^b(\sigma')\}_2 = e^a_\mu(\sigma)e^b_\mu(\sigma')\nu(\sigma'-\sigma),$$
$$\{x^a(\sigma),p_b(\sigma'\}_2 = -e^a_\mu(\sigma)\frac{\partial e^c_\mu(\sigma'}{\partial x^b}p_c(\sigma')\nu(\sigma'-\sigma),$$
$$\{p_a(\sigma),p_b(\sigma')\}_2 = \frac{\partial e^c_\mu(\sigma)}{\partial x^a}p_c(\sigma)\frac{\partial e^d_\mu(\sigma')}{\partial x^b}p_d(\sigma')\nu(\sigma'-\sigma) - e^\mu_a(\sigma)e^\mu_b(\sigma')\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma)$$
$$+\left[\frac{\partial e^\mu_a}{\partial x^c}e^\mu_b - \frac{\partial e^\mu_b}{\partial x^c}e^\mu_a - \frac{\partial e^\mu_c}{\partial x^a}e^\mu_b + \frac{\partial e^\mu_c}{\partial x^b}e^\mu_a\right]x'^c(\sigma)\delta(\sigma'-\sigma), \quad (10)$$

where veilbein e^a_μ is satisfied on the additional conditions:

$$g^{ab} = \eta^{\mu\nu}e^a_\mu e^b_\nu, \qquad g_{ab} = \eta_{\mu\nu}e^\mu_a e^\nu_b$$

and $\eta^{\mu\nu}$ is the metric tensor of the flat space.

The particular solution of the Jacobi identity is

$$\frac{\partial e^a_\mu(\sigma)}{\partial x^d} e^b_\mu(\sigma') e^d_\nu(\sigma) e^c_\nu(\sigma'') - \frac{\partial e^a_\mu(\sigma)}{\partial x^d} e^c_{mu}(\sigma'') e^d_\nu(\sigma) e^b_\nu(\sigma') = [T^b, T^c]T^a] f(\sigma, \sigma', \sigma'')\nu(\sigma'' - \sigma)\nu(\sigma' - \sigma). \tag{11}$$

As example let me consider the the constant curvature space.

Example 1. The constant curvature space is described by the metric tensor $g_{ab}(x(\sigma))$ and by it inverse tensor g^{-1}_{ab}:

$$g_{ab} = \eta_{ab} + \frac{k x_a x_b}{1 - kx^2}, \qquad g^{ab} \equiv g^{-1}_{ab} = \eta_{ab} - k x_a x_b.$$

Proposition 3. *Dual PB* $\{\,,\,\}_2$ *are:*

$$\begin{aligned}
&\{x_a(\sigma), x_b(\sigma')\} = [\eta_{ab} - k x_a(\sigma) x_b(\sigma')]\nu(\sigma' - \sigma),\\
&\{x_a(\sigma), p_b(\sigma')\} = k x_a(\sigma) p_b(\sigma')\nu(\sigma' - \sigma),\\
&\{p_a(\sigma), p_b(\sigma')\} = -k p_a(\sigma) p_b(\sigma')\nu(\sigma' - \sigma)\\
&\quad - \frac{1}{2}\left[2\eta_{ab} + \frac{k x_a x_b}{1 - kx^2}(\sigma) + \frac{k x_a x_b}{1 - kx^2}(\sigma')\right] \frac{\partial}{\partial \sigma'}\delta(\sigma' - \sigma) + \frac{x_a x'_b - x_b x'_a}{2(1 - kx^2)}\delta(\sigma' - \sigma).
\end{aligned} \tag{12}$$

The Jacobi identity (7) is led to the equation

$$\begin{aligned}
&[\eta_{ab} x_c(\sigma'') - \eta_{ac} x_b(\sigma')]\nu(\sigma' - \sigma)\nu(\sigma - \sigma'') + [\eta_{bc} x_a(\sigma) - \eta_{ba} x_c(\sigma'')]\nu(\sigma - \sigma')\nu(\sigma' - \sigma'')\\
&\quad + [\eta_{ca} x_b(\sigma') - \eta_{cb} x_a(\sigma)]\nu(\sigma' - \sigma'')\nu(\sigma'' - \sigma) = 0.
\end{aligned}$$

The particular solution of this equation is:

$$\eta_{ab} x_c(\sigma'') - \eta_{ac} x_b(\sigma') = [T_b, T_c]T_a] f(\sigma, \sigma', \sigma'')\nu(\sigma'' - \sigma)\nu(\sigma' - \sigma). \tag{13}$$

Consequently, the space-time coordinate $x_a(\sigma)$ is the matrix representation of the simple Lie algebra. The Jacobi identity $\{x_a(\sigma), x_b(\sigma')\}p_c(\sigma'')\}_J$ is led to the equation

$$k\eta_{ab} p_c(\sigma'')\nu(\sigma' - \sigma)[\nu(\sigma'' - \sigma) + \nu(\sigma'' - \sigma')] = 0. \tag{14}$$

These results can be obtained from the veilbein formalism under the following ansatz for the veilbein of the constant curvature space:

$$e^{a(s)}_\mu = n_\mu\big(m^{(s)}_1 n^a + \sqrt{k} m^{(s)}_2 x^a\big), \qquad e^{\mu(s)}_a = n^\mu g_{ab}\big(m^{(s)}_1 n^b + \sqrt{k} m^{(s)}_2 x^b\big),$$

where $n^2_\mu = 1$, $m^{(s)}_1 m^{(s)}_1 = 1$, $m^{(s)}_2 m^{(s)}_2 = -1$, $m^{(s)}_1 m^{(s)}_2 = 0$, $n^a n^b = \delta^{ab}$ and (s) is number of the solution of the equations

$$e^a_\mu e^b_\mu = g^{ab}, \qquad e^\mu_a e^\mu_b = g_{ab}, \qquad e^a_\mu e^\mu_b = \delta^a_b.$$

The following example is time-dependent metric space.

Example 2. The time-dependent metric in the light-cone variables has form:

$$ds^2 = g_{ik}(x^+) dx^i dx^k + g_{++}(x^+) dx^+ dx^+ + 2 g_{+-} dx^+ dx^-.$$

We are used Poisson brackets (5) for the space coordinates $x^a = \{x^i, x^+, x^-\}$, $i = 1, 2, \ldots, D-2$. We introduced the light-cone gauge as two first kind constraints:

$$F_1(\sigma) = x'^+ \approx 0, \qquad F_2(\sigma) = p'_- \approx 0,$$

and we imposed them on the equations of motion and on the Jacobi identities. The Jacobi identities are reduced to the simple equation

$$\frac{\partial\omega^{ab}}{\partial x^+}\omega^{+c} - \frac{\partial\omega^{ac}}{\partial x^+}\omega^{+b} = 0.$$

We obtained following result from this equation and additional condition (6): there is constant background gravity field only for the non-degenerate metric.

2.1 Constant background fields

In this subsection we are supplemented the bi-Hamiltonity condition (4) by the mirror transformations of the integrals of motion.

$$\dot{x}^a = \left\{x^a, \int_0^\pi h_1 d\sigma\right\}_1 = \left\{x^a, \int_0^\pi \pm h_2 d\sigma'\right\}_{\pm 2}.$$

The dual PB are

$$\{x^a(\sigma), x^b(\sigma')\}_{\pm 2} = \pm g^{ab}\nu(\sigma'-\sigma), \qquad \{x^a(\sigma), p_b(\sigma')\}_{\pm 2} = 0,$$
$$\{p_a(\sigma), p_b(\sigma')\}_{\pm 2} = \mp\frac{1}{2} g_{ab}\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma).$$

The dual dynamical system

$$\dot{x}^a = \{x^a, \pm H_2\}_1 = \{x^a, H_1\}_{\pm 2}.$$

is the left(right) chiral string

$$\dot{x}^a = \pm x'^a, \qquad \dot{p}_a = \pm p_a'.$$

In the terms of the Virasoro operators

$$L_k = \frac{1}{4\pi}\int_0^\pi (h_1+h_2)e^{ik\sigma}d\sigma, \qquad \bar{L}_k = \frac{1}{4\pi}\int_0^\pi (h_1-h_2)e^{ik\sigma}d\sigma,$$

the first kind constraints form the Vir $\oplus$ Vir algebra under the PB $\{\,,\,\}_1$.

$$\{L_n, L_m\}_1 = -i(n-m)L_{n+m}, \qquad \{\bar{L}_n, \bar{L}_m\}_1 = -i(n-m)\bar{L}_{n+m}, \qquad \{L_n, \bar{L}_m\}_1 = 0.$$

The dual set of the integrals of motion is obtained from initial it by the mirror transformations

$$H_1 \to \pm H_2, \qquad L_0 \to \pm L_o, \qquad \bar{L}_0 \to \mp\bar{L}_0, \qquad \tau \to \sigma.$$

and by the contraction of the first kind constraints algebra $L_n = 0$, or $\bar{L}_n = 0$, $n \neq 0$. Another way to obtain the dual brackets is the imposition of the second kind constraints on the initial dynamical system, by such manner, that $F_i = F_k$ for $i \neq k$, $i,k = 1,2,\ldots$ on the constraints surface $f(x^a, p_a) = 0$. The constraints $f_a^{(-)}(x,p) = p_a - g_{ab}x'^b \approx 0$ or $f_a^{(+)} = p_a + g_{ab}x'^b \approx 0$ (do not simultaneously) are the second kind constraints.

$$\{f_a^{(\pm)}(\sigma), f_b^{(\pm)}(\sigma')\}_1 = C_{ab}^{(\pm)}(\sigma-\sigma') = \pm 2 g_{ab}\frac{\partial}{\partial\sigma'}\delta(\sigma'-\sigma).$$

The inverse matrix $(C^{(\pm)})^{-1}$ has following form $C^{(\pm)ab}(\sigma-\sigma') = \pm\frac{1}{2}g^{ab}\nu(\sigma'-\sigma)$. There is only one set of the constraints, because consistency condition

$$\{f^{(\pm)}(\sigma), H_1\}_1 = f'^{(\pm)}(\sigma) \approx 0, \quad \ldots, \quad \{f^{(\pm)(n)}(\sigma), H_1\}_1 = f^{(\pm)(n+1)}(\sigma) \approx 0.$$

is not produce the new sets of constraints. By using the standard definition of the Dirac bracket, we are obtained following Dirac brackets for the phase space coordinates

$$\{x^a(\sigma), x^b(\sigma')\}_D = \pm\frac{1}{2}g^{ab}\nu(\sigma'-\sigma), \qquad \{p_a(\sigma), p_b(\sigma')\}_D = \mp\frac{1}{2}g_{ab}\frac{\partial}{\sigma'}\delta(\sigma'-\sigma),$$
$$\{x^a(\sigma), p_b(\sigma')\}_D = \frac{1}{2}\delta^a_b\delta(\sigma'-\sigma).$$

The equations of motion under the Hamiltonians $H_1 = h_1$, $H_2 = h_2$ and Dirac bracket

$$\dot{x}^a = \{x^a, H_1\}_D = \{x^a, H_2\}_D = g^{ab}p_b = \pm x'^a,$$
$$\dot{p}_a = \{p_a, H_1\}_D = \{p_a, H_2\}_D = g_{ab}x'^b = \pm p'_a.$$

are coincide on the constraints surface. The dual brackets $\{\ ,\ \}_{\pm 2}$ are coincide with the Dirac brackets also. The contraction of the algebra of the first kind constraints means that the integrals of motion $H_1 = H_2$ are coincide on the constraints surface too.

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Towards Noncommutative Integrable Equations[1]

Masashi HAMANAKA † *and Kouichi TODA* ‡

† *Institute of Physics, University of Tokyo, Komaba, Meguro-ku, Tokyo 153-8902, Japan*
E-mail: *hamanaka@hep1.c.u-tokyo.ac.jp*

‡ *Department of Math. Phys., Toyama Prefectural University, Toyama 939-0398, Japan*
E-mail: *kouichi@yukawa.kyoto-u.ac.jp*

We study an extension of integrable equations which possess the Lax representations to noncommutative spaces. We construct various noncommutative Lax equations by Lax-pair generating technique and Sato theory. Sato theory has revealed essential aspects of the integrability of commutative soliton equations and the noncommutative extension is worth studying. We succeed in deriving various noncommutative hierarchy equations in the framework of Sato theory, which is brand-new. The existence of the hierarchy would suggest a hidden infinite-dimensional symmetry in the noncommutative Lax equations. We finally show that a noncommutative version of Burgers equation is completely integrable because it is linearizable via noncommutative Cole–Hopf transformation. These results are expected to lead to the completion of noncommutative Sato theory.

1 Introduction

The extension of ordinary integrable systems to noncommutative (NC) spaces is one of hot topics in the recent study of integrable systems [1–26]. NC extension in gauge theories corresponds to the presence of background magnetic fields and leads to the discovery of many new physical objects and successful applications to string theories [27]. In particular, NC (anti-)self-dual Yang–Mills (YM) equations are integrable and important [28].

On the other hand, many typical integrable equations such as the Korteweg–de Vries (KdV) equation contain no gauge field and the NC extension of them perhaps might have no physical picture. NC extension of $(1+1)$-dimensional nonlinear equations introduces infinite number of time derivatives and it becomes very hard to define the integrability. Nevertheless, some of them actually possess integrable properties, such as the existence of infinite number of conserved quantities [3, 14]. Furthermore, a few of them can be derived from NC (anti-)self-dual YM equations by suitable reductions [8, 24]. This fact may give some physical meanings and good properties to the lower-dimensional NC field equations. Now it is time to study in detail whether they are actually integrable or not.

In this article, we present various NC equations which possess the Lax representations. We mainly discuss the Lax-pair generating technique and applications of Sato theory. Sato theory is one of the most beautiful soliton theories and reveals various integrable aspects of soliton equations, such as the existence of multi-soliton solutions, the structure of the solution space and the hidden symmetry of them. Hence NC extension of Sato theory is worth studying and the present discussions on it are all new. Here we prove the existence of various NC hierarchy equations, which would suggest hidden infinite-dimensional symmetries of NC equations. Finally we discuss the integrability of NC $(1+1)$-dimensional Burgers equation and prove that it is completely integrable due to the linearizability.

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2 Noncommutative field equations

NC spaces are defined by the noncommutativity of the coordinates: $[x^i, x^j] = i\theta^{ij}$, where θ^{ij} are real constants and are called the *NC parameters*. NC field theories can be defined by replacement of ordinary products of fields in the commutative theories with the *star-product*. The star-product is defined for ordinary fields on commutative spaces and explicitly given by

$$f \star g(x) := \exp\left(\frac{i}{2}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x}, \tag{1}$$

where $\partial_i^{(x')} := \partial/\partial x'^i$ and so on. The star-product has associativity: $f \star (g \star h) = (f \star g) \star h$ and returns to the ordinary product in the commutative limit: $\theta^{ij} \to 0$. The modification of the product makes the ordinary spatial coordinate "noncommutative", that is, $[x^i, x^j]_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}$.

We note that the fields themselves take c-number values as usual and the differentiation and the integration for them are well-defined as usual. A nontrivial point is that NC field equations contain infinite number of derivatives in the nonlinear terms. Hence the integrability of the equations is not so trivial as in commutative cases. For detailed discussion on it, see [24].

3 Noncommutative Lax equations

A given NC differential equation is said to have the Lax representation if there exists a suitable pair of operators (L, B) so that the following equation:

$$[\partial_t - B, L]_\star = 0, \tag{2}$$

is equivalent to the given NC differential equation. Here the star-product does not affect the derivative operator, for example, $\partial_t \star \partial_x = \partial_t \partial_x$. The pair of operators (L, B) and the equation (2) are called the *Lax pair* and the *NC Lax equation*, respectively.

On NC spaces, the meaning of Lax representations would be vague [24]. However, they actually have close connections with the bi-complex method [2] which leads to infinite number of conserved quantities, and the (anti)-self-dual YM equation which is integrable in the context of twistor descriptions and ADHM constructions [28].

Now let us construct NC Lax equations by the *Lax-pair generating technique*. The technique is a method to find a corresponding B-operator for a given L-operator and based on the following ansatz for the B-operator:

$$B = \partial_i^n L^m + B'. \tag{3}$$

Then the problem reduces to that for the B'-operator which is determined by hand so that the Lax equation should be a differential equation without bare differential ∂_i.

In order to explain the steps, for example, let us consider the KdV equation on NC $(1+1)$-dimensional space-time where the coordinate and the noncommutativity are denoted by (x, t) and $[t, x] = i\theta$, respectively.

NC KdV equation [3, 8, 17, 21]. The L-operator for KdV equation is given by $L_{\rm KdV} = \partial_x^2 + u(x,t)$. The ansatz for the operator B is of the following type:

$$B_{\rm KdV} = \partial_x L_{\rm KdV} + B' = \partial_x^3 + u\partial_x + u_x + B', \tag{4}$$

where $u_x := \partial u/\partial x$. The Lax equation (2) leads to the equation for the unknown operator B':

$$[\partial_x^2 + u, B'] = u_x\partial_x^2 + u_x \star u - u_t. \tag{5}$$

Here the bare derivative term $u_x\partial_x^2$ is troublesome. In order to eliminate it, let us take the following ansatz for B': $B' = X\partial_x + Y$, where X and Y are polynomial of $u, u_x, u_t, u_{xx} := \partial^2 u/\partial x^2$ etc. The Lax equation (2) reduces to

$$(2X-u)_x\partial_x^2 + (X_{xx} + [u,X] + 2Y_x)\partial_x + (Y_{xx} + [u,Y] - X\star u_x + u_t - u_x\star u) = 0. \tag{6}$$

The condition that the coefficients of ∂_x^2 and ∂_x should vanish yields differential equations for X and Y, which are easily solved by $X = \frac{1}{2}u$, $Y = -\frac{1}{4}u_x$. Now the Lax equation (2) becomes a differential equation, that is, the NC KdV equation:

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u\star u)_x, \tag{7}$$

where $(u\star u)_x := u_x\star u + u\star u_x$. The nonlinear term becomes symmetric and the equation shows just a conservation law. In the star-product formalism, the spatial integration is well-defined. Hence the spatial integrations of current densities are conserved quantities as in commutative cases [24]. Here $Q := \int dx u$ is conserved, that is, $\partial_t Q = 0$.

In this way, we can generate a wide class of Lax equations on NC (2+1) and (1+1)-dimensional space-times. In particular, this method is suitable for higher-dimensional extension both on commutative spaces [29] and NC spaces [17]. Here we present some results of (2+1)-dimensional NC Lax equations where the coordinate is denoted by (x,y,t). The noncommutativity is basically introduced in space-space directions: $[x,y] = i\theta$. For more discussions and examples, see [17,21].

NC Kadomtsev–Petviashvili (KP) equation [11, 17, 21]

$$u_t = \frac{1}{4}u_{xxx} + \frac{3}{4}(u\star u)_x + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}\big[u, \partial_x^{-1}u_y\big]_\star, \tag{8}$$

where $\partial_x^{-1}f(x) := \int^x dx' f(x')$. The Lax pair is given by

$$\begin{aligned} L_{\rm KP} &= \partial_x^2 + u(x,y,t) + \partial_y =: L'_{\rm KP} + \partial_y, \\ B_{\rm KP} &= \partial_x L'_{\rm KP} + X\partial_x + Y = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x - \frac{3}{4}\partial_x^{-1}u_y. \end{aligned} \tag{9}$$

There is seen to be a nontrivial deformed term $[u, \partial_x^{-1}u_y]_\star$ in the equation (8), which vanishes in the commutative limit.

NC Bogoyavlenski–Calogero–Schiff (BCS) equation [17, 21]

$$u_t = \frac{1}{4}u_{xxy} + \frac{1}{2}(u\star u)_y + \frac{1}{4}u_x\star(\partial_x^{-1}u_y) + \frac{1}{4}(\partial_x^{-1}u_y)\star u_x + \frac{1}{4}[u, \partial_x^{-1}[u, \partial_x^{-1}u_y]_\star]_\star, \tag{10}$$

whose Lax pair and the ansatz are

$$\begin{aligned} L_{\rm BCS} &= \partial_x^2 + u(x,y,t), \\ B_{\rm BCS} &= \partial_y L_{\rm BCS} + X\partial_x + Y = \partial_x^2\partial_y + u\partial_y + \frac{1}{2}(\partial_x^{-1}u_y)\partial_x + \frac{3}{4}u_y - \frac{1}{4}\partial_x^{-1}[u, \partial_x^{-1}u_y]_\star. \end{aligned} \tag{11}$$

This time, a nontrivial term is found even in the B-operator. In commutative limit, this coincides with the BCS equation which has multi-soliton solutions [30].

4 Noncommutative hierarchy equations and Sato theory

In this section, we derive various noncommutative hierarchy equations in the framework of Sato theory [31] introducing the pseudo-differential operator.

Let us introduce the following Lax operator as a (first-order) pseudo-differential operator:

$$L = \partial_x + u_2\partial_x^{-1} + u_3\partial_x^{-2} + u_4\partial_x^{-3} + \cdots, \qquad u_k = u_k(t_1, t_2, t_3, \ldots). \tag{12}$$

The action of the operator ∂_x^n on a multiplicity operator f is given by

$$\partial_x^n \cdot f := \sum_{i\geq 0} \left(\begin{array}{c} n \\ i \end{array} \right) (\partial_x^i f)\partial^{n-i} := \sum_{i\geq 0} \frac{n(n-1)\cdots(n-i+1)}{i(i-1)\cdots 1}(\partial_x^i f)\partial^{n-i}. \tag{13}$$

We note that the definition can be extended to negative n. The examples are,

$$\begin{aligned}
&\partial_x^{-1} \cdot f = f\partial_x^{-1} - f_x\partial_x^{-2} + f_{xx}\partial_x^{-3} - \cdots, \\
&\partial_x^{-2} \cdot f = f\partial_x^{-2} - 2f_x\partial_x^{-3} + 3f_{xx}\partial_x^{-4} - \cdots, \\
&\partial_x^{-3} \cdot f = f\partial_x^{-3} - 3f_x\partial_x^{-4} + 6f_{xx}\partial_x^{-5} - \cdots,
\end{aligned}$$

where ∂_x^{-1} in the RHS acts as an integration operator $\int^x dx$. Products of pseudo-differential operators are also well-defined and the total set of pseudo-differential operators forms an operator algebra. For more on pseudo-differential operators and Sato theory, see e.g. [32].

The Lax representation for a hierarchy in Sato's framework is defined as

$$[\partial_{t_m} - B_m, L]_\star = 0, \qquad m = 1, 2, \ldots, \tag{14}$$

where B_m is given here by

$$B_m := (\underbrace{L \star \cdots \star L}_{m \text{ times}})_{\geq 0} =: (L^m)_{\star \geq 0}. \tag{15}$$

The suffix "≥ 0" represents the positive and 0-th power part of L^m. The examples are

$$B_1 = \partial_x, \quad B_2 = \partial_x^2 + 2u_2, \quad B_3 = \partial_x^3 + 3u_2\partial_x + 3(u_3 + u_{2x}), \quad \ldots. \tag{16}$$

The noncommutativity is introduced for infinite number of "time variables" $(t_1, t_2, \ldots)$. As it can be taken arbitrarily, we do not fix the noncommutativity here.

NC KP hierarchy. The hierarchy (14) gives rise to NC KP hierarchy which contains the NC KP equation (8). The coefficients of each powers of (pseudo-)differential operators yield infinite series of NC "evolution equations", that is, for $m = 1$

$$\partial_x^{1-k}) \quad u_{kt_1} = u_{kx}, \quad k = 2, 3, \ldots \quad \Rightarrow \quad t_1 \equiv x, \tag{17}$$

for $m = 2$

$$\begin{aligned}
&\partial_x^{-1}) \quad u_{2t_2} = u_{2xx} + 2u_{3x}, \\
&\partial_x^{-2}) \quad u_{3t_2} = u_{3xx} + 2u_{4x} + 2u_2 \star u_{2x} + 2[u_2, u_3]_\star, \\
&\partial_x^{-3}) \quad u_{4t_2} = u_{4xx} + 2u_{5x} + 4u_3 \star u_{2x} - 2u_2 \star u_{2xx} + 2[u_2, u_4]_\star, \quad \ldots
\end{aligned} \tag{18}$$

and for $m = 3$

$$\begin{aligned}
&\partial_x^{-1}) \quad u_{2t_3} = u_{2xxx} + 3u_{3xx} + 3u_{4x} + 3u_{2x} \star u_2 + 3u_2 \star u_{2x}, \\
&\partial_x^{-2}) \quad u_{3t_3} = 3u_{4xx} + 6u_2 \star u_{3x} + 3u_{2x} \star u_3 + 3u_3 \star u_{2x} + 3[u_2, u_4]_\star, \quad \ldots.
\end{aligned} \tag{19}$$

These just imply the NC KP equation (8) with $2u_2 \equiv u$, $t_2 \equiv y$, $t_3 \equiv t$. Important point is that infinite kind of fields $u_3, u_4, u_5, \ldots$ are represented in terms of one kind of field $2u_2 \equiv u$ as is seen in equation (18). This guarantees the existence of NC KP hierarchy and the infinite differential equations are called the *NC (KP) hierarchy equations*.

Putting the constraint $L^l = B_l$ on the NC KP hierarchy (14), we get infinite set of NC hierarchies. Then the following NC hierarchy:

$$\left[\partial_{t_m} - B_m, L^l\right]_\star = 0, \tag{20}$$

becomes a NC differential equation. We can easily show

$$\frac{\partial u}{\partial t_{Nl}} = 0, \qquad N = 1, 2, \dots \tag{21}$$

because of $B_{Nl} = L^{Nl}$. The reduced NC hierarchy is called the *l-reduction* of NC KP hierarchy.

NC KdV hierarchy (2-reduction of NC KP). Taking the constraint $L^2 = B_2$, we get the KdV hierarchy. The Lax equation:

$$\frac{\partial u}{\partial t_m} = \left[B_m, L^2\right]_\star, \tag{22}$$

gives rise to the m-th KdV hierarchy equation which becomes, for example, the (third) NC KdV equation (7) with $t_3 \equiv t$ and the 5-th NC KdV equation [17]:

$$u_{t_5} = \frac{1}{16}u_{xxxxx} + \frac{5}{16}(u \star u_{xxx} + u_{xxx} \star u) + \frac{5}{8}(u_x \star u_x + u \star u \star u)_x. \tag{23}$$

In Lax-pair generating technique, the B_m-operator is given by the ansatz $B_m = \partial_x^{m-2} L_{\mathrm{KdV}} + B'$.

NC Boussinesq hierarchy (3-reduction of NC KP). The 3-reduction $L^3 = B_3$ yields the NC Boussinesq hierarchy. The Lax equation:

$$\frac{\partial u}{\partial t_m} = \left[B_m, L^3\right]_\star, \tag{24}$$

leads to the m-th Boussinesq hierarchy equation which is, for $m = 3$, the NC Boussinesq equation [17]:

$$u_{tt} = \frac{1}{3}u_{xxxx} + (u \star u)_{xx} + ([u, \partial_x^{-1} u_t]_\star)_x. \tag{25}$$

In this way, we can generate infinite set of the l-reduced NC hierarchies. Furthermore, if we take other set-up for the definition of pseudo-differential operator L and "time-evolution" operator B_m, we can get many other hierarchies as follows.

NC modified KdV (mKdV) hierarchy. If we take

$$L = \partial_x + u_1 + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots, \qquad B_m = (L^m)_{\star \geq 1}, \tag{26}$$

and put the constraint $L^2 = B_2$, infinite kind of fields $u_2, u_3, \dots$ are represented in terms of one kind of field $2u_1 \equiv v$. We can easily see $\partial_{t_{2N}} v = 0$. The set of Lax equations:

$$\left[B_m - \partial_{t_m}, L^2\right]_\star = 0 \tag{27}$$

yields NC mKdV hierarchy which implies the NC mKdV equation for $m = 3$ with $t_3 \equiv t$:

$$\frac{\partial v}{\partial t} = \frac{1}{4}v_{xxx} - \frac{3}{8}v \star v_x \star v + \frac{3}{8}[v, v_{xx}]_\star. \tag{28}$$

We note that the NC mKdV equation (28) is different from that in [3] from NC KdV equation (7) via NC Miura map $u = -v \star v - v_x$. The NC mKdV hierarchy can be considered as the 2-reduction of NC mKP hierarchy. However the existence of the NC mKP hierarchy seems to be nontrivial because it cannot be represented in terms of one kind of field. (The NC mKP equation given in early versions of [21] is incorrect.) For the same reason, the higher reduction of the NC mKP hierarchy seems to be hard to obtain.

NC Burgers hierarchy [24]. If we take

$$L = \partial_x + u_1 + u_2\partial_x^{-1} + u_3\partial_x^{-2} + \cdots, \qquad B_m = (L^m)_{\star \geq 1}, \tag{29}$$

and put the constraint $L = B_1 =: \partial_x + v$, the hierarchy:

$$\partial_{t_m} v = [B_m, L]_\star \tag{30}$$

yields the NC Burgers hierarchy which implies the NC Burgers equation:

$$\frac{\partial v}{\partial t_2} = [B_2, L]_\star = \left[\partial_x^2 + 2v\partial_x, \partial_x + v\right]_\star = v_{xx} + 2v \star v_x. \tag{31}$$

and the third-order NC Burgers equation:

$$\frac{\partial v}{\partial t_3} = [B_3, L]_\star = v_{xxx} + 3v \star v_{xx} + 3v_x \star v_x + 3v \star v \star v_x \tag{32}$$

and so on. The nonlinear terms are not symmetric, which will be proved to be a key point in the linearization in the next section. The B_m-operator is given in Lax-pair generating technique by $B_m = \partial_x^{m-1} L + B'$. This time, the generated equations contain some parameters and thus covers wider class of Lax equations [24].

5 Integrability of noncommutative Burgers equation

In this section, we discuss the integrability of the Burgers equation (31) on NC (1+1)-dimensional space-time where the coordinate and the noncommutativity are denoted by $(x, t) \equiv (t_1, t_2)$ and $[t, x] = i\theta$, respectively.

In commutative case, it is well known that the Burgers equation is linearized by the Cole–Hopf transformation. The discussion can be extended to noncommutative case [24, 25]. NC Burgers equation (31) can be linearized by the following noncommutative analogue of the Cole–Hopf transformation: $v = \psi^{-1} \star \psi_x$. The linearized equation is a (NC) diffusion equation: $\psi_t = \psi_{xx}$. The naive solution of the above NC diffusion equation is

$$\psi(t, x) = 1 + \sum_{i=1}^{N} h_i e^{k_i^2 t} \star e^{\pm k_i x} = 1 + \sum_{i=1}^{N} h_i e^{\frac{i}{2} k_i^3 \theta} e^{k_i^2 t \pm k_i x}, \tag{33}$$

where h_i, k_i are complex constants. In the commutative limit, this reduces to the N-shock wave solution in fluid dynamics. The final form in (33) shows that the N-shock wave solution is deformed by $e^{\frac{i}{2} k_i^3 \theta}$ due to the noncommutativity. The explicit representation in terms of v is hard to obtain because the derivation of ψ^{-1} in the NC Cole–Hopf transformation is nontrivial. However we can discuss the asymptotic behaviors at $t \to \pm\infty$ and actually see the effect of the NC deformation. In fact, the exact solutions for $N = 1, 2$ are obtained in [25] and nontrivial effects of the NC-deformation are reported. We note that NC one shock-wave solutions can always reduce to the commutative ones because $f(t - x) \star g(t - x) = f(t - x)g(t - x)$ [24].

The results show that the NC Burgers equation (31) is completely integrable even though the NC Burgers equation contains infinite number of time-derivatives in the nonlinear term. The linearized equation is a differential equation of first order with respect to time and the initial value problem is well-defined. This is a surprising result. The (NC) diffusion equation can be solved for arbitrarily boundary conditions by the Fourier transformation. Furthermore, we note that the form of the nonlinear term in the NC Burgers equation (31) is crucial for the linearization. If it becomes symmetric like the NC KdV equation (7), the linearization is proved to be impossible [24].

6 Conclusion and discussion

In this article, we presented various NC Lax equations and proved the existence of many NC hierarchies. The NC extension of Sato theory is new. We also confirmed that NC Burgers equation is linearizable and completely integrable even though it is a differential equation of infinite order with respect to time. The linearized equation is a (NC) diffusion equation and can be solved in usual ways.

NC extension of Ward conjecture [33] would be very interesting [21] though we have omitted it in this article because of limitations of space. Some NC equations are actually derived from NC (anti-)self-dual YM equations by reduction [8, 24] and embedded [9, 12, 34] in $N = 2$ string theory [35]. This guarantees that NC integrable equations would have a physical meaning and might be helpful to understand new aspects of the corresponding string theory.

The next step is NC extension of Hirota's bilinearization [36]. This could be realized as a simple generalization of the Cole–Hope transformation whose extension to NC spaces are already successful as shown in Section 5. Hirota's bilinearization leads to the theory of tau-functions. Sato theory is based on the existence of hierarchies and tau-functions. We have just revealed the existence of hierarchies and the completion of NC Sato theory would be drawing near at hand.

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Nonlocal Constructions in the Geometry of PDE

Paul KERSTEN †, *Iosif KRASIL'SHCHIK* ‡ *and Alexander VERBOVETSKY* ‡

† *Faculty of Mathematical Sciences, University of Twente,*
P.O. Box 217, 7500 AE Enschede, The Netherlands
E-mail: *kersten@math.utwente.nl*

‡ *Independent University of Moscow, 11 B. Vlasevsky Srt., 119002 Moscow, Russia*
E-mail: *josephk@diffiety.ac.ru, verbovet@mccme.ru*

We give an overview of recent developments of nonlocal constructions in the geometry of partial differential equations and their applications to Bäcklund transformations, Hamiltonian and symplectic structures, recursion operators, etc.

This is an overview of recent results obtained by the authors and related to using of nonlocal constructions in geometry of partial differential equations. For general references concerning geometry of PDE see [1, 15].

Let $\mathscr{E} \subset J^\infty(\pi) \xrightarrow{\pi_\infty} M$ be an infinitely prolonged differential equation considered as a submanifold in an appropriate manifold of infinite jets. Then $\mathscr{E}$ is endowed with a natural finite-dimensional integrable distribution[1] (the *Cartan distribution* denoted by $\mathscr{C}$) locally spanned by the total derivatives. Invariantly, at any point $\theta \in \mathscr{E}$ the space $\mathscr{C}_\theta$ is the span of all tangent planes to the graphs of jets passing through θ. A fiber bundle $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is called a *covering* over $\mathscr{E}$ if (a) $\tilde{\mathscr{E}}$ is endowed with an integrable distribution $\tilde{\mathscr{C}}$, $\dim \tilde{\mathscr{C}} = \dim \mathscr{C}$ and (b) $\tau_* \tilde{\mathscr{C}}_y = \mathscr{C}_{\tau(y)}$ for any $y \in \tilde{\mathscr{E}}$. A $\pi_\infty \circ \tau$-vertical vector field X is called a *nonlocal symmetry* of $\mathscr{E}$ if it preserves $\tilde{\mathscr{C}}$. Note that since the distributions $\mathscr{C}$ and $\tilde{\mathscr{C}}$ are *horizontal*, they determine flat connections in the bundles π_∞ and $\pi_\infty \circ \tau$ respectively. We call these connections the *Cartan connections* and use the same notation, $\mathscr{C}$ and $\tilde{\mathscr{C}}$, for them.

A $\tau \circ \pi_\infty$-vertical vector field X on $\tilde{\mathscr{E}}$ is called a *nonlocal τ-symmetry* if $[X, \tilde{\mathscr{C}}] \subset \tilde{\mathscr{C}}$. They form a Lie algebra denoted by $\operatorname{sym}_\tau \mathscr{E}$. Nonlocal symmetries can be expressed in finite terms rather rarely. A good (and rather useful) substitute is the notion of a *shadow* (that is often mixed up with nonlocal symmetries themselves!). Naively, in local coordinates a shadow can be introduced as follows. Note that any vector field X lying in the Cartan distribution can be uniquely lifted to a field $\tilde{X}$ lying in $\tilde{\mathscr{C}}$ such that $\tau_* \tilde{X} = X$. Moreover, one has $\widetilde{[X, Y]} = [\tilde{X}, \tilde{Y}]$. Consequently, any linear differential operator Δ on $\mathscr{E}$ expressed in total derivatives (a so-called *$\mathscr{C}$-differential operator*) is lifted to an operator $\tilde{\Delta}$ on $\tilde{\mathscr{E}}$. In particular, assume that $\mathscr{E}$ is given by a differential operator F and $\ell_{\mathscr{E}}$ is the restriction of its linearization to $\mathscr{E}$. Then $\ell_{\mathscr{E}}$ can be lifted to an operator $\tilde{\ell}_{\mathscr{E}}$ on $\tilde{\mathscr{E}}$. Consider a solution φ of the equation $\tilde{\ell}_{\mathscr{E}}(\varphi) = 0$. Then to such a φ there corresponds a derivation $S_\varphi\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$. This derivation (or φ itself) is called a shadow in the covering τ, or a *τ-shadow*. The *Reconstruction Theorem* (see [10, 14] and below) states that for any shadow S_φ in τ there exist a covering $\mathscr{E}_\varphi \xrightarrow{\tau_\varphi} \tilde{\mathscr{E}} \xrightarrow{\tau} \mathscr{E}$ and a nonlocal symmetry S in this covering such that $S|_{C^\infty(\mathscr{E})} = S_\varphi$. Vice versa, any derivation $S\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$ that takes the Cartan distribution on $\mathscr{E}$ to that on $\tilde{\mathscr{E}}$ is locally identified with a τ-shadow. Thus, this property of S can be taken for the global definition of shadows.

In a similar way, one can consider the adjoint operator $\ell^*_{\mathscr{E}}$ and its lifting $\tilde{\ell}^*_{\mathscr{E}}$ to $\tilde{\mathscr{E}}$. In what follows, we assume that the equation under consideration satisfies the conditions of *Vinogradov's Two-Line Theorem* [18–20]. Then solutions of the equation $\tilde{\ell}^*_{\mathscr{E}}(\psi) = 0$ are called *nonlocal gen-*

[1] Integrability in this case means that for any two vector fields $X, Y \in \mathscr{C}$ their commutator $[X, Y]$ lies in $\mathscr{C}$ as well.

erating functions. Nonlocal symmetries (and their shadows) and generating functions naturally arise in various problems related to geometry of PDE. We shall consider some of them below.

1 The long exact sequence of a covering

Let $\mathscr{E} \subset J^\infty(\pi)$, as before, be an infinitely prolonged differential equation understood as a submanifold in the manifold of infinite jets, $\pi_\infty\colon \mathscr{E} \to M$ be the natural projection. Consider a covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ and the *structure element*[2] $U_\tau \in \mathrm{D}^{\mathrm{v}}(\Lambda^1(\tilde{\mathscr{E}})) \subset \mathrm{D}(\Lambda^1(\tilde{\mathscr{E}}))$, where (and below) $\mathrm{D}(\Lambda^i(\tilde{\mathscr{E}}))$ denotes the module of $\Lambda^i(\tilde{\mathscr{E}})$-valued derivations $C^\infty(\tilde{\mathscr{E}}) \to \Lambda^i(\tilde{\mathscr{E}})$ while $\mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}))$ consists of π_∞-vertical derivation. Then (see [12]) the *$\mathscr{C}$-complex*

$$0 \to \mathrm{D}^{\mathrm{v}}(\tilde{\mathscr{E}}) \xrightarrow{\partial_\tau} \mathrm{D}^{\mathrm{v}}(\Lambda^1(\tilde{\mathscr{E}})) \to \cdots \to \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) \xrightarrow{\partial_\tau} \mathrm{D}^{\mathrm{v}}(\Lambda^{i+1}(\tilde{\mathscr{E}})) \to \cdots \tag{1}$$

arises, where the differential $\partial_\tau = [\![U_\tau, \cdot]\!]^{\mathrm{n}}$ is defined by the *Frölicher–Nijenhuis bracket*. The corresponding cohomology is denoted by $H^i_{\mathscr{C}}(\tilde{\mathscr{E}})$ and called the *$\mathscr{C}$-cohomology* of the covering τ.

Consider the module $\mathrm{D}^{\mathrm{g}}(\Lambda^i(\tilde{\mathscr{E}})) = \{X \in \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}) \mid X(C^\infty(\mathscr{E})) = 0\}$, where $C^\infty(\mathscr{E})$ is understood as a subalgebra in $C^\infty(\tilde{\mathscr{E}})$. Let also $\mathrm{D}^{\mathrm{s}}(\Lambda^i(\tilde{\mathscr{E}})) = \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}))/\mathrm{D}^{\mathrm{g}}(\Lambda^i(\tilde{\mathscr{E}}))$ be the quotient module. By basic properties of the Frölicher–Nijenhuis bracket, $\partial_\tau(\mathrm{D}^{\mathrm{g}}(\Lambda^i(\tilde{\mathscr{E}}))) \subset \mathrm{D}^{\mathrm{g}}(\Lambda^{i+1}(\tilde{\mathscr{E}}))$ and thus the short exact sequence of complexes

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D}^{\mathrm{g}}(\Lambda^i(\tilde{\mathscr{E}})) & \longrightarrow & \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) & \longrightarrow & \mathrm{D}^{\mathrm{s}}(\Lambda^i(\tilde{\mathscr{E}})) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_\tau} & & \downarrow{\scriptstyle \partial_\tau} & & \downarrow{\scriptstyle \partial_\tau} & & \\
0 & \longrightarrow & \mathrm{D}^{\mathrm{g}}(\Lambda^{i+1}(\tilde{\mathscr{E}})) & \longrightarrow & \mathrm{D}^{\mathrm{v}}(\Lambda^{i+1}(\tilde{\mathscr{E}})) & \longrightarrow & \mathrm{D}^{\mathrm{s}}(\Lambda^{i+1}(\tilde{\mathscr{E}})) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{2}$$

arises (we preserve the notation ∂_τ for the differential in both quotient and subcomplexes). Denote by $H^i_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^i_{\mathrm{g}}(\tilde{\mathscr{E}})$ the cohomology groups of the quotient and subcomplexes, respectively.

Definition 1. The groups $H^i_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^i_{\mathrm{g}}(\tilde{\mathscr{E}})$ are called *shadow* and *gauge* $\mathscr{C}$-cohomologies of the covering τ, respectively. The cohomological sequence

$$\begin{aligned}
&0 \to H^0_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^1_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots \\
&\quad \to H^i_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^i_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^i_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{i+1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots
\end{aligned} \tag{3}$$

corresponding to (2) is called the *long exact sequence* of the covering τ.

Remark 1. Recall (see [12]) that the modules $\Lambda^i(\tilde{\mathscr{E}})$ split into the direct sum

$$\Lambda^i(\tilde{\mathscr{E}}) = \oplus_{p+q=i}\Lambda^{p,q}(\tilde{\mathscr{E}}), \qquad \Lambda^{p,q}(\tilde{\mathscr{E}}) = \mathscr{C}\Lambda^p(\tilde{\mathscr{E}}) \otimes \Lambda^q_h(\tilde{\mathscr{E}}), \tag{4}$$

where $\mathscr{C}\Lambda^p(\tilde{\mathscr{E}})$ and $\Lambda^q_h(\tilde{\mathscr{E}})$ are modules of *Cartan* and *horizontal* forms, respectively. To (4) there corresponds the splitting $\mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) = \oplus_{p+q=i}\mathrm{D}^{\mathrm{v}}(\Lambda^{p,q}(\tilde{\mathscr{E}}))$ and the differential ∂_τ takes

[2]The structure element U_τ may be understood as the *connection form* of the Cartan connection $\tilde{\mathscr{E}}$. In a similar way, we call the connection form of $\mathscr{C}$ the structure element of $\mathscr{E}$.

elements $\mathrm{D}^{\mathrm{v}}(\Lambda^{p,q}(\tilde{\mathscr{E}}))$ to $\mathrm{D}^{\mathrm{v}}(\Lambda^{p,q+1}(\tilde{\mathscr{E}}))$. This leads to the cohomology groups $H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}})$, $H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}})$ and $H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}})$ and (3) actually splits into the series of the following sequences

$$0 \to H^{p,0}_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^{p,0}_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^{p,0}_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{p,1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots$$
$$\cdots \to H^{p,q}_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^{p,q}_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{p,q+1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \to \cdots$$

The groups $H^{p,q}_{\mathscr{C}}(\tilde{\mathscr{E}})$ are identified with the *horizontal cohomology* of $\tilde{\mathscr{E}}$ with coefficients in $\mathrm{D}^{\mathrm{v}}(\mathscr{C}\Lambda^p(\tilde{\mathscr{E}}))$ [17].

1.1 Interpretations

Before discussing particular applications of the above constructions, let us describe the geometrical meaning of some of the groups in (3) that we shall need below.

$H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$**:** elements of this group are identified with *nonlocal τ-symmetries*;

$H^0_{\mathrm{g}}(\tilde{\mathscr{E}})$**:** these are *gauge symmetries* in τ, i.e., τ-vertical symmetries (the corresponding diffeomorphisms, if they exist, are automorphisms of τ);

$H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$**:** this cohomology group consists of *τ-shadows* of nonlocal symmetries, i.e., π_∞-vertical derivations $X\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$ preserving the Cartan distributions. As it was already mentioned, in local coordinates, shadows are described by vector-functions $\varphi = (\varphi^1, \dots, \varphi^m)$, $\varphi^l \in C^\infty(\tilde{\mathscr{E}})$, $m = \dim \pi$, satisfying the equation $\tilde{\ell}_{\mathscr{E}}(\varphi) = 0$, where $\tilde{\ell}_{\mathscr{E}}$ is the linearization of $\mathscr{E}$ naturally lifted to $\tilde{\mathscr{E}}$;

$H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$**:** these are equivalence classes of nontrivial *infinitesimal deformations* of U_τ, i.e., of the element defining the basic geometrical structure on $\tilde{\mathscr{E}}$; a deformation is infinitesimally trivial if and only if its cohomological class in $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$ vanishes. On the other hand, elements of $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$ act on $H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) = \mathrm{sym}_\tau \mathscr{E}$ by contraction: $R(X) = \mathrm{i}_X(R)$, $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$, $R \in H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$. Nontrivial actions may correspond to elements of $H^{1,0}_{\mathscr{C}}(\tilde{\mathscr{E}})$ only;

$H^1_{\mathrm{g}}(\tilde{\mathscr{E}})$**:** those cohomological classes are *τ-vertical*, or *gauge*, deformations. They deform the covering structure itself only and do not change the structure of the underlying equation $\mathscr{E}$;

$H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$**:** Similar to $H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$, these are also shadows of $H^1_{\mathscr{C}}(\tilde{\mathscr{E}})$, i.e., classes of π_∞-vertical derivations $C^\infty(\mathscr{E}) \to \Lambda^1(\tilde{\mathscr{E}})$ that preserve the Cartan distributions. Elements of $H^{1,0}_{\mathrm{s}}(\tilde{\mathscr{E}})$ have the following local description: $\mu = (\mu^1, \dots, \mu^m)$, $\mu^l \in \mathscr{C}\Lambda^1(\tilde{\mathscr{E}})$ lies in $H^{1,0}_{\mathrm{s}}(\tilde{\mathscr{E}})$ if and only if $\tilde{\ell}_{\mathscr{E}}(\mu) = 0$.

1.2 Reconstruction of shadows

Consider again a covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ and a shadow $\varphi\colon C^\infty(\mathscr{E}) \to C^\infty(\tilde{\mathscr{E}})$. We say that φ is *reconstructible* in τ if there exists a symmetry $X\colon C^\infty(\tilde{\mathscr{E}}) \to C^\infty(\tilde{\mathscr{E}})$ such that $X\big|_{C^\infty(\mathscr{E})} = \varphi$. Of course, given a covering τ, not any shadow can be reconstructed in this covering, but a weaker result was found by N. Khor'kova and proved in [14]:

Theorem 1 (Reconstruction Theorem). *Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering and φ be a τ-shadow. Then there exists a covering $\tau_\varphi\colon \mathscr{E}_\varphi \to \tilde{\mathscr{E}}$ such that φ is reconstructible in the composition covering $\tau \circ \tau_\varphi\colon \mathscr{E}_\varphi \to \mathscr{E}$.*

Remark 2. How the theorem works in practice may be seen in the classical example of the KdV equation

$$u_t = uu_x + u_{xxx}. \tag{5}$$

It is known that this equation possesses two infinite series of symmetries – a local one, independent of x and t and consisting of higher KdV equations, and another, (x,t)-dependent series whose first two terms are local (the Galilean boost and a scaling symmetry) and all others are nonlocal. Take the first nonlocal one. It has the form $\psi_5 = tu_{xxxxx} + \cdots$ and contains the nonlocal variable w^1 defined by the relations

$$w^1_x = u, \qquad w^1_t = \tfrac{1}{2}u^2 + u_{xx}. \tag{6}$$

Actually, ψ_5 is not a symmetry but only a shadow in the covering corresponding to (6). To reconstruct the symmetry, one needs to find the coefficient at $\partial/\partial w^1$. But when we do this, a new nonlocal variable w^2 arises that is to satisfy the relation $w^2_x = u^2$, etc. As it was shown in [10], this process 'stops' at infinity only: to reconstruct the initial shadow, one has to add all nonlocal variables of the form $w^j_x = c^j$, where c^j is the density of the jth conservation law. As we shall see, this situation is somewhat typical.

Proof of Theorem 1. Consider the following part of (3) $H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^1_{\mathrm{g}}(\tilde{\mathscr{E}})$. Let $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$ be a shadow in the covering τ. It is reconstructible if there exists a symmetry $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$ such that $\beta(X) = \varphi$. Due to the exactness, this is equivalent to $\partial(\varphi) = 0$. Thus the element $\omega_\varphi = \partial(\varphi)$ is the obstruction to reconstructibility of the shadow φ. So, the intermediate result is

Proposition 1. *A shadow $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$ is reconstructible in the covering τ if and only if the obstruction $\omega_\varphi \in H^{0,1}_{\mathrm{g}}(\tilde{\mathscr{E}})$ vanishes.*

The obstruction ω_φ is represented by a vector-valued horizontal 1-form on $\tilde{\mathscr{E}}$, the number of its components ω^l_φ equals the dimension of the covering τ. Each form ω^l_φ is closed and thus determines a 1-dimensional covering τ^l over $\tilde{\mathscr{E}}$, see [14]. This covering is trivial if and only if this form is exact (with respect to the horizontal de Rham differential) and $\omega_\varphi = 0$ if and only if all ω^l_φ are exact. Thus, if we choose those ω^l_φ whose horizontal cohomology classes are nontrivial, take the corresponding 1-dimensional coverings and consider the Whitney product $\tilde{\tau}\colon \tilde{\tilde{\mathscr{E}}} \to \tilde{\mathscr{E}}$ then the obstruction ω_φ will vanish in this covering, but a new obstruction of the same nature may arise. If the latter vanishes the shadow is reconstructible in $\tilde{\tau} = \tau_\varphi$, otherwise we must repeat the construction, etc. Eventually, we shall either stop at some finite step, or shall arrive to an infinite covering τ_φ. In both cases, there will be no obstruction to reconstruct the shadow φ. ■

Remark 3. From the proof of the theorem it can be seen that the shadow φ completely determines the 'minimal' covering, where it reconstructs to a symmetry. In the case of a two-dimensional base M (i.e., when the equation $\mathscr{E}$ is in two independent variables) the cohomology classes of the forms ω^l_φ are identified with *conservation laws* of $\mathscr{E}$, generally nonlocal. But in some cases (for the KdV and similar equations) the reconstruction procedure deals with local conservation laws only, see [10].

Remark 4. Though the covering, where the shadow at hand reconstructs, is well defined (but not uniquely!) algorithmically, the symmetry that corresponds to this shadow is definitely not unique: even when a covering τ_φ is chosen, a symmetry corresponding to φ is defined up to elements of $H^0_{\mathrm{g}}(\mathscr{E}_\varphi)$, i.e., up to gauge symmetries in τ_φ. This means that to deal with nonlocal symmetries is not the same as to deal with their shadows.

1.3 Toward Bäcklund transformations [3]

Geometrically, a *Bäcklund transformation* between two differential equations $\mathscr{E}$ and $\mathscr{E}'$ is a pair of coverings $\mathscr{E} \xleftarrow{\tau} \tilde{\mathscr{E}} \xrightarrow{\tau'} \mathscr{E}'$. This geometrical construction may be understood as a differential

relation between solutions of $\mathscr{E}$ and $\mathscr{E}'$. If $s \subset \mathscr{E}$ is a solution of $\mathscr{E}$ then $\tau^{-1}(s)$ is fibered by a $\dim\tau$-dimensional family of solutions to $\tilde{\mathscr{E}}$ while $\tau'(\tau^{-1}(s))$ delivers solutions to $\mathscr{E}'$, and vice versa.

In applications, one of the most important cases is when $\mathscr{E} = \mathscr{E}'$ and τ, τ' belong to a smooth family of coverings τ_t, $t \in \mathbb{R}$ being a nonremovable[3] parameter. Such a family may be regarded as a *deformation* of the covering τ_{t_0}. These deformations may be treated as follows.

Essentially, we repeat here the main result given in [3]. Let us first recall the following construction, see [14]. Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering and X be a symmetry of $\mathscr{E}$ possessing a one-parameter group of transformations $A_t\colon \mathscr{E} \to \mathscr{E}$. Consider an arbitrary lift of A_t to $\tilde{\mathscr{E}}$ such that the diagram

$$\begin{array}{ccc} \tilde{\mathscr{E}} & \xrightarrow{\tilde{A}_t} & \tilde{\mathscr{E}} \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle\tau} \\ \mathscr{E} & \xrightarrow{A_t} & \mathscr{E} \end{array} \tag{7}$$

is commutative. Let us define on $\tilde{\mathscr{E}}$ a t-parameter family of distributions $\tilde{\mathscr{C}}_t$ by setting

$$(\tilde{\mathscr{C}}_t)_{\tilde{\theta}} = (\tilde{A}_t)_*^{-1}\tilde{\mathscr{C}}_{\tilde{A}_t\tilde{\theta}}, \qquad \tilde{\theta} \in \tilde{\mathscr{E}}.$$

All distributions $\tilde{\mathscr{C}}_t$ are integrable, i.e., $[\tilde{\mathscr{C}}_t, \tilde{\mathscr{C}}_t] \subset \tilde{\mathscr{C}}_t$, and thus $\tilde{\mathscr{E}}_t = (\tilde{\mathscr{E}}, \tilde{\mathscr{C}}_t)$ covers $\mathscr{E}$ by means of τ. Denote this covering by $\tau_t\colon \tilde{\mathscr{E}}_t \to \mathscr{E}$ and notice that if $\tilde{A}'_t$ is another lift satisfying (7) then for any t the covering τ_t is either equivalent to τ'_t or not. It for all t (sufficiently small) τ_t and τ'_t are equivalent then this means that X can be lifted to a symmetry of $\tilde{\mathscr{E}}$. Thus we obtain

Proposition 2. *Let $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering and X be a symmetry of $\mathscr{E}$ possessing a one-parameter group of transformations and such that it cannot be lifted to a symmetry of $\tilde{\mathscr{E}}$. Then X generates a one-parameter family of (equivalence classes of) coverings $\tau_t\colon \tilde{\mathscr{E}}_t \to \tilde{\mathscr{E}}$ such that*

1. *$\tilde{\mathscr{E}}_t$ and $\tilde{\mathscr{E}}_{t'}$ isomorphic as manifolds with distributions;*
2. *τ_t and $\tau_{t'}$ are pair-wise inequivalent for sufficiently small t and t'.*

We say that a family τ_t satisfying properties 1 and 2 above is *irreducible*.

Remark 5. Let $\mathfrak{g}$ be a finite-dimensional Lie subalgebra in the algebra of classical symmetries of $\mathscr{E}$. Then we obtain a G/H-irreducible family, where G is the Lie group corresponding to $\mathfrak{g}$ and H is the stabilizer of τ under the above described action.

Remark 6. Proposition 2 means that if we have a covering τ and a shadow that cannot be reconstructed to a symmetry in this covering[4] then, under certain conditions, this shadow generates an irreducible family of coverings. As we see below, this is, in a sense, a general way to obtain irreducible families.

Consider an irreducible family of coverings $\tau_t\colon \tilde{\mathscr{E}} \to \mathscr{E}$, $\tau_0 = \tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$, and the following part of (3)

$$H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \xrightarrow{\beta} H^0_{\mathrm{s}}(\tilde{\mathscr{E}}) \xrightarrow{\partial} H^{0,1}_{\mathrm{g}}(\tilde{\mathscr{E}}) \xrightarrow{\alpha} H^{0,1}_{\mathscr{C}}(\tilde{\mathscr{E}}).$$

The family τ_t may be considered as a deformation of τ; hence its infinitesimal part μ lies in $H^{0,1}_{\mathrm{g}}(\tilde{\mathscr{E}})$. By Property 1 from Proposition 2, the corresponding deformation of $\tilde{\mathscr{E}}$ is trivial and thus $\alpha(\mu) = 0$. By the exactness, $\mu = \partial(\varphi)$, $\varphi \in H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$. The deformation τ_t is infinitesimally nontrivial if and only if $\mu \neq 0$ and, again by the exactness, if and only if $\varphi \neq \beta(X)$, $X \in H^0_{\mathscr{C}}(\tilde{\mathscr{E}})$, i.e., if and only if φ is not reconstructible to a τ-symmetry. To state the final result, it remains to note that the action of ∂ is given by $[\![U_\tau, \cdot]\!]^{\mathrm{n}}$.

[3] I.e., for any $t_1 \neq t_2$ the coverings τ_{t_1} and τ_{t_2} are inequivalent.

[4] For the problem of reconstruction see Subsection 1.2.

Theorem 2. *Any irreducible family of coverings τ_t, $\tau_0 = \tau$, is infinitesimally generated by a τ-shadow that cannot be reconstructed to a nonlocal τ-symmetry.*

Remark 7. Of course, not to any shadow φ there corresponds an irreducible family of coverings. Anyway, we can construct a *formal deformation* $\exp(t\varphi)U_\tau$ and use it in some applications.

1.4 Action of recursion operators

We now pass to the last topic of this section. Let us first briefly recall the cohomological theory of recursion operators as it was exposed in [13]. Consider an equation $\mathscr{E}$ and the $\mathscr{C}$-complex (1) associated to it. Then, as it was already mentioned, the group $H^0_{\mathscr{C}}(\mathscr{E})$ is identified with the Lie algebra $\operatorname{sym}(\mathscr{E})$ of higher symmetries of $\mathscr{E}$. The contraction (or inner product) operation determines an action of elements of $H^1_{\mathscr{C}}(\mathscr{E})$ on $\operatorname{sym}(\mathscr{E})$ and nontrivial actions may correspond to the elements of $H^{1,0}_{\mathscr{C}}(\mathscr{E})$ only[5]. Locally, these elements are represented as Cartan vector-forms of degree 1, $\omega = (\omega_1, \dots, \omega_m)$, $\omega_l \in \mathscr{C}\Lambda^1(\mathscr{E})$, $m = \dim \pi$, satisfying the equation

$$\ell_{\mathscr{E}}(\omega) = 0. \tag{8}$$

Solving (8) and applying these solutions to the known symmetries we, in principle, can generate new symmetries. The same remains valid for nonlocal symmetries if we take $\tilde{\mathscr{E}}$ instead of $\mathscr{E}$. It is natural to anticipate that in this way we find recursion operators for symmetries of the equation at hand.

But in practice the picture is more complicated. If, for example, we realize the above procedure for the KdV equation (5) we shall obtain trivial solutions only. On the other hand, this equation possesses the recursion operator

$$R = D_x^2 + \frac{2}{3}u + \frac{1}{3}D_x^{-1},$$

where D_x is *total derivative* with respect to x. Of course, this operator could not be found by the described above method because of the nonlocal term D_x^{-1}. Such a term may arise in the covering given by (6), but if we try the same in this nonlocal setting the result will be negative again.

What actually gives nontrivial results is the following (see all the examples in [13]). Let us show how it works for the KdV equation. Namely, we do the following:

1. We take the covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ given by (6). Note that $\tilde{\mathscr{E}}$ is isomorphic to the infinite prolongation of the equation

   $$w_t = w_{xxx} + \tfrac{1}{2}u_x^2.$$

2. Then we consider the module of Cartan 1-forms on $\tilde{\mathscr{E}}$. It is generated by the forms

   $$\omega_k = du_k - u_{k+1}\,dx - D_x^k(u_3 + u_1)\,dt, \qquad \omega_{-1} = dw^1 - u\,dx - \left(u_2 + \tfrac{1}{2}u^2\right)\,dt,$$

   where $k = 0, 1, \dots$ and $u_k = u\underbrace{x\dots x}_{k \text{ times}}$, $u_0 = u$.

3. The next step is to solve the equation

   $$\tilde{\ell}_{\mathscr{E}}(\Omega) = 0, \tag{9}$$

   where $\tilde{\ell}_{\mathscr{E}}$ is the lift of the linearization to τ and $\Omega = \sum\limits_{k\geq -1} f_k\omega_k$.

[5]Note also that the contraction determines a structure of an associative algebra with unit on $H^1_{\mathscr{C}}(\mathscr{E})$.

4. Solving (9), we obtain two independent solutions: $\Omega_1 = \omega_0$ and $\Omega_2 = \omega_2 + \frac{2}{3}u\omega_0 + \frac{1}{3}\omega_{-1}$. The first one corresponds to the identical action while the second one gives the classical recursion operator for the KdV equation (see above).

So, from this scheme we see that construction of recursion operators amounts to computation of the group $H_1^{1,0}(\tilde{\mathscr{E}})$. Indeed, we have the following

Theorem 3. *Let $\mathscr{E}$ be an equation and $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ be a covering. Then the contraction operation generates the action $R\colon H^0_{\mathscr{C}}(\tilde{\mathscr{E}}) \to H^0_{\mathrm{s}}(\tilde{\mathscr{E}})$, where $R \in H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$, i.e., elements of $H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$ take τ-symmetries to τ-shadows*[6].

Remark 8. The result of this action, in general, is really a shadow. For example, applying the recursion operator of the KdV equation to the scaling symmetry $\psi_3 = tu_{xxx} + \cdots$, we obtain the element ψ_5 (see above) that reconstructs to a symmetry in the infinite-dimensional covering, where all conservation laws of the equation are 'killed'.

Proof of Theorem 3. The proof is very simple: consider the standard contraction

$$\mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}})) \lrcorner\, \mathrm{D}^{\mathrm{v}}(\Lambda^j(\tilde{\mathscr{E}})) \to \mathrm{D}^{\mathrm{v}}(\Lambda^{i+j-1}(\tilde{\mathscr{E}})).$$

It remains to note that if $X \in \mathrm{D}^{\mathrm{v}}(\Lambda^i(\tilde{\mathscr{E}}))$ and $Y \in \mathrm{D}^{\mathrm{s}}(\Lambda^j(\tilde{\mathscr{E}}))$ then $X \lrcorner\, Y \in \mathrm{D}^{\mathrm{s}}(\Lambda^{i+j-1}(\tilde{\mathscr{E}}))$ and that the differential in (1) preserves the inner product and thus the latter is inherited in the cohomology groups. ■

Remark 9. In some cases action of recursion operator on local symmetries leads to local symmetries again. For example, this is the case for classical hierarchies of integrable equations. Some remarks on how to establish locality may be found in [11].

2 Δ-coverings [7]

A covering $\tau\colon \tilde{\mathscr{E}} \to \mathscr{E}$ is said to be *linear* if τ is a vector bundle and for any vector field X on M the field $\tilde{X}$ lifted to $\tilde{\mathscr{E}}$ by the Cartan connection $\tilde{\mathscr{C}}$ preserves the subspace of fiber-wise linear functions in $C^\infty(\tilde{\mathscr{E}})$. If Δ is a $\mathscr{C}$-differential operator over $\mathscr{E}$, then there exists a canonical way to construct a linear covering associated to Δ. This covering is called Δ-*covering*. From *technical* point of view, Δ-coverings are useful to solve the following *factorization problem*: take another $\mathscr{C}$-differential operator ∇ and find $\mathscr{C}$-differential operators V and V' such that the equation

$$\nabla \circ V = V' \circ \Delta \tag{10}$$

holds.

Theorem 4. *Solutions $V \mod (\square \circ \Delta)$ of (10), where $\square$ is a $\mathscr{C}$-differential operator, are in one-to-one correspondence with fiber-wise linear solutions s of the equation $\tilde{\nabla}(s) = 0$, where $\tilde{\nabla}$ is the lifting of ∇ to the Δ-covering.*

There are two canonical coverings associated to a given equation. Namely, taking the operator $\ell_{\mathscr{E}}$, we obtain the $\ell_{\mathscr{E}}$-*covering* and for $\ell^*_{\mathscr{E}}$ one gets the $\ell^*_{\mathscr{E}}$-*covering*. The first one plays the role of the tangent bundle to $\mathscr{E}$ while the second is the counterpart of the cotangent bundle in the category of differential equations. This determines *conceptual* importance of the introduced constructions.

[6] Actually, nontrivial actions may correspond to the elements of $H^{1,0}_{\mathrm{s}}(\tilde{\mathscr{E}}) \subset H^1_{\mathrm{s}}(\tilde{\mathscr{E}})$ only.

In the simplest case of a scalar evolution equation $u_t = f(x, t, u, \dots, u_k)$ the ℓ-covering is described by additional coordinates $\omega = \omega_0, \omega_1, \dots$ that satisfy the relations

$$(\omega_i)_x = \omega_{i+1}, \qquad \omega_t = \sum_{i=0}^{k} \frac{\partial f}{\partial u_i}\omega_i,$$

while the ℓ^*-covering is described by the coordinates $p = p_0, p_1, \dots$ satisfying

$$(p_i)_x = p_{i+1}, \qquad p_t = \sum_{i=0}^{k}(-D_x)^i \left(\frac{\partial f}{\partial u_i}p\right).$$

Similar formulas can be obtained for equations of general form.

2.1 Recursion operators for symmetries [7, 13]

Let $\Delta = \nabla = \ell_{\mathscr{E}}$. Then equation (10) takes the form

$$\ell_{\mathscr{E}} \circ V = V' \circ \ell_{\mathscr{E}},$$

i.e., V takes $\ker \ell_{\mathscr{E}}$ (symmetries) to itself. In other words, V is a recursion operator for symmetries. From Theorem 4 we obtain an efficient method to compute such operators:

Corollary 1. *Recursion operators for symmetries are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}_{\mathscr{E}}(s) = 0$, i.e., with linear shadows in the $\ell_{\mathscr{E}}$-covering.*

Remark 10. Evidently, this result is in full agreement with what we saw in Subsection 1.4.

2.2 Recursion operators for conservation laws [7]

Let $\Delta = \nabla = \ell^*_{\mathscr{E}}$. Then equation (10) takes the form $\ell^*_{\mathscr{E}} \circ V = V' \circ \ell^*_{\mathscr{E}}$, i.e., V takes $\ker \ell^*_{\mathscr{E}}$ (generating functions) to itself. In other words, V is a recursion operator for generating functions. From Theorem 4 we obtain an efficient method to compute such operators:

Corollary 2. *Recursion operators for generating functions are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}^*_{\mathscr{E}}(s) = 0$, i.e., with nonlocal linear generating functions in the $\ell^*_{\mathscr{E}}$-covering.*

To obtain a recursion procedure for conservation laws, recall that for 'two-line equations' [18–20] their conservation laws (the term $E_1^{0,n-1}$ of the Vinogradov $\mathscr{C}$-spectral sequence) are related to generating functions (the term $E_1^{1,n-1}$) by the injective differential $d_1^{0,n-1}$, which, in the simplest case of evolution equations, coincides with the Euler operator $\mathcal{E}$, and a generating function ψ corresponds to a conservation law if and only if $\ell_\psi = \ell^*_\psi$.

2.3 Hamiltonian structures [7]

Let $\mathscr{E}$ be an evolution equation, $\Delta = \ell^*_{\mathscr{E}}$, and $\nabla = \ell_{\mathscr{E}}$. Then equation (10) takes the form

$$\ell_{\mathscr{E}} \circ V = V' \circ \ell^*_{\mathscr{E}}, \tag{11}$$

i.e., V takes $\ker \ell^*_{\mathscr{E}}$ (generating functions) to $\ker \ell_{\mathscr{E}}$ (symmetries). We call such operators *pre-Hamiltonian*. As above, Theorem 4 provides an efficient method to compute these operators:

Corollary 3. *Pre-Hamiltonian operators are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}_{\mathscr{E}}(s) = 0$, i.e., with linear shadows in the $\ell^*_{\mathscr{E}}$-covering.*

A pre-Hamiltonian structure is *Hamiltonian* if (a) $V + V^* = 0$ and (b) $[\![s, s]\!]^{\mathrm{s}} = 0$, where $[\![\cdot, \cdot]\!]^{\mathrm{s}}$ is the *variational Schouten bracket* [4]. If $\mathscr{E}$ is of the form

$$u_t = f(x, t, u, \dots, u_k), \qquad u = (u^1, \dots, u^m), \qquad f = (f^1, \dots, f^m), \qquad u_l = \frac{\partial^l u}{\partial x^l}, \tag{12}$$

and $V = \| \sum_l a_{ij}^l D_x^l \|$, where D_x is the total derivative with respect to x, then conditions (a) and (b) can be reformulated in terms of the function[7] $W_V = \sum_{ijl} a_{ij}^l p_l^i p_0^j$ as follows

$$\sum_i \frac{\delta W_V}{\delta p^i} p_0^i = -2W_V, \qquad \mathcal{E} \sum_i \left(\frac{\delta W_V}{\delta u^i} \frac{\delta W_V}{\delta p^i} \right) = 0$$

and are easily checked. Here $\delta/\delta u$, $\delta/\delta p$ are variational derivatives[8].

2.4 Symplectic structures [7]

Let again $\mathscr{E}$ be an evolution equation, $\Delta = \ell_{\mathscr{E}}$, and $\nabla = \ell_{\mathscr{E}}^*$. Then equation (10) takes the form

$$\ell_{\mathscr{E}}^* \circ V = V' \circ \ell_{\mathscr{E}}, \tag{13}$$

i.e., V takes $\ker \ell_{\mathscr{E}}$ (symmetries) to $\ker \ell_{\mathscr{E}}^*$ (generating functions). We call such operators *presymplectic*. Again, Theorem 4 gives an efficient method to compute these operators:

Corollary 4. *Presymplectic operators are in one-to-one correspondence with fiber-wise linear solutions of the equation $\tilde{\ell}_{\mathscr{E}}^*(s) = 0$, i.e., with linear generating functions in the $\ell_{\mathscr{E}}$-covering.*

A presymplectic structure is *symplectic* if condition (a) holds and s is 'variationally closed', that for equation (12) means that

$$\mathcal{E} \sum_i \frac{\delta W_V}{\delta u^i} p_0^i = 0.$$

Remark 11. As it is shown in [8], any skew-adjoint operator V that satisfies the condition (11) *is automatically Hamiltonian*, provided that the symbol of the right-hand side of the corresponding evolution equation is nondegenerate. A similar statement is valid for symplectic structures if we take skew-adjoint solutions of (13).

2.5 Canonical representation for nonlocalities

As it was mentioned above, efficient construction of recursion operators and similar structures needs introduction of nonlocal variables. In all applications we used the following *canonical* choice of nonlocal variables:

for the ℓ-covering: ω_ψ defined by $(\omega_\psi)_x = \psi\omega_0$, where ψ is a generating function, i.e., satisfies the equation $\ell_{\mathscr{E}}^*(\psi) = 0$;

for the ℓ^*-covering: p_φ defined by $(p_\varphi)_x = \varphi p_0$, where φ is a symmetry, i.e., satisfies the equation $\ell_{\mathscr{E}}(\varphi) = 0$.

[7] The variables p_l^i, $l = 0, \dots, k$, $i = 1, \dots, m$, should be *odd*.

[8] In the formula above, as well as in the condition for symplectic structures below, the Euler operator contains variational derivatives both in u and in p.

These nonlocal variables lead to the operators of the form

$$\sum_{\alpha\geq 0} c_{ij}^{\alpha} D_y^{\alpha} + \sum_{\beta} d_j^{\beta} D_y^{-1} \circ e_i^{\beta},$$

where $\left\|c_{ij}^{\alpha}\right\|$ is an $m\times m$-matrix, $\left\|d_j^{\beta}\right\|$ is an $m\times l$-matrix, and $\left\|e_i^{\beta}\right\|$ is an $l\times m$-matrix for some $l>0$ (matrix-valued functions, to be more precise). In the table it is shown how the matrices d and e look for different types of operators.

Type of operator	Lines of matrix d	Columns of matrix e
Recursions for symmetries	Symmetry	Generating function
Recursions for generating functions	Generating function	Symmetry
Hamiltonian structures	Symmetry	Symmetry
Symplectic structures	Generating function	Generating function

2.6 Example: the coupled KdV-mKdV system

We shall now describe a Hamiltonian structure for the coupled KdV-mKdV system of the form

$$\begin{aligned} u_t &= -u_{xxx} + 6uu_x - 3vv_{xxx} - 3v_xv_{xx} + 3u_xv^2 + 6uvv_x,\\ v_t &= -v_{xxx} + 3v^2v_x + 3uv_x + 3u_xv. \end{aligned} \tag{14}$$

This system arises as the so-called *bosonic limit* of the $N=2$, $a=1$ supersymmetric extension of the KdV equation [6]; integrability properties of this system (existence of a recursion operator) were studied in [9]. In [5], by means of the prolongations structure techniques, a Lax pair for (14) was constructed.

2.6.1 Hamiltonian structures

Using the methods described above, we found the following Hamiltonian operator for system (14):

$$V = \begin{pmatrix} -D_x^3 + 4uD_x + 2u_1 & 2vD_x \\ 2vD_x + 2v_1 & D_x \end{pmatrix}. \tag{15}$$

Moreover, the following result is valid:

Theorem 5. *The coupled KdV-mKdV system* (14) *is Hamiltonian with respect to the the Hamiltonian operator* (15) *and possesses the Hamiltonian* $H = \frac{1}{2}(u^2+uv^2-vv_2)$. *The corresponding energy is given by the form*

$$\begin{aligned} \eta = {}& \tfrac{1}{2}\big(u^2+uv^2-vv_2\big)\,dx + \tfrac{1}{2}\big(4u^3+9u^2v^2-2uu_2+3uv^4-11uvv_2+uv_1^2\\ &+u_1^2-u_1vv_1-4u_2v^2-6v^3v_2-3v^2v_1^2+vv_4-v_1v_3+v_2^2\big)\,dt. \end{aligned}$$

This structure is unique in the class of Hamiltonian structures independent of x and t and polynomial in u_k, v_k.

2.6.2 Recursion operators

To describe the recursion operator for (14), let us first introduce the necessary nonlocal variables. They are denoted by w, w_1, w_{11}, w_{12} and defined by the equations

$$w_x = v, \qquad w_t = 3uv + v^3 - v_2;$$

$$
\begin{aligned}
&w_{1,x} = u, \qquad w_{1,t} = 3u^2 + 3uv - u_2 - 3vv_2;\\
&w_{11,x} = \cos(2w)w_1v + \sin(2w)v^2,\\
&w_{11,t} = \cos(2w)(3wuv + wv^3 - wv_2 + uv_1 - u_1v - v^2v_1) + \sin(2w)(4uv^2 + v^4 - vv_2 - v_1^2);\\
&w_{12,x} = \cos(2w)v^2 - \sin(2w)wv,\\
&w_{12,t} = \cos(2w)(4uv^2 - v^4 - 2vv_2 + v_1^2) + \sin(2w)(-3wuv - wv^3 + wv_2 - uv_1 + v^2v_1).
\end{aligned}
$$

Then the recursion operator is given by a 2×2-matrix operator R with the entries

$$
\begin{aligned}
R_{11} &= D_x^2 - 4u - v^2 - Y^u_{1,1}D_x^{-1} \circ \psi^u_{1,2} + Y^u_{1,2}D_x^{-1} \circ \psi^u_{1,1} - \tfrac{3}{2}Y^u_{1,0}D_x^{-1},\\
R_{12} &= 2vD_x^2 + v_1D_x - 3uv + 2v_2 - Y^u_{1,1}D_x^{-1} \circ \psi^v_{1,2} + Y^u_{1,2}D_x^{-1} \circ \psi^v_{1,1} + Y^u_{2,1}D_x^{-1},\\
R_{21} &= -2v - Y^v_{1,1}D_x^{-1} \circ \psi^u_{1,2} + Y^v_{1,2}D_x^{-1} \circ \psi^u_{1,1} - \tfrac{3}{2}Y^v_{1,0}D_x^{-1},\\
R_{22} &= D_x^2 - 2u - v^2 - Y^v_{1,1}D_x^{-1} \circ \psi^v_{1,2} + Y^v_{1,2}D_x^{-1} \circ \psi^v_{1,1} + Y^v_{2,1}D_x^{-1},
\end{aligned}
$$

where $Y_{1,0}$, $Y_{1,1}$, $Y_{1,2}$, and $Y_{2,1}$ are symmetries of (14) with the components

$$
\begin{aligned}
&Y^u_{1,0} = u_1, \qquad Y^v_{1,0} = v_1;\\
&Y^u_{1,1} = -\cos(2w)\left(\tfrac{1}{2}u_1 + vv_1\right) + \sin(2w)\left(uv - \tfrac{1}{2}v_2\right), \qquad Y^v_{1,1} = \tfrac{1}{2}\cos(2w)v_1 - \tfrac{1}{2}\sin(2w)u;\\
&Y^u_{1,2} = \cos(2w)(uv - \tfrac{1}{2}v_2) + \sin(2w)\left(\tfrac{1}{2}u_1 + vv_1\right), \qquad Y^v_{1,2} = -\tfrac{1}{2}\cos(2w)u - \tfrac{1}{2}\sin(2w)v_1;\\
&Y^u_{2,1} = \cos(2w)(-w_{11}u_1 - 2w_{11}vv_1 + 2w_{12}uv - w_{12}v_2)\\
&\qquad + \sin(2w)(2w_{11}uv - w_{11}v_2 + w_{12}u_1 + 2w_{12}vv_1) - 2uv_1 - 3u_1v - 2v^2v_1 + v_3,\\
&Y^v_{2,1} = \cos(2w)(w_{11}v_1 - w_{12}u) - \sin(2w)(w_{11}u + w_{12}v_1) - (u_1 + vv_1),
\end{aligned}
$$

while

$$
\begin{aligned}
&\psi^u_{1,1} = -\sin(2w), \qquad \psi^v_{1,1} = 2(2\sin(2w)v - w_{12});\\
&\psi^u_{1,2} = -\cos(2w), \qquad \psi^v_{1,2} = 2(2\cos(2w)v + w_{11}).
\end{aligned}
$$

are the generating functions (cf. the remark above on the canonical representation of nonlocal objects).

2.7 Back to Bäcklund transformations

The above described scheme needs one generalization at least. Since recursion operators arising in practice usually contain terms of the D_x^{-1} type, these nonlocalities should be added in the initial setting[9]. This is being done by substituting equation $\mathscr{E}$ with its covering $\tilde{\mathscr{E}}$ in all constructions. Consequently, the operators obtained cease to be differential, but can be transformed to *$\mathscr{C}$-differential relations* between the objects where they act. Geometrically, these relations are realized as Bäcklund transformations relating:

- $\ell_{\mathscr{E}}$-covering with itself (recursion operators for symmetries);
- $\ell^*_{\mathscr{E}}$-covering with itself (recursion operators for generating functions);
- $\ell_{\mathscr{E}}$-covering with $\ell^*_{\mathscr{E}}$ covering (pre-Hamiltonian and presymplectic structures).

The first step in this direction was made in [16], where recursion operators for symmetries were treated as Bäcklund transformations. Clearly, generalized recursion operators for generating functions can be constructed in a similar way. As for the third type of Bäcklund transformations, one can see that they unify the concepts of pre-Hamiltonian and presymplectic structures. Following [2], these transformations may be called *pre-Dirac structures*. Their study is a subject for future research.

[9] Such terms arise also in *nonlocal* Hamiltonian and symplectic structures.

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On Calculation of Finite-Gap Spectra for One-Dimensional Schrödinger Operator

Andrii A. KOROSTIL and Igor A. KOROSTIL

Institute of Magnetism of NAS Ukraine, 36-b Vernadskii Ave., 03142 Kyiv-142, Ukraine
E-mail: *amk@imag.kiev.ua*

The general self-consistent approach for calculating of spectral parameters of the one-dimensional Schrödinger operator with the finite-gap Abelian potential is proposed. It is based on use of finite-band equations and solution of the Jacobi inverse problem in terms of hyperelliptic Weierstrass functions.

1 Introduction

General solutions of completely integrable systems are expressed in terms of the Riemann theta functions containing unknown parameters. Obtaining these solutions in an explicit form is reduced to the problem of calculation of the mentioned parameters. Usually, solving of this problem is realized in the framework of two following approaches. The first algebro-geometric approach is based on an use of the so-called Thomae formulae describing relations between the unknown parameters and so-called theta-constants (see [1–3]). In so doing, it is assumed that spectral branch points are given, and their calculation is a special problem. Role of independent parameters in this formulae is played by the period matrix τ generated by the hyperelliptic Riemann curve $\Gamma_g : y^2 = \prod_{i=1}^{2g+1} (\varepsilon - \varepsilon_i)$ (see [2]). The second approach is based on construction of algebraic equations for the unknown parameters by forward substitution of the general solutions in to an original integrable equation (see [3]). These two approaches assume that the explicit form of the generated Riemann curve is given and do not provide their calculation.

In the same time, the full and self-consistent solving of the considered problem assumes construction of a unique system of equations determining all unknown parameters. This can be realized in the form of so-called finite-band equations in the framework of the spectral problem for auxiliary linear operators (see [4–7]) corresponding to a given integrable equation.

In the case of the KdV equation under consideration this problem is connected with the spectral problem for the finite-gap Schrödinger operator. In the presented paper the effective method for full solving of the spectral problem for the one-dimensional Schrödinger operator is proposed on the basis of finite-band equations. As distinct from the above-mentioned second approach, the substitution of solutions (the Schrödinger potential) is realized with respect to the finite-band equations. In so doing, the problem is reduced to solving of a self-consistent system of algebraic equations determining all unknown parameters through theta-constants (see [1]). This provides calculation of both parameters of the potential and spectral parameters of the Schrödinger operator.

The presented paper consists of three sections. General analytic solving of the spectral problem for the finite-gap Schrödinger operator in terms of hyperelliptic Weierstrass functions is considered in Section 2. In Section 3 we obtain the system of algebraic equations for the unknown parameters including spectral parameters and consider particular cases of one- and two-gap spectra.

2 Finite-gap solutions

Eigenfunctions of linear differential operators with a finite-gap spectrum are characterized by their dependence on a spectral variable ε through polynomials in ε [5]. Therefore solving of the corresponding spectral problem is based on the comparison of asymptotic expansions in ε of general and finite-gap eigenfunctions. This is expressed in the form of so-called finite-band equations describing all unknown parameters of finite-gap solutions.

The spectral equation ($H\Psi = \varepsilon\Psi$) of the general and finite-gap one-dimensional Schrödinger operator (H) determines its eigenfunction in the form

$$\Psi_{\pm}(x) = C\sqrt{\chi(\varepsilon,x)}\exp\left(\pm\imath\int_{x_0}^{x} dx\chi(\varepsilon,x)\right)$$

with $\chi(\varepsilon,x)$ determined by the general spectral equation and $\chi(\varepsilon,x) = y(\varepsilon)/X(\varepsilon,x)$ for the finite-gap spectra. Here polynomial functions $y(\varepsilon)$ and $X(\varepsilon,x)$ are of $(2g+1)$-th and g-th orders in ε, respectively. Here the X-polynomial is characterized by a dependence on x of coefficients at powers of ε.

The function $X(\varepsilon,x)$ is the product of two eigenfunctions ($X(\varepsilon,x) = \Psi_+(\varepsilon,x)\Psi_-(\varepsilon,x)$) and satisfies to the equation, which in terms of X-polynomial roots $\mu_j(x)$ take the form (see [8])

$$\partial_x X(\varepsilon,x)|_{\varepsilon=\mu_j(x)} = 2\imath y(\mu_j(x)).$$

This equation is integrated by the Abelian map $\zeta_j = \sum\limits_{k=1}^{g}\int_{a_k}^{\mu_k(x)} dv_j(\varepsilon)$, where $dv_j = \sum\limits_{i=1}^{g} C_{ij}d\omega_i$ is normalized holomorphic differential on the Riemann curve Γ_g: $y^2 = \prod\limits_{i=1}^{2g+1}(\varepsilon-\varepsilon_i)$ and $d\omega_j = \varepsilon^{j-1}d\varepsilon/y(\varepsilon)$. Corresponding solutions have the form $\zeta_j = 2\imath C_{gj}x$. Thus calculation of symmetrized products of the set $\{\mu_j(x)\}$ is reduced to inverse of the Abelian map (see for instance [1, 2]). Symmetrized products of $\{\mu_j(x)\}$, considered as the point set on the Riemann curve Γ_g, can be calculated with help of the Riemann theta functions

$$\theta[\boldsymbol{\alpha}](\boldsymbol{z}|\boldsymbol{\tau}) = \sum_{n\in\mathbb{Z}^g}\exp[\pi\imath(\boldsymbol{n}+\boldsymbol{\alpha}_1)\tau(\boldsymbol{n}+\boldsymbol{\alpha}_1)+2\pi\imath\boldsymbol{z}\cdot(\boldsymbol{n}+\boldsymbol{\alpha}_2)], \tag{1}$$

where

$$\boldsymbol{z} = \int_{x_0}^{x} d\boldsymbol{\omega} - \boldsymbol{C}\boldsymbol{u} - \boldsymbol{K}, \qquad u_j = \sum_{k=1}^{g}\int_{a_k}^{x_k} d\omega_j(x)$$

which vanish in points $\{\mu_j(x)\}$.

Here α is the $2\times g$ half-integer matrix with rows represented by vectors α_1 and α_2; τ is the period matrix on the Riemann curve Γ_g, $\boldsymbol{K}$ is a so-called Riemann constant vector.

In the framework of the Kleinian construction (see [9]) the so-called hyperelliptic σ-function

$$\sigma(\boldsymbol{u}) = C\exp\left(\boldsymbol{u}\boldsymbol{\kappa}\boldsymbol{u}+2\pi\imath\boldsymbol{q}'(-\boldsymbol{C}\boldsymbol{u}+\boldsymbol{r}\boldsymbol{q}'/2-\boldsymbol{q}')\right)\theta\left(\int_{x_0}^{x}\boldsymbol{C}d\omega-\boldsymbol{C}\boldsymbol{u}-\boldsymbol{K}|\boldsymbol{\tau}\right), \tag{2}$$

plays the role of the generating functions for hyperelliptic Abelian functions. Absence of $\theta(z)$ implies the function (1) at $\alpha = 0$.

Here $\boldsymbol{r}_j$ is determined by the expression $dr_j = r_j dx$ ($r_j = \sum\limits_{k=j}^{2g+1-j}(k+1-j)\lambda_{k+1+j}x^k/2y$), which describes canonical Abelian differential of second kind on the Riemann curve Γ_g; $\boldsymbol{q}'$ and $\boldsymbol{q}$ are half-integer characteristics of the Riemann constant vector $\boldsymbol{K}$.

Hyperelliptic zeta and Weierstrass functions are defined as logarithmic derivatives of the function (2) by the formulae

$$\zeta_{j_1}(u) = \frac{\partial}{\partial u_{j_1}} \ln \sigma(\boldsymbol{u}), \qquad \wp_{j_1 j_2 \cdots j_n}(\boldsymbol{u}) = -\frac{\partial^{j_1+j_2+\cdots+j_n}}{\partial u_{j_1} \partial u_{j_2} \ldots \partial u_{j_n}} \ln \sigma(\boldsymbol{u}), \quad n \geq 2. \tag{3}$$

The Weierstrass functions in (3) implicitly contain unknown parameters $\{C_{ij}\}$ and the period matrix $\boldsymbol{\tau}$ as an independent parameter. Taking into account the dependence of $\boldsymbol{u}$ on points $\{x_i\}$ on the Riemann curve Γ_g it is see that second derivative of the σ-function (2) represents a polynomial in x_i with coefficient functions $\wp_{j_1 j_2}(\boldsymbol{u})$. Evidently the equality for a such polynomial enable to determine symmetrized products of $\mu_j(x)$ through hyperelliptic functions $\wp_{j_1 j_2}(\boldsymbol{u})$, which, with account of above-mentioned relations between vectors $\boldsymbol{\zeta}$ and $\boldsymbol{u}$, will contain unknown parameters C_{ij}.

This equality can be obtained on the basis of the known Baker relation (see [9])

$$\int_\nu^x \sum_{i=1}^g \int_{\nu_i}^{x_i} d\Omega(x, x_i) = \ln \left(\sigma \left(\int_{x_0}^x d\omega - \sum_{i=1}^g \int_{x_0}^{x_i} d\omega \right) \sigma^{-1} \left(\int_{x_0}^x d\omega - \sum_{i=1}^g \int_{x_0}^{\nu_i} d\omega \right) \right) \Bigg|_{x=\nu}^x, \tag{4}$$

where the fundamental 2-differential of the second kind $d\Omega(x, x_i)$ is defined by the expression

$$d\Omega(x, x_i) = \frac{\partial}{\partial_x} \left(\frac{y(x) + y(x_i)}{2y(x)(x - x_i)} \right) dx dx_i + d\omega(x) d\boldsymbol{r}(x_i).$$

The partial derivation of (4) with respect to x and x_k taking into account the definition (3) leads to the relation

$$\sum_{i,j=1}^g \wp_{ij} \left(\int_{x_0}^x d\omega - \sum_{i=1}^g \int_{x_0}^{x_i} d\,\omega \right) x^i x_k^j = \frac{F(x, x_k) - 2y y_k}{4(x - x_k)^2}, \tag{5}$$

with known [9] F-function which has the form of polynomial in variables x and x_k. Using the known relation $F(x, x_k) \to x^{g+1} x_k^{g+1}$ at $x \to \infty$ from (5) in the asymptotic limit $x \to \infty$ we obtain the equation

$$x_k^g - \sum_{j=1}^g \wp_{g,g+1-j}(\boldsymbol{u}) x_k^{g-j} = 0. \tag{6}$$

In view of (6) the coefficient function $\wp_{g,g+1-j}(u)$ equals to the symmetrized product of a j degree of the above mentioned point set $\{x_k\}$ of the Riemann curve Γ_g. This is true also with respect to the point set $\{\mu_k(x)\}$.

Thus coefficient functions $b_j(x)$ are expressed through generalized Weierstrass $\wp$-functions by the relation

$$b_j(x) = -\wp_{g,g+1-j}(\boldsymbol{u}). \tag{7}$$

In terms of the above-mentioned normalized vectors $\zeta = \boldsymbol{C}\boldsymbol{u}$ belonging to Abelian map on the basis of the normalized first-kind Abelian holomorphic differentials v, the relation (6) takes the form

$$b_j(x) = -\sum_{n,m=1}^g C_{gn} C_{g,g+1-j} \wp_{nm}(\boldsymbol{\alpha} x), \qquad (\boldsymbol{\alpha})_j = C_{gj}. \tag{8}$$

Thus the spectral problem for the one-dimensional Schrödinger operator with a finite-gap spectrum is reduced to calculation of the normalized coefficient matrix C. Normalized coefficients in (8) are described by algebraic equations, which can be obtained by substitution of the expression (8) into known finite-band equations with transition to the theta function representation.

3 Calculation of unknown parameters

The system of finite-band equations represents relations between coefficient functions $b_j(x)$ and polynomials in derivatives of the Schrödinger potential $U(x)$ with respect to x. This system follows from comparison of asymptotic expansions in ε for general and finite-gap χ-functions entering in the above-mentioned expression for the eigenfunction $\Psi(\varepsilon, x)$ of the Schrödinger spectral equation (see [4]).

For the general solution, coefficients of the power expansion

$$\chi = \sqrt{\varepsilon}\left(1 + \sum_{n=0}^{\infty}\left((-1)^n/2^{2n+1}\right)\chi_{2n+1}\varepsilon^{-(n+1)}\right)$$

are polynomial in derivatives of $U(x)$. For the finite-gap solutions, coefficients of the power expansion $\chi(\varepsilon, x) = y(\varepsilon)/X(\varepsilon, x)$ are represented polynomial in spectral coefficients $\{a_j\}$ and coefficient functions $b_j(x)$. Comparison of power expansions for both cases leads to the so-called finite-band system of equations

$$\begin{aligned}
&\frac{1}{n!}\frac{\partial^n}{\partial_z^n}\left(\frac{\left(\sum\limits_{k=0}^{2g+1} a_k z^k\right)^{1/2}}{\sum\limits_{k=0}^{g} b_k(x) z^k}\right)_{z=0} = \frac{(-1)^{n-1}}{2^{2n-1}}\chi_{2n-1}(x), \qquad a_0, b_0 = 1,\\
&\chi_{n+1} = \frac{\partial}{\partial_x}\chi_n + \sum_{k=1}^{n-1}\chi_k\chi_{n-k}, \qquad \chi_1 = -U(x),
\end{aligned} \tag{9}$$

in which $b_k(x) = 0$ at $k > g$. Feature of this equations is that the coefficient function $b_j(x)$ enters linearly to each equation (9) at $n = j$ and is algebraically expressed through all previous coefficient functions $b_i(x)$, $i = \overline{1, j-1}$. The same regularity takes place for spectral parameters a_j at $j > g$.

Therefore all coefficient functions b_j at $j \le g$ are expressed as polynomials in derivatives of potential $U(x)$. The system of spectral parameters a_j at $j = \overline{g+1, 2g+1}$ is algebraically expressed through first g parameters $a_1, \ldots, a_g$ and polynomials in derivatives $U^{(i)}$.

Thus, use of an eliminating method concerning a_j and $b_j(x)$ reduces equations (9) at $n = \overline{2g+2, 3g+1}$ to the closed system (RS) of algebraic equations for g unknowns a_j, $j = \overline{1, g}$.

The potential $U(x)$ in (9) is connected with the coefficient function $b_1(x)$ by the relation

$$b_1(x) = \frac{1}{2}\left(U(x) + a_1\right),$$

which together with the equality (7) result in the explicit expression

$$U(x) = -2\sum_{j,j'=1}^{g}\alpha_j\alpha_{j'}\wp_{jj'}(\boldsymbol{\alpha}x|\boldsymbol{\tau}) - a_1. \tag{10}$$

describing the Schrödinger potential in terms of generalized Weierstrass $\wp$-functions at the given module matrix $\boldsymbol{\tau}$ of the Riemann curve Γ (in future for simplicity $\boldsymbol{\tau}$ will be omitted). Here components α_j, $j = \overline{1, g}$ must be calculated from the above-mentioned reduced system.

Substitution of the expression (10) into equations (9) results in a system of nonlinear equations for unknowns a_j, $j = \overline{1, 2g+1}$ and α_j, $j = \overline{1, g}$ with coefficients expressed through derivatives of $\wp$-function with respect to x. In view of independence on x this system is represented by algebraic nonlinear equations with respect to a_j and α_j. In representation of above-mentioned

theta functions such system is reduced to algebraic nonlinear equations with coefficients which are expressed through theta-constants (i.e. theta functions with zero argument). Correspondingly, the above-mentioned RC system is transformed in to the algebraic system of equations concerning unknowns a_j and α_j at $j=\overline{1,g}$.

For obtaining above-mentioned equations it is convenient to introduce j-component vector $\boldsymbol{n}_j=(n_1,\dots,n_j)$ and notations $\alpha_{\boldsymbol{n}_j}=2\prod\limits_{\boldsymbol{n}_i=1}^{j}\alpha_{n_i}$ and $\wp_{\boldsymbol{n}_j}=\wp_{n_1\cdots n_j}$. Then the expression (10) can be rewritten more compactly as

$$U(x)=\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}}(\boldsymbol{\alpha}x)-a_1, \tag{11}$$

where summing by repetitive subscript components is implied.

Substitution of the expression (11) into the equations (9) results in the equation

$$\begin{aligned} b_2(x)=&\frac{1}{8}\big(3\left(\alpha_{\boldsymbol{n}_2g}\wp_{\boldsymbol{n}_2}-a_1\right)^2-\left(\alpha_{\boldsymbol{n}_4g}\wp_{\boldsymbol{n}_4}\right)\big)\\ &-\frac{1}{4}a_1\left(\alpha_{\boldsymbol{n}_2g}\wp_{\boldsymbol{n}_2}-a_1\right)+\frac{1}{2}a_2-\frac{1}{8}a_1^3 \end{aligned} \tag{12}$$

and

$$\begin{aligned} b_3(x)=&\frac{1}{32}\big(\alpha_{\boldsymbol{n}_4g}\wp_{\boldsymbol{n}_4}+10\left(\alpha_{\boldsymbol{n}_2g}\wp_{\boldsymbol{n}_2}-a_1\right)^3-5\left(C_{\boldsymbol{n}_3g}\wp_{\boldsymbol{n}_3}\right)^2\\ &-10\left(\alpha_{\boldsymbol{n}_2g}\wp_{\boldsymbol{n}_2}-a_1\right)\alpha_{\boldsymbol{n}_4g}\wp_{\boldsymbol{n}_4}\big)-\frac{1}{16}a_1\big(3\left(\alpha_{\boldsymbol{n}_2g}\wp_{\boldsymbol{n}_2}-a_1\right)^2-\alpha_{\boldsymbol{n}_4g}\wp_{\boldsymbol{n}_4}\big)\\ &+\frac{1}{16}(\alpha_{\boldsymbol{n}_2g}\wp_{\boldsymbol{n}_2}-a_1)\left(a_1^2-4a_2\right)+\frac{1}{2}a_3+\frac{1}{4}a_1a_2+\frac{1}{16}a_1^3. \end{aligned} \tag{13}$$

Finite-gap equations (12), (13) and of higher order are valid with any arguments of generalized Weierstrass $\wp$-functions. Therefore these equations can be represented in terms of theta-constants (theta-functions on zero argument) by the formal substitute

$$\wp_{\boldsymbol{n}_j}\to(\ln\theta)_{\boldsymbol{n}_j},\qquad(\ln\theta)_{\boldsymbol{n}_j}=\partial_{n_1,\dots,n_j}\ln\theta(z|\boldsymbol{\tau})_{z=0}). \tag{14}$$

In the case of the one-gap spectrum, the generalized Weierstrass functions transform in to the elliptic Weierstrass function ($\wp_{\boldsymbol{n}_j}\to\wp_{11}$) and $\alpha_1=1$. In this case $b_j(x)=0$ at $j>1$. Therefore reducing in (12) and (13) similar terms and vanishing coefficients at degrees of $\wp_{11}$ we immediately obtain the following solutions for spectral parameters a_1, a_2, a_3

$$\begin{aligned} &a_1=0,\qquad a_2=\wp(\omega_1)\wp(\omega_2)+\wp(\omega_1)\wp(\omega_3)+\wp(\omega_2)\wp(\omega_3),\\ &a_3=\wp(\omega_1)\wp(\omega_2)\wp(\omega_3),\qquad \omega_1=\omega,\qquad \omega_2=\omega+\omega',\qquad \omega_3=\omega', \end{aligned}$$

where ω and ω' are real and imaginary self-periods of the elliptic Weierstrass function.

In the two-gap case ($g=2$) normalized constants $\{C_j\}$ are not known and are determined by equations (13) and (14). Unknowns a_j, α_j, $j=1,2$ are solutions of the above-mentioned RC system (corresponding to (9) at $n=\overline{6,9}$). This system takes the form of algebraic nonlinear equations which can be written as

$$\begin{aligned} 0=&B_{610}^{22}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2^2+B_{61j}^{21}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2a_1^j|_0^1+B_{61j}^{20}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_1^j|_0^4,\\ 0=&B_{71j}^{22}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2^2a_1^j|_0^1+B_{71j}^{21}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2a_1^j|_0^3+B_{71j}^{20}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_1^j|_0^5,\\ 0=&B_{810}^{23}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2^3+B_{81j}^{22}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2^2a_1^j|_0^2+B_{81j}^{21}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2a_1^j|_0^4+B_{81j}^{20}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_1^j|_0^6,\\ 0=&B_{91j}^{23}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2^3a_1^j|_0^1+B_{91j}^{22}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2^2a_1^j|_0^3\\ &+B_{91j}^{21}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_2a_1^j|_0^5+B_{81j}^{20}(\alpha_{\boldsymbol{n}_2}\wp_{\boldsymbol{n}_2})a_1^j|_0^8, \end{aligned}$$

where summing by repetitive subindexes in boundaries which are denoted at vertical hyphens is implied. Also the formal transfer (14) is suggested. Here $B_{m1j}^{2i}(z)$ implies coefficient function at $a_2^i a_1^j$ corresponding to the equation (9) at $n = m$. This coefficient function is a polynomial in derivatives of its argument with respect to x and therefore contains terms $\partial_x^k \wp_{\boldsymbol{n}_2} = \alpha_{\boldsymbol{n}_{2+k}} \wp_{\boldsymbol{n}_{2+k}}$ in accordance with (11). The latter system is represented by the closed system of the four algebraic nonlinear equations which determines two spectral parameters a_1, a_2 and components α_1, α_2. The remaining unknowns a_j, $j = \overline{3,5}$ are calculated by forward substitution of obtained a_1, a_2 and α_1, α_2 into the reduced equations (9) at $n = \overline{3,5}$. In so doing, the matrix $\boldsymbol{\tau}$ plays a role of a independent parameter.

Similar calculation can be applied in the case of the arbitrary number of spectral gaps in the spectrum of the one-dimensional Schrödinger operator. Then unknown spectral parameters $a_1, \ldots, a_g$ and the parameters $\alpha_1, \ldots, \alpha_g$ will be determined by reduced equations a system of $2g$ reduced algebraic equations (9) at $n = \overline{2g+2, 4g+1}$. The rest spectral parameters a_j, $j = \overline{g+1, 2g+1}$ will be calculated by forward substitution of computed $a_1, \ldots, a_g$ and $\alpha_1, \ldots, \alpha_g$ into the reduced equations (9) at $n = \overline{g+1, 2g+1}$.

The general reduced algebraic equations are represented by the necessary conditions that the Riemann τ-matrix will be the period matrix of holomorphic differentials on the Riemann curve Γ. Thus there are no restrictions on the number of gaps in spectra.

The suggested finite-gap method gives simple and common algorithm for solving the spectral problem for the finite-gap Schrödinger operator. This problem is reduced to calculation of the closed system of algebraic nonlinear equations. The method is represented by the hyperelliptic generalization of the earlier suggested approach [10] for solution of the spectral problem in the one-gap case for the one-dimensional elliptic Schrödinger operator.

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Finite-Dimensional Reductions of 2D dToda Hierarchy Constrained by the String Equation

Igor LOUTSENKO † *and Oksana YERMOLAYEVA* ‡

† *SISSA, Via Beirut 2-4, 34014, Trieste, Italy*
E-mail: *loutseni@fm.sissa.it*

‡ *Concordia University, 1455 de Maisonneuve Blvd. W. Montreal, Quebec, H3G 1M8, Canada*
E-mail: *yermolae@crm.umontreal.ca*

The paper concerns the study of rational and logarithmic reductions of the 2D dispersionless Toda hierarchy of integrable equations. The subject is motivated by important applications to problems in interface dynamics and statistical physics. We prove the consistency of such reductions with respect to the "Toda–Krichever" flows of the 2dToda hierarchy constrained by a string equation.

1 Introduction

Our paper concerns with the study of rational solutions of the dToda hierarchy of integrable equations. The subject is motivated by numerous important applications to problems of interface dynamics and statistical physics. We outline them briefly in this section and widen the discussion in the summary of [1] of the present volume.

The Laplacian growth is a process of propagation of a boundary between two phases on the plane [6, 4]. The both phases are described by scalar harmonic fields (e.g. electrostatic potential, density or pressure), while the speed of the interface is proportional to the gradient of the difference between the fields. In the case where the boundary is an analytic curve it is usual to introduce a time-dependent conformal mapping from the exterior of the unit circle in the "mathematical" (w) plane $z = z(w, x)$, $w = \exp(\sqrt{-1}\phi)$, $0 < \phi < 2\pi$ to the boundary curve $z = X + \sqrt{-1}Y$ on the "physical" plane. It is shown in [6, 7, 4] that the Laplacian growth problem is equivalent to solution of the following equation

$$\mathrm{Im}\left(\frac{\partial z}{\partial \phi}\frac{\partial \bar{z}}{\partial x}\right) = w\left(\frac{\partial z(w,x)}{\partial w}\frac{\partial \bar{z}(1/w,x)}{\partial x} - \frac{\partial z(w,x)}{\partial x}\frac{\partial \bar{z}(1/w,x)}{\partial w}\right) = 1, \tag{1}$$

where bar stands for the complex conjugation (and $\bar{w} = w^{-1}$ at the curve). Equation (1) is called the Laplacian Growth equation [8, 7].

It turns out that (1) plays an essential role in the theory of infinite-dimensional integrable hierarchies. In particular, the equivalence of the contour dynamics to the dispersionless limit of the integrable Toda hierarchy constrained by (1) was established in [4]. Equation (1) may be interpreted as a constraint on an infinite commuting set of dynamical systems defined in the space of one-parameter families of conformal maps. This constraint characterizes the fixed points of an "additional symmetry" [5] in which x is interpreted as the flow parameter.

In what follows, we consider finite-dimensional solutions of (1) in a context of 2dToda hierarchy of integrable equations. We study formal algebraic solutions of the problem, forgetting about complex structure and treating z, $\bar{z}$ as independent functions, and w as a formal variable.

2 Reductions of 2dToda hierarchy constrained by string equation

2.1 2dToda hierarchy and string equation

The dToda hierarchy is defined in terms of two functions $z(w,x)$ and $\bar{z}(w^{-1},x)$ of the form:

$$z(w,x)=r(x)w+\sum_{k=0}^{\infty}u_k(x)w^{-k},\qquad \bar{z}\left(w^{-1},x\right)=r(x)w^{-1}+\sum_{k=0}^{\infty}\bar{u}_k(x)w^{k}, \tag{2}$$

where the coefficients $r(x)$, $u_k(x)$, $\bar{u}_k(x)$ are viewed as coordinate functionals on the phase space consisting of such pairs $z(w,x)$, $\bar{z}(w^{-1},x)$. The dToda flow equations are

$$\partial_{t_k}z=\{H_k,z\},\qquad \partial_{\bar{t}_k}\bar{z}=\{\bar{H}_k,\bar{z}\},\qquad \partial_{t_k}\bar{z}=\{H_k,\bar{z}\},\qquad \partial_{\bar{t}_k}z=\{\bar{H}_k,z\}, \tag{3}$$

$$H_k=(z^k)_+ +1/2\big(z^k\big)_0,\qquad \bar{H}_k=\big(\bar{z}^k\big)_- +1/2\big(\bar{z}^k\big)_0 \tag{4}$$

with subscripts $\pm$, 0 denoting the negative/positive and zero parts of the formal Laurent expansion in w. The Poisson–Lax bracket notation here stays for

$$\{f,g\}:=w\frac{\partial f}{\partial w}\frac{\partial g}{\partial x}-w\frac{\partial f}{\partial x}\frac{\partial g}{\partial w}. \tag{5}$$

The dispersionless limit of the "string equation" is the constraint

$$\left\{z(w,x),\bar{z}\left(w^{-1},x\right)\right\}=1. \tag{6}$$

Equation (6) is invariant under (3) and so defines an invariant under the dToda flows (3) manifold and a reduction of the full dToda hierarchy. The reduction of the dToda hierarchy by the string equation is still an infinite, compatible set of infinite-dimensional dynamical systems. In what follows we are interested in further "functional" reductions where z, $\bar{z}$ are polynomial, rational or logarithmic functions of w. As we show below such reductions are consistent with 2D dToda flows (3) if the string equation (6) holds. Thus, for the consistency we need a double ("functional" plus "string") reduction, which defines a finite-dimensional invariant sub-manifold on the phase space of general 2d Toda hierarchy.

2.2 Polynomial reductions

We begin with polynomial reductions of the 2D Toda chain

$$z(w)=rw+\sum_{i=-N}^{0}u_iw^i,\qquad \bar{z}\left(w^{-1}\right)=rw^{-1}+\sum_{i=0}^{N}\bar{u}_iw^i \tag{7}$$

constrained by the string equation (6). The following proposition states consistency of the polynomial reductions under the Toda flows:

Proposition 1. *If the string equation* (6) *holds, then* (7) *belong to a manifold invariant under the 2dToda flows* $\partial/\partial t_i$, $\partial/\partial\bar{t}_i$, $0<i<N+2$ (3). *This manifold has dimension* $2N+3$ *with* $r=r(T)$, $u=u(T)$, $\bar{u}=\bar{u}(T)$, $T=\{t_i,\bar{t}_i\}$, $i=1,\dots,2N+1$ *being solution of the dynamical system induced by 2dToda flows* (3).

Proof. We must prove that z remains of the form (7) under the flows generated by H_k, $\bar{H}_k$ provided the string equation (6) holds. (The proof for $\bar{z}$ is similar).

1. First we proceed with the flows generated by H_k. The lowest degree of z is $-N$. Since

$$H_k = \left(z^k\right)_+ + 1/2\left(z^k\right)_0 = h_k w^k + h_{k-1}w^{k-1} + \cdots + h_0 \tag{8}$$

is a Laurent polynomial of positive degree, the lowest degree term in the bracket $\{z, H_k\}$ is not less than $-N$. Therefore the lowest degree term in z remains $\geq -N$ under the flows. On the other hand, the complement $z^k - H_k$ of (8) is a polynomial in $1/w$ and so, $\{H_k, z\} = -\{z^k - H_k, z\}$ is a Laurent polynomial with the highest degree in w not exceeding 1. Therefore the highest degree term in z remains ≤ 1.

2. Unlike the evolution under the H_k-flows, the form-invariance (7) of z under the flows generated by the $\bar{H}_k$'s requires extra restrictions on the derivatives $\partial_{t_k} u_i$, stemming from the string equation. Since $\bar{H}_k = (\bar{z}^k)_- + 1/2(\bar{z}^k)_0$ is a Laurent polynomial of non-positive degree, we again see that the highest degree in $\{z, H_k\}$ is 1, and hence this degree cannot increase under the flow. However, since $\bar{H}_k = \bar{z}^k - ((\bar{z}^k)_+ + 1/2(\bar{z}^k)_0)$ is the difference between $\bar{z}^k$ and a polynomial of nonnegative degree, $\{\bar{H}_k, z\} = \{\bar{z}^k, z\} - \{((\bar{z}^k)_+ + 1/2(\bar{z}^k)_0), z\}$, the second bracket in this expression has lowest degree $\geq -N$, but not, in general, the first. However, since $\{\bar{z}^k, z\} = k\bar{z}^{k-1}\{\bar{z}, z\}$, imposing the extra restriction (6) (string equation), again implies that the lowest degree term in this bracket does not exceed $-N$, provided $k \leq N+1$. ■

2.3 Rational reductions

Consider now the space of rational functions $z(w)$ and $\bar{z}(w)$ of the form

$$z(w) = \frac{q_{N+1}(w)}{p_N(w)} = \frac{r w^{N+1} + \sum_{i=0}^{N} a_i w^i}{w^N + \sum_{i=0}^{N-1} b_i w^i}, \tag{9}$$

$$\bar{z}(w^{-1}) = \frac{\bar{q}_{N+1}(w^{-1})}{\bar{p}_N(w^{-1})} = \frac{r w^{-(N+1)} + \sum_{i=0}^{N} \bar{a}_i w^{-i}}{w^{-N} + \sum_{i=0}^{N-1} \bar{b}_i w^{-i}}, \tag{10}$$

where the $4N+3$ coefficients r, a_i, $\bar{a}_i$, b_i, $\bar{b}_i$ are functions of x. The following Lemma states onvariance of such rational reductions of $z(w)$ and $\bar{z}(w)$ under the dToda flows.

Lemma 1. *The space of functions $z(w)$ of the form* (9) *is invariant under the dToda flows ∂_{t_i}, $i > 0$. and, similarly, the space of functions $\bar{z}(w^{-1})$ of the form* (10) *is invariant under the $\partial_{\bar{t}_i}$, $i > 0$ flows.*

Proof. Consider the flows generated by H_k. The proof for $\bar{z}$ is similar. Since

$$H_k = \left(z^k\right)_+ + 1/2\left(z^k\right)_0 \tag{11}$$

is a polynomial of nonnegative degree in w, its complement $z^k - H_k$ is a polynomial in $1/w$. Then the Laurent expansion of brackets (5) $\{H_k, z\} = -\{z^k - H_k, z\}$ around infinity has the following form

$$\{H_k, z\} = k_1(T)w + k_0(T) + k_{-1}(T)w^{-1} + \cdots . \tag{12}$$

While the Lax bracket (5) implies that it is rational of the form $\{H_k, z\} = Q(w)/p_N(w)^2$, it follows from (12) that the highest degree in polynomial $Q(w)$ does not exceed $2N+1$. On the other hand, since it is clear that $\partial_{t_k} z = P(w)/{p_N}^2(w)$, $P(w) = p_N \partial_{t_k} q_{N+1} - q_{N+1}\partial_{t_k} p_N$,

where the highest degree in the polynomial $P(w)$ also does not exceed $2N+1$, we may therefore equate to zero the coefficients of polynomial $P(w)-Q(w)$, and thereby get a system of differential equations for r, a, b. The number of equations obtained is $2N+2$. Thus we get a compatible system of differential equations for $2N+2$ unknowns r, a, b. ■

The consistency of rational solutions for the whole Toda hierarchy requires some extra restrictions. This is the point where the string equation plays an essential role.

Proposition 2. *The string equation* (6) *is a sufficient condition for the rational functions* $z(w)$ *and* $\bar{z}(w)$ (9), (10) *to belong to a manifold, which is only invariant under the first two-Toda flows* $\partial/\partial t_1$, $\partial/\partial \bar{t}_1$ *(see* (3)*).*

Proof is by argument similar to those used for Lemma (1) and Proposition (1).

So, there are only two flows compatible with the rational ansatz. In fact, we could not expect more invariant flows associated to the two simple poles at $w=0$ and $w=\infty$. In the polynomial case, the number of invariant flows was equal to the number of variables (polynomial coefficients), since one can associate n invariant flows to the pole of the n th order, and poles at zero and infinity are immovable. Below, we introduce additional flows, related to movable singularities of $z(w)$ and $\bar{z}(w^{-1})$ using a result by Krichever.

2.4 Additional flows for rational reductions of the dKP hierarchy

As mentioned in [2], on the phase space of extended Benney systems, i.e. rational dKP reductions admitting poles of arbitrary degree, there arise some new flows related to the pole structure of the corresponding maps. These additional flows were introduced by Krichever (see [3]).

Consider a representation of rational maps in pole-residue form for simple poles along with a half (1dToda or dKP) of flows. In the Takasaki gauge [9, 2]

$$z(w) = w + u_0 + \sum_{\alpha=1}^{N} \frac{u_\alpha}{w - w_\alpha}. \tag{13}$$

The new flows attached to the poles are defined as before

$$\partial_{t_{k,\alpha}} z = \{B_{k,\alpha}, z\}, \qquad \alpha = \infty, 1, 2, \ldots, \quad k = 0, 1, 2, \ldots \tag{14}$$

with the evolution operators being associated with the pole structure of z as follows :

$$B_{k,\infty} = \big(z(w)^k\big)_{\geq 0}$$

for an immovable pole at infinity, while for each finite-distance pole there appear additional flows with evolution operators as follows:

$$B_{k,\alpha} = \big(z(w)^k\big)_\alpha, \qquad B_{0,\alpha} = \log(w_\alpha - w).$$

Here, $z(w)_\alpha$ stays for the negative part of a formal expansion of $z(w)$ near its poles w_α:

$$f(w)_\alpha = \sum_{i>0} \frac{f_i}{(w-w_\alpha)^i} \qquad \text{if} \qquad f = \sum_{i \in Z} \frac{f_i}{(w-w_\alpha)^i}. \tag{15}$$

These additional flows commute amongst themselves and with the ordinary (associated with poles at infinity) 1dToda or dKP flows [3].

2.5 Additional invariant flows of the 2dToda system

The dKP hierarchy can be extended to the 2dToda system by introducing an infinite set of $\bar{t}$-flows associated to the poles at $w=0$ and $w=\infty$, similarly the Krichever–Benney system has the following 2dToda extension:

$$z(w)=rw+u_0+\sum_{\alpha=1}^{N}\frac{u_\alpha}{w-w_\alpha},\qquad \bar{z}(1/w)=r/w+\bar{u}_0+\sum_{\beta=1}^{N}\frac{\bar{u}_\beta}{1/w-\bar{w}_\beta}, \tag{16}$$

$$\partial_{t_{k,\alpha}}z=\{H_{k,\alpha},z\},\quad \partial_{\bar{t}_{k,\beta}}\bar{z}=\{\bar{H}_{k,\beta},\bar{z}\},\quad \partial_{t_{k,\alpha}}\bar{z}=\{H_{k,\alpha},\bar{z}\},\quad \partial_{\bar{t}_{k,\beta}}z=\{\bar{H}_{k,\beta},z\}. \tag{17}$$

In (17) we transformed the flows from the Takasaki to the Lax–Sato gauge needed for our purposes. The relation between the evolution operators in different gauges is found to be:

$$H_{k,\alpha}(w)=B_{k,\alpha}(w)-\tfrac{1}{2}B_{k,\alpha}(w=0),\qquad \bar{H}_{k,\beta}(y)=\bar{B}_{k,\beta}(y)-\tfrac{1}{2}\bar{B}_{k,\beta}(y=0), \tag{18}$$

where

$$\begin{aligned}&B_{k,\infty}(w)=(z(w)^k)_{\geq 0},\quad B_{0,\alpha}=\log\left(r(w_\alpha-w)\right),\quad B_{k,\alpha}(w)=(z(w)^k)_\alpha,\\ &\bar{B}_{k,\infty}(y)=(\bar{z}(y)^k)_{\geq 0},\quad \bar{B}_{0,\beta}=\log\left(r(\bar{w}_\beta-y)\right),\quad \bar{B}_{k,\beta}(y)=(\bar{z}(y)^k)_\beta,\quad y=1/w.\end{aligned} \tag{19}$$

Lemma 2. *All vector fields attached to the pole structure of rational maps formally commute, i.e. the introduced evolution operators* (18) *satisfy zero-curvature conditions.*

It is important to note that in the system (17), the equations for the flows $\partial_{t_k}\bar{z}$, $\partial_{\bar{t}_k}z$ do not make a sense fully, since these flows do not preserve the rational ansatz for z, $\bar{z}$ in (13) and (16) until the additional reduction is made. As we have seen before, to be consistent, these systems must be restricted by a string equation, which makes (17) a finite-dimensional dynamical system.

Proposition 3. *The* $4N+2$ *commuting Toda–Krichever flows*

$$\begin{aligned}&\partial_{\tau_i}z=\{h_i,z\},\qquad \partial_{\bar{\tau}_i}z=\left\{\bar{h}_i,z\right\},\\ &\partial_{\tau_i}\bar{z}=\{h_i,\bar{z}\},\qquad \partial_{\bar{\tau}_i}\bar{z}=\left\{\bar{h}_i,\bar{z}\right\},\qquad i=0,\dots,2N,\end{aligned} \tag{20}$$

where

$$\begin{aligned}&h_0=H_{1,\infty}=rw+u_0/2, && \bar{h}_0=\bar{H}_{1,\infty}=r/w+\bar{u}_0/2,\\ &h_{2i-1}=H_{1,i}=\frac{u_i}{w-w_i}+\frac{u_i}{2w_i}, && \bar{h}_{2i-1}=\bar{H}_{1,i}=\frac{\bar{u}_i}{\bar{w}_i-1/w}+\frac{1}{2}\bar{u}_i/\bar{w}_i,\\ &h_{2i}=H_{0,i}=\log(w_i-w)+1/2\log(r/w_i), && \bar{h}_{2i}=\bar{H}_{0,i}=\log(\bar{w}_i-1/w)+1/2\log(r/\bar{w}_i)\end{aligned} \tag{21}$$

and

$$\tau_0=t_{1,\infty},\quad \tau_{2i-1}=t_{1,i},\quad \tau_{2i}=t_{0,i},\quad \bar{\tau}_0=\bar{t}_{1,\infty},\quad \bar{\tau}_{2i-1}=\bar{t}_{1,i},\quad \bar{\tau}_{2i}=\bar{t}_{0,i} \tag{22}$$

preserve the rational form of $z(w)$ (13) *and* $\bar{z}(1/w)$ (16) *(or equally* (10), (9)*) provided the string equation* (6) *holds.*

As in the polynomial case, the total number of invariant flows equals the dimension of the dynamical system minus one. In what follows we show that these flows are Hamiltonian. Since the dimension of the phase space is odd and is equal to $4N+3$, it is, in fact a Poisson manifold where the dimension of the symplectic leaf is $4N+2$. This last number is exactly equal to the number of commuting Toda–Krichever flows.

Now we conjecture that the stated result is true for the more general setting. We introduce a logarithmic ansatz for the 2dToda hierarchy and prove Lemma 2 and Proposition 3 for logarithms followed by a limiting procedure.

2.6 Logarithmic flows

As was mentioned above, it is easier to prove consistency of the rational ansatz with the dynamics of the 2dToda system using the more general Logarithmic solutions. Let us set:

$$z = r(x)w + u(x) + \sum_{i=1}^{n+1} a_i \log(w_i(x) - w), \qquad \sum_{i=1}^{n+1} a_i = 0,$$
$$\bar{z} = r(x)w^{-1} + \bar{u}(x) + \sum_{i=1}^{n+1} \bar{a}_i \log\left(\bar{w}_i(x) - w^{-1}\right), \qquad \sum_{i=1}^{n+1} \bar{a}_i = 0, \qquad i = 1, \dots, n+1, \quad (23)$$

where a_i, $\bar{a}_i$ are arbitrary constants, subject to conditions $\sum\limits_{i=1}^{n+1} a_i = 0$, $\sum\limits_{i=1}^{n+1} \bar{a}_i = 0$ which ensure the absence of logarithmic singularities at infinity. For the introduced ansatz (23) we claim the following result:

Proposition 4. *Let us generalize the evolution operators to be as follows:*

$$\mathcal{H}_0 = r(x)w + \tfrac{1}{2}u(x), \qquad \bar{\mathcal{H}}_0 = \bar{r}(x)w^{-1} + \tfrac{1}{2}\bar{u}(x),$$
$$\mathcal{H}_i = \log(w_i(x) - w) + \tfrac{1}{2}\log(r(x)/w_i(x)),$$
$$\bar{\mathcal{H}}_i = \log\left(\bar{w}_i(x) - w^{-1}\right) + \tfrac{1}{2}\log(r(x)/\bar{w}_i(x)), \qquad i = 1, \dots, n+1. \quad (24)$$

Then, the $2n+4$ flows generated by the Lax equations

$$\partial_{\tau_i} z = \{\mathcal{H}_i, z\}, \qquad \partial_{\bar{\tau}_i} z = \left\{\bar{\mathcal{H}}_i, z\right\},$$
$$\partial_{\tau_i} \bar{z} = \{\mathcal{H}_i, \bar{z}\}, \qquad \partial_{\bar{\tau}_i} \bar{z} = \left\{\bar{\mathcal{H}}_i, \bar{z}\right\}, \qquad i = 0, \dots, n+1 \quad (25)$$

commute. They preserve the logarithmic ansatz (23) *provided the string equation* (6) *holds.*

In other words, the 2dToda flows are tangent to the manifold of logarithmic functions if the string condition is imposed and we again have $2n+4$ flows leaving invariant a $2n+5$ dimensional sub-manifold of the 2dToda system.

It is easier to prove the proposition in the Takasaki gauge. For this purpose we give the transition formulas between different gauges before the proof. For the general 2dToda hierarchy, the Lax functions $z^{(g)}$ and $\bar{z}^{(g)}$ in an arbitrary g-gauge are expressed through the Lax–Sato gauge which corresponds to $g = 0$, $z = z^{(0)}$, $\bar{z} = \bar{z}^{(0)}$ as follows

$$z^{(g)}(w) = z\big(w/r^{2g}\big) = r(x)^{1-2g}w + \sum_{i=-\infty}^{0} u_i^{(g)}(x)w^i,$$
$$\bar{z}^{(g)}\big(w^{-1}\big) = \bar{z}\big(r^{2g}/w\big) = r(x)^{1+2g}w + \sum_{i=0}^{\infty} \bar{u}_i^{(g)}(x)w^i$$

while, (omitting evident subscripts)

$$H^{(g)}(w) = B^{(g)}(w) - \left(\tfrac{1}{2} - g\right) B^{(g)}(w = 0),$$
$$\bar{H}^{(g)}(1/w) = \bar{B}^{(g)}(1/w) - \left(\tfrac{1}{2} + g\right) \bar{B}^{(g)}(1/w = 0). \quad (26)$$

Here, $B^{(g)}$ and $\bar{B}^{(g)}$ are specified similarly to those in (19). In particular, in the Takasaki gauge $g = 1/2$ (below we omit superscripts (1/2))

$$z = w + u(x) + \sum_{i=1}^{n+1} a_i \log(w_i(x) - w),$$

$$\bar z = r(x)^2 w^{-1} + \bar u(x) + \sum_{i=1}^{n+1} \bar a_i \log\left(\bar w_i(x) - w^{-1}\right), \tag{27}$$

$$\mathcal{H}_0 = w + u(x), \qquad \bar{\mathcal{H}}_0 = r(x)^2 w^{-1},$$

$$\mathcal{H}_i = \log(w_i(x) - w), \qquad \bar{\mathcal{H}}_i = \log\left(\bar w_i(x) - w^{-1}\right) + \log(w_i(x)/r(x)). \tag{28}$$

Proof of Proposition 4.

1. *Formal commutativity of flows.* We demonstrate the commutativity of $\partial_{\tau_i} z$, and $\partial_{\tau_j} z$. The proof is similar for the rest of the flows.

The commutativity of the flows is equivalent to the zero-curvature equation

$$\{\mathcal{H}_i, \mathcal{H}_j\} - \frac{\partial \mathcal{H}_j}{\partial \tau_i} + \frac{\partial \mathcal{H}_i}{\partial \tau_j} = 0. \tag{29}$$

Equations (28), (27) and (25) imply that

$$\frac{\partial \mathcal{H}_j}{\partial \tau_i} = \frac{\partial_{\tau_i} w_j}{w_j - w} = \frac{1}{a_j}(\partial_{\tau_i} z)_j = \frac{1}{a_j}\{\mathcal{H}_i, z\}_j,$$

where subscript j stands for the singular part of the expansion around w_j (see (15)). Thus instead of the lhs of (29) we have

$$\{\mathcal{H}_i, \mathcal{H}_j\} - \frac{1}{a_j}\{\mathcal{H}_i, \mathcal{H}_j\}_j + \frac{1}{a_i}\{\mathcal{H}_i, \mathcal{H}_j\}_i$$

which is zero by direct calculation applying (28) and (5). This proves the formal commutativity of the flows.

2. *Consistency of* (27) *with equations of motion* (25). In a way similar to that of the polynomial and rational cases the consistency of the half-flows $\partial_{\tau_i} z$, and $\partial_{\bar\tau_i} \bar z$ for logarithmic solutions is automatic and does not require any extra restrictions on the coefficients (the proof is also similar). In the logarithmic ansatz the main obstacle appears for the flows $\partial_{\tau_i} \bar z$ and $\partial_{\bar\tau_i} z$. We prove the consistency for $\partial_{\bar\tau_i} z$. The proof for $\partial_{\tau_i} \bar z$ is similar.

Differentiating z directly with respect to $\bar\tau_i$ and using the equation of motion (25) we get

$$\partial_{\bar\tau_i} z = \partial_{\bar\tau_i} u + \sum_{j=1}^{n+1} \frac{a_i \partial_{\bar\tau_i} w_j}{w - w_j} = \{\bar{\mathcal{H}}_i, z\}.$$

The lhs of the equation contains only singularities of z. Using definition (5) and equations (27), (28), we see that the rhs contains both singularities of z and one singularity of $\bar{\mathcal{H}}_i$. In order for the equation of motion to be consistent with the logarithmic ansatz, the rhs, $\{\bar{\mathcal{H}}_i, z\}$, must contain the same singularities as those in the lhs, i.e. it must not contain a singularity of $\bar{\mathcal{H}}_i$ at $w = 1/\bar w_i$.

Let us exploit the fact that in the Takasaki gauge (27), (28), z and $\bar z$ can be represented as sums over the corresponding $\mathcal{H}$:

$$z = \mathcal{H}_0 + \sum_{i=1}^{n+1} a_i \mathcal{H}_i, \qquad \bar z = \bar{\mathcal{H}}_0 + \sum_{i=1}^{n+1} \bar a_i \bar{\mathcal{H}}_i + f(r, w_1, \dots, w_{n+1}), \tag{30}$$

where f is a w-independent function. Note that the conditions $\sum\limits_{i=1}^{n+1} a_i = 0$, $\sum\limits_{i=1}^{n+1} \bar a_i = 0$ of Proposition 4 ensure that there are no logarithmic singularities at infinity.

Now, using (30) we obtain the following expression

$$\bar{a}_i\{\bar{\mathcal{H}}_i, z\} = \Big\{\bar{z} - f - \sum_{j\neq i} \bar{a}_j\bar{\mathcal{H}}_j, z\Big\} = \{\bar{z}, z\} - \{f, z\} - \sum_{j\neq i} \bar{a}_j\{\bar{\mathcal{H}}_j, z\}.$$

The term $\sum\limits_{j\neq i} \bar{a}_j\{\bar{\mathcal{H}}_j, z\}$ of the rhs of the last expression does not contain singularities of $\bar{H}_i$ since it does not contain the index i in the sum. The next term $\{f, z\}$ does not contain them either, since f is independent of w. The only term which may contain undesired singularities is $\{\bar{z}, z\}$, however the string equation (6) holds and therefore $\{\bar{\mathcal{H}}_i, z\}$ is free of singularities of $\bar{z}$. Thus the equation of motion is consistent with the logarithmic ansatz due to the string equation. ■

Corollary 1. *The consistency of the rational ansatz follows from the consistency of the logarithmic ansatz. (Proposition 3 is a consequence of Proposition 4).*

Proof. Choosing $n = 2N - 1$ and substituting the values

$$\begin{aligned} &a_{2i-1} = 1/\epsilon, \qquad a_{2i} = -1/\epsilon, \qquad w_{2i} = w_{2i-1} + \epsilon u_i, \\ &\bar{a}_{2i-1} = 1/\epsilon, \qquad \bar{a}_{2i} = -1/\epsilon, \qquad \bar{w}_{2i} = \bar{w}_{2i-1} + \epsilon \bar{u}_i \end{aligned} \tag{31}$$

for z and $\bar{z}$ into (23) we get rational solutions (16), (13) in the $\epsilon \to 0$ limit.

The Hamiltonians are then related as follows:

$$\begin{aligned} &h_0 = \lim_{\epsilon=0} \mathcal{H}_0, && \bar{h}_0 = \lim_{\epsilon=0} \bar{\mathcal{H}}_0, \\ &h_{2i-1} = \lim_{\epsilon=0} \frac{1}{\epsilon}(\mathcal{H}_{2i} - \mathcal{H}_{2i-1}), && \bar{h}_{2i} = \lim_{\epsilon=0} \frac{1}{\epsilon}(\bar{\mathcal{H}}_{2i} - \bar{\mathcal{H}}_{2i-1}), \\ &h_{2i} = \lim_{\epsilon=0} \mathcal{H}_{2i-1}, && \bar{h}_{2i-1} = \lim_{\epsilon=0} \bar{\mathcal{H}}_{2i-1}. \end{aligned}$$ ■

Thus we have obtained a rational ansatz as a limiting case of the logarithmic ansatz, merging pairs of logarithmic singularities together, in which case they become simple poles. In a similar way one may deduce any kind of rational maps containing a combination of poles of any degrees absorbing different numbers of logarithmic singularities.

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Poisson Structure of Rational Reductions of the 2D dToda Hierarchy

Igor LOUTSENKO † *and Oksana YERMOLAYEVA* ‡

† *SISSA, Via Beirut 2-4, 34014, Trieste, Italy*
E-mail: *loutseni@fm.sissa.it*

‡ *Concordia University, 1455 de Maisonneuve Blvd. W. Montreal, Quebec, H3G 1M8, Canada*
E-mail: *yermolae@crm.umontreal.ca*

The paper concerns the Hamiltonian structure of the finite-dimensional reductions 2D dispersionless Toda hierarchy constrained by the string equation. We derive the Hamiltonian structure of the reduced dynamics and show connections of integrals of "multi-finger" solutions of the Laplacian growth problem with the "Toda–Krichever" flows of the 2dToda hierarchy constrained by a string equation. The Poisson structure of the rationally reduced 1dToda hierarchy is also derived.

1 Hamiltonians and action-angle variables

In this paper we study the Poisson structure of rational reductions of the 2dToda hierarchy, introduced in the previous paper of the present volume [1]. These reductions, as seen from the above considerations, form a finite-dimensional completely integrable system. We find explicit expressions leading to a canonical Hamiltonian structure on the phase space of rational reductions of the 2D dToda system. Such reductions are interesting from the point of view of the applications mentioned in the Introduction.

A reader interested in the Poisson structure of rational reductions of the 1dToda hierarchy (which are infinite-dimensional, unrestricted by string equation systems) will find it in the second part of the paper. This structure, which turns out to be a quadratic algebra, is interesting by itself.

Since the Toda flows commute, the Toda times are actually the action-angle variables of the system.

Thus we have only to express the times through coordinates of the phase space. For the logarithmic ansatz [1, Proposition 4] the following result holds[1]:

Proposition 1. *The times τ_i, $\bar\tau_i$ of equation* ([1].25) *are expressed in terms of z and $\bar z$* ([1].23) *as*

$$\tau_0 = \frac{1}{2\pi\sqrt{-1}}\oint_\infty \bar z d\ln z + \text{const}, \qquad \bar\tau_0 = \frac{1}{2\pi\sqrt{-1}}\oint_0 z d\ln\bar z + \text{const},$$
$$\tau_i = \frac{1}{2\pi\sqrt{-1}}\oint_{w_i} \bar z dz + \text{const}, \qquad \bar\tau_i = \frac{1}{2\pi\sqrt{-1}}\oint_{1/\bar w_i} z d\bar z + \text{const}, \qquad i = 1,\dots,n+1$$

or, in other words, the following values

$$I_0 = \bar z(1/w=0) = \bar u + \sum_{i=1}^{n+1}\bar a_i \ln(\bar w_i), \qquad \bar I_0 = z(0) = u + \sum_{i=1}^{n+1} a_i \ln(w_i),$$

[1]Here and below equations from [1] are cited in the form ([1].N), where N is the number of corresponding equation.

$$I_i = a_i \bar{z}(w_i^{-1}) = a_i \left(r w_i^{-1} + \bar{u} + \sum_{j=1}^{n+1} \bar{a}_j \ln(\bar{w}_j - w_i^{-1}) \right),$$
$$\bar{I}_i = \bar{a}_i z(\bar{w}_i^{-1}) = \bar{a}_i \left(r \bar{w}_i^{-1} + u + \sum_{j=1}^{n+1} a_j \ln(w_j - \bar{w}_i^{-1}) \right), \qquad i = 1, \dots, n+1, \tag{1}$$

are the action-angle variables and

$$\begin{aligned} Q &= \frac{1}{4\pi\sqrt{-1}} \sum_{i=0}^{n+1} \oint_{1/\bar{w}_i} z d\bar{z} + \oint_{w_i} \bar{z} dz \\ &= \frac{1}{2} r \left(\left(\frac{\partial z(w)}{\partial w} \right)_{w=0} + \left(\frac{\partial \bar{z}(1/w)}{\partial (1/w)} \right)_{1/w=0} \right) - \frac{1}{2} \sum_{i=1}^{n+1} \left(I_i + \bar{I}_i \right) \\ &= r^2 - \frac{1}{2} \sum_{i=1}^{n+1} \left(r \left(\frac{a_i}{w_i} + \frac{\bar{a}_i}{\bar{w}_i} \right) + I_i + \bar{I}_i \right) \end{aligned} \tag{2}$$

is a Casimir for ([1].25).

Proof. We first calculate the derivatives of each I_i with respect to the times τ_i, $\bar{\tau}_i$, $i = 1, \dots, n$. Using integration by parts, we get

$$2\pi\sqrt{-1} \frac{\partial I_i}{\partial \tau_j} = \frac{\partial}{\partial \tau_j} \oint_{w_i} \bar{z} \frac{\partial z}{\partial w} dw = \oint_{w_i} \frac{\partial}{\partial w} \left(\bar{z} \frac{\partial z}{\partial \tau_j} \right) dw + \oint_{w_i} \left(\frac{\partial \bar{z}}{\partial \tau_j} \frac{\partial z}{\partial w} - \frac{\partial z}{\partial \tau_j} \frac{\partial \bar{z}}{\partial w} \right) dw.$$

By equations of motion ([1].25), definition of the Lax–Poisson brackets ([1].5) and by the string equation ([1].6) this value reduces to:

$$\begin{aligned} &\oint_{w_i} \frac{\partial}{\partial w} \left(\bar{z} \frac{\partial z}{\partial \tau_j} \right) dw + \oint_{w_i} \left(\{\mathcal{H}_j, \bar{z}\} \frac{\partial z}{\partial w} - \{\mathcal{H}_j, z\} \frac{\partial \bar{z}}{\partial w} \right) dw \\ &\quad = \oint_{w_i} \frac{\partial}{\partial w} \left(\bar{z} \frac{\partial z}{\partial \tau_j} \right) dw + \oint_{w_i} \frac{\partial \mathcal{H}_j}{\partial w} w \left(\frac{\partial z}{\partial w} \frac{\partial \bar{z}}{\partial x} - \frac{\partial z}{\partial x} \frac{\partial \bar{z}}{\partial w} \right) dw \\ &\quad = \oint_{w_i} \frac{\partial}{\partial w} \left(\bar{z} \frac{\partial z}{\partial \tau_j} \right) dw + \oint_{w_i} \frac{\partial \mathcal{H}_j}{\partial w} dw = 2\pi\sqrt{-1} \delta_{ij}. \end{aligned}$$

We get the Kronecker δ-symbol as the final answer, since the first term in the last line vanishes. Indeed, this occurs because $\bar{z} \frac{\partial z}{\partial \tau_j}$ is a meromorphic in w in the vicinity of any singularity w_i of z and the second term gives the δ_{ij} contribution by definition ([1].24) of $\mathcal{H}_j$.

Using similar reasoning we prove the rest of the proposition. ■

The values (1) are similar to those appearing in the work [8] in connection with the Laplacian Growth problem as integrals of the Laplacian Growth (string) equation. The following proposition (although not assuming complex structure) confirms this fact.

Proposition 2. *The values* (1) *form a set of integrals of the string equation* ([1].6)*, i.e., provided the string equation holds.*

$$dI_i/dx = 0$$

while for the Casimir (2) *we have*

$$dQ/dx = 1.$$

Proof. This is similar to the proof of proposition (1); one has to take the same steps substituting x for τ_i. ■

As was mentioned in the Introduction the Laplacian growth problem is endowed with a complex structure, with $\bar{z}$ being the complex conjugate of z (and bar denoting complex conjugation). Then, as may be seen from the proposition, the Casimir Q plays the role of the area (which grows with the unit speed), while I are functions of the harmonic moments of the boundary curve.

Since the equations of motion ([1].25) have the following form in I, $\bar{I}$

$$\begin{aligned} &\partial_{\tau_j} Q = 0, \qquad \partial_{\bar{\tau}_j} Q = 0, \qquad \partial_{\tau_j} I_i = \delta_{ij}, \\ &\partial_{\bar{\tau}_j} I_i = 0, \qquad \partial_{\tau_j} \bar{I}_i = 0, \qquad \partial_{\bar{\tau}_j} \bar{I}_i = -\delta_{ij} \end{aligned} \tag{3}$$

these variables are canonical. For instance, we may choose the following Poisson structure

$$\{I_i, \bar{I}_j\}_p = \delta_{ij}, \qquad \{I_i, I_j\}_p = \{\bar{I}_i, \bar{I}_j\}_p = 0, \qquad \{Q, \bar{I}_j\}_p = \{Q, I_j\}_p = 0, \tag{4}$$

$$\mathrm{H}_i = \bar{I}_i, \qquad \bar{\mathrm{H}}_i = I_i. \tag{5}$$

It is important to note that the Poisson brackets $\{\ ,\ \}_p$ in (4) are different from $\{\ ,\ \}$ in ([1].5): The former defines the Poisson structure on the space of rational solutions of the 2dToda hierarchy, while the latter (Lax–Poisson bracket) is a dispersionless limit of the commutator.

To get similar results for the rational case we have to take a limit as in Corollary 1 of [1].

Corollary 1. *Let z, $\bar{z}$ be represented as in* ([1].13)–([1].16). *The following $2N+1$ values $(i = 1, \dots, N)$*

$$I_0 = \bar{u}_0 - \sum_{i=1}^{N} \bar{u}_i/\bar{w}_i, \qquad \bar{I}_0 = u_0 - \sum_{i=1}^{N} u_i/w_i,$$

$$I_{2i-1} = r w_i^{-1} + \bar{u}_0 + \sum_{j=1}^{N} \frac{\bar{u}_j}{1/w_i - \bar{w}_j}, \qquad \bar{I}_{2i-1} = r \bar{w}_i^{-1} + u_0 + \sum_{j=1}^{N} \frac{u_j}{1/\bar{w}_i - w_j},$$

$$I_{2i} = \left(r - \sum_{j=1}^{N} \frac{\bar{u}_j}{(1/w_i - \bar{w}_j)^2} \right) \frac{u_i}{w_i^2}, \qquad \bar{I}_{2i} = \left(r - \sum_{j=1}^{N} \frac{u_j}{(1/\bar{w}_i - w_j)^2} \right) \frac{\bar{u}_i}{\bar{w}_i^2}$$

$$Q = r^2 - \frac{1}{2} \sum_{i=1}^{N} \left(r \left(\frac{\bar{u}_i}{\bar{w}_i^2} + \frac{u_i}{w_i^2} \right) + \bar{I}_{2i} + I_{2i} \right)$$

are the action-angle variables in the rational case [1, Proposition 3], i.e. the variables for which equations in this Proposition have the form (3).

Again, in the rational limit we may choose the Poisson structure as in (4), (5).

2 Poisson structure of the 1dToda hierarchy

In this section we consider the Poisson structure of rational reductions of the 1dToda system.

Recall, that for the 1dToda system one takes into account only a "half" of flows, connected with z (but not $\bar{z}$, or vice verse).

$$\partial_{t_i} z = \{H_i, z\}, \qquad H_i = \big(z(w)^i\big)_+ + \tfrac{1}{2}\big(z(w)^i\big)_0, \qquad i = 0, \dots, \infty. \tag{6}$$

This system is bi-Hamiltonian (for general information e.g. see [3]) with two (linear and quadratic) compatible Poisson structures. As it was first found in [2] for the generic Toda system ([1].2),

the dispersionless linear Poisson brackets for the "field variables" u_i, $i = 1, \ldots, \infty$ have the following form

$$\{u_n(x), u_m(y)\}_1 = 2(c_n + c_m - 1)\big[(n+m)u_{n+m}(x)\delta'(x-y) + mu'_{n+m}(x)\delta(x-y)\big], \tag{7}$$

where

$$c_k = \begin{cases} 1, & \text{if } k > 0, \\ \frac{1}{2}, & \text{if } k = 0, \\ 0, & \text{if } k < 0 \end{cases}$$

while the quadratic brackets are

$$\begin{aligned}\{u_n(x), u_m(y)\}_2 = &\left[\frac{1}{2}(n-m)u_n(x)u'_m(x) + \left(\sum_{k=1}^{1-n}(n-m+k)u_{n+k}(x)u'_{m-k}(x)\right.\right. \\ &\left.\left. + ku'_{n+k}(x)u_{m-k}(x)\right)\right]\delta(x-y) \\ &+ \left[\frac{1}{2}(n-m)u_nu_m + \sum_{k-1}^{1-n}(n-m+2k)u_{n+k}u_{m-k}\right]\delta(x-y). \end{aligned}\tag{8}$$

As seen from the Lemma 1 of [1], the rational functions

$$z(w) = \frac{q_{N+1}(w)}{p_N(w)} = \frac{w^{N+1} + \sum\limits_{i=0}^{N} a_i w^i}{\sum\limits_{i=0}^{N} b_i w^i} \tag{9}$$

are form-invariant under all of the 1dToda flows ∂_{t_i} (6), without any extra restriction (i.e. the string equation is not needed).

We obtain the corresponding Poisson structures for coefficients a_i, b_i, by using result (8), expressing u_i in terms of a_i, b_i, $i = 0, \ldots, N$. Both the linear (7) and quadratic (8) Poisson structures lead to the quadratic brackets for a_i, b_i. Namely, the second Poisson structure for (9) reads as follows

$$\begin{aligned}\{a_k(x), a_l(y)\}_2 = &\left[\sum_{n=1}(l+n-k)a_{k-n}(x)a_{l+n}(y) + na_{k-n}(y)a_{l+n}(x))\right. \\ &\left. + (l-N-1)a_k(x)a_l(y)\right]\delta'(x-y), \end{aligned}\tag{10}$$

$$\begin{aligned}\{b_k(x), b_l(y)\}_2 = &\left[\sum_{n=1}(k-l-n)b_{k-n}(x)b_{l+n}(y) - nb_{k-n}(y)b_{l+n}(x))\right. \\ &\left. + \frac{k-l}{2}b_k(x)b_l(y)\right]\delta'(x-y), \end{aligned}\tag{11}$$

$$\{a_k(x), b_l(y)\}_2 = \frac{k-N-1}{2}a_k(x)b_l(y)\delta'(x-y). \tag{12}$$

The first Poisson structure can be obtained from (10)–(12) with the help of the linear transformation (shift by a constant)

$$a_i \to a_i + \lambda b_i, \qquad z(w,x) \to z(w,x) + \lambda$$

and using the bi-Hamiltonian nature of (7), (8).

$$\{a_k(x), a_l(y)\}_1 = \Bigg[\sum_{n=1}\Big((k-l-n)(a_{k-n}(x)b_{l+n}(y)+b_{k-n}(x)a_{l+n}(y)) - n(a_{k-n}(y)b_{l+n}(x)+b_{k-n}(y)a_{l+n}(x))\Big) + \frac{N+1-l}{2}b_k(x)a_l(y) + \frac{k+N+1-2l}{2}a_k(x)b_l(y)\Bigg]\delta'(x-y), \tag{13}$$

$$\{a_k(x), b_l(y)\}_1 = \Bigg[\sum_{n=1}\Big((k-l-n)b_{k-n}(x)b_{l+n}(y) - nb_{k-n}(y)b_{l+n}(x)\Big) + \frac{N+1-l}{2}b_k(x)b_l(y)\Bigg]\delta'(x-y), \tag{14}$$

$$\{b_k(x), b_l(y)\}_1 = 0 \tag{15}$$

in all the above expressions $a_{N+1} = 1$ and $a_i = 0$ if i goes beyond the range $i = 0, \ldots, N+1$ (and $b_j = 0$ if $j \neq 0, \ldots, N$).

These brackets form a bi-Hamiltonian structure for rational reductions of the 1dToda hierarchy:

$$\partial_{t_i} z = \{\mathrm{H}_i, z\}_1 = \{\mathrm{H}_{i-1}, z\}_2 \tag{16}$$

with the following Hamiltonians:

$$\mathrm{H}_i = \frac{1}{i+1}\int (z^{i+1}(x))_0 dx. \tag{17}$$

3 Summary and perspectives

We have established the Hamiltonian structure on the space of rational solutions of the 2dToda hierarchy connected with the problem of ideal interface dynamics. A further application of this result can become quite challenging in dealing with the problem of encountering a surface tension for the Laplacian growth process.

The Laplacian growth equation ([1].1), ([1].6) describes the propagation of the boundary with zero surface tension between the fields. Although, addressing a number of important questions (e.g. finger width selection problem in [7]), such an idealized model does not account for essential physical features, such as fractal formation, stability etc. Thus, the inclusion of tension effects is important for the solution of the problem.

There are two approaches to look for in such a generalization. The first one is to introduce tension terms in the theory, destroying the integrability of the problem. In another approach one might look for integrable deformations of the idealized model, simulating surface effects and stabilizing interface dynamics.

Another feature that gives our result a certain interest lies in the possible investigation of the perturbed (by small surface tension) system given in terms of separated variables (4). In other words, is it possible, in some situations, to approximate the perturbed solution by the multi-finger ansatz ([1].23) with different dependence of coordinates on time x? If the answer is affirmative we will get a finite-dimensional dynamical system, conveniently written in terms of I, $\bar{I}$, Q.

Finally, returning to the idealized problem, one can further elaborate the theory of rational reductions from the point of view of symmetries. An important observation of our study is

that once a rational reduction is compatible with 2dToda dynamics, the string condition is satisfied automatically. In other words *the string (Laplacian growth) equation* ([1].6) *turns out to be a consequence of rationality* in the context of the 2dToda hierarchy. On the other hand, it is well known that the string equation is connected with additional (Orlov–Schulman [3, 9]) symmetries of the 2dToda hierarchy [10]. It would be interesting to see how the string constraint can be an element of a set of the symmetry constraints leading to the rationality of solutions.

Amongst the other open questions and future directions is the following problem: to investigate Hamiltonian structure of rational reductions in the context of the two-matrix [4] model, whose partition function is a tau-function of the 2dToda system constrained by the string equation. The applicability of our study to the models of normal matrices is another interesting aspect worthy of further analysis [6].

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A Conservative Numerical Integration Algorithm for Integrable Henon–Heiles System

Yukitaka MINESAKI and Yoshimasa NAKAMURA

Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan
E-mail: *minesaki@amp.i.kyoto-u.ac.jp, ynaka@i.kyoto-u.ac.jp*

A new conservative numerical integration algorithm for the integrable Henon–Heiles system is presented which conserves *all* of the constants of motion.

1 Introduction

The Henon–Heiles system in celestial mechanics is a well-known chaotic Hamiltonian system [4]. Since this two dimensional system has only one constant of motion, it is a non-integrable system. By simple changes of the Hamiltonian we obtain some completely integrable Henon–Heiles type systems. They are completely integrable in the sense of Liouville–Arnold. Namely, they are two-dimensional systems having two constants of motion cf. [8]. To solve numerically the equations of motion, one often tries to discretize them. Even for the integrable cases a good numerical integration scheme will be useful for describing orbits for arbitrary given initial data.

The symplectic integrators [3, 9] and the energy preserving methods [2, 1] are well-known as good numerical integration schemes. The symplectic integrators conserve the symplectic form in the phase space, so that the resulting discrete-time evolution is regarded as a canonical transformation. Though the symplectic integrators do not conserve Hamiltonian and other additional constants of motion, for example, the angular momentum and the Runge–Lenz vector for the Kepler motion, in general, they are widely used in numerical simulation for the solar system. This is because the symplectic integrators give a good approximation of the orbits of the Hamiltonian system in the sense in which they conserve a modified (or approximate) Hamiltonian. However, a non-existence theorem of modified constant of motion besides the modified Hamiltonian is proved recently in [10,11]. The energy preserving methods originated by Greenspan [2] keep the value of the Hamiltonian constant. These methods are based on discrete variational derivatives [1, 5]. Generally, they conserve only energy and do not preserve other additional constants of motion. It has not been known for long time how to design an energy preserving method which conserves all of the additional constants of motion of certain class of integrable Hamiltonian systems. Especially, it is impossible to trace an ellipse of the Kepler motion, since the Runge–Lentz vector is not conserved by any known numerical integrators including the symplectic integrators and the usual energy preserving methods.

In the previous paper [6] the authors have established a new numerical integration algorithm which conserves all of the constants of motion of the two-dimensional Kepler motion including the Runge–Lenz vector. In this paper we consider one of the integrable Henon–Heiles type systems discussed in [8]. We discretize the Henon–Heiles system which preserves all constants of motion as an application of the numerical integration algorithm.

It is also shown in this paper that 1) a time step size, which is larger than those of other known integrators, is easily used, 2) a variable time step is naturally introduced. These basic properties underlie the efficiency of the new numerical integration algorithm.

2 Integrable Henon–Heiles system and canonical transformation

The integrable Henon–Heiles system is defined by the Hamiltonian

$$\mathcal{H}_{\mathrm{hh}-1}(p_x, p_y, x, y) = p_x^2 + p_y^2 + x^2 + y^2 + \frac{2}{3}x^3 + 2xy^2 - E_{\mathrm{hh}} \tag{1}$$

on a subset of $\mathbf{R}^4 = \{p_x, p_y, x, y\}$, where E_{hh} is an arbitrary constant. The time variable is t. When the positive sign of $\frac{2}{3}x^3$ in (1) becomes changed to negative, (1) is the Hamiltonian of a chaotic dynamical system in celestial mechanics [4]. The equation of motion for (1) are expressed with

$$\frac{dx}{dt} = 2p_x, \qquad \frac{dy}{dt} = 2p_y, \qquad \frac{dp_x}{dt} = -2x^2 - 2x - 2y^2, \qquad \frac{dp_y}{dt} = -4xy - 2y. \tag{2}$$

After a canonical transformation

$$q_1 = \frac{1}{2}(x + y), \qquad q_2 = \frac{1}{2}(x - y), \qquad p_1 = p_x + p_y, \qquad p_2 = p_x - p_y, \tag{3}$$

the Hamiltonian (1) leads to the Hamiltonian

$$\mathcal{H}_{\mathrm{hh}-2}(p_1, p_2, q_1, q_2) = \frac{1}{2}\left(p_1^2 + p_2^2 + U(q_1) + U(q_2)\right) \tag{4}$$

on a subset of $\mathbf{R}^4 = \{p_1, p_2, q_1, q_2\}$, where the potential functions are

$$U(q_j) = \frac{16}{3}q_j^3 + 4q_j^2 - E_{j,\mathrm{hh}}, \qquad j = 1, 2. \tag{5}$$

The variables p_k and q_k are now separated. Note that the canonical transformation (3) is a mapping from $\mathbf{R}^4$ to $\mathbf{R}^4$. The equations of motion are then

$$\frac{dq_k}{dt} = p_k, \qquad \frac{dp_k}{dt} = -8q_k^2 - 4q_k, \qquad k = 1, 2. \tag{6}$$

They are just the equations of the two-dimensional anharmonic oscillator with the Hamiltonian (4) on $\mathbf{R}^4 = \{p_1, p_2, q_1, q_2\}$ with the time t. Let us set the constant E_{hh} in $\mathcal{H}_{\mathrm{hh}-2}$ such that

$$\mathcal{H}_{\mathrm{hh}-2}(p_1, p_2, q_1, q_2) = 0. \tag{7}$$

Obviously, $\mathcal{H}_{\mathrm{hh}-1}(p_x, p_y, x, y) = 0$ for such E_{hh}. Since $\mathcal{H}_{\mathrm{hh}-2}$ is a constant of motion of (6), the value of the constant E_{hh} reflects the initial values of the variables p_k and q_k. Namely, E_{hh} is expressed as

$$E_{\mathrm{hh}} = E_{1,\mathrm{hh}} + E_{2,\mathrm{hh}}, \tag{8}$$

where

$$E_{j,\mathrm{hh}} = \frac{1}{2}\left(p_j(0)^2 + U(q_j(0))\right), \qquad j = 1, 2. \tag{9}$$

We see that the canonical transformation (3) generates a transformation that maps the integrable Henon–Heiles system to a system of anharmonic oscillators.

The integrable Henon–Heiles system has an additional constant of motion. This constant is the angular momentum

$$I_{2,\mathrm{hh}-1}(p_x, p_y, x, y) = 2p_xp_y + 4x^2y + \frac{4}{3}y^3 + 4xy - E_{1,\mathrm{hh}} + E_{2,\mathrm{hh}}. \tag{10}$$

The constants of motion are expressed by using the canonical variables p_k and q_k as follows. The Hamiltonian $\mathcal{H}_{\mathrm{hh}-2}(p_k, q_k)$ is as (4), where $(p_k, q_k) = (p_1, p_2, q_1, q_2)$, for simplicity. Note that (10) is expressed as

$$I_{2,\mathrm{hh}-2} = \frac{1}{2}\left(p_1^2 - p_2^2\right) + \frac{8}{3}\left(q_1^3 - q_2^3\right) + 2\left(q_1^2 - q_2^2\right). \tag{11}$$

3 Discrete integrable Henon–Heiles system

The equations of motion of the two-dimensional oscillator (6) are discretized by the energy preserving method (cf. [2,5]) as follows,

$$\begin{aligned}
&\frac{Q_k^{(j+1)}-Q_k^{(j)}}{t^{(j+1)}-t^{(j)}}=\frac{P_k^{(j)}+P_k^{(j+1)}}{2},\\
&\frac{P_k^{(j+1)}-P_k^{(j)}}{t^{(j+1)}-t^{(j)}}=-\frac{8\left(\left(Q_k^{(j+1)}\right)^2+Q_k^{(j+1)}Q_k^{(j)}+\left(Q_k^{(j)}\right)^2\right)+6\left(Q_k^{(j+1)}+Q_k^{(j)}\right)}{3},\\
&t^{(0)}<\cdots<t^{(j-1)}<t^{(j)}<t^{(j+1)}<\cdots,
\end{aligned}\tag{12}$$

where $t^{(j)}$ is an arbitrarily increasing sequence which indicates a discrete-time, $P_k^{(j)}$, $Q_k^{(j)}$ are the values of the variables P_k, Q_k at the time $t^{(j)}$, respectively. Here P_k and Q_k are discrete analogues of the canonical variables p_k and q_k, respectively, where we set

$$P_k^{(0)}=p_k(0),\qquad Q_k^{(0)}=q_k(0),\qquad k=1,2.\tag{13}$$

On the orbit of the two-dimensional discrete-time harmonic oscillator (12) the Hamiltonian (4) is kept constant for any $t^{(j)}$, namely,

$$\mathcal{H}_{\mathrm{hh}-2}\big(P_k^{(j+1)},Q_k^{(j+1)}\big)=\mathcal{H}_{\mathrm{hh}-2}\big(P_k^{(j)},Q_k^{(j)}\big),\qquad j=0,1,\ldots,\tag{14}$$

where $\big(P_k^{(j)},Q_k^{(j)}\big)=\big(P_1^{(j)},P_2^{(j)},Q_1^{(j)},Q_2^{(j)}\big)$. This is a direct consequence of the energy preserving method. Since $\mathcal{H}_{\mathrm{hh}-2}(p_k,q_k)=0$, $P_k^{(0)}=p_k(0)$ and $Q_k^{(0)}=q_k(0)$, we obtain

$$\begin{aligned}
&\mathcal{H}_{\mathrm{hh}-2}\big(P_k^{(j)},Q_k^{(j)}\big)=0,\quad j=0,1,\cdots,\\
&E_{k,\mathrm{hh}}=\frac{1}{2}\big(P_k^{(0)}\big)^2+\frac{8}{3}\big(Q_k^{(0)}\big)^3+2\big(Q_k^{(0)}\big)^2,\qquad k=1,2.
\end{aligned}\tag{15}$$

Through the inverse of the canonical transformation (3) the discrete-time system (12) is converted to

$$\begin{aligned}
&\frac{X^{(j+1)}-X^{(j)}}{t^{(j+1)}-t^{(j)}}=P_X^{(j+1)}+P_X^{(j)},\qquad \frac{Y^{(j+1)}-Y^{(j)}}{t^{(j+1)}-t^{(j)}}=P_Y^{(j+1)}+P_Y^{(j)},\\
&\frac{P_X^{(j+1)}-P_X^{(j)}}{t^{(j+1)}-t^{(j)}}=-\frac{2}{3}\Big(\Big(\big(X^{(j+1)}\big)^2+X^{(j+1)}X^{(j)}+\big(X^{(j)}\big)^2\Big)\\
&\qquad\qquad+\Big(\big(Y^{(j+1)}\big)^2+Y^{(j+1)}Y^{(j)}+\big(Y^{(j)}\big)^2\Big)\Big)-\big(X^{(j+1)}+X^{(j)}\big),\\
&\frac{P_Y^{(j+1)}-P_Y^{(j)}}{t^{(j+1)}-t^{(j)}}=-\frac{2}{3}\Big(2X^{(j+1)}Y^{(j+1)}+2X^{(j)}Y^{(j)}\\
&\qquad\qquad+X^{(j+1)}Y^{(j)}+X^{(j)}Y^{(j+1)}\Big)-\big(Y^{(j+1)}+Y^{(j)}\big).
\end{aligned}\tag{16}$$

Equations (16) preserve the value of $\mathcal{H}_{\mathrm{hh}-1}\big(P_X^{(j)},P_Y^{(j)},X_1^{(j)},Y^{(j)}\big)$ as zero. We call (16) the *discrete integrable Henon–Heiles system*.

4 Discrete integrable Henon–Heiles system as exactly conservative integrator

In the previous section the Hamiltonian $\mathcal{H}_{\mathrm{hh}-1}(P_X^{(j)},P_Y^{(j)},X^{(j)},Y^{(j)})$ is shown to be constant under the time evolution of the discrete integrable Henon–Heiles system (16). The other constant

of motion (10) of the continuous-time integrable Henon–Heiles is also conserved. Now we are in a position to state the main result.

Theorem 1. *The integrable Henon–Heiles system* (16) *has the following two constants of motion. The first is the Hamiltonian*

$$\mathcal{H}_{\mathrm{d-hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big) = \mathcal{H}_{\mathrm{hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big). \tag{17}$$

The second constant is a discrete analogue of the angular momentum (10)

$$\begin{aligned} &I_{2,\mathrm{d-hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big) \\ &\quad = 2P_X^{(j)}P_Y^{(j)} + 4\big(X^{(j)}\big)^2 Y^{(j)} + \frac{4}{3}\big(Y^{(j)}\big)^3 + 4X^{(j)}Y^{(j)} - E_{1,\mathrm{hh}} + E_{2,\mathrm{hh}}. \end{aligned} \tag{18}$$

The proof follows by a direct calculation with an alternative expression of the discrete integrable Henon–Heiles system (16). We here give an important remark.

Remark 1. The form of the constants (17) and (18) of motion is consistent with that of the continuous-time integrable Henon–Heiles system. Namely,

$$\begin{aligned} &\mathcal{H}_{\mathrm{d-hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big) = \mathcal{H}_{\mathrm{hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big), \\ &I_{2,\mathrm{d-hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big) = I_{2,\mathrm{hh-1}}\big(P_X^{(j)}, P_Y^{(j)}, X^{(j)}, Y^{(j)}\big). \end{aligned} \tag{19}$$

Finally in this section we give a numerical example for the discrete integrable Henon–Heiles system. Fig. 1[1] indicates the orbits of the symplectic Euler scheme with step-size $t^{(j+1)} - t^{(j)} = dt = 0.1,\ 0.2,\ 0.4$, the initial value is $X^{(0)} = 0.0$, $Y^{(0)} = 0.0$, $P_X^{(0)} = 0.35$, $P_Y^{(0)} = -0.15$. Especially, the orbit with step-size $t^{(j+1)} - t^{(j)} = dt = 0.4$ is quite different from that of the continuous-time integrable Henon–Heiles system. In contract to the orbits given by the symplectic Euler scheme, the discrete integrable Henon–Heiles system with the constant discrete step-size $t^{(j+1)} - t^{(j)} = \delta = 0.1,\ 0.2,\ 0.4$ draws orbits which give a better approximation than those of the symplectic Euler scheme in Fig. 2. Moreover, the discrete integrable Henon–Heiles system with variable step-size gives the almost same orbit as the continuous-time integrable Henon–Heiles system has. Fig. 3 displays the orbit of the discrete integrable Henon–Heiles system with a step-size changing periodically from 0.1 to 0.2.

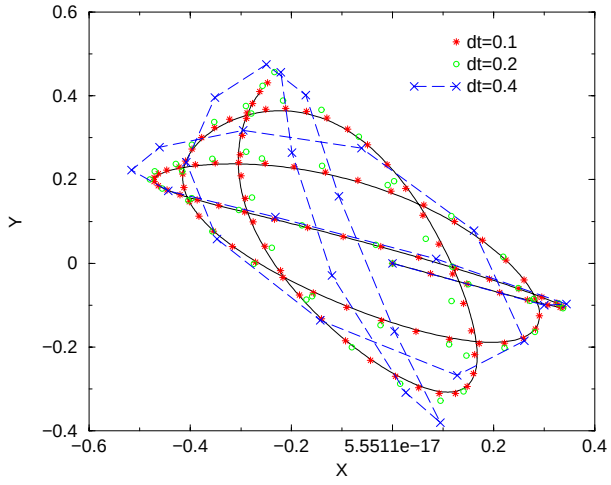

Figure 1. Symplectic Euler scheme $\delta = 0.1,\ 0.2,\ 0.4$.

[1]Figures in colour will be available only in electronic version.

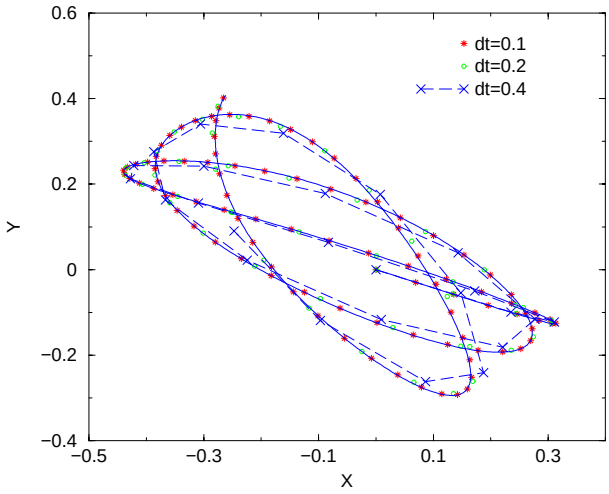

Figure 2. Discrete integrable Henon–Heiles $\delta = 0.1,\ 0.2,\ 0.4$.

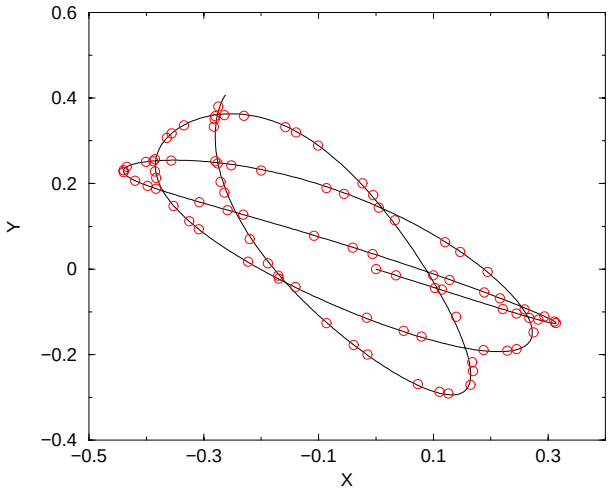

Figure 3. Discrete Integrable Henon–Heiles with a variable step-size.

5 Conclusion

In this paper a new numerical integration algorithm for the integrable Henon–Heiles system is designed with the help of canonical transformation. All of the constants of motion are then exactly conserved. This property is rather different from the known numerical integrators of the integrable Henon–Heiles system. Other types of the integrable Henon–Heiles systems [8] can be discretized in the same method.

There are some additional advantages of the integrable Henon–Heiles system. As is observed in numerical examples, the orbit of the discrete integrable Henon–Heiles gives a better approximation than the symplectic Euler scheme. Secondly, a variable step time is naturally assigned to the discrete integrable Henon–Heiles system. Using the new stable numerical integration algorithm, the authors wish to discretize some other integrable Hamiltonian systems besides the integrable Henon–Heiles system and the two-dimensional Kepler motion.

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Toda Chain, Sheffer Class of Orthogonal Polynomials and Combinatorial Numbers

Yoshimasa NAKAMURA † *and Alexei ZHEDANOV* ‡

† *Department of Applied Mathematics and Physics, Graduate School of Informatices, Kyoto University, Kyoto 606-8501, Japan*
E-mail: *ynaka@amp.i.kyoto-u.ac.jp*

‡ *Donetsk Institute for Physics and Technology, 83114 Donetsk, Ukraine*
E-mail: *zhedanov@kinetic.ac.donetsk.ua*

A classification of Hankel determinant solutions of the restricted Toda chain equations is presented through polynomial Ansatz for moments. Each solution corresponds to the Sheffer class orthogonal polynomials. In turn, these solutions are equivalent to solutions with separated variables in Toda chain. These solutions lead naturally to explicit Hankel determinants of some combinatorial numbers.

1 Introduction

The Toda chain equations [17]

$$\dot{u}_n = u_n(b_n - b_{n-1}), \quad n = 1, 2, \dots, \qquad \dot{b}_n = u_{n+1} - u_n, \quad n = 0, 1, \dots \tag{1}$$

with additional condition

$$u_0 = 0 \tag{2}$$

has the well-known relation with the theory of orthogonal polynomials, where the dot indicates the differentiation with respect to t. In what follows we will call equations (1) with restriction (2) *the restricted Toda chain* (TC) equations.

Let $P_n(x;t)$ be orthogonal polynomials satisfying the three-term recurrence relation

$$P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x), \qquad n = 1, 2, \dots \tag{3}$$

with initial conditions

$$P_0 = 1, \qquad P_1(x) = x - b_0. \tag{4}$$

We will assume that $u_n \neq 0$, $n = 1, 2, \dots$. Then, by the Favard theorem [4], there exists a nondegenerate linear functional σ such that the polynomials $P_n(x)$ are orthogonal with respect to it:

$$\sigma(P_n(x)P_m(x)) = h_n\delta_{nm}, \tag{5}$$

where h_n are normalization constants. The linear functional σ can be defined through its moments

$$c_n = \sigma(x^n), \qquad n = 0, 1, \dots. \tag{6}$$

It is usually assumed that $c_0 = 1$ (standard normalization condition), but we will not assume this condition in the followings. So we will assume that c_0 is an arbitrary nonzero parameter.

Introduce the Hankel determinants

$$D_n = \det(c_{i+j})_{i,j=0,\dots,n-1}, \qquad D_0 = 1, \qquad D_1 = c_0. \tag{7}$$

Then the polynomials $P_n(x)$ can be uniquely represented as [4]

$$P_n(x) = \frac{1}{D_n} \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}. \tag{8}$$

The normalization constants are expressed as

$$h_n = \frac{D_{n+1}}{D_n}, \qquad h_0 = D_1 = c_0. \tag{9}$$

While the recurrence coefficients u_n satisfy the relation

$$u_n = \frac{h_n}{h_{n-1}} = \frac{D_{n-1}D_{n+1}}{D_n^2}. \tag{10}$$

Thus we have

$$h_n = c_0 u_1 u_2 \cdots u_n. \tag{11}$$

Then we have

Theorem 1. *The following statements are equivalent.*

(i) *The recurrence coefficients u_n, b_n satisfy the TC equations* (1) *with the restriction $u_0 = 0$ (i.e. $\dot b_0 = u_1$).*

(ii) *The corresponding orthogonal polynomials $P_n(x;t)$ satisfy the relation*

$$\dot P_n(x;t) = -u_n P_{n-1}(x;t). \tag{12}$$

(iii) *The moments c_n satisfy the relation*

$$\dot c_n = c_{n+1} + \frac{\dot c_0 - c_1}{c_0} c_n, \tag{13}$$

where $c_0(t)$ is an arbitrary differentiable function of t.

See [1, 13], for the proof of this theorem.

We note only that it is commonly assumed that $c_0(t) \equiv 1$, but in what follows we will choose another normalization condition

$$\dot c_0 = c_1. \tag{14}$$

Then the condition (13) becomes very simple

$$\dot c_n = c_{n+1}, \tag{15}$$

or, equivalently,

$$c_n(t) = \frac{d^n c_0(t)}{dt^n}. \tag{16}$$

Hence, for the Toda chain case, the Hankel determinants $D_n = D_n(t)$ have the form

$$D_n = \det(c_0^{(i+k)})_{i,k=0,\dots,n-1}, \qquad D_0 = 1, \qquad D_1 = c_0, \tag{17}$$

where $c_0^{(j)}$ means the j-th derivative of $c_0(t)$ with respect to t.

Now we have

Proposition 1. *The restricted TC equations are equivalent also to the equations*

$$\frac{d^2 \log D_n}{dt^2} = \frac{D_{n-1} D_{n+1}}{D_n^2}, \qquad n = 1, 2, \dots . \tag{18}$$

Proof of this proposition is almost obvious. Equations (18) are equivalent to the Hirota bilinear form [6] for the restricted TC equations which was analyzed by many authors (see, e.g. [2, 11]). The parametric determinants such as $D_n(t)$ have played a very fundamental role in the Hirota–Sato theory of integrable dynamical systems as tau-functions. As was noticed in [15,18] the relation (18) for the Hankel determinants of type (17) with (15) was firstly obtained by Sylvester and is known today as the Sylvester theorem [9].

Note also that for the Hankel determinants of the form (17) we have two useful relations

$$b_n = \frac{\dot{D}_{n+1}}{D_{n+1}} - \frac{\dot{D}_n}{D_n} \tag{19}$$

and

$$\dot{h}_n = h_n b_n. \tag{20}$$

In particular, for $n = 0$ we have from (20)

$$b_0 = \frac{\dot{c}_0}{c_0}. \tag{21}$$

The relation (21) allows us to restore $c_0(t)$ if the recurrence coefficient $b_0 = b_0(t)$ is known explicitly from Toda chain solutions (1).

In this paper we consider a class of explicit solutions of the restricted TC equations through a separation of variables $c_n(t)$. Such solutions correspond to the Sheffer class orthogonal polynomials such as the Meixner, Pollaczek, Laguerre, Charlier and Hermite polynomials. It is shown that Hankel determinants of some combinatorial numbers, such as the Euler, the binomial coefficients and Bell numbers, are then presented.

2 Generating functions of the moments

In the theory of orthogonal polynomials the Stieltjes function $F(z)$ is defined as a generating function of the moments [4]

$$F(z) = \frac{c_0}{z} + \frac{c_1}{z^2} + \cdots + \frac{c_n}{z^{n+1}} + \cdots . \tag{22}$$

If moments c_n depend on t according to the Toda Ansatz (15), we then have

$$\dot{F}(z;t) = \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_n}{z^n} + \cdots = zF(z) - c_0. \tag{23}$$

In fact, the relation (23) is equivalent to restricted TC equations (15).

We consider also a generating function of another type:

$$\Phi(p) = \sum_{k=0}^{\infty} c_k \frac{p^k}{k!}. \tag{24}$$

Note that in number theory for a given sequence of numbers c_n the generating function of type (22) is called a G-function, and the generating function of type (24) is called an E-function [5]. The relationship between functions $F(z)$ and $\Phi(p)$ is known and is given by the (formal) Laplace transform:

$$F(z) = \sum_{k=0}^{\infty} c_k z^{-k-1} = \sum_{k=0}^{\infty} c_k \int_0^{\infty} \frac{p^k e^{-pz}}{k!} dp = \int_0^{\infty} e^{-pz} \Phi(p) dp. \tag{25}$$

For the case of the restricted TC equations with condition (15) we see that generating function $\Phi(p)$ is given automatically by the formal Taylor expansion

$$\Phi(p;t) = \sum_{k=0}^{\infty} c_k(t)\frac{p^k}{k!} = \sum_{k=0}^{\infty} c_0^{(k)}(t)\frac{p^k}{k!} = c_0(t+p) \tag{26}$$

of $c_0(t+p)$. Thus the E-generating function is given just by the shifted $c_0(t+p)$ zero-moment function. The Stieltjes function is given then as the Laplace transform

$$F(z;t) = \int_0^{\infty} e^{-pz} c_0(t+p)dp. \tag{27}$$

3 Special polynomial Ansatz and separated variables

In this section we describe concrete examples of the E-generating functions $c_0(t)$ (26) which stem from special Ansatz for functional structure of the moments $c_n(t)$. Namely, we assume a separation of variables as follows

$$c_n(t) = T_n(y(t))c_0(t), \qquad n = 0, 1, \ldots, \tag{28}$$

where $T_n(y(t))$ is a polynomial of exactly degree n of some (unknown) variable $y(t)$. Note that in [7,8] some systems of orthogonal polynomials were considered having moments as orthogonal polynomials from some variable. In our approach we *do not* require that $T_n(y)$ be orthogonal polynomials.

The main result is

Theorem 2. *In order to Ansatz* (28) *be compatible with the restricted TC equations it is necessary and sufficient that the function $y(t)$ be a solution of the equation*

$$\dot{y}(t) = \sigma(y), \tag{29}$$

where $\sigma(y)$ is a (nonzero) polynomial in y with degree less or equal 2,

$$\sigma(y) = \xi y^2 + \eta y + \zeta, \tag{30}$$

and the function $\phi(y) = c_0(t(y))$ be a solution of the equation

$$\phi'(y) = \frac{\tau(y)}{\sigma(y)}\phi(y), \tag{31}$$

where $\tau(y)$ is a polynomial of exactly first degree,

$$\tau(y) = \alpha y + \beta \tag{32}$$

with $\alpha \neq 0$ and β arbitrary parameters, and $t(y)$ is the inverse function with respect to $y(t)$. The restriction between α and ξ, $\xi \neq -\alpha/n$, $n = 1, 2, \ldots$, is assumed.

It is possible also to find explicit expression for the recurrence coefficients:

Proposition 2. *If moments $c_n(t)$ satisfy the polynomial Ansatz (28) then the recurrence coefficients $u_n(t)$, $b_n(t)$ have the explicit expressions*

$$\begin{aligned} b_n(t(y)) &= \tau(y) + n\sigma'(y), \\ u_n(t(y)) &= n\sigma(y)\left(\tau'(y) + \frac{1}{2}(n-1)\sigma''(y)\right) = n\sigma(y)(\alpha + (n-1)\xi). \end{aligned} \tag{33}$$

Proof of this proposition is an elementary application of induction.

We see that the recurrence coefficient $u_n(t)$ has expression with separated variables:

$$u_n(t) = q(t)\kappa_n, \qquad n = 0, 1, \ldots, \tag{34}$$

where $q(t) = \sigma(y(t))$ depends only on t and $\kappa_n = n(\alpha + (n-1)\xi)$ depends only on n.

It is possible to prove an inverse theorem

Theorem 3. *Two Ansatzes* (28) *and* (34) *(with the restriction* $\kappa_0 = 0$*) for solutions of the restricted TC equations are equivalent.*

Such solutions were constructed in [12] and [19]. They were also rediscovered in [3]. In [19] and [3] it was established, that corresponding polynomials belong to the Sheffer class.

4 Hankel determinants of some combinatorial numbers

In this section we consider some special cases of the general scheme of separated variables. Corresponding Toda solutions lead to simple determinantal formulas containing classical combinatorial numbers.

Case 1. $\sigma(y) = 1 - y^2$. For a special choice of parameters we have $\tau(y) = -y$ and $c_0 = 2e^t/(e^{2t}+1)$. For recurrence coefficients we have

$$b_n(t) = -(1+2n)\tanh(t), \qquad u_n(t) = -n^2\mathrm{sech}^2(t). \tag{35}$$

As discussed in [19, 3] this solution is related to the Meixner orthogonal polynomials.

It is to be noted that the moment $c_0(t)$ has an intimate relationship to the Euler numbers E_k, $k = 0, 1, 2, \ldots$, which are combinatorial numbers defined by

$$E_k = i^k \sum_{m=0}^{k} 2^m a_m \left(\begin{array}{c} k \\ m \end{array} \right), \qquad \frac{2}{e^t+1} = \sum_{k=0}^{\infty} a_m \frac{t^m}{m!}, \qquad i = \sqrt{-1}. \tag{36}$$

Here a_m are rational numbers and $\left(\begin{array}{c} k \\ m \end{array} \right)$ are binomial coefficients. Thus $c_0(t) = \Phi(2t; 1/2)$ and the coefficients $E_k(1/2)$ of the expansion

$$c_0(t) = \sum_{k=0}^{\infty} E_k(1/2) \frac{2^k t^k}{k!} \tag{37}$$

give the Euler numbers E_k through the relation $(2i)^k E_k(1/2) = E_k$.

Proposition 3. *The Hankel determinant of the Euler numbers is given by*

$$\det(E_{i+j})_{i,j=0,\ldots,n-1} = \left(\prod_{k=0}^{n-1} k! \right)^2. \tag{38}$$

Radoux [15] presented the Hankel determinant $\det(E_{2i+2j})_{i,j=0,\ldots,n-1} = \left(\prod_{k=0}^{n-1} (2k)! \right)^2$ of nonzero Euler numbers E_{2i+2j}.

Case 2. $\sigma(y) = -(y-1)^2$. In this case $\sigma(y)$ has a real root of multiplicity 2. For a special choice of parameters we have

$$b_n(t) = -z + \frac{\alpha - 2n}{t+1}, \qquad u_n(t) = \frac{n(n-\alpha-1)}{(t+1)^2}. \tag{39}$$

with some parameter z. These coefficients correspond to the Laguerre polynomials. Then we obtain

$$c_0(t) = (t+1)^\alpha e^{-zt+z}. \tag{40}$$

Expression $(t+1)^\alpha e^{-zt}$ is a generating function of the Laguerre polynomials ${L_k}^{\alpha-k}(z)$:

$$(t+1)^\alpha e^{-zt} = \sum_{k=0}^{\infty} {L_k}^{\alpha-k}(z) t^k.$$

From this relation it follows that ${L_k}^{\alpha-k}(0) = \binom{\alpha}{k}$ are (generalized) binomial coefficients.

Let us consider the Hankel determinant of the generating function of the generalized binomial coefficients ${L_k}^{\alpha-k}(0)$

$$\begin{aligned} &D_n(t) = \det(d_{i+j}(t))_{i,j=0,\dots,n-1}, \qquad D_0(t) = 1, \qquad D_1(t) = d_0, \\ &d_0(t) = (t+1)^\alpha, \qquad d_n(t) = \frac{d^n d_0(t)}{dt^n}. \end{aligned} \tag{41}$$

Proposition 4. *The Hankel determinant of the generalized binomial coefficients* ${L_k}^{\alpha-k}(0)$ *is given by*

$$D_n(0) = (-1)^{[n/2]} \prod_{k=1}^{n-1} k!(\alpha-k+1)^{n-k}. \tag{42}$$

Case 3. $\sigma(y) = y+1$. For special choice of parameters the solution of the restricted TC equations is

$$b_n(t) = u_n(t) = ne^t \tag{43}$$

which corresponds to the Charlier orthogonal polynomials [19].

For $c_0(t)$ we have

$$c_0(t) = \exp\left(e^t - 1\right).$$

This expression coincides with the generating function of the Bell numbers B_k. Namely,

$$c_0(t) = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \qquad B_k = \sum_{m=0}^{k} S(k,m), \tag{44}$$

where $S(k,m)$ is the Stirling numbers of the second kind. Thus $B_k = c_0^{(k)}(t)$. It is known [15] that the associated Hankel determinants

$$D_n(t) = \det(c_0^{(i+j)}(t))_{i,j=0,\dots,n-1} \qquad \text{and} \qquad D_n(0) = \det(B_{i+j})_{i,j=0,\dots,n-1}$$

are

$$\begin{aligned} &D_n(t) = \prod_{k=0}^{n-1} k! \cdot \exp\left(\frac{n(n-1)}{2} t + ne^t - n\right), \\ &D_n(0) = \prod_{k=0}^{n-1} k!, \end{aligned} \tag{45}$$

respectively.

Case 4. $\sigma(y) = 1$. In this case we have (shifted) Hermite polynomials:

$$b_n(t) = -2t + 2z, \qquad u_n(t) = -2n. \tag{46}$$

The E-generating function

$$c_0(t) = \exp\left(-t^2 + 2zt\right) \tag{47}$$

of the Hermite polynomials $H_k(z)$ is derived, namely,

$$c_0(t) = \sum_{k=0}^{\infty} H_k(z)\frac{t^k}{k!}. \tag{48}$$

The Hankel determinants $D_n = \det(H_{i+j}(z))_{i,j=0,\ldots,n-1}$ of Hermite polynomials is found in Radoux [14] by an alternative way. The result is

$$D_n = \det(H_{i+j}(z))_{i,j=0,\ldots,n-1} = (-2)^{n(n-1)} \prod_{k=0}^{n-1} k!. \tag{49}$$

Note that D_n is independent of z.

Thus the Hankel determinants of some combinatorial numbers can be easily calculated explicitly from corresponding Toda solutions with separated variables.

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Operators of Binary Darboux Transformations for Dirac's System

Marta POCHYNAYKO † *and Yuriy SYDORENKO* ‡

† *Institute of Applied Mathematics and Fundamental Sciences, National University "Lvivska Politechnika", 12 S. Bandery Str., 79013 Lviv, Ukraine*

‡ *Franko Lviv National University, 1 Universytets'ka Str., 79000 Lviv, Ukraine*
E-mail: *matmod@franko.lviv.ua*

It is shown that the binary Darboux transformations (BDT) [1–4] for Dirac's operator generate transformations operators obtained by L.P. Nizhnik by considering inverse scattering problem of Dirac's system [5,6]. We found a wide set of explicit solutions of the space-two-dimensional non-linear Schrödinger equation. It is shown that solutions of those equations obtained by inverse scattering method are contained among those obtained by binary Darboux transformations method.

Let us consider Dirac's operator L of the form

$$L = \begin{pmatrix} \partial_x & u_1 \\ u_2 & \partial_y \end{pmatrix}, \quad u_1 = u_1(x,y), \quad u_2 = u_2(x,y), \quad \partial_x := \frac{\partial}{\partial x}, \quad \partial_y := \frac{\partial}{\partial y}. \tag{1}$$

Let 1. $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ be arbitrary and

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} := \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1K} \\ \varphi_{21} & \cdots & \varphi_{2K} \end{pmatrix}$$

some fixed $(2 \times K)$-matrix solutions of Dirac's system

$$LY = 0. \tag{2}$$

2. $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$ be arbitrary and

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := \begin{pmatrix} \psi_{11} & \cdots & \psi_{1K} \\ \psi_{21} & \cdots & \psi_{2K} \end{pmatrix}$$

some fixed $(2 \times K)$-matrix solutions of transposed system

$$L^\tau Z = 0, \; L^\tau = \begin{pmatrix} -\partial_x & u_2 \\ u_1 & -\partial_y \end{pmatrix}. \tag{3}$$

3.

$$\Omega[Z, Y, M_0, M] := \Omega[Z, Y] \tag{4}$$

is a matrix potential which satisfies the condition $\Omega[Z, Y, M_0, M_0] = 0$, where $M = (x, y)$, $M_0 = (x_0, y_0) \in \overline{\mathbb{R}^2}$.

Remark 1. From equations (2), (3) for arbitrary solutions Y, Z the relation $(Z_1^\top Y_1)_x = -(Z_2^\top Y_2)_y$ follows, which ensures the existence (up to an arbitrary constant) of the matrix potential

$$\Omega: \; \Omega_x = -Z_2^\top Y_2, \;\; \Omega_y = Z_1^\top Y_1.$$

4. Let C be some $(K \times K)$-constant matrix and the potential $C + \Omega[\psi, \varphi]$ is non-generated in a neighborhood of $M_0 = (x_0, y_0) \in \overline{\mathbb{R}^2}$.

By the direct computation we prove the following proposition.

Proposition 1. *The integral operator W defined on the space of solutions of Dirac's system by the formula*

$$WY := Y - \varphi(C + \Omega[\psi, \varphi])^{-1}\Omega[\psi, Y] \tag{5}$$

transforms Dirac's operator (1) *into Dirac's operator $\hat{L} = WLW^{-1}$ of the form*

$$\hat{L} = \begin{pmatrix} \partial_x & \hat{u}_1 \\ \hat{u}_2 & \partial_y \end{pmatrix},$$

where

$$\begin{aligned} \hat{u}_1 &= u_1 - \varphi_1(C + \Omega[\psi, \varphi])^{-1}\psi_2^\top, \\ \hat{u}_2 &= u_2 + \varphi_2(C + \Omega[\psi, \varphi])^{-1}\psi_1^\top. \end{aligned} \tag{6}$$

The functions $\hat{Y} := WY$ is the general solution of Dirac's system

$$\hat{L}\hat{Y} = 0$$

with the coefficients (potentials) $\hat{u}_1$, $\hat{u}_2$ (6).

Remark 2. The (2×2)-matrix integral operator $W - I$ is of the form

$$W - I := \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}, \tag{7}$$

where operators W_{ij} are defined by their kernels $W_{ij}(x, y, s)$, $i, j = 1, 2$ depending on concrete realization of potential $\Omega[Z, Y]$, are the Volterra operators of variables x and y. The coefficients $\hat{u}_1$, $\hat{u}_2$ (6) are defined as the diagonals of kernels of the operators W_{12}, W_{21} respectively.

Remark 3. It is not difficult to show that formulas (6) give solutions of space-two-dimensional non-linear Schrödinger equations [6, 7]

$$\begin{aligned} &u_{1t} + u_{1xx} + u_{1yy} - 2(v_1 - v_2)u_1 = 0, \\ &u_{2t} - u_{2xx} - u_{2yy} - 2(v_2 - v_1)u_2 = 0, \\ &v_{1x} = (u_1 u_2)_y, \qquad v_{2y} = (u_1 u_2)_x, \end{aligned} \tag{8}$$

which has the Lax representation $[L, A] = 0$, where L is Dirac's operator (1), and A is some matrix evolution operator [2]. The functions φ, ψ in solutions (6) depend on parameter t in view of the equations

$$A\varphi = 0, \qquad A^\tau \psi = 0.$$

In this paper we show that certain realization of the operator W (5), which is generated by concrete choice of potential (4), coincides with transformation operators for Dirac's systems which were obtained in the papers [5, 6].

Let us consider the operator W defined by the following formula

$$\begin{aligned} \hat{Y} = WY = Y - \varphi &\left[C + \int_{x_0}^{x} (-\psi_2^\top \varphi_2)\, dx + \int_{y_0}^{y} \psi_1^\top \varphi_1\, dy\right]^{-1} \\ &\times \left(\int_{x_0}^{x} (-\psi_2^\top Y_2)\, dx + \int_{y_0}^{y} \psi_1^\top Y_1\, dy\right). \end{aligned} \tag{9}$$

Respectively, the kernels of the operators W_{12}, W_{21} (7) have the form

$$W_{12}(x,y,s) = \varphi_1(x,y)\left[C - \int_{x_0}^{x} \psi_2^\top \varphi_2(x,y_0)\,dx + \int_{y_0}^{y} \psi_1^\top \varphi_1(x,y)\,dy\right]^{-1} \psi_2^\top(s,y),$$
$$W_{21}(x,y,s) = -\varphi_2(x,y)\left[C - \int_{x_0}^{x} \psi_2^\top \varphi_2(x,y_0)\,dx + \int_{y_0}^{y} \psi_1^\top \varphi_1(x,y)\,dy\right]^{-1} \psi_1^\top(x,s), \qquad (10)$$

and the coefficients $\hat{u}_1$, $\hat{u}_2$ can be written

$$\hat{u}_1 = u_1(x,y) - W_{12}(x,y,x),$$
$$\hat{u}_2 = u_2(x,y) + W_{21}(x,y,y). \qquad (11)$$

In the case when Dirac's operator is non-perturbed ($u_1 = u_2 \equiv 0$), the solutions φ, ψ of equations (2), (3) admit evidently, such forms

$$\varphi = \begin{pmatrix} -f_1(y) & 0 \\ 0 & g_2(x) \end{pmatrix}, \qquad f_1 = (f_{11},\dots,f_{1n}), \qquad g_2 = (g_{21},\dots,g_{2n}),$$
$$\psi = \begin{pmatrix} g_1(y) & 0 \\ 0 & -f_2(x) \end{pmatrix}, \qquad g_1 = (g_{11},\dots,g_{1n}), \qquad f_2 = (g_{21},\dots,f_{2n}).$$

Let a $(2n \times 2n)$-matrix C be of the form

$$C = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \qquad I_n := \operatorname{diag}(1,\dots,1),$$
$$M_0 := (x_0,y_0) = (-\infty,+\infty), \qquad Y_1 = Y_1(y) := Y_1(-\infty,y), \qquad Y_2 = Y_2(x) := Y_2(x,+\infty).$$

By formula (9) we obtain

$$\begin{pmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{pmatrix} := WY = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} - \begin{pmatrix} -f_1(y) & 0 \\ 0 & g_2(x) \end{pmatrix}$$
$$\times \begin{pmatrix} -\int_{+\infty}^{y} g_1^\top f_1(y)\,dy & I_n \\ I_n & \int_{-\infty}^{x} f_2^\top g_2(x)\,dx \end{pmatrix}^{-1} \left(\int_{-\infty}^{x} (f_2^\top Y_2)\,dx + \int_{+\infty}^{y} g_1^\top Y_1\,dy\right).$$

By using the known formula

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}^{-1}(I + A_{12}T^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}T^{-1} \\ -T^{-1}A_{21}A_{11}^{-1} & T^{-1} \end{pmatrix}$$

for a block matrix A, where $T = A_{22} - A_{21}A_{11}^{-1}A_{12}$ and provided that $A_{12} = A_{21} = I_n$, we obtain

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} -[I_n - A_{22}A_{11}]^{-1}A_{22} & [I_n - A_{22}A_{11}]^{-1} \\ [I_n - A_{11}A_{22}]^{-1} & -[I_n - A_{11}A_{22}]^{-1}A_{11} \end{pmatrix}.$$

Respectively, the operator W is of the form

$$W = I - \begin{pmatrix} -f_1 & 0 \\ 0 & g_2 \end{pmatrix} \begin{pmatrix} -[I_n - A_{22}A_{11}]^{-1}A_{22} & [I_n - A_{22}A_{11}]^{-1} \\ [I_n - A_{11}A_{22}]^{-1} & -[I_n - A_{11}A_{22}]^{-1}A_{11} \end{pmatrix}$$
$$\times \begin{pmatrix} -\int_{+\infty}^{y} g_1^\top \cdot\, dy & 0 \\ 0 & \int_{-\infty}^{x} f_2^\top \cdot\, dx \end{pmatrix}, \qquad (12)$$

which coincides with respective transformation operators for Dirac's system obtained in paper [6]. The kernels (10) of the operators W_{12}, W_{21} in formula (12) are

$$W_{12}(x,y,s) = f_1(y)\left[I + \int_{-\infty}^{x} f_2^{\top} g_2(x)\,dx \int_{+\infty}^{y} g_1^{\top} f_1(y)\,dy\right]^{-1} f_2^{\top}(s),$$
$$W_{21}(x,y,s) = g_2(x)\left[I + \int_{+\infty}^{y} g_1^{\top} f_1(y)\,dy \int_{-\infty}^{x} f_2^{\top} g_2(x)\,dx\right]^{-1} g_1^{\top}(s).$$

The solution $\hat{u}_1$, $\hat{u}_2$ (6) of equations (8) by the formulas (11) coincide with the corresponding solution obtained by inverse scattering method [6, 7].

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Quasigraded Lie Algebras and Matrix Generalization of Landau–Lifshitz Equation

Taras V. SKRYPNYK

Bogoliubov Institute for Theoretical Physics, 14-b Metrologichna Str., 03143 Kyiv-143, Ukraine
E-mail: *tskrypnyk@imath.kiev.ua*

Using special quasigraded Lie algebras we obtain new hierarchies of integrable equations in partial derivatives admitting zero-curvature representations. In particular, we obtain generalization of Landau–Lifshitz equation for the Lie algebras $so(2n)$, $sp(n)$ and $gl(n)$.

1 Introduction

Integrability of equations of $1+1$ field theory and condensed matter physics is based on the possibility to represent them in the form of the so-called zero-curvature equations [1, 2]:

$$\frac{\partial U(x,t,\lambda)}{\partial t} - \frac{\partial V(x,t,\lambda)}{\partial x} + [U(x,t,\lambda), V(x,t,\lambda)] = 0. \tag{1}$$

The most productive interpretation of zero-curvature equation (see [6, 5]) is to consider them as a consistency condition for a set of a commuting Hamiltonian flows on the dual space to some infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ of matrix-valued function of λ written in the Euler–Arnold (generalized Lax) form:

$$\frac{\partial L(\lambda)}{\partial t_l} = \mathrm{ad}^*_{\nabla I_l(L(\lambda))} L(\lambda), \qquad \frac{\partial L(\lambda)}{\partial t_k} = \mathrm{ad}^*_{\nabla I_k(L(\lambda))} L(\lambda), \tag{2}$$

where $L(\lambda) \in \widetilde{\mathfrak{g}}^*$ is the generic element of the dual space, $\nabla I_k(L(\lambda)) \in \widetilde{\mathfrak{g}}$ is algebra-valued gradient of $I_k(L(\lambda))$, and the "Hamiltonians" $I_k(L(\lambda))$, $I_l(L(\lambda))$ belong to the set of mutually commuting with respect to the natural Lie–Poisson bracket functions on $\widetilde{\mathfrak{g}}^*$.

Consistency condition of two commuting flows given by equations (2) yields equation (1) with $U \equiv \nabla I_k$, $V \equiv \nabla I_l$, $t_k \equiv x$, $t_l \equiv t$. Hence in order to construct new integrable hierarchies in the framework of the described approach it is necessary to have some infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ admitting an algorithm of construction of infinite family of commuting Hamiltonians on its dual space. Such the algorithm is famous Kostant–Adler scheme [3, 4].

In our previous paper [9] we proposed a multiparametric family of special quasigraded Lie algebras $\widetilde{\mathfrak{g}}_A$ parametrized by numerical matrices A. The main feature of the constructed Lie algebras is the property to admit Kostant–Adler scheme, i.e. to admit decomposition into the sum of two subalgebras $\widetilde{\mathfrak{g}}_A = \widetilde{\mathfrak{g}}_A^+ + \widetilde{\mathfrak{g}}_A^-$. In the present paper we use them in order to construct new sets of integrable equations in partial derivatives. In the result we obtain an infinite set of equations (1) with $\widetilde{\mathfrak{g}}_A^{\pm}$-valued $U - V$ pairs constituting so-called integrable hierarchies. We concentrate our attention on one of these hierarchies connected with the subalgebras $\widetilde{\mathfrak{g}}_A^-$.

In the present paper we show that explicit form of the corresponding equations depends on the form of the additional covariant constraint on the matrices entering into the U-operators. We consider the simplest U-operators and the simplest covariant matrix constraint:

$$U = \lambda^{-1} S, \qquad S^2 = \frac{1}{4}\mathbf{1}.$$

Such covariant constraints exist for the cases of underlying simple (reductive) matrix Lie algebras $\mathfrak{g}$ are $so(2n)$, $sp(n)$ and $gl(n)$. In the result we obtain the following equation on the matrix S:

$$\frac{\partial S}{\partial t} = \left[S, \frac{\partial^2 S}{\partial x^2}\right] + \frac{1}{2}\frac{\partial}{\partial x}(AS + SA) + \left[S, \left[S, \frac{\partial S}{\partial x}\right]\right]_A + \frac{1}{2}[S, AS + SA]_A. \tag{3}$$

We show that this equation is the direct matrix generalization of the Landau–Lifshiz equation, where the role of the anisotropy tensor is played by the matrix A. In the case of $\mathfrak{g} = so(4)$ it could be reduced to the standard Landau–Lifshitz equation [15, 5].

The volume of the paper gives no possibility to consider other equations connected with algebras $\widetilde{\mathfrak{g}}_A$. Interested reader may consult our papers [10, 11] for further examples.

2 Special quasi-graded algebras

In this section we describe a new class of infinite-dimensional Lie algebras $\widetilde{\mathfrak{g}}$ that could be used to generate classical integrable systems. These algebras satisfy the following integrability requirements (IR):

- (IR1) they possess infinite number of algebraically independent invariants of coadjoint representation,
- (IR2) they are decomposable into the direct sum of two subalgebras: $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}^- + \widetilde{\mathfrak{g}}^+$,
- (IR3) subalgebras $\widetilde{\mathfrak{g}}^+$, $\widetilde{\mathfrak{g}}^-$ possess infinite set of embedded ideals $\mathcal{J}_{\pm n}$ of finite co-dimensions.

2.1 General construction

Definition 1. Infinite-dimensional Lie algebra $\widetilde{\mathfrak{g}}$ is called $\mathbb{Z}$-quasigraded of type (p, q) if it admits the decomposition:

$$\widetilde{\mathfrak{g}} = \sum_{j\in\mathbb{Z}} \mathfrak{g}_j, \quad \text{such that} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \sum_{k=-p}^{q} \mathfrak{g}_{i+j+k}.$$

The following proposition holds true [9]:

Proposition 1. *Let $\widetilde{\mathfrak{g}}$ be $\mathbb{Z}$-quasi graded of type $(0, 1)$, or $(1, 0)$. Then $\widetilde{\mathfrak{g}}$ satisfies conditions* (IR2) *and* (IR3).

So our aim in this section will be a construction of $\mathbb{Z}$-quasigraded algebras of type $(0, 1)$. For this purpose we will deform Lie algebraic structure in loop algebras. We will introduce into $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathrm{Pol}\,(\lambda, \lambda^{-1})$ new Lie bracket:

$$[X \otimes p(\lambda), Y \otimes q(\lambda)]_F = [X, Y] \otimes p(\lambda)q(\lambda) - F(X, Y) \otimes \lambda p(\lambda)q(\lambda), \tag{4}$$

where $X, Y \in \mathfrak{g}$, $p(\lambda), q(\lambda) \in \mathrm{Pol}\,(\lambda, \lambda^{-1})$, $[\,,\,]$ in the righthand side of this identity denotes ordinary Lie bracket in $\mathfrak{g}$ and map $F : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is skew. It is evident by the very construction that the Lie algebras with the so defined bracket are $\mathbb{Z}$-quasigraded Lie algebras of type $(0, 1)$ with the quasigrading being defined by degrees of the spectral parameter λ.

The following Propositions answer the question of when bracket (4) satisfy the Jacobi identity:

Proposition 2. *For bracket* (4) *to satisfy the Jacobi identities the cochain F should satisfy the following two requirements:*

$$\text{(J1)} \quad \sum_{\text{c.p. } \{i,j,k\}} (F([X_i, X_j], X_k) + [F(X_i, X_j), X_k]) = 0,$$

$$\text{(J2)} \quad \sum_{\text{c.p. } \{i,j,k\}} F(F(X_i, X_j), X_k) = 0.$$

In the case of classical matrix Lie algebras it is possible to give an explicit construction of large (multiparametric) family of cochains F, that satisfy conditions (J1)–(J2). Let $\mathfrak{g}$ be hereafter a classical matrix Lie algebra of the type $gl(n)$, $so(n)$ and $sp(n)$ over the field of the complex or real numbers. We will realize the algebra $so(n)$ as algebra of skew-symmetric matrices: $so(n) = \{X \in gl(n) | X = -X^\top\}$ and the algebra $sp(n)$ as the following matrix algebra: $sp(n) = \{X \in gl(n) | X = sX^\top s\}$, where n is an even number, $s \in so(n)$ and $s^2 = -1$.

The following Proposition holds true:

Proposition 3. *Let $\mathfrak{g}$ be a classical matrix Lie algebra over the field $\mathbb{K}$ of complex or real numbers. Let us define the numerical ($\mathbb{K}$-valued) $n \times n$ matrix A of the following type:*

1) *A is arbitrary for $\mathfrak{g} = gl(n)$,*
2) *$A = A^\top$ for $\mathfrak{g} = so(n)$,*
3) *$A = -sA^\top s$ for $\mathfrak{g} = sp(n)$.*

Then maps $F_A : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of the form $F_A(X, Y) = XAY - YAX$ are correctly defined skew-symmetric maps that satisfy conditions (J1)–(J2).

Remark 1. Cocycle F_A defines a second Lie bracket in the finite-dimensional Lie algebras $\mathfrak{g}$, that is compatible with the standard one. This fact was noticed in the paper [12] and was used to construct compatible Poisson brackets on the finite-dimensional Lie algebras $\mathfrak{g}$. The idea to use the same cocycle to generate infinite-dimensional Lie algebras with Kostant–Adler decomposition was proposed in [9] as a natural generalization of semi-geometric construction of [7, 8] of special quasigraded Lie algebras on the higher genus curves.

Definition 2. We will denote the Lie bracket in $\mathfrak{g}$ defined by the cocycles F_A by $[\,,\,]_A$, the corresponding finite-dimensional Lie algebra by $\mathfrak{g}_A$ and the infinite-dimensional Lie algebra with the Lie bracket given by (4) by $\widetilde{\mathfrak{g}}_A$.

The Lie bracket in the algebra $\widetilde{\mathfrak{g}}_A$ will have the following form:

$$[X(\lambda), (\lambda)]_{F_A} \equiv [X(\lambda), Y(\lambda)]_{A(\lambda)} = [X(\lambda), Y(\lambda)] - \lambda [X(\lambda), Y(\lambda)]_A, \tag{5}$$

where $X(\lambda), Y(\lambda) \in L(\mathfrak{g}) = \mathfrak{g} \otimes \mathrm{Pol}\,(\lambda, \lambda^{-1})$, $A(\lambda) \equiv 1 - \lambda A$, and we extend brackets $[\,,\,]$ and $[\,,\,]_A$ from the Lie algebra $\mathfrak{g}$ to the Lie algebra of $\mathfrak{g}$-valued functions $L(\mathfrak{g})$ in a natural way.

Now we can introduce convenient bases in the algebras $\widetilde{\mathfrak{g}}_A$. Due to the fact that we are dealing with matrix Lie algebras $\mathfrak{g}$, we will denote their basic elements as X_{ij}. For example, for the case $\mathfrak{g} = gl(n)$ we will have that $X_{ij} = I_{ij}$, where $(I_{ij})_{ab} = \delta_{ai}\delta_{bj}$ for the case $\mathfrak{g} = so(n)$ we will have that $X_{ij} = I_{ij} - I_{ji}$ etc. Let

$$X_{ij}^m \equiv X_{ij} \otimes \lambda^m$$

be the natural basis in $\widetilde{\mathfrak{g}}_A$. Commutation relations (5) in this basis have the following form:

$$[X_{ij}^r, X_{kl}^m]_{F_A} = \sum_{p,q} C_{ij,kl}^{pq} X_{pq}^{r+m} - \sum_{p,q} C_{ij,kl}^{pq}(A) X_{pq}^{r+m+1}, \tag{6}$$

where $C_{ij,kl}^{pq}$ and $C_{ij,kl}^{pq}(A)$ are the structure constants of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_A$ respectively.

Remark 2. Algebra $\widetilde{\mathfrak{g}}_A$ could be realized also in the space of special matrix valued functions of λ with an ordinary Lie bracket $[\,,\,]$. Nevertheless we consider realization in the space $\mathfrak{g} \otimes \mathrm{Pol}\,(\lambda, \lambda^{-1})$ with the "deformed" bracket to be the most convenient.

2.2 Coadjoint representation and its invariants

In this subsection we define dual spaces, coadjoint representations and their invariants for the Lie algebras $\widetilde{\mathfrak{g}}_A$. Let us at first explicitly describe the dual space $\widetilde{\mathfrak{g}}_A^*$ of $\widetilde{\mathfrak{g}}_A$. For this purpose we will define the pairing between $\widetilde{\mathfrak{g}}_A$ and $\widetilde{\mathfrak{g}}_A^*$ in the following standard way:

$$\langle X, L\rangle = \operatorname*{res}_{\lambda=0} \operatorname{Tr}\left(X(\lambda)L(\lambda)\right). \tag{7}$$

The generic element of the dual space $L(\lambda) \in \widetilde{\mathfrak{g}}_A^*$ is written as follows:

$$L(\lambda) = \sum_{k\in\mathbb{Z}} \sum_{i,j=1,n} l_{ij}^{(k)} \lambda^{-(k+1)} X_{ij}^*. \tag{8}$$

The following proposition is true [9]:

Proposition 4. *The coadjoint action of $\widetilde{\mathfrak{g}}_A$ on $\widetilde{\mathfrak{g}}_A^*$ is written as follows:*

$$\operatorname{ad}^*_{X(\lambda)} \circ L(\lambda) = A(\lambda)X(\lambda)L(\lambda) - L(\lambda)X(\lambda)A(\lambda), \tag{9}$$

where $X(\lambda), Y(\lambda) \in \widetilde{\mathfrak{g}}_A$, $L(\lambda) \in \widetilde{\mathfrak{g}}_A^*$.

Remark 3. Note that linear spaces $\widetilde{\mathfrak{g}}_A$ and $\widetilde{\mathfrak{g}}_A^*$ do not coincide as $\widetilde{\mathfrak{g}}_A$-modules. Moreover, rigorously speaking, they also do not coincide as linear spaces, because $\widetilde{\mathfrak{g}}_A^*$ contains formal power series, and $\widetilde{\mathfrak{g}}_A$, by the very definition, consists of the Loran polynomials.

Proposition 4 has the following important corollary:

Corollary 1. *Let $L(\lambda)$ be the generic element of $\widetilde{\mathfrak{g}}_A^*$. Then the functions*

$$I_k^m(L(\lambda)) = \frac{1}{m} \operatorname*{res}_{\lambda=0} \lambda^{-(k+1)} \operatorname{Tr}\left(L(\lambda)A(\lambda)^{-1}\right)^m. \tag{10}$$

are invariants of the coadjoint representation of $\widetilde{\mathfrak{g}}_A$.

Remark 4. The matrix $A(\lambda)^{-1} \equiv (1-\lambda A)^{-1}$ has to be understood as a power series in λ in the neighborhood of 0 or ∞: $A(\lambda)^{-1} = (1+A\lambda+A^2\lambda^2+\cdots)$ or $A(\lambda)^{-1} = -(A^{-1}\lambda^{-1}+A^{-2}\lambda^{-2}+\cdots)$.

2.3 Lie–Poisson structure

Let us define the Poisson structures in the space $\widetilde{\mathfrak{g}}_A^*$ using the pairing $\langle\,,\,\rangle$ defined above. It defines Lie–Poisson (Kirillov–Kostant) bracket on $P(\widetilde{\mathfrak{g}}_{\mathcal{A}}^*)$ in the following standard way:

$$\{F(L(\lambda)), G(L(\lambda))\} = \langle L(\lambda), [\nabla F(L(\lambda)), \nabla G(L(\lambda))]\rangle, \tag{11}$$

where $\nabla F(L(\lambda)) = \sum\limits_{k\in\mathbb{Z}} \sum\limits_{i,j=1}^{n} \dfrac{\partial F}{\partial l_{ij}^{(k)}} X_{ij}^k$, $\nabla G(L) = \sum\limits_{m\in\mathbb{Z}} \sum\limits_{k,l=1}^{n} \dfrac{\partial G(\lambda)}{\partial l_{kl}^{(m)}} X_{kl}^m$.

From the Corollary 1 and standard arguments the next statement follows:

Proposition 5. *Functions $I_\phi^m(L(\lambda))$ are central for the Lie–Poisson bracket* (11).

Let us explicitly calculate Poisson bracket (11). It is easy to show that for the coordinate functions $l_{ij}^{(m)}$ these brackets will have the following form:

$$\{l_{ij}^{(n)}, l_{kl}^{(m)}\} = \sum_{p,q} C_{ij,kl}^{pq} l_{pq}^{(n+m)} - \sum_{p,q} C_{ij,kl}^{pq}(A) l_{pq}^{(n+m+1)}. \tag{12}$$

It is evident, that this bracket determine in the space of linear functions $\{l_{ij}^{(n)}\}$ a structure of the Lie algebra isomorphic to $\widetilde{\mathfrak{g}}_A$. That is why the corresponding Poisson algebra possesses decomposition into the direct sum of two Poisson subalgebras or by other words subspaces $(\widetilde{\mathfrak{g}}_A^{\pm})^*$ are Poisson.

3 Infinite-dimensional Hamiltonian systems via K-A admissible Lie algebras

In the previous section we have constructed infinite-dimensional Lie algebras $\widetilde{\mathfrak{g}}_A$ that have a decomposition into direct sum of two subalgebras and possess infinite set of invariants of coadjoint representation, i.e. admit so-called Kostant–Adler scheme. In the first subsection of this section we apply Kostant–Adler scheme to $\widetilde{\mathfrak{g}}_A$ in order to construct an infinite set of mutually commuting (with respect to the natural Lie–Poisson bracket) functions on the infinite-dimensional subalgebras $\widetilde{\mathfrak{g}}_A^{\pm}$. In the second subsection we will obtain zero-curvature type equations with $\widetilde{\mathfrak{g}}_A^{\pm}$-valued $U-V$ pairs.

3.1 Integrable Hamiltonian systems connected with algebras $\widetilde{\mathfrak{g}}_A^{\pm}$

Let $L^{\mp}(\lambda) \equiv \sum\limits_{i,j=1,n} L_{ij}^{\mp}(\lambda) X_{ji} = \sum\limits_{k\in\mathbb{Z}_{\pm}} \sum\limits_{i,j=1,n} l_{ij}^{(k)} \lambda^{-(k+1)} X_{ji}$ be the generic elements of the spaces $(\widetilde{\mathfrak{g}}_A^{\pm})^*$. Let us consider restriction of the invariant functions $\{I_k^m(L(\lambda))\}$ onto these subspaces. Note, that although Poisson subspaces $(\widetilde{\mathfrak{g}}_A^{\pm})^*$ are infinite-dimensional, all functions $\{I_k^m(L^{\pm}(\lambda))\}$ are polynomials, i.e. after restriction to $(\widetilde{\mathfrak{g}}_A^{\pm})^*$ no infinite sums appear in their explicit expressions. Corresponding Hamiltonian equations are written as:

$$\frac{\partial L_{ij}^{\mp}(\lambda)}{\partial t_k^m} = \{L_{ij}^{\mp}(\lambda), I_k^m(L^{\mp}(\lambda))\}. \tag{13}$$

The following important theorem holds true:

Theorem 1. (i) *Time flows defined by Hamiltonian equations* (13) *mutually commute.*
(ii) *Hamiltonian equations* (13) *are written in the "deformed" Lax form:*

$$\frac{\partial L^{\mp}(\lambda)}{\partial t_k^m} = A(\lambda) M_k^m(\lambda) L^{\mp}(\lambda) - L^{\mp}(\lambda) M_k^m(\lambda) A(\lambda), \tag{14}$$

where $M_k^m(\lambda) = \nabla I_k^m(L^{\mp}(\lambda)) \equiv \sum\limits_{s\in\mathbb{Z}_{\pm}} \sum\limits_{i,j=1}^{n} \dfrac{\partial I_k^m}{\partial l_{ij}^{(s)}} X_{ij}^s$ *is an algebra-valued gradient of* $I_k^m(L^{\mp}(\lambda))$.

(Proof of the theorem follows from the standard framework of the Kostant–Adler scheme [4].)

Remark 5. In this subsection we have obtained Hamiltonian systems of the Euler–Arnold type on the special infinite-dimensional Lie algebras possessing infinite number of the commuting integrals of motion. These Hamiltonian systems are "mechanical" because they are described by ordinary differential equations. Nevertheless we can consider our dynamical variables $l_{ij}^{(p)}$ to be functions of all time variables $t_k^{m\pm}$ and using the commutativity of all time flows obtain differential identities on functions $l_{ij}^{(p)}(t_k^{m\pm})$ that coincide with the integrable equations in partial derivatives. For this purpose in the next subsection we will derive zero-curvature equations.

3.2 "Deformed" zero curvature equations

In this section we will obtain zero curvature-type equations as compatibility conditions for the set of the commutative Hamiltonian flows constructed in the previous section. The following theorem holds true:

Theorem 2. *Let infinite-dimensional Lie algebras $\widetilde{\mathfrak{g}}_A$, $\widetilde{\mathfrak{g}}_A^{\pm}$, their dual spaces and polynomial Hamiltonians $I_k^m(L^{\pm}(\lambda))$, $I_l^n(L^{\pm}(\lambda))$ on them be defined as in previous sections. Then algebra-valued gradients of these functions satisfy the "deformed" zero-curvature equations:*

$$\frac{\partial \nabla I_k^m(L^{\pm}(\lambda))}{\partial t_l^{n\pm}} - \frac{\partial \nabla I_l^n(L^{\pm}(\lambda))}{\partial t_k^{m\pm}} + [\nabla I_k^m(L^{\pm}(\lambda)), \nabla I_l^n(L^{\pm}(\lambda))]_{A(\lambda)} = 0. \tag{15}$$

(Proof follows from the commutativity of two time flows defined by equations (14).)

Remark 6. By means of other realizations of $\widetilde{\mathfrak{g}}_A$ deformed zero-curvature equations can be rewritten in the form of the standard zero-curvature equations, but in this case corresponding $U-V$ pairs will be more complicated as functions of the spectral parameter λ.

Theorem 2 provides us with an infinite number of $\widetilde{\mathfrak{g}}_A^{\pm}$-valued $U-V$ pairs that satisfy zero curvature-type equations. The latter are non-linear equations in the partial derivatives on the dynamical variables – matrix elements of the matrices $L^{\pm}(\lambda)$. In the terminology of [2] equations generated by the infinite set of $U-V$ pairs are called "integrable in the kinematic sense". In the next subsections we will consider the simplest examples of such integrable equations and their hierarchies.

Now let us explain the technique of obtaining integrable equation in partial derivatives starting from zero-curvature equations. Let us at first note that in the described approach no "space" variable x is *a priori* singled out: all times $t_k^{m\pm}$ are equivalent. Fixation of the "space" flow is equivalent to the fixation of integrable hierarchy. For this purpose one should fix a Hamiltonian that generate x-flow. For the case of integrable systems, connected with algebras $\widetilde{\mathfrak{g}}_A^{\pm}$, this choice yields fixation of dynamical variables. In more details, for the dynamical variables in this case serve the matrix elements of $\nabla I_k^m(L^{\mp}(\lambda))$, where the Hamiltonian $I_k^m(L^{\mp}(\lambda))$ is chosen to generate an x-flow. Using zero-curvature conditions one can express matrix elements of all other matrix gradients $\nabla I_l^n(L^{\mp}(\lambda))$ via these dynamical variables and their derivatives with respect to the "space" coordinate. Substituting these expressions back to zero-curvature condition we obtain the desired equation in partial derivatives on the matrix elements of $\nabla I_k^m(L^{\mp}(\lambda))$.

4 Matrix generalization of L-L hierarchy

In this section we will obtain integrable hierarchies of differential equations in partial derivatives, admitting a zero curvature type representation (15) with the values in $\widetilde{\mathfrak{g}}_A^-$.

Let us consider dual space $(\widetilde{\mathfrak{g}}_A^-)^*$. Its generic element has the following form:

$$L^+(\lambda) = \sum_{k<0} \sum_{i,j=1,n} l_{ij}^{(k)} \lambda^{-(k+1)} X_{ji} = L^{(-1)} + \lambda L^{(-2)} + \lambda^2 L^{(-3)} + \lambda^3 L^{(-4)} + \cdots, \tag{16}$$

where $L^{(-k)} \equiv \sum\limits_{i,j=1,n} l_{ij}^{(-k)} X_{ji}$. Let us now calculate the Hamiltonians $I_k^m(L^+(\lambda))$. In order for the Hamiltonians $I_k^m(L^+(\lambda))$ to be polynomials we have to expand expression $A(\lambda)^{-1}$ in the power series in the neighborhood of zero:

$$A(\lambda)^{-1} = 1 + A\lambda + A^2\lambda^2 + \cdots.$$

From the results of the of the previous section it follows that the matrix gradients of Hamiltonians $I_k^m(L^+(\lambda))$ satisfy "deformed" zero-curvature equation (15). We will be interested in the two simplest Hamiltonians of the set $I_k^2(L^+(\lambda))$. By the direct calculations we obtain for them the following expressions:

$$I_0^2(L^+(\lambda)) = 1/2 \operatorname{Tr}\left(L^{(-1)}\right)^2, \qquad I_1^2(L^+(\lambda)) = \operatorname{Tr}\left(A\left(L^{(-1)}\right)^2\right) + \operatorname{Tr}\left(L^{(-1)}L^{(-2)}\right). \tag{17}$$

The corresponding matrix gradients are:

$$\nabla I_0^2(L^+(\lambda)) = L^{(-1)}\lambda^{-1},$$
$$\nabla I_1^2(L^+(\lambda)) = L^{(-1)}\lambda^{-2} + \big(\big(AL^{(-1)} + L^{(-1)}A\big) + L^{(-2)}\big)\lambda^{-1}. \tag{18}$$

As it follows from the said above, in order to fix an integrable hierarchy we should choose the Hamiltonian that generates an x-flow. In order to choose the simplest hierarchy we have to choose the Hamiltonian with the simplest matrix gradient. That is why we will take for such Hamiltonian the function $I_0^2(L^+(\lambda))$, putting $t_0^{2+} \equiv x$; $\nabla I_0^2(L^+(\lambda)) \equiv U(x,\lambda)$, $\nabla I_k^m(L^+(\lambda)) \equiv V_k^m(x,\lambda)$, $m,k > 0$ as the basic $U-V$ pairs that generate this hierarchy. In this case the role of the dynamical variables is played by the matrix elements of the matrix $L^{(-1)}$.

Let us obtain an explicit form of the simplest equation of the hierarchy described above. For this purpose we have to choose a Hamiltonian that generates the "time" flow in the simplest possible way. We take for such Hamiltonian the function $I_1^2(L^+(\lambda))$, i.e. $t \equiv t_1^{2+}$. In the result we obtain that zero-curvature equation (15) is equivalent the following λ-independent equations:

$$\frac{\partial S}{\partial t} - \frac{\partial M}{\partial x} = [S,M]_A, \tag{19a}$$
$$\frac{\partial S}{\partial x} = [S,M], \tag{19b}$$

where $S \equiv L^{(-1)}$, $M \equiv L^{(-2)} + \big(AL^{(-1)} + L^{(-1)}A\big)$.

In order to obtain equations in partial derivatives for the dynamical variables – matrix elements of the matrix S, it is necessary to solve equation (19b), i.e. to express M via S and S_x and then substitute this expression into equation (19a). We will illustrate this procedure by the simplest, but most interesting example.

Let us now consider the case of the higher-rank algebras. In this case in order to solve equation (19b), i.e. in order to obtain one matrix equation in partial derivatives instead of two equations (19) it is necessary to impose additional constraints on the matrix S. Let $\mathfrak{g} = gl(n)$, $so(2n)$ or $sp(n)$ and for the matrix S to satisfy G-invariant constraint:

$$S^2 = \frac{\alpha}{4}\mathbf{1}. \tag{20}$$

This constraint means that S belongs to the degenerated coadjoint orbits of G of the following type: $Gl(n)/GL(p)\times GL(q)$, $SO(2n)/Gl(n)$ or $SP(n)/GL(n)$. On this orbits we may solve equation (19b) in the following way:

$$M = \frac{1}{\alpha}\left[S, \frac{\partial S}{\partial x}\right] + M', \quad \text{where} \quad M' \in \ker \mathrm{ad}_L.$$

Ambiguity connected with the existence of $\ker \mathrm{ad}_S$ is removed by the requirement that the constraint (20) is consistent with equations (19). It is easy to show, that for $M' = 1/2(AS+SA)$ the constraint (20) is consistent with equation (19), i.e. $\left(\frac{\partial S^2}{\partial x}\right)|_{S^2=\frac{\alpha}{4}\mathbf{1}} = 0$ and $\left(\frac{\partial S^2}{\partial t}\right)_{S^2=\frac{\alpha}{4}\mathbf{1}} = 0$. Resulting matrix equation acquires the following form:

$$\frac{\partial S}{\partial t} = \frac{1}{\alpha}\left[S, \frac{\partial^2 S}{\partial x^2}\right] + \frac{1}{2}\frac{\partial}{\partial x}(AS+SA) + \frac{1}{\alpha}\left[S, \left[S, \frac{\partial S}{\partial x}\right]\right]_A + \frac{1}{2}[S, AS+SA]_A, \tag{21}$$

where $S^2 = \frac{\alpha}{4}\mathbf{1}$. In the case $\alpha = 1$ we obtain equation (3). In the limit $A \to 0$ this equation goes to standard higher-rank generalization of Heisenberg magnet equation [13, 14].

Let us now show, that this equation is direct matrix generalization of the well-known Landau–Lifshitz equation. For this purpose we consider the following example:

Example. Let $\mathfrak{g} = so(4)$. and matrix S satisfy G-invariant constraint (20) with $\alpha = -1$. In this case it is easy to show that matrix S can be written in the form:

$$S = \begin{pmatrix} 0 & -s_3 & s_2 & s_1 \\ s_3 & 0 & -s_1 & s_2 \\ -s_2 & s_1 & 0 & s_3 \\ -s_1 & -s_2 & -s_3 & 0 \end{pmatrix},$$

where three-component vector $\vec{s}$ belongs to $S^2 = SO(3)/SO(2) = SO(4)/GL(2)$:

$$\langle \vec{s}, \vec{s} \rangle = 1/4.$$

As a result of the special form of matrix S the second and third item of the right-hand side of equation (21) are eliminated, and it could be rewritten as equation for the vector $\vec{s}$ as follows:

$$\frac{\partial \vec{s}}{\partial t} = \left[\vec{s} \times \frac{\partial^2 \vec{s}}{\partial x^2} \right] + \left[\vec{s} \times J(\vec{s}) \right].$$

It coincides with the well-known Landau–Lifshiz equation (here $J = 1/4\big(\widehat{A}^2 - 2a_4\widehat{A}\big)$, $\widehat{A} = \mathrm{diag}\,(a_1, a_2, a_3)$).

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Generalized Binary Darboux-like Theorem for Constrained Kadomtsev–Petviashvili (cKP) Flows

Yuriy SYDORENKO

Franko Lviv National University, 1 Universytets'ka Str., 79000 Lviv, Ukraine
E-mail: *matmod@franko.lviv.ua*

Generalizations of the classical Darboux theorem are established for pseudo–differential scattering operators of the form $L = \sum_{i=0}^{n} u_i \mathcal{D}^i + \boldsymbol{q}\mathcal{M}\mathcal{D}\boldsymbol{r}^\top$. Iteration of the simplest binary Darboux-like transformations (BDT) leads to a gauge transformed operator with coefficients given by Grammian-type formulas involving a set of eigenfunctions of an operator L and adjoint eigenfunctions (eigenfunctions of the transposed operator L^τ). Nonlinear integrable partial differential equations are associated with the scattering operator L which arise as a symmetry reduction of the matrix KP hierarchy. With a suitable linear time evolution for the eigenfunctions the binary Darboux-like transformation is used to obtain solutions of the integrable equations in terms of Grammian-type determinants.

1 Introduction

Since the mid-1970s, Darboux transformations (DTs) have been shown to be a powerful tool for studying of the nonlinear integrable systems of soliton theory. Darboux's original result [1] is a transformation of the Schrödinger equation $f_{xx} + uf = \lambda f$ with different potentials. With a first eigenfunction for a potential u given, it can be used to map all other eigenfunctions to new eigenfunctions for a Schrödinger operator with a modified potential. Crum [2] considered the iteration of the DT, when a collection of eigenfunctions is given, and Crum's formulae correspond to compact Wronskian representations of the n-soliton solutions of the KdV [3].

Generalizations of these ideas for other spectral equations and their associated soliton systems have been successfully applied to a variety of systems. An excellent overview summarizing various forms of DTs and their applications in soliton theory is the book [3] by Matveev and Salle.

We show that not only differential Lax operators but also pseudo-differential scattering problems may be considered. Our aim is to apply the BDT to Lax operators of the form

$$L = \sum_{i=0}^{n} u_i \mathcal{D}^i + \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{r}^\top \tag{1}$$

(where $\boldsymbol{q} = (q_1, \dots, q_l)$, $\boldsymbol{r} = (r_1, \dots, r_l)$, $\mathcal{M}$ is $(l \times l)$-matrix of constants) and obtain Grammian type solution formulas for the associated nonlinear evolution equations.

Scattering operators of the class (1) were encountered in the context of reductions of the KP hierarchy [4–12], which generalizes a reduction concept originally used to reduce soliton systems in $1 + 1$ dimensions to finite dimensional equations [13–15].

In Section 2 we briefly review the essentials of pseudo-differential symbols to be used for the description of the (constrained) KP hierarchy. We review the fact that for each pair of eigenfunctions φ and adjoint eigenfunctions ψ (eigenfunctions of the transposed operator) the squared eigenfunction can be integrated to a potential $\Omega[\psi, \varphi]$ satisfying $\Omega_x[\psi, \varphi] = \psi^\top \varphi$ and a corresponding time evolution [10, 16]. With these potentials a proper spectral problem can be formulated for pseudo-differential operators of the form (1).

In Section 3 general formulation for the BDT for Lax pairs is presented in terms of pseudo-differential symbols. The resulting Lax pairs are characterized by a gauge operator. Also we show that the BDT will leave the class of operators (1) invariant, if the triggering eigenfunctions are solutions of a spectral problem involving the potential associated with eigenfunctions and adjoint eigenfunctions. This, in combinations with the result on the temporal aspects of the BDTs, provides an efficient tool to generate exact solutions of the nonlinear equations associated with operators of the form (1).

2 The algebra ζ of the pseudo-differential operators. Integrable hierarchies in $2+1$ and $1+1$ dimensions

We consider the algebra

$$\zeta = \left\{ \sum_{i \ll \infty} u_i(x) \mathcal{D}^i \right\}, \tag{2}$$

of pseudo-differential operators of a "space" variable $x \in \mathbb{R}$ with coefficients u_i being (in general case) $(N \times N)$-matrix-valued functions of x. For positive powers $m \in \mathbb{Z}_+$ of the differential operator $\mathcal{D}$, the algebraic multiplication with a multiplication operator represented by a function $u = u(x)$ is given by the usual Leibnitz rule

$$\mathcal{D}^m u := \sum_{j \geq 0} \left(\begin{array}{c} m \\ j \end{array} \right) \frac{\partial^j u}{\partial x^j} \mathcal{D}^{m-j}, \qquad \left(\begin{array}{c} m \\ j \end{array} \right) = \frac{m(m-1)\cdots(m-j+1)}{j!}, \qquad 0! = 1$$

with the same definition for negative powers, so that in particular $\mathcal{D}^{-1} u := \sum\limits_{j \geq 0} (-1)^j u^{(j)} \mathcal{D}^{-1-j}$, we obtain a well defined associative algebraic structure on ζ (2).

The integrable scalar KP hierarchy can be formulated through linear system [14]

$$f_{t_n} = B_n\{f\}, \qquad g_{t_n} = -B_n^\tau\{g\}, \qquad n \in \mathbb{N}, \qquad t_1 := x, \quad t_2 := y, \quad t_3 := t, \quad \ldots \tag{3}$$

with an infinite hierarchy of operators $B_n := (L_{\mathrm{KP}}^n)_+$, where subscript + denotes the projection of the powers of the micro-differential Lax operator

$$L := L_{KP} = \mathcal{D} + \sum_{i \geq 1} U_i \mathcal{D}^{-1} \tag{4}$$

onto its differential part and B_n^τ is a transposed operator. The first of the operators B_n are computed as

$$B_1 = \mathcal{D}, \qquad B_2 = \mathcal{D}^2 + 2U_1, \qquad B_3 = \mathcal{D}^3 + 3U_1\mathcal{D} + 3U_{1x} + U_2.$$

The Lax equation and the implying compatibility conditions of (3) are as follow:

$$L_{t_n} = [B_n, L], \qquad n \in \mathbb{N}, \tag{5}$$

$$B_{n,t_k} - B_{k,t_n} + [B_n, B_k] = 0. \tag{6}$$

They yield the field equations along with differential relations for the fields U_i such as

$$U_2 = -\frac{1}{2} u_x + \frac{1}{2} \partial_x^{-1} u_y,$$

$$U_3 = \frac{1}{4} u_{xx} - \frac{1}{2} u_y - \frac{1}{2} u^2 + \frac{1}{4} \partial_x^{-2} u_{yy}, \qquad \partial_x^{-1} f := \int^x f(s)\, ds,$$

where the leading coefficient $u := U_1$ plays a distinguished role, since it satisfies the KP equation (9) ($n = 2$, $k = 3$ in the (6)) and its higher flows. Consequently all equations of this hierarchy can be expressed via the single field $u := U_1$.

We will refer to solutions of (3) as eigenfunctions and adjoint eigenfunctions respectively.

It has been observed that the product qr of an eigenfunction q and an adjoint eigenfunction r represents a conserved covariant of the KP hierarchy. Hence, one may impose constraints on the KP flows by expressing the dynamical field u via qr [4,5,8,9]. In terms of the Lax operator (4) these constraints are characterized by the requirement that the negative differential orders of a power L^k have the specific form

$$\left(L^k\right)_- = q\mathcal{D}^{-1}r,$$

or, in general,

$$\left(L^k\right)_- = \sum_{i=1}^{m} q_i\mathcal{D}^{-1}r_i := \boldsymbol{q}\mathcal{D}^{-1}\boldsymbol{r}^\top, \tag{7}$$

where m pairs of (adjoint) eigenfunctions are considered [6,7]. For given k this constraint leaves the coefficients $u := U_1, U_2, \dots, U_{k-1}$ in (4) and q_i, r_i, $i = \overline{1,m}$ as independent fields, whereas $U_k, \dots, U_{k+1}, \dots$ become differential expressions of these functions. One may replace L by the new Lax operator

$$L_{k\text{-cKP}} := L^k = \mathcal{D}^k + ku\mathcal{D}^{k-2} + \sum_{i=0}^{k-3} u_i\mathcal{D}^i + \boldsymbol{q}\mathcal{D}^{-1}\boldsymbol{r}^\top, \tag{8}$$

where $u_0, \dots, u_{k-3}$ are differential expressions of $u, U_2, \dots, U_{k-1}$. Thus the fields $u_0, \dots, u_{k-3}, u$ may be regarded as new independent fields related to $u = U_1, U_2, \dots, U_{k-1}$ by a coordinate transformation. The constraints (7) may be regarded as multi-component symmetry reductions of the KP hierarchy (5)–(6), since

$$u_{t_k} = \operatorname{res} L_{t_k} = \operatorname{res}\left[\left(L^k\right)_+, L\right] = \operatorname{res}\left(L^k\right)_x, \qquad \operatorname{res}\left(\sum_{i\ll\infty} a_i\mathcal{D}^i\right) := a_{-1},$$

so that (7) implies the relation [6,7]

$$u_{t_k} = \left(\boldsymbol{q}\boldsymbol{r}^\top\right)_x$$

between the k-th flow of the KP hierarchy and the symmetry generated by square eigenfunctions.

Remark 1. If $\boldsymbol{q} = (q_1, \dots, q_l)$ are eigenfunctions then the product $\boldsymbol{q}\mathcal{M}$ with a constant $(l \times l)$-matrix $\mathcal{M}$ is a vector of the eigenfunctions for KP hierarchy (5)–(6) too.

Remark 2. If we eliminate U_i, $i \geq 2$ from (5), the remaining equations for the function $u := U_1$ in (6) represent the $(2+1)$-dimensional KP equation

$$4u_{t_3} = u_{xxx} + 12uu_x + 3\partial_x^{-1}u_{t_2t_2} \tag{9}$$

and its higher versions.

Remark 3. The so-called "k-constrained KP (k-cKP) hierarchy" (scalar) is the ordinary KP hierarchy (5)–(6) restricted to a pseudo-differential operator $L_{k\text{-cKP}}$ of the form (8)

$$L := L^k = \mathcal{D}^k + \sum_{i=0}^{k-2} u_i\mathcal{D}^i + \boldsymbol{q}\mathcal{D}^{-1}\boldsymbol{r}^\top, \qquad u_{k-2} = ku,$$

$$L_{t_n} = [B_n, L], \qquad B_n := \left(L^{\frac{n}{k}}\right)_+, \qquad n = 2, 3, \dots,$$

where u_i, $\boldsymbol{q}=(q_1,\dots,q_l)$, $\boldsymbol{r}=(r_1,\dots,r_l)$ are all functions of $x:=t_1$ and higher time evolution variables $t_2:=y$, $t_3=t$, ... In particular, the constraints

$$u_x=\left(\boldsymbol{q}\mathcal{M}\boldsymbol{r}^\top\right)_x,\qquad u_y=\left(\boldsymbol{q}\mathcal{M}\boldsymbol{r}^\top\right)_x\qquad\text{and}\qquad u_t=\left(\boldsymbol{q}\mathcal{M}\boldsymbol{r}^\top\right)_x,$$

are known to produce vector versions of $(1+1)$-dimensional of the AKNS ($k=1$), the Yajima–Oikawa ($k=2$) and the Melnikov hierarchy ($k=3$), respectively [6, 7].

3 BDT for the $(N\times N)$-matrix cKP hierarchy

Theorem 1. *Let φ, ψ be $(N\times K)$ matrix solutions of the scattering problem*

$$\sum_{i=0}^{n}u_i\varphi^{(i)}+\boldsymbol{q}\mathcal{M}_0\Omega[\boldsymbol{r},\varphi]:=L\{\varphi\}=\varphi\Lambda,$$

and the transposed scattering problem

$$\sum_{i=0}^{n}(-1)^i\left(u_i^\top\psi\right)^{(i)}-\boldsymbol{r}\mathcal{M}_0^\top\Omega[\boldsymbol{q},\psi]:=L^\tau\{\psi\}=\psi\tilde{\Lambda},$$

where Λ, $\tilde{\Lambda}$ are constant spectral matrices and $\Omega[\boldsymbol{r},\varphi]$, $\Omega[\boldsymbol{q},\psi]$ are functions satisfying $\Omega_x[\boldsymbol{r},\varphi]=\boldsymbol{r}^\top\varphi$, $\Omega_x[\boldsymbol{q},\psi]=\boldsymbol{q}^\top\psi$. Then

1. *Let f be a solution of the spectral problem $\sum\limits_{i=0}^{n}u_if^{(i)}+\boldsymbol{q}\mathcal{M}_0\Omega[\boldsymbol{r},f]:=L\{f\}=f\hat{\Lambda}$ with some constant spectral matrix $\hat{\Lambda}$ and functions $\Omega_x[\boldsymbol{r},f]=\boldsymbol{r}^\top f$.*

Then $F:=W\{f\}=f-\Phi\Omega[\psi,f]$ satisfies the spectral problem

$$\sum_{i=0}^{n}\hat{u}_iF^{(i)}+\Phi\mathcal{M}\Omega[\Psi,F]+\hat{\boldsymbol{q}}\mathcal{M}_0\Omega[\hat{\boldsymbol{r}},F]=F\hat{\Lambda},\tag{10}$$

where $\Omega[\Psi,F]$ and $\Omega[\hat{\boldsymbol{r}},F]$ are functions satisfying $\Omega_x[\Psi,F]=\Psi^\top F$ and $\Omega_x[\hat{\boldsymbol{r}},F]=\hat{\boldsymbol{r}}^\top F$.

2. *The gauge transformed operator $\hat{L}:=WLW^{-1}$ with $W=I-\varphi(C+\Omega[\psi,\varphi])^{-1}\mathcal{D}^{-1}\psi^\top$, where C is a constant $(K\times K)$ matrix, has the form*

$$\hat{L}=\sum_{i=0}^{n}\hat{u}_i\mathcal{D}^i+\Phi\mathcal{M}\mathcal{D}^{-1}\Psi^\top+\hat{\boldsymbol{q}}\mathcal{M}_0\mathcal{D}^{-1}\hat{\boldsymbol{r}}^\top,\tag{11}$$

with $\mathcal{M}=C\Lambda-\tilde{\Lambda}^\top C$, $\Phi=\varphi(C+\Omega[\psi,\varphi])^{-1}$,$\Psi^\top=(C+\Omega[\psi,\varphi])^{-1}\psi^\top$, $\hat{\boldsymbol{q}}=\boldsymbol{q}-\Phi\Omega[\psi,\boldsymbol{q}]$, $\hat{\boldsymbol{r}}=\boldsymbol{r}-\Psi\Omega[\varphi,\boldsymbol{r}]$. The coefficients $\hat{u}_i$, $i=\overline{0,n}$ can be expressed in the form $\hat{u}_n=u_n$, $\hat{u}_i=u_i+P_i(u_{i+1},\dots,u_n,\varphi,\psi)$ with suitable differential expressions P_i of the indicated arguments.

3. *If we have a Lax equation $L_t=[M,L](:=ML-LM)$, $M=\sum\limits_{i=0}^{m}v_i\mathcal{D}^i$ and φ, ψ are solutions of the evolution equations*

$$\varphi_t=M\{\varphi\},\qquad\psi_t=-M^\tau\{\psi\}\tag{12}$$

then BDT triggered by φ and ψ will map any eigenfunction f satisfying both equations

$$f_t=M\{f\},\qquad L\{f\}=f\hat{\Lambda}\tag{13}$$

to a new eigenfunction F for the transformed Lax pair $\hat{L}$ (11), $\hat{M}=-W(\partial_t-M)W^{-1}+\partial_t$, which satisfies both the new evolution equations $F_t=\hat{M}\{F\}$ as well as the new spectral problem $\hat{L}\{F\}=F\hat{\Lambda}$ (10).

Proof. 1. Let $\hat{L} := WLW^{-1}$ and $L\{f\} = f\hat{\Lambda}$ then $F := W\{f\} = f - \Phi\Omega[\boldsymbol{r}, f]$ satisfies

$$\hat{L}\{F\} := WLW^{-1}\{W\{f\}\} = WL\{f\} = W\{f\hat{\Lambda}\} = W\{f\}\hat{\Lambda} = F\hat{\Lambda},$$

and an explicit form of the operator $\hat{L}$ is given in the second part of the Theorem 1.

2. For proving the second part of the Theorem 1 we use helpful formulas

$$1)\ \Psi^\top\Phi = (C + \Omega[\psi,\varphi])^{-1}\psi^\top\varphi(C + \Omega[\psi,\varphi])^{-1} = [(C + \Omega[\psi,\varphi])^{-1}]_x$$
$$\Rightarrow \Omega[\Psi,\Phi] := -(C + \Omega[\psi,\varphi])^{-1}, \tag{14}$$
$$2)\ \Psi^\top F = (C + \Omega[\psi,\varphi])^{-1}\psi^\top[f - \varphi(C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f]] = [(C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f]]_x$$
$$\Rightarrow \Omega[\Psi,F] := -(C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f], \tag{15}$$
$$3)\ (I + \varphi\mathcal{D}^{-1}\Psi^\top)\{F\} := F + \varphi\Omega[\Psi,F] := W\{f\} + \varphi(C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f]$$
$$= f - \Phi\Omega[\psi,f] + \Phi\Omega[\psi,f]] = f$$
$$\Rightarrow W^{-1} = I + \varphi\mathcal{D}^{-1}\Psi^\top := I + \varphi\Omega[\Psi,\cdot] \tag{16}$$

and W^{-1} (16) is the inverse operator to $W = I - \Phi\mathcal{D}^{-1}\psi^\top := I - \Phi\Omega[\psi,\cdot]$.

Analogously to the statement of part 1 of Theorem 1 we prove the next statement.

4) Let g be a solution of the spectral problem $L^\tau\{g\} = g\tilde{\hat{\Lambda}}$ with some constant spectral matrix $\tilde{\hat{\Lambda}}$

$$\sum_{i=0}^{n}(-1)^i\big(u_i^\top g\big)^{(i)} - \boldsymbol{r}\mathcal{M}_0^\top\Omega[\boldsymbol{q},g] = g\tilde{\hat{\Lambda}} \tag{17}$$

and functions $\Omega_x[\boldsymbol{q},g] = \boldsymbol{q}^\top g$.

Then

$$G := W^{-1,\tau}\{g\} = (W^\tau)^{-1}\{g\} = \big(I - \Psi\mathcal{D}^{-1}\varphi^\top\big)\{g\} = g - \Psi\Omega[\varphi,g]$$

satisfies the spectral problem

$$\hat{L^\tau}\{G\} = G\tilde{\hat{\Lambda}}, \qquad \hat{L^\tau} := W^{-1,\tau}L^\tau W^\tau = (\hat{L})^\tau. \tag{18}$$
$$5)\ G^\top F = [g^\top - \Omega[g,\varphi](C + \Omega[\psi,\varphi])^{-1}\psi^\top][f - \varphi(C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f]]$$
$$= g^\top f - g^\top\varphi(C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f] - \Omega[g,\varphi](C + \Omega[\psi,\varphi])^{-1}\psi^\top f$$
$$+ \Omega[g,\varphi](C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f](C + \Omega[\psi,\varphi])^{-1}\psi^\top f$$
$$= [\Omega[g,f] - \Omega[g,\varphi](C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f]]_x$$
$$\Rightarrow \Omega[G,F] := \Omega[g,f] - \Omega[g,\varphi](C + \Omega[\psi,\varphi])^{-1}\Omega[\psi,f]. \tag{19}$$

6) For functions h_i, $i = \overline{1,4}$ and the differential operator $A \in \zeta_+$ next formulas hold (see e.g. [16, 17])

$$Ah_1\mathcal{D}^{-1}h_2 = \big(Ah_1\mathcal{D}^{-1}h_2\big)_+ + A\{h_1\}\mathcal{D}^{-1}h_2, \tag{20}$$
$$h_1\mathcal{D}^{-1}h_2A = \big(h_1\mathcal{D}^{-1}h_2A\big)_+ + h_1\mathcal{D}^{-1}[A^\tau\{h_2\}]^\tau, \tag{21}$$
$$h_1\mathcal{D}^{-1}h_2h_3\mathcal{D}^{-1}h_4 = h_1\Omega[h_2^\top,h_3]\mathcal{D}^{-1}h_4 - h_1\mathcal{D}^{-1}\Omega[h_2^\top,h_3]h_4, \tag{22}$$

and for the operator

$$L = \sum_{i=0}^{n}u_i\mathcal{D}^i + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top := L_+ + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top,$$

using formulas (16), (20), (21) by direct calculations we obtain

$$\begin{aligned}
L \to WLW^{-1} &= \big(I - \Phi\mathcal{D}^{-1}\psi^\top\big)\big(L_+ + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\big)\big(I + \varphi\mathcal{D}^{-1}\Psi^\top\big)\\
&= \big(L - \Phi\mathcal{D}^{-1}\psi^\top L\big)\big(I + \varphi\mathcal{D}^{-1}\Psi^\top\big) = L + L\varphi\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}\psi^\top L\\
&- \Phi\mathcal{D}^{-1}\psi^\top L\varphi\mathcal{D}^{-1}\Psi^\top = L_+ + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top + L_+\varphi\mathcal{D}^{-1}\Psi^\top\\
&+ \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\varphi\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}\psi^\top L_+ - \Phi\mathcal{D}^{-1}\psi^\top\boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\\
&- \Phi\mathcal{D}^{-1}\psi^\top L_+\varphi\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}\psi^\top\boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\varphi\mathcal{D}^{-1}\Psi^\top\\
&= L_+ + \big(L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_+ - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\big)_+ - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_+\\
&+ \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top + L_+\{\varphi\}\mathcal{D}^{-1}\Psi^\top + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\varphi\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}(L_+^\tau\{\psi\})^\top\\
&- \Phi\mathcal{D}^{-1}\psi^\top\boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_-\\
&- \Phi\mathcal{D}^{-1}\psi^\top\boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\varphi\mathcal{D}^{-1}\Psi^\top = \hat{L}_+ + \hat{L}_-,
\end{aligned}$$

where

$$\begin{aligned}
\hat{L}_+ &= L_+ + \big(L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_+ - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\big)_+ - \big(\Phi\mathcal{D}^{-1}\psi^\top L_+\varphi\mathcal{D}^{-1}\Psi^\top\big)_+\\
&:= L_+ + U_{+1} - U_{+2} - U_{+3} = \sum_{i=0}^{n}\hat{u}_i\mathcal{D}^i, \qquad \hat{u}_n = u_n.
\end{aligned} \tag{23}$$

We obtain the exact form of the differential operators U_{1+}, U_{2+}, U_{3+} after using the Leibnitz rule for a composition of pseudo-differential symbols. These operators are obtained in [16] in explicit forms.

The integral part $\hat{L}_-$ of the operator $\hat{L}$ may be constructed after long transformation using formulas (20)–(22) and the definition of the functions Φ and Ψ.

As a result we obtain

$$\begin{aligned}
\hat{L}_- &= (L\{\varphi\} - \Phi\Omega[\psi, L\{\varphi\}])\mathcal{D}^{-1}\Psi^\top - \Phi\mathcal{D}^{-1}(L^\tau\{\psi\} - \Psi\Omega[\varphi, L^\tau\{\psi\}])^\top + \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top\\
&\quad - \boldsymbol{q}\mathcal{M}_0\mathcal{D}^{-1}(\Psi\Omega[\varphi, \boldsymbol{r}])^\top - \Phi\Omega[\psi, \boldsymbol{q}]\mathcal{M}_0\mathcal{D}^{-1}\boldsymbol{r}^\top + \Phi\Omega[\psi, \boldsymbol{q}]\mathcal{M}_0\mathcal{D}^{-1}(\Psi\Omega[\varphi, \boldsymbol{r}])^\top.
\end{aligned}$$

From the formulas

$$\Omega[\psi, L\{\varphi\}] = \Omega[\psi, \varphi\Lambda] = \Omega[\psi, \varphi]\Lambda, \qquad \Omega[\varphi, L^\tau\{\psi\}] = \Omega[\varphi, \psi\tilde{\Lambda}] = \Omega[\varphi, \psi]\tilde{\Lambda} = (\tilde{\Lambda}^\top\Omega[\varphi, \psi])^\top$$

and the definition of the functions

$$\Phi := \varphi(C + \Omega[\psi, \varphi])^{-1}, \qquad \Psi := \psi\big(C^\top + \Omega[\varphi, \psi]\big)^{-1}$$

it follows

$$\hat{L}_- = \Phi\big(C\Lambda - \tilde{\Lambda}^\top C\big)\mathcal{D}^{-1}\Psi^\top + \big(\boldsymbol{q} - \Phi\Omega[\psi, \boldsymbol{q}]\big)\mathcal{D}^{-1}\big(\boldsymbol{r} - \Psi\Omega[\varphi, \boldsymbol{r}]\big)^\top, \tag{24}$$

thus the gauge transformed operator $\hat{L} := \hat{L}_+ + \hat{L}_-$ (23), (24) has the form (11) and F solves the new spectral problem (10) with the potentials $\Omega[\Psi, F]$ (15) and

$$\Omega[\hat{\boldsymbol{r}}, F] = \Omega[\boldsymbol{r}, f] - \Omega[\hat{\boldsymbol{r}}, \varphi](C + \Omega[\psi, \varphi])^{-1}\Omega[\psi, f], \qquad \Omega_x[\hat{\boldsymbol{r}}, F] = \hat{\boldsymbol{r}}^\top F. \tag{25}$$

Corollary 1. 1. *From formulas* (17)–(18) *we see that function* $G = W^{-1,\tau}\{g\}$ *solves the new adjoint spectral problem*

$$\hat{L^\tau}\{G\} := \sum_{i=0}^{n}(-1)^i\big(\hat{u}_i^\top G\big)^{(i)} - \Psi\mathcal{M}\Omega[\Phi, G] - \hat{\boldsymbol{r}}\mathcal{M}_0\Omega[\hat{\boldsymbol{q}}, G] = G\tilde{\hat{\Lambda}}$$

with the potentials $\Omega[\Phi,G]$, $\Omega[\hat{\boldsymbol{q}},G]$, *which can be calculated similarly to formulas* (14)–(15), (24)*:*

$$\Omega_x[\Phi,G]:=\Phi^{\top}G=\big[\big(C^{\top}+\Omega[\varphi,\psi]\big)^{-1}\Omega[\varphi,g]\big]_x$$
$$\Rightarrow\ \Omega[\Phi,G]=\big(C^{\top}+\Omega[\varphi,\psi]\big)^{-1}\Omega[\varphi,g], \tag{26}$$
$$\Omega[\hat{\boldsymbol{q}},G]=\Omega[\boldsymbol{q},g]-\Omega[\boldsymbol{q},\psi]\big(C^{\top}+\Omega[\varphi,\psi]\big)^{-1}\Omega[\varphi,g]. \tag{27}$$

2. *The functions* F, G *are integrated in the potential* $\Omega[G,F]$ *according to formula* (19).

3. *Let us have a differential operator* $M\in\zeta_+$. *Suppose that equations* (12)–(13) *hold and the adjoint eigenfunction* g *satisfies the linear evolution equation*

$$g_t=-M^{\tau}\{g\}, \tag{28}$$

then F, G *solve the new linear evolution systems*

$$F_t=\hat{M}\{F\},\qquad G_t=-\hat{M}^{\tau}\{G\}, \tag{29}$$

where $\hat{M}=-W(\partial_t-M)W^{-1}+\partial_t=\sum\limits_{i=0}^{m}\hat{v}_i\mathcal{D}^i$.

A proof of this fact and exact formulas for the coefficients $\hat{v}_i$, $i=\overline{0,m-1}$; $\hat{v}_m=v_m$ can be found in [16]. From the compatibility conditions $L_t=[M,L]$, $L_t^{\tau}=[L^{\tau},M^{\tau}]$ for the linear systems $L\{f\}=f\hat{\Lambda}$, $f_t=M\{f\}$ and $L^{\tau}\{g\}=g\tilde{\hat{\Lambda}}$, $g_t=-M^{\tau}\{f\}$ respectively compatibility conditions for the new linear equations (10), (18), (29) follow:

$$\hat{L}_t=[\hat{M},\hat{L}],\qquad \hat{L}_t^{\tau}=[\hat{L}^{\tau},\hat{M}^{\tau}].$$

From the Lagrange formula (see [16]) for the systems (13), (28)

$$(g^{\top}f)_t=\left[\sum_{i=1}^{m}\sum_{j=1}^{i-1}(-1)^j\big(g^{\top}v_i\big)^{(i)}f^{(i-j-1)}\right]_x \tag{30}$$

we obtain a possibility to express all potentials (14), (15), (19), (25), (26), (27) in terms of fixed potential

$$\Omega[g,f]=\int_{(x_0,t_0)}^{(x,t)}P[g,f]\,dx+Q[g,f]\,dt,\qquad \Omega[g,f](x_0,t_0)=0, \tag{31}$$

where $P[g,f]=\Omega_x[g,f]=g^{\top}f$, $Q[g,f]_x=P_t[g,f]$ (see (30)). ■

4 Conclusion remarks

I. In paper [18] a general $(N\times N)$-matrix operator W (16) of the BDT was obtain as a composition of K simplest Darboux-type transformations and K simplest inverse Darboux-type transformations. In this paper it is shown too that the operator W admits a so-called canonical factorization by the K simplest elementary Darboux-like transformations.

II. We used a particular case of the general dressing operator W for integration of the Lax–Zakharov–Shabat equations in a matrix algebra of differential [16] and pseudo-differential (in the scalar case cKP hierarchy, see [17, 19]) operators [20]. This particular case corresponds to a $(K\times K)$-matrix potential function Ω (31) of the form

$$\Omega[\psi,\varphi]=\mathcal{D}^{-1}\big\{\psi^{\top}\varphi\big\}:=C+\int_{\pm\infty}^{x}\psi^{\top}(s,t)\varphi(s,t)\,ds.$$

Wide classes of exact solutions for the integrable systems of the soliton theory in terms of Grammian–type determinants were obtained in these papers.

III. The general BDT Theorem 1 will be applied to constructing the solutions of matrix cKP hierarchy in forthcoming publications.

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Soliton Systems as $\star$-Deformed Dispersionless Equations

Błażej M. SZABLIKOWSKI and Maciej BŁASZAK

Institute of Physics, A. Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland
E-mail: *bszablik@amu.edu.pl, blaszakm@amu.edu.pl*

The formalism of quantization deformation is reviewed and the Weyl–Moyal like deformation is applied to systematic construction of the field and lattice integrable soliton systems from Poisson algebras of dispersionless systems.

1 Introduction

Recently, various aspects of the Moyal deformation theory and its application to the integrable field systems, which leads to the so-called Moyal type Lax dynamics, have become of increasing interest [1–5]. The aim of this paper is to present a complete picture of construction of the field and lattice soliton systems by Weyl–Moyal like deformations from Poisson algebras of underlying dispersionless systems. The Weyl–Moyal like deformation is the special case of the deformation quantization.

In the theory of evolutionary systems (dynamical systems) one of the most important issues is a systematic method for construction of integrable systems. As integrable systems we understand those which have infinite hierarchy of symmetries and conservation laws. It is well known that a very powerful tool, called the classical R-matrix formalism, proved to be very fruitful in systematic construction of the field and lattice soliton systems as well as dispersionless systems (see [6–16] and the references there).

As well known, a quasi-classical limit of field and lattice soliton systems gives related integrable dispersionless systems. We would like to inverse this procedure and construct field and lattice soliton systems from some classes of integrable dispersionless systems through a Weyl–Moyal like deformation quantization procedure. Actually, we will do it at the level of their Lax representations.

2 Classical R-matrix formalism

The crucial point of the formalism is the observation that integrable dynamical systems can be obtained from the Lax equations. Let $\mathfrak{g}$ be a Lie algebra, equipped with the Lie bracket $[\cdot,\cdot]$, for which its dual $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$. So, we assume the existence of a symmetric, non-degenerate and ad-invariant scalar product $(\cdot,\cdot)_{\mathfrak{g}}$. We assume additionally that the scalar product is given by a trace form $\mathrm{tr}:\mathfrak{g}\to\mathbb{R}$, i.e. $(a,b)_{\mathfrak{g}}=\mathrm{tr}\,(ab)$. A linear map $R:\mathfrak{g}\to\mathfrak{g}$, such that the bracket $[a,b]_R:=[Ra,b]+[a,Rb]$ is a second Lie product on $\mathfrak{g}$, is called the classical R-matrix. Then, the vector fields on $\mathfrak{g}$

$$L_{t_n}=[R\,(dC_n)\,,L]\qquad n\in\mathbb{Z}, \tag{1}$$

generated by the differentials of Casimir functions $dC_n\in\mathfrak{g}$, such that

$$[dC_n,L]=0\qquad \forall\, L\in\mathfrak{g},$$

commute mutually. The hierarchy of evolution equations (1) is the Lax hierarchy with common infinite set of symmetries and conserved quantities. In this sense (1) represents a hierarchy of integrable evolution equations. Moreover, it is known that (1) are tri-Hamiltonian systems [17].

To construct the simplest R-structure let us assume that the Lie algebra $\mathfrak{g}$ can be split into a direct sum of Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, i.e. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $[\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm$. Denoting the projections onto these subalgebras by $P_\pm$, we define the R-matrix as

$$R = \frac{1}{2}(P_+ - P_-) = P_+ - \frac{1}{2} = \frac{1}{2} - P_-. \tag{2}$$

Following the above scheme, we are able to construct in a systematic way integrable Hamiltonian systems, with infinite hierarchy of involutive constants of motion and infinite hierarchy of related commuting symmetries, once we fix a Lie algebra. For example the Lie algebra of pseudo-differential operators with the commutator leads to construction of soliton systems [6–10]. The Lie algebra of shift operators leads to lattice field systems [11, 12]. On the other hand, the Poisson algebras (which are Lie algebras with associative, commutative multiplication) of formal Laurent series lead to the construction of dispersionless systems [13–16].

3 Star products and deformation quantizable Poisson brackets

The idea behind the deformation quantization theory [18–21] is that a classical system can be obtained from quantum system by the quasi-classical limit $\hbar \to 0$, where $\hbar$ is the Planck constant divided by 2π. Therefore, the quantization of classical systems should be done by appropriate deformations depending on a formal parameter $\hbar$.

Let $\mathcal{A} = \mathcal{C}^\infty(M)$ be the space of all smooth ($\mathbb{R}$ or $\mathbb{C}$ valued) functions on $2n$-dimensional smooth manifold M. Let $\{\cdot,\cdot\}_{\rm PB}$ be the classical Poisson bracket, which is bilinear, skew-symmetric and satisfies the Jacobi identity. Obviously $\mathcal{A}$ is a commutative, associative algebra over $\mathbb{R}$ or $\mathbb{C}$ with the standard multiplication. Let $\star$ be the deformed associative non-commutative multiplication on $\mathcal{A}$ given by the following formula

$$f \star g = \sum_{k\geq 0} \hbar^k B_k(f,g), \qquad f, g \in \mathcal{A}, \tag{3}$$

where $\hbar$ is the formal parameter and $B_k : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are bidifferential (bilinear) operators. We also define deformed bracket as a commutator

$$\{f,g\}_\star := \frac{1}{\hbar}(f \star g - g \star f). \tag{4}$$

Definition 1. An associative deformed multiplication $\star$, given by the formula (3), is a formal quantization of the algebra $\mathcal{A}$ and is called the $\star$-product if

1. $\lim\limits_{\hbar\to 0} f \star g = fg$,
2. $c \star f = f \star c = cf$ for $c \in \mathbb{R}$ or $\mathbb{C}$,
3. $\lim\limits_{\hbar\to 0}\{f,g\}_\star = \{f,g\}_{\rm PB}$.

Lemma 1. *The bracket* (4) *defined by the $\star$-product is bilinear, skew-symmetric and satisfies the Jacobi identity. So, it is well defined Lie bracket.*

The proof is obvious as the Jacobi identity is a consequence of an associativity of multiplication $\star$. Hence, the bracket (4) is called the deformation quantization of the underlying classical Poisson bracket $\{\cdot,\cdot\}_{\rm PB}$. Let $D : \mathcal{A} \longrightarrow \mathcal{A}$ be a linear automorphism parameterized by $\hbar$, such that

$$Df = \sum_{k\geq 0} \hbar^k D_k f, \qquad D_0 = 1, \tag{5}$$

where D_k are differential operators. Such an automorphism produces a new $\star'$-product in $\mathcal{A}$ in the following way

$$f \star' g := D\left(D^{-1}f \star D^{-1}g\right). \tag{6}$$

Definition 2. Two $\star$-products: $\star$ and $\star'$ are called gauge equivalent or simply equivalent if there exist a linear automorphism $D : \mathcal{A} \longrightarrow \mathcal{A}$ (5) such that (6) is satisfied.

Let us consider now a Weyl–Moyal like deformations. It is well known that an arbitrary classical Poisson bracket can be presented in the following form

$$\{f,g\}_{\mathrm{PB}} = f\left(\sum_{i=1}^{n} Y_i \wedge X_i\right) g = f\left(\sum_{i=1}^{n} (Y_i \otimes X_i - X_i \otimes Y_i)\right) g$$
$$= \sum_{i=1}^{n} \left[Y_i(f)X_i(g) - X_i(f)Y_i(g)\right], \tag{7}$$

where X_i, Y_i, $i = 1,\dots,n$ are pair-wise commuting vector fields on $2n$ dimensional smooth manifold M and $f, g \in \mathcal{A}$. The Jacobi identity for (7) follows from the commutativity of vector fields X_i, Y_i. In what follows, we will use the Einstein summation convention in the case of repeating indices i, j at the vectors X, Y and a standard convention (with the summation symbols) otherwise.

Infinitely many Weyl–Moyal like deformed multiplications are of the following form [17]

$$f \star^{\alpha} g = f \exp\left[\frac{\hbar}{2}\left((2-\alpha)\, Y_i \otimes X_i - \alpha X_i \otimes Y_i\right)\right] g. \tag{8}$$

If the classical Poisson bracket (7) is a canonical one, i.e. $Y_i = \partial_{p_i}$, $X_i = \partial_{x_i}$ ($\partial_x = \frac{\partial}{\partial x}$, $\partial_p = \frac{\partial}{\partial p}$), then for $\alpha = 1$ the product (8) is the Groenewold product [22] and the deformed bracket (4) is the well known Moyal bracket [23] and for $\alpha = 0$ the product (8) is the Kupershmidt–Manin (KM) product and the deformed bracket (4) is the KM bracket [24, 25]. Notice that all (8) are well defined quantization deformation of (7). Moreover, they are gauge equivalent by the linear automorphism $D : \mathcal{A} \longrightarrow \mathcal{A}$

$$D^{\alpha} = \exp\left[-\alpha \frac{\hbar}{2} Y_i X_i\right], \qquad \alpha \in \mathbb{R}. \tag{9}$$

Then, $f \star^{\alpha'} g := D^{\alpha'-\alpha}\big(D^{-(\alpha'-\alpha)} f \star^{\alpha} D^{-(\alpha'-\alpha)} g\big)$. Now, we impose the Lie algebra structure on the algebra $\mathcal{A}$, denoting it by $\mathcal{A}_\alpha = (\mathcal{A}, \star^\alpha)$, with the commutator $\{f,g\}_{\star^\alpha} := \frac{1}{\hbar}(f \star^{\alpha} g - g \star^{\alpha} f)$. Obviously, the automorphism (9) induces the isomorphisms between the Lie algebras $D^{\alpha'-\alpha} : \mathcal{A}_\alpha \longrightarrow \mathcal{A}_{\alpha'}$. We will call the Lie algebras $\mathcal{A}_\alpha$ gauge equivalent as one can choose freely the algebra one wants to work with.

4 Poisson algebras of formal Laurent series

Consider the simplest possible case of $\dim M = 2$, when M is parametrized by a pair of coordinates (x,p). The Poisson bracket on $\mathcal{A}$ can be introduced in infinitely many ways as

$$\{f,g\}^r_{\mathrm{PB}} := p^r \left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p}\right), \qquad r \in \mathbb{Z}. \tag{10}$$

Moreover, in $\mathcal{A}$ there exists the following Poisson subalgebra of formal Laurent series (Lax polynomials)

$$A = \left\{ L = \sum_{i\in\mathbb{Z}} u_i(x) p^i \right\}, \tag{11}$$

where the coefficients u_i are smooth functions of x. An appropriate non-degenerate symmetric ad-invariant product on A is given by a trace form $\operatorname{Tr} L = \int_\Omega \operatorname{res}_r L \; dx$, where $\operatorname{res}_r L \equiv$

$u_{r-1}(x)$ [15]. We construct the simplest R-matrix, through a decomposition of A into a direct sum of Lie subalgebras. For a fixed r let $A_{\geqslant -r+k} = \{\sum_{i\geqslant -r+k} u_i(x)p^i\}$ and $A_{<-r+k} = \{\sum_{i<-r+k} u_i(x)p^i\}$. As presented in [15], $A_{\geqslant -r+k}, A_{<-r+k}$ are Lie subalgebras in the following cases: $k=0$, $r=0$; $k=1,2$, $r\in\mathbb{Z}$ and $k=3$, $r=2$[1]. Then, the R-matrix (2) is given by $R=\frac{1}{2}(P_{\geqslant -r+k}-P_{<-r+k})$. Hence, the hierarchy of evolution equations (1) for Casimir functionals

$$C_n(L)=\frac{1}{n+1}\operatorname{Tr}\left(L^{n+1}\right), \qquad dC_n(L)=L^n,$$

is given by

$$L_{t_q}=\{(L^q)_{\geqslant -r+k},L\}^r_{\mathrm{PB}}=-\{(L^q)_{<-r+k},L\}^r_{\mathrm{PB}}, \qquad L\in A, \tag{12}$$

which are Lax hierarchies. Notice that (12) are multi-Hamiltonian systems [16]. To construct (1+1)-dimensional dispersionless systems we have to choose properly restricted Lax operators L in such way that L contains the finite number of dynamical fields and gives consistent equations (12). They are given in the form [16]

$$\begin{aligned}
&k=0,\ r=0\ :\quad L=p^N+u_{N-2}p^{N-2}+\cdots+u_1p+u_0,\\
&k=1,\ r\in\mathbb{Z}\ :\quad L=p^N+u_{N-1}p^{N-1}+\cdots+u_{1-m}p^{1-m}+u_{-m}p^{-m},\\
&k=2,\ r\in\mathbb{Z}\ :\quad L=u_Np^N+u_{N-1}p^{N-1}+\cdots+u_{1-m}p^{1-m}+p^{-m}.
\end{aligned} \tag{13}$$

5 Weyl–Moyal like deformation of Poisson algebras of formal Laurent series

The Poisson brackets (10) on $\mathcal{A}$ can be presented in the following form

$$\{f,g\}^r_{\mathrm{PB}}=f\left(p^r\partial_p\wedge\partial_x\right)g, \qquad r\in\mathbb{Z}. \tag{14}$$

Notice that this is a special case of (7), when $Y_1=p^r\partial_p$ and $X_1=\partial_x$ with $[Y_1,X_1]=0$. For a fixed r, the Poisson bracket (14) on $\mathcal{A}$ can be quantized in infinitely many equivalent ways via the $\star^\alpha$-product (8)

$$f\star^\alpha g=f\exp\left[\frac{\hbar}{2}\left((2-\alpha)\,p^r\partial_p\otimes\partial_x-\alpha\partial_x\otimes p^r\partial_p\right)\right]g.$$

One finds that $(p^r\partial_p)^s p^m=c_s^m(r)\ p^{m-s(1-r)}$, $s\in\mathbb{Z}_+$, where for $k\in\mathbb{Z}$

$$c_s^{k(1-r)}(r)=\begin{cases}(1-r)^s\dfrac{k!}{(k-s)!} & \text{for } k\geq s \text{ and } r\neq 1,\\ 0 & \text{for } s>k\geq 0 \text{ and } r\neq 1,\\ (-1+r)^s\dfrac{(s-k-1)!}{s!} & \text{for } k<0 \text{ and } r\neq 1,\end{cases}$$

for $m\neq k(1-r)$: $c_s^m(r)=m(m-(1-r))\cdots(m-(s-1)(1-r))$ and for an arbitrary $m\in\mathbb{Z}$: $c_s^m(1)=m^s$.

We can quantize separately the Poisson subalgebra A (11) to the following Lie subalgebras $A_\alpha=(A,\star^\alpha)\subset\mathcal{A}_\alpha$. Obviously, the Lie algebras A_α for a fixed value of r are gauge equivalent under the isomorphism

$$D^{\alpha'-\alpha}:A_\alpha\longrightarrow A_{\alpha'}, \qquad D^{\alpha'-\alpha}=\exp\left[(\alpha-\alpha')\frac{\hbar}{2}p^r\partial_p\partial_x\right]. \tag{15}$$

[1] In [15–17] there was omitted the case $r=2$, $k=3$, but it does not lead to any new dispersionless systems as it is related to the case $r=0$, $k=0$ by transformation $p\to p^{-1}$. In further considerations we will skip this case.

Let $L = \sum\limits_{m=-\infty}^{+\infty} u_m p^m \in A_\alpha$ and $L' = \sum\limits_{n=-\infty}^{+\infty} v_n p^n \in A_{\alpha'}$. Then $L' = D^{\alpha'-\alpha} L$ and fields u_m, v_n are mutually related in the following way $v_n = \sum_{s\geq 0}((\alpha-\alpha')\frac{\hbar}{2})^s \frac{1}{s!}\ c_s^{s(1-r)+n}(r)\ (u_{s(1-r)+n})_{sx}$.

On the other hand one can show the following relations

$$u \star^\alpha v = uv, \qquad p^m \star^\alpha p^n = p^{m+n}, \tag{16}$$

$$p^m \star^\alpha u = \sum_{s\geq 0} \frac{\hbar^s}{s!} u_{sx} \star^\alpha (p^r \partial_p)^s\, p^m, \qquad u \star^\alpha p^m = \sum_{s\geq 0} \frac{\hbar^s}{s!} (p^r \partial_p)^s\, p^m \star^\alpha u_{sx}. \tag{17}$$

As all relations (16)–(17) have the same form independently of α we skip this index in further considerations. Hence, we can quantize separately the algebra A to the following special algebra of Lax operators:

$$\mathfrak{a} = \left\{ L = \sum_{i\in\mathbb{Z}} u_i(x) \star p^i \right\}. \tag{18}$$

It is obviously associative algebra under commutation rules (17). The algebra $\mathfrak{a}$ in the case of $r = 0$ was considered for the first time in [4]. Then, the Lie-bracket on $\mathfrak{a}$ is given by

$$\begin{aligned} \{u\star p^m, v\star p^n\}_\star &= \frac{1}{\hbar}\left(u\star p^m \star v \star p^n - v\star p^n \star u \star p^m\right) \\ &= \sum_{s=0}^{\infty} \frac{\hbar^{s-1}}{s!}\left[c_s^m(r)\ uv_{sx} - c_s^n(r)\ u_{sx}v\right] \star p^{m+n-s(1-r)}. \end{aligned} \tag{19}$$

Notice that the algebra $\mathfrak{a}$ differs from that defined in the third section, where we introduced deformation quantization, as in (18) we also deformed the Lax polynomials. Let us remark that the algebra $\mathfrak{a}$ is naturally isomorphic to the algebra A_0 as $u \star^0 p^m = up^m$. Hence, in the further considerations we will concentrate only on the algebra $\mathfrak{a}$, as the results for the algebras A_α for all values of α can be obtained simply from A_0. The second reason is that $\mathfrak{a}$ can be considered as the generalization of the algebra of the pseudo-differential operators and the algebra of the shift operators in the following sense. Let us consider the case of $r = 0$, then the rules (17) take the particular form

$$p^m \star u = \sum_{s=0} \hbar^s \binom{m}{s} u_{sx} \star p^{m-s}, \qquad u \star p^m = \sum_{s=0} (-\hbar)^s \binom{m}{s} p^{m-s} \star u_{sx}$$

and the Lie bracket (19) is

$$\{u\star p^m, v\star p^n\} = \sum_{s=0}^{\infty} \hbar^{s-1}\left[\binom{m}{s} uv_{sx} - \binom{n}{s} u_{sx}v\right] \star p^{m+n-s}.$$

Notice, that the algebra $\mathfrak{a}$ for fixed $r = 0$ is isomorphic to the algebra of pseudo-differential operators. In the case of $r = 1$ the rules (17) become

$$\begin{aligned} p^m \star u(x) &= \sum_{s=0} \frac{\hbar^s}{s!} m^s \left(u(x)\right)_{sx} \star p^m = u(x+m\hbar) \star p^m, \\ u(x) \star p^m &= \sum_{s=0} \frac{(-\hbar)^s}{s!} m^s p^m \star \left(u(x)\right)_{sx} = p^m \star u(x-m\hbar), \end{aligned}$$

where we use the formula for Taylor expansion. The Lie bracket (19) is

$$\{u(x)\star p^m, v(x)\star p^n\}_\star = \frac{1}{\hbar}\left[u(x)v(x+m\hbar) - u(x+n\hbar)v(x)\right] \star p^{m+n}.$$

The algebra $\mathfrak{a}$ for a fixed $r = 1$ is isomorphic to the algebra of shift operators.

The following property for $\mathfrak{a}$ is very important.

Lemma 2 ([17]). *An appropriate symmetric, non-degenerate and* ad*-invariant scalar product on* $\mathfrak{a}$ *is given by a trace form* $(L_1, L_2)_{\mathfrak{a}} := \mathrm{Tr}\,(L_1 \star L_2)$*, where* $\mathrm{Tr}\,L = \int_\Omega \mathrm{res}_r L\, dx$ *and* $\mathrm{res}_r L \equiv u_{r-1}(x)$*.*

To proof Lemma 2 one needs the following useful relation: $c_s^m(r) = (-1)^s\, c_s^{(s-1)(1-r)-m}(r)$. Notice that one can simply rewrite the trace formula from $\mathfrak{a} \cong A_0$ to A_α as the trace formula is invariant under isomorphism (15).

6 *R*-matrix formalism and Lax hierarchies for Lie algebra $\mathfrak{a}$

To construct the integrable field systems one has to split the algebra $\mathfrak{a}$ into a direct sum of Lie subalgebras. Observing (19) one finds that in general it can be done only for $r = 0$, $r = 1$ or $r = 2$. Now, we decompose $\mathfrak{a}$ for $r = 0, 1, 2$ into a direct sum of Lie subalgebras in the following way. Let $\mathfrak{a}_{\geqslant -r+k} = \{\sum_{i \geqslant -r+k} u_i(x) \star p^i\}$ and $\mathfrak{a}_{<-r+k} = \{\sum_{i<-r+k} u_i(x) \star p^i\}$. Then, $\mathfrak{a}_{\geqslant -r+k}$ and $\mathfrak{a}_{<-r+k}$ are Lie subalgebras in the case of $r = 0$ for $k = 0, 1, 2$; in the case of $r = 1$ for $k = 1, 2$ and in the case of $r = 2$ for $k = 1, 2, 3$[2]. Hence, the R-matrix (2) is given by $R = \frac{1}{2}(P_{\geqslant -r+k} - P_{<-r+k})$. Then,

The hierarchy of evolution equations are generated by the Casimir functionals

$$C_n(L) = \frac{1}{n+1}\,\mathrm{Tr}\left(L^{n+1}\right), \qquad dC_n(L) = L^n, \qquad L^n = L \star L \star \cdots \star L,$$

and for appropriate k has the form

$$L_{t_n} = \{(L^n)_{\geqslant -r+k}, L\}_\star = -\{(L^q)_{<-r+k}, L\}_\star \tag{20}$$

which are Lax hierarchies. Notice, that (20) are in general tri-Hamiltonian systems [17].

The Lie algebras A_α can be decomposed into direct sum of Lie subalgebras exactly in the same way as $A_0 \cong \mathfrak{a}$. Hence, the R-matrix is invariant under transformations (15). Moreover, as transformations (15) are Lie algebra isomorphisms the Lax hierarchies (20) are also invariant with respect to them.

To construct (1+1)-dimensional closed systems with a finite number of fields we have to choose properly restricted Lax operators L which give consistent Lax equations (20). In the case of $r = 0$ the admissible simplest restricted Lax operators are given in the form

$$\begin{aligned}
&k = 0: \quad L = p^N + u_{N-2} \star p^{N-2} + \cdots + u_1 \star p + u_0,\\
&k = 1: \quad L = p^N + u_{N-1} \star p^{N-1} + \cdots + u_0 + p^{-1} \star u_{-1},\\
&k = 2: \quad L = u_N \star p^N + u_{N-1} \star p^{N-1} + \cdots + u_0 + p^{-1} \star u_{-1} + p^{-2} \star u_{-2}.
\end{aligned} \tag{21}$$

In the case of $r = 1$ the admissible simplest restricted Lax operators are

$$\begin{aligned}
&k = 1: \quad L = p^N + u_{N-1} \star p^{N-1} + \cdots + u_{1-m} \star p^{1-m} + u_{-m} \star p^{-m},\\
&k = 2: \quad L = u_N \star p^N + u_{N-1} \star p^{N-1} + \cdots + u_{1-m} \star p^{1-m} + p^{-m}.
\end{aligned} \tag{22}$$

Comparing the Lax operators related to soliton systems (21) and (22) with the Lax operators related to the dispersionless systems (13) one observes that dispersionless systems for $r \neq 0, 1$ and some for $r = 0$ do not have counterpart soliton systems in the quantization scheme considered.

We will now display examples of some field and lattice soliton systems calculated in proposed quantization scheme. We consider the Lax hierarchy (20) with little changed numerations of

[2] In [17] there was omitted the case $r = 2$, $k = 1, 2, 3$, but it does not lead to any new soliton systems as it is related to the case $r = 0$, $k = 2, 1, 0$, respectively, by transformation $p \to p^{-1}$. In further considerations we will skip the case of $r = 2$.

evolution variables: $L_{t_n} = \left\{\left(L^{\frac{n}{N}}\right)_{\geqslant -r+k}, L\right\}_\star$, where N is the highest order of the Lax operator L. We will exhibit the first nontrivial equation of the Lax hierarchy. The advantage of the use of $\mathfrak{a}$ algebra is that during whole calculations there is no need of using the $\star^\alpha$-product in explicit form and we only use the commutation relations (17). As a result, one gets the same equations and Poisson structures as these obtained from quantized algebra A_0. The Hamiltonian systems related to quantized algebras A_α are simply reconstructed via the linear transformation (15). Such a procedure of calculations is applied in present examples and we have written down only the final results for A_α.

Example 1. The case of $r = 0$ and $k = 0$. The dispersionless Boussinesq system is given in the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ -\frac{2}{3}uu_x \end{pmatrix}. \tag{23}$$

It has the following Lax representation [16] $L_{t_2} = \left\{\left(L^{\frac{2}{3}}\right)_{\geqslant 0}, L\right\}^0_{\mathrm{PB}}$ for Lax operator in the form $L = p^3 + up + v \in A$. The quantization procedure leads now to the following Lax operator $L = p^3 + u \star p + v \in \mathfrak{a}$. Then, one can derive the Boussinesq system from $L_{t_2} = \{(L^{\frac{2}{3}})_{\geqslant 0}, L\}_\star$. Now, by the transformation to the algebras A_α one finds the following systems

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x + (\alpha - 1)\,\hbar u_{2x} \\ -\frac{2}{3}uu_x + (1-\alpha)\,\hbar v_{2x} - \left(\frac{\alpha^2}{2} - \alpha + \frac{2}{3}\right)\hbar^2 u_{3x} \end{pmatrix}. \tag{24}$$

In the case of $\alpha = 0$ (24) is obviously the standard case of Boussinesq system obtained from Gel'fand–Dikii hierarchy, for $\alpha = 1$ it is the Moyal case. The limit $\hbar \to 0$ of (24) gives (23) as it should be.

Example 2. The case of $r = 1$ and $k = 1$. The dispersionless Toda system has the form

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} v_x \\ u_x v \end{pmatrix}. \tag{25}$$

The Lax representation for (25) is [16] $L_{t_2} = \{(L^2)_{\geqslant 0}, L\}^1_{\mathrm{PB}}$, where the Lax operator is $L = p + u + vp^{-1}$. Then, the quantization scheme leads to the following Lax operator in $\mathfrak{a}$ in the form $L = p + u(x) + v(x) \star p^{-1}$. One derives the Toda system from $L_{t_2} = \{(L^2)_{\geqslant 0}, L\}_\star$. Next, by the transformation to A_α one finds

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \frac{1}{\hbar}\begin{pmatrix} v\left(x + \left(1 - \frac{\alpha}{2}\right)\hbar\right) - v\left(x - \frac{\alpha}{2}\hbar\right) \\ v(x)\left[u\left(x + \frac{\alpha}{2}\hbar\right) - u\left(x - \left(1 - \frac{\alpha}{2}\right)\hbar\right)\right] \end{pmatrix}. \tag{26}$$

The case of $\alpha = 0$ of (26) is the standard case of Toda system, the case of $\alpha = 1$ is the Moyal case. Notice that in our construction Toda equation depends on continuous coordinate x contrary to a standard case when x is integer.

7 Conclusions

In a previous paper [16] we have constructed Lax representations for a wide class of dispersionless systems, derived from classical R-matrix theory. The number of the constructed dispersionless systems is much greater then the number of known soliton systems (dispersive integrable systems). So, the question rises whether for any dispersionless Lax hierarchy one can construct a related soliton hierarchy. We have tried to obtain an answer to this question via the procedure

of deformation quantization for Poisson algebras of dispersionless systems and appropriate R-matrix theory. We have managed to quantize all Poisson algebras (with arbitrary r (10)) but the R-matrix after quantization, at least of the form (2), exists only in the case of two algebras, i.e. for $r = 0$ ($r = 2$) and $r = 1$, respectively. The first case leads to soliton field systems related to the algebra of pseudo-differential operators, and the second one leads to lattice soliton systems related to the algebra of shift operators. In that sense, although we have failed to construct new soliton equations through presented deformation procedure, nevertheless we have found a unified procedure of the construction of field and lattice Hamiltonian soliton systems in one scheme.

For more details and proofs we send the reader to [17].

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Relation between Periodic Soliton Resonance and Instability

Masayoshi TAJIRI

Graduate School of Engineering, Osaka Prefecture University, Sakai, Osaka, 599-8531, Japan
E-mail: *tajiri@ms.osakafu-u.ac.jp*

There are periodic soliton resonances in interactions between two periodic solitons and between periodic soliton and another kind of soliton. A close relation exists between the periodic soliton resonance and soliton instability to transverse disturbances. It is shown that the instability occurred in the line soliton is relaxed by emission of the periodic soliton and the periodic soliton resonance is inverse process of the instability.

1 Introduction

The dynamics of nonlinear waves in higher-dimensional space have richer phenomena than one-dimensional case, since various localized structures may be considered as candidates for solitons. The two-dimensional generalization of the Korteweg–de Vries (KdV) equation was given by Kadomtsev and Petviashvili to discuss the stability of one-dimensional KdV soliton (line soliton in two dimensions) against transverse long-wave disturbances, which is known as the Kadomtsev–Petviashvili (KP) equation and is expressed as follows [1],

$$(u_t + 6uu_x + u_{xxx})_x + 3su_{yy} = 0, \qquad s = \pm 1. \tag{1}$$

The propagation property of solitons depends essentially on the sign of s in equation (1). In the media with negative dispersion ($s = +1$), line solitons are stable to long transverse perturbations. On the other hand, line solitons are unstable for positive dispersion ($s = -1$) [1]. This leads to the conjecture that solutions having two-dimensional localized structures should be found in the positive dispersion case. Such solitons were found by Manakov *et al.* [2] and Ablowitz and Satsuma [3], which have no longer exponential tails, but instead take the form of rational functions in both space and time variables.

The N line soliton solution was obtained by Zakharov and Shabat [4] and Satsuma [5]. Miles [6] found the existence of the singular parameter regime for multi-soliton solutions in the negative dispersion case, where the solutions become singular. He showed that the two-soliton solution reveal a resonant interaction when it is just on the borderline between regular and singular regimes in the parameter space. In this case, when the relative inclination between wave normals is at a certain small critical angle, two solitons interact strongly to make a resonant soliton from a point at which two incident solitons meet together. Such a resonant interaction is called "soliton resonance". On the other hand, in the positive dispersion media, the solution which describes the interaction between two line solitons never satisfies the resonant condition.

Another kind of two-dimensional localized soliton in the positive dispersion media is a periodic soliton, which was found by Zaitsev at first by superposition of rational solitons (algebraic solitons) [7]. The periodic soliton solutions which describe the multi-soliton interactions have been obtained by Tajiri and Murakami [8]. The stability for the y-periodic and x-periodic solitons has been discussed by Zhdanov [9], using the dressing method. He also obtained the solution which describes the nonlinear stage of instability of the line soliton propagating in the x-direction and pointed out the decay of a one-dimensional soliton into a soliton of lower amplitude and a periodic soliton.

As there are three kinds of solitons, line soliton, algebraic soliton and periodic soliton, various interactions between two different kinds of solitons may happen in the positive dispersion media. The interactions between two periodic solitons, periodic soliton and line soliton, line soliton and algebraic soliton and periodic soliton and algebraic soliton have been studied by several authors [10–16]. Thing to be most emphasized among them is the existence of resonant interactions between two periodic solitons and between periodic soliton and another kind of soliton. Here we call them "periodic soliton resonances", which are qualitatively different from the resonant interaction between two line solitons (soliton resonances) of the KP equation with negative dispersion. Periodic soliton resonances are irrelevant to the divergence of soliton solutions. This is crucially different from the case of soliton resonances.

Another equation having two-dimensional localized soliton solutions is the Davey–Stewartson (DS) equation [17]:

$$\begin{aligned} &iu_t + pu_{xx} + u_{yy} + r|u|^2u - 2uv = 0, \\ &v_{xx} - pv_{yy} - r\left(|u|^2\right)_{xx} = 0, \end{aligned} \tag{2}$$

where $p = \pm 1$, r is constant. This is the two-dimensional generalization of the nonlinear Schrödinger (NLS) equation. Equation (2) with $p = +1$ and $p = -1$ are called the DSI and DSII equations, respectively. Anker and Freeman [18] studied the interactions between two dark line solitons which are skewed with respect to each other and found the existence of soliton resonances. The DS equation has algebraic soliton and periodic soliton solutions in addition to the dark line soliton solution [19]. In the previous paper, the interactions between two periodic solitons, between dark line soliton and periodic soliton, and between algebraic soliton and periodic soliton were investigated [20, 21]. We found the existence of periodic soliton resonances in each case.

The existence of solitons having the structures peculiar to high-dimensionality may contribute to the variety of the dynamics of nonlinear waves. Especially, we expect that the periodic soliton resonances play fundamental role in the nonlinear development of high-dimensional wave fields as the existence of periodic solitons is related to the instability of solitons.

The purpose of this study is to show that we can discuss the problem of the stability of solitons by making use of the solutions of periodic soliton resonance.

2 Soliton stability theory due to periodic soliton resonance solutions

2.1 Stability of the line soliton of the KP equation with positive dispersion

At first, we consider the resonant interaction between line soliton and y-periodic soliton which propagate in the same direction. Although we can obtain the expressions that describe the general interaction between solitons in arbitrary directions, we limit our concern into the solitons that extend in the y-direction and propagate in the x-direction. The solutions that describe the interaction between line soliton and y-periodic soliton are given by [13]

$$u = 2(\log f)_{xx}, \tag{3}$$

with

$$f = 1 - \mathrm{e}^{\xi_P}\cos\eta + \frac{M}{4}\mathrm{e}^{2\xi_P} + \mathrm{e}^{\xi_L}\left(1 - N\mathrm{e}^{\xi_P}\cos\eta + \frac{MN^2}{4}\mathrm{e}^{2\xi_P}\right), \tag{4}$$

where

$$\xi_P = \alpha x - \Omega_P t + \sigma_P, \qquad \xi_L = Px - \Omega_L t + \sigma_L, \qquad \eta = \delta y + \theta,$$

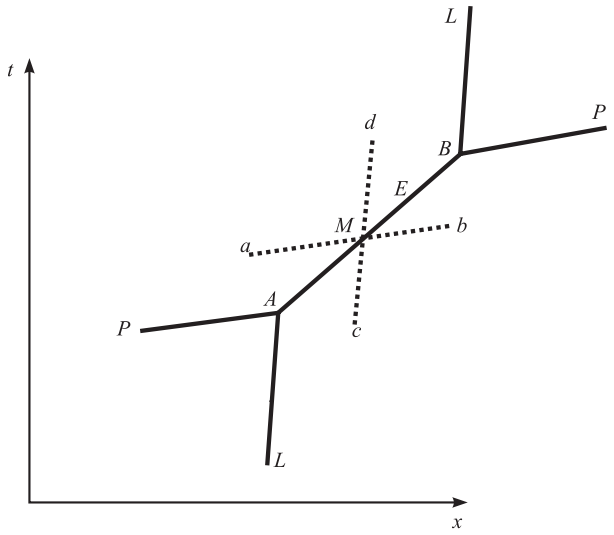

Figure 1. The schematic diagram of the world line of the soliton humps in the $x-t$ plane. AB corresponds to the quasi-resonant state. On line ab; $\xi_P + (1/2)\Gamma \simeq O(1)$ and on line cd; $\xi_L + \Gamma \simeq O(1)$.

$$\Omega_P = \alpha^3 + 3\frac{\delta^2}{\alpha}, \qquad \Omega_L = P^3, \qquad M = \frac{\delta^2}{\delta^2 - \alpha^4}, \qquad N = \frac{(\alpha - P)^2 - (\delta/\alpha)^2}{(\alpha + P)^2 - (\delta/\alpha)^2}, \tag{5}$$

where σ_P, σ_L and θ arbitrary real constants. The phase shift due to the interaction between line soliton and y-periodic soliton is determined only by N. In the case $\alpha P > 0$, the phase shift in the propagating direction is given by $\Gamma = \log|N|$ while that in the transverse direction is determined by the sign of N.

The resonant conditions are obtained by equating the denominator of N to zero as follows,

$$\frac{\delta}{\alpha^2} = \frac{P}{\alpha} + 1, \tag{6}$$

$$\frac{\delta}{\alpha^2} = -\frac{P}{\alpha} - 1. \tag{7}$$

Fig. 1 is the schematic diagram of the world lines of the soliton humps in the $x-t$ plane for the attractive interaction ($\Gamma > 0$). The line AB in Fig. 1 corresponds to the quasi-resonant state and at the midpoint M of the world line AB, the quasi-resonant state reaches the most resonant state. When we replace $\xi_P + \frac{1}{2}\ln|N|$ and $\xi_L + \ln|N|$ with $\tilde{\xi}_P$ and $\tilde{\xi}_L$, respectively, so as to $\tilde{\xi}_P \sim \tilde{\xi}_L \sim O(1)$ at the most resonant point M, that is,

$$\xi_P + \frac{1}{2}\log|N| = \tilde{\xi}_P \sim O(1), \qquad \sigma_P + \frac{1}{2}\log|N| = \tilde{\sigma}_P,$$
$$\xi_L + \log|N| = \tilde{\xi}_L \sim O(1), \qquad \sigma_L + \log|N| = \tilde{\sigma}_L,$$

(see Fig. 1) equation (4) is rewritten as follows;

$$f = f^{(0)} + \frac{1}{\sqrt{|N|}} f^{(1)} + \frac{1}{|N|} f^{(2)}, \tag{8}$$

with

$$f^{(0)} = 1 + \frac{M}{4} \mathrm{e}^{\tilde{\xi_L} + 2\tilde{\xi_P}}, \tag{9}$$

$$f^{(1)} = -\mathrm{e}^{\tilde{\xi_P}} \left\{ \cos\eta + \mathrm{e}^{\tilde{\xi_L}} \cos(\eta + \epsilon\pi) \right\}, \tag{10}$$

$$f^{(2)} = \mathrm{e}^{\tilde{\xi}_L} + \frac{M}{4}\mathrm{e}^{2\tilde{\xi}_P}, \tag{11}$$

where $\epsilon = 0$ for $N > 0$ and $\epsilon = 1$ for $N < 0$. Taking the limit $|N| \to \infty$, we see that the resonant soliton is the line soliton with wave number $(P/2 + \alpha)$. If we take $|N| \gg 1$, equation (8) shows that the quasi-resonant state consists of the resonant line soliton and small disturbance. The disturbance becomes minimum at midpoint M. The line ME in Fig. 1 corresponds to the period of the linear development of the disturbance on the resonant line soliton. Substituting equation (8) into $u = 2(\log f)_{xx}$, and neglecting the terms of $O(1/|N|)$, we have

$$u = u^{(0)} + \frac{1}{\sqrt{|N|}}\left\{-2u^{(0)}\frac{f^{(1)}}{f^{(0)}} + \frac{2}{f^{(0)2}}\left(f^{(0)}_{xx}f^{(1)} + f^{(0)}f^{(1)}_{xx} - 2f^{(0)}_{x}f^{(1)}_{x}\right)\right\}, \tag{12}$$

where

$$u^{(0)} = \frac{1}{2}P_0^2\,\mathrm{sech}^2\frac{1}{2}\left(P_0 x - \Omega_0 t + \sigma_0\right),$$

which is resonant line soliton solution, where $\sigma_0 = 2\tilde{\sigma}_P + \tilde{\sigma}_L + \log\left(\frac{M}{4}\right)$. Substituting equations (9) and (10) into equation (12) and replacing $\tilde{\xi}_L$ with

$$\tilde{\xi}_L = \frac{P}{P_0}\xi_0 + \left(\frac{P}{P_0}\Omega_0 - \Omega_L\right)t - \frac{P}{P_0}\sigma_0 + \tilde{\sigma}_L, \tag{13}$$

we have [22]

$$\begin{aligned} u = u^{(0)} + \frac{u^{(0)}}{\sqrt{MN_L}}\Bigg[&\left\{2\,\mathrm{sech}\frac{\xi_0}{2} - \frac{4\alpha\,(\alpha+P)}{P_0^2}\left(A\cosh\frac{1}{2}\xi_0 - B\sinh\frac{1}{2}\xi_0\right)\right\} \\ &\times \exp\left(\frac{P}{2P_0}\xi_0 + \frac{1}{2}\left(-\frac{P}{P_0}\sigma_0 + \tilde{\sigma}_L\right)\right)\mathrm{e}^{\gamma t} \\ &+ \left\{2\,\mathrm{sech}\frac{\xi_0}{2} - \frac{4\alpha\,(\alpha+P)}{P_0^2}\left(A\cosh\frac{1}{2}\xi_0 + B\sinh\frac{1}{2}\xi_0\right)\right\} \\ &\times \exp\left(-\left\{\frac{P}{2P_0}\xi_0 + \frac{1}{2}\left(-\frac{P}{P_0}\sigma_0 + \tilde{\sigma}_L\right)\right\}\right)\mathrm{e}^{-\gamma t}\Bigg]\cos\eta, \end{aligned} \tag{14}$$

where

$$\gamma = \frac{1}{2}\left(\frac{P}{P_0}\Omega_0 - \Omega_L\right). \tag{15}$$

Substituting the relation $P{=}P_0{-}2\alpha$ into the resonant condition, $\alpha + P - (\delta/\alpha) = 0$, we have

$$\alpha = \frac{1}{2}\left(P_0 - \sqrt{P_0^2 - 4\delta}\right), \tag{16}$$

$$P = \sqrt{P_0^2 - 4\delta}. \tag{17}$$

When we substitute equations (16) and (17) into the dispersion relations of line soliton and y-periodic soliton, Ω_P and Ω_L can be expressed by P_0 and δ. Then we have the growth rate γ,

$$\gamma = 2\delta\sqrt{P_0^2 - 4\delta}, \tag{18}$$

which is the same as the result obtained by Zhdanov by using the dressing method [9]. Taking the long limit $|\delta| \ll 1$ $(\alpha \ll 1)$, we have

$$\alpha \simeq \frac{\delta}{P_0}, \qquad \frac{4\alpha(\alpha+P)}{P_0^2}A \simeq \frac{4\alpha(\alpha+P)}{P_0^2}B \simeq 2\left(1 - \frac{2\delta}{P_0^2}\right),$$

and then

$$\gamma \simeq 2\delta P_0, \tag{19}$$

$$\begin{aligned} u \simeq u^{(0)} &- \frac{2}{\sqrt{MN_L}}\frac{1}{P_0}\left[C\left\{-\frac{P_0^3}{2}\operatorname{sech}^2\left(\frac{1}{2}\xi_0\right)\tanh\left(\frac{1}{2}\xi_0\right)\right.\right. \\ &- \frac{\gamma}{2}\left(\operatorname{sech}^2\left(\frac{1}{2}\xi_0\right) - \frac{\xi_0}{2}\operatorname{sech}^2\left(\frac{1}{2}\xi_0\right)\tanh\left(\frac{1}{2}\xi_0\right)\right)\Big\}\mathrm{e}^{\gamma t} \\ &- \frac{1}{C}\left\{-\frac{P_0^3}{2}\operatorname{sech}^2\left(\frac{1}{2}\xi_0\right)\tanh\left(\frac{1}{2}\xi_0\right) - \left(\frac{-\gamma}{2}\right)\left(\operatorname{sech}^2\left(\frac{1}{2}\xi_0\right)\right.\right. \\ &\left.\left.\left. - \frac{\xi_0}{2}\operatorname{sech}^2\left(\frac{1}{2}\xi_0\right)\tanh\left(\frac{1}{2}\xi_0\right)\right)\right\}\mathrm{e}^{-\gamma t}\right]\cos\eta, \end{aligned} \tag{20}$$

where $C = \exp\left\{-(1/2)\left(\sqrt{P_0^2 - 4\delta}\sigma_0/P_0 - \tilde{\sigma}_L\right)\right\}$. It is very interesting to note that these equations are expressed only by the wave number of resonant line soliton, P_0, and the wave number of the disturbance δ. We can see that the solution (20) has the growing term and decaying term.

2.2 Stability of the line soliton of the DSI equation

In this section, the stability of the line soliton to the DSI equation is studied by making use of the solution of periodic soliton resonance. The solution describing the interaction between line soliton and y-periodic soliton is given by

$$u = u_0 e^{i\zeta}\frac{g}{f}, \qquad v = -2\left(\ln f\right)_{xx} \tag{21}$$

with

$$f(\xi_P, \xi_L) = 1 + e^{\xi_P}\cos\eta + \frac{M}{4}e^{2\xi_P} + e^{\xi_L}\left\{1 + Ne^{\xi_P}\cos\eta + \frac{MN^2}{4}e^{2\xi_P}\right\}, \tag{22}$$

$$g(\xi_P, \xi_L, \phi_P, \phi_L) = f(\xi_P + i\phi_P, \xi_L + i\phi_L), \tag{23}$$

where

$$\begin{aligned} &\zeta = kx + ly - \omega t, \qquad \xi_P = \alpha x - \Omega_P t + \sigma_P, \\ &\xi_L = Px - \Omega_L t + \sigma_L, \qquad \eta = \delta y - \gamma_P t + \theta, \\ &\sin^2(\phi_P/2) = (\alpha^2 + \delta^2)/(2ru_0^2), \qquad \sin^2(\phi_L)/2) = P^2/(2ru_0^2), \\ &\Omega_P = 2k\alpha - (\alpha^2 - \delta^2)\cot(\phi_P/2), \qquad \gamma_P = 2l\delta, \\ &\Omega_L = P\{2k - P\cot(\phi_L/2)\}, \qquad M = \{1 - (\alpha^2 + \delta^2)^2/(2\delta^2 ru_0^2)\}^{-1}, \\ &N = \frac{2ru_0^2\sin(\phi_P/2)\sin(\phi_L/2)\cos\{(\phi_P - \phi_L)/2\} - \alpha P}{2ru_0^2\sin(\phi_P/2)\sin(\phi_L/2)\cos\{(\phi_P + \phi_L)/2\} - \alpha P}. \end{aligned} \tag{24}$$

When we replace $\xi_P + \frac{1}{2}\ln|N|$ and $\xi_L + \ln|N|$ with $\tilde{\xi}_P$ and $\tilde{\xi}_L$, respectively, so as to $\tilde{\xi}_P \sim \tilde{\xi}_L \sim O(1)$ at the most resonant point M, equations (22) and (23) are rewritten as follows [23]:

$$\begin{aligned} f &= f^{(0)} + \frac{1}{\sqrt{|N|}}f^{(1)}\cos\eta + \frac{1}{|N|}f^{(2)}, \\ g &= g^{(0)} + \frac{1}{\sqrt{|N|}}g^{(1)}\cos\eta + \frac{1}{|N|}g^{(2)}, \end{aligned} \tag{25}$$

where

$$f^{(0)} = 1 + \frac{M}{4}\exp(\tilde{\xi}_0), \tag{26}$$

$$g^{(0)} = 1 + \frac{M}{4}\exp(\tilde{\xi}_0 + \mathrm{i}\Phi), \tag{27}$$

$$f^{(1)}(\tilde{\xi}_P, \tilde{\xi}_L) = \exp(\tilde{\xi}_P)\left\{1 + \varepsilon\exp(\tilde{\xi}_L)\right\}, \tag{28}$$

$$f^{(2)}(\tilde{\xi}_P, \tilde{\xi}_L) = \exp(\tilde{\xi}_L) + \frac{M}{4}\exp(2\tilde{\xi}_P), \tag{29}$$

$$g^{(j)} = f^{(j)}(\tilde{\xi}_P + \mathrm{i}\phi_P, \tilde{\xi}_L + \mathrm{i}\phi_L) \qquad (j = 1, 2), \tag{30}$$

where $\Phi = 2\phi_P + \phi_L$, $P_0 = 2\alpha + P$, $\Omega_0 = 2\Omega_P + \Omega_L$, $\tilde{\xi_0} = 2\tilde{\xi}_P + \tilde{\xi}_L = P_0 x - \Omega_0 t + 2\tilde{\sigma}_P + \tilde{\sigma}_L$ and $\varepsilon = 1$ for $0 < N$, $\varepsilon = -1$ for $N < 0$. Taking the limit $|N| \to \infty$, we have the resonant line soliton solution with wave number P_0 and phase Φ as follows,

$$u_0 = u_0 \mathrm{e}^{\mathrm{i}\zeta}\frac{1 + \mathrm{e}^{\xi_0 + \mathrm{i}\Phi}}{1 + \mathrm{e}^{\xi_0}}, \qquad v_0 = -\frac{P_0^2}{2}\mathrm{sech}^2\frac{1}{2}\xi_0, \tag{31}$$

where $\xi_0 = \tilde{\xi}_0 + (1/2)\log(M/4)$. Using the resonant condition, and dispersion relations of y-periodic soliton and line soliton, we obtain

$$P^2 = P_0^2 - 4\delta^2 \pm 4\left(\sqrt{2ru_0^2 - P_0^2}\right)\delta. \tag{32}$$

Taking account of the relations, $\alpha = (P_0 - P)/2$, $\sin^2(\phi_L/2) = P^2/2ru_0^2$ and $\phi_P = (\Phi - \phi_L)/2$, we see that P, α, ϕ_L, ϕ_P, Ω_P and Ω_L are expressed by P_0, Φ and δ. Therefore, the quasi-resonant solution is expressed by parameters (P_0, Φ, δ), which are the wave number and the phase of resonant line soliton and the wave number of transverse disturbance on the resonant line soliton, respectively. Neglecting the terms of $O(1/N)$, we have [23].

$$u = u_0 \mathrm{e}^{\mathrm{i}\zeta}\left[\frac{g^{(0)}}{f^{(0)}} + \frac{1}{\sqrt{|N|}}\frac{g^{(0)}}{f^{(0)}}\left(\frac{g^{(1)}}{g^{(0)}} - \frac{f^{(1)}}{f^{(0)}}\right)\cos\eta\right], \tag{33}$$

$$v = -2\frac{f_{xx}^{(0)}f^{(0)} - f_x^{(0)^2}}{f^{(0)^2}} + \frac{1}{\sqrt{|N|}}\left[\left\{4\frac{f_{xx}^{(0)}f^{(0)} - f_x^{(0)^2}}{f^{(0)^2}}\frac{f^{(1)}}{f^{(0)}} - \frac{2}{f^{(0)^2}}\left[f_{xx}^{(0)}f^{(1)} - 2f_x^{(0)}f_x^{(1)} + f^{(0)}f_{xx}^{(1)}\right]\right\}\cos\eta\right]. \tag{34}$$

It should be noted that the magnitude of the initial disturbance (the magnitude of the disturbance on the resonant line soliton at the most resonant state) is of the order $(1/\sqrt{|N|})$. The second term of equations (33) and (34) describes the linear development of the disturbance on the resonant line soliton. Regarding the quasi-resonant soliton solution at the most resonant state as the initial state, we can size the full-time evolution up to the nonlinear stage of the soliton with a small disturbance undergoing the instability by using the quasi-resonant solution. From these facts, we can expect that the instability occurred in the line soliton is relaxed by the emission of the periodic soliton and the periodic soliton resonance is inverse process of the instability.

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The Connection of the Degasperis–Procesi Equation with the Vakhnenko Equation

Vyacheslav O. VAKHNENKO † *and John PARKES* ‡

† *Institute for Geophysics, 32 Palladin Ave., 03680 Kyiv, Ukraine*
E-mail: *vakhnenko@bitp.kiev.ua*

‡ *University of Strathclyde, Richmond St., Glasgow G1 1XH, UK*
E-mail: *ejp@maths.strath.ac.uk*

Travelling-wave solutions of the Degasperis–Procesi equation (DPE) are investigated. The solutions are characterized by two parameters. Hump-like, loop-like and coshoidal periodic-wave solutions are found; hump-like, loop-like and peakon solitary-wave solutions are obtained as well. Hone and Wang showed a connection between the DPE and the Vakhnenko equation (VE). Comparing the solutions of the DPE and the VE, we observe that, for both equations at interaction of waves, there are three kinds of phaseshift that depend on the ratio of wave amplitudes. In particular, there is a case when two interacted waves have phaseshifts in the positive direction.

1 Introduction

This report deals with the Degasperis–Procesi equation (DPE) [1–3]

$$u_t - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx} \tag{1}$$

which is contained in the family of equations [1]

$$u_t - u_{txx} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{2}$$

(parameter b is constant). When $b = 2$, equation (2) reduces to the well-known Cammasa–Holm equation (CH) [4, 5]

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}. \tag{3}$$

Mikhailov and Novikov developed a powerful extension of the symmetry classification method [6], and applying this to the equation (2) they found that only the cases $b = 2$, 3 could possess infinitely many commuting symmetries, and so only these two cases are integrable. In a recent paper Degasperis, Holm and Hone [2] constructed a Lax pair of the equation (1), and hence proved the integrability of the Degasperis–Procesi equation. Unfortunately, there are no solutions of the direct problem and the inverse problem of the spectral equation from this Lax pair.

Hone and Wang [7] have shown the connection of the DPE (1) with the integrable Vakhnenko equation (VE) [8–10]

$$(u_t + uu_x)_x + u = 0. \tag{4}$$

Indeed, the transformation $x \to \varepsilon x - t/3\varepsilon$, $t \to \varepsilon t$, $u \to u - 1/3\varepsilon^2$ reduces the DPE to the VE when $\varepsilon \to 0$

$$[(u_t + uu_x)_x + u]_x = \varepsilon^2(u_t + 4uu_x). \tag{5}$$

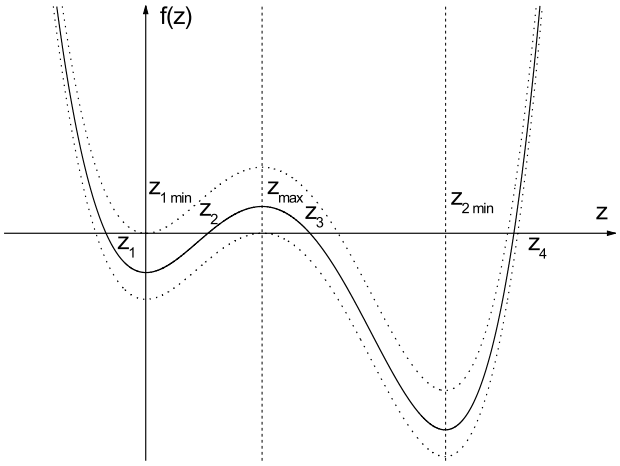

Figure 1. The polynomial $f(z)$. The interval of integration is between the roots z_2 and z_3.

Recently, we have investigated equation (4) by direct integration [8,9], by Hirota's method [11] as well as by the inverse scattering transform method [10]. In the context of this paper, it is of interest that the VE has two families of travelling-wave solutions (see Figs. 1, 2 in [8]); in particular, there is a soliton solution in loop-like form.

Comparing the properties of the solutions of the DPE (1) and the VE (4), we observe that for both equations under interaction of waves, there are three kinds of phaseshift that depend on the ratio of wave amplitudes [2, 12, 13]. In particular, there is a case when two interacted waves have phaseshifts in the positive direction.

However, it engages our attention that soliton solutions have not been observed for the DPE recently. If the VE has a loop-like solution, it follows that the DPE should admit a loop-like solution. In this paper we investigate the travelling-wave solutions of the DPE. The solutions are characterized by two parameters. Hump-like, loop-like and coshoidal periodic-wave solutions are found; hump-like, loop-like and peakon solitary-wave solutions are obtained as well.

2 Travelling-wave solutions

We consider the DPE (1). Restricting our attention to travelling waves, we introduce new variables

$$z = (u - v)/|v|, \qquad \eta = x - vt - x_0, \qquad \tau = |v|t, \tag{6}$$

where v and x_0 are arbitrary constants with $v \neq 0$. In this case the initial equation is reduced to the ODE

$$(zz_\eta)_{\eta\eta} = (4z + 3c)z_\eta \qquad \text{with} \qquad c := v/|v| = \pm 1. \tag{7}$$

After two integrations we have

$$(zz_\eta)^2 = f(z) \tag{8}$$

with

$$f(z) = z^4 + 2cz^3 + Az^2 + B = (z - z_1)(z - z_2)(z - z_3)(z - z_4),$$

where $f(z)$ is a polynomial of fourth order and is shown in Fig. 1.

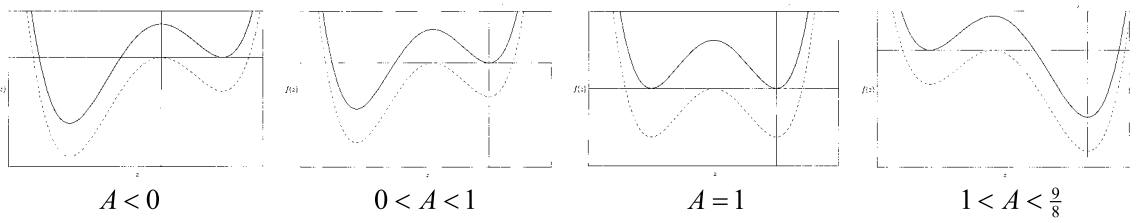

Figure 2. Four possible cases for the polynomial $f(z)$.

Analyzing equation (8), we observe that the roots of the polynomial $f(z)$ are real and the interval of integration is between the roots z_2 and z_3. We can write two forms of solution. The first form of solution is

$$z = \frac{z_2 - z_1 \operatorname{sn}^2(w|m)}{1 - n \operatorname{sn}^2(w|m)} \qquad \text{with} \qquad n = \frac{z_3 - z_2}{z_3 - z_1}, \tag{9}$$

where

$$w = p\zeta, \qquad p = \tfrac{1}{2}\sqrt{(z_4 - z_2)(z_3 - z_1)}, \qquad m = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_4 - z_2)(z_3 - z_1)}$$

and

$$\eta = [wz_1 + (z_2 - z_1)\Pi(n; w|m)]/p. \tag{10}$$

Here $\operatorname{sn}(w|m)$ is a Jacobian elliptic function and $\Pi(n; w|m)$ is the elliptic integral of the third kind. The second form of solution is

$$z = \frac{z_3 - z_4 \operatorname{sn}^2(w|m)}{1 - n \operatorname{sn}^2(w|m)} \qquad \text{with} \qquad n = \frac{z_3 - z_2}{z_4 - z_2} \tag{11}$$

and

$$\eta = [wz_4 + (z_4 - z_3)\Pi(n; w|m)]/p. \tag{12}$$

Hence the solution is given in parametric form with w as a parameter. The solutions are characterized by the two parameters A and B, or equivalently A and m.

3 Classification of solutions

The equation (7) is invariant under the transformation $z \to -z$, $c \to -c$; this corresponds to the transformation $u \to -u$, $v \to -v$. Hence we need to consider only the case $c = 1$ (i.e. $v > 0$).

Four cases are possible for the polynomial $f(z)$ corresponding to different ranges of values of A (see Fig. 2). Examples of the corresponding solutions are illustrated in Fig. 3. In the first column are periodic solutions with $m \neq 1$. In the second column are the corresponding limiting cases with $m = 1$. These are solitary wave solutions, except for the figure in the second line which is a periodic peakon solution.

4 Conclusion

Travelling-wave solutions of the Degasperis–Procesi equation are investigated. The solutions are characterized by two parameters. Hump-like, loop-like and coshoidal periodic-wave solutions are found; hump-like, loop-like and peakon solitary-wave solutions are obtained as well.

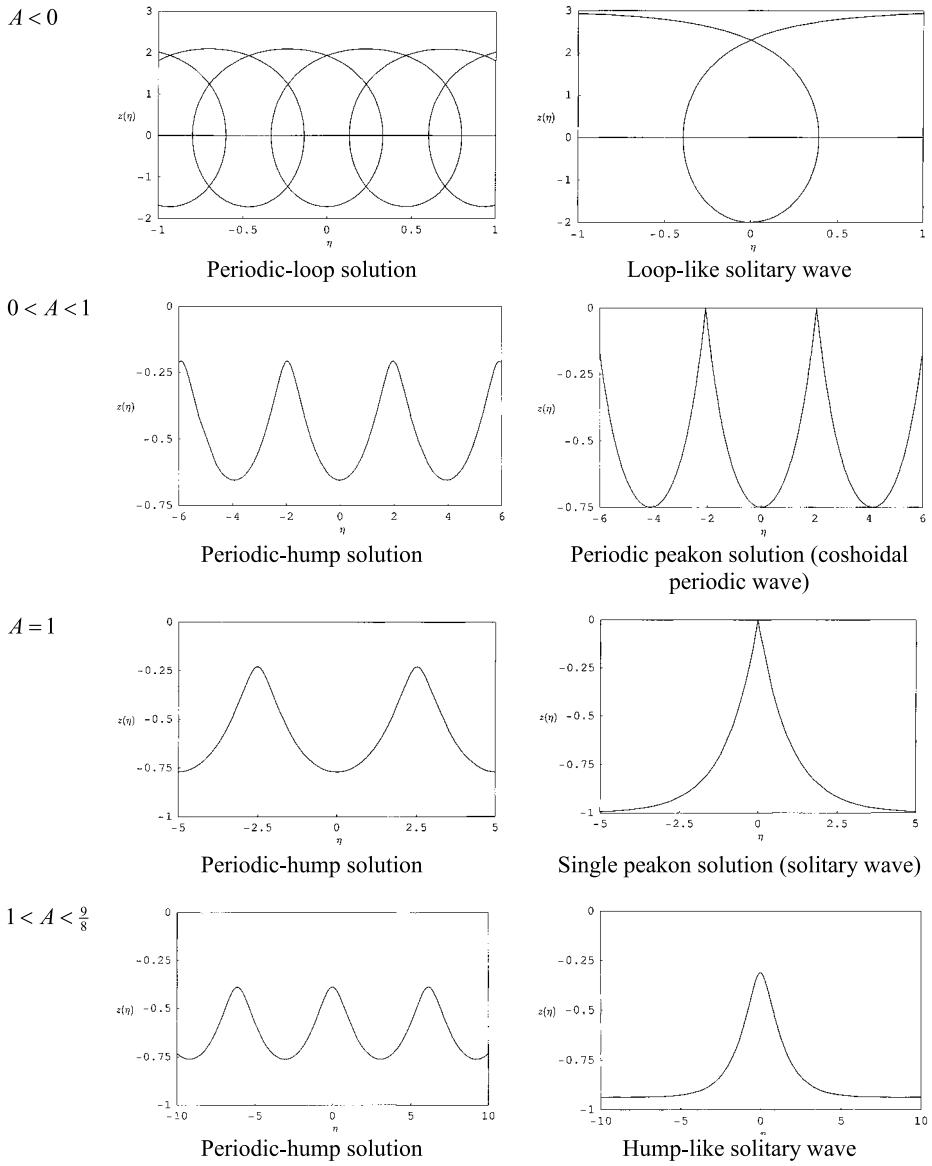

Figure 3. Travelling-wave solutions.

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Higher Conserved Quantities for the Multi-Centre Metrics

Galliano VALENT

Laboratoire de Physique Theorique et des Hautes Energies, Unité associée au CNRS UMR 7589, 2 Place Jussieu, 75251 Paris Cedex 05, France;
Centre de Physique Theorique, CNRS Luminy Case 907, F-13288 Marseille Cedex 9, France
E-mail: *gvalent@cpt.univ-mrs.fr*

The multi-centre metrics are a family of Euclidean solutions of the empty space Einstein equations with self-dual curvature. We consider the Hamiltonian dynamics generated by the geodesic flow along the $U(1)$ fibers and obtain several metrics in this class which do exhibit an extra conserved quantity quadratic in the momenta. The corresponding systems are shown to be classically integrable.

1 Introduction

The integrability of the geodesic flow for the multi-centre metrics began with the discovery of the generalized Runge–Lenz vector for the Taub–NUT metric [1] and the derivation of its Killing–Stäckel tensor in [2]. It was generalized to the Eguchi–Hanson metric in [3] where the Hamilton–Jacobi equation was separated. A further progress led to the integrability proof of the full 2-centre metric [2] which includes Taub–NUT and Eguchi–Hanson as particular cases. Despite these successes, a systematic analysis of the full family of the multi-centre metrics was still lacking. We will present a new approach which will determine which metrics, in this class, do exhibit a quadratic Killing–Stäckel tensor leading to classical integrability.

2 Basic facts on the multi-centre metrics

These Euclidean and four dimensional metrics on M_4 have at least one Killing vector $\widetilde{\mathcal{K}} = \partial_t$. Fibrating along this Killing vector, we have for metric

$$g = \frac{1}{V}\,(dt + \Theta)^2 + V\,d\vec{x}\cdot d\vec{x}, \qquad V = V(x), \qquad \Theta = \Theta_i(x)\,dx^i, \tag{1}$$

where the x^i are the coordinates of the subspace orthogonal to the $U(1)$ fibers. The metric (1) is a solution of the empty space Einstein equations provided that we have the monopole equation

$$dV = \star\, d\Theta, \tag{2}$$

where the Hodge star refers to the flat 3-dimensional metric. Notice that the integrability condition for the monopole equation is $\Delta V = 0$. So to each solution of Laplace equation in a 3-dimensional *flat* space there corresponds a Ricci-flat four-dimensional metric: we have an exact linearization of the empty space Einstein equations!

These metrics have a richer geometry since they are endowed with a triplet of complex structures, with associated closed 2-forms

$$\Omega_i^{(-)} = E_0 \wedge E_i - \frac{1}{2}\,\epsilon_{ijk}\,E_j \wedge E_k = (dt + \Theta) \wedge dx_i - \frac{1}{2}\,V\,\epsilon_{ijk}\,dx_j \wedge dx_k. \tag{3}$$

It follows that the multi-centre metrics are hyper-Kähler, hence Ricci-flat.

As shown by [4], in the hyper-Kähler case, any Killing vector has to be either tri-holomorphic or holomorphic with respect to the triplet of complex structures (3). For instance $\widetilde{\mathcal{K}}$ is tri-holomorphic and this means that

$$\underset{\widetilde{\mathcal{K}}}{\mathcal{L}}\,\Omega_i^{(-)} = 0, \qquad i = 1,2,3,$$

whereas for an holomorphic Killing vector K we would have

$$\underset{K}{\mathcal{L}}\,\Omega_1^{(-)} = \Omega_2^{(-)}, \qquad \underset{K}{\mathcal{L}}\,\Omega_2^{(-)} = -\Omega_1^{(-)}, \qquad \underset{K}{\mathcal{L}}\,\Omega_3^{(-)} = 0.$$

The following drawings display the potentials for the general 2-centre metrics. For generic parameters we have two Killing vectors (the subscripts H or T refer to its holomorphic or tri-holomorphic nature). For special values of the parameters the isometry algebra is enhanced to $u(2)$ leading to either Taub–NUT or to Eguchi–Hanson metrics.

$$\begin{cases} \bullet \quad V = v_0 + \dfrac{m_1}{|\vec{x} - c\vec{e}_z|} + \dfrac{m_2}{|\vec{x} + c\vec{e}_z|} \\ \bullet \quad \text{isometries :} \;\; u(1)_{\mathrm{H}} \oplus u(1)_{\mathrm{T}} \end{cases}$$

$m_1 = m_2 \to m$

$c \to 0$

Taub–NUT

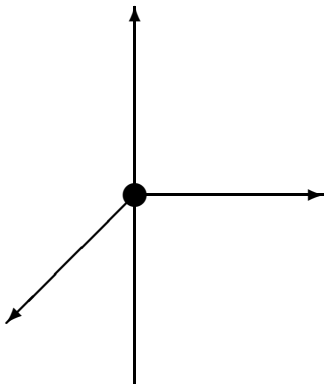
$$\begin{cases} \bullet \quad V = v_0 + \dfrac{m}{|\vec{x}|} \\ \bullet \quad \text{isometries :} \;\; su(2)_{\mathrm{H}} \oplus u(1)_{\mathrm{T}} \\ \bullet \quad a \;\text{ breaks }\; su(2)_{\mathrm{T}} \\ \bullet \quad \text{flat iff} \;\; v_0 = 0 \;\; \text{or} \;\; m = 0 \end{cases}$$

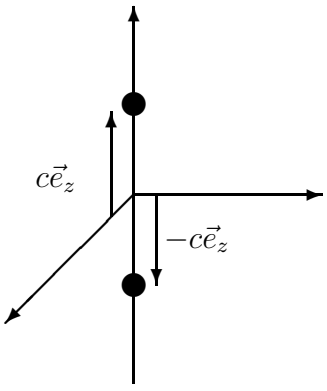
$m_1 = m_2 \to 1$

$v_0 \to 0$

Eguchi–Hanson

$$\begin{cases} \bullet \quad V = \dfrac{1}{|\vec{x} - c\vec{e}_z|} + \dfrac{1}{|\vec{x} + c\vec{e}_z|} \\ \bullet \quad \text{isometries :} \;\; u(1)_{\mathrm{H}} \oplus su(2)_{\mathrm{T}} \\ \bullet \quad c \;\text{ breaks }\; su(2)_{\mathrm{H}} \\ \bullet \quad \text{flat iff} \;\; c = 0 \end{cases}$$





2.1 Geodesic flow

The geodesic flow is the Hamiltonian flow of the metric considered as a function on the cotangent bundle of M_4. Using the coordinates (t, x_i) we will write a cotangent vector as

$$\Pi_i\, dx_i + \Pi_0\, dt,$$

and the symplectic form is

$$dx_i \wedge d\Pi_i + dt \wedge d\Pi_0.$$

We take for Hamiltonian

$$H = \frac{1}{2}\, g^{\mu\nu}\, \Pi_\mu\, \Pi_\nu == \frac{1}{2}\left(\frac{1}{V}\,(\Pi_i - \Pi_0\,\Theta_i)^2 + V\,\Pi_0^2\right).$$

For geodesics orthogonal to the $U(1)$ fibers and affinely parametrized by λ the equations for the flow allow on the one hand to express the velocities

$$\begin{aligned} \dot t &\equiv \frac{dt}{d\lambda} = \frac{\partial H}{\partial \Pi_0} = \left(V + \frac{\Theta^2}{V}\right)\Pi_0 - \frac{\Theta_i \Pi_i}{V}, \\ \dot x_i &\equiv \frac{dx_i}{d\lambda} = \frac{\partial H}{\partial \Pi_i} = \frac{1}{V}\, p_i, \qquad p_i = \Pi_i - \Pi_0\,\Theta_i, \end{aligned} \tag{4}$$

and on the other hand to get the dynamical evolution equations

$$\dot\Pi_0 = -\frac{\partial H}{\partial t} = 0, \qquad q \equiv \Pi_0 = \frac{(\dot t + \Theta_i\, \dot x_i)}{V}, \tag{5}$$

and

$$\dot\Pi_i = -\frac{\partial H}{\partial x_i} \quad \Longrightarrow \quad \dot p_i = \left(\frac{H}{V} - q^2\right)\partial_i V + \frac{q}{V}\,(\partial_i\Theta_s - \partial_s\Theta_i)\, p_s. \tag{6}$$

Relation (5) expresses the conservation of the charge q, a consequence of the $U(1)$ isometry of the metric. For the multi-centre metrics, use of relation (2) brings the equations of motion (6) to the nice form

$$\dot{\vec p} = \left(\frac{H}{V} - q^2\right)\vec\nabla\, V + \frac{q}{V}\,\vec p \wedge \vec\nabla V. \tag{7}$$

The conservation of the energy

$$H = \frac{1}{2}\left(\frac{p_i^2}{V} + q^2\, V\right) = \frac{V}{2}\left(\dot x_i^2 + q^2\right) = \frac{1}{2}\, g_{\mu\nu}\, \dot x^\mu\, \dot x^\nu$$

is obvious since it expresses the constancy of the length of the tangent vector $\dot x^\mu$ along a geodesic.

Hence at this stage we have already obtained two conserved quantities: $q = \Pi_0$ and H. Two more quantities are needed for integrability.

2.2 Killing–Stäckel tensors and their conserved quantities

A Killing–Stäckel (KS) tensor is a symmetric tensor $S_{\mu\nu}$ which satisfies

$$\nabla_{(\mu}\, S_{\nu\rho)} = 0. \tag{8}$$

Let us observe that if K and L are two (possibly different) Killing vectors their symmetrized tensor product $K_{(\mu}\, L_{\nu)}$ is a KS tensor. So we will define *irreducible* KS tensors as the ones

which cannot be written as linear combinations, with constant coefficients, of symmetrized tensor products of Killing vectors.

For a given KS tensor $S_{\mu\nu}$ the quadratic form of the velocities:

$$\mathcal{S} = S_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu} \tag{9}$$

is preserved by the geodesic flow.

In all what follows we will look for KS tensors preserved by Lie dragging along the tri-holomorphic Killing vector:

$$\underset{\widetilde{\mathcal{K}}}{\mathcal{L}}\, S_{\mu\nu} = 0, \qquad \widetilde{\mathcal{K}} = \partial_t,$$

and we will consider *generic* values of H and $q \neq 0$.

Let us examine more closely the structure of the conserved quantity induced by such a KS tensor. Expanding the sums in (9) and using the relations (4) we obtain the following structure for the conserved quantity:

$$\mathcal{S} = A_{ij}(x_k)\,p_i\,p_j + 2q\,B_i(x_k)\,p_i + C(x_k), \tag{10}$$

where the various unknown functions, as a consequence of our hypotheses, are independent of the coordinate on the $U(1)$ fiber.

It is interesting to notice that the knowledge of $\mathcal{S}$ is *equivalent* to the knowledge of the KS tensor.

Imposing the conservation of $\mathcal{S}$ under the geodesic flow gives the following equations[1]

$$\begin{aligned}
&a) \quad q \cdot \underset{B}{\mathcal{L}}\, V = 0,\\
&b) \quad \partial_{(k}\,A_{ij)} = 0,\\
&c) \quad q(\partial_{(i}\,B_{j)} - A_{s(i}\,\epsilon_{j)su}\,\partial_u V) = 0,\\
&d) \quad \partial_i\,C + 2\left(H - q^2\,V\right)A_{is}\,\partial_s\,V - 2\,q^2\,\epsilon_{ist}\,B_s\,\partial_t V = 0.
\end{aligned} \tag{11}$$

The relation (11a) calls for a spatial Killing vector for the potential V. However it could well be that V has no symmetry at all. In this event we must have $B_i = 0$. Then relation (11c) can be written

$$[A, R] = 0, \qquad (R)_{ij} = \epsilon_{isj}\,\partial_s\,V.$$

Since V has no Killing the matrix R is a generic matrix in the Lie algebra $so(3)$. By Schur lemma it follows that A has to be proportional to the identity matrix and this does trivialize the corresponding conserved quantity $\mathcal{S}$. So the potential must have at least one extra spatial Killing vector L (notice that L lifts up to an isometry of the 4 dimensional metric).

The full discussion and integration of the system (11) has been worked out in [5] and the results can be classified according to wether the extra Killing vector is holomorphic or tri-holomorphic. For each potential the detailed form of the conserved quantity induced by the Killing–Stäckel tensor has been determined. Let us state these results.

3 One extra holomorphic Killing vector

We can take for extra spatial Killing vector $\widehat{\mathcal{L}} = x\partial_y - y\partial_x$. Then the metrics with the following potentials will exhibit a quadratic Killing–Stäckel tensor:

[1] Since H and q take generic values, the quantity $H - q^2V$ does not vanish identically.

1. The 2-centre metric with

$$V = v_0 + \frac{m_1}{r_+} + \frac{m_2}{r_-}, \qquad r_\pm = \sqrt{x^2 + y^2 + (z \pm c)^2}.$$

The conserved quantity is

$$\mathcal{S}_I = \vec{L}^{\,2} + c^2\, p_z^2 + 2\, qc\, \Delta\, L_z + 2cz\, \Delta \left(H - q^2 V\right) - q^2 r^2 \Delta^2, \qquad r^2 = x^2 + y^2 + z^2,$$

with

$$L_i = \epsilon_{ijk}\, x_j\, p_k, \qquad \Delta = \frac{m_1}{r_+} - \frac{m_2}{r_-}.$$

which we have checked to be in full agreement with [2].

2. A first dipolar breaking of Taub–NUT with

$$V = v_0 + \frac{m}{r} + F\frac{z}{r^3}, \qquad r = \sqrt{x^2 + y^2 + z^2},$$

which is, in fact, the singular limit $c \to 0$ of the previous case. The conserved quantity is

$$\mathcal{S}_{II} = \vec{L}^{\,2} - 2\, q\, \frac{F}{r}\, L_z - 2F\, \frac{z}{r}\, \left(H - q^2 v_0\right) + q^2\, F^2\, \frac{\left(x^2 + y^2\right)}{r^4}.$$

3. A second dipolar breaking of Taub–NUT with

$$V = v_0 + \frac{m}{r} + Ez.$$

The conserved quantity is

$$\mathcal{S}_{III} = (\vec{p} \wedge \vec{L})_z - q\, \left(\frac{m}{r} - E\, z\right) L_z - 2U\, \left(H - q^2 v_0\right) - 2q^2\, mE\, \frac{\left(x^2 + y^2\right)}{r}.$$

In the limiting case of Taub–NUT, we have in fact 3 extra Killing vectors, which are holomorphic. It follows that we will have 3 Killing–Stäckel tensors, giving rise to the generalized Runge–Lenz vector

$$\vec{\mathcal{S}} = \vec{p} \wedge \vec{J} + m\, \left(q^2 v_0 - H\right) \frac{\vec{r}}{r}, \qquad \vec{J} = \vec{L} + q\, \frac{m}{r}\, \vec{r},$$

discovered by Gibbons and Manton [1]. The conserved quantity $\mathcal{S}_{III}$ does reduce to $\mathcal{S}_z$ in the limit $E \to 0$.

All of these metrics have at least the 4 conserved quantities

$$q = \Pi_0, \qquad H, \qquad L^\mu \Pi_\mu, \qquad \mathcal{S},$$

which are easily checked to be in involution, thus establishing the Liouville integrability of their corresponding geodesic flows.

Let us consider the class of metrics with an extra tri-holomorphic Killing.

4 One extra tri-holomorphic Killing

Here we can take the Killing vector to be $\widehat{L} = \partial_z$ and since the potential does depend only on the coordinates x and y we will state the results using the complex coordinate $z = x + iy$. Then the metrics with the following potentials will exhibit a quadratic Killing–Stäckel tensor:

1. First case:

$$V = v_0 + m\frac{z}{\sqrt{z^2+c^2}} + \text{c.c.},$$

where c.c. means complex conjugation. The conserved quantity is

$$\mathcal{S}_1 = L_z^2 - c^2\,\Pi_x^2 - 2c^2F\,\Pi_0\,\Pi_z + c^2(2v_0U + D)\Pi_0^2 - 2c^2UH, \qquad c \neq 0,$$

with

$$U + iF = -m\,\frac{z+\overline{z}}{\sqrt{z^2+c^2}}, \qquad D = -2|m|^2\,\frac{\left(z^2+\overline{z}^2+|z|^2+c^2\right)}{|\sqrt{z^2+c^2}|^2}.$$

The Bianchi VII_0 and Bianchi VI_0, which appear as particular cases of these metrics for $v_0 = 0$ were already known to be integrable from [6].

2. Second case:

$$V = v_0 + \frac{m}{2z^2} + \text{c.c.}, \qquad m \in \mathbb{C},$$

with the conserved quantity

$$\mathcal{S}_2 = L_z^2 - 2F\,\Pi_0\,\Pi_z + 2v_0U\,\Pi_0^2 - 2UH, \qquad U + iF = m\,\frac{\overline{z}}{z}.$$

3. Third case:

$$V = v_0 + \frac{m}{\sqrt{z}} + \text{c.c.}, \qquad m \in \mathbb{C},$$

with the conserved quantity

$$\mathcal{S}_3 = \Pi_y\,L_z - 2F\,\Pi_0\,\Pi_z + (2v_0U + D)\Pi_0^2 - 2UH,$$

and

$$U + iF = -\frac{m}{2}\,\frac{z-\overline{z}}{\sqrt{z}}, \qquad D = |m|^2\,\frac{z+\overline{z}}{|\sqrt{z}|^2}.$$

4. The fourth case:

$$V = v_0 + m(z+\overline{z})/2, \qquad m \in \mathbb{R},$$

is super-integrable, with two conserved quantities

$$\mathcal{S}_4^{(1)} = \Pi_y^2 + (\Pi_z - my\,\Pi_0)^2, \qquad \mathcal{S}_4^{(2)} = \Pi_x\,\Pi_y - V\,\Pi_0(\Pi_z - my\,\Pi_0) - my\,H.$$

The parameter v_0 is always real.

Here too, all of these metrics have at least the 4 conserved quantities

$$q = \Pi_0, \qquad H, \qquad L^\mu\Pi_\mu, \qquad \mathcal{S},$$

which are easily checked to be in involution, thus establishing the Liouville integrability of their corresponding geodesic flows.

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Construction of Special Solutions for Nonintegrable Dynamical Systems with the Help of the Painlevé Analysis

Sergey Yu. VERNOV

Skobeltsyn Institute of Nuclear Physics, Moscow State University, Vorob'evy Gory, 119992 Moscow, Russia
E-mail: *svernov@theory.sinp.msu.ru*

The generalized Hénon–Heiles system has been considered. In two nonintegrable cases with the help of the Painlevé test new special solutions have been found as Laurent series, depending on three parameters. The obtained series converge in some ring. One of parameters determines the singularity point location, other parameters determine coefficients of series. For some values of these parameters the obtained Laurent series coincide with the Laurent series of the known exact solutions. The Painlevé test can be used not only to construct local solutions as the Laurent series but also to find elliptic solutions.

1 The Painlevé property and integrability

A Hamiltonian system in a $2s$-dimensional phase space is called *completely integrable* (Liouville integrable) if it possesses s independent integrals that commute with respect to the associated Poisson bracket. When this is the case, the equations of motion are (in principal, at least) separable and solutions can be obtained by the method of quadratures.

When some mechanical problem is studied, time is assumed to be real, whereas the integrability of motion equations depends on the behavior of their solutions as functions of complex time. S.V. Kovalevskaya was the first who proposed [1] to interpret time as a complex variable and to require that solutions of mechanical problems have to be single-valued functions meromorphic in the entire complex plane. This idea gave a remarkable result: S.V. Kovalevskaya discovered a new integrable case (nowadays known as the Kovalevskaya's case) for the motion of a heavy rigid body about a fixed point [1] (see also [2,3]). The Kovalevskaya's result demonstrated that the analytic theory of differential equations can be fruitfully applied to mechanical and physical problems. The important stage of development of this theory was the Painlevé classification of ordinary differential equations (ODEs) with respect to the types of singularities of their solutions [4].

Let us formulate the Painlevé property for ODEs. Solutions of a system of ODEs regarded as analytic functions may have isolated singularity points [5,6]. A singularity point of a solution is said to be *critical* (as opposed to *noncritical*) if the solution is multivalued (single-valued) in its neighborhood and *movable* if its location depends on initial conditions. The *general solution* of an ODE of order N is the set of all solutions mentioned in the existence theorem of Cauchy, i.e. determined by the initial values. It depends on N arbitrary independent constants. A *special solution* is any solution obtained from the general solution by giving values to the arbitrary constants. A *singular solution* is any solution which is not special, i.e. which does not belong to the general solution.

Definition 1. A system of ODEs has ***the Painlevé property*** if its general solution has no movable critical singularity point [4].

Investigations of many dynamical systems [7] show that systems with the Painlevé property are completely integrable. Arguments, which clarify the connection between the Painlevé analysis and the existence of motion integrals, are presented in [8, 9]. At the same time the integrability of an arbitrary system with the Painlevé property has yet to be proved. There is no algorithm for construction of additional integrals by the Painlevé analysis. It is easy to give an example of an integrable system without the Painlevé property [10]. The system with the Hamiltonian $H = \frac{1}{2}p^2 + f(x)$, where $f(x)$ is a polynomial whose power is not lower than five, is trivially integrable, but its general solution is not a meromorphic function.

The study of complex-time singularities is a useful tool for analysis of not only integrable systems, but also of chaotic dynamics [11]. The Painlevé analysis can be connected to the normal form theory [12].

The ***Painlevé test*** is any algorithm that checks some necessary conditions for a differential equation to have the Painlevé property. The original algorithm, developed by P. Painlevé and used by him to find all the second order ODEs with the Painlevé property, is known as the α-method. The method of S.V. Kovalevskaya is not as general as the α-method, but much simpler. The remarkable property of this test is that it can be checked in a finite number of steps. This test can only detect the occurrence of logarithmic and algebraic branch points. Up to the present there is no general finite algorithmic method to detect the occurrence of essential singularities. Different variants of the Painlevé test are compared in [13, R. Conte paper].

Developing the Kovalevskaya method [1] further, M.J. Ablowitz, A. Ramani and H. Segur constructed a new algorithm of the Painlevé test for ODEs [14]. They also were the first to point out the connection between the nonlinear partial differential equations (PDEs) that are solvable by the inverse scattering transform method, and ODEs with the Painlevé property. Subsequently the Painlevé property for PDEs was defined and the corresponding Painlevé test (the WTC procedure) was constructed [15, 16] (see also [13, 17]). With the help of this test it was found that all PDEs solvable by inverse scattering transforms, have the Painlevé property, maybe after some change of variables. For many integrable PDEs, for example, the Korteweg–de Vries equation [7], the Bäcklund transformations and the Lax representations result from the WTC procedure [16, 18]. For certain nonintegrable PDEs special solutions were constructed using this algorithm [19, 20].

The algorithm for finding of special solutions for ODEs into the form of a finite expansion in powers of unknown function $\varphi(t - t_0)$ was constructed in [21, 22]. The function $\varphi(t - t_0)$ and coefficients have to satisfy some system of ODE, often simpler than an initial one. This method has been used to construct exact special solutions for some nonintegrable ODEs [23, 24]. With the help of the perturbative Painlevé test [17] a four-parameter generalization of an exact three-parameter solution of the Bianchi IX cosmological model was constructed [25].

2 The Hénon–Heiles Hamiltonian

In the 1960s the models of the star motion in an axial-symmetric and time-independent potentials were developed to show either existence or absence of the third integral for some polynomial potentials. Due to the symmetry of the potential the considered system is equivalent to two-dimensional one. To clarify the question of the existence of the third integral Hénon and Heiles [26] considered behavior of numerically integrated trajectories. Emphasizing that their choice does not proceed from experimental data, they have proposed the Hamiltonian

$$H = \frac{1}{2}\left(x_t^2 + y_t^2 + x^2 + y^2\right) + x^2 y - \frac{1}{3}y^3,$$

because on the one hand, it is analytically simple; this makes the numerical computations of trajectories easy; on the other hand, it is sufficiently complicated to give trajectories that are

far from trivial. Indeed, for low energies the Hénon–Heiles system appears to be integrable, in so much as trajectories (numerically integrated) always lay on well-defined two-dimensional surfaces. On the other hand, for high energies many of these integral surfaces are destroyed, it points to absence of the third integral.

The generalized Hénon–Heiles system is described by the Hamiltonian:

$$H = \frac{1}{2}\left(x_t^2 + y_t^2 + \lambda x^2 + y^2\right) + x^2 y - \frac{C}{3}\, y^3 \tag{1}$$

and the corresponding system of the motion equations:

$$\begin{aligned} x_{tt} &= -\lambda x - 2xy, \\ y_{tt} &= -y - x^2 + Cy^2, \end{aligned} \tag{2}$$

where $x_{tt} \equiv \frac{d^2x}{dt^2}$ and $y_{tt} \equiv \frac{d^2y}{dt^2}$, λ and C are numerical parameters.

The generalized Hénon–Heiles system is a model, not only actively investigated by various mathematical methods (see [27] and references therein), but also widely used in physics, in particular, in gravitation [28, 29]. The models, described by the Hamiltonian (1) with some additional nonpolynomial terms, are actively studied as well [30–32].

Due to the Painlevé analysis the following integrable cases of (2) were found:

(i) $C = -1$, $\lambda = 1$,
(ii) $C = -6$, λ is an arbitrary number,
(iii) $C = -16$, $\lambda = 1/16$.

The general solutions in the analytic form are known only in integrable cases [31, 32], in other cases not only four-, but even three-parameter exact solutions have yet to be found. In nonintegrable cases local four-parameter solutions as converging psi-series solutions were found [33] for all values of the parameter C, except $C = -2$. The Ablowitz–Ramani–Segur algorithm of the Painlevé test appears very useful to find such values of parameter at which three-parameter solutions can be expanded into formal Laurent series and to construct these local solutions. The knowledge of local solutions can be used to find solutions in the analytical form.

Let us assume that the behavior of solutions in a sufficiently small neighborhood of the singularity point is algebraic, i.e., x and y tend to infinity as

$$x = a_\alpha (t - t_0)^\alpha \qquad \text{and} \qquad y = b_\beta (t - t_0)^\beta,$$

where α, β, a_α and b_β are some constants.

The Painlevé test gives all information about behavior of solutions in the neighborhood of the singularity point (see, for example, [7]). There exist two possible variants of dominant behavior and resonance structure of solutions of the generalized Hénon–Heiles system [7, 33]:

Case 1:	*Case 2*: ($\beta < \Re e(\alpha)$)
$\alpha = -2$,	$\alpha = \frac{1 \pm \sqrt{1-48/C}}{2}$,
$\beta = -2$,	$\beta = -2$,
$a_\alpha = \pm 3\sqrt{2+C}$,	$a_\alpha = c_1$ (an arbitrary number),
$b_\beta = -3$,	$b_\beta = \frac{6}{C}$,
$r = -1,\ 6,\ \frac{5}{2} - \frac{\sqrt{1-24(1+C)}}{2},\ \frac{5}{2} + \frac{\sqrt{1-24(1+C)}}{2}$	$r = -1,\ 0,\ 6,\ \mp\sqrt{1 - \frac{48}{C}}$

The values of r denote resonances: $r = -1$ corresponds to arbitrary parameter t_0; $r = 0$ (in the *Case 2*) corresponds to arbitrary parameter c_1. Other values of r determine powers of t, to be exact, $t^{\alpha+r}$ for x and $t^{\beta+r}$ for y, at which new arbitrary parameters can appear.

For integrability of system (2) all values of r have to be integer and all systems with zero determinants have to have solutions at any values of free parameters. It is possible only in the integrable cases (i)–(iii).

Those values of C, at which r are integer numbers either only in the *Case 1* or only in the *Case 2*, are of interest for search of special three-parameter solutions. Those cases where an additional negative resonance is present likely correspond to singular, rather the general, solutions [7].

Let us consider all cases, when there exist special (not singular) solutions, representable as three-parameter Laurent series (maybe, multiplied on $\sqrt{t-t_0}$). From the requirement that all values of r but one are integer and nonnegative numbers, we obtain the following values of C: $C=-1$, $C=-4/3$ (the *Case 1*), $C=-16/5$, $C=-6$, $C=-16$ (the *Case 2*, $\alpha=\frac{1-\sqrt{1-48/C}}{2}$) and $C=-2$, when two types of singular behavior coincide.

At $C=-2$ (in the *Case 1*) $a_\alpha=0$. This is the consequence of the fact that, contrary to our assumption, the behavior of the solution in the neighborhood of a singularity point is not algebraic, because its dominant term includes logarithm [7]. At $C=-6$ and any value of λ exact four-parameter solutions are known. In cases $C=-1$ and $C=-16$ substitution of unknown function as Laurent series gives the equations in λ: accordingly $\lambda=1$ and $\lambda=1/16$; hence, in nonintegrable cases special three-parameter local solutions have to include logarithmic terms. Single-valued three-parameter special solutions can exist only in two nonintegrable cases, at $C=-16/5$ and at $C=-4/3$.

3 New solutions

Let us consider the Hénon–Heiles system with $C=-16/5$. In the *Case 1* some values of r are not rational. To find special three-parameter solutions we consider the *Case 2*. In this case $\alpha=-3/2$ and $r=-1,\ 0,\ 4,\ 6$, hence, in the neighborhood of the singularity point t_0 we have to seek x in such a form that x^2 can be expanded into Laurent series, beginning with $(t-t_0)^{-3}$. Let $t_0=0$, substituting

$$x=\sqrt{t}\left(c_1t^{-2}+\sum_{j=-1}^{\infty}a_jt^j\right)\qquad\text{and}\qquad y=-\frac{15}{8}t^{-2}+\sum_{j=-1}^{\infty}b_jt^j \tag{3}$$

in (2), we obtain the following sequence of linear system in a_k and b_k:

$$\begin{aligned}&\left(k^2-4\right)a_k+2c_1b_k=-\lambda a_{k-2}-2\sum_{j=-1}^{k-1}a_jb_{k-j-2},\\&((k-1)k-12)b_k=-b_{k-2}-\sum_{j=-2}^{k-1}a_ja_{k-j-3}-\frac{16}{5}\sum_{j=-1}^{k-1}b_jb_{k-j-2}.\end{aligned} \tag{4}$$

The determinants of the systems (4) corresponding to $k=2$ and $k=4$ are equal to zero. To determine a_2 and b_2 we have the following system:

$$\begin{aligned}&c_1\big(557056c_1^8+(15552000\lambda-4860000)c_1^4+864000000b_2\\&\qquad+108000000\lambda^2-67500000\lambda+10546875\big)=0,\\&818176c_1^8+(15660000\lambda-4893750)c_1^4-810000000b_2-6328125=0.\end{aligned} \tag{5}$$

It is easy to see that this system contains no terms proportional to a_2, therefore, a_2 is the new constant of integration. We discard the solution with $c_1=0$ and obtain the system in c_1^4

and b_2. System (5) has solutions only if

$$c_1^4 = \frac{1125\left(525 - 1680\lambda \pm 4\sqrt{35(2048\lambda^2 - 1280\lambda + 387)}\right)}{167552}.$$

We obtain new constant of integration a_2, but we must fix c_1, so number of constants of integration is equal to 2. It is easy to verify that b_4 is an arbitrary parameter, because the corresponding system is equivalent to one linear equation. System (2) is invariant under exchange x to $-x$, so we obtain four different local solutions which depend on three parameters, namely t_0, a_2 and b_4. With the help of some computer algebra system, for example, **REDUCE** [34], these solutions can be obtained with arbitrary accuracy. For the case $\lambda = 1/9$ the obtained Laurent series are presented in [35].

At $C = -4/3$ the situation is similar. In the *Case 1* we have $r = -1,\ 1,\ 4,\ 6$. Substituting

$$x = \sqrt{6}t^{-2} + \sum_{k=-1}^{\infty} d_k t^k \qquad \text{and} \qquad y = -3t^{-2} + \sum_{k=-1}^{\infty} f_k t^k \tag{6}$$

in system (2), we receive a sequence of linear systems in d_k and f_k:

$$\begin{aligned}
&((k-1)k-6)d_k + 2\sqrt{6}f_k = -\lambda d_{k-2} - 2\sum_{j=-1}^{k-1} d_j f_{k-j-2},\\
&2\sqrt{6}d_k + ((k-1)k-8)f_k = -f_{k-2} - \sum_{j=-1}^{k-1} d_j d_{k-j-2} - \frac{4}{3}\sum_{j=-1}^{k-1} f_j f_{k-j-2}.
\end{aligned} \tag{7}$$

The determinants of the systems (7) corresponding to $k = -1,\ 2,\ 4$ are equal to zero. The first system ($k = -1$) always has infinite number of solutions and f_{-1} is a parameter. We have to fix this parameter to solve the system corresponding to $k = 2$. This system has solutions only if

$$f_{-1}^2 = \frac{105 - 140\lambda \pm \sqrt{7\big(1216\lambda^2 - 1824\lambda + 783\big)}}{385} \qquad \text{or} \qquad f_{-1} = 0.$$

At $k = 4$ system (7) is reduced to one equation. Thus, at $C = -4/3$ we have five three-parameter (t_0, f_2 and f_4) solutions.

The convergence of all the Laurent series solutions on some real time interval have been proved in [33]. For the obtained solutions it is easy to find conditions at which the series converge at $0 < |t| \leqslant 1 - \varepsilon$, where ε is any positive number. Our series converge in the above-mentioned ring, if $\exists\, N$ and $\exists\, M$ such that $\forall\, n > N \ \ |a_n| \leqslant M$ and $|b_n| \leqslant M$. Let $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for all $-1 < n < k$, then (in the case $C = -16/5$) from (4) we obtain:

$$|a_k| \leqslant \frac{2M(k+1) + |\lambda| + 2|c_1|}{|k^2 - 4|}M, \qquad |b_k| \leqslant \frac{21Mk + 26M + 5}{5|k^2 - k - 12|}M.$$

It is easy to see that there exists such N that if $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for $-1 \leqslant n \leqslant N$, then $|a_n| \leqslant M$ and $|b_n| \leqslant M$ for $-1 \leqslant n < \infty$. So one can prove the convergence, analyzing values of a finite number of the first coefficients of series. For $C = -4/3$ it is easy to obtain the analogous result.

4 Global single-valued solutions

We have found some local three-parameter solutions. To seek for the global single-valued solutions we transform system (2) into the fourth order equation [36, 37]:

$$y_{tttt} = (2C-8)y_{tt}y - (4\lambda+1)y_{tt} + 2(C+1)y_t^2 + \frac{20C}{3}y^3 + (4C\lambda - 6)y^2 - 4\lambda y - 4H. \tag{8}$$

Point out that the energy of the system H and initial data y_0, y_{0t}, y_{0tt} and y_{0ttt} are dependent. If $y_{0tt} = Cy_0^2 - y_0$, then there exists the connection $y_{0ttt} = 2Cy_0y_{0t} - y_{0t}$, in opposite case H is a function of initial data.

There are some reasons to seek three-parameter solutions of equation (8) in terms of elliptic functions. In 1999 E.I. Timoshkova [37] found that the general solution of the following equation:

$$y_t^2 = \tilde{\mathcal{A}}y^3 + \tilde{\mathcal{B}}y^2 + \tilde{\mathcal{C}}y + \tilde{\mathcal{D}} + \tilde{\mathcal{G}}y^{5/2} + \tilde{\mathcal{E}}y^{3/2}$$

with some values of constants $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{D}}$, $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{E}}$, is one-parameter solution of equation (8) in each of two above-mentioned nonintegrable cases ($C = -4/3$ or $C = -16/5$, λ is an arbitrary number). If $\tilde{\mathcal{G}} = 0$ and $\tilde{\mathcal{E}} = 0$ we obtain the well-known solutions in terms of the Weierstrass elliptic function. Solutions with $\tilde{\mathcal{G}} \neq 0$ or $\tilde{\mathcal{E}} \neq 0$ are derived only at $\tilde{\mathcal{D}} = 0$, therefore, substitution $y(t) = \varrho(t)^2$ gives:

$$\varrho_t^2 = \frac{1}{4}\big(\tilde{\mathcal{A}}\varrho^4 + \tilde{\mathcal{G}}\varrho^3 + \tilde{\mathcal{B}}\varrho^2 + \tilde{\mathcal{E}}\varrho + \tilde{\mathcal{C}}\big). \tag{9}$$

Two-parameter solutions $y(t) = \varrho(t)^2 + P_0$, where P_0 is an arbitrary constant and $\varrho(t)$ satisfies equation (9), were obtained in [38,39]. These solutions are the following elliptic functions

$$y(t - t_0) = \left(\frac{a\wp(t - t_0) + b}{c\wp(t - t_0) + d}\right)^2 + P_0, \qquad ad - bc = 1, \tag{10}$$

where $\wp(t - t_0)$ is the Weierstrass elliptic function, a, b, c and d are some constants. The parameter P_0 defines the energy of the system. There exist two different elliptic solutions for each possible pair of values of C and λ.

Let us consider the three-parameter solutions that were obtained at $C = -4/3$. If we choose $f_{-1} = 0$, then we obtain the solution which generalizes the well-known two-parameter solution in terms of Weierstrass elliptic functions. Other solutions generalize two-parameter solutions, obtained in [39]. The coefficient f_{-1} is a residue of y. The sum of residues of an elliptic function in its parallelogram of periods has to be zero [40], hence, two local solutions with opposite signs of f_{-1} correspond to one global elliptic solution. The obtained local three-parameter solution generalizes the Laurent series of the two-parameter elliptic solutions in the form (10).

For $C = -16/5$ we obtained four local solutions, which generalize two global elliptic solutions in the form (10). So, each obtained local three-parameter solution generalize the Laurent series of some two-parameter elliptic solution, and we can assume that unknown global three-parameter solutions are elliptic functions.

Of course, solutions, which are single-valued in the neighborhood of one singularity point, can be multivalued in the neighborhood of another singularity point. So, we can only assume that global three-parameter solutions are single-valued. If we assume this and, moreover, that these solutions are elliptic functions (or some degenerations of them), then we can seek them as solutions of some polynomial first order equations. The classical theorem, which was established by Briot and Bouquet [41], proves that if the general solution of the autonomous polynomial first order ODE is single-valued, then this solution is either an elliptic function, or a rational function of $e^{\gamma x}$, γ being some constant, or a rational function of x. Note that the third case is a degeneracy of the second one that in its turn is a degeneracy of the first one. At the same time, there exist elementary functions, for example, the function $f(t) = t + \sin(t)$ that are not solutions of any first order polynomial ODE.

A second result, of immediate practical use, is due to P. Painlevé [4]. He has proved that if the general solution of the autonomous polynomial first order ODE is single-valued, then the necessary form of this ODE is

$$F(y, y_t) \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} h_{jk} y^j {y_t}^k = 0, \qquad h_{0m} = 1, \tag{11}$$

in which m is a positive integer number and the h_{jk} are constants.

In 2003 R. Conte and M. Musette have proposed a new method to find elliptic solutions [42]. This method is based on the Painlevé test and uses the Laurent series expansion to find the analytic form of elliptic solutions. Rather than substitute the first-order equation (11) into equation (8) one can substitute the found Laurent series of solutions of equation (8), for example, either solution (3) or solution (6), into equation (11) and obtain a linear system in h_{jk}. This method is more powerful than the traditional method and allows in principle to find all elliptic solutions. I hope that the use of this method allows finding of the three-parameter elliptic solutions.

5 Conclusion

Using the Painlevé analysis one can not only find integrable cases of dynamical systems, but also construct special solutions in nonintegrable cases.

We have found the special solutions of the Hénon–Heiles system with $C = -16/5$ and $C = -4/3$ as Laurent series depending on three parameters. For some values of these parameters the obtained solutions coincide with the known exact periodic solutions. There are no obstacles to existence of three-parameter single-valued solutions, so the probability of finding of exact three-parameter solutions that generalize the solutions obtained in [37, 39] is high.

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Explicitly Reducible Integrable Natural Systems with the Integral of Full Momentum

Andrij VUS and Sergiy PIDKUYKO

Franko Lviv National University, 1 Universytets'ka Str., 79000 Lviv, Ukraine
E-mail: *matmod@franko.lviv.ua*

This paper is devoted to the problem of exact reduction of natural Hamiltonian systems. The integrable Hamiltonian systems with the full collection of first integrals being polynomial in the momenta, are considered. It is proved, that the corresponding reduced (with respect to the full momentum) Hamiltonian system possesses the full collection of first integrals, which are also polynomial in the momenta.

1 Introduction

Numerous achievements in the problem of complete integrability of the finite-dimensional Hamiltonian systems are connected with the projection method, i.e. using the known first integral one can reduce the above mentioned system to system with less number of degrees of freedom. Unfortunately, the symmetry of the reduced system often possesses imperceptible symmetry, which is hard to observe. A lot of corresponding examples for this problem are considered in [1,2]. The correspondence between the full collections of involutory integrals for the initial and reduced systems is not still described. In this paper we propose the description of such correspondence for the Hamiltonian systems admitting the full collection of first integrals, which are polynomial in the momenta. This result was first considered for the case $n = 2$ and proved in [3]. We generalize it for any $n \in \mathbb{N}$. Suppose the natural Hamiltonian system with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + W(x_1, x_2, \ldots, x_n) \tag{1}$$

is integrable by Liouville and also admits the first integral (the full momentum)

$$\mathcal{P} = \sum_{i=1}^{n} p_i.$$

The following problem arises: is it possible to state that the reduced (in regard to $\mathcal{P}$) Hamiltonian system (now with $n-1$ degrees of freedom) is also integrable by Liouville? The equivalent formulation is: *does this system also admit a full collection of integrals containing the full momentum* $\mathcal{P}$? In this paper we consider the class of n-dimensional Hamiltonian natural systems with the Hamiltonian (1).

2 Principles of exact reduction

Lemma 1. *There exists the linear canonical transformation* $y = Ax$, $p = A^t q$, *such that in new phase coordinates* (y, q) *the Hamiltonian* (1) *takes the form*

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} q_i^2 + W(y_1, y_2, \ldots, y_{n-1}),$$

where the full momentum is now $\mathcal{P} = q_n$.

Proof. The proof is based on the following remark. Obviously, there exists the orthogonal matrix $A \in SL_n(\mathbb{R})$ with the last row $a_{n1} = a_{n2} = \cdots = a_{nn} = 1/\sqrt{n}$. Therefore the above-stated transformation is canonical as the expansion of the punctual transformation of the configurational variables. ■

Further on we shall denote the new canonical coordinates and momenta as (x, p) and consider the system with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + W(x_1, x_2, \ldots, x_{n-1}). \tag{2}$$

Lemma 2. *Let $F(x,p)$ be a polynomial in the momenta first integral of the system with the Hamiltonian* (2)*. Then*

1) $\frac{\partial F}{\partial p_n}$ *is also the first integral of this system;*
2) F *is a polynomial in* x_n*.*

Proof. Obviously, $\{\frac{\partial F}{\partial x_n}, \mathcal{H}\} = \frac{\partial}{\partial x_n}\{F, \mathcal{H}\} = 0$. As $F(x,p)$ is a polynomial in the momenta, it can be represented in the form

$$F = F_m + F_{m-1} + \cdots + F_0,$$

where the component F_k is a homogeneous polynomial of degree k in the momenta $p_1, \ldots, p_n$. Then for the upper part F_m we obtain that the Poisson bracket $\{F_m, \frac{1}{2}\sum\limits_{i=1}^{n} p_i^2\} = 0$, i.e. $\sum\limits_{i=1}^{n} \frac{\partial F}{\partial x_i} = 0$. Therefore F_m is a polynomial in the collection of its arguments

$$\{\, p_i \mid i \in \overline{1,n}\,\} \cup \{(x_i p_j - x_j p_i) \mid i, j \in \overline{1,n}\},$$

hence the degree in the momenta of the polynomial $(\frac{\partial}{\partial x_n})^{n+1} F$ is less than n. Consequently, $(\frac{\partial}{\partial x_n})^{n(n+1)/2} F = 0$. ■

Now we consider the possibility of the reduction of Hamiltonian system with the Hamiltonian (2) to the problem with $(n-1)$ degrees of freedom and the Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2}\sum_{i=1}^{n-1} p_i^2 + W(x_1, x_2, \ldots, x_{n-1}). \tag{3}$$

in the following sense:

1) the initial system also admits the existing of the involutory collection of first integrals in the form $\Omega = (\mathcal{H} = F_1, F_2, \ldots, F_{n-1}, p_n)$;
2) $\Omega_0 = (\mathcal{H}_0 = F_1, F_2, \ldots, F_{n-1})$ is the involutory collection of first integrals for the reduced system with the Hamiltonian (3).

Notice that the integrals $F_1, \ldots, F_{n-1}$ have to be independent of x_n.

Theorem 1. *Let the Hamiltonian system with Hamiltonian* (2) *admit a full collection of the first integrals*

$$\{\mathcal{H} = F_1, F_2, \ldots, F_n\}$$

(that is, it is functionally independent and in involution), where the integrals $F_2, \ldots, F_n$ are polynomials in the momenta $p_1, p_2, \ldots, p_n$. Then this system also admits a full collection of integrals $\{\mathcal{H}_0, R_1, \ldots, R_{n-1}, p_n\}$, where the integrals R_i are polynomials in the momenta $p_1, p_2, \ldots, p_n$, independent of x_n.

Proof. Under conditions of the Theorem, rank $(\mathcal{H} = F_1, \dots, F_n) = n$ in all points of the phase space. Obviously, the elements of the collection Ω can be described in the form

$$F_i = F_{im_i} x_n^{m_i} + F_{im_i-1} x_n^{m_i-1} + \cdots + F_{i1} x_n + F_{i0}, \tag{4}$$

i.e. F_i are polynomials of degree m_i in the coordinate x_n, in addition $0 = m_1 \le m_2 \le \cdots \le m_n$. Denote

$$F_i^{(s)} = \frac{(m_i - s)!}{m_i!} \left(\frac{\partial}{\partial x_n} \right)^{(m_i - s)} F_i.$$

Then, evidently, $F_i^{(0)}$ is the coefficient F_{im_i} in the expansion (4).

Definition 1. The collection of the first integrals

$$\widetilde{\Omega} = \{\mathcal{H} = \widetilde{F}_1, \widetilde{F}_2, \dots, \widetilde{F}_n\}$$

is called to be a *minimal* collection of first integrals, if the following conditions hold:

1) the integrals $\widetilde{F}_i$ are put in ascending order of its degrees $\widetilde{m}_i$;
2) exactly k $(k > 0)$ of first integrals $\widetilde{F}_i$ are independent of x_n, i.e. $\widetilde{m}_k = 0 < \widetilde{m}_{k+1}$;
3) all the leading components $F_i^{(0)}$ $(k + 1 \le i \le n)$ are functionally independent with the collection $\{\mathcal{H} = \widetilde{F}_1, \widetilde{F}_2, \dots, \widetilde{F}_k\}$;
4) all the leading components $F_i^{(0)}$ (including those for $1 \le i \le k$) are in involution;
5) Any collection $\widetilde{\widetilde{\Omega}}$, which satisfies the conditions 1–4, is not smaller (in lexicographic order with respect to the ascending sequence $\{m_i\}$) than the collection $\widetilde{\Omega}$.

Definition 2. The collection Ω of the first integrals is called to be *almost involutory*, if for any pair of the first integrals F_s and F_t, such that their orders m_s and m_t are nonzero, the following identity holds:

$$\{ (\partial/\partial x_n)^{m_s - 1} F_s, (\partial/\partial x_n)^{m_t} F_t \} = \{ (\partial/\partial x_n)^{m_t - 1} F_t, (\partial/\partial x_n)^{m_s} F_s \}.$$

It is clear that the class of the collections $\widetilde{\Omega}$, which satisfy the Definitions 1, 2, is nonempty. Therefore further the collection Ω will be considered to satisfy the Definitions 1, 2. Note that according to the Definition 2, we remove the restriction for all the first integrals F_i to be in involution.

Lemma 3. *If the collection Ω of n functionally independent first integrals is minimal and almost involutory, then $k > n - 3$.*

Proof. Assume that there is more than 2 first integrals, dependent on x_n, belonging to the collection Ω. It easy to show that the degrees of these integrals are equal to 1. Indeed, consider the collection $\omega = \{F_i \in \Omega \,|\, \deg_{x_n} F_i > 0\}$, i.e. $\omega = \{F_{k+1}, \dots, F_n\}$. Consider now the following two cases:

Case 1. $m_{k+1} = 1$. If $\deg_{x_n} F_n > 1$, then, according to the definition of minimal collection, $(\partial/\partial x_n) F_n$ is functionally dependent with $\{F_1, F_2, \dots, F_{n-1}\}$. At the same time, the collection $\widetilde{\omega} = \{F_{k+1}, \dots, F_n + F_{k+1}\}$ is also minimal, and for this modified set we also can state, that $(\partial/\partial x_n)(F_n + F_{k+1}$ is functionally dependent with $\{F_1, F_2, \dots, F_{n-1}\}$. It follows from two previous statements, that $(\partial/\partial x_n) F_{k+1} = F_{k+1}^{(0)}$ is functionally dependent with $\{F_1, F_2, \dots, F_{n-1}\}$. As $F_{k+1}^{(0)}$ does not depend on x_n, is in involution with the first integrals from the set $(\Omega \backslash \omega)$, then it

functionally depends on at least one (say, the integral F_s) of the integrals $\{F_{k+2}, \dots, F_n\}$. Therefore, after substitution $(\partial/\partial x_n)F_{k+1}$ instead of F_s we obtain the collection, which is smaller (in lexicographic order) then Ω. More precisely, the new collection has smaller number of the integrals, essentially dependent on x_n.

Case 2. $m_{k+1} > 1$. In this case, according to the definition of minimal collection, $(\partial/\partial x_n)F_{k+1}$ is functionally dependent with $\{F_1, F_2, \dots, F_k, F_{k+2}, \dots, F_n\}$. As $(\partial/\partial x_n)F_{k+1}$ depends on x_n, it essentially depends on at least one (say, the integral F_s) of the integrals $\{F_{k+2}, \dots, F_n\}$. Therefore, after substitution $(\partial/\partial x_n)F_{k+1}$ instead of F_s we obtain the collection, which is smaller (in lexicographic order) then Ω. This contradiction with assumption of the minimality of Ω proves the Lemma 3. ■

To complete the proof of the Theorem, we notice, that for the case $k > n - 2$ only one or two of the first integrals in the collection Ω essentially depend on the coordinate x_n. Therefore, the statement of the main theorem in [3] can be easy transformed to be applied for n degrees of freedom. ■

The similar statement for the system with the Hamiltonian (1) with a full collection of the first integrals being analytical in the momenta is still open.

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Symmetry Breaking and AC-Driven Dynamics of Solitons

Yaroslav ZOLOTARYUK † *and Mario SALERNO* ‡

† *Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine*
E-mail: *yzolo@bitp.kiev.ua*

‡ *Dipartamento di Fisica "E. R. Caianiello" and Istituto Nazionale di Fisica della Materia (INFM), Universitá di Salerno, I-84081 Baronissi, Salerno, Italy*
E-mail: *salerno@sa.infn.it*

We report on the possibility of unidirectional motion of a topological soliton in a dissipative AC-driven sine-Gordon equation. This motion is due to breaking of certain space-time symmetries. It can be achieved by the proper choice of the potential or the AC force. We formulate the necessary symmetry conditions for the AC force and confirm them with the first order perturbation theory and numerical simulations. The same symmetry considerations are applied to the discrete sine-Gordon equation.

1 Introduction

Transport phenomena based on nonlinear effects are at the heart of many problems in physics. In this context the problem of creation of a net DC current from signals of zero mean is of particular interest [1]. Historically this problem has been posed by Smolouchowski and Feynmann (see [2]) under the name of the *ratchet* effect. It can be described as a Brownian particle in an asymmetric periodic potential, moving in a specific direction in presence of damping under the action of AC forces of zero average. The origin of a net motion is associated with the breaking of space-temporal symmetries of the system [3, 4]. The effect, originally studied for Brownian particles, was generalized to soliton systems with asymmetric potentials [5]. On the other hand, it is known that for a particle an *unidirectional* transport is possible also in symmetric periodic potentials, provided the driving force breaks certain *temporal* symmetries which relate orbits of opposite velocities in phase space [3]. Since in applications it is easier to act on the temporal part (by using external forces) than on the spatial part (by inducing distortion of the potential) of the system, we aim to explore the effect of the temporal symmetry breaking on soliton dynamics.

2 Sine-Gordon equation

Solitons are nonlinear waves that are spatially localized, move with constant velocity and collide elastically. Among the equations that have soliton solutions, the sine-Gordon (SG) equation

$$u_{tt} - u_{xx} = -\sin u, \tag{1}$$

takes a special place. This equation has a number of applications in condensed matter physics, quantum field theory and other areas of modern science. It has soliton solutions in the form of kinks and antikinks: $u(x,t) = 4\tan^{-1}\left[\exp\left(\pm(x-vt)/\sqrt{1-v^2}\right)\right]$. The sign "+" stands for the kink while "−" stands for the antikink. Kinks and antikinks come as a one-parametric family of solutions with the velocity v as a parameter. The velocity spectrum for kinks and antikinks is symmetric with respect to zero: $-1 < v < 1$. In the unperturbed version the solitons of the

SG equation have well-defined particle properties with their mass, energy, momentum

$$P = -\int_{-\infty}^{+\infty} u_x u_t dx, \tag{2}$$

and topological charge

$$Q = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u_x dx. \tag{3}$$

For kinks $Q = 1$ while for antikinks $Q = -1$. For brevity, further on we will consider only kinks. If a weak perturbation is added the particle properties of the topological soliton will survive. The dynamics of the soliton can be approximately described by the dynamics of its center of mass. Thus, one could define the soliton velocity as

$$v = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x u_{xt} dx, \tag{4}$$

bearing in mind that it works only if the soliton can be approximated as a particle, e.g. when the perturbation is weak enough.

We are interested in the AC-driven and damped sine-Gordon (SG) equation

$$u_{tt} - u_{xx} + \sin u = -\alpha u_t + E(t) \equiv \epsilon f(t), \qquad \langle E(t)\rangle_t = 0, \tag{5}$$

where $E(t) = E(t+T)$ is a periodic field with zero mean and α stands for the damping coefficient. For our problem the boundary conditions can be either periodic: $u(x+L,t) = u(x,t) + 2\pi$, $u_x(x+L,t) = u_x(x,t)$, or free ends: $u_x(0,t) = u_x(L,t) = 0$. If the size of the system L is much larger then the size of the soliton, for our purposes both cases should yield the same result. For practical tasks, mostly connected with soliton dynamics in Josephson junctions, the choice of boundary conditions means different physical situations, but this is out of scope of this paper.

Now our aim is to formulate general conditions on the driving field $E(t)$ that will provide directed motion of a soliton. All symmetry operations (see for details [6, 7]) that can relate a soliton solution $u(x,t;v)$ having velocity v with the identical solution but with velocity $-v$

$$\hat{\boldsymbol{S}} u(x,t;v) = u(x,t;-v),$$

must be destroyed. These symmetries can be combinations of all possible shifts and reflections in t, x and u. We also should keep in mind that these symmetries should not change the sign of the topological charge (3) in order to avoid transformations from a kink to an antikink (or vice versa). For the unperturbed SG equation (1) there are only two such operations:

$$\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{X}}}: \; x \to -x, \quad t \to t + T/2, \quad u \to -u + 2\pi, \qquad \hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{T}}}: \; t \to -t + 2t_0.$$

The parameter t_0 is defined by the shape of the function $E(t)$. If the perturbation is added as written in equation (5), the time-reversal symmetry $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{T}}}$ is destroyed due to damping. Thus, the only symmetry left is the $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{X}}}$ symmetry. The SG equation will be invariant under this symmetry if the driving fields satisfy the following condition:

$$E(t + T/2) = -E(t). \tag{6}$$

It is easy to see that if $E(t) = E(t+T)$ and $\langle E(t)\rangle_t = 0$, the only way to change the sign of the function $E(t)$ is to shift it by $T/2$. In the *underdamped* limit $\alpha = 0$ the equation of motion is also invariant under the time reversal symmetry $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{T}}}$ if

$$E(t + t_0) = E(-t + t_0). \tag{7}$$

Thus, in order to achieve directed soliton propagation, one must apply the driving field $E(t)$ that violates the condition of equation (6) in the general dissipative case ($\alpha \neq 0$) and in the non-dissipative case ($\alpha = 0$) both (6) and (7) should be violated.

It has been shown in [8] that equation (5) with the single harmonic drive [$E(t) = E_0 \cos(\omega t + \theta_0)$] does not yield propagating kinks if the the driving amplitude is small enough. Kinks can propagate either if damping is very weak (see [8]) or when the driving amplitude exceeds certain threshold and the driving frequency is close to the resonance with the phonons $\omega \sim 1$ (see [9]). But in both cases this motion is not *directed*: kinks can move in either direction and proper averaging with respect to the initial phase will yield zero average kink velocity. For our problem we choose the driving field in the following form:

$$E(t) = E_1 \cos \omega t + E_2 \cos (m\omega t + \theta). \tag{8}$$

For this choice condition (6) is violated for any θ if $E_1, E_2 \neq 0$ and m is even, and is not violated for any E_1, E_2 and θ if m is odd. Also, condition (7) is violated if $\theta \neq 0, \pm\pi$ and $E_1, E_2 \neq 0$.

2.1 Perturbation theory

In order to get an idea how the driving field (8) affects the soliton dynamics, we use the soliton perturbation theory (see [10, 8] for details). The first order of the perturbation theory takes an advantage of the particle properties of the SG kink. It is assumed that perturbation $\epsilon f(t)$ of equation (5) is weak, so that the kink shape is not changed under the perturbation and its parameters change in time adiabatically. Therefore the initial problem can be mapped into the dynamics of the kink mass center coordinates ($X(t), v(t) \equiv \dot{X}(t)$), that can be introduced as $u(x,t) = 4\tan^{-1}(\exp\{[x - X(t)]/[1 - \dot{X}^2(t)]^{1/2}\})$. An ordinary differential equation for $X(t)$ can be obtained by differentiating the momentum (2) with respect to time, and using equation (5):

$$\dot{P} = -\alpha P + 2\pi E(t). \tag{9}$$

Assuming $P(t) = 8v(t)/\sqrt{1 - v(t)^2}$, equation (9) can be solved for $P(t)$, and eventually analytical expression of the average kink velocity $\langle v \rangle$, valid in the limit $E_j/\sqrt{\alpha^2 + (j\omega)^2} \ll 1$, $j = 1, 2$, can be obtained (see [7] for details) for $m = 2$:

$$\langle v \rangle = \frac{1}{T}\int_0^T v(\tau) d\tau \simeq \frac{3\pi^3 E_1^2 E_2 \sin(\theta - \theta_0)}{512(\alpha^2 + \omega^2)\sqrt{\alpha^2 + 4\omega^2}}, \qquad \tan\theta_0 = \frac{\alpha(\alpha^2 + 3\omega^2)}{2\omega^3}. \tag{10}$$

Similarly, for $m = 3$ we obtain that $\langle v \rangle = 0$ independently on θ and for an arbitrary initial phase. These results can be easily understood from the symmetry properties of $E(t)$. In the underdamped limit $\alpha \to 0$ we obtain $\theta_0 \to 0$, and, consequently $\langle v \rangle \sim \sin\theta$. Thus, directed soliton motion ceases to exist if $\theta = 0, \pm\pi$. This is not surprising since for these values of θ the driving field equals $E(t) = E_1 \cos(\omega t) \pm E_2 \cos(2\omega t)$ and the symmetry $\hat{\boldsymbol{S}}_{\boldsymbol{T}}$ is restored because $E(t) = E(-t)$.

2.2 Numerical simulations

In order to check the above considerations we have performed numerical simulations of the perturbed SG equation (5). In Fig. 1 the dynamics of a SG kink, initially at rest, driven by a bi-harmonic driver with $m = 2$ and by a single harmonic driver (i.e. $E_2 = 0$), are reported in Figs. 1(a,b), respectively (note that we use contour plots to show the time evolution surface generated by the kink profile). From these figures we see that, in the case $m = 2$ the soliton center of mass moves with a constant velocity, while for the single harmonic driver, it oscillates around the initial position. Numerical results of [7] show that an antikink with parameters as in Fig. 1a will move in the opposite direction.

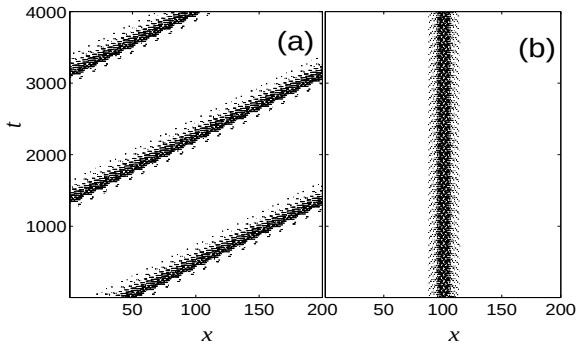

Figure 1. Contour plots of the velocity field $u_x(x,t)$ for (a): $E_1 = 0.4$, $E_2 = 0.26$ and (b): $E_1 = 0.66$, $E_2 = 0$ under periodic boundary conditions. Other parameters are $\alpha = 0.12$, $\omega = 0.25$, $\theta = 1.61$.

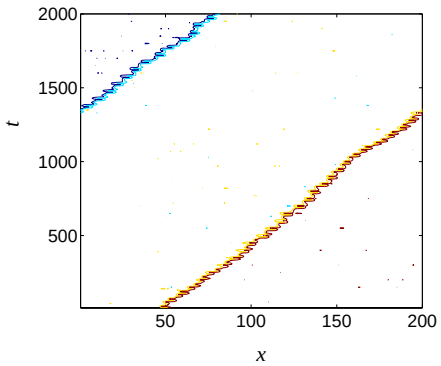

Figure 2. Contour plot for the displacement field $u(x,t)$ in the presence of white noise for parameters as in Fig. 1a.

Let us now briefly investigate the influence of a white noise on the kink dynamics. In order to accomplish this, we have simulated equation (5) with an additional stochastic term $\xi(x,t)$ which satisfies $\langle \xi(x,t) \rangle = 0$, $\langle \xi(x,t)\xi_m(x',t') \rangle = 2\alpha D\delta(x-x')\delta(t-t')$. In Fig. 2[1] we report the contour plot for the kink motion in the case of a bi-harmonic driver with $m = 2$. We see that the noise introduces disturbances on the profile but does not destroy the directed motion of the soliton. We checked that this property remains true also if we increase the amplitude of the noise up to the value when kink-antikink pairs start to nucleate. Moreover, the existence of the phenomena in presence of noise shows the validity of the above mechanism also for the non-deterministic soliton ratchets. Since the simulations with white noise can also be considered as "averaging" over the phase space, our simulations also have demonstrated that there is only one attractor of the system and the computed soliton velocity indeed can be considered and an average one. Similar simulations for $m = 3$ (see [7] for details) show that no directed motion of a kink is possible. This is because for $m = 3$ the $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{X}}}$ symmetry is not broken.

In Fig. 3 we show how the average kink velocity changes as a function of the phase difference between the driving fields, θ. Numerical results clearly confirm analytical approximation of the previous subsection (equation (10)). Another important observation is confirmation of the symmetry argument. As long as the underdamped limit is approached, the curve $\langle v \rangle(\theta)$ crosses the line $\langle v \rangle = 0$ closer and closer to the points $\theta = 0, \pm\pi$. This is in accordance with the fact that the symmetry $\hat{\boldsymbol{S}}_{\boldsymbol{\mathcal{T}}}$ is restored in this limit and (7) is no longer violated.

3 Discrete sine-Gordon equation

Discrete sine-Gordon (DSG) equation is a modification of its continuous counterpart for the media with spatial discreteness. It is used to model phenomena on lattices, for example fluxon

[1] Figures in colour will be available only in electronic version.

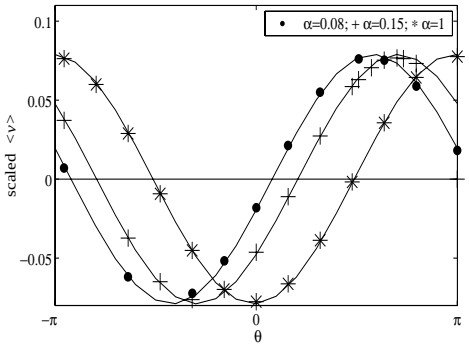

Figure 3. Scaled average kink velocity as a function of the phase difference θ for $E_1 = 0.3$, $E_2 = 0.195$, $\omega = 0.25$; $\alpha = 0.08$ ($\bullet$), $\alpha = 0.15$ ($+$) and $\alpha = 1$ ($*$). Solid lines are the best fit to the $A\sin(\theta + \theta_0)$ function.

dynamics in parallel Josephson junction arrays, dislocation motion in crystals, etc. The driven and damped DSG has the following form:

$$\ddot{u}_n + \alpha\dot{u}_n - \kappa(u_{n+1} - 2u_n + u_{n-1}) + \sin u_n + E(t) = 0, \qquad n = 1, 2, \ldots, N, \tag{11}$$

Here all the terms and parameters have the same meaning as in the SG counterpart. The parameter κ measures the discreteness of the system, in the limit $\kappa \to \infty$ the above equation reaches the continuum limit. One must mention that DSG is not integrable as compared to the continuous one and there is certain principal difference in its dynamics: kinks in this model cannot propagate with constant velocity (in the unperturbed limit, $\alpha = 0$, $E(t) = 0$). Kinks in DSG are pinned by the lattice (see [11] for details). Similarly to the case of the continuous sine-Gordon, two cases of boundary conditions can be considered: periodic $u_{n+N} = u_n + 2\pi$, $n = 1, 2, \ldots, N$ or free ends.

Discrete kink dynamics can be mapped (with a certain degree of approximation) into dynamics of a single particle. As shown by Braun and Kivshar in [11] this dynamics is described by the dynamics of the kink center $X(t)$: $u_n(t) = 4\arctan[\exp[n - X(t)]]$. Using this ansatz, the effective equation of motion for the kink center can be obtained:

$$\ddot{X} + \alpha\dot{X} + V'_{\rm PN}(X) + \tilde{E}(t) = 0, \qquad \tilde{E}(t) \sim E(t). \tag{12}$$

Here the Peierls–Nabarro (PN) potential $V_{\rm PN}(X) = V_{\rm PN}(X + 1)$, $V'_{\rm PN}(X) \sim \sin 2\pi X$, takes into account the discreteness of the lattice. Progressive kink motion is analogous to rotations of an AC-driven and damped pendulum. It will be locked to the frequency of the drive and will correspond to limit cycles of equation (12). The average kink velocity thus can be expressed as:

$$\langle v\rangle = \langle\dot{X}(t)\rangle = (m/n)(\omega/2\pi), \qquad m, n \in \mathbb{Z}.$$

Here m corresponds to the number of sites which kink has traveled during the time $nT = 2\pi n/\omega$.

Similarly to the previous case of the continuous SG, in order to achieve directed kink motion, we should destroy all symmetries which relate solutions with opposite velocities. We can do this analysis on the basis of equation (12) and this brings us to the problem, solved some time ago (see papers [3,4]). Operations which change the sign of the velocity are either shift in space and reflection in time or shift in time and reflection is space. In the underdamped case $\alpha = 0$ there are two such symmetries

$$\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{X}}}: \ t \to t + T/2, \quad X \to -X, \qquad \hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{T}}}: \ t \to -t + 2t_0.$$

Situation here is very similar to the continuous case. Equation (12) is invariant under the operation $\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{X}}}$ always only if $E(t) = -E(t + T/2)$ is satisfied (along with $V_{\rm PN}(-X) = V_{\rm PN}(X)$

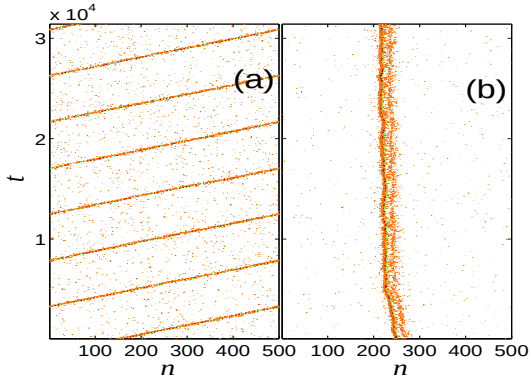

Figure 4. Numerical integration of equation (11) for (a) $E_1 = 0.3$, $E_2 = 0.15$ and (b) $E_1 = 0.45$, $E_2 = 0$. Other parameters are $\kappa = 1$, $\alpha = 0.05$, $\omega = 0.1$, $\theta = 2$. Periodic boundary conditions have been applied.

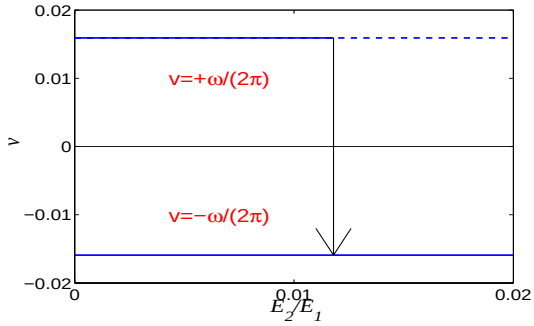

Figure 5. Velocities of two kinks, running into the opposite directions at $\alpha = 0.05$, $\omega = 0.1$, $E_1 = 0.21$, $\theta = 0$ as a function of E_2/E_1. Dashed line denotes that the given solution is no longer stable.

which is always satisfied because the sine-Gordon potential is symmetric). In the non-damping case is also invariant under $\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{T}}}$ if $E(t + t_0) = E(-t + t_0)$. A new symmetry appears in the overdamped limit ($\alpha \to \infty$). In that case the equation of motion can be rewritten as $\dot{X} + V'_{\mathrm{PN}}(X) + \tilde{E}(t) = 0$. The symmetry can be written as follows:

$$\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{T}}} : \; t \to -t + t_0, \quad X \to X + 1/2, \tag{13}$$

and equation (12) is invariant under this symmetry if

$$E(t + t_0) = -E(-t + t_0), \qquad V'_{\mathrm{PN}}(X) = -V'_{\mathrm{PN}}(X + 1/2). \tag{14}$$

If $V'_{\mathrm{PN}}(X) \sim \sin 2\pi X$ the second condition is satisfied. Thus, in the overdamped case both (6) and (14) must be violated if we want to obtain directed kink motion.

3.1 Numerical simulations

Verification of the above considerations has been performed via numerical integration of the DSG equation (11) in the presence of weak noise (similarly as in the previous section for the continuous SG) by adding an stochastic term $\xi_n(t)$ $[\langle \xi_n(t) \rangle = 0$, $\langle \xi_n(t)\xi_m(t') \rangle = 2\alpha D \delta_{mn} \delta(t - t')]$ to each of equations (11). Results of simulations are shown in Fig. 4. We clearly observe directed motion of a kink in the case of two mixed harmonics and absence of such motion when the driving field $E(t)$ consists from only one harmonic. Motion of an antikink for the same parameters occurs in the opposite direction with the same absolute value of the average velocity.

Emergence of the directed kink motion is caused by desymmetrization of basins of attraction of two limit cycles with opposite velocities. For one-particle ratchets it has been shown in [3,12].

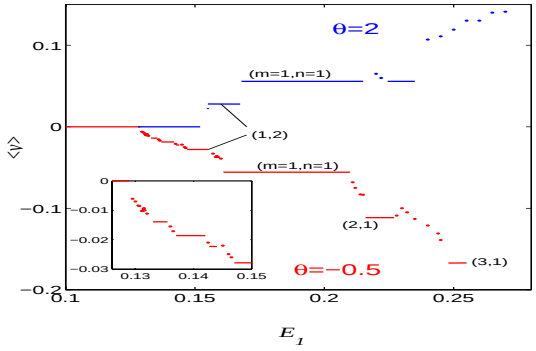

Figure 6. Kink velocity as a function of the driving amplitude $E_1 = E_2$ for $\theta = 2$ and $\theta = -0.5$. Other parameters are: $\omega = 0.35$, $\alpha = 0.1$, $\kappa = 1$. The inset shows more detailed picture for the $\theta = -0.5$ case.

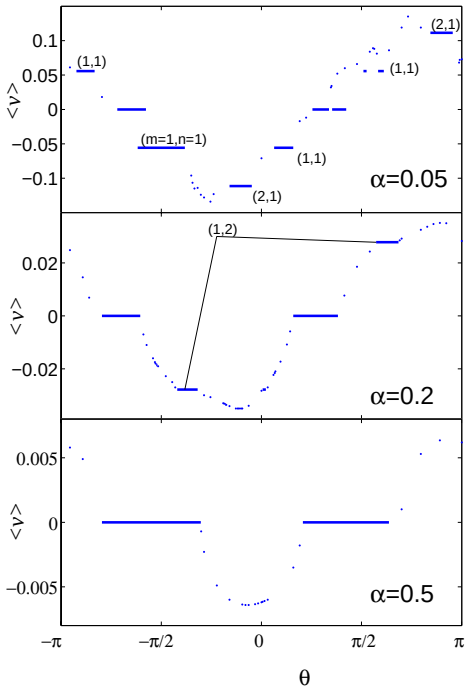

Figure 7. Average kink velocity as a function of the phase θ at $\omega = 0.35$, $E_1 = E_2 = 0.2$, and different values of damping: $\alpha = 0.05$ (upper panel),$\alpha = 0.2$ (middle panel) and $\alpha = 0.5$ (bottom panel).

As long as symmetry is broken (by switching to E_2), one of the basins of attraction begins to shrink and eventually disappears. Thus, only limit cycles which correspond to motion in one direction survive, as shown in Fig. 5. We see that already for rather weak desymmetrizations only one attractor survives.

In Fig. 6 we plot the kink velocity as a function of the driving amplitude. We observe certain differences compared to the case of continuous SG (see also [13] for details on damped DSG): a depinning threshold exists, while for the continuous case $\langle v \rangle \sim E_1^2 E_2$, e.g. kink can move for an arbitrarily small applied field. Since kink dynamics is locked to the driving frequency ω, the steps in the figure correspond to the rational multipliers of $\omega/(2\pi)$ and the dependence has a form of "devils staircases". After certain critical threshold $E_1 = E_2 \sim 0.3$ is passed, dynamics becomes chaotic and there is no kink solutions.

In Fig. 7 we plot average kink velocity as a function of the phase θ. Note that dependence of the kink velocity on the system parameters is no longer smooth. This is the result that dynamics is locked to the time-periodic driving field. Thus, the function $\langle v \rangle(\theta)$ only slightly resembles the

sine function, especially for strong dampings when depinning of the kink requires very strong fields. In the underdamped limit $\alpha \to 0$ directed motion disappears approximately at $\theta = 0, \pm\pi$, when $\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{T}}}$ is restored: $E(t) = E(-t)$. Note that the same happens in the continuum case. In the overdamped limit $\alpha \to \infty$ directed motion disappears approximately at $\theta = \pm\pi/2$, when the symmetry $\hat{\boldsymbol{D}}_{\boldsymbol{\mathcal{I}}}$ is restored: $E(t) = E_1 \cos\omega t + E_2 \cos(2\omega t + \pi/2) = -E(\pi/\omega - t)$.

4 Conclusions

We have considered a way to produce a directed motion of a topological soliton (in both discrete and continuous media) in the presence of damping, using an appropriate AC driving force of zero mean. The effect has been demonstrated for both continuous and discrete SG equations, but can be extended to other kink-bearing models. We have shown that in order to get directed soliton propagation it is necessary to break certain symmetries of the driving field $E(t)$ and have confirmed our findings both analytically and numerically. The phenomenon can be observed experimentally in an AC biased annular long Josephson junction or annular 1D array of inductively coupled parallel small Josephson junctions.

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web-page: www.imath.kiev.ua/~appmath

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Дане видання є другою частиною і включає праці, присвячені суперсиметрії та застосуванню симетрійних методів у фізиці.

Розраховано на наукових працівників, аспірантів, які цікавляться новими тенденціями симетрійного аналізу та його застосувань.

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This issue is the second part which contains papers devoted to supersymmetry and applications of symmetry methods in physics.

The book may be useful for researchers and post graduate students who are interested in modern trends in symmetry analysis and its applications.

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Supersymmetry

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Nonlinear Supersymmetry in Quantum Mechanics

Alexandr A. ANDRIANOV †‡ *and Andrey V. SOKOLOV* ‡

† *INFN, Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy*
E-mail: *andrianov@bo.infn.it*

‡ *V.A. Fock Institute of Physics, St.-Petersburg State University,*
1 Ulianovskaya Str., Petrodvorets, 198504 St.-Petersburg, Russia
E-mail: *sokolov@mph.phys.spbu.ru*

We study the Nonlinear (Polynomial, N-fold, ...) SUSY algebra in one-dimensional QM. Its structure is determined by the type of conjugation operation (Hermitian conjugation or transposition) and described with the help of the Super-Hamiltonian projection on the zero-mode subspace of a supercharge. We show that the SUSY algebra with transposition symmetry is always polynomial in the Super-Hamiltonian if supercharges represent differential operators of finite order. The appearance of the extended SUSY with several supercharges is analyzed and it is established that no more than two independent supercharges may generate a nonlinear superalgebra. In the case with two independent supercharges we find a nontrivial hidden symmetry operator. It is revealed that wave functions of all Super-Hamiltonian bound states or, in the case of Super-Hamiltonian with periodic potential(s), all periodic wave functions corresponding to boundaries between allowed and forbidden energy bands are zero-modes of the hidden symmetry operator.

1 Introduction

In our report we consider the following topics: (i) Nonlinear SUSY algebras [1–3] with Hermitian conjugation and transposition symmetries of supercharges, (ii) extended SUSY algebra and appearance of antisymmetric symmetry operator (hidden symmetry operator), (iii) "strip-off" problem, (iv) (in)dependence of supercharges, maximal number of coexisting independent supercharges and optimal basis of supercharges, and (v) example: two independent supercharges of 2nd and 1st orders. The new (with respect to [4]) result presented in the paper is a description of properties of antisymmetric symmetry operator for a Hamiltonian with periodic potential. The proofs of presented results and a more complete list of references can be found in [4].

2 Four types of SUSY algebras with complex supercharges

Let h^+ and h^- be components of the matrix one-dimensional Schrödinger operator, a Super-Hamiltonian,

$$H = \begin{pmatrix} h^+ & 0 \\ 0 & h^- \end{pmatrix} = \begin{pmatrix} -\partial^2 + V_1(x) & 0 \\ 0 & -\partial^2 + V_2(x) \end{pmatrix} \equiv -\partial^2 \boldsymbol{I} + \boldsymbol{V}(x),$$

where $\partial \equiv d/dx$. The isospectral connection between Hamiltonians h^+ and h^- is provided by intertwining relations with the help of Crum–Darboux differential operators $q_N^\pm$,

$$h^+ q_N^+ = q_N^+ h^-, \qquad q_N^- h^+ = h^- q_N^-, \qquad q_N^\pm = \sum_{k=0}^{N} w_k^\pm(x)\partial^k, \qquad w_N^\pm = \text{const} \equiv (\mp 1)^N, \quad (1)$$

which, in the framework of SUSY QM, are components of the supercharges,

$$Q = \begin{pmatrix} 0 & q_N^+ \\ 0 & 0 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} 0 & 0 \\ q_N^- & 0 \end{pmatrix}, \qquad Q^2 = \bar{Q}^2 = 0, \qquad [H, Q] = [H, \bar{Q}] = 0.$$

According to known operator q_N^+ we can choose as q_N^- both $(q_N^+)^t$ and $(q_N^+)^\dagger$ and *vice versa*, where ^{t} and † denote transposition and Hermitian conjugation respectively. Thus, if coefficients of q_N^+ are real then we have unique $q_N^- = (q_N^+)^t = (q_N^+)^\dagger$, but if coefficients of q_N^+ are not real then we have two different q_N^- type operators, $(q_N^+)^t$ and $(q_N^+)^\dagger$. Hence, we have in general four different supercharges

$$Q, \qquad \bar{Q} = Q^t, \qquad \bar{Q}_c = Q^\dagger, \qquad Q_c = \bar{Q}_c^t,$$

pairs of which generates four SUSY algebras:

$$\mathcal{A}_1 \longleftrightarrow (\bar{Q}_c, Q), \qquad \mathcal{A}_2 \longleftrightarrow (\bar{Q}, Q), \qquad \mathcal{A}_3 \longleftrightarrow (\bar{Q}_c, Q_c), \qquad \mathcal{A}_4 \longleftrightarrow (\bar{Q}, Q_c).$$

Certain properties of $\mathcal{A}_{1,4}$ type complex SUSY algebras were considered in [2].

3 Superalgebras with transposition symmetry

The structure of SUSY algebras with real coefficient functions as well as the structure of the $\mathcal{A}_{2,3}$ type complex SUSY algebras is described by the following theorem.

Theorem 1 (on SUSY algebras with T-symmetry). *Let us introduce two sets of N linearly independent functions $\phi_n^\pm(x)$ $(n = 1, \ldots, N)$ which represent complete sets of zero-modes of the supercharge components* (1)*,*

$$q_N^\pm \phi_n^\pm = 0, \qquad q_N^- = (q_N^+)^t.$$

Then[1]*:*

1) *the Hamiltonians $h^\pm$ have finite matrix representations when acting on the set of functions $\phi_n^\pm(x)$,*

$$h^\pm \phi_n^\mp = \sum_m S_{nm}^\pm \phi_m^\mp,$$

2) *the SUSY algebra closure with $\bar{Q} = Q^t$ takes the polynomial form,*

$$\{Q, Q^t\} = \det\left[E\boldsymbol{I} - \boldsymbol{S}^+\right]_{E=H} = \det\left[E\boldsymbol{I} - \boldsymbol{S}^-\right]_{E=H} \equiv \mathcal{P}_N(H), \tag{2}$$

irrespectively on whether the Q type supercharge of order N is unique or there exist several such supercharges for a given Super-Hamiltonian H.

Corollary 1. *From* (2) *it is evident that eigenvalues of $\boldsymbol{S}^+$ and $\boldsymbol{S}^-$ and their corresponding degeneracies coincide.*

4 Several supercharges and appearance of symmetry operators

Let us now examine the case when for the Super-Hamiltonian H there are two different supercharges K and P of the type Q and of the orders N and N_1 respectively,

$$K = \begin{pmatrix} 0 & k_N^+ \\ 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & p_{N_1}^+ \\ 0 & 0 \end{pmatrix}, \tag{3}$$

where k_N^+ and $p_{N_1}^+$ have real coefficient functions and $N > N_1$. In particular, in the case when complex supercharge Q exists, we can choose K and P as $(Q + Q^*)/2$ and $(Q - Q^*)/(2i)$

[1]The first proposition is a necessary condition for the Hamiltonian system to be quasi-exactly solvable and it was investigated recently [5] within the notion of "conditional symmetry".

respectively, where * denotes complex conjugation of coefficient functions. Thus, each of supercharges K and P generates in our case a unique supercharge of type $\bar{Q}$:

$$\bar{K} = K^t = K^\dagger = \begin{pmatrix} 0 & 0 \\ k_N^- & 0 \end{pmatrix}, \qquad k_N^- = (k_N^+)^t,$$
$$\bar{P} = P^t = P^\dagger = \begin{pmatrix} 0 & 0 \\ p_{N_1}^- & 0 \end{pmatrix}, \qquad p_{N_1}^- = (p_{N_1}^+)^t.$$

The existence of two supercharges of type Q (*i.e.* K and P) conventionally implies the extension of SUSY algebra. To close the algebra one has to include all anticommutators between supercharges. Two supercharges K and P generate two Polynomial SUSY,

$$\left\{K, K^\dagger\right\} = \tilde{\mathcal{P}}_N(H), \qquad \left\{P, P^\dagger\right\} = \tilde{\mathcal{P}}_{N_1}(H),$$

which have to be embedded into a $\mathcal{N} = 2$ SUSY algebra. The closure of the extended, $\mathcal{N} = 2$ SUSY algebra is given by

$$\left\{P, K^\dagger\right\} \equiv \mathcal{R} = \begin{pmatrix} p_{N_1}^+ k_N^- & 0 \\ 0 & k_N^- p_{N_1}^+ \end{pmatrix}, \qquad \left\{K, P^\dagger\right\} \equiv \bar{\mathcal{R}} = \begin{pmatrix} k_N^+ p_{N_1}^- & 0 \\ 0 & p_{N_1}^- k_N^+ \end{pmatrix}.$$

Evidently the components of operators $\mathcal{R}$, $\bar{\mathcal{R}} = \mathcal{R}^\dagger = \mathcal{R}^t$ are differential operators of $N + N_1$ order commuting with the Hamiltonians $h^\pm$, hence they form symmetry operators $\mathcal{R}$, $\bar{\mathcal{R}}$ for the Super-Hamiltonian. However, in general, they are not polynomials of the Hamiltonians $h^\pm$ and these symmetries impose certain constraints on potentials.

Let us find the formal relation between the symmetry operators $\mathcal{R}$, $\bar{\mathcal{R}}$ and the Super-Hamiltonian. These operators can be decomposed into a Hermitian and an anti-Hermitian parts,

$$\mathcal{B} \equiv \frac{1}{2}(\mathcal{R} + \bar{\mathcal{R}}) \equiv \begin{pmatrix} b^+ & 0 \\ 0 & b^- \end{pmatrix}, \qquad i\mathcal{E} \equiv \frac{1}{2}(\mathcal{R} - \bar{\mathcal{R}}) \equiv i \begin{pmatrix} e^+ & 0 \\ 0 & e^- \end{pmatrix}.$$

It can be shown that the Hermitian operator $\mathcal{B}$ is a polynomial of the Super-Hamiltonian of the order $N_b \leq [(N+N_1)/2] \leq N-1$. The second Hermitian symmetry operator $\mathcal{E}$ is antisymmetric, $\mathcal{E}^t = -\mathcal{E}$. Hence, if $\mathcal{E}$ does not vanish identically it is a differential operator of *odd* order and cannot be realized by a polynomial of H. But at the same time

$$\mathcal{E}^2(H) \equiv \frac{1}{4}\left[2(\mathcal{R}\bar{\mathcal{R}} + \bar{\mathcal{R}}\mathcal{R}) - (\mathcal{R} + \bar{\mathcal{R}})^2\right] = \tilde{\mathcal{P}}_N(H)\tilde{\mathcal{P}}_{N_1}(H) - \mathcal{B}^2(H) \equiv \mathcal{P}_e(H) \tag{4}$$

is polynomial of H. Thus the nontrivial operator $\mathcal{E}(H)$ is a non-polynomial function of H — square root of equation (4) in an operator sense.

Closures of SUSY algebras $\mathcal{A}_1$–$\mathcal{A}_4$ can be written in terms of notation relating to real supercharges $K = (Q + Q^*)/2$ and $P = (Q - Q^*)/(2i)$ in the following form:

$$\mathcal{A}_1: \ \{Q, \bar{Q}_c\} = \tilde{\mathcal{P}}_N(H) + \tilde{\mathcal{P}}_{N_1}(H) - 2\mathcal{E}(H),$$
$$\mathcal{A}_2: \ \{Q, \bar{Q}\} = \tilde{\mathcal{P}}_N(H) - \tilde{\mathcal{P}}_{N_1}(H) + i2\mathcal{B}(H),$$
$$\mathcal{A}_3: \ \{Q_c, \bar{Q}_c\} = \tilde{\mathcal{P}}_N(H) - \tilde{\mathcal{P}}_{N_1}(H) - i2\mathcal{B}(H),$$
$$\mathcal{A}_4: \ \{Q_c, \bar{Q}\} = \tilde{\mathcal{P}}_N(H) + \tilde{\mathcal{P}}_{N_1}(H) + 2\mathcal{E}(H),$$
$$\bar{Q}_c = Q^\dagger = K^\dagger - iP^\dagger, \qquad Q_c = Q^* = K - iP, \qquad \bar{Q} = Q^t = K^\dagger + iP^\dagger.$$

One can show that components of $\mathcal{E}(H)$ have the following important properties.

1. In the case when $h^\pm$ has bound states, wave functions $\Psi_i^\pm$ of all these states are zero-modes of $e^\pm(h^\pm)$ and corresponding energies $E_i^\pm$ are roots of the polynomial $\mathcal{P}_e(E)$:

$$e^\pm(h^\pm)\Psi_i^\pm = e^\pm(E_i^\pm)\Psi_i^\pm = 0, \qquad \mathcal{P}_e(E_i^\pm) = 0. \tag{5}$$

Among solutions of (5) one reveals also a zero-energy state at the bottom of continuum spectrum.

2. In the case, when the potential of $h^{\pm}$ is periodic, all boundaries between allowed and forbidden energy bands $E_i^{\pm}$ are roots of $\mathcal{P}_e(E)$ and corresponding to these boundaries periodic wave functions $\Psi_i^{\pm}$ are zero-modes of $e^{\pm}(h^{\pm})$ (see (5) again).

Thus (5) represents an algebraic equation on bound state energies or on energy-band boundaries of a system possessing two supersymmetries. On the other hand one could find also the solutions of (5) which are not associated to any bound state or to any energy-band boundary. The very appearance of such unphysical solutions is accounted for by the trivial possibility to replicate supercharges by their multiplication on the polynomials of the Super-Hamiltonian and it is discussed below.

5 "Strip-off" problem

The pair of two supersymmetries analyzed before may rigidly determine the class of potentials $V_{1,2}$ contracting the freedom in their choice from a functional one to a parametric one. On the other hand, there exists a trivial possibility when the intertwining operators $k_N^{\pm}$ and $p_{N_1}^{\pm}$ are related by a factor depending on the Hamiltonian,

$$k_N^{\pm} = F(h^{\pm})p_{N_1}^{\pm} = p_{N_1}^{\pm}F(h^{\mp}),$$

where $F(x)$ is assumed to be a polynomial. Obviously in this case the symmetry operator $\mathcal{E}(H)$ identically vanishes and the appearance of the second supercharge does not result in any restrictions on potentials.

More generally the orders of polynomial superalgebras and some of the roots of associated polynomials may not be involved in determination of the structure of the potentials. In particular, let the operators $k_N^{\pm}$ and $p_{N_1}^{\pm}$ be reducible to some lower-order ones $\tilde{k}_{\tilde{N}}^{\pm}$ and $\tilde{p}_{\tilde{N}_1}^{\pm}$,

$$k_N^{\pm} = F_k(h^{\pm})\tilde{k}_{\tilde{N}}^{\pm} = \tilde{k}_{\tilde{N}}^{\pm}F_k(h^{\mp}), \qquad p_{N_1}^{\pm} = F_p(h^{\pm})\tilde{p}_{\tilde{N}_1}^{\pm} = \tilde{p}_{\tilde{N}_1}^{\pm}F_p(h^{\mp}), \tag{6}$$

where $F_k(y)$ and $F_p(y)$ are polynomials. Then evidently the superalgebra generated by $\tilde{k}_{\tilde{N}}^{\pm}$ and $\tilde{p}_{\tilde{N}_1}^{\pm}$ equally well characterizes the Super-Hamiltonian system with the same potentials.

Thus, we have come to the problem of how to discern the nontrivial part of a supercharge and avoid multiple SUSY algebras generated by means of "dressing" (6). It can be systematically performed with the help of the following theorem.

Theorem 2 ("strip-off" theorem). *Let us admit the construction of the Theorem on SUSY algebras with T-symmetry. Then the requirement that Jordan form of the matrix $\boldsymbol{S}^-$ (or $\boldsymbol{S}^+$) contains n pairs (and no more) of Jordan cells with equal eigenvalues λ_l and the sizes ν_l of a smallest cell in the l-th pair is necessary and sufficient to ensure for the intertwining operator q_N^+ (or q_N^-) to be represented in the factorized form:*

$$q_N^{\pm} = \tilde{q}_{\tilde{N}}^{\pm}\prod_{l=1}^{n}(\lambda_l - h^{\mp})^{\nu_l},$$

where $\tilde{q}_{\tilde{N}}^{\pm}$ are intertwining operators of order $\tilde{N} = N - 2\sum_{l=1}^{n}\nu_l$ which cannot be decomposed further on in the product similar to (6) *with $F_q(x) \neq \text{const}$.*

Corollary 2. *Jordan forms of $\boldsymbol{S}^+$ and $\boldsymbol{S}^-$ are identical up to transposition of certain cells.*

6 Optimization of supercharges

Still the stripped-off supercharges $k_N^{\pm}$ and $p_{N_1}^{\pm}$ do not necessarily represent an optimal set of them and provide an optimal structure of the symmetry operator $\mathcal{E}(H)$. Let us illustrate it with the help of the following example. One can show that there is the case when intertwining operators of both pairs $k_N^{\pm}$, $p_{N_1}^{\pm}$ and $t^{\pm} \equiv p_{N_1}^{\pm} k_N^{\mp} p_{N_1}^{\pm}$, $p_{N_1}^{\pm}$ cannot be stripped-off, but at the same time the latter pair is evidently more complex than the former. In addition, the composed of $t^{\pm}$ and $p_{N_1}^{\pm}$ symmetry operator $\mathcal{E}_t(H)$ with components,

$$e_t^{\pm} = \pm i\frac{1}{2}(t^{\pm}p_{N_1}^{\mp} - p_{N_1}^{\pm}t^{\mp}) = \mp i\frac{1}{2}(k_N^{\pm}p_{N_1}^{\mp} - p_{N_1}^{\pm}k_N^{\mp})p_{N_1}^{\pm}p_{N_1}^{\mp} = -e^{\pm}\tilde{\mathcal{P}}_{N_1}(h^{\pm})$$

is not optimal because of the superfluous polynomial factor $\tilde{\mathcal{P}}_{N_1}(H)$.

Thus, if there are several SUSY generators we have not only such fundamental problems as (i) to introduce the notion of (in)dependence of intertwining operators and (ii) to find out how many independent intertwining operators can coexist, but also and the problem (iii) to define an optimal basis of intertwining operators.

Let us define the intertwining operators $q_i^{\pm}$, $i = 1, \ldots, n$ to be dependent if and only if the polynomials $\alpha_i^{\pm}(y)$ exist such that not all of them are vanishing and

$$\sum_{i=1}^{n} \alpha_i^{\pm}(h^{\pm}) q_i^{\pm} = 0. \tag{7}$$

If the relation (7) results in $\alpha_i^{\pm}(y) = 0$ for all i the corresponding SUSY generators are independent. Evidently the (in)dependence of q_i^{+} entails the (in)dependence of q_i^{-} and *vice versa*.

The following theorem plays a key role in resolution of how many independent supercharges can commute with a given Super-Hamiltonian.

Theorem 3 (on (in)dependence of supercharges). *Consider two non-trivial intertwining operators $q_i^{\pm}$, $i = 1, 2$ with transposition symmetry $q_i^{+} = (q_i^{-})^t$ which in general may have complex coefficients and let us normalize them in accordance to* (1)*. Then the stripped-off intertwining operators $\tilde{q}_i^{\pm}$ coincide if and only if the symmetry operator made of $q_i^{\pm}$ vanishes, $q_1^{+}q_2^{-} - q_2^{+}q_1^{-} = 0$ (or equivalently $q_1^{-}q_2^{+} - q_2^{-}q_1^{+} = 0$).*

With the help of this Theorem one can show the following:

a) any symmetric (self-transposed) symmetry operator $B^{\pm} = (B^{\pm})^t$, $B^{\pm}h^{\pm} = h^{\pm}B^{\pm}$ is a polynomial of the Hamiltonian $h^{\pm}$;

b) any two antisymmetric symmetry operators $e_i^{\pm} = -(e_i^{\pm})^t$, $e_i^{\pm}h^{\pm} = h^{\pm}e_i^{\pm}$, $i = 1, 2$ are dependent, *i.e.* being stripped off coincide;

c) the maximal number of independent intertwining operators is two.

Using the fact that maximal number of independent intertwining operators is two, one can show the next.

1. In the case when any two intertwining operators are dependent, every intertwining operator q^{+} can be represented in the form

$$q^{+} = \alpha_q(h^{+})p^{+},$$

where $\alpha_q(y)$ is polynomial and p^{+} is normalized intertwining operator of minimal order. It is evident that p^{+} is unique and real.

2. In the case when there are two independent intertwining operators, every intertwining operator q^{+} can be represented in the form

$$q^{+} = \alpha_q(h^{+})p^{+} + \beta_q(h^{+})k^{+},$$

where $\alpha_q(y)$ and $\beta_q(y)$ are polynomials, p^+ is normalized intertwining operator of minimal order and k^+ is independent of p^+ normalized intertwining operator of minimal order. It is evident that p^+ is unique and real and that k^+ can be chosen with real coefficients. It can be shown that sum of orders of p^+ and k^+ is odd.

Thus the set of p^+ and k^+ (or only p^+ in the case 1) form an optimal basis of intertwining operators. As all $q^- = (q^+)^t$ the same results are translated to the set of $p^- = (p^+)^t$ and $k^- = (k^+)^t$.

7 Example: $N = 2$, $N_1 = 1$

Let us examine the algebraic structure of the simplest non-linear SUSY with two supercharges K of the 2nd order and P of the 1st order. The supersymmetries generated by K, $\bar{K}$ and P, $\bar{P}$ with components

$$k_2^{\pm} \equiv \partial^2 \mp 2f(x)\partial + \tilde{b}(x) \mp f'(x), \qquad p_1^{\pm} \equiv \mp\partial + \chi(x)$$

prescribe that

$$V_{1,2} = \chi^2 \mp \chi' = \mp 2f' + f^2 + \frac{f''}{2f} - \left(\frac{f'}{2f}\right)^2 - \frac{d}{4f^2} - a, \qquad \tilde{b} = f^2 - \frac{f''}{2f} + \left(\frac{f'}{2f}\right)^2 + \frac{d}{4f^2},$$

where χ, f are real functions and a, d are real constants. The related superalgebra closure for K, $\bar{K}$ and P, $\bar{P}$ takes the form,

$$\{K, K^\dagger\} = (H + a)^2 + d, \qquad \{P, P^\dagger\} = H,$$

the latter one clarifies the role of constants a, d.

The compatibility of two supersymmetries is achieved on solutions of the following equations

$$\chi = 2f + \chi_0, \qquad f^2 + \frac{f''}{2f} - \left(\frac{f'}{2f}\right)^2 - \frac{d}{4f^2} - a = \chi^2 = (2f + \chi_0)^2, \tag{8}$$

where χ_0 is an arbitrary real constant. The latter one represents a nonlinear second-order differential equation which solutions are parameterized by two integration constants. Therefore as it was advertised the existence of two SUSY constrains substantially the class of potentials for which they may hold.

Let us use the freedom to redefine the higher-order supercharge $k_2^{\pm} \to k_2^{\pm} + \chi_0 p_1^{\pm}$ for eliminating the constant χ_0 in (8). After this simplification the equation (8) is integrated into the following, first-order one,

$$\chi = 2f, \qquad (f')^2 = 4f^4 + 4af^2 + 4G_0 f - d \equiv \Phi_4(f),$$

where G_0 is a real constant.

The solutions of this equation are elliptic functions which can be easily found in the implicit form,

$$\int_{f_0}^{f(x)} \frac{df}{\sqrt{\Phi_4(f)}} = \pm(x - x_0),$$

where the lower limit of integration f_0 and x_0 are real constants.

It can be shown that they may be nonsingular in three situations.

a) The polynomial $\Phi_4(f)$ has four different real roots $f_1 < f_2 < f_3 < f_4$ and f_0 is chosen between f_2 and f_3. The corresponding potentials are periodic. This case will not be examined here.

b) $\Phi_4(f)$ has three different real roots and the double root $\beta/2$ is either the maximal one or a minimal one,

$$\Phi_4(f) = 4\left(f - \frac{\beta}{2}\right)^2 \left[\left(f + \frac{\beta}{2}\right)^2 - (\beta^2 - \epsilon)\right], \qquad 0 < \epsilon < \beta^2. \tag{9}$$

Then there exists a relation between constants a, d, G_0 in terms of parameters β, ϵ,

$$a = \epsilon - \frac{3\beta^2}{2} < 0, \qquad G_0 = \beta(\beta^2 - \epsilon), \qquad d = \beta^2\left(\frac{3\beta^2}{4} - \epsilon\right). \tag{10}$$

Besides, the constant f_0 is taken between the double root and a nearest simple root.

c) $\Phi_4(f)$ has two different real double roots which correspond in (9), (10) to $G_0 = 0$, $\beta^2 = \epsilon > 0$, $a = -\beta^2/2$, $d = -\beta^4/4$. The constant f_0 is taken between the roots.

The corresponding potentials $V_{1,2}$ are well known and in the cases b) and c) are reflectionless, with one bound state at the energy $E_b = (\beta^2 - \epsilon)$ and with the continuum spectrum starting from $E_c = \beta^2$. In particular, in the case b) the potentials coincide in their form and differ only by shift in the coordinate (about the latter phenomenon see [6]),

$$V_{1,2} = \beta^2 - \frac{2\epsilon}{\mathrm{ch}^2\left(\sqrt{\epsilon}(x - x_0^{(1,2)})\right)}, \qquad x^{(1,2)} = x_0 \pm \frac{1}{4\sqrt{\epsilon}} \ln\frac{\beta - \sqrt{\epsilon}}{\beta + \sqrt{\epsilon}},$$

and in the case c) one of the potentials can be chosen constant,

$$V_1 = \beta^2, \qquad V_2 = \beta^2\left(1 - \frac{2}{\mathrm{ch}^2\left(\beta(x - x_0)\right)}\right).$$

For these potentials one can illustrate all the relations of extended SUSY algebra. Thus, in particular, the polynomial symmetry operator $\mathcal{B}(H)$ turns out to be constant, $\mathcal{B}(H) = G_0$, the second symmetry operator reads,

$$\mathcal{E}(H) = i\left[\boldsymbol{I}\,\partial^3 - \left(a\boldsymbol{I} + \frac{3}{2}\boldsymbol{V}(x)\right)\partial - \frac{3}{4}\boldsymbol{V}'(x)\right]$$

and, finally, $\mathcal{E}^2(H)$ takes the form

$$\mathcal{E}^2(H) \equiv \mathcal{P}_e(H) = H\left[(H + a)^2 + d\right] - G_0^2 \equiv (H - E_b)^2(H - E_c).$$

Hence, in considered case both roots of $\mathcal{P}_e(E)$ characterize spectrum of H: one of them (E_b) characterizes the energy of a bound state and another (E_c) characterizes the energy of a state in the bottom of continuum spectrum.

8 More about symmetry operators

Let us suppose that:

a) the Hamiltonian h_0 commutes with an antisymmetric real operator R_0 of order $2n + 1$ which cannot be stripped off;

b) h_0 has n bound states with wave functions Ψ_l and energies E_l, $l = 0, \ldots, n - 1$, where $E_{l+1} > E_l$, $l = 0, \ldots, n - 2$.

Then one can show that:

a) the (normalized) symmetry operator R_0 can be factorized in the form,

$$R_0 = r_0^t \cdots r_{n-1}^t\, \partial\, r_{n-1} \cdots r_0, \qquad r_l \equiv \partial + \chi_l, \tag{11}$$

with non-singular real superpotentials χ_l such that

$$r_l \cdots r_0 \Psi_l = 0, \qquad l = 0, \ldots, n-1;$$

b) the ladder (dressing chain) relations hold,

$$h_{l+1} r_l = r_l h_l, \qquad l = 0, \ldots, n-1,$$
$$h_l \equiv r_{l-1} r_{l-1}^t + E_{l-1} = r_l^t r_l + E_l, \qquad l = 1, \ldots, n-1,$$
$$h_0 = r_0^t r_0 + E_0, \qquad h_n = r_{n-1} r_{n-1}^t + E_{n-1};$$

c) the antisymmetric symmetry operators arise for each intermediate Hamiltonian,

$$R_l = r_l^t \cdots r_{n-1}^t \, \partial \, r_{n-1} \cdots r_l, \qquad R_n = \partial, \qquad R_l h_l = h_l R_l, \qquad l = 0, \ldots, n.$$

Evidently the Hamiltonian h_n describes a free particle and therefore the Hamiltonian h_0 is intertwined with the Hamiltonian of free particle.

The described result can be illustrated with the help of the case b) of Section 7, where each component of the symmetry operator $\mathcal{E}(H)$ can be represented in the canonical factorized form (11),

$$e^{\pm} = i \left[\partial^3 - \left(a + \frac{3}{2} V_{1,2} \right) \partial - \frac{3}{4} V_{1,2}' \right] = -i \left(-\partial - \frac{\Psi_{01,2}'}{\Psi_{01,2}} \right) \partial \left(\partial - \frac{\Psi_{01,2}'}{\Psi_{01,2}} \right),$$

by means of the bound-state wave functions, $\Psi_{01,2} = C\sqrt{V_{1,2} - \beta^2}$.

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D-branes on Calabi–Yau Hypersurfaces

Ivan M. BURBAN

Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine
E-mail: *mmtpitp@bitp.kiev.ua*

The construction of BPS D-branes of the type IIB superstring theory on two- and three-parameter elliptic and $K3$ fibered Calabi–Yau hypersurfaces is discussed. The explicit maps between the characteristic classes of the Chan–Paton fiber bundle and the D-brane charges is established.

1 Introduction

In this note we study the description of BPS D-branes of the type IIB superstring theory compactified on Calabi–Yau manifolds. It is well know that superstring theory is not only theory of superstrings but also another of extended objects, D-branes, which although are not perturbative string states themselves but allow description in the perturbative string theory.

In spite of significant progress in the study of D-branes on the curved space-time spaces, in particular, on Calabi–Yau manifolds, situation is far from being understood. In the arsenal of the theoretical physics there is a number of simple Calabi–Yau manifolds, which are convenient for studying superstring theories and their D-branes. In this note we consider D-branes on the Calabi–Yau hypersurfaces of weighted projective spaces with two- and three-parameter Kähler moduli spaces. These surfaces have orbifold singularities. After resolving these singularities the hypersurfaces exhibit structures of fibered Calabi–Yau manifolds. It helps us to make the description of D-branes on these manifolds in more detail. The analysis of string dualities in four space-time dimensions requires to research D-branes wrapped on supersymmetric cycles of compactifying manifold. The study of boundary states of D-branes wrapped around of supersymmetric cycles in the quintic hypersurface of four dimension projective space was done in [1]. The detailed information on a part of spectrum of the D-branes on Calabi–Yau manifold in the geometric phase was presented in [2]. In [3] the relevant aspects were extended to some other models.

2 Weighted projective spaces

We consider weighted projective spaces $P^4_{(k_1,k_2,k_3,k_4,k_5)}$ with weights (k_1,k_2,k_3,k_4,k_5) described in five complex "homogeneous coordinates" (z_1,z_2,z_3,z_4,z_5) not all of them vanishing, which are subject to identification

$$(z_1,z_2,z_3,z_4,z_5)\simeq\left(\lambda^{k_1}z_1,\lambda^{k_2}z_2,\lambda^{k_3}z_3,\lambda^{k_4}z_4,\lambda^{k_5}z_5\right) \tag{1}$$

for nonzero λ. Thus, a weighted projective space is a generalization of ordinary projective space and in this notation $P^4=P^4_{(1,1,1,1,1)}$. These spaces, except of $P^4_{(k_1,k_2,k_3,k_4,k_5)}=P^4_{(1,1,1,1,1)}$, have orbifold singularities due to the identification (1).

Indeed, if we put $z_i=(\zeta_i)^{k_i}$ then from relation (1) we obtain

$$(\zeta_1^{k_1},\zeta_2^{k_2},\zeta_3^{k_3},\zeta_4^{k_4},\zeta_5^{k_5})\simeq\left((\lambda\zeta_1)^{k_1},(\lambda\zeta_2)^{k_2},(\lambda\zeta_3)^{k_3},(\lambda\zeta_4)^{k_4},(\lambda\zeta_5)^{k_5}\right) \tag{2}$$

that is

$$\zeta_i^{k_i} \simeq (\lambda\zeta_i)^{k_i},$$

or

$$(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5) \simeq \lambda\,(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5)$$

and

$$(\zeta_1,\zeta_2,\zeta_3,\zeta_4,\zeta_5) \simeq \left(\zeta_1 e^{\frac{2\pi i}{k_1}},\zeta_2 e^{\frac{2\pi i}{k_2}},\zeta_3 e^{\frac{2\pi i}{k_3}},\zeta_4 e^{\frac{2\pi i}{k_4}},\zeta_5 e^{\frac{2\pi i}{k_5}}\right).$$

Therefore,

$$P^4_{(k_1,k_2,k_3,k_4,k_5)} = \frac{P^4_{(1,1,1,1,1)}}{Z_{k_1}\times Z_{k_2}\times Z_{k_3}\times Z_{k_4}\times Z_{k_5}}$$

and we have obtained a manifold $P^4_{(k_1,k_2,k_3,k_4,k_5)}[d], d = k_1+k_2+k_3+k_4+k_5$, with orbifold singularities.

We wish to study D-branes on the simplest Calabi–Yau hypersurfaces defined by polynomials on the homogeneous coordinates in weighted projective spaces.

The one-parameter Calabi–Yau hypersurfaces M [4] are given by

$$z_1^5+z_2^5+z_3^5+z_4^5+z_5^5=0, \qquad (z_1:z_2:z_3:z_4:z_5)\in P^4_{(1,1,1,1,1)}[5], \tag{3}$$

$$z_1^6+z_2^6+z_3^6+z_4^6+z_5^3=0, \qquad (z_1:z_2:z_3:z_4:z_5)\in P^4_{(1,1,1,1,2)}[6], \tag{4}$$

$$z_1^8+z_2^8+z_3^8+z_4^8+z_5^2=0, \qquad (z_1:z_2:z_3:z_4:z_5)\in P^4_{(1,1,1,1,4)}[8], \tag{5}$$

$$z_1^{10}+z_2^{10}+z_3^{10}+z_4^5+z_5^2=0, \qquad (z_1:z_2:z_3:z_4:z_5)\in P^4_{(1,1,1,2,5)}[10]. \tag{6}$$

In order to study the stringy geometry of these manifolds we consider one-parameter family of mirror manifolds W given by the Greene–Plesser orbifold construction $\{p_i=0\}/G_i$

$$z_1^5+z_2^5+z_3^5+z_4^5+z_5^5-5\psi z_1z_2z_3z_4z_5=0, \qquad G=Z_5\times Z_5, \tag{7}$$

$$z_1^6+z_2^6+z_3^6+z_4^6+z_5^3-6\psi z_1z_2z_3z_4z_5=0, \qquad G=Z_3\times Z_6^2, \tag{8}$$

$$z_1^8+z_2^8+z_3^8+z_4^8+z_5^2-8\psi z_1z_2z_3z_4z_5=0, \qquad G=Z_2\times Z_8^2, \tag{9}$$

$$z_1^{10}+z_2^{10}+z_3^{10}+z_4^5+z_5^2-10\psi z_1z_2z_3z_4z_5=0, \qquad G=Z_{10}^2. \tag{10}$$

The stringy geometry of the Calabi–Yau hypersurfaces M [5,6].

$$8z_1^8+z_2^8+z_3^4+z_4^4+z_5^4=0, \qquad (z_1:z_2:z_3:z_4:z_5)\in P^4_{(1,1,2,2,2)}, \tag{11}$$

$$z_1^{12}+z_2^{12}+z_3^6+z_4^6+z_5^2=0, \qquad (z_1:z_2:z_3:z_4:z_5)\in P^4_{1,1,2,2,6}, \tag{12}$$

$$z_1^{18}+z_2^{18}+z_3^{18}+z_4^3+z_5^2=0, \qquad (z_1:z_2:z_3:z_4:z_5)\in P^4_{1,1,1,6,9}. \tag{13}$$

are defined by the two-parameter families of the mirror manifolds W obtained by the Greene–Plesser orbifold construction $\{p_i=0\}/G_.$

$$z_1^8+z_2^8+z_3^4+z_4^4+z_5^4-8\psi z_1z_2z_3z_4z_5-2\phi z_1^4z_2^4=0, \qquad G=Z_4^3, \tag{14}$$

$$z_1^{12}+z_2^{12}+z_3^6+z_4^6+z_5^2-12\psi z_1z_2z_3z_4z_5-2\phi z_1^6z_2^6=0, \qquad G=Z_6^2\times Z_2, \tag{15}$$

$$z_1^{18}+z_2^{18}+z_3^{18}+z_4^3+z_5^2-18\psi z_1z_2z_3z_4z_5-3\phi x_1x_2x_3=0, \qquad G=Z_{18}. \tag{16}$$

The Calabi–Yau hypersurface M defined by the equation

$$z_1^{24}+z_2^{24}+z_3^{12}+z_4^3+z_5^2=0, \qquad (z_1:z_2:z_3:z_4:z_5)\in P^4_{(1,1,2,8,12)} \tag{17}$$

has its three-parameter mirror family of W obtained by the Greene–Pleser orbifold construction

$$z_1^{24}+z_2^{24}+z_3^{12}+z_4^3+z_5^2-12\psi_0 z_1z_2z_3z_4z_5 \\ -2\psi_1(z_3z_4z_5)^6-\psi_2(z_4z_5)^{12}=0, \qquad G=Z_6\times Z_{12}. \tag{18}$$

3 Classical geometry of elliptic and $K3$ fibrations

The hypersurface $P^4_{(1,1,1,6,9)}[18]$ has two divisors E and L, which generate $H_4(M,Z)$. The exceptional divisor E in M is generated by blowing up singular line $z_2 = z_3 = 0$, which intersects M in a single point. The divisor L is defined by first order polynomial in z_1, z_2 and z_3.

The two homology classes h, l generate homology group $H_2(M,Z)$. The elliptic fibration structure is induced by linear system $|L|$ which maps M to P^2. The generic fiber can be proved to be an elliptic curve. The homology class l is a hyperplane class of E.

The choice (E,L) as generators of the complexified Kähler cone leads to generic Kähler class $K = t_1E + t_2L$, where (t_1,t_2) are classical coordinates on Kähler moduli space of M.

The models $P^4_{(1,1,2,2,2)}[8]$ and $P^4_{(1,1.2,2,6)}[12]$ given by degree 8 and 12 hypersurfaces both at $z_1 = z_2 = 0$ have a curve C of singularities blowing up of which leads to exceptional divisor E in M. The second divisor class L of both models is $K3$ fiber of fibration of M over P^1, generated by a linear system $|L|$ which is generated by the degree one polynomials. The divisor classes E and L together generate $H^4(M,Z)$. In both models the degree two polynomials generate the linear system $|H| = |2L+E|$. The homology group $H_2(M,Z)$ is generated by the classes l and h, which are defined by $h = \frac{1}{4}H\cdot L$, $l = \frac{1}{4}H\cdot E$ for degree 8 model and $h = \frac{1}{2}H\cdot L$, $l = \frac{1}{2}H\cdot E$ for degree 12 model. The intersection relations for these models are

$$\begin{aligned}
&L^3 = 0, \qquad E\cdot h = 1,\\
&E^3 = 9, \qquad E^2\cdot L = -3, \qquad E\cdot L^2 = 1,\\
&h = L^2, \qquad l = E\cdot H\cdot L, \qquad E\cdot l = -2,
\end{aligned} \tag{19}$$

where $H = E + 3L$, for $P^4_{(1,1,1,6,9)}[18]$ hypersurface,

$$\begin{aligned}
&H^3 = 8, \qquad H^2\cdot L = 4, \qquad H\cdot L^2 = 0, \qquad L^3 = 0,\\
&E^3 = 16, \qquad E^2\cdot L = 4, \qquad E\cdot L^2 = 0, \qquad H\cdot E\cdot L = 4,\\
&h = 1/4H\cdot L, \qquad l = 1/4H\cdot E, \qquad H\cdot h = 1, \qquad E\cdot l = -2,\\
&L\cdot l = 1, \qquad L\cdot h = 0, \qquad H\cdot l = 0, \qquad E\cdot h = 1,\\
&c_2(M)\cdot E = 8, \qquad c_2(M)\cdot H = 56,
\end{aligned} \tag{20}$$

where $|H| = |2L + E|$, for $P^4_{(1,1,2,2,6)}[18]$ and

$$\begin{aligned}
&H^3 = 4, \qquad H^2\cdot L = 2, \qquad H\cdot L^2 = 0, \qquad L^3 = 0,\\
&E^3 = -8, \qquad E^2\cdot L = 2, \qquad E\cdot L^2 = 0, \qquad H\cdot E\cdot L = 2,\\
&h = 1/4H\cdot L, \qquad l = 1/4H\cdot E, \qquad H\cdot h = 1, \qquad E\cdot l = -2,\\
&L\cdot l = 1, \qquad L\cdot h = 0, \qquad H\cdot l = 0, \qquad E\cdot h = 1,\\
&c_2(M)\cdot E = 8, \qquad c_2(M)\cdot H = 56,
\end{aligned} \tag{21}$$

where $|H| = |2L + E|$, for $P^4_{(1,1,2,6,9)}[18]$ models. The choice (E,L) as generators of the complexified Kähler cone leads to generic Kähler class $K = t_1E + t_2L$, where (t_1,t_2) are classical coordinates on the Kähler moduli space of M.

4 D-branes and periods

In the points of large volume limit of Kähler moduli space we can relate the BPS charge lattice of the low energy effective theory (the integral symplectic lattice $H^3(W,Z)$ of the middle cohomology of mirror manifold W) with BPS charge lattice of microscopic D-brane charges (the integer

quadratic lattice $K(M)$ of K-theory of manifold M). An every vector $\boldsymbol{n} = (n_6, n_4^1, n_4^2, n_0, n_2^1, n_2^2)$ of middle cohomology $H^3(W, Z)$ of the mirror manifold W defines central charge

$$Z(\boldsymbol{n}) = n_6\Pi_1 + n_4^1\Pi_2 + n_4^2\Pi_3 + n_0\Pi_4 + n_2^1\Pi_5 + n_2^2\Pi_6. \tag{22}$$

On the other hand every element $\eta \in K(M)$ corresponding to topological invariant $\mathrm{ch}\,(\eta)$ defines the effective charge (the Mukai vector) of D-brane

$$Q = \mathrm{ch}\,(\eta)\sqrt{\mathrm{Td}\,(\mathcal{E})} \in H^0(M) \oplus H^2(M) \oplus H^4(M) \oplus H^6(M). \tag{23}$$

The central charge associated to the state η is

$$Z(t) = \int_M \frac{t^3}{6}Q^0 - \frac{t^2}{2}Q^2 + tQ^4 - Q^6. \tag{24}$$

Equating (22) and (24) we obtain relations between the topological invariant of the K-theory class η and low energy charges of a D-brane $\boldsymbol{n} = (n_6, n_4^1, n_4^2, n_0, n_2^1, n_2^2)$ in the large radius limit.

5 Examples

For example, in type IIB superstring theory we consider the topological invariants of D6-branes wrapped on $P^4_{(1,1,1,6,9)}[18]$.

In this case $\mathcal{E}$ is the vector bundle (the coherent sheaf) on the manifold M. The standard formulae

$$\mathrm{ch}\,(\mathcal{E}) = r + c_1 + \frac{1}{2}\left(c_1^2 - 2c_2\right) + \frac{1}{6}\left(c_1^2 - 3c_1c_2 + c_3\right) + \cdots,$$
$$\mathrm{Td}\,(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{24}c_1c_2 + \cdots.$$

give possibility to rewrite the charge (23) in the form

$$Q = \left(r, c_1(\mathcal{E}), \mathrm{ch}_2(\mathcal{E}) + \frac{r}{24}c_2(M), \mathrm{ch}_3(\mathcal{E}) + \frac{1}{24}c_1(\mathcal{E})c_2(M)\right). \tag{25}$$

At the large radius limit we have following expression for these periods of holomorphic three form $\hat{\Omega}$ of the mirror manifolds W.

$$\begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \\ \Pi_5 \\ \Pi_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(3t_1^3 + 3t_1^2t_2 + t_1t_2^2) + \frac{17}{4}t_1 + \frac{3}{2}t_2, \\ -\frac{1}{2}t_2^2 + \frac{3}{2}t_1 + \frac{3}{2}, \\ -\frac{1}{2}(3t_1^2 + 2t_1t_2) + \frac{3}{2}t_1 + \frac{3}{2}, \\ 1 \\ t_1 \\ t_2 \end{pmatrix}, \tag{26}$$

for $P^4_{(1,1,2,6,9)}[18]$ surface,

$$\begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \\ \Pi_5 \\ \Pi_6 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}t_1^3 + t_1^2t_2 + \frac{13}{6}t_1 + t_2 \\ \frac{1}{6} - 2t_1t_2 \\ 1 - t_1^2 \\ 1 \\ t_1 \\ t_2 \end{pmatrix}, \quad \begin{pmatrix} \Pi_1 \\ \Pi_2 \\ \Pi_3 \\ \Pi_4 \\ \Pi_5 \\ \Pi_6 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}(3t_1^3 + 2t_1t_2 + \frac{7}{3}t_1 + t_2 \\ -4t_1t_2 + 4t_1 - 2t_2 + \frac{1}{3} \\ -2t_1^2 - 2t_1 + 1 \\ 1 \\ t_1 \\ t_2 \end{pmatrix}. \tag{27}$$

for $P^4_{(1,1,2,2,2)}[8]$ and $P^4_{(1,1,2,2,6)}[12]$ surfaces, respectively. Using formulae (22), (23), (24), (26), (27) we obtain the topological invariants of D6-branes

$$\begin{aligned}
&r = n_6,\\
&\mathrm{ch}_1(\mathcal{E}) = n_4^1 E + n_4^2 L,\\
&\mathrm{ch}_2(\mathcal{E}) = \left(\frac{3}{2}n_4^2 + n_2^1\right) h + \left(\frac{3}{2}n_4^1 + n_2^2\right) l\\
&\mathrm{ch}_3(\mathcal{E}) = -n_0 + \frac{1}{2}n_4^1 - 3n_4^2,
\end{aligned} \tag{28}$$

wrapped on $P^4_{(1,1,1,6,9)}[18]$ surface,

$$\begin{aligned}
&r = n_6,\\
&\mathrm{ch}_1(\mathcal{E}) = n_4^1 E + n_4^2 L,\\
&\mathrm{ch}_2(\mathcal{E}) = (4n_4^1 - 2n_4^2 + n_2^1)h + (-2n_4^1 + n_2^2)l,\\
&\mathrm{ch}_2(\mathcal{E}) = -n_0 - \frac{2}{3}n_4^1 - 2n_4^2,
\end{aligned} \tag{29}$$

wrapped on $P^4_{(1,1,2,2,2)}[8]$,

$$\begin{aligned}
&r = n_6,\\
&\mathrm{ch}_1(\mathcal{E}) = n_4^1 E + n_4^2 L,\\
&\mathrm{ch}_2(\mathcal{E}) = 4n_2^1 h + n_2^2 l,\\
&\mathrm{ch}_2(\mathcal{E}) = -n_0 - \frac{1}{3}n_4^1 - 2n_4^2,
\end{aligned} \tag{30}$$

wrapped on $P^4_{(1,1,2,2,6)}[12]$.

6 Localized D-branes on hypersurfaces

Let $i : D \hookrightarrow M$ be embedding of an even projective surface D in a Calabi–Yau manifold M. We can obtain class of D-brane states by wrapping D-branes on submanifold D. Multiple brane configurations are described by coherent sheaf $\mathcal{E}$ on D. The charge of this configuration is defined by torsion coherent sheaf $i_*\mathcal{E}$ which is the extension $\mathcal{E}$ by zero to the manifold M. The Grothendieck–Riemann–Roch formula for the embedding $i : D \hookrightarrow M$ gives Mukai vector

$$i_*(\mathrm{ch}\,(\mathcal{E})\,\mathrm{Td}\,(D)) = \mathrm{ch}\,(i_*\mathcal{E})\,\mathrm{Td}\,(M).$$

From this formula and from the topological invariants of the sheaf $\mathcal{E}$ on D we obtain topological invariants of the torsion sheaf $i_*\mathcal{E}$

$$\begin{aligned}
&\mathrm{ch}_1(i_*\mathcal{E}) = rD,\\
&\mathrm{ch}_2(i_*\mathcal{E}) = i_*c_1(\mathcal{E}) + \frac{r}{2}i_*c_1(\mathcal{E}),\\
&\mathrm{ch}_2(i_*\mathcal{E}) = i_*\left(\mathrm{ch}_2(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{E})c_1(D) + \frac{1}{12}(c_1(D))^2 + c_2(D)\right) - \frac{r}{12}Dc_2(M).
\end{aligned} \tag{31}$$

and Mukai charge vector

$$\begin{aligned}
Q = \Big(&0, rD, i_*c_1(\mathcal{E}) + \frac{r}{2}i_*c_1(D),\\
&\mathrm{ch}_2(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{E})c_1(D) + \frac{r}{12}\big(c_1(D) + c_2(D)\big) - \frac{r}{24}Dc_2(D)\Big).
\end{aligned} \tag{32}$$

By means of an adjunction formula

$$Dc_2(M) = c_2(D) - D^2 = c_2(D) - D^3 = c_2(D) - c_1(D)^2$$

this Mukai charge vector can be represented as

$$Q = \left(0, rD, i_*c_1(\mathcal{E}) + \frac{r}{2}i_*c_1(D), \mathrm{ch}_2(\mathcal{E}) + \frac{1}{2}c_1(\mathcal{E})c_1(D) + \frac{r}{8}c_1(D)^2 + \frac{r}{24}Dc_2(D)\right) \tag{33}$$

and central charge associated to it can be rewritten

$$\begin{aligned} Z(Q) = &-\frac{r}{2}t^2D + \left(i_*c_1(\mathcal{E}) + \frac{r}{2}i_*c_1(D)\right)t - \mathrm{ch}_2(\mathcal{E}) \\ &- \frac{1}{2}c_1(\mathcal{E})c_1(D) - \frac{r}{8}c_1(D)^2 - \frac{r}{24}c_1(D). \end{aligned} \tag{34}$$

In type IIB superstring theory the topological invariants D-4 branes wrapped on some hypersurfaces of the Calabi–Yau manifold can be defined. It is easy to obtain topological invariants of the D4-branes wrapped on the exceptional divisor E corresponding to BPS state with the charge vector $\boldsymbol{n} = (0, n_4^1, 0, n_0, 0, n_2^2)$ and with the central charge $Z(\boldsymbol{n}) = n_4^1\mathcal{F}^1 + n_2^2t^2 + n_0$ for the $P^4_{(1,1,2,6,9)}$ model

$$\begin{aligned} &r = n_4^1, \\ &c_1(\mathcal{E}) = n_l^2, \\ &c_2(\mathcal{E}) = \frac{1}{2}n_2^2(n_2^2+3) + n_4^1 + n_0), \end{aligned} \tag{35}$$

and with the charge $\boldsymbol{n} = (0, n_4^1, 0, n_0, n_2^1, n_2^2)$ and the central charges $Z(\boldsymbol{n}) = n_4^1\mathcal{F}^1 + n_2^1t^1 + n_2^2t_2 + n_0$ for the $P^4_{(1,1,2,2,2)}[8]$ and $P^4_{(1,1,2,2,6)}[12]$ models

$$\begin{aligned} &r(\mathcal{E}) = n_4^1, \\ &c_1(\mathcal{E}) = (n_2^1 + \nu n_4^1)h + \left(n_2^2 - \frac{\nu}{2}n_4^1\right)l, \\ &\mathrm{ch}(\mathcal{E}) = -\frac{3}{2}\nu n_4^1 - \frac{1}{2}n_2^1 + n_2^2 - n_0. \end{aligned} \tag{36}$$

where $\nu = -2, -4$. The same method may be used to define the topological invariants of D-branes on the surface (17). This surface is an elliptic fibration over a Hirzebruch surface which itself is a P^1-fibration over P^1. As in the cases considered above this fibration structure is essentially for investigation of the monodromy around singular points in the moduli space.

7 Monodromy

Consider complex function $f(z) = \sqrt{z}$. If we circle once the origin $z \to z\exp 2i\pi$, then $f(z\exp 2i\pi) = -f(z)$. The function does not return back to itself. This is signal that the function at the point $z = 0$ is singular and in this case it is the start of a branch cut.

The behavior of a complex function or set of functions after transport around a singular point (singularity) is connected with their monodromy. In general, a set of function transported once around the singular point t_0 return to the linear combination of themselves $f_i((z-z_0)\exp 2i\pi) = m_{ij}(z_0)f_j(z)$. The matrix m_{ij} depend on the singular point and it is called by the monodromy matrix. It does not change under smooth deformations of the contour. The monodromy has topological characteristic.

In general, one of the important problems of algebraic geometry is to study families of algebraic variety parameterized by another variety. A subvariety in moduli space parameterizing the singular fibers has the special interest. This subvariety is called the discriminant locus. One of the main its topological invariants is so-called monodromy group. It is defined by the action of homotopy group of the complement of discriminant locus on cohomology ring of fixed nonsingular fiber. The large radius limit point is situated in the intersection of the two divisors H and L on the boundary of the moduli space [6]. The monodromy matrices of the two divisors E, H, are expressed with respect to the (E, L) basis [5, 6] by

$$S_L = \begin{pmatrix} 1 & -1 & -3 & 10 & 9 & 3 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_H = \begin{pmatrix} 1 & 0 & -1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{37}$$

for $P^4_{(1,1,1,6,9)}[18]$ model,

$$S_L = \begin{pmatrix} 1 & 0 & -1 & 2 & -2 & 0 \\ 0 & 1 & 0 & -2 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_H = \begin{pmatrix} 1 & -1 & -2 & 6 & 4 & 0 \\ 0 & 1 & 0 & 4 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \tag{38}$$

for $P^4_{(1,1,2,2,2)}[8]$ model,

$$S_L = \begin{pmatrix} 1 & 0 & -1 & 20 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad S_H = \begin{pmatrix} 1 & -1 & -2 & 5 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{39}$$

for $P^4_{(1,1,2,2,6)}[12]$ model. We can use above formulae to convert the monodromy transformations (37), (38), (39) into automorphisms of K-group $K(M)$

$$[\mathcal{E}] \to [\mathcal{E} \otimes \mathcal{O}_M(L)], [\mathcal{E}] \to [\mathcal{E} \otimes \mathcal{O}_M(H)], \tag{40}$$

where $[\mathcal{E}] \in K(M)$. The topological invariants of the sheaf $\mathcal{E}' = \mathcal{E} \otimes \mathcal{O}_M(D)$ under the transformation (40) change to the form

$$\begin{aligned} &r(\mathcal{E}') = r(\mathcal{E}), \\ &\mathrm{ch}_1(\mathcal{E}') = \mathrm{ch}_1(\mathcal{E}) + rD, \\ &\mathrm{ch}_2(\mathcal{E}') = \mathrm{ch}_2(\mathcal{E}) + \mathrm{ch}_1(\mathcal{E})D + \frac{r}{2}D^2, \\ &\mathrm{ch}_3(\mathcal{E}') = \mathrm{ch}_3(\mathcal{E}) + \mathrm{ch}_2(\mathcal{E})D + \frac{1}{2}\mathrm{ch}_1(\mathcal{E})D^2 + \frac{r}{6}D^3, \end{aligned} \tag{41}$$

where $D = H$, L. The representations of these transformations on the space of charge vectors $\boldsymbol{n} = (n_0, n_4^1, n_4^2, n_0, n_2^1, n_2^2)$ and transformations (37), (38), (39) are connected by the relation

$$M(D) = S_D^{-1}. \tag{42}$$

8 Conclusions

In this paper, we have considered the method of the D-brane constructions in superstring theory corresponding to Calabi–Yau compactification where Calabi–Yau manifolds appear as hypersurfaces in the weighted projective space. In the large volume "phase" of the Kähler moduli space this construction provides clear geometric picture of the D-brane spectrum. We find that this construction gives clear picture of the large volume monodromy transformations. All the above is a simple particular case of a more ambitious program which proposes for type IIB superstring theory an identification of the D-branes by objects of a derived category of the category of coherent sheaves [7, 8] (see also [9]).

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DGLAP and BFKL Equations in $N = 4$ Supersymmetric Gauge Theory

A.V. KOTIKOV †, *L.N. LIPATOV* ‡ *and V.N. VELIZHANIN* ‡

† *Univ. Karlsruhe, D-76128 Karlsruhe, Germany and BLThPh, JINR, 141980 Dubna, Russia*
E-mail: *kotikov@thsun1.jinr.ru*

‡ *Theoretical Physics Department, PNPI, Gatchina, 188300 St. Petersburg, Russia*
E-mail: *lipatov@thd.pnpi.spb.ru, velizh@thd.pnpi.spb.ru*

We show the results for the DGLAP and BFKL evolution equations in the $N = 4$ supersymmetric gauge theory obtained earlier in the next-to-leading approximation. The eigenvalue of the BFKL kernel in this model turns out to be an analytic function of the conformal spin $|n|$. The corresponding kernel for the Bethe–Salpeter equation has the property of the Hermitian separability. The anomalous dimension matrix can be transformed to a triangle form with the use of the similarity transformation for the diagonalization of the anomalous dimension matrix in the leading order. The eigenvalues of these matrices can be expressed in terms of a universal function by an integer shift of its argument. We also investigate in this approximation possible relations between the DGLAP and BFKL equations.

1 Introduction

The Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation [1] and the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equation [2] are used now for a theoretical description of structure functions of the deep-inelastic ep scattering at small values of the Bjorken variable x. The higher-order QCD corrections to the splitting kernels of the DGLAP equation are well known. But the calculation of the next-to-leading order (NLO) corrections to the BFKL kernel was completed recently [3, 4].

In supersymmetric gauge theories the structure of the BFKL and DGLAP equations is simplified significantly. In the case of an extended $N = 4$ SUSY the NLO corrections to the BFKL equation were calculated in ref. [4] for arbitrary values of the conformal spin n. Below these results are presented in the dimensional reduction ($\overline{\rm DR}$) scheme [5] which does not violate the supersymmetry. The analyticity of the eigenvalue of the BFKL kernel as a function of the conformal spin $|n|$ gives a possibility to relate in the leading logarithmic approximation (LLA) the DGLAP and BFKL equations in this model (see [4]). It was shown [6] that the eigenvalue of the BFKL kernel has the property of Hermitian separability similar to the holomorphic separability.

Let us introduce the unintegrated parton distributions $\varphi_a(x, k_\perp^2)$ (hereafter $a = q, g, \varphi$ for the spinor, vector and scalar particles, respectively) and the (integrated) parton distributions $n_a(x, Q^2)$. In DIS $Q^2 = -q^2$ and $x = Q^2/(2pq)$ are the Bjorken variables, $k_\perp$ is the transverse component of the parton momentum and q and p are the photon and hadron momenta, respectively.

After the Mellin transformation of partonic distributions $n_a\big(x, Q^2\big)$ the DGLAP equation can be written as follows [2]

$$\frac{d}{d\ln Q^2}\, n_a\big(j, Q^2\big) = \sum_b \gamma_{ab}(j)\, n_b\big(j, Q^2\big) \qquad \left(n_a\big(j, Q^2\big) = \int_0^1 dx\, x^{j-1} n_a\big(x, Q^2\big)\right).$$

The Mellin moment of the splitting kernel $\gamma_{ab}(j)$ coincides with the anomalous dimension

(AD) matrix for the twist-2 operators[1]. These operators are constructed as bilinear combinations of the fields which describe corresponding partons

$$
\begin{aligned}
&O^g_{\mu_1,\ldots,\mu_j} = \hat{S} G_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \cdots D_{\mu_{j-1}} G_{\rho\mu_j}, \qquad && \tilde{O}^g_{\mu_1,\ldots,\mu_j} = \hat{S} G_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \cdots D_{\mu_{j-1}} \tilde{G}_{\rho\mu_j},\\
&O^q_{\mu_1,\ldots,\mu_j} = \hat{S} \overline{\Psi} \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_j} \Psi, && \tilde{O}^q_{\mu_1,\ldots,\mu_j} = \hat{S} \overline{\Psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_j} \Psi,\\
&O^\varphi_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Phi} D_{\mu_1} D_{\mu_2} \cdots D_{\mu_j} \Phi, && \qquad (1)
\end{aligned}
$$

where the spinor Ψ and field tensor $G_{\rho\mu}$ describe gluinos and gluons, respectively. The last expression is constructed from the covariant derivatives D_μ of the scalar field Φ appearing in extended supersymmetric models. The symbol $\hat{S}$ implies a symmetrization of the tensor in the Lorenz indices $\mu_1, \ldots, \mu_j$ and a subtraction of its traces.

On the other hand, the BFKL equation relates the unintegrated gluon distributions with small values of the Bjorken variable x:

$$
\frac{d}{d\ln(1/x)} \varphi_g(x, k_\perp^2) = 2\omega(-k_\perp^2) \varphi_g(x, k_\perp^2) + \int d^2 k'_\perp K(k_\perp, k'_\perp) \varphi_g\big(x, (k'_\perp)^2\big),
$$

where $\omega(-k_\perp^2) < 0$ is the gluon Regge trajectory [1].

The matrix elements of $O^a_{\mu_1,\ldots,\mu_j}$ and $\tilde{O}^a_{\mu_1,\ldots,\mu_j}$ are related to the moments of the parton distributions $n_a\big(x, Q^2\big)$ and $\Delta n_a\big(x, Q^2\big)$ in a hadron h in the following way

$$
\begin{aligned}
&\int_0^1 dx\, x^{j-1} n_a\big(x, Q^2\big) = \langle h \,|\, \tilde{n}^{\mu_1} \cdots \tilde{n}^{\mu_j} O^a_{\mu_1,\ldots,\mu_j} \,|\, h \rangle, \qquad a = (q, g, \varphi),\\
&\int_0^1 dx\, x^{j-1} \Delta n_a\big(x, Q^2\big) = \langle h \,|\, \tilde{n}^{\mu_1} \cdots \tilde{n}^{\mu_j} \tilde{O}^a_{\mu_1,\ldots,\mu_j} \,|\, h \rangle, \qquad a = (q, g). \qquad (2)
\end{aligned}
$$

Here the vector $\tilde{n}^\mu$ is light-like: $\tilde{n}^2 = 0$. Note, that in the deep-inelastic ep scattering we have $\tilde{n}^\mu \sim q^\mu + x p^\mu$.

The quantum numbers appeared in the solution of the BFKL equation being the integer conformal spin $|n|$ and the quantity $1 + \omega$ (ω is an eigenvalue of the BFKL kernel) coincides respectively with the total numbers of transversal and longitudinal Lorentz indices of the tensor $O^a_{\mu_1,\ldots,\mu_J}$ with the rank $J = 1 + \omega + |n|$. The corresponding matrix elements can be expressed through the solution of the BFKL equation

$$
\tilde{n}^{\mu_1} \cdots \tilde{n}^{\mu_{1+\omega}} l_\perp^{\mu_{2+\omega}} \cdots l_\perp^{\mu_j} \langle P \,|\, O^g_{\mu_1,\ldots,\mu_j} \,|\, P \rangle \sim \int_0^1 dx x^\omega \int d^2 k_\perp \left(\frac{k_\perp}{|k_\perp|} \right)^n \varphi_g\big(x, k_\perp^2\big).
$$

It is important that the AD matrices $\gamma_{ab}(j)$ and $\tilde{\gamma}_{ab}(j)$ for the twist-2 operators $O^a_{\mu_1,\ldots,\mu_j}$ and $\tilde{O}^a_{\mu_1,\ldots,\mu_j}$ do not depend on various projections of indices due to the Lorentz invariance. But generally the Lorentz spin j is less then J. Thus, it looks reasonable to extract some additional information concerning the parton x-distributions satisfying the DGLAP equation from the analogous $k_\perp$-distributions satisfying the BFKL equation.

In LLA the integral kernel of the BFKL equation is the same in all supersymmetric gauge theories. Due to the Möbius invariance in the impact parameter representation the solution of the homogeneous BFKL equation has the form (see [7])

$$
E_{\nu,n}(\overrightarrow{\rho_{10}}, \overrightarrow{\rho_{20}}) \equiv \langle \phi(\overrightarrow{\rho_1}) O_{m,\tilde{m}}(\overrightarrow{\rho_0})\, \phi(\overrightarrow{\rho_2}) \rangle = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^m \left(\frac{\rho^*_{12}}{\rho^*_{10}\rho^*_{20}} \right)^{\widetilde{m}},
$$

[1] As in ref. [3,4], the anomalous dimensions differ from those used usually in the description of DIS by a factor (-2), i.e. $\gamma_{ab}(j) = (-1/2)\gamma^{\rm DIS}_{ab}(j)$.

where $m = 1/2 + i\nu + n/2$ and $\widetilde{m} = 1/2 + i\nu - n/2$ are conformal weights related to the eigenvalues of the Casimir operators of the Möbius group. We introduced also the complex variables $\rho_k = x_k + iy_k$ in the transverse subspace and used the notation $\rho_{kl} = \rho_k - \rho_l$.

For the principal series of the unitary representations the quantities ν and n are respectively real and integer numbers. The projection n of the conformal spin $|n|$ can be positive or negative, but the eigenvalue of the BFKL equation in LLA [1]

$$\omega = \omega^0(n,\nu) = 8\overline{a}\left(\Psi(1) - \mathrm{Re}\,\Psi\left(\frac{1}{2} + i\nu + \frac{|n|}{2}\right)\right), \qquad \overline{a} = \frac{g^2 N_c}{16\pi^2} = \frac{\alpha_s N_c}{4\pi} \tag{3}$$

depends only on $|n|$. The Möbius invariance takes place also for the Schrödinger equation describing the composite states of several reggeized gluons [8]. As a consequence of the relation

$$\omega^0(n,\nu) = \omega^0(m) + \omega^0(\widetilde{m}), \qquad \omega^0(m) = 2\overline{a}\big(2\Psi(1) - \Psi(m) - \Psi(1-m)\big).$$

one obtains the property of the holomorphic separability $H = h + h^*$, $[h, h^*] = 0$ for the Hamiltonian of an arbitrary number of reggeized gluons in the multi-colour QCD [9]. In the same limit the BFKL dynamics is completely integrable [10, 11] and the holomorphic Hamiltonian h coincides with the local Hamiltonian for an integrable Heisenberg spin model [12]. Moreover the theory turns out to be invariant under the duality transformation [13]. Presumably the remarkable mathematical properties of the reggeon dynamics in LLA are consequences of the extended $N = 4$ supersymmetry [14]. It was argued also in ref. [14] that generalized DGLAP equations for the matrix elements of quasi-partonic operators [15] for $N = 4$ SUSY are integrable.

The solution of the inhomogeneous BFKL equation in the LLA approximation can be constructed in the impact parameter representation and for $\overrightarrow{\rho_{1'}} \to \overrightarrow{\rho_{2'}}$ we obtain [7]

$$\langle \phi(\overrightarrow{\rho_1})\,\phi(\overrightarrow{\rho_2})\,\phi(\overrightarrow{\rho_{1'}})\,\phi(\overrightarrow{\rho_{2'}})\rangle \sim \sum_n C(\nu_\omega, |n|)\frac{E_{\nu_\omega,|n|}(\overrightarrow{\rho_{11'}}, \overrightarrow{\rho_{21'}})}{\omega'(|n|,\nu_\omega)}|\rho_{1'2'}|^{1+2i\nu_\omega}\left(\frac{\rho_{1'2'}}{\rho^*_{1'2'}}\right)^{|n|/2},$$

where ν_ω is a solution of the equation $\omega = \omega^0(|n|,\nu)$ with $\mathrm{Im}\,\nu_\omega < 0$.

The above asymptotics has a simple interpretation in terms of the Wilson operator-product expansion of two reggeon fields produced the local operator $O_{\nu_\omega,|n|}(\overrightarrow{\rho_{1'}})$ having the transverse dimension $\Gamma_\omega = 1 + i\nu_\omega$ calculated in units of a squared mass. The corresponding tensor has a mixed projections of a gauge-invariant tensor O with $J = 1 + \omega + |n|$ indices. Note, that because Γ_ω is real in the deep-inelastic regime $\rho_{12} \to 0$, the operator $O_{\nu_\omega,|n|}(\overrightarrow{\rho_{1'}})$ belongs to an exceptional series of unitary representations of the Möbius group (see [16]).

The AD $\gamma(j)$ obtained from the BFKL equation in LLA (3) has the poles

$$\Gamma_\omega = 1 + \frac{|n|}{2} - \gamma(j), \qquad \gamma(j)|_{\omega\to 0} = \frac{4\overline{a}}{\omega}. \tag{4}$$

The operator $O_{\nu_\omega,|n|}$ for $|n| = 1, 2, \ldots$ has the twist higher than 2 because its AD γ is singular at $\omega \to 0$. In the paper [4] the analytic continuation of the BFKL AD $\gamma(|n|,\omega)$ to the points $|n| = -r - 1$ $(r = 0, 1, 2, \ldots)$ was suggested to calculate the AD singularities in the twist-2 operators in negative integer $J = 1 + \omega + |n| \to -r$. Because for positive $|n|$ and $\omega \to 0$ the quantity $\gamma(|n|,\omega)$ corresponds to higher twist operators, to obtain γ for the twist-2 operators, one should push $\Delta|n| = |n| + r + 1$ to zero at $\omega \to 0$ sufficiently rapidly $\Delta|n| = C(r)\omega^2$. In LLA the results obtained from the BFKL equation $\gamma(j)|_{j\to -r} = \frac{4\overline{a}}{j+r}$ for $r = -1, 0, 1, \ldots$ coincide with the direct calculation of the eigenvalues of the AD matrix $\gamma_{a,b}$ for $N = 4$ SUSY in accordance with the fact, that only in this theory the eigenvalue of the BFKL equation is an analytic function of $|n|$ [4]. It is important that the AD is the same for all twist-2 operators entering in the $N = 4$ supermultiplet up to a shift of its argument by an integer number, because this

property leads to an integrability of the evolution equations for quasi-partonic operators [15] in the multi-color limit $N_c \to \infty$ (see [14]).

In the NLO approximation for the BFKL equation there is a difficulty related to an appearance of the double-logarithmic (DL) terms leading to triple poles at $j = -r$ for even r. The origin of the DL terms can be understood in a simple way using as an example the process of the forward annihilation of the e^+e^- pair in the $\mu^+\mu^-$ pair in QED [17]. For this process the t-channel partial wave f_ω ($\omega = j$) in the DL approximation can be written as follows $f_\omega = \frac{\gamma^{-1}}{\omega} - \frac{\alpha_{em}}{2\pi\left(\frac{1}{\omega-\gamma}+\frac{1}{\gamma}\right)}$. By expanding perturbatively the position of the pole in γ we shall generate the triple pole term in the AD $\Delta\gamma \sim \alpha_{em}^2/\omega^3$.

It is important to note that the equation for the pole position of f_ω is similar to the BFKL equation in the modified leading logarithmic approximation

$$\omega \sim 2\Psi(1) - \Psi(1 + |n| + \omega - \gamma) - \Psi(\gamma), \qquad |n| = -1, -2, \dots .$$

2 NLO corrections to the BFKL kernel in the $N = 4$ SUSY

Let us introduce the new variables $M = \gamma + \frac{|n|}{2}$, $\widetilde{M} = \gamma - \frac{|n|}{2}$, $\gamma = \frac{1}{2} + i\nu$. Then the eigenvalue relation for the BFKL equation in the $\overline{\rm DR}$-scheme can be written in the Hermitially separabel form [6]

$$1 = \frac{4\widehat{a}}{\omega}\left(2\Psi(1) - \Psi(M) - \Psi(1-\widetilde{M}) + \widehat{a}\left(\phi(M) + \phi(1-\widetilde{M})\right)\right) - 2\widehat{a}\left(\rho(M) + \rho(1-\widetilde{M})\right), \qquad \widehat{a} = \overline{a} + \frac{\overline{a}^2}{3}, \qquad \overline{a} = \frac{g^2 N_c}{16\pi^2}, \tag{5}$$

where $\overline{a}$ (see (3)) and $\widehat{a}$ are expressed through the Yang–Mills constants g in the $\overline{\rm MS}$ and $\overline{\rm DR}$-schemes, respectively, and

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2), \quad \phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi_2(M) + 2\beta'(M)\big(\Psi(1) - \Psi(M)\big). \tag{6}$$

Here

$$\Psi(M) = \frac{\Gamma'(M)}{\Gamma(M)}, \quad \beta'(z) = \sum_{l=0}^{\infty}\frac{(-1)^{l+1}}{(l+z)^2}, \quad \Phi_2(M) = \sum_{k=0}^{\infty}\frac{\big(\beta'(k+1) + (-1)^k\Psi'(k+1)\big)}{k+M}.$$

In the right-hand side of the eigenvalue equation the first contribution corresponds to the singularity at $l = -1$ generated by the pole of the Legendre function $Q_l(x)$ in the kernel at $\omega = 1 + l \to 0$ and the last term appears from the regular part of the Born contribution. Note that M and $1 - \widetilde{M}$ coincide with the anomalous dimensions appearing in the asymptotical expressions for the BFKL kernel in the limits when the gluon virtualities are large: $q_1^2 \to \infty$ and $q_2^2 \to \infty$, respectively. Because $1 - \widetilde{M} = M^*$, the Hermitian separability guarantees the symmetry of ω for the principal series of the unitary representations of the Möbius group to the substitution $\nu \to -\nu$ and the Hermicity of the BFKL Hamiltonian. It turns out, however, that the holomorphic separability corresponding to the symmetry $m \leftrightarrow \tilde{m}$ is violated in the NLO approximation [6].

Analogously to refs. [3, 4] one can calculate the eigenvalues of the BFKL kernel in the case of the non-symmetric choice for the energy normalization s_0 in equation (15) related to the interpretation of the NLO corrections in the framework of the renormalization group. For the scale $s_0 = q^2$ natural for the deep-inelastic scattering we obtain the corresponding eigenvalue equation

$$1 = 4\hat{a}\omega^{-1}\left(2\Psi(1) - \Psi(\gamma) - \Psi(J-\gamma) + \hat{a}\left(\phi(\gamma) + \phi(J-\gamma)\right)\right) + 2\hat{a}\left(\Psi'(\gamma) - \rho(\gamma) + \Psi'(J-\gamma) - \rho(J-\gamma)\right). \tag{7}$$

3 Anomalous dimension matrix in the $N = 4$ SUSY

In LLA AD matrices in the N=4 SUSY have the following form (see [18]):
for tensor twist-2 operators (hereafter $\Psi(j+1) - \Psi(1) \equiv S_1(j)$)

$$\gamma_{gg}^{(0)}(j) = 4\left(-S_1(j-2) - \frac{2}{j} + \frac{1}{j+1} - \frac{1}{j+2}\right), \qquad \gamma_{qg}^{(0)}(j) = 8\left(\frac{1}{j} - \frac{2}{j+1} + \frac{2}{j+2}\right),$$
$$\gamma_{\varphi g}^{(0)}(j) = 12\left(\frac{1}{j+1} - \frac{1}{j+2}\right), \qquad \gamma_{gq}^{(0)}(j) = 2\left(\frac{2}{j-1} - \frac{2}{j} + \frac{1}{j+1}\right), \qquad \gamma_{q\varphi}^{(0)}(j) = \frac{8}{j},$$
$$\gamma_{qq}^{(0)}(j) = 4\left(-S_1(j-1) + \frac{1}{j} - \frac{2}{j+1}\right), \qquad \gamma_{\varphi q}^{(0)}(j) = \frac{6}{j+1},$$
$$\gamma_{\varphi\varphi}^{(0)}(j) = -4S_1(j), \qquad \gamma_{g\varphi}^{(0)}(j) = 4\left(\frac{1}{j-1} - \frac{1}{j}\right), \tag{8}$$

for the pseudo-tensor operators:

$$\tilde\gamma_{gg}^{(0)}(j) = 4\left(-S_1(j) - \frac{2}{j+1} + \frac{2}{j}\right), \qquad \tilde\gamma_{qq}^{(0)}(j) = 4\left(-S_1(j) + \frac{1}{j+1} - \frac{1}{j}\right),$$
$$\tilde\gamma_{qg}^{(0)}(j) = 8\left(-\frac{1}{j} + \frac{2}{j+1}\right), \qquad \tilde\gamma_{gq}^{(0)}(j) = 2\left(\frac{2}{j} - \frac{1}{j+1}\right). \tag{9}$$

Note that in the $N = 4$ SUSY multiplet there are twist-2 operators with fermion quantum numbers but their AD are the same as for the bosonic components of the corresponding supermultiplet (cf. ref. [15]). It is possible to construct 5 independent twist-two operators with a multiplicative renormalization. The corresponding parton distribution momenta and their LLA AD have the form [18]:

$$n_I(j) = n_g^j + n_q^j + n_\varphi^j, \qquad \gamma_I^{(0)}(j) = -4S_1(j-2) \equiv \gamma_+^{(0)}(j),$$
$$n_{II}(j) = -2(j-1)n_g^j + n_q^j + \frac{2}{3}(j+1)n_\varphi^j, \qquad \gamma_{II}^{(0)}(j) = -4S_1(j) \equiv \gamma_0^{(0)}(j),$$
$$n_{III}(j) = -\frac{j-1}{j+2}n_g^j + n_q^j - \frac{j+1}{j}n_\varphi^j, \qquad \gamma_{III}^{(0)}(j) = -4S_1(j+2) \equiv \gamma_-^{(0)}(j),$$
$$n_{IV}(j) = 2\Delta n_g^j + \Delta n_q^j, \qquad \gamma_{IV}^{(0)}(j) = -4S_1(j-1) \equiv \tilde\gamma_+^{(0)}(j),$$
$$n_V(j) = -(j-1)\Delta n_g^j + \frac{j+2}{2}\Delta n_q^j, \qquad \gamma_V^{(0)}(j) = -4S_1(j+1) \equiv \tilde\gamma_-^{(0)}(j).$$

Thus, we have one supermultiplet of operators with the same AD $\gamma^{\rm LLA}(j)$ proportional to $\Psi(1) - \Psi(j-1)$. The momenta of the corresponding linear combinations of the parton distributions can be obtained from the above expressions by an appropriate shift of their argument j to obtain this universal anomalous dimension $\gamma^{\rm LLA}(j)$. Moreover, the coefficients in these linear combinations for $N = 4$ SUSY can be found from the super-conformal invariance (cf. ref. [15]). However, in two-loop approximation these coefficients are slightly renormalized [19] due to the breaking of the conformal invariance [20]. In the paper [6] using some plausible arguments an universal AD in two-loops for $N = 4$ SUSY in the $\overline{\rm DR}$-scheme was suggested. Other AD are obtained by an integer shift of its arguments. These results were justified by a direct calculation of the AD matrix in ref. [19][2]. With the use of the basis for the multiplicatively renormalizable operators obtained in LLA in [6] one can transform this matrix to a triangle form. The diagonal elements of the triangle matrix are expressed in terms of the universal AD $\gamma(j)$ by an appropriate integer shift of its argument:

$$\gamma(j) = -4\hat a\big(S_1(j-2) + 4\hat a Q(j-2)\big), \tag{10}$$

[2]Note that in [19] the results have been obtained in $\overline{\rm DR}$-scheme but with the $\overline{\rm MS}$ coupling constant $\overline{a}$ that leads to the additional term $-1/3S_1(j)$ in the NLO correction $Q(j)$.

where

$$Q(j) = S_{-2,1}(j) - S_1(j)\big(S_2(j) + S_{-2}(j)\big) - \big(S_3(j) + S_{-3}(j)\big)/2,$$
$$S_k(n) = \sum_{i=1}^{n} \frac{1}{i^k}, \qquad \zeta(k) = \sum_{i=1}^{\infty} \frac{1}{i^k}, \qquad S_{-k}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{i^k}, \qquad S_{-k,l}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{i^k} S_l(i).$$

The analytical continuation of functions $\gamma_{ab}^{(1)}$ $(a, b = g, q, \varphi)$ and $\tilde{\gamma}_{ab}^{(1)}$ $(a, b = g, q)$ to the complex values of j can be done analogously to refs. [21, 6].

4 Relation between the DGLAP and BFKL equations

As we have discussed already above, in the case of $N = 4$ SUSY the BFKL eigenvalue is analytic in $|n|$ and one can continue the AD to the negative values of $|n|$. It gives a possibility to find the AD singular contributions in the twist-2 operators not only at $j = 1$ but also at other integer points $j = 0, -1, -2, \ldots$. As it was discussed already in the Introduction, in the Born approximation we obtain $\gamma = 4\hat{a}(\Psi(1) - \Psi(j-1))$, which coincides with the result of the direct calculations (see [14, 18]). Thus, in the case of $N = 4$ the BFKL equation presumably contains the information sufficient for restoring the kernel of the DGLAP equation. Below we investigate the relation between these equations in the NLO approximation.

Let us start with an investigation of AD singularities obtained from the DGLAP equation. By presenting the Lorentz spin j as $\omega - r$, where $r = -1, 0, 1, \ldots$ and pushing $\omega \to 0$ we can calculate the singular behavior of the universal anomalous dimension $\gamma(j)$

$$\gamma(j) = 4\hat{a}\left[\frac{1}{\omega} - S_1(r+1) + O(\omega)\right] \tag{11}$$
$$+ (4\hat{a})^2 \begin{cases} \dfrac{1}{\omega^3} - 2S_1(r+1)\dfrac{1}{\omega^2} - (\zeta(2) + S_2(r+1))\dfrac{1}{\omega} + O\big(\omega^0\big) & \text{if } r = 2m, \\ S_2(r+1)\dfrac{1}{\omega} + O\big(\omega^0\big) & \text{if } r = 2m+1. \end{cases}$$

Let us consider initially the BFKL equation in a modified LLA, i.e. when $\omega^{\rm MLLA} = 4\,\hat{a}\big(2\Psi(1) - \Psi(\gamma) - \Psi(J-\gamma)\big)$. In the limit $J = 1 + |n| + \omega \to -r + \omega$ by inverting this equation one can obtain

$$\gamma = \frac{4\hat{a}}{\omega} + (4\hat{a})^2\left[\frac{1}{\omega^3} - S_1(r)\frac{1}{\omega^2} - (\zeta(2) + S_2(r))\frac{1}{\omega}\right] + O\big(\hat{a}^3\big), \tag{12}$$

i.e. this result coincides after the shift $r \to r+1$ with the singular part of the corresponding DGLAP result for the even values of r with the exception of the coefficient in the front of $\hat{a}^2/\omega^2$.

In a general case the next-to-leading corrections contain the divergencies at $|n| \to -r - 1$. Their appearance is related to the presence of the double-logarithms. Indeed the eigenvalue relation for the Bethe–Salpeter equation can be written near $\gamma = 0$ and $J \simeq j = -r + \omega$ in the form

$$1 = \frac{4\hat{a}}{\gamma(\omega - \gamma)} + O\big(\hat{a}^2\big).$$

Because the first contribution in the right-hand side contains additional singularities in comparison with the pole $1/\omega$ in the physical case of the positive $|n|$, we should subtract from the correction $O\big(\hat{a}^2\big)$ the terms appeared in its first iteration:

$$(4\hat{a})^2 \frac{1}{\gamma^2(\omega - \gamma)^2} \simeq (4\hat{a})^2 \left(\frac{1}{\gamma^2\omega^2} + \frac{2}{\gamma\omega^3}\right).$$

This subtraction leads to the final result for even r, which is in an agreement with the fact that the double-logarithms in the universal AD $\gamma(j)$ exist at even negative j. For the odd r the divergency $\hat{a}^2/(\gamma\omega^3)$ is absent in accordance with the absence in $\gamma(j)$ of the DL terms $\sim \hat{a}^2/\omega^3$ at odd negative j. For a more accurate comparison of the singularities of the BFKL and DGLAP equations in two-loop approximation one needs to calculate in the BFKL kernel non-singular terms at $j \to -r$.

5 Conclusion

Above we reviewed the LLA and NLL results for the eigenvalue of the kernels of the BFKL and DGLAP equations in the $N = 4$ supersymmetric gauge theory and constructed the operators with a multiplicative renormalization [6]. These AD can be obtained from the universal AD $\gamma(j)$ by a shift of its argument $j \to j + k$. The NLO corrections to the AD matrix were found with the use of the plausible arguments [6] and by the direct methods [19].

Note that recently the LLA AD in this theory for large $\alpha_s N_c$ were constructed in ref. [22] in the limit $j \to \infty$ from the superstring model with the use of the Maldacena correspondence [23]. Also in $N = 4$ SUSY at large $\alpha_s N_c$ the Pomeron coincides with the graviton [24]. It will be interesting to obtain these results directly from the DGLAP and BFKL equation. Already in the perturbation theory as it was demonstrated above, the BFKL dynamics has remarkable properties: analyticity in the conformal spin $|n|$, Möbius invariance, holomorphic (and Hermitian) separability and integrability in a generalized LLA. On the other hand, for the DGLAP dynamics the AD for all twist-2 operators are proportional in LLA to the function $\Psi(1) - \Psi(j-1)$ up to an integer shift of its arguments, which corresponds to the eigenvalue of a pair Hamiltonian in the integrable Heisenberg spin model [4, 25]. The investigation of the $N = 4$ supersymmetric model should be continued in the perturbation theory and for large $\alpha_s N_c$ because it is helpful for understanding of QCD.

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Three-Dimensional Integrable and Superintegrable Isospectral Potentials

Şengül KURU and Abdullah VERÇÍN

Ankara University, Faculty of Science, Department of Physics, 06100 Tandoğan/Ankara, Turkey
E-mail: *kuru@science.ankara.edu.tr, vercin@science.ankara.edu.tr*

As an extension of the intertwining operator idea a constructive method for establishment of families of three-dimensional (super)integrable and isospectral potentials having higher-order dynamical symmetries is developed.

In analogy to integrability concepts of classical mechanics [1–3], a quantum mechanical system described in N dimensional (ND) Euclidean space by a stationary Hamiltonian operator H is called to be completely integrable if there exists a set of $N-1$ (together with H, N) algebraically independent linear operators commuting with H and among each other [4–8]. If there exist k ($0 < k \leq N-1$), additional operators commuting with H it is said to be superintegrable. The superintegrability is said to be minimal if $k = 1$ and maximal if $k = N-1$. Recently, the intertwining operator method has been systematically used in establishing families of $2D$ superintegrable potentials that are at the same time isospectral [9, 10]. In this letter we shall extend this program in constructing $3D$ integrable and superintegrable isospectral potentials.

The method of intertwining is a unified approach widely used in various fields of physics and mathematics [11–20]. To see the usefulness of this method let us consider two Hamiltonian operators $\mathcal{H}_0$ and $\mathcal{H}_1$ that are Hermitian (on some common function space) and intertwined by a linear intertwining operator $\mathcal{L}$ such that $\mathcal{L}\mathcal{H}_0 = \mathcal{H}_1\mathcal{L}$. Two dimension and form independent general properties [14, 15] immediately follow from such a relation; (i) If ψ^0 is an eigenfunction of $\mathcal{H}_0$ with eigenvalue of E^0 then $\psi^1 = \mathcal{L}\psi^0$ is an (unnormalized) eigenfunction of $\mathcal{H}_1$ with the same eigenvalue E^0. That is $\mathcal{L}$ transforms one solvable problem into another. (ii) $\mathcal{L}^\dagger$ intertwines in the other direction $\mathcal{H}_0\mathcal{L}^\dagger = \mathcal{L}^\dagger\mathcal{H}_1$ and this in turn provides two hidden dynamical symmetries of $\mathcal{H}_0$ and $\mathcal{H}_1$ in terms of $\mathcal{L}$: $[\mathcal{H}_0, \mathcal{L}^\dagger\mathcal{L}] = 0 = [\mathcal{H}_1, \mathcal{L}\mathcal{L}^\dagger]$, where † and $[\,,\,]$ stand for Hermitian conjugation and commutator.

In this letter we shall first apply this method to a pair of $3D$ systems described by

$$\mathcal{H}_0 = -\nabla^2 + V_0, \qquad \mathcal{H}_1 = -\nabla^2 + V_1, \tag{1}$$

and take $\mathcal{L}$ to be the following first order linear operator

$$\mathcal{L} = L_0 + \boldsymbol{L}\cdot\boldsymbol{\nabla}. \tag{2}$$

∇^2 is the Laplace operator in the Cartesian coordinates (x_1, x_2, x_3), $\boldsymbol{L} = (L_1, L_2, L_3)$ and "$\cdot$" denotes the usual inner product of $\mathbb{R}^3$. The potentials V_0, V_1 and L_0, $\boldsymbol{L}$ are some functions of coordinates that are to be determined from consistency equations of $\mathcal{L}\mathcal{H}_0 = \mathcal{H}_1\mathcal{L}$ which for (1) and (2) takes the form:

$$[\nabla^2, \boldsymbol{L}\cdot\boldsymbol{\nabla}] = -[\nabla^2, L_0] + [V_0, \boldsymbol{L}\cdot\boldsymbol{\nabla}] + \mathcal{P}\,\mathcal{L}, \tag{3}$$

where $\mathcal{P} = V_1 - V_0$.

By equating the second power of partial derivatives $\partial_j \equiv \partial/\partial x_j$ in equation (3) (these come only from $[\nabla^2, \boldsymbol{L}\cdot\boldsymbol{\nabla}]$) we obtain

$$\boldsymbol{L} = \boldsymbol{a} - \boldsymbol{b}\times\boldsymbol{r}, \tag{4}$$

where $\boldsymbol{a}=(a_1,a_2,a_3)$, $\boldsymbol{b}=(b_1,b_2,b_3)$ are arbitrary constant vectors and "$\times$" stands for the usual cross product of $\mathbb{R}^3$. In terms of vector operators $\boldsymbol{T}=\boldsymbol{\nabla}$, $\boldsymbol{J}=-(\boldsymbol{r}\times\boldsymbol{\nabla})$ whose components close into defining relations of six dimensional Euclidean algebra $e(3)$

$$[T_j,T_k]=0,\qquad [J_j,J_k]=\epsilon_{jkl}J_l,\qquad [J_j,T_k]=\epsilon_{jkl}T_l, \tag{5}$$

the differential part of $\mathcal{L}$ can be written as

$$\boldsymbol{L}\cdot\boldsymbol{\nabla}=\boldsymbol{a}\cdot\boldsymbol{T}+\boldsymbol{b}\cdot\boldsymbol{J}. \tag{6}$$

T_j's and J_k's generate, respectively, the translation subalgebra $t(3)$ and rotation subalgebra $so(3)$ of $e(3)$ which accepts $\nabla^2=\boldsymbol{T}\cdot\boldsymbol{T}$ as its Casimir.

The first and zeroth powers of derivatives of (3) gives the following set of consistency equations

$$\boldsymbol{\nabla}L_0=\mathcal{P}\boldsymbol{L}, \tag{7}$$

$$(-\nabla^2+\mathcal{P})L_0=\boldsymbol{L}\cdot\boldsymbol{\nabla}V_0. \tag{8}$$

Equation (8) is nonlinear and the integrability conditions $\partial_j\partial_k L_0=\partial_k\partial_j L_0$ for the linear set (7) are equivalent to $\boldsymbol{L}\times\boldsymbol{\nabla}\mathcal{P}=-2\boldsymbol{b}\mathcal{P}$, where we have used the relation $\boldsymbol{\nabla}\times\boldsymbol{L}=-2\boldsymbol{b}$. Scalar multiplication of the both sides of this relation by $\boldsymbol{L}$ implies, for $\mathcal{P}\neq 0$, that

$$\boldsymbol{b}\cdot\boldsymbol{L}=\boldsymbol{a}\cdot\boldsymbol{b}=0. \tag{9}$$

Equations (7) and (8) must be solved after choosing $\boldsymbol{a}$ and $\boldsymbol{b}$ in accordance with condition (9). In doing that we shall make use of the orbit structure of $e(3)$ under the adjoint action of the Euclidean group $E(3)$ [20]. Under such a transformation, generated by

$$U=e^{\eta_1 J_3}e^{\eta_2 J_1}e^{\eta_3 J_3}e^{\boldsymbol{\xi}\cdot\boldsymbol{T}}$$

$e(3)$ has three orbit types with representatives T_3, J_3 and $\kappa_1T_3+\kappa_2J_3$, where η_i, ξ_j, κ_1 and κ_2 are real group parameters. That is, any element of $e(3)$ can be transformed by a similarity transformation U to one of the orbit representatives. Evidently, such a transformation leaves the intertwining relation form invariant; $\bar{\mathcal{L}}\bar{\mathcal{H}}_0=\bar{\mathcal{H}}_1\bar{\mathcal{L}}$, where $\bar{X}=UXU^\dagger$ and $U^{-1}=U^\dagger$ stands for the inverse of $U\in E(3)$. Thus, if we choose the differential part of $\mathcal{L}$ as one of the orbit representatives the so found potentials and L_0 will be unique up to a similarity action of $E(3)$. Noting that only the choices T_3 or J_3 for $\boldsymbol{L}\cdot\boldsymbol{\nabla}$ respect the condition (9) below we shall follow this procedure in two steps.

Step I. First we take $\boldsymbol{L}\cdot\boldsymbol{\nabla}=T_3$. In that case equation (7) gives the equations: $\partial_1L_0=0=\partial_2L_0$, $2\partial_3L_0=\mathcal{P}$ which imply $L_0=g_1(x_3)$ and $\mathcal{P}=2g_1'(x_3)$, where g_1 is an arbitrary differentiable function of x_3 (we use prime to denote derivative with respect argument). Then from the nonlinear equation (8) which takes the form $(-\partial_3^2+\mathcal{P})L_0=\partial_3V_0$, we obtain

$$V_0=V(x_1,x_2)+V_-,\qquad V_1=V(x_1,x_2)+V_+, \tag{10}$$

where

$$V_\pm=g_1^2(x_3)\pm g_1'(x_3). \tag{11}$$

Hence, we have found two isospectral Hamiltonians $H_0=-\nabla^2+V_0$ and $H_1=-\nabla^2+V_1$ that are intertwined by $\mathcal{L}_{10}=g_1(x_3)+T_3$ as follows $\mathcal{L}_{10}H_0=H_1\mathcal{L}_{10}$.

Step II. Fixing the form of H_1 found above we shall now intertwine it to an another Hamiltonian H_2 with $\mathcal{L}_{21}$. This will amount to the fact that we have three pairwise intertwined Hamiltonians

$$\mathcal{L}_{10}H_0=H_1\mathcal{L}_{10},\qquad \mathcal{L}_{21}H_1=H_2\mathcal{L}_{21},\qquad \mathcal{L}_{20}H_0=H_2\mathcal{L}_{20},$$

where $\mathcal{L}_{20} \equiv \mathcal{L}_{21}\mathcal{L}_{10}$ and the last relation is a direct result of the first two. Moreover, since each Hamiltonian will be double intertwined we will have two additional symmetries for each one.

For $H_2 = -\nabla^2 + V_2$ and $\mathcal{L}_{21} = K_0 + J_3$ the linear set of consistency equations can be read from (7) to be

$$2\partial_1 K_0 = Px_2, \qquad 2\partial_2 K_0 = -Px_1, \qquad \partial_3 K_0 = 0. \tag{12}$$

Now $P = V_2 - V_1$ and solutions are $K_0 = g_2(u)$ and $P = -2g_2'(u)/x_1^2$, where g_2 is a differentiable function of $u = x_2/x_1$. In that case the nonlinear equation is $(-\nabla^2 + P)K_0 = J_3 V_1$ and for the found K_0, P takes the form

$$\partial_u \left[(1+u^2)g_2'(u) + g_2^2(u)\right] = -x_1^2 J_3 V(x_1, x_2). \tag{13}$$

The most general form of V that makes the right-hand side of equation (13) a function of u is

$$V(x_1, x_2) = F(\rho) + \frac{1}{x_1^2} h(u), \tag{14}$$

where F and h are arbitrary functions of $\rho = (x_1^2 + x_2^2)^{1/2}$ and u respectively. Note that $J_3 F = 0$. Using (14) in (13) we see that g_2 and h have to satisfy the equation

$$\left(1+u^2\right) g_2'(u) + g_2^2(u) = \left(1+u^2\right) h(u). \tag{15}$$

Combining the results of above two steps we get the following triplet of isospectral potentials

$$V_0 = V + V_-, \qquad V_1 = V + V_+, \qquad V_2 = V_1 - \frac{2g_2'(u)}{x_1^2}. \tag{16}$$

$V_\pm$ are given by (11) and g_2, h are any solutions of equation (15). The corresponding symmetry generators are collected as follows

$$\begin{aligned}
X_0 &= \mathcal{L}_{10}^\dagger \mathcal{L}_{10} = V_- - T_3^2, \\
Y_0 &= \mathcal{L}_{20}^\dagger \mathcal{L}_{20} = \left[g_2^2 - (1+u^2)g_2' - J_3^2\right] X_0, \\
X_1 &= \mathcal{L}_{10}\mathcal{L}_{10}^\dagger = V_+ - T_3^2, \\
Y_1 &= \mathcal{L}_{21}^\dagger \mathcal{L}_{21} = g_2^2 - \left(1+u^2\right) g_2' - J_3^2, \\
X_2 &= \mathcal{L}_{21}\mathcal{L}_{21}^\dagger = g_2^2 + (1+u^2)g_2' - J_3^2, \\
Y_2 &= \mathcal{L}_{20}\mathcal{L}_{20}^\dagger = \left[g_2^2 + \left(1+u^2\right) g_2' - J_3^2\right] X_1,
\end{aligned} \tag{17}$$

where the subscripts of X_j, Y_j indicate the Hamiltonians they belong to. By construction, all these operators are factorized, have even orders (Y_2 and Y_0 are both the fourth order and X_1, Y_1, X_0, X_2 are second order) and $[H_i, X_i] = 0 = [H_i, Y_i]$; $i = 0, 1, 2$. One can also easily verify that $[X_i, Y_i] = 0$. As we have, together with the Hamiltonians, three independent, pairwise commuting symmetry generators for each Hamiltonian, equation (16) describes 3D integrable isospectral systems. From (17) we also observe that the fourth order symmetries are of the form $Y_0 = Y_1 X_0$, $Y_2 = X_2 X_1$. This is due to the fact that $\mathcal{L}_{10}$ and $\mathcal{L}_{21}$ are commuting.

We shall now introduce two additional methods which produce minimally superintegrable (having four independent symmetries) subclasses of the above integrable $3D$ systems.

Method I. In this method we shall construct an additional symmetry generator Z_1 of H_1. Since H_1 is intertwined to H_0 and H_2 the existence of such symmetry will provide $Z_0 = \mathcal{L}_{10}^\dagger Z_1 \mathcal{L}_{10}$ and $Z_2 = \mathcal{L}_{21} Z_1 \mathcal{L}_{21}^\dagger$ as new symmetry generators of H_0 and H_2. This method is another general property (independent from the dimension and Hamiltonians) of the intertwining operator idea.

As a simple realization of this method we shall take $Z_1 = \boldsymbol{c}\cdot\boldsymbol{J}$, where $\boldsymbol{c} = (c_1, c_2, c_3)$ is a constant vector. Then the condition $[Z_1, H_1] = 0$ yields the consistency equation $\boldsymbol{c}\cdot\boldsymbol{K} = 0$, where

$$K_1 = x_3\partial_2 V - x_2\partial_3 V_+(x_3), \qquad K_2 = x_3\partial_1 V - x_1\partial_3 V_+(x_3), \qquad K_3 = -J_3[\frac{h(u)}{x_1^2}], \tag{18}$$

and V_+, V are given by (11) and (14). Below three special cases are considered.

Case I. In the case of $Z_1 = J_3$ we have the equation $K_3 = 0$, which yields $h = \beta/(1+u^2)$, where β is an arbitrary constant. Since there is no restriction on F and $V_\pm$, the potentials are given by (16) with $V(\rho) = F(\rho) + \beta\rho^{-2}$. The condition (15) now is as follows

$$(1+u^2)g_2'(u) + g_2^2(u) = \beta. \tag{19}$$

The new symmetry generators of H_0 and H_2 are the following third-order operators

$$Z_0 = \mathcal{L}_{10}^\dagger Z_1 \mathcal{L}_{10} = (g_1 - T_3)J_3(g_1 + T_3) = X_0 J_3,$$
$$Z_2 = \mathcal{L}_{21} Z_1 \mathcal{L}_{21}^\dagger = (g_2 + J_3)J_3(g_2 - J_3).$$

In addition to $[Z_0, H_0] = 0 = [Z_2, H_2]$, we have

$$\begin{aligned}
&[X_1, Z_1] = 0,\\
&[Y_1, Z_1] = 4(1+u^2)g_2 g_2',\\
&[X_0, Z_0] = \mathcal{L}_{10}^\dagger [X_1, Z_1]\mathcal{L}_{10} = 0,\\
&[Y_0, Z_0] = \mathcal{L}_{10}^\dagger [Y_1, Z_1] X_1 \mathcal{L}_{10},\\
&[X_2, Z_2] = \mathcal{L}_{21}[Y_1, Z_1]\mathcal{L}_{21}^\dagger,\\
&[Y_2, Z_2] = \mathcal{L}_{21}[Y_1, Z_1] X_1 \mathcal{L}_{21}^\dagger.
\end{aligned} \tag{20}$$

Provided that $[Y_1, Z_1] \neq 0$, that is $g \neq$ const, we have $[Y_i, Z_i] \neq 0$ for $i = 0, 2$, and $[X_2, Z_2] \neq 0$. Each one of H_i corresponds to a minimally superintegrable system, with four symmetry generators (H_i, X_i, Y_i, Z_i).

Case II. If we take $Z_1 = J_2$ then the separable equation $K_2 = 0$ gives

$$V_+ = \frac{1}{2}\alpha x_3^2, \qquad h = \frac{\beta}{1+u^2} + \frac{\gamma}{u^2}, \qquad F = \frac{1}{2}\alpha\rho^2 - \frac{\beta}{\rho^2}, \tag{21}$$

where α, β, γ are arbitrary constants. In that case the potentials given by (16) are restricted to the following forms

$$V_0 = V_1 - 2g_1', \qquad V_1 = \frac{\alpha}{2}r^2 + \frac{\gamma}{x_2^2}, \qquad V_2 = V_1 - \frac{2}{x_1^2}g_2', \tag{22}$$

where $r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$, and g_1, g_2 are any solutions of

$$g_1' + g_1^2 = \frac{1}{2}\alpha x_3^2, \qquad \left(1+u^2\right)g_2' + g_2^2 = \beta + \gamma + \frac{\gamma}{u^2}. \tag{23}$$

It is easy to verify that $[X_1, Z_1]$ and $[Y_1, Z_1]$ are always different from zero. The new third-order symmetry generators of H_0 and H_2 satisfy the commutators given by (20).

Case III. Finally we take $Z_1 = kJ_2 + J_1$. In that case V_+, F are the same as in (21) and

$$h = \frac{\beta}{(1+u^2)} + \frac{\gamma}{(1-ku)^2}.$$

On substituting these solutions into (16), we obtain

$$V_0 = V_1 - 2g_1', \qquad V_1 = \frac{\alpha}{2}r^2 + \frac{\gamma}{x_1^2(1-ku)^2}, \qquad V_2 = V_1 - \frac{2}{x_1^2}g_2', \tag{24}$$

where g_1 is determined from the first equation of (23) and by equation (15), g_2 is any solution of

$$\left(1+u^2\right)g_2' + g_2^2 = \beta + \gamma\frac{1+u^2}{(1-ku)^2}. \tag{25}$$

Other choices of $\boldsymbol{c}$ produce similar potentials.

Method II. By fixing the form of H_1 found in the Step I we shall now intertwine it to another Hamiltonian H_3 with $\mathcal{L}_{31}$. This will provide four pairwise intertwined Hamiltonians with three additional symmetries since each one will be three times intertwined as follows:

$$\begin{aligned}
&\mathcal{L}_{10}H_0 = H_1\mathcal{L}_{10}, \qquad \mathcal{L}_{21}H_1 = H_2\mathcal{L}_{21}, \qquad \mathcal{L}_{20}H_0 = H_2\mathcal{L}_{20},\\
&\mathcal{L}_{30}H_0 = H_3\mathcal{L}_{30}, \qquad \mathcal{L}_{31}H_1 = H_3\mathcal{L}_{31}, \qquad \mathcal{L}_{32}H_2 = H_3\mathcal{L}_{32},
\end{aligned}$$

where $\mathcal{L}_{20} \equiv \mathcal{L}_{21}\mathcal{L}_{10}$, $\mathcal{L}_{30} \equiv \mathcal{L}_{31}\mathcal{L}_{10}$ and $\mathcal{L}_{32} \equiv \mathcal{L}_{31}\mathcal{L}_{21}^{\dagger}$.

Let us take $H_3 = -\nabla^2 + V_3$ and $\mathcal{L}_{31} = M_0 + a_1T_1 + J_2$. From equation (7) the linear set of consistency equations are

$$2\partial_1 M_0 = P(a_1 - x_3), \qquad \partial_2 M_0 = 0, \qquad 2\partial_3 M_0 = Px_1, \tag{26}$$

where $P = V_3 - V_1$ and the solutions are $M_0 = g_3(v)$ and $P = -2g_3'(v)/x_1^2$, where g_3 is a differentiable function of $v = (a_1 - x_3)/x_1$. Now the nonlinear equation (corresponding to equation (8)) is $(-\nabla^2 + P)M_0 = (a_1T_1 + J_2)(V + V_+)$, and for M_0 and P found above this equation takes the form

$$\partial_v\left[\left(1+v^2\right)g_3'(v) + g_3^2(v)\right] = -x_1^2(a_1 - x_3)\partial_1\left[F(\rho) + \frac{h(u)}{x_1^2}\right] - x_1^3\partial_3 V_+(x_3), \tag{27}$$

where V_+ is given by equations (11) and we made use of equation (14). It is not hard to see that the most general forms of F, h and V_+ which make the right-hand side of equation (27) only a function of v are as follows

$$F = \mu\rho^2 - \frac{\gamma}{\rho^2} + \lambda, \qquad h = \frac{\gamma}{1+u^2} + \frac{\mu_2}{u^2} + \mu_1, \qquad V_+ = \mu(a_1 - x_3)^2 + \frac{\mu_3}{(a_1 - x_3)^2},$$

where μ, γ, μ_i; $i = 1, 2, 3$ are arbitrary constants. On substituting these in (11), (15) and (27), we see that g_1, g_2 and g_3 have to satisfy the following Riccati equations

$$\begin{aligned}
&g_1'(x_3) + g_1^2(x_3) = \mu(a_1 - x_3)^2 + \frac{\mu_3}{(a_1 - x_3)^2},\\
&\left(1+u^2\right)g_2'(u) + g_2^2(u) = \gamma + \mu_2 + \mu_1\left(1+u^2\right) + \mu_2\frac{1}{u^2},\\
&\left(1+v^2\right)g_3'(v) + g_3^2(v) = \mu_1 v^2 + \frac{\mu_3}{v^2} + \lambda_1.
\end{aligned} \tag{28}$$

Provided that g_1, g_2, g_3 are any solutions of these equations the potentials are

$$V_0 = V_1 - 2g_1'(x_3), \qquad V_1 = V + V_+, \qquad V_2 = V_1 - \frac{2g_2'(u)}{x_1^2}, \qquad V_3 = V_1 - \frac{2g_3'(v)}{x_1^2}, \tag{29}$$

where

$$V=\mu\left(x_1^2+x_2^2\right)+\left(\frac{\mu_1}{x_1^2}\right)+\left(\frac{\mu_2}{x_2^2}\right)+\lambda,$$

and λ_1, λ are arbitrary constants. The old symmetry generators X_i, Y_i, $i=0,1,2$ are given by (17) and the new ones are as follows

$$\begin{aligned}&Z_0=\mathcal{L}_{30}^{\dagger}\mathcal{L}_{30}, \qquad Z_1=\mathcal{L}_{31}^{\dagger}\mathcal{L}_{31}, \qquad Z_2=\mathcal{L}_{32}^{\dagger}\mathcal{L}_{32},\\ &X_3=\mathcal{L}_{31}\mathcal{L}_{31}^{\dagger}, \qquad Y_3=\mathcal{L}_{30}\mathcal{L}_{30}^{\dagger}, \qquad Z_3=\mathcal{L}_{32}\mathcal{L}_{32}^{\dagger}.\end{aligned} \tag{30}$$

By construction, all generators commute with the corresponding Hamiltonians, $(Y_0, Z_0, Y_2, Z_2, Y_3, Z_3)$ are of the order four (in derivative, or equivalently, in the generators of $e(3)$ algebra) and the remaining ones are the second order. H_i's are minimally superintegrable since $[X_i, Y_i]=0$, $[X_i, Z_i]\neq 0$ and $[Y_i, Z_i]\neq 0$, for $i=0,1,2$. But, because of the relations $[X_3, Y_3]\neq 0$, $[X_3, Z_3]\neq 0$ and $[Y_3, Z_3]\neq 0$ the new Hamiltonian H_3 is not even integrable. However, by applying the method I we have introduced above to the potentials given by (29), H_3 (and the other Hamiltonians) can be made maximally superintegrable (another way which may produce different potentials is to use the intertwining method once again). Finally we would like to emphasise that a hierarchy of potentials can be constructed by first linearizing the Riccati equations found above and then using the solutions of resulting $1D$ Schrödinger equations [10].

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Symmetries and Supersymmetries in Trapped Ion Hamiltonian Models

Benedetto MILITELLO [†], *Anatoly NIKITIN* [‡] *and Antonino MESSINA* [†]

[†] *INFM, MIUR and Dipartimento di Scienze Fisiche ed Astronomiche dell'Universitá di Palermo, Via Archirafi, 36 - 90123 Palermo, Italy*
E-mail: *bdmilite@fisica.unipa.it*

[‡] *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *nikitin@imath.kiev.ua*

The implementation of a vibronic coupling in the context of trapped ions, and its diagonalization are reported. The specific spin-boson-like Hamiltonian model is studied exploiting its symmetries and supersymmetries. Applications are discussed.

1 Introduction

Trapped ions provide an interesting physical scenario wherein fundamental aspects of quantum mechanics may be *tested* and useful applications in *quantum computation* may be realized [1].

Generally speaking, a time dependent quadrupolar electric field is responsible for a charged particle dynamics which may be *effectively described* as the motion of a point into a quadratic potential [2]. When the harmonically confined particle is an ion, the system we obtain possesses both bosonic and fermionic degrees of freedom, the first corresponding to the ion center of mass motion, the second describing the atomic internal state, i.e. the electronic configuration. In most of the cases, in trapped ion dynamics only two atomic levels are involved, making the Pauli operator two-level system description effective.

Once the ion has been confined, acting upon it via laser field it is possible to implement vibronic couplings (i.e. involving both bosonic and fermionic degrees of freedom) practically at will [3]. Unfortunately, sometimes, the vibronic coupling problem is not solvable even if implementable.

The study of symmetries plays a central role in the analysis of trapped ion dynamical problems. Indeed, in most of the cases, symmetries provide an elegant and effective way for finding Hamiltonian eigensolutions.

In this paper we describe a trapped ion physical situation described by a Hamiltonian model possessing interesting and useful symmetries. In detail, conservation of total excitation number and rotational invariance are found. Moreover a Second Order Hidden Supersymmetry is discovered. Symmetry based diagonalization provides the possibility of writing down the time evolution concerning simple initial conditions. Interesting nonclassical behaviors like the generation of GHZ [7] states are predicted.

2 The physical system

The physical system on which we focus our attention consists in a two-level ion confined into a three dimensional isotropic trap, described by the unperturbed Hamiltonian

$$\hat{H}_0 = \frac{\hbar\omega_A}{2}\hat{\sigma}_3 + \hbar\omega_T \sum_{j=x,y,z} \hat{a}_j^\dagger \hat{a}_j,$$

where ω_A is the relevant atomic Bohr frequency, $\hat{\sigma}_3$, third component of the Pauli vector, describes the two level atomic system, ω_T is the common frequency of the degenerate trap whose annihilation (creation) operators are $\hat{a}_j$ ($\hat{a}_j^\dagger$). The total system of Bohr frequencies involving atomic transitions are given by

$$\omega = \omega_A + (m_x + m_y + m_z)\,\omega_T,$$

where m_x, m_y, m_z are relative numbers. Considering a one-dimensional dynamics, the frequency $\omega = \omega_A + m\omega_T$ is referred to as the *carrier*, *a red sideband* or a *blue sideband* depending on $m = 0$, $m < 0$ or $m > 0$ respectively.

2.1 Implementation of the Hamiltonian model

Acting upon the system via a laser field of amplitude $\vec{E}_{0,x}$, propagating along x and tuned to the second red sideband, an interaction term is added to the Hamiltonian:

$$\hat{H}_{\rm int}^{(S)} = -\vec{d}\cdot\vec{E}(\vec{r}),$$

where $\vec{d} = \vec{d}_{\pm}\hat{\sigma}_+ + \vec{d}_{\pm}^{*}\hat{\sigma}_-$ is the two-level restriction of the atomic dipole operator and $\vec{E}(\hat{\vec{r}}) = \vec{E}_{0,x}e^{i\left(k_x\hat{X}-\omega_L t\right)} + \vec{E}_{0,x}^{*}e^{-i\left(k_x\hat{X}-\omega_L t\right)}$, $\hat{X}$ being the trapped ion center of mass position operator, describes the field.

Passing to the interaction picture and expanding the exponentials using the Baker–Campbell–Hausdorff formula, the interaction Hamiltonian in the Rotating Wave approximation is

$$\hat{H}_{\rm int} = -\hbar g_x\left[e^{-\frac{\eta_L^2}{2}}\sum_{s,j=0}^{\infty}\frac{(-i\eta_x)^{s+j}}{s!j!}(\hat{a}_x^\dagger)^j\;\hat{a}_x^s e^{i(j-s)\omega_T t}e^{-i(\omega_A-\omega_L)t}\hat{\sigma}_-\right] + \text{h.c.}, \tag{1}$$

where $g_x \equiv \vec{d}_{\pm}\cdot\vec{E}_{0,x}$, $\eta_x \equiv \frac{k_x}{2\pi}\sqrt{\frac{\hbar}{\mu\omega_T}}$, μ being the mass of the ion is the so called Lamb–Dicke parameter η_x expressing the ratio between the spatial dimensions of the bosonic ground state wave function and the laser field wavelength, and hence governing the influence of the e.m. field gradient to the laser-driven trapped ion dynamics.

Retaining only stationary terms in equation (1), and observing that

$$e^{i(j-s)\omega_T t}e^{-i(\omega_A-\omega_L)t} = e^{-i(j-s-2)\omega_T t}$$

one obtains

$$\hat{H}_{\rm int} \approx -\hbar g_x\left[e^{-\frac{\eta_x^2}{2}}\sum_{j=0}^{\infty}\frac{(-i\eta_x)^{2(j+1)}}{j!(j+2)!}(\hat{a}_x^\dagger)^{j+2}\;\hat{a}_x^j\right]\hat{\sigma}_- + \text{h.c.} \tag{2}$$

In the so-called Lamb–Dicke limit, meaning that $\eta_x \ll 1$, the Hamiltonian in equation (2) may be cast in the form

$$\hat{H}_{\rm int} \approx \hbar\gamma_x\left[(\hat{a}_x^\dagger)^2\hat{\sigma}_- + (\hat{a}_x)^2\hat{\sigma}_+\right] \qquad \text{with} \quad \gamma_x \equiv g_x e^{-\frac{\eta_x^2}{2}}\frac{\eta_x^2}{2}.$$

Simultaneously acting with three laser fields tuned to the second red sideband and directed along the three x, y and z axis directions one obtains

$$\hat{H}_{\rm int} \approx \hbar\left[\gamma_x\hat{a}_x^2 + \gamma_y\hat{a}_y^2 + \gamma_z\hat{a}_z^2\right]\hat{\sigma}_+ + \text{h.c.}$$

Suitably adjusting the laser field amplitudes and the Lamb–Dicke parameters one can set equal the three γ-parameters, obtaining

$$\hat{H}_{\rm int} \approx \hbar\gamma\left(\hat{a}_x^2 + \hat{a}_y^2 + \hat{a}_z^2\right)\hat{\sigma}_+ + \text{h.c.}, \tag{3}$$

where $\gamma \equiv \gamma_x = \gamma_y = \gamma_z$.

2.2 Symmetries

The interaction picture Hamiltonian model in equation (3) describing two boson trimodal Jaynes–Cummings-like transitions, possesses interesting symmetries useful in view of $\hat{H}_{\rm int}$ diagonalization.

First of all observe that the total excitation number operator is conserved during the dynamics. Indeed, the operator

$$\hat{N} = \sum_{j=x,y,z} \hat{a}_j^\dagger \hat{a}_j + \hat{\sigma}_z + 1$$

is a constant of motion, meaning that the *phase changes*

$$\hat{a}_j \to e^{i\phi}\hat{a}_j, \qquad \hat{\sigma}_\pm \to e^{\mp i2\phi}\hat{\sigma}_\pm$$

leave unchanged the Hamiltonian operator.

Rotational invariance is the second important symmetry. In fact the center of mass angular momentum does commute with $\hat{H}_{\rm int}$:

$$[\hat{H}_{\rm int}, \hat{\vec{L}}] = \vec{0},$$

where the three components of $\hat{\vec{L}}$ are given by

$$\hat{L}_k = i(\hat{a}_j^\dagger \hat{a}_l - \hat{a}_l^\dagger \hat{a}_j) \qquad (j,l,k = x,y,z \ \text{ and cyclic}).$$

The operators $\hat{N}$, $\hat{L}^2$, $\hat{L}_z$ all commute each other and with $\hat{H}_{\rm int}$. It is worth noting that such three operators also commute with the unperturbed Hamiltonian $\hat{H}_0$, so that they really are constants of motion, in the sense that no explicit time dependence has been introduced in the passage to the interaction picture.

3 Hamiltonian model eigensolutions

To solve the interaction picture Hamiltonian eigensolution problem

$$\hat{H}_{\rm int}|\psi\rangle = E_\psi|\psi\rangle$$

observe first of all that

$$\hat{H}_{\rm int}^2 = (\hbar\gamma)^2\left[\hat{N}^2 + \hat{N} - \hat{L}^2\right]. \tag{4}$$

Such a formula determines, up to a sign, the eigenvalues of $\hat{H}_{\rm int}$ as

$$|e_{n,l}| = \hbar\gamma\sqrt{[n(n+1) - l(l+1)]},$$

$n = 2N + L + \frac{5}{2} + \hat{\sigma}_z$ and l being the quantum numbers related to the excitation number operator and to the square of the angular momentum, $N, l = 0, 1, \ldots$, $\hat{\sigma}_z = \pm 1$.

In addition to the individualization of the eigenvalue structure, equation (4) strongly suggests the existence of a Foldy–Wouthuysen-like transformation able to diagonalize the Hamiltonian model [4,5]. Such a circumstance is related to a *Second Order Hidden Supersymmetry* the system possesses.

Generally speaking, a Hamiltonian $\hat{H}_S$ is said to be supersymmetric if two operators $\hat{Q}$ and $\hat{Q}^\dagger$ exist satisfying the following algebra [6]

$$[\hat{H}_S, \hat{Q}] = [\hat{H}_S, \hat{Q}^\dagger] = 0, \qquad \{\hat{Q}, \hat{Q}^\dagger\} = \hat{H}_S, \qquad \{\hat{Q}, \hat{Q}\} = \{\hat{Q}^\dagger, \hat{Q}^\dagger\} = 0. \tag{5}$$

Starting from this, it is possible to introduce the Hermitian supercharges

$$\hat{Q}_+ = \hat{Q} + \hat{Q}^\dagger, \qquad \hat{Q}_- = i\big(\hat{Q} - \hat{Q}^\dagger\big) \tag{6}$$

satisfying the relation

$$\hat{Q}_-^2 = \hat{Q}_+^2 = \hat{H}_S. \tag{7}$$

Usually, supersymmetry is used to solve the $\hat{H}_S$ Hamiltonian eigenvalue problem or at least to predict the specific degeneration of the Hamiltonian spectrum. Nevertheless, if our attention is focused on one of the supercharges, the knowledge of relations in equation (5) and (7) proves to be useful. More in detail, a unitary transformation having $\hat{Q}_-$ as generator is able to *diagonalize* $\hat{Q}_+$ (*partially* or *completely* depending on the specific problem) and vice versa.

Since in our case we find an operator $\hat{K}_- = i\left\{\left[\hat{a}_x^2 + \hat{a}_y^2 + \hat{a}_z^2\right]\hat{\sigma}_+ - \text{h.c.}\right\}$ such that $\hat{Q}_-^2 \equiv \left(\hbar\gamma\hat{K}\right)^2 = \hat{H}_{\text{int}}^2$ we can set $\hat{Q}_+ = \hat{H}_{\text{int}}$, $\hat{Q}_- = \hbar\gamma\hat{K}$.

The unitary operator

$$\hat{U} = \exp\left(-\frac{\pi}{4}\frac{\hat{Q}_-}{|\hat{Q}_-|}\right) = \exp\left(-\frac{\pi\hat{K}}{4\left(\hat{N}^2 + \hat{N} - \hat{L}^2\right)^{1/2}}\right). \tag{8}$$

transforms $\hat{H}_{\text{int}}$ as follows:

$$\hat{U}^\dagger\hat{H}_{\text{int}}\hat{U} = \hbar\gamma\big(\hat{N}^2 + \hat{N} - \hat{L}^2\big)^{1/2}\hat{\sigma}_3. \tag{9}$$

Observe the accordance of equation (9) with equation (4) due to $\hat{\sigma}_3^2 = 1$, taking in mind that the unitary operator in equation (8) does not transform the square of $\hat{H}_{\text{int}}$ despite $\hat{H}_{\text{int}}$ itself is modified. Hence squaring equation (9) gives exactly equation (4), which is not sensible to the anti-transformation induced by $\hat{U}^\dagger$.

Standard algebraic manipulations leading to the expansion of the angular momentum eigenfunctions in the Fock basis complete the diagonalization procedure.

4 Applications and conclusive remarks

The investigated features of the Hamiltonian Model $\hat{H}_{\text{int}}$ lead to the possibility of inducing quantum dynamics satisfying certain symmetry conditions, for instance, rotational invariance. As an example assume the system prepared in the vacuum vibrational state and in the excited atomic level:

$$|\psi(t=0)\rangle = |0,0,0,+\rangle, \tag{10}$$

where the first three numbers refer to the x, y, and z vibrational excitation numbers, while the sign indicates the atomic ground $(-)$ or excited $(+)$ state.

In the light of the previous results, it is straightforward to find that once the laser interaction is turned on the system evolves in the following way:

$$|\psi(t)\rangle = \cos\left(\sqrt{6}\gamma t\right)|0,0,0,+\rangle - \frac{i}{\sqrt{3}}\sin\left(\sqrt{6}\gamma t\right)\left[|2,0,0,+\rangle + |0,2,0,+\rangle + |0,0,2,+\rangle\right]. \tag{11}$$

After a $\frac{\pi}{2}$-pulse, i.e. at the instant of time t_0 such that $\sqrt{6}\gamma t_0 = \frac{\pi}{2}$, up to a global phase factor, the system is found to be in the GHZ state [7]

$$|\psi(t_0)\rangle = \frac{1}{\sqrt{3}}|2,0,0,+\rangle + |0,2,0,+\rangle + |0,0,2,+\rangle.$$

The importance of such a state relies in its applications in the study of the Bell's inequality violations [7]. In passing we observe that, as expected, starting from the spherically symmetric wavefunction given in equation (10), the dynamics preserves such a symmetry at any instant of time. As another example consider an non-spherically symmetric initial state which distinguishes the z-direction from the other two. Let be

$$|\psi(t=0)\rangle = |0,0,2,-\rangle.$$

The relevant time evolution is given by

$$|\psi(t)\rangle = \frac{1}{3}\big[\cos\big(\sqrt{6}\gamma t\big)+2\big]|0,0,2,-\rangle + \frac{1}{3}\big[\cos\big(\sqrt{6}\gamma t\big)-1\big]\left[|2,0,0,-\rangle + |0,2,0,-\rangle\right] - \frac{i}{\sqrt{3}}\sin\big(\sqrt{6}\gamma t\big)|0,0,0,+\rangle. \tag{12}$$

Due to the fact that $\{|2,0,0,-\rangle, |0,2,0,-\rangle, |0,0,2,-\rangle, |0,0,0,+\rangle\}$ is an excitation number of invariant subspace, the states involved in the two considered dynamics, the ones in equation (11) and equation (12), are the same. Nevertheless, the weights are significantly different. In particular, as well and easily seen, the initial symmetry breaking is preserved by the time evolution. Indeed, for instance, despite $|0,0,0,+\rangle$ appears, no instant of time exists when the state $|\psi(t)\rangle$ coincides with it.

Summarizing, in this paper we have described the trapped ion physical scenario and focused our attention to a specific experimental configuration (three lasers suitably tuned and so on) corresponding to a second order vibronic Jaynes–Cummings like interaction model.

The Hamiltonian exhibits interesting and useful symmetry properties: total excitation number conservation, spherical invariance and a hidden supersymmetry. Exploiting all such features we succeed in finding Hamiltonian Model eigensolutions and, hence, in describing the dynamics it is responsible for, with respect to some simple initial conditions.

As expected, the degree of spherical symmetry is found to be preserved during all the time evolution. Moreover, in correspondence to a vacuum vibrational state initial state, a GHZ state is obtained.

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Generalized Self-Duality in Superfield Formalism

Anatoli PAVLYUK

Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine
E-mail: *a_pavlyuk@ukr.net*

The generalized super self-duality equations for $N = 2$ Super Yang–Mills theory in $N = 1$ superspace, which are first-order equations, in terms of component fields as well as in terms of superfields are presented.

In the pure $SU(2)$ Yang–Mills theory the self-duality equations read

$$iF_{mn} = \widetilde{F}_{mn} \equiv \frac{1}{2}\epsilon_{mnkl}F^{kl}, \tag{1}$$

where

$$F_{mn} = \partial_m V_n - \partial_n V_m + ig[V_m, V_n], \qquad \eta_{mn} = \mathrm{diag}\,(-1, 1, 1, 1).$$

Going from vector indices $(m, n = 0, 1, 2, 3)$ to spinor indices $(\alpha, \beta = 1, 2)$

$$x_m \to x_{\alpha\dot{\alpha}} = \sigma^m{}_{\alpha\dot{\alpha}} x_m,$$

where $\sigma^m = (-1, \vec{\sigma})$, (1 is a unit 2×2 matrix, $\vec{\sigma}$ are Pauli matrices), one obtains the Yang–Mills strength in the following form

$$F_{\alpha\dot{\alpha},\beta\dot{\beta}} \equiv \sigma^m{}_{\alpha\dot{\alpha}}\sigma^n{}_{\beta\dot{\beta}}F_{mn} = \frac{1}{2}\varepsilon_{\alpha\beta}f_{\dot{\alpha}\dot{\beta}} + \frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta},$$

where

$$f_{\dot{\alpha}\dot{\beta}} \equiv \epsilon^{\alpha\gamma}F_{\gamma\dot{\alpha},\alpha\dot{\beta}}, \qquad f_{\alpha\beta} \equiv \epsilon^{\dot{\alpha}\dot{\gamma}}F_{\alpha\dot{\gamma},\beta\dot{\alpha}}.$$

Now the self-duality equations (1) take the form

$$f_{\alpha\beta} = 0. \tag{2}$$

The two-spinor indices are raised and lowered by means of the two-dimensional Levi-Civita tensors

$$\varepsilon^{12} = -\varepsilon_{12} = \varepsilon^{\dot{1}\dot{2}} = -\varepsilon_{\dot{1}\dot{2}} = +1.$$

The raising and lowering of spinor indices are defined by

$$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta, \qquad \lambda^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\lambda_{\dot{\beta}}, \qquad \lambda_\alpha = \varepsilon_{\alpha\beta}\lambda^\beta, \qquad \lambda_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\lambda^{\dot{\beta}}.$$

In the $SU(2)$ $N = 1$ Super Yang–Mills theory [1]

$$L = \mathrm{Tr}\left\{-\frac{1}{4}F_{mn}F^{mn} - i\bar{\lambda}\overline{\sigma}^m\mathcal{D}_m\lambda + \frac{1}{2}D^2\right\}, \tag{3}$$

where with help of supersymmetric transformations of (2) one obtains the system of super self-duality equations in component fields [2]

$$f_{\alpha\beta} = 0, \qquad D = 0, \qquad \mathcal{D}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} = 0, \qquad \lambda_\alpha = 0. \tag{4}$$

The system of super self-duality equations (4) is a supersymmetric generalization of self-duality equations (2) and satisfies the equations of motion of theory (3). It is invariant under $N = 1$ supersymmetric transformations.

Let us proceed to the $N = 2$ supersymmetric Yang–Mills theory in $N = 1$ superspace, which is described by vector and scalar multiplets of component fields,

$$V = (V_m; \lambda_\alpha; \bar\lambda_{\dot\alpha}; D), \qquad \Phi = (A, B; \psi_\alpha; \bar\psi_{\dot\alpha}; F, G).$$

Gauge-invariant supersymmetric Lagrangian of $N = 2$ Yang–Mills theory in $N = 1$ superspace, whose fields belong to the adjoint representation of $SU(2)$ gauge group, has the form

$$\begin{aligned} L = \mathrm{Tr}\Big\{ & -\frac{1}{4}F_{mn}F^{mn} - \frac{1}{2}\mathcal{D}_m A\mathcal{D}^m A - \frac{1}{2}\mathcal{D}_m B\mathcal{D}^m B + igD[A,B] \\ & - i\bar\lambda\overline{\sigma}^m\mathcal{D}_m\lambda - i\bar\psi\overline{\sigma}^m\mathcal{D}_m\psi + ig(A+iB)\{\lambda^\alpha,\psi_\alpha\} \\ & + ig(A-iB)\{\bar\lambda_{\dot\alpha},\bar\psi^{\dot\alpha}\} + \frac{1}{2}D^2 + \frac{1}{2}F^2 + \frac{1}{2}G^2\Big\}. \end{aligned} \tag{5}$$

The equations of motion of theory (5) are as follows

$$\begin{aligned} & \varepsilon^{\dot\alpha\dot\gamma}\mathcal{D}_{\alpha\dot\gamma}f_{\dot\alpha\dot\beta} + \varepsilon^{\beta\gamma}\mathcal{D}_{\gamma\dot\beta}f_{\alpha\beta} + 4g\big(\{\lambda_\alpha,\bar\lambda_{\dot\beta}\} + \{\psi_\alpha,\bar\psi_{\dot\beta}\}\big) \\ & \qquad - ig\big([A-iB,\mathcal{D}_{\alpha\dot\beta}(A+iB)] + [A+iB,\mathcal{D}_{\alpha\dot\beta}(A-iB)]\big) = 0, \\ & \mathcal{D}_{\alpha\dot\beta}\mathcal{D}^{\alpha\dot\beta}(A-iB) - 4ig\{\lambda^\alpha,\psi_\alpha\} - 2g[D,A-iB] = 0, \\ & \mathcal{D}_{\alpha\dot\beta}\mathcal{D}^{\alpha\dot\beta}(A+iB) - 4ig\{\bar\lambda_{\dot\alpha},\bar\psi^{\dot\alpha}\} + 2g[D,A+iB] = 0, \\ & D + ig[A,B] = 0, \qquad F = G = 0, \\ & \mathcal{D}^{\alpha\dot\beta}\lambda_\alpha - g[\bar\psi^{\dot\beta},A-iB] = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} - g[\psi_\alpha,A+iB] = 0, \\ & \mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar\lambda^{\dot\beta},A-iB] = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\psi^{\dot\beta} + g[\lambda_\alpha,A+iB] = 0. \end{aligned} \tag{6}$$

Lagrangian (5) and the equations of motion (6) are invariant under the following $N = 1$ supersymmetric transformations

$$\begin{aligned} & \delta_\xi(A-iB) = 2\xi^\alpha\psi_\alpha, \qquad \delta_\xi(A+iB) = 2\bar\xi_{\dot\alpha}\bar\psi^{\dot\alpha}, \\ & \delta_\xi V_{\alpha\dot\alpha} = -2i(\xi_\alpha\bar\lambda_{\dot\alpha} + \bar\xi_{\dot\alpha}\lambda_\alpha), \qquad \delta_\xi D = -\xi^\alpha\mathcal{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} + \bar\xi_{\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}\lambda_\alpha, \\ & \delta_\xi(F+iG) = 2i\bar\xi_{\dot\beta}\big(\mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar\lambda^{\dot\beta},A-iB]\big), \\ & \delta_\xi(F-iG) = 2i\xi^\alpha\big(\mathcal{D}_{\alpha\dot\beta}\bar\psi^{\dot\beta} + g[\lambda_\alpha,A+iB]\big), \\ & \delta_\xi\lambda_\alpha = \frac{1}{2}\xi^\beta f_{\alpha\beta} + i\xi_\alpha D, \qquad \delta_\xi\bar\lambda_{\dot\alpha} = \frac{1}{2}\bar\xi^{\dot\beta}f_{\dot\alpha\dot\beta} - i\bar\xi_{\dot\alpha}D, \\ & \delta_\xi\psi_\alpha = i\bar\xi^{\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}(A-iB) + \xi_\alpha(F+iG), \\ & \delta_\xi\bar\psi_{\dot\alpha} = -i\xi^\alpha\mathcal{D}_{\alpha\dot\alpha}(A+iB) + \bar\xi_{\dot\alpha}(F-iG), \end{aligned} \tag{7}$$

where ξ_α, $\bar\xi_{\dot\alpha}$ are the parameters of $N = 1$ supersymmetric transformations.

In the theory (5) there is no supersymmetric generalization of self-duality equations (2) to super self-duality equations. We propose the system of first-order equations, which satisfies the second-order equations of motion (6) [3]

$$\begin{aligned} & f_{\alpha\beta} = 2igc_{\alpha\beta}[A,B], \\ & \mathcal{D}_{1\dot\alpha}(A-iB) = k_1\mathcal{D}_{2\dot\alpha}(A-iB), \qquad \mathcal{D}_{1\dot\alpha}(A+iB) = k_2\mathcal{D}_{2\dot\alpha}(A+iB), \\ & D + ig[A,B] = 0, \qquad F = G = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta} - g[\psi_\alpha,A+iB] = 0, \\ & \mathcal{D}^{\alpha\dot\beta}\psi_\alpha + g[\bar\lambda^{\dot\beta},A-iB] = 0, \qquad \lambda_\alpha = \bar\psi_{\dot\alpha} = 0, \qquad \psi_1 = k_1\psi_2, \end{aligned} \tag{8}$$

where constant complex bispinor coefficients $c_{\alpha\beta}$ are symmetric and unimodular

$$c_{\alpha\beta} = c_{\beta\alpha}, \qquad \det \|c_{\alpha\beta}\| \equiv c_{11} \cdot c_{22} - c_{12}^2 \equiv \frac{1}{2} c^{\alpha\beta} c_{\alpha\beta} = 1.$$

The coefficients k_1, k_2 can be calculated through $c_{\alpha\beta}$:

$$k_1 = \frac{c_{11}}{c_{12} - i} = \frac{c_{12} + i}{c_{22}}, \qquad k_2 = \frac{c_{11}}{c_{12} + i} = \frac{c_{12} - i}{c_{22}}.$$

In turn, the coefficients $c_{\alpha\beta}$ are expressed through k_1, k_2:

$$c_{11} = \frac{2ik_1k_2}{k_1 - k_2}, \qquad c_{12} = \frac{i(k_1 + k_2)}{k_1 - k_2}, \qquad c_{22} = \frac{2i}{k_1 - k_2}.$$

The system of first-order equations (8) is invariant under $N = 1$ supersymmetric transformations (7) in the subspace of the parameters of transformations, which is defined by the following condition

$$\xi_1 = k_2 \xi_2.$$

In the case of vanishing component fields of scalar multiplet

$$A = B = \psi_\alpha = \bar{\psi}_{\dot\alpha} = F = G = 0$$

the system (8) reduces to the system of super self-duality equations (4). Because of these properties we call (8) the system of super quasi-self-duality equations. Considering only bosonic sector of theory (5), we obtain from (8) the system of quasi-self-duality equations. The concept of quasi-self-duality was introduced by V.A. Yatsun [4, 5].

The fact that the solutions of the system (8) are the solutions of the equations of motion (6) is immediately proved by taking covariant derivatives from the first three equations of the system (8). To see this we take a covariant derivative $\epsilon^{\beta\gamma}\mathcal{D}_{\gamma\dot\beta}$ from the first equation of the system (8):

$$f_{\alpha\beta} - g c_{\alpha\beta}[A - iB, A + iB] = 0.$$

Then by using the second and third equations of the system (8) and taking into account the Bianchi identity $\epsilon^{\dot\alpha\dot\gamma}\mathcal{D}_{\alpha\dot\gamma} f_{\dot\alpha\dot\beta} = \epsilon^{\beta\gamma}\mathcal{D}_{\gamma\dot\beta} f_{\alpha\beta}$ and $\lambda_\alpha = \bar{\psi}_{\dot\alpha} = 0$, we obtain the first equation of the system (6). Taking a covariant derivative $\mathcal{D}^{1\dot\alpha}$ and $\mathcal{D}^{2\dot\alpha}$ from the second and third equations of (8) and using the first equation of this system, we obtain corresponding equations of motion from the system (6).

A superfield formulation of the super quasi-self-duality equations (8) can be obtained in the superspace $(x^m; \theta_\alpha; \bar\theta_{\dot\alpha})$ with the fixed Grassmann variable $\theta_1 = k_2\theta_2$. A spinor chiral superfield is defined by [6]

$$W_\alpha = -\frac{1}{8g}\overline{\mathcal{D}\mathcal{D}}\big(e^{-2gV}\mathcal{D}_\alpha e^{2gV}\big),$$

where V is a vector superfield in the Wess–Zumino gauge

$$V = -\theta^\alpha\bar\theta^{\dot\alpha}V_{\alpha\dot\alpha} + i\theta\theta\bar\theta\bar\lambda - i\bar\theta\bar\theta\theta\lambda + \frac{1}{2}\theta\theta\bar\theta\bar\theta D,$$

and the covariant derivatives are

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}, \qquad \overline{\mathcal{D}}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\partial_{\alpha\dot\alpha}.$$

In component fields it is given by

$$W_\alpha = -i\lambda_\alpha - \frac{i}{2}\theta^\beta(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D) + \theta^\beta\bar\theta^{\dot\beta}\partial_{\beta\dot\beta}\lambda_\alpha + \theta\theta\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta}$$
$$- \frac{1}{4}\theta\theta\bar\theta^{\dot\alpha}\epsilon^{\beta\gamma}\partial_{\gamma\dot\alpha}(f_{\alpha\beta} + 2i\varepsilon_{\alpha\beta}D) - \frac{i}{4}\theta\theta\bar\theta\bar\theta\Box\lambda_\alpha, \tag{9}$$

where

$$\Box = \partial_m\partial^m = -\frac{1}{2}\partial_{\alpha\dot\beta}\partial^{\alpha\dot\beta}.$$

The Hermitian adjoint superfield is

$$\big(W_\alpha\big)^\dagger = \overline{W}_{\dot\alpha} = -\frac{1}{8g}\mathcal{D}\mathcal{D}\big(\overline{\mathcal{D}}_{\dot\alpha}e^{2gV}\cdot e^{-2gV}\big).$$

A scalar chiral superfield is defined by the condition $\overline{\mathcal{D}}_{\dot\alpha}\Phi = 0$ and in components has the form

$$\Phi = \frac{1}{2}(A - iB) + \theta\psi + \frac{i}{2}\theta^\alpha\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}(A - iB) + \frac{i}{2}\theta\theta\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\psi^\alpha$$
$$+ \frac{1}{8}\theta\theta\bar\theta\bar\theta\Box(A - iB) + \frac{1}{2}\theta\theta(F + iG).$$

Consequently, its Hermite conjugate $\Phi^\dagger$ is given by $\mathcal{D}_\alpha\Phi^\dagger = 0$. The gauge-covariant spinor derivatives are

$$\nabla_\alpha = \mathcal{D}_\alpha + i[\mathcal{A}_\alpha, \cdot\,], \qquad \nabla_{\dot\alpha} = \overline{\mathcal{D}}_{\dot\alpha} + i[\mathcal{A}_{\dot\alpha}, \cdot\,],$$

where

$$\mathcal{A}_\alpha = -ie^{-2gV}\mathcal{D}_\alpha e^{2gV}, \qquad \mathcal{A}_{\dot\alpha} = i\overline{\mathcal{D}}_{\dot\alpha}e^{2gV}\cdot e^{-2gV}.$$

The system of super quasi-self-duality equations (8) can be presented in superfield formalism in the following form

$$\nabla_\alpha W_\beta = -2ig(c_{\alpha\beta} - i\varepsilon_{\alpha\beta})[\Phi, \Phi^\dagger - 2g[V, \Phi^\dagger]],$$
$$\nabla_1\Phi = k_1\nabla_2\Phi, \qquad \nabla_{\dot\alpha}\Phi^\dagger = 0, \qquad \theta_1 = k_2\theta_2. \tag{10}$$

The system of superfield quasi-self-duality equations (10) includes the equation for Grassmann coordinates

$$\theta_1 = k_2\theta_2. \tag{11}$$

To obtain component system (8) from superfield system (10) we have to calculate first three superfield equations of (10) in component fields, and only after that we have to impose the condition (11) on the Grassmann variables in these superfield equations.

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Four Dimensional Cubic Supersymmetry

M. RAUSCH de TRAUBENBERG

Laboratoire de Physique Théorique, Université Louis Pasteur,
CNRS UMR 7085, 3 rue de l'Université, 67084 Strasbourg, France
E-mail: *rausch@lpt1.u-strasbg.fr*

A four dimensional non-trivial extension of the Poincaré algebra different from supersymmetry is explicitly studied. Representation theory is investigated and an invariant Lagrangian is exhibited. Some discussion on the Noether theorem is also given.

1 Introduction

The concept of symmetries is a central tool in the description of physical systems. One of the main questions is, of course, what are the mathematical structures which are useful in describing the laws of physics, and, in particular, useful in particle physics or quantum field theories. Among others, finite-dimensional Lie algebras become essential for the description of space-time symmetries and fundamental interactions. On the other hand, it was the discovery of supersymmetry in relativistic quantum field theory or as a possible non-trivial extension of Poincaré invariance [1] which gave rise to the concept of Lie superalgebras. One natural question one should address is the possibility to weigh up, in relativistic quantum field theory, algebraic structures which are not Lie (super)algebras. *A priori* this should be a difficult task. Indeed, according to the Noether theorem, to all (Noetherian) symmetries correspond a conserved currents. These symmetries are then generated by charges which are expressed in terms of the fields. But, having two kinds of fields, of integer or half-integer spin, which, by spin-statistics theorem will close with commutators or anticommutators, *a priori* one should obtain only Lie and Lie superalgebras. Furthermore, the Coleman–Mandula [2] and the Haag–Lopuszanski–Sohnius [3] theorems state that within the framework of Lie algebras one obtains only the description of space-time and/or internal symmetries, while within Lie superalgebras supersymmetry is the *unique* non-trivial extension of the Poincaré algebra which is possible.

But, if one examines the hypotheses of the above theorems, one sees that it is possible to imagine symmetries which go beyond supersymmetry. Several possibilities have been considered in the literature, the intuitive idea being that the generators of the Poincaré algebra are obtained as an appropriate product of more fundamental additional symmetries. These new generators are in an appropriate representation of the Poincaré algebra.

In this contribution we would like to study one of the possible non-trivial extension of the Poincaré algebra, different from supersymmetry, named fractional supersymmetry (FSUSY) [4–20] and its associated underlying algebraic structure named F-Lie algebras [17,19]. In supersymmetric theories, the extensions of the Poincaré algebra are obtained from a "square root" of the translations, "$QQ \sim P$". In this paper, new algebras are obtained from yet higher order roots. We mainly focus on the simplest alternative where "cubic roots" are involved "$QQQ \sim P$" [20]. It is important to stress that such structures are not Lie (super)algebras and as such escape *a priori* the Coleman–Mandula [2] as well as the Haag–Lopuszanski–Sohnius [3] no-go theorems. Furthermore, as far as we know, no no-go theorem associated to such types of extensions has been considered in the literature. This can open interesting possibilities to search for a field theoretic realization of a non-trivial extension of the Poincaré algebra that *is not the supersymmetric one.*

The aim of this paper is to summarize some results already obtained in [20], *i.e.* to construct explicitly the first field theoretic construction in $(1+3)$ dimensions of an FSUSY with $F = 3$,

which we will refer to as *cubic supersymmetry*, or 3SUSY. Representation of our algebra leads to fermionic or bosonic multiplets. We find that the fermion multiplets are made of three definite chirality fermions that are degenerate in mass, while the boson multiplets contain Lorentz scalars, vectors and two-forms. A striking feature for the boson multiplets is the compatibility of 3SUSY with gauge symmetry only when the latter is gauge fixed in the usual way. Some discussions on Noether theorem in relation with 3SUSY are also given. The paper is organized as follow. In Section 2 some results on the algebraic extension of the Poincaré algebra are given. In Section 3 representations of the 3SUSY algebra are exhibited. In Section 4, we construct an invariant Lagrangian in the case of the bosonic multiplet. Section 5 is devoted to some discussions on Noether theorem. Finally, some conclusions are given in Section 6.

2 Non trivial extension of the Poincaré algebra

A natural generalization of Lie (super)algebras which is relevant for the algebraic description of FSUSY was defined in [18, 19] and called an F-Lie algebra. An $F-$Lie algebra admits a $\mathbb{Z}_F$-gradation, the zero-graded part being a Lie algebra. An F-fold symmetric product (playing the role of the anticommutator in the case $F = 2$) expresses the zero graded part in terms of the non-zero graded part. The first examples of F-Lie algebras where infinite-dimensional [17]. It was then established that one of these examples of infinite-dimensional algebras was relevant to apply FSUSY on relativistic anyons in $(1 + 2)D$ [16]. Later on, it was shown how to construct finite-dimensional F-Lie algebras with $F > 2$ by an inductive process starting from Lie algebras and Lie superalgebras [19]. Among these families of examples one can identify F-Lie algebras that could generate extensions of the Poincaré algebra. One of these examples is given by

$$\mathfrak{g} = \mathfrak{sp}(4, \mathbb{R}) \oplus \mathrm{ad}\,(\mathfrak{sp}(4, \mathbb{R})) \tag{1}$$

with $\mathrm{ad}\,(\mathfrak{sp}(4, \mathbb{R}))$ the adjoint representation of $\mathfrak{sp}(4, \mathbb{R})$. $\mathfrak{g}_0 = \mathfrak{sp}(4, \mathbb{R})$ is the zero graded part ("bosonic") of $\mathfrak{g}$ and $\mathfrak{g}_1 = \mathrm{ad}\,(\mathfrak{sp}(4, \mathbb{R}))$ is the graded part of the algebra. If we denote J_a, $a = 1, \ldots, 10$ a basis of $\mathfrak{sp}(4, \mathbb{R})$, A_a the corresponding basis for $\mathrm{ad}\,(\mathfrak{sp}(4, \mathbb{R}))$, and $g_{ab} = \mathrm{Tr}\,(A_a A_b)$ the Killing form of $\mathfrak{sp}(4, \mathbb{R})$, then the F-Lie algebra of order 3 $\mathfrak{g}$ reads [19, 20][1]

$$[J_a, J_b] = f_{ab}{}^c J_c, \qquad [J_a, A_b] = f_{ab}{}^c A_c, \qquad \{A_a, A_b, A_c\} = g_{ab} J_c + g_{ac} J_b + g_{bc} J_a, \tag{2}$$

where $f_{ab}{}^c$ are the structure constant of $\mathfrak{sp}(4, \mathbb{R})$ and $\{A_a, A_b, A_c\}$ is given by the symmetric three-fold product $A_a A_b A_c + A_a A_c A_b + A_b A_c A_a + A_b A_a A_c + A_c A_a A_b + A_c A_b A_a$.

Observing that $\mathfrak{so}(1, 3) \subset \mathfrak{so}(2, 3) \cong \mathfrak{sp}(4)$, and that the $(1 + 3)D$ Poincaré algebra is related to $\mathfrak{sp}(4)$ through an Inönü–Wigner contraction, from the F-Lie algebra (2), an extension of the Poincaré algebra can be constructed [20]:

$$\begin{aligned}
&[L_{mn}, L_{pq}] = \eta_{nq} L_{pm} - \eta_{mq} L_{pn} + \eta_{np} L_{mq} - \eta_{mp} L_{nq}, \qquad [L_{mn}, L_p] = \eta_{np} P_m - \eta_{mp} P_n, \\
&[L_{mn}, Q_p] = \eta_{np} Q_m - \eta_{mp} Q_n, \qquad [P_m, Q_n] = 0, \\
&\{Q_m, Q_n, Q_r\} = \eta_{mn} P_r + \eta_{mr} P_n + \eta_{rn} P_m,
\end{aligned} \tag{3}$$

where η_{mn} is the Minkowski metric, L_{mn}, P_m are the Poincaré generators and Q_m are the "supercharges" in the vector representation of $\mathfrak{so}(1, 3)$.

3 Representations

Representations of (3) were also studied in [20]. It turns out that, for algebras defined by cubic relations, the situation is a more difficult task than in usual supersymmetric theories. Indeed,

[1] In addition to these relations, one has also some appropriate Jacobi identities. See [18–20] for details.

representations of supersymmetry are related to representations theory of the well-known Clifford algebra, while representation theory of FSUSY is related to Clifford algebras of polynomial [21]. To obtain representations of the algebra (3), we rewrite the RHS of the trilinear bracket as $\{Q_m, Q_n, Q_r\} = f_{mnr} = f_{mnr}{}^s P_s$, with $f_{mnr}{}^s = \eta_{mn}\delta_r{}^s + \eta_{mr}\delta_n{}^s + \eta_{rn}\delta_m{}^s$. This substitution shows that the cubic polynomial $f(v^0, v^1, v^2, v^3) = f_{mnr}v^m v^n v^r = 3(v.P)(v.v)$ is associated to the symmetric tensor f_{mnr}. Moreover, the algebra (3) simply means that $f(v) = (v^m Q_m)^3$, as can be verified by developing the cube and identifying all terms, using the trilinear bracket. The generators Q_m, $m = 0, \dots, 3$, which are associated with the variables v^m, $m = 0, \dots, 3$, then generate an extension of the Clifford algebra called Clifford algebra of the polynomial f (denoted $\mathcal{C}_f$). This means that the Q's allow to "linearize" f. The general representation theory of $\mathcal{C}_f$ is not known, however, a systematic method to represent $\mathcal{C}_f$ by appropriate matrices has been given [22]. For the algebra (3) one obtains [20]

$$Q_m = \begin{pmatrix} 0 & \Lambda^{1/3}\gamma_m & 0 \\ 0 & 0 & \Lambda^{1/3}\gamma_m \\ \Lambda^{-2/3}P_m & 0 & 0 \end{pmatrix}. \tag{4}$$

with γ_m being the $4D$ Dirac matrices and $P_m = -i\frac{\partial}{\partial x^m}$. It is interesting to notice that because P is dimensionful, a parameter with a mass dimension appears naturally in (4).

The Q being in the vector representation of $\mathfrak{so}(1,3)$, it is easy to see that

$$J_{mn} = \frac{1}{4}(\gamma_m\gamma_n - \gamma_n\gamma_m) + i(x_m P_n - x_n P_m) \tag{5}$$

are the appropriate Lorentz generators acting on Q: $[J_{mn}, Q_r] = \eta_{nr}Q_m - \eta_{mr}Q_m$.

Introducing the $4D$ Dirac matrices in the Weyl representation, $\gamma_m = \begin{pmatrix} 0 & \sigma_m \\ \bar{\sigma}_m & 0 \end{pmatrix}$, with $\sigma_{m\,\alpha\dot{\alpha}} = (1, \sigma_i)$, $\bar{\sigma}_m{}^{\dot{\alpha}\alpha} = (1, -\sigma_i)$ and σ_i the Pauli matrices (the convention for dotted and undotted indices are those conventionally used in SUSY, – see *e.g.* appendix B of [20]), shows that the representation (4) is reducible and leads to the two inequivalent $6D$ representations:

$$Q_m = \begin{pmatrix} 0 & \Lambda^{1/3}\sigma_m & 0 \\ 0 & 0 & \Lambda^{1/3}\bar{\sigma}_m \\ \Lambda^{-2/3}P_m & 0 & 0 \end{pmatrix}, \qquad Q_m = \begin{pmatrix} 0 & \Lambda^{1/3}\bar{\sigma}_m & 0 \\ 0 & 0 & \Lambda^{1/3}\sigma_m \\ \Lambda^{-2/3}P_m & 0 & 0 \end{pmatrix}. \tag{6}$$

In this representation the trilinear part of the algebra (3) is realized as

$$\begin{aligned} &Q_mQ_nQ_r + Q_mQ_rQ_n + Q_nQ_mQ_r + Q_nQ_rQ_m + Q_rQ_mQ_n + Q_rQ_nQ_m \\ &\quad = \eta_{mn}P_r + \eta_{nr}P_m + \eta_{mr}P_n. \end{aligned} \tag{7}$$

3.1 Fermionic multiplet

As usual, the content of the representation is not only specified by the form of the matrix representation, but also by the behavior of the vacuum under Lorentz transformations. If we denote by Ω the vacuum, which is in some specified representation of the Lorentz algebra, with Σ_{mn} the corresponding Lorentz generators, then J_{mn} given in (5) is replaced by $J_{mn} + \Sigma_{mn}$. In the case, where Ω is a Lorentz scalar, one sees that the multiplet of the representations (6) contains two left-handed and one right-handed fermions for the first matrices, while the multiplet of the representation contains one left-handed and two right-handed fermions for the second matrices. These two multiplets are CPT conjugate. In the first case, if we denote $\mathbf{\Psi} = \begin{pmatrix} \psi_{1\alpha} \\ \bar{\psi}_2^{\dot{\alpha}} \\ \psi_{3\alpha} \end{pmatrix}$,

then under a 3SUSY transformation we have $\delta_\varepsilon \mathbf{\Psi} = \varepsilon^m Q_m \mathbf{\Psi}$ and we obtain

$$\delta_\varepsilon \psi_{1\alpha} = \varepsilon^n \Lambda^{1/3} \sigma_{n\alpha\dot\alpha} \bar\psi_2{}^{\dot\alpha}, \qquad \delta_\varepsilon \bar\psi_2{}^{\dot\alpha} = \varepsilon^n \Lambda^{1/3} \bar\sigma_n{}^{\dot\alpha\alpha} \psi_{3\alpha}, \qquad \delta_\varepsilon \psi_{3\alpha} = \varepsilon^n \Lambda^{-2/3} P_n \psi_{1\alpha} \tag{8}$$

with ε a pure imaginary number.

3.2 Bosonic multiplet

In the previous subsection, we were considering the fundamental representation associated to the matrices (6), say $\mathbf{\Psi} = \begin{pmatrix} \psi_{1\alpha} \\ \bar\psi_2{}^{\dot\alpha} \\ \psi_{3\alpha} \end{pmatrix}$ and $\mathbf{\Psi}' = \begin{pmatrix} \bar\psi_1'{}^{\dot\alpha} \\ \psi'_{2\alpha} \\ \bar\psi_3'{}^{\dot\alpha} \end{pmatrix}$, *i.e.* with the vacuum Ω in the trivial representation of the Lorentz algebra. In this section boson multiplets, will be introduced, corresponding to a vacuum in the spinor representations of the Lorentz algebra. This means that four types of boson multiplets can be introduced: $\mathbf{\Psi} \otimes \Omega^\alpha$, $\mathbf{\Psi}' \otimes \Omega^\alpha$, with Ω^α a left-handed spinor and $\mathbf{\Psi} \otimes \bar\Omega_{\dot\alpha}$, $\mathbf{\Psi}' \otimes \bar\Omega_{\dot\alpha}$ with $\bar\Omega_{\dot\alpha}$ a right-handed spinor.

For the multiplet associated to $\mathbf{\Psi}^\beta = \mathbf{\Psi} \otimes \Omega^\beta$, we have $\mathbf{\Psi}^\beta = \begin{pmatrix} \rho_{1\alpha}{}^\beta \\ \bar\rho_2{}^{\dot\alpha\beta} \\ \rho_{3\alpha}{}^\beta \end{pmatrix}$ and for the one (CPT conjugate of the previous) $\mathbf{\Psi}_{\dot\beta} = \mathbf{\Psi}' \otimes \Omega_{\dot\beta} = \begin{pmatrix} \bar\rho_1{}^{\dot\alpha}{}_{\dot\beta} \\ \rho_{2\alpha\dot\beta} \\ \bar\rho_3{}^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}$. The transformation under 3SUSY is $\delta_\varepsilon \mathbf{\Psi}^\beta = \varepsilon^m Q_m \mathbf{\Psi}^\beta$ with Q_m given in (6) and similarly for $\mathbf{\Psi}_{\dot\beta}$.

Notice that ρ_1, $\bar\rho_1$, $\bar\rho_2$, ρ_2 and ρ_3, $\bar\rho_3$ are not irreducible representations of $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(1,3)$, we therefore define

$$\begin{aligned} &\rho_1 = \varphi\, I_2 + \frac{1}{2} B_{mn}\, \sigma^{nm}, \qquad \bar\rho_1 = \varphi'\, \bar I_2 + \frac{1}{2} B'_{mn} \bar\sigma^{nm}, \qquad \bar\rho_2 = A^m\, \bar\sigma_m, \qquad \rho_2 = A'^m\, \sigma_m, \\ &\rho_3 = \tilde\varphi\, I_2 + \frac{1}{2} \tilde B_{mn}\, \sigma^{nm}, \qquad \bar\rho_3 = \tilde\varphi'\, \bar I_2 + \frac{1}{2} \tilde B'{}_{mn}\, \bar\sigma^{nm} \end{aligned} \tag{9}$$

with I_2 and $\bar I_2$ the two by two identity matrices, σ_{mn} and $\bar\sigma_{mn}$ the Lorentz generators for the two spin representations, A^m and A'^m two vectors, φ, $\tilde\varphi$ and φ', $\tilde\varphi'$ four scalars, B_{mn}, $\tilde B_{mn}$ two self-dual two-forms and $B'{}_{mn}$, $\tilde B'{}_{mn}$ two anti-self-dual two-forms. Then one can show that the transformations read [20]

$$\begin{aligned} &\delta_\varepsilon \varphi = \Lambda^{1/3} \varepsilon^m A_m, \qquad \delta_\varepsilon \varphi' = \Lambda^{1/3} \varepsilon^m A'_m, \\ &\delta_\varepsilon B_{mn} = -\Lambda^{1/3} \left(\varepsilon_m A_n - \varepsilon_n A_m\right) + \Lambda^{1/3} i \varepsilon_{mnpq} \varepsilon^p A^q, \\ &\delta_\varepsilon B'_{mn} = -\Lambda^{1/3} \left(\varepsilon_m A'_n - \varepsilon_n A'_m\right) - \Lambda^{1/3} i \varepsilon_{mnpq} \varepsilon^p A'^q, \\ &\delta_\varepsilon A_m = \Lambda^{1/3} \left(\varepsilon_m \tilde\varphi + \varepsilon^n \tilde B_{mn}\right), \qquad \delta_\varepsilon A'_m = \Lambda^{1/3} \left(\varepsilon_m \tilde\varphi' + \varepsilon^n \tilde B'_{mn}\right), \\ &\delta_\varepsilon \tilde\varphi = \Lambda^{-2/3} \varepsilon^m P_m \varphi, \qquad \delta_\varepsilon \tilde\varphi' = \Lambda^{-2/3} \varepsilon^m P_m \varphi', \\ &\delta_\varepsilon \tilde B_{mn} = \Lambda^{-2/3} \varepsilon^p P_p B_{mn}, \qquad \delta_\varepsilon \tilde B'_{mn} = \Lambda^{-2/3} \varepsilon^p P_p B'_{mn}. \end{aligned} \tag{10}$$

The second bosonic multiplet being CPT conjugate to the first one, we have $\left(\bar\rho_1{}^{\dot\alpha}{}_{\dot\beta}\right)^\star = \rho_1{}^\alpha{}_\beta$, $\left(\rho_{2\alpha\dot\beta}\right)^\star = \bar\rho_{2\dot\alpha\beta}$ and $\left(\bar\rho_3{}^{\dot\alpha}{}_{\dot\beta}\right)^\star = \rho_3{}^\alpha{}_\beta$. [$B^\star$, the complex conjugate of B, is not to be confused with ${}^\star B$, the dual of B, see after.] This means, paying attention to the position of the indices, that we have

$$\varphi'^\star = -\varphi, \qquad \tilde\varphi'^\star = -\tilde\varphi, \qquad B'^\star_{mn} = -B_{mn}, \qquad \tilde B'^\star_{mn} = -\tilde B_{mn}, \qquad A'^\star_m = A_m. \tag{11}$$

These relations are compatible with the transformations laws given in (10) since ${\varepsilon_n}^\star = -\varepsilon_n$ and $P_n = -i \frac{\partial}{\partial x^n}$.

4 Invariant action

To construct a real invariant action involving the bosonic multiplets (9), we introduce the real fields

$$
\begin{aligned}
&A_- = i\frac{A-A'}{\sqrt{2}}, \qquad A_+ = \frac{A+A'}{\sqrt{2}},\\
&B_- = \frac{B-B'}{\sqrt{2}}, \qquad B_+ = i\frac{B+B'}{\sqrt{2}}, \qquad \tilde{B}_- = \frac{\tilde{B}-\tilde{B}'}{\sqrt{2}}, \qquad \tilde{B}_+ = i\frac{\tilde{B}+\tilde{B}'}{\sqrt{2}},\\
&\varphi_- = \frac{\varphi-\varphi'}{\sqrt{2}}, \qquad \varphi_+ = i\frac{\varphi+\varphi'}{\sqrt{2}}, \qquad \tilde{\varphi}_- = \frac{\tilde{\varphi}-\tilde{\varphi}'}{\sqrt{2}}, \qquad \tilde{\varphi}_+ = i\frac{\tilde{\varphi}+\tilde{\varphi}'}{\sqrt{2}}.
\end{aligned}
\tag{12}
$$

These new fields form now one (reducible) multiplet of 3SUSY, with ${}^\star B_- = B_+$ (${}^\star B_-$ is the dual of B_- [20]). The corresponding two- and three-form field strengths read

$$
\begin{aligned}
&F_{\pm mn} = \partial_m A_{\pm n} - \partial_n A_{\pm m},\\
&H_{\pm mnp} = \partial_m B_{\pm np} + \partial_n B_{\pm pm} + \partial_p B_{\pm mn}.
\end{aligned}
\tag{13}
$$

They are invariant under the gauge transformations

$$
\begin{aligned}
&\varphi_\pm \to \varphi_\pm,\\
&A_{\pm m} \to A_{\pm m} + \partial_m \chi_\pm,\\
&B_{\pm mn} \to B_{\pm mn} + (\partial_m \chi_{\pm\, n} - \partial_n \chi_{\pm\, m}),
\end{aligned}
\tag{14}
$$

where $\chi_\pm$ ($\chi^m_\pm$) are arbitrary scalar (vector) functions (χ^m_- and χ^m_+ can still be related in order to preserve the duality relations between B_- and B_+)[2].

In a similar way we introduce the field strength $\tilde{H}_{-mnp}$, $\tilde{H}_{+mnp}$, as well as the dual fields ${}^\star H_{-m}$, ${}^\star H_{+m}$, ${}^\star \tilde{H}_{-m}$, ${}^\star \tilde{H}_{+m}$ (where ${}^\star H_m \equiv \frac{1}{6}\varepsilon_{mnpq}H^{npq}$. For instance ${}^\star \tilde{H}_{-m} = i\partial^n B_{+mn}$). We consider now the following local gauge invariant and zero graded Lagrangian,

$$
\begin{aligned}
\mathcal{L} = {}& \partial_m\varphi_-\partial^m\tilde{\varphi}_- - \partial_m\varphi_+\partial^m\tilde{\varphi}_+\\
&- \frac{1}{4}F_{-mn}F_-{}^{mn} + \frac{1}{4}F_{+mn}F_+{}^{mn} - \frac{1}{2}\left(\partial_m A_-{}^m\right)^2 + \frac{1}{2}\left(\partial_m A_+{}^m\right)^2\\
&- \frac{1}{12}H_{-mnp}\tilde{H}_-{}^{mnp} + \frac{1}{12}H_{+mnp}\tilde{H}_+{}^{mnp} + \frac{1}{2}{}^\star H_{-m}{}^\star\tilde{H}_-{}^m - \frac{1}{2}{}^\star H_{+m}{}^\star\tilde{H}_+{}^m.
\end{aligned}
\tag{15}
$$

By means of (12), a direct calculation shows that (15) is invariant under the transformations (10), up to a surface term. It is interesting to notice that the 3SUSY invariance is compatible with gauge symmetries if the latter are gauged fixed. A usual 't Hooft Feynman gauge fixing term $(-\frac{1}{2}\left(\partial_m A_-{}^m\right)^2 + \frac{1}{2}\left(\partial_m A_+{}^m\right)^2)$ is required for the vector fields. For the two forms, due to the relation ${}^\star B_- = B_+$ also some gauge fixing terms *à la* 't Hooft Feynman are present. Developing all terms in (15) the Lagrangian can be rewritten *à la* "Fermi-like"

$$
\begin{aligned}
\mathcal{L} = {}& \partial_m\varphi_-\partial^m\tilde{\varphi}_- - \frac{1}{2}\partial_m A_{-n}\partial^m A_-{}^n + \frac{1}{4}\partial_m B_{-np}\partial^m\tilde{B}_{-np}\\
&- \partial_m\varphi_+\partial^m\tilde{\varphi}_+ + \frac{1}{2}\partial_m A_{+n}\partial^m A_+{}^n - \frac{1}{4}\partial_m B_{+np}\partial^m\tilde{B}_{+np}
\end{aligned}
$$

(a similar Lagrangian appears in the action-at-a-distance formalism for vector and two-forms – see *e.g.* [23]).

[2]Note that the gauge transformations (14) correspond naturally to the zero-, one- and two-form character of the components of the 3SUSY gauge multiplet.

One should note, though, the relative minus signs in front of the kinetic terms of the vector fields in (15) which endanger *a priori* the boundedness from below of the density energy of the "electromagnetic" fields. This difficulty does not have a clear physical interpretation as long as interaction terms have not been included, and necessitates a more careful study of the field manifold associated to the density energy.

5 Noether currents

The 3SUSY algebra we have studied has one main difference from the usual Lie (super)algebra: it does not close through quadratic, but rather cubic, relations. Moreover, it might be possible that some usual results of Lie (super)algebra do not apply straightforwardly. One example is the Noether currents and their associated algebra. This interesting point was studied in [20].

In this section we would like, however, to construct explicitly the Noether current. Using (15) and (10) we obtain

$$\begin{aligned} J_{mn} = &-i\Lambda^{1/3}(A_{-n}\partial_m\tilde{\varphi}_- - \partial_m A_{-n}\tilde{\varphi}_-) - i\Lambda^{1/3}(A_{+n}\partial_m\tilde{\varphi}_+ - \partial_m A_{+n}\tilde{\varphi}_+) \\ &+ i\Lambda^{1/3}(\tilde{B}_{-rn}\partial_m A^r_- - \tilde{B}_{-rn}\partial_m A^r_-) + i\Lambda^{1/3}(\tilde{B}_{+rn}\partial_m A^r_+ - \tilde{B}_{+rn}\partial_m A^r_+) \\ &- i\Lambda^{-2/3}(\partial_m\varphi_-\partial_n\varphi_- - \partial_m\varphi_+\partial_n\varphi_+) - \frac{i}{4}\Lambda^{-2/3}(\partial_m B^{rs}_-\partial_n B_{-rs} - \partial_m B^{rs}_+\partial_n B_{+rs}), \end{aligned} \tag{16}$$

which is, due to the equations of motion, conserved $\partial_m J^{mn} = 0$ (up to a surface term). Then, the conserved charges are obtained as usual $\hat{Q}_m = \int d^3x J_{0m}$. Introducing the conjugate momentum Π of the fields Ψ (where Ψ is one of the fields of Section 4): $\Pi = \frac{\delta\mathcal{L}}{\delta\partial_0\Psi}$ after the usual quantization (equal-time commutation relations) one easily obtains

$$\delta_\varepsilon\Psi = \left[\varepsilon_n\hat{Q}^n, \Psi\right] \qquad \left(\delta_{Q_m}\Psi = \left[\hat{Q}^m, \Psi\right]\right). \tag{17}$$

In particular, this means that the algebra (3) is realized through

$$\begin{aligned} (\delta_{Q_m}.\delta_{Q_n}.\delta_{Q_r} + \text{perm})\,\Psi &= \left[\hat{Q}_m, \left[\hat{Q}_n, \left[\hat{Q}_r, \Psi\right]\right]\right] + \text{perm} \\ &= \eta_{mn}\left[\hat{P}_r, \Psi\right] + \eta_{mr}\left[\hat{P}_n, \Psi\right] + \eta_{rn}\left[\hat{P}_m, \Psi\right] = (\eta_{mn}\delta_{P_r} + \eta_{mr}\delta_{P_n} + \eta_{rn}\delta_{P_m})\,\Psi \end{aligned} \tag{18}$$

with $\hat{P}$ the generators of the Poincaré translations. Indeed, starting with the abstract algebra (3), we can represent it by some matrices as in Section 3 (see *e.g.* (4)). In this case the product of two transformations will be given by $\delta_n\delta_m\Psi = Q_nQ_m\Psi$ and the algebra will be realized as in (7). But, we can also represent (3) with commutators (17) acting on some Hilbert space, thus the product of two transformations will be given by $\delta_n\delta_m\Psi = \left[\hat{Q}_n, \left[\hat{Q}_m, \Psi\right]\right]$ leading to the realization (18) of the algebra (3). For a more general discussion one can see [20].

6 Conclusion

In this paper we have studied some four-dimensional realizations of 3SUSY, a non-trivial extension of the Poincaré algebra different from supersymmetry. Representation theory was explicitly constructed. Then, an invariant Lagrangian involving bosonic fields was given (some Lagrangian involving fermionic fields was also considered in [20]). The next step will be to construct an interacting theory.

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On Linearization of Superon-Graviton Model (SGM)

Kazunari SHIMA †, *Motomu TSUDA* † *and Manabu SAWAGUCHI* ‡

† *Laboratory of Physics, Saitama Institute of Technology, Okabe-machi, Saitama 369-0293, Japan*
E-mail: *shima@sit.ac.jp, tsuda@sit.ac.jp*

‡ *High-Tech Research Center, Saitama Institute of Technology, Okabe-machi, Saitama 369-0293, Japan*
E-mail: *sawa@sit.ac.jp*

Referring to the supermultiplet of $N = 1$ supergravity(SUGRA) the linearization of $N = 1$ SGM action describing the nonlinear supersymmetric (NL SUSY) gravitational interaction of superon (Nambu–Goldstone (N-G) fermion) is attempted. The field contents of on-shell SUGRA supermultiplet are realized as the composites, though they have new SUSY transformations which closes on super-Poincaré (SP) algebra. Particular attentions are paid to the local Lorentz invariance.

1 Introduction

Extending the geometrical arguments of Einstein general relativity theory (EGRT) on Riemann space-time to new space-time where the coset space coordinates of $\frac{\text{superGL(4,R)}}{\text{GL(4,R)}}$ turning to the N-G fermion degrees of freedom (d.o.f.) are attached at every Riemann space-time point, we have proposed a new Einstein–Hilbert (E-H) type action [1]. The new E-H type action describes the NLSUSY [2] invariant gravitational interaction of N-G fermion *superon* in Riemann space-time.

In this paper we would like to discuss the linearization of the new E-H type action ($N = 1$ SGM action) to obtain the equivalent linear (L) SUSY [3] theory in the low energy, which is renormalizable. Considering a phenomenological potential of SGM, though qualitative and group theoretical, discussed in [4] based upon the composite picture of L SUSY representation and the recent interest in NLSUSY in superstring (membrane) world, the linearization of NLSUSY in curved space-time may be of some general interest.

The linearization of SGM is physically interesting in general, even if the consequent theory were a existing SUGRA-like, for the equivalence among them gives a new insight into the fundamental structure of nature. For the self-contained arguments we review the SGM action briefly. ($N = 1$) SGM action is given by [1];

$$L_{\text{SGM}} = -\frac{c^3}{16\pi G}|w|(\Omega + \Lambda), \tag{1}$$

$$|w| = \det w^a{}_\mu = \det(e^a{}_\mu + t^a{}_\mu), \qquad t^a{}_\mu = \frac{i}{2}\kappa^4(\bar{\psi}\gamma^a\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^a\psi), \tag{2}$$

where $e^a{}_\mu$ is the vierbein of EGRT, ψ is N-G fermion (superon), $\kappa^4 = \left(\frac{c^3\Lambda}{16\pi G}\right)^{-1}$ is the fundamental volume of four-dimensional space-time of Volkov–Akulov (V-A) model [2], and Λ is the *small* cosmological constant related to the strength of the superon-vacuum coupling constant. Ω is a new scalar curvature analogous to the Ricci scalar curvature R of EGRT, whose explicit expression is obtained by just replacing $e^a{}_\mu(x)$ by $w^a{}_\mu(x)$ in Ricci scalar R. These results can be understood intuitively by observing that $w^a{}_\mu(x) = e^a{}_\mu(x) + t^a{}_\mu(x)$ defined by $\omega^a = w^a{}_\mu dx^\mu$, where ω^a is the NL SUSY invariant differential one-form of V-A [2], is invertible,

and $s^{\mu\nu}(x) \equiv w_a{}^{\mu}(x)w^{a\nu}(x)$ are a unified vierbein and a unified metric tensor in SGM space-time [1,5]. The SGM action (1) is invariant at least under the following symmetry [6]; ordinary GL(4,R), the following new NLSUSY transformation:

$$\delta^{NL}\psi(x) = \frac{1}{\kappa^2}\zeta + i\kappa^2(\bar{\zeta}\gamma^{\rho}\psi(x))\partial_{\rho}\psi(x), \qquad \delta^{NL}e^{a}{}_{\mu}(x) = i\kappa^2(\bar{\zeta}\gamma^{\rho}\psi(x))\partial_{[\rho}e^{a}{}_{\mu]}(x), \tag{3}$$

where ζ is a constant spinor and $\partial_{[\rho}e^{a}{}_{\mu]}(x) = \partial_{\rho}e^{a}{}_{\mu} - \partial_{\mu}e^{a}{}_{\rho}$, the following GL(4,R) transformations due to (3):

$$\delta_{\zeta}w^{a}{}_{\mu} = \xi^{\nu}\partial_{\nu}w^{a}{}_{\mu} + \partial_{\mu}\xi^{\nu}w^{a}{}_{\nu}, \qquad \delta_{\zeta}s_{\mu\nu} = \xi^{\kappa}\partial_{\kappa}s_{\mu\nu} + \partial_{\mu}\xi^{\kappa}s_{\kappa\nu} + \partial_{\nu}\xi^{\kappa}s_{\mu\kappa}, \tag{4}$$

where $\xi^{\rho} = i\kappa^2\bar{\zeta}\gamma^{\rho}\psi(x)$ and $s_{\mu\nu} = w^{a}{}_{\mu}w_{a\nu}$, and the following local Lorentz transformation on $w^{a}{}_{\mu}$:

$$\delta_L w^{a}{}_{\mu} = \epsilon^{a}{}_{b}w^{b}{}_{\mu} \tag{5}$$

with the local parameter $\epsilon_{ab} = (1/2)\epsilon_{[ab]}(x)$ or equivalently on ψ and $e^{a}{}_{\mu}$

$$\delta_L\psi(x) = -\frac{i}{2}\epsilon_{ab}\sigma^{ab}\psi, \qquad \delta_L e^{a}{}_{\mu}(x) = \epsilon^{a}{}_{b}e^{b}{}_{\mu} + \frac{\kappa^4}{4}\varepsilon^{abcd}\bar{\psi}\gamma_5\gamma_d\psi(\partial_{\mu}\epsilon_{bc}). \tag{6}$$

The local Lorentz transformation forms a closed algebra, for example, on $e^{a}{}_{\mu}(x)$

$$[\delta_{L_1}, \delta_{L_2}]e^{a}{}_{\mu} = \beta^{a}{}_{b}e^{b}{}_{\mu} + \frac{\kappa^4}{4}\varepsilon^{abcd}\bar{\psi}\gamma_5\gamma_d\psi(\partial_{\mu}\beta_{bc}), \tag{7}$$

where $\beta_{ab} = -\beta_{ba}$ is defined by $\beta_{ab} = \epsilon_{2ac}\epsilon_1{}^{c}{}_{b} - \epsilon_{2bc}\epsilon_1{}^{c}{}_{a}$. The commutators of two new NLSUSY transformations (3) on $\psi(x)$ and $e^{a}{}_{\mu}(x)$ are GL(4,R), i.e. new NLSUSY (3) is the square-root of GL(4,R):

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\psi = \Xi^{\mu}\partial_{\mu}\psi, \qquad [\delta_{\zeta_1}, \delta_{\zeta_2}]e^{a}{}_{\mu} = \Xi^{\rho}\partial_{\rho}e^{a}{}_{\mu} + e^{a}{}_{\rho}\partial_{\mu}\Xi^{\rho}, \tag{8}$$

where $\Xi^{\mu} = 2i(\bar{\zeta}_2\gamma^{\mu}\zeta_1) - \xi_1^{\rho}\xi_2^{\sigma}e_a{}^{\mu}(\partial_{[\rho}e^{a}{}_{\sigma]})$. They show the closure of the algebra. SGM action (1) is invariant at least under [6]

$$[\text{global NLSUSY}] \otimes [\text{local GL}(4,\text{R})] \otimes [\text{local Lorentz}], \tag{9}$$

which is isomorphic to SP whose single irreducible representation gives the group theoretical description of SGM [4].

2 Linearization of SGM

Here we just mention that the SGM action of equation (1) is a nontrivial generalization of E-H action. Interestingly, the following local spinor translation with a local parameter $\epsilon(x)$, $\delta\psi = \epsilon$, $\delta e^{a}{}_{\mu} = -i\kappa^4(\bar{\epsilon}\gamma^{a}\partial_{\mu}\psi + \bar{\psi}\gamma^{a}\partial_{\mu}\epsilon)$, gives $\delta w^{a}{}_{\mu} = 0 = \delta w_a{}^{\mu}$. However, this local spinor transformation cannot transform away the d.o.f. of ψ. Indeed, ψ seems to be transformed away if we choose $\delta\psi = \epsilon = -\psi$, but it is restored precisely in the unified vierbein $w^{a}{}_{\mu}$ by simultaneously transforming $e^{a}{}_{\mu}$, i.e., $w(e,\psi) = w(e+\delta e, \psi+\delta\psi) = w(e+t, 0)$ as indicated by $\delta w^{a}{}_{\mu} = 0$. And also the above local spinor transformation is a fake gauge transformation in a sense that, in contrast with the local Lorentz transformation on the coordinates in the vierbein formalism of EGRT, it cannot eliminate the d.o.f. of ψ since the unified vierbein $w^{a}{}_{\mu} = e^{a}{}_{\mu} + t^{a}{}_{\mu}$ is the only gauge field on SGM space-time and contains only integer spin. This confusive situation comes from the new geometrical formulation of SGM on unfamiliar SGM space-time, where

besides the Minkowski coordinates x^a, ψ is a Grassmann coordinate (i.e. the fundamental d.o.f.) defining the tangential space-time with $\mathrm{SO}(3,1) \times \mathrm{SL}(2,\mathrm{C})$ d.o.f. inspired by NLSUSY. The local spinor transformation ($\delta\psi = \epsilon(x)$) is just a coordinate transformation (redefinition) on SGM space-time. This situation can be understood easily by observing that the unified vierbein $w^a{}_\mu = e^a{}_\mu + t^a{}_\mu$ is defined by $\omega^a = dx^a + i\kappa^4\bar{\psi}\gamma^a d\psi = w^a{}_\mu dx^\mu$, where ω^a is the NLSUSY invariant differential one-form of V-A [2] and (x^a, ψ) are coordinates specifying the (SGM) flat space-time inspired by NLSUSY.

From these geometrical viewpoints (in SGM space-time) we can understand that ψ is a coordinate and would not be transformed away and the initial SGM space-time is preserved. Eliminating ψ by some arguments regarding the above local spinor translation as a gauge transformation leads to a different theory (ordinary E-H action) with a different vacuum (Minkowski flat space-time), which is another from SGM scenario assuming that the SGM space-time is an ultimate physical entity.

The linearization of such a highly nonlinear theory is interesting and inevitable for obtaining of a renormalizable field theory which is equivalent and describes the observed low energy physics.

The flat space linearization of $N = 1$ V-A model has been carried out, and it was proved that $N = 1$ V-A model is equivalent to $N = 1$ scalar supermultiplet [7] or $N = 1$ axial vector gauge supermultiplet of linear SUSY [8]. As a flat space exercise for the extended SGM linearization, we have carried out the linearization of $N = 2$ V-A model and have shown that it is equivalent to the spontaneously broken $N = 2$ linear SUSY *vector* $J^P = 1^-$ gauge supermultiplet model with SU(2) structure [9]. Interestingly, SU(2) algebraic gauge structure of the electroweak standard model (SM) may be explained for the first time provided that the electroweak gauge bosons are the composite fields of this (SGM) type in the low energy.

In these work the linearization is carried out by using the superfield formalism and/or by the heuristic and intuitive arguments on the relations between the component fields of L SUSY and NLSUSY. In either case it is crucial to discover the SUSY invariant relations, which connect the supermultiplets of L and NL theories and reproduce the SUSY transformations.

In above-mentioned cases of the global SUSY in flat space-time the SUSY invariant relations are obtained straightforwardly, for L and NL supermultiplets are well understood and the algebraic structures are the same SP.

The situation is rather different in SGM, for (i) the supermultiplet structure of the linearized theory of SGM is unknown except if it is expected to be a broken SUSY SUGRA-like theory containing graviton and a (massive) spin 3/2 field as dynamical d.o.f. and (ii) the algebraic structure (9) is changed into SP.

Therefore, by the heuristic arguments and referring to SUGRA we discuss for the moment the linearization of $N = 1$ SGM.

At first, we assume faithfully the SGM scenario that:

(i) the linearized theory should contain the spontaneously broken *global* (at least) SUSY;

(ii) graviton is an elementary field (not composite of superons corresponding to the vacuum of the Clifford algebra) in both L and NL theories;

(iii) the NLSUSY supermultiplet of SGM ($e^a{}_\mu(x)$, $\psi(x)$) should be connected to the composite supermultiplet ($\tilde{e}^a{}_\mu(e(x), \psi(x))$, $\tilde{\lambda}_\mu(e(x), \psi(x))$) for elementary graviton field and a composite (massive) spin 3/2 field of the SUGRA-like linearized theory.

From these assumptions and following the arguments used in the flat space cases we require that the SUGRA gauge transformation [10] with the global spinor parameter ζ should hold for the supermultiplet ($\tilde{e}^a{}_\mu(e, \psi)$, $\tilde{\lambda}_\mu(e, \psi)$) of the (SUGRA-like) linearized theory, i.e.,

$$\delta\tilde{e}^a{}_\mu(e, \psi) = i\kappa\bar{\zeta}\gamma^a\tilde{\lambda}_\mu(e, \psi), \tag{10}$$

$$\delta\tilde{\lambda}_\mu(e,\psi) = \frac{2}{\kappa}D_\mu\zeta = -\frac{i}{\kappa}\tilde{\omega}(e,\psi)_\mu{}^{ab}\sigma_{ab}\zeta, \tag{11}$$

where $\sigma^{ab} = \frac{i}{4}[\gamma^a,\gamma^b]$, $D_\mu = \partial_\mu - \frac{i}{2}\omega_\mu{}^{ab}(e,\psi)\sigma_{ab}$, ζ is a global spinor parameter and the variations in the left-hand side are induced by NLSUSY (3).

We put the following SUSY invariant relations, which connect $e^a{}_\mu$ to $\tilde{e}^a{}_\mu(e,\psi)$:

$$\tilde{e}^a{}_\mu(e,\psi) = e^a{}_\mu(x). \tag{12}$$

This relation (12) is the assumption (ii) and holds simply from the metric conditions. Consequently, the following covariant relation is obtained by substituting (12) into (10) and computing the variations under (3) [11]:

$$\tilde{\lambda}_\mu(e,\psi) = \kappa\gamma_a\gamma^\rho\psi(x)\partial_{[\rho}e^a{}_{\mu]}. \tag{13}$$

(As discussed later, these may be considered as the leading order of the expansions in κ of SUSY invariant relations. The expansions terminate with $(\psi)^4$. Now we see LSUSY transformation induced by (3) on the (composite) supermultiplet $(\tilde{e}^a{}_\mu(e,\psi),\ \tilde{\lambda}_\mu(e,\psi).)$

The LSUSY transformation on $\tilde{e}^a{}_\mu$ becomes as follows. The left-hand side of (10) gives

$$\delta\tilde{e}^a{}_\mu(e,\psi) = \delta^{NL}e^a{}_\mu(x) = i\kappa^2(\bar{\zeta}\gamma^\rho\psi(x))\partial_{[\rho}e^a{}_{\mu]}(x). \tag{14}$$

While substituting (13) into the right-hand side of (10) we obtain

$$i\kappa^2(\bar{\zeta}\gamma^\rho\psi(x))\partial_{[\rho}e^a{}_{\mu]}(x) + \cdots \text{ (extra terms)}. \tag{15}$$

These results show that (12) and (13) are not SUSY invariant relations and reproduce (10) with unwanted extra terms, which should be identified with the auxiliary fields. The commutator of the two LSUSY transformations induces GL(4,R) with the field dependent parameters as follows:

$$[\delta_{\zeta_1},\delta_{\zeta_2}]\tilde{e}^a{}_\mu(e,\psi) = \Xi^\rho\partial_\rho\tilde{e}^a{}_\mu(e,\psi) + \tilde{e}^a{}_\rho(e,\psi)\partial_\mu\Xi^\rho, \tag{16}$$

where $\Xi^\mu = 2i(\bar{\zeta}_2\gamma^\mu\zeta_1) - \xi_1^\rho\xi_2^\sigma e_a{}^\mu(\partial_{[\rho}e^a{}_{\sigma]})$.

On $\tilde{\lambda}_\mu(e,\psi)$, the left-hand side of (11) becomes apparently rather complicated:

$$\begin{aligned}\delta\tilde{\lambda}_\mu(e,\psi) &= \kappa\delta(\gamma_a\gamma^\rho\psi(x)\partial_{[\rho}e^a{}_{\mu]}) \\ &= \kappa\gamma_a[\delta^{NL}\gamma^\rho\psi(x)\partial_{[\rho}e^a{}_{\mu]} + \gamma^\rho\delta^{NL}\psi(x)\partial_{[\rho}e^a{}_{\mu]} + \gamma^\rho\psi(x)\partial_{[\rho}\delta^{NL}e^a{}_{\mu]}].\end{aligned} \tag{17}$$

However, the commutator of the two LSUSY transformations induces the similar GL(4,R):

$$[\delta_{\zeta_1},\delta_{\zeta_2}]\tilde{\lambda}_\mu(e,\psi) = \Xi^\rho\partial_\rho\tilde{\lambda}_\mu(e,\psi) + \tilde{\lambda}_\rho(e,\psi)\partial_\mu\Xi^\rho. \tag{18}$$

These results indicate that it is necessary to generalize (10), (11) and (13) for obtaining SUSY invariant relations and for the closure of the algebra. Furthermore, due to the complicated expression of LSUSY (17) that makes the physical and mathematical structures obscure, we can hardly guess a linearized invariant action equivalent to SGM.

Now we generalize the linearization by considering the auxiliary fields such that LSUSY transformation on the linearized fields induces SP transformation.

By comparing (11) with (17) we understand that the local Lorentz transformation plays a crucial role. As for the local Lorentz transformation on the linearized asymptotic fields corresponding to the observed particles (in the low energy), it is natural to take (irrespective of (6)) the following forms

$$\delta_L\tilde{\lambda}_\mu(x) = -\frac{i}{2}\epsilon_{ab}\sigma^{ab}\tilde{\lambda}_\mu(x), \qquad \delta_L\tilde{e^a}{}_\mu(x) = \epsilon^a{}_b\tilde{e}^b{}_\mu, \tag{19}$$

where $\epsilon_{ab} = (1/2)\epsilon_{[ab]}(x)$ is a local parameter. The relation between (6), i.e. the ultimate Lorentz invariance encoded geometrically in SGM space-time, and (19), i.e. the Lorentz invariance defined on the (composite) asymptotic field in Riemann space-time, is unclear. In SGM the local Lorentz transformations (5) and (6), i.e. the local Lorentz invariant gravitational interaction of superon, are introduced by the geometrical arguments in SGM space-time [6] following EGRT. While in SUGRA theory the local Lorentz transformation invariance (19) is realized as usual by introducing the Lorentz spin connection $\omega_\mu{}^{ab}$. And the LSUSY transformation is defined successfully by the (Lorentz) covariant derivative containing the spin connection $\tilde{\omega}_\mu{}^{ab}(e,\psi)$ as seen in (11), which causes the super-Poincaré algebra on the commutator of SUSY and is convenient for constructing the invariant action. Therefore, in the linearized (SUGRA-like) theory the local Lorentz transformation invariance is expected to be realized as usual by defining (19) and introducing the Lorentz spin connection $\omega_\mu{}^{ab}$. We investigate how the spin connection $\tilde{\omega}_\mu{}^{ab}(e,\psi)$ appears in the linearized (SUGRA-like) theory through the linearization process. This is also crucial for constructing a nontrivial (interacting) linearized action, which has manifest invariances.

We discuss the Lorentz covariance of the transformation by comparing (17) with the right-hand side of (11). The direct computation of (11) by using SUSY invariant relations (12) and (13) under (3) produces complicated redundant terms as read off from (17). The local Lorentz invariance of the linearized theory may become ambiguous and lose the manifest invariance.

For a simple restoration of the manifest local Lorentz invariance we survey the possibility that such redundant terms may be adjusted by the d.o.f. of the auxiliary fields in the linearized supermultiplet. As for the auxiliary fields it is necessary for the closure of the off-shell superalgebra to include the equal number of the fermionic and the bosonic d.o.f. in the linearized supermultiplet. As new NLSUSY is a global symmetry, $\tilde{\lambda}_\mu$ has 16 fermionic d.o.f.. Therefore at least 4 bosonic d.o.f. must be added to the off-shell SUGRA supermultiplet with 12 d.o.f. [12] and a vector field may be a simple candidate. However, counting the bosonic d.o.f. present in the redundant terms corresponding to $\tilde{\omega}_\mu{}^{ab}(e,\psi)$, we may need a bigger supermultiplet e.g. $16 + 4 \cdot 16 = 80$ d.o.f., to carry out the linearization, in which case a rank-3 tensor $\phi_{\mu\nu\rho}$ and a rank-2 tensor-spinor $\lambda_{\mu\nu}$ may be candidates for the auxiliary fields.

Now we consider the simple modification of SUGRA transformations (algebra) by adjusting the (composite) structure of the (auxiliary) fields. We take, instead of (10) and (11),

$$\delta\tilde{e}^a{}_\mu(x) = i\kappa\bar{\zeta}\gamma^a\tilde{\lambda}_\mu(x) + \bar{\zeta}\tilde{\Lambda}^a{}_\mu, \tag{20}$$

$$\delta\tilde{\lambda}_\mu(x) = \frac{2}{\kappa}D_\mu\zeta + \tilde{\Phi}_\mu\zeta = -\frac{i}{\kappa}\tilde{\omega}_\mu{}^{ab}\sigma_{ab}\zeta + \tilde{\Phi}_\mu\zeta, \tag{21}$$

where $\tilde{\Lambda}^a{}_\mu$ and $\tilde{\Phi}_\mu$ represent symbolically the auxiliary fields $80+80$ and are functionals of $e^a{}_\mu$ and ψ. We need $\tilde{\Lambda}^a{}_\mu$ term in (20) to alter (14), (16), (17) and (18) toward that of super-Poincaré algebra of SUGRA. We attempt the restoration of the manifest local Lorentz invariance order by order by adjusting $\tilde{\Lambda}^a{}_\mu$ and $\tilde{\Phi}_\mu$. In fact, the Lorentz spin connection $\omega_\mu{}^{ab}(e)$ (i.e. the leading order terms of $\tilde{\omega}_\mu{}^{ab}(e,\psi)$) of (21) is reproduced by taking the following one

$$\tilde{\Lambda}^a{}_\mu = \frac{\kappa^2}{4}[ie_b{}^\rho\partial_{[\rho}e^b{}_{\mu]}\gamma^a\psi - \partial_{[\rho}e_{|b|\sigma]}e^b{}_\mu\gamma^a\sigma^{\rho\sigma}\psi], \tag{22}$$

which holds (16). Accordingly $\tilde{\lambda}_\mu(e,\psi)$ is determined up to the first order in ψ as follows:

$$\tilde{\lambda}_\mu(e,\psi) = \frac{1}{2i\kappa}(i\kappa^2\gamma_a\gamma^\rho\psi(x)\partial_{[\rho}e^a{}_{\mu]} - \gamma_a\tilde{\Lambda}^a{}_\mu) = -\frac{i\kappa}{2}\omega_\mu{}^{ab}(e)\sigma_{ab}\psi, \tag{23}$$

which indicates the minimal Lorentz covariant gravitational interaction of superon. Substituting (23) into (21) we obtain the following new LSUSY transformation of $\tilde{\lambda}_\mu$ (after Fiertz transformations)

$$\delta\tilde{\lambda}_\mu(e,\psi) = -\frac{i\kappa}{2}\{\delta^{NL}\omega_\mu{}^{ab}(e)\sigma_{ab}\psi + \omega_\mu{}^{ab}(e)\sigma_{ab}\delta^{NL}\psi\}$$

$$= -\frac{i}{2\kappa}\omega_\mu{}^{ab}(e)\sigma_{ab}\zeta + \frac{i\kappa}{2}\{\tilde{\epsilon}^{ab}(e,\psi)\sigma_{ab}\cdot\omega_\mu{}^{cd}(e)\sigma_{cd}\psi + \cdots\}. \tag{24}$$

Remarkably, the local Lorentz transformations of $\tilde{\lambda}_\mu(e,\psi)$ (i.e. the second term) with the field dependent antisymmetric parameters $\tilde{\epsilon}^{ab}(e,\psi)$ are induced in addition to the intended ordinary global SUSY transformation. (23) are the SUSY invariant relations for $\tilde{\lambda}_\mu(e,\psi)$ in the lowest order with ψ, for the SUSY transformation of (23) gives the right hand side of (21) with the consequent auxiliary fields. Interestingly the commutator of the two LSUSY transformations on (23) induces GL(4,R):

$$[\delta_{\zeta_1},\delta_{\zeta_2}]\tilde{\lambda}_\mu(e,\psi) = \Xi^\rho\partial_\rho\tilde{\lambda}_\mu(e,\psi) + \partial_\mu\Xi^\rho\tilde{\lambda}_\rho(e,\psi), \tag{25}$$

where Ξ^ρ is the same field dependent parameter as given in (16). (16) and (25) show the closure of the algebra on SP algebra provided that the SUSY invariant relations (12) and (23) are adopted. These phenomena coincide with SGM scenario [1,4] from the algebraic point of view, i.e. they are the superon-graviton composite (eigenstates) corresponding to the linear representations of SP algebra. As for the redundant terms in (24) with $(\psi)^2$ the SUSY transformations we can recast them by considering the modified spin connection $\tilde{\omega}_\mu{}^{ab}(e,\psi)$ particularly with the contorsion terms and (the auxiliary field) $\tilde{\Phi}_\mu(e,\psi)$. In fact, we have proved that the contributions to (24) from the SUGRA-inspired contorsion terms:

$$\begin{aligned} K_{\mu ab}\sigma^{ab}\psi &= \frac{i\kappa^2}{4}(\bar{\tilde{\lambda}}_a\gamma_b\tilde{\lambda}_\mu - \bar{\tilde{\lambda}}_b\gamma_a\tilde{\lambda}_\mu + \bar{\tilde{\lambda}}_a\gamma_\mu\tilde{\lambda}_b)\sigma^{ab}\psi \\ &= \frac{i\kappa^4}{16}\{(e_a{}^\mu\bar{\psi}\sigma^{cd}\omega_{\nu cd}(e)\gamma_b\omega_{\mu fg}(e)\sigma^{fg}\psi - [a\leftrightarrow b] + \cdots\}\sigma^{ab}\psi, \end{aligned} \tag{26}$$

satisfies

$$[\delta_{\zeta_1},\delta_{\zeta_2}](K_{\mu ab}\sigma^{ab}\psi) = \Xi^\rho\partial_\rho(K_{\mu ab}\sigma^{ab}\psi) + \partial_\mu\Xi^\rho(K_{\rho ab}\sigma^{ab}\psi). \tag{27}$$

We can obtain the complete linearized (off-shell) supermultiplets of the super-Poincaré algebra by repeating the similar procedures (including the auxiliary fields $80+80$) order by order which terminates with $(\psi)^4$. The complicated procedures has been carried out successfully up to $O(\psi^2)$ [13].

Finally, we mention another way of the systematic linearization by using the superfield formalism applied to the coupled system of V-A action with SUGRA [14]. We can define on such a coupled system a local spinor gauge symmetry, which induces a super-Higgs mechanism [15] converting V-A field to the longitudinal component of massive spin 3/2 field. The consequent Lagrangian obtained may be an analogue of what we anticipated in above discussions but with the elementary spin 3/2 field.

The linearization of the new E-H type action (1) with the extra dimensions, which gives another unification framework describing the observed particles as elementary fields, is open. And the linearization of SGM action for spin 3/2 N-G fermion field [16] (with extra dimensions) may be in the same scope.

3 Discussion

Now we summarize the results as follows. Referring to SUGRA transformations we have attempted explicitly the linearization of $N=1$ SGM up to $O(\psi^2)$ in the (SUGRA-like) LSUSY transformations. The closure of the new LSUSY transformations (20) and (24) on the linearized supermultiplet, which are different from SUGRA transformations, can be proved order by order with ψ by introducing the auxiliary fields. It is interesting that the simple relation

$\lambda_\mu = e^a{}_\mu \gamma_a \psi + \cdots$, which is suggested by the flat space-time linearization, seems disfavour with the SGM linearization. As conjectured before, what LSUSY SP may be to SGM in quantum field theory, what O(4) symmetry is to the relativistic hydrogen model in quantum mechanics, which is tested by the linearization. The linearization of SGM is physically interesting in general, even if it were a existing SUGRA-like theory, for the consequent broken LSUSY theory is shown to be equivalent and gives a new insight into the fundamental structure of nature (like the relation between Landau–Ginzberg theory and BCS theory for superconductivity). The complete linearization to all orders, which can be anticipated by the systematics emerging in the present study, needs specifications of the auxiliary fields and remains to be studied. They are now in progress.

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Hamiltonian of Superstring and Super p-Branes with Enhanced Supersymmetry

D.V. UVAROV † *and A.A. ZHELTUKHIN* †‡

† *Kharkiv Institute of Physics and Technology, 61108 Kharkiv, Ukraine*
E-mail: *uvarov@kipt.kharkov.ua*

‡ *Institute of Theoretical Physics, University of Stockholm, SCFAB, SE-10691 Stockholm, Sweden*
E-mail: *aaz@physto.se*

The Hamiltonian and covariant generators of κ-symmetry and extra bosonic gauge symmetries of superstring and super p-branes preserving 3/4 of the $D = 4$ $N = 1$ supersymmetry are presented.

1 Introduction

As shown in [1] exotic BPS states preserving $\frac{\mathsf{M}-1}{\mathsf{M}}$ fraction of $N = 1$ supersymmetry can be realized by static configurations of free tensionless super p-branes ($p = 1, 2, \ldots$) with the action linear in derivatives[1]. These static configurations were described by general solutions of the equations of motion of super p-branes evolving in superspace extended by tensor central charge (TCC) coordinates. Because of the $OSp(1|2\mathsf{M})$ global symmetry of the model, its static p-brane solution was formulated in terms of supertwistor previously used to formulate superparticle models [3–5] and forming a subspace of the $Sp(2\mathsf{M})$ invariant symplectic space [6, 7]. As a result, the static form of the discussed supertwistor representation of the BPS brane solution is not static in terms of the original superspace-time and TCC coordinates. It is static only modulo transformations of enhanced κ-symmetry and its accompanying local symmetries, since the supertwistor components are invariant under these gauge symmetries, as shown in [8]. In the bosonic sector the unphysical p-brane motions related to the gauge symmetries were geometrically realized as the Abelian shifts [8] of the space-time and TCC coordinates by the Lorentz bivectors (generally multivectors) generalizing vector light-like Penrose shifts of the standard space-time coordinates [9]. Being inessential on the classical level of consideration, these shifts may turn out to be essential in the quantum dynamics of strings and branes. This necessitates quantum treatment of the model [1] in the original variables that belong to the superspace extended by TCC coordinates and auxiliary spinor fields. Interest to this problem is stimulated by a conjectured relation of the tensionless strings with higher spin theories and free conformal SYM theories [10, 11, 7], as well as by the presence of higher spin states in the quantized $OSp(1|2\mathsf{M})$ invariant model of superparticle [12].

In this talk we start study of the formulated problem on the example of $D = 4$ $N = 1$ super p-brane model [1] and consider its Hamiltonian description, based on the covariant approach to the first- and second-class constraints division [13, 14], correlating with the description [15] in the fixed light-cone gauge. The covariant first-class constraints which are generators of the local symmetries of the model, as well as, its Hamiltonian are presented. Further development of the results towards the exploration of quantum dynamics and symmetries of the string/brane model is discussed.

[1] New Wess–Zumino like super p-brane models nonlinear in derivatives and preserving $\frac{\mathsf{M}-1}{\mathsf{M}}$ fraction of supersymmetry were recently proposed in [2].

2 Lagrangians for strings and branes with enhanced supersymmetry and symplectic twistor

A new simple model [1] describes tensionless strings and p-branes spreading in the symplectic superspace $\mathcal{M}^{\rm susy}_{\mathsf{M}}$. For $\mathsf{M} = 2^{[\frac{D}{2}]}$ ($D = 2, 3, 4$ mod 8) this superspace naturally associates with D-dimensional Minkowski space-time extended by the Majorana spinor θ_a ($a = 1, 2, \ldots, 2^{[\frac{D}{2}]}$) and the tensor central charge coordinates z_{ab} additively unified with the standard space-time coordinates $x_{ab} = x^m(\gamma_m C^{-1})_{ab}$ in the symmetric spin-tensor Y_{ab}. The supersymmetric and reparametrization invariant action of the model [1]

$$S_p = \frac{1}{2}\int d\tau d^p\sigma\ \rho^\mu U^a W_{\mu ab} U^b \tag{1}$$

includes the world-volume pullback

$$W_{\mu ab} = \partial_\mu Y_{ab} - 2i(\partial_\mu\theta_a\theta_b + \partial_\mu\theta_b\theta_a),$$

of the supersymmetric Cartan differential one-form $W_{ab} = W_{\mu ab}d\xi^\mu$, where $\partial_\mu \equiv \frac{\partial}{\partial\xi^\mu}$ and $\xi^\mu = (\tau, \sigma^M) \equiv (\tau, \vec{\sigma})$, ($M = 1, 2, \ldots, p$) are world-volume coordinates. The local auxiliary Majorana spinor $U^a(\tau, \sigma^M)$ parametrizes the generalized momentum $P^{ab} = \rho^\tau U^a U^b$ of the tensionless p-brane and $\rho^\mu(\tau, \sigma^M)$ is the world-volume vector density providing the reparametrization invariance of S_p. This action has $(\mathsf{M}-1)$ κ-symmetries and consequently preserves $\frac{\mathsf{M}-1}{\mathsf{M}}$ fraction of the original global supersymmetry.

By the generalized Penrose transformation of variables

$$Y_{ab}U^b = i\tilde{Y}_a + \tilde{\eta}\theta_a, \qquad \tilde{\eta} = -2i(U^a\theta_a), \tag{2}$$

where $\tilde{\eta}$ is real Goldstone fermion associated with the spontaneous breakdown of $\frac{1}{\mathsf{M}}$ supersymmetry, the differential one-form $U^aW_{ab}U^b$ is presented as

$$U^aW_{ab}U^b = i\{U^a d\tilde{Y}_a - dU^a\tilde{Y}_a + d\tilde{\eta}\tilde{\eta}\} \equiv dY^\Lambda G_{\Lambda\Sigma}Y^\Sigma. \tag{3}$$

The new object $Y^\Lambda = (iU^a, \tilde{Y}_a, \tilde{\eta})$ in (2), (3) is $OSp(1|2\mathsf{M})$ supertwistor and $G_{\Lambda\Sigma} = (-)^{\Lambda\Sigma+1}G_{\Sigma\Lambda}$ is $OSp(1|2\mathsf{M})$ invariant supersymplectic metric

$$G_{\Lambda\Sigma} = \begin{pmatrix} \omega^{(2\mathsf{M})} & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 0 & -\delta_a{}^b & 0 \\ \delta^a{}_b & 0 & 0 \\ 0 & 0 & i \end{pmatrix},$$

which is the supersymmetric generalization of $Sp(2\mathsf{M})$ symplectic metric $\omega^{(2\mathsf{M})}$. In view of (2) and (3), the action S_p (1) is presented in the supertwistor form

$$S_p = \frac{1}{2}\int d\tau d^p\sigma\ \rho^\mu\partial_\mu Y^\Lambda G_{\Lambda\Sigma}Y^\Sigma \tag{4}$$

that is apparently invariant under global $OSp(1|2\mathsf{M})$ symmetry. For the particular case of $D = 11$ the action (4) is invariant under $OSp(1|64)$ generalized superconformal symmetry [16].

The original action (1) is invariant under $(\mathsf{M}-1)$ κ-symmetries since the transformation parameter $\kappa_a(\tau, \vec{\sigma})$ is restricted by only one real condition

$$U^a\kappa_a = 0, \tag{5}$$

as it follows from the transformation rules of the primary variables

$$\delta_\kappa\theta_a = \kappa_a, \qquad \delta_\kappa Y_{ab} = -2i(\theta_a\kappa_b + \theta_b\kappa_a), \qquad \delta_\kappa U^a = 0. \tag{6}$$

It is easy to show [8] that all components of the supertwistor $Y^\Lambda = (iU^a, \tilde{Y}_a, \tilde{\eta})$ are invariant under κ-symmetry transformations (5), (6)

$$\delta_\kappa \tilde{Y}_a = 0, \qquad \delta_\kappa \tilde{\eta} = 0, \qquad \delta_\kappa U^a = 0,$$

so that new representation of S_p (4) includes only κ-invariant variables. Note that in 4-dimensional space-time Y^Λ contains only 9 real variables that is twice less than the number of the original variables Y_{ab}, θ_a, U^a.

3 Example of $OSp(1|8)$ string/brane. Primary constraints

$OSp(1|8)$ is the global supersymmetry of the massless fields of all spins in $D = 4$ space-time extended by TCC coordinates [6, 7]. Therefore, we study $D = 4$ example of the string/brane model (1) formulated in generalized $(4 + 6)$-dimensional space $\mathcal{M}_{4+6}$ extended by the Grassmannian Majorana bispinor θ_a. In this case the $D = 4$ $N = 1$ superalgebra

$$\{Q_a, Q_b\} = \left(\gamma^m C^{-1}\right)_{ab} P_m + i\left(\gamma^{mn} C^{-1}\right)_{ab} Z_{mn}$$

includes the TCC 2−form generator Z_{mn}, and the matrix coordinates Y_{ab} are

$$Y_{ab} = x_{ab} + z_{ab},$$

where

$$x_{ab} = x_m \left(\gamma^m C^{-1}\right)_{ab}, \qquad z_{ab} = z_{mn} \left(\gamma^{mn} C^{-1}\right)_{ab}$$

with the charge conjugation matrix C chosen to be imaginary in the Majorana representation. Here we use the same agreements about the spinor algebra as in [1]. In the Weyl basis real symmetric 4×4 matrix Y_{ab} is presented as

$$Y_a{}^b = Y_{ad} C^{db} = \begin{pmatrix} z_\alpha{}^\beta & x_{\alpha\dot{\beta}} \\ \tilde{x}^{\dot{\alpha}\beta} & \bar{z}^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}, \qquad C^{ab} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$

In the $D = 4$ case the auxiliary Majorana spinor $U^a(\tau, \sigma^M)$ complemented by another auxiliary Majorana spinor $V^a(\tau, \sigma^M)$ forms a spinor basis together with the spinors $(\gamma_5 U)_a$ and $(\gamma_5 V)_a$

$$U_a = \begin{pmatrix} u_\alpha \\ \bar{u}^{\dot{\alpha}} \end{pmatrix}, \qquad V_a = \begin{pmatrix} v_\alpha \\ \bar{v}^{\dot{\alpha}} \end{pmatrix}, \qquad (U\gamma_5 V) = -2i, \qquad (UV) = 0,$$

where the γ_5-matrix is

$$\gamma_5 = \begin{pmatrix} -i\delta^\beta_\alpha & 0 \\ 0 & i\delta^{\dot{\alpha}}_{\dot{\beta}} \end{pmatrix}.$$

Respectively the linear independent Weyl spinors u^α and v^α attached to the string/brane world volume may be identified with the local Newman–Penrose dyad [9]

$$u^\alpha v_\alpha \equiv u^\alpha \varepsilon_{\alpha\beta} v^\beta = 1, \qquad u^\alpha u_\alpha = v^\alpha v_\alpha = 0.$$

In the Weyl basis the action (1) acquires the form

$$S_p = \frac{1}{2} \int d\tau d^p\sigma \ \rho^\mu \left(2u^\alpha \omega_{\mu\alpha\dot{\alpha}} \bar{u}^{\dot{\alpha}} + u^\alpha \omega_{\mu\alpha\beta} u^\beta + \bar{u}^{\dot{\alpha}} \bar{\omega}_{\mu\dot{\alpha}\dot{\beta}} \bar{u}^{\dot{\beta}}\right), \tag{7}$$

where the supersymmetric one-forms $\omega_{\mu\alpha\dot\alpha}$ and $\omega_{\mu\alpha\beta}$ are

$$\omega_{\mu\alpha\dot\alpha} = \partial_\mu x_{\alpha\dot\alpha} + 2i(\partial_\mu\theta_\alpha\bar\theta_{\dot\alpha} + \partial_\mu\bar\theta_{\dot\alpha}\theta_\alpha),$$
$$\omega_{\mu\alpha\beta} = -\partial_\mu z_{\alpha\beta} - 2i(\partial_\mu\theta_\alpha\theta_\beta + \partial_\mu\theta_\beta\theta_\alpha),$$
$$\bar\omega_{\mu\dot\alpha\dot\beta} = -\partial_\mu\bar z_{\dot\alpha\dot\beta} - 2i(\partial_\mu\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} + \partial_\mu\bar\theta_{\dot\beta}\bar\theta_{\dot\alpha}).$$

The momenta densities $\mathcal{P}^{\mathfrak{M}}(\tau,\sigma^M)$

$$\mathcal{P}^{\mathfrak{M}} = \frac{\partial L}{\partial\dot{\mathcal{Q}}_{\mathfrak{M}}} = \left(P^{\dot\alpha\alpha}, \pi^{\alpha\beta}, \bar\pi^{\dot\alpha\dot\beta}, \pi^\alpha, \bar\pi^{\dot\alpha}, P_u^\alpha, \bar P_u^{\dot\alpha}, P_v^\alpha, \bar P_v^{\dot\alpha}, P_\mu^{(\rho)}\right)$$

are canonically conjugate to the coordinates

$$\mathcal{Q}_{\mathfrak{M}} = (x_{\alpha\dot\alpha}, z_{\alpha\beta}, \bar z_{\dot\alpha\dot\beta}, u_\alpha, \bar u_{\dot\alpha}, v_\alpha, \bar v_{\dot\alpha}, \rho^\mu)$$

with respect to the Poisson brackets

$$\{\mathcal{P}^{\mathfrak{M}}(\vec\sigma), \mathcal{Q}_{\mathfrak{N}}(\vec\sigma')\}_{\text{P.B.}} = \delta^{\mathfrak{M}}_{\mathfrak{N}}\delta^p(\vec\sigma - \vec\sigma')$$

with the periodic δ-function $\delta^p(\vec\sigma - \vec\sigma')$, where $\vec\sigma = (\sigma^1, \dots, \sigma^p)$, for the case of closed string/brane studied here.

As far as S_p (7) is linear in the proper time derivatives, it is characterized by the presence of the primary constraints. These constraints may be divided into four sectors.

The bosonic Φ-sector includes the constraints $\Phi \equiv \left(\Phi^{\dot\alpha\alpha}, \Phi^{\alpha\beta}, \bar\Phi^{\dot\alpha\dot\beta}\right)$ with

$$\begin{aligned}
&\Phi^{\dot\alpha\alpha} = P^{\dot\alpha\alpha} - \rho^\tau u^\alpha\bar u^{\dot\alpha} \approx 0,\\
&\Phi^{\alpha\beta} = \pi^{\alpha\beta} + \frac{1}{2}\rho^\tau u^\alpha u^\beta \approx 0,\\
&\bar\Phi^{\dot\alpha\dot\beta} = \bar\pi^{\dot\alpha\dot\beta} + \frac{1}{2}\rho^\tau\bar u^{\dot\alpha}\bar u^{\dot\beta} \approx 0.
\end{aligned} \tag{8}$$

The constraints from the Grassmannian Ψ-sector, where $\Psi = (\Psi^\alpha, \bar\Psi^{\dot\alpha})$, are given by

$$\begin{aligned}
&\Psi^\alpha = \pi^\alpha - 2i\bar\theta_{\dot\alpha}P^{\dot\alpha\alpha} - 4i\pi^{\alpha\beta}\theta_\beta \approx 0,\\
&\bar\Psi^{\dot\alpha} = -(\Psi^\alpha)^* = \bar\pi^{\dot\alpha} - 2iP^{\dot\alpha\alpha}\theta_\alpha - 4i\bar\pi^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta} \approx 0.
\end{aligned} \tag{9}$$

The dyad or (u,v)-sector is formed by the constraints

$$\begin{aligned}
&P_u^\alpha \approx 0, \qquad \bar P_u^{\dot\alpha} \approx 0, \qquad P_v^\alpha \approx 0, \qquad \bar P_v^{\dot\alpha} \approx 0,\\
&\Xi \equiv u^\alpha v_\alpha - 1 \approx 0, \qquad \bar\Xi \equiv \bar u^{\dot\alpha}\bar v_{\dot\alpha} - 1 \approx 0.
\end{aligned} \tag{10}$$

Finally, the $\rho-$sector includes the constraints

$$P_\mu^{(\rho)} \approx 0, \qquad \mu = (\tau, M), \qquad M = (1, \dots, p). \tag{11}$$

To find all local symmetries of the brane action, it is necessary to split the constraints (8)–(11) into the first- and the second-class sets. Then the first-class constraints will generate the local symmetries on the Poisson brackets in accordance with the Dirac prescription.

4 The first-class constraints

To begin with consider the first-class constraints from the Ψ-sector (9) that generate κ-symmetry transformations (6). To find their explicit form it is necessary to solve equation (5) restricting the transformation parameter $\kappa_a(\tau,\vec{\sigma})$. Taking into account its decomposition over the dyad basis

$$\kappa_\alpha = \kappa u_\alpha + \tilde{\kappa} v_\alpha, \qquad \bar{\kappa}_{\dot{\alpha}} = \bar{\kappa}\bar{u}_{\dot{\alpha}} + \bar{\tilde{\kappa}}\bar{v}_{\dot{\alpha}}$$

it follows from equation (5) that $\tilde{\kappa} = \bar{\tilde{\kappa}}$ thus $\tilde{\kappa} = \tilde{\kappa}_R$ is real. Then the 3-parametric κ-symmetry transformation laws acquire the form

$$\begin{aligned}
&\delta_\kappa\theta_\alpha = \kappa u_\alpha, \qquad \delta_\kappa\bar{\theta}_{\dot{\alpha}} = \bar{\kappa}\bar{u}_{\dot{\alpha}},\\
&\delta_\kappa x_{\alpha\dot{\alpha}} = -2i(\kappa u_\alpha\bar{\theta}_{\dot{\alpha}} + \bar{\kappa}\bar{u}_{\dot{\alpha}}\theta_\alpha), \qquad \delta_\kappa z_{\alpha\beta} = -2i\kappa(u_\alpha\theta_\beta + u_\beta\theta_\alpha);\\
&\delta_{\tilde{\kappa}_R}\theta_\alpha = \tilde{\kappa}_R v_\alpha, \qquad \delta_{\tilde{\kappa}_R}\bar{\theta}_{\dot{\alpha}} = \tilde{\kappa}_R\bar{v}_{\dot{\alpha}},\\
&\delta_{\tilde{\kappa}_R} x_{\alpha\dot{\alpha}} = -2i\tilde{\kappa}_R(v_\alpha\bar{\theta}_{\dot{\alpha}} + \bar{v}_{\dot{\alpha}}\theta_\alpha), \qquad \delta_{\tilde{\kappa}_R} z_{\alpha\beta} = -2i\tilde{\kappa}_R(v_\alpha\theta_\beta + v_\beta\theta_\alpha).
\end{aligned} \tag{12}$$

Transformations (12) are generated on the Poisson brackets by three constraints

$$\Psi^{(u)} \equiv \Psi^\alpha u_\alpha \approx 0, \qquad \bar{\Psi}^{(u)} \equiv \bar{\Psi}^{\dot{\alpha}}\bar{u}_{\dot{\alpha}} \approx 0, \qquad \Psi_R^{(v)} \equiv \Psi^\alpha v_\alpha + \bar{\Psi}^{\dot{\alpha}}\bar{v}_{\dot{\alpha}} \approx 0$$

that belong to the first-class. The Poisson brackets of these κ-symmetry generators

$$\begin{aligned}
&\{\Psi^{(u)}(\vec{\sigma}), \bar{\Psi}^{(u)}(\vec{\sigma}')\}_{\text{P.B.}} = -4i\Phi^{(u)}\delta^p(\vec{\sigma}-\vec{\sigma}') \approx 0,\\
&\{\Psi^{(u)}(\vec{\sigma}), \Psi^{(u)}(\vec{\sigma}')\}_{\text{P.B.}} = -8iT^{(u)}\delta^p(\vec{\sigma}-\vec{\sigma}') \approx 0,\\
&\{\bar{\Psi}^{(u)}(\vec{\sigma}), \bar{\Psi}^{(u)}(\vec{\sigma}')\}_{\text{P.B.}} = -8i\bar{T}^{(u)}\delta^p(\vec{\sigma}-\vec{\sigma}') \approx 0,\\
&\{\Psi_R^{(v)}(\vec{\sigma}), \Psi_R^{(v)}(\vec{\sigma}')\}_{\text{P.B.}} = -8i\tilde{T}_R^{(v)}\delta^p(\vec{\sigma}-\vec{\sigma}') \approx 0,\\
&\{\Psi_R^{(v)}(\vec{\sigma}), \Psi^{(u)}(\vec{\sigma}')\}_{\text{P.B.}} = -2i(\tilde{T}^{(+)} - i\tilde{T}^{(-)})\delta^p(\vec{\sigma}-\vec{\sigma}') \approx 0,\\
&\{\Psi_R^{(v)}(\vec{\sigma}), \bar{\Psi}^{(u)}(\vec{\sigma}')\}_{\text{P.B.}} = -2i(\tilde{T}^{(+)} + i\tilde{T}^{(-)})\delta^p(\vec{\sigma}-\vec{\sigma}') \approx 0
\end{aligned}$$

are closed by 6 bosonic first-class constraints from the Φ-sector that were chosen as follows

$$T^{(u)} \equiv u_\alpha\Phi^{\alpha\beta}u_\beta \approx 0, \qquad \bar{T}^{(u)} = (T^{(u)})^*, \tag{13}$$

$$\Phi^{(u)} \equiv u_\alpha\Phi^{\alpha\dot{\beta}}\bar{u}_{\dot{\beta}}, \tag{14}$$

$$\tilde{T}_R^{(v)} \equiv v_\alpha\Phi^{\alpha\beta}v_\beta + \bar{v}_{\dot{\alpha}}\bar{\Phi}^{\dot{\alpha}\dot{\beta}}\bar{v}_{\dot{\beta}} + \bar{v}_{\dot{\alpha}}\Phi^{\dot{\alpha}\alpha}v_\alpha \approx 0, \tag{15}$$

$$\tilde{T}^{(+)} \equiv \Phi^{\dot{\alpha}\alpha}(u_\alpha\bar{v}_{\dot{\alpha}} + v_\alpha\bar{u}_{\dot{\alpha}}) + 2\left(\Phi^{\alpha\beta}u_{\{\alpha}v_{\beta\}} + \bar{\Phi}^{\dot{\alpha}\dot{\beta}}\bar{u}_{\{\dot{\alpha}}\bar{v}_{\dot{\beta}\}}\right) \approx 0, \tag{16}$$

$$\tilde{T}^{(-)} \equiv i\left[\Phi^{\dot{\alpha}\alpha}(u_\alpha\bar{v}_{\dot{\alpha}} - v_\alpha\bar{u}_{\dot{\alpha}}) + 2\left(\Phi^{\alpha\beta}u_{\{\alpha}v_{\beta\}} - \bar{\Phi}^{\dot{\alpha}\dot{\beta}}\bar{u}_{\{\dot{\alpha}}\bar{v}_{\dot{\beta}\}}\right)\right] \approx 0. \tag{17}$$

The meaning of the constraints (13)–(17) is that they generate local shifts of space-time and TCC coordinates along the directions defined by the moving tetrade attached to super p-brane world volume [8]. Indeed, constraints (13) correspond to local shifts of TCC coordinates along the conjugate momentum

$$\delta_{T^{(u)}} z_{\alpha\beta} = \epsilon_{T^{(u)}} u_\alpha u_\beta \approx -2\frac{\epsilon_{T^{(u)}}}{\rho^\tau}\pi_{\alpha\beta}, \qquad \delta_{\bar{T}^{(u)}}\bar{z}_{\dot{\alpha}\dot{\beta}} = \bar{\epsilon}_{\bar{T}^{(u)}}\bar{u}_{\dot{\alpha}}\bar{u}_{\dot{\beta}} \approx -2\frac{\bar{\epsilon}_{\bar{T}^{(u)}}}{\rho^\tau}\bar{\pi}_{\dot{\alpha}\dot{\beta}},$$

whereas the constraint (14) generates analogous shift for ordinary space-time coordinates

$$\delta_{\Phi^{(u)}} x_{\alpha\dot{\alpha}} = \epsilon_{\Phi^{(u)}} u_\alpha\bar{u}_{\dot{\alpha}} \approx \frac{\epsilon_{\Phi^{(u)}}}{\rho^\tau}P_{\alpha\dot{\alpha}}.$$

The local symmetry generated by the constraint (15) is the shift of space-time and TCC coordinates along another light-like direction parametrized by the dyad component v_α

$$\delta_{\tilde{T}_R^{(v)}} x_{\alpha\dot{\alpha}} = \epsilon_{\tilde{T}_R^{(v)}} v_\alpha \bar{v}_{\dot{\alpha}}, \qquad \delta_{\tilde{T}_R^{(v)}} z_{\alpha\beta} = \epsilon_{\tilde{T}_R^{(v)}} v_\alpha v_\beta, \qquad \delta_{\tilde{T}_R^{(v)}} \bar{z}_{\dot{\alpha}\dot{\beta}} = \epsilon_{\tilde{T}_R^{(v)}} \bar{v}_{\dot{\alpha}} \bar{v}_{\dot{\beta}}.$$

The action (7) is invariant under this transformation due to mutual cancellation of contributions to the variation coming from space-time and TCC coordinates. Finally constraints (16), (17) correspond to local shifts of space-time coordinates along the directions transverse to the world volume

$$\delta_{\tilde{T}^{(+)}} x_{\alpha\dot{\alpha}} = \epsilon_{\tilde{T}^{(+)}} m^{(+)}_{\alpha\dot{\alpha}}, \qquad \delta_{\tilde{T}^{(-)}} x_{\alpha\dot{\alpha}} = \epsilon_{\tilde{T}^{(-)}} m^{(-)}_{\alpha\dot{\alpha}}$$

compensated by appropriate shifts of TCC coordinates

$$\delta_{\tilde{T}^{(+)}} z_{\alpha\beta} = 2\epsilon_{\tilde{T}^{(+)}} u_{\{\alpha} v_{\beta\}}, \qquad \delta_{\tilde{T}^{(+)}} \bar{z}_{\dot{\alpha}\dot{\beta}} = 2\epsilon_{\tilde{T}^{(+)}} \bar{u}_{\{\dot{\alpha}} \bar{v}_{\dot{\beta}\}};$$
$$\delta_{\tilde{T}^{(-)}} z_{\alpha\beta} = 2i\epsilon_{\tilde{T}^{(-)}} u_{\{\alpha} v_{\beta\}}, \qquad \delta_{\tilde{T}^{(-)}} \bar{z}_{\dot{\alpha}\dot{\beta}} = -2i\epsilon_{\tilde{T}^{(-)}} \bar{u}_{\{\dot{\alpha}} \bar{v}_{\dot{\beta}\}},$$

where $u_{\{\alpha} v_{\beta\}} = \frac{1}{2}(u_\alpha v_\beta + u_\beta v_\alpha)$.

In the dyad sector there present are two first-class constraints

$$P^{(u)}_v \equiv P^\alpha_v u_\alpha \approx 0, \qquad \bar{P}^{(u)}_v \equiv \bar{P}^{\dot{\alpha}}_v \bar{u}_{\dot{\alpha}} \approx 0$$

generating shifts of v_α ($\bar{v}_{\dot{\alpha}}$) dyad components along u_α ($\bar{u}_{\dot{\alpha}}$). These shifts constitute apparent local symmetry of the action (7) as S_p does not contain v_α ($\bar{v}_{\dot{\alpha}}$).

The first-class constraints of the ρ-sector are

$$P^{(\rho)}_M \approx 0, \qquad M = (1, \ldots, p).$$

They commute with all other constraints since canonically conjugate variables ρ^M do not enter the primary constraints (8)–(11). $P^{(\rho)}_M$ constraints correspond to redefinition of p space components ρ^M of the $(p+1)$-dimensional world-volume vector density $\rho^\mu(\tau, \vec{\sigma})$

$$\delta_\epsilon \rho^M = \epsilon^M(\tau, \vec{\sigma}). \tag{18}$$

The transformations (18) are new local symmetries of the action S_p due to arbitrariness in the definition of ρ^M.

The first-class constraint that includes primary ones from the different sectors is provided by the Weyl symmetry generator

$$\Delta_W \equiv (P^\alpha_u u_\alpha + \bar{P}^{\dot{\alpha}}_u \bar{u}_{\dot{\alpha}}) - (P^\alpha_v v_\alpha + \bar{P}^{\dot{\alpha}}_v \bar{v}_{\dot{\alpha}}) - 2\rho^\mu P^{(\rho)}_\mu \approx 0.$$

The local transformation generated by Δ_W is the dilation that acts only on auxiliary variables u_α, v_α and ρ^μ

$$\rho'^\mu = e^{-2\Lambda}\rho^\mu, \qquad u'_\alpha = e^\Lambda u_\alpha, \qquad v'_\alpha = e^{-\Lambda} v_\alpha,$$
$$x'_{\alpha\dot{\alpha}} = x_{\alpha\dot{\alpha}}, \; z'_{\alpha\beta} = z_{\alpha\beta}, \qquad \theta'_\alpha = \theta_\alpha. \tag{19}$$

The transformation (19) is identified with the Weyl symmetry of the p-brane action. From the string point of view, the Weyl invariants $\rho^\mu u_\alpha \bar{u}_{\dot{\alpha}}$ and $\rho^\mu u_\alpha u_\beta$ constructed from auxiliary world-volume fields are similar to conventional Weyl invariant of tensile string

$$\sqrt{-g} g^{\mu\nu} \Leftrightarrow \rho^\mu u_\alpha \bar{u}_{\dot{\alpha}},$$

but here (19) is the symmetry the tensionless super p-brane action.

Finally the set of the first-class constraints comprises p space-like reparametrization generators

$$L_M = P^{\dot\alpha\alpha}\omega_{M\alpha\dot\alpha} - \pi^{\alpha\beta}\omega_{M\alpha\beta} - \bar\pi^{\dot\alpha\dot\beta}\bar\omega_{M\dot\alpha\dot\beta} + \partial_M\theta_\alpha\Psi^\alpha + \partial_M\bar\theta_{\dot\alpha}\bar\Psi^{\dot\alpha} + (P_u^\alpha\partial_M u_\alpha + P_v^\alpha\partial_M v_\alpha) + (\bar P_u^{\dot\alpha}\partial_M\bar u_{\dot\alpha} + \bar P_v^{\dot\alpha}\partial_M\bar v_{\dot\alpha}) - \partial_M P_\nu^{(\rho)}\rho^\nu \approx 0$$

that are the secondary constraints. The remaining time-like τ-reparametrization constraint is not independent and is constructed from the constraints $T^{(u)}$ (13), $\Phi^{(u)}$ (14) for $z_{\alpha\beta}$, $x_{\alpha\dot\alpha}$ and some algebraic combinations of the other first-class constraints for the remaining generalized coordinates.

5 The Hamiltonian

Having completed the construction of the first-class constraints we present the total gauge independent Hamiltonian density of the super p-brane in the form of their linear combination

$$H_T(\tau,\vec\sigma) = \kappa_u\Psi^{(u)} + \bar\kappa_u\bar\Psi^{(u)} + \kappa_R\Psi_R^{(v)} + a_u\Phi^{(u)} + b_uT^{(u)} + \bar b_u\bar T^{(u)} + c^{(+)}\tilde T^{(+)} + c^{(-)}\tilde T^{(-)} + c_R^{(v)}\tilde T_R^{(v)} + eP_v^{(u)} + \bar e\bar P_v^{(u)} + \omega\Delta_W + f^MP_M^{(\rho)} + \tilde\rho^ML_M \approx 0,$$

where the functions κ, a, b, c, e, f, ω and $\tilde\rho$ form the set of $(9+2p)_B + 3_F$ real Lagrange multipliers. The above Hamiltonian density yields the covariant equations of motion.

6 Conclusion

The Hamiltonian of the simplest $D = 4$ $N = 1$ super p-brane model of which general solution describes the BPS state with exotic 3/4 fraction of supersymmetry was constructed. To separate the first-class constraints in the manifestly Lorentz covariant way there has been introduced additional Weyl spinor field $v_\alpha(\tau,\vec\sigma)$ forming the local Newman–Penrose dyad [9] together with $u_\alpha(\tau,\vec\sigma)$. It was shown that the model possesses 3 fermionic first-class constraints that are generators of enhanced κ-symmetry transformations and $2p+9$ bosonic first-class constraints corresponding to space-like reparametrizations (p constraints), redefinitions of p space components of the auxiliary world-volume vector density $\rho^\mu = (\rho^\tau, \rho^M)$, shifts of the dyad components v_α along u_α (2 constraints), local Weyl symmetry (1 constraint) and 6 Abelian shifts of the space-time and TCC coordinates along the directions specified by the bilinear combinations of the dyad components. The first-class constraints fix the number of physical degrees of freedom of the p-brane model to be $n_{\rm phys} = 2(4-p)_B + 1_F$ $(0 \le p \le 4)$ and encode them by the $OSp(1|8)$ supertwistor, which is invariant under $8_B + 3_F$ gauge symmetries, as was proved in [8]. So, in the supertwistor representation $8_B + 3_F$ redundant superspace-time degrees of freedom are covariantly excluded just by the variables transformation without any gauge fixing.

The results obtained here constitute a reliable basis for the covariant separation of the second-class constraints, construction of the Dirac brackets and performing the canonical quantization of super p-brane model [1]. These results are presented in the recent papers [19].

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Symmetries and Supersymmetries of the Dirac-Type Operators on Curved Spaces

Mihai VISINESCU

Department of Theoretical Physics, Institute for Physics and Nuclear Engineering, Magurele, P.O.Box MG-6, Bucharest, Romania
E-mail: *mvisin@theory.nipne.ro*

The role of the Killing–Yano tensors in the construction of the Dirac-type operators is pointed out. The general results are applied to the case of the four-dimensional Euclidean Taub–Newman–Unti–Tamburino space. Three new Dirac-type operators, equivalent to the standard Dirac operator, are constructed from the covariantly constant Killing–Yano tensors of this space. Finally the Runge–Lenz operator for the Dirac equation in this background is expressed in terms of the fourth Killing–Yano tensor that is not covariantly constant.

1 Introduction

The (skew-symmetric) Killing–Yano (K-Y) tensors that were first introduced by Yano [1] from purely mathematical reasons, are profoundly connected to the supersymmetric classical and quantum mechanics on curved manifolds where such tensors do exist [2]. The K-Y tensors play an important role in theories with spin and especially in the Dirac theory on curved space-times where they produce first order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one D [3]. Another virtue of the K-Y tensors is that they enter as square roots in the structure of several second rank Stäckel–Killing tensors that generate conserved quantities in classical mechanics or conserved operators which commute with D. The construction of Carter and McLenaghan depended upon the remarkable fact that the (symmetric) Stäckel–Killing tensor $K_{\mu\nu}$ involved in the constant of motion quadratic in the four-momentum p_μ

$$Z = \frac{1}{2} K^{\mu\nu} p_\mu p_\nu \tag{1}$$

has a certain square root in terms of K-Y tensors $f_{\mu\nu}$:

$$K_{\mu\nu} = f_{\mu\lambda} f^{\lambda}_{\cdot\nu}. \tag{2}$$

The K-Y tensor here is a 2-form $f_{\mu\nu} = -f_{\nu\mu}$ which satisfies the equation

$$f_{\mu\nu;\lambda} + f_{\mu\lambda;\nu} = 0.$$

These attributes of the K-Y tensors lead to an efficient mechanism of supersymmetry, especially when the Stäckel–Killing tensor $K_{\mu\nu}$ in equation (1) is proportional to the metric tensor $g_{\mu\nu}$ and the corresponding K-Y tensors in equation (2) are covariantly constant. Then each tensor of this type f^i gives rise to a Dirac-type operator D^i representing a supercharge of the superalgebra $\{D^i, D^j\} \propto D^2 \delta_{ij}$.

The general results are applied to the case of the four-dimensional Euclidean Taub–Newman–Unti–Tamburino (Taub–NUT) space. The Euclidean Taub-NUT metric is involved in many modern studies in physics. This metric might give rise to the gravitational analog of the Yang–Mills instanton [4]. The Kaluza–Klein monopole of Gross and Perry [5] and of Sorkin [6] was

obtained by embedding the Taub–NUT gravitational instanton into five-dimensional Kaluza–Klein theory. On the other hand, in the long-distant limit, the relative motion of two monopoles is approximately described by the geodesics of this space [7].

The Euclidean Taub–NUT space which is a hyper-Kähler manifold possessing three covariantly constant K-Y tensors. By means of these covariantly constant Killing–Yano tensors it is possible to construct new Dirac-type operators [3] which anticommute with the standard Dirac operator.

We show that the representation of the whole theory can be changed using the $U(2)$ transformations among them the $SU(2)$ ones are generated just by the spin-like operators constructed using the above-mentioned three Killing–Yano tensors [8, 9]. On the basis of these results, we define the parity transformation and a discrete group with eight elements formed by the transformations that relate to each other the four Dirac operators and their parity is transformed as well. We show that this discrete group is a realization of the quaternion group which is isomorphic to the dicyclic group of order eight.

The Taub–NUT space also possesses a Killing–Yano tensor which is not covariantly constant. The corresponding non-standard operator, constructed with the general rule [3] anticommutes with the standard Dirac operator but is not equivalent to it [10]. This non-standard Dirac operator is connected to the hidden symmetries of the space allowing construction of a conserved vector operator analogous to the Runge–Lenz vector of the Kepler problem. We shall discuss the behavior of this operator under discrete transformations pointing out that the hidden symmetries are in some sense decoupled from the discrete symmetries studied here [10–15].

2 Dirac equation on a curved background

In what follows we shall consider the Dirac operator on a curved background that has the form

$$D_s = \gamma^\mu \hat{\nabla}_\mu. \tag{3}$$

In this expression the Dirac matrices γ_μ are defined in local coordinates by the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I$ and $\hat{\nabla}_\mu$ denotes the canonical covariant derivative for spinors.

Carter and McLenaghan showed that in the theory of Dirac fermions for any isometry with Killing vector R_μ there is an appropriate operator [3]:

$$X_k = -i\left(R^\mu \hat{\nabla}_\mu - \frac{1}{4}\gamma^\mu\gamma^\nu R_{\mu;\nu}\right)$$

which commutes with the *standard* Dirac operator (3).

Moreover, each Killing–Yano tensor $f_{\mu\nu}$ produces a *non-standard* Dirac operator of the form

$$D_f = -i\gamma^\mu\left(f_\mu{}^\nu \hat{\nabla}_\nu - \frac{1}{6}\gamma^\nu\gamma^\rho f_{\mu\nu;\rho}\right)$$

which anticommutes with the standard Dirac operator D_s.

3 Dirac operators of the Taub–NUT space

Let us consider the Taub–NUT space and the chart with Cartesian coordinates x^μ ($\mu, \nu, \ldots = 1, 2, 3, 4$) having the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{V}dl^2 + V(dx^4 + A_i dx^i)^2,$$

where $dl^2 = (d\vec{x})^2 = (dx^1)^2+(dx^2)^2+(dx^3)^2$ is the Euclidean three-dimensional line element and $\vec{A}$ is the gauge field of a monopole. Another chart suitable for applications is that of spherical coordinates $(r,\, \theta,\, \phi,\, \chi)$ among them the first three are the spherical coordinates commonly associated with the Cartesian space ones x^i $(i, j, \ldots = 1, 2, 3)$, while $\chi + \phi = -x^4/\mu$. The real number μ is the parameter of the theory which enters in the form of the function $1/V(r) = 1 + \mu/r$. The only non-vanishing component of the vector potential in spherical coordinates is $A_\phi = \mu(1 - \cos\theta)$. This space has the isometry group $G_s = SO(3) \otimes U(1)_4$ formed by rotations of the Cartesian space coordinates and x^4 translations. The $U(1)_4$ symmetry is important since that eliminates so called NUT singularity if x^4 has the period $4\pi\mu$.

For the theory of the Dirac operators in Cartesian charts of the Taub–NUT space, it is convenient to consider the local frames given by tetrad fields $e(x)$ and $\hat{e}(x)$ as defined in [16] while the four Dirac matrices $\gamma^{\hat{\alpha}}$ that satisfy $\{\gamma^{\hat{\alpha}},\, \gamma^{\hat{\beta}}\} = 2\delta^{\hat{\alpha}\hat{\beta}}$ have to be written in the following representation

$$\gamma^i = -i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \tag{4}$$

where all of them are self-adjoint. In addition we consider the matrix

$$\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}$$

which is denoted by γ^0 in Kaluza–Klein theory explicitly involving the time [12].

The *standard* Dirac operator of the theory without explicit mass term is defined as $D_s = \gamma^{\hat{\alpha}}\hat{\nabla}_{\hat{\alpha}}$ [12, 10] where the spin covariant derivatives with local indices $\hat{\nabla}_{\hat{\alpha}}$ depend on the momentum operators, $P_i = -i(\partial_i - A_i\partial_4)$ and $P_4 = -i\partial_4$, and spin connection [12], such that the Hamiltonian operator [12, 13]

$$H = \gamma^5 D_s = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$$

can be expressed in terms of Pauli operators,

$$\alpha = \sqrt{V}\left(\vec{\sigma}\cdot\vec{P} - \frac{iP_4}{V}\right), \qquad \alpha^* = V\left(\vec{\sigma}\cdot\vec{P} + \frac{iP_4}{V}\right)\frac{1}{\sqrt{V}},$$

involving the Pauli matrices σ_i. These operators give the (scalar) Klein–Gordon operator of the Taub–NUT space [12, 13], $\Delta = -\nabla_\mu g^{\mu\nu}\nabla_\nu = \alpha^*\alpha$. We specify that here the star superscript is a mere notation that does not represent the Hermitian conjugation, because we are using a non-unitary representation of the algebra of Dirac operators. Of course, this is *equivalent* to the unitary representation where all of these operators are self-adjoint [12].

The first three Killing–Yano tensors of the Taub–NUT space [17],

$$f^i = f^i_{\hat{\alpha}\hat{\beta}}\hat{e}^{\hat{\alpha}} \wedge \hat{e}^{\hat{\beta}} = 2\hat{e}^4 \wedge \hat{e}^i + \varepsilon_{ijk}\hat{e}^j \wedge \hat{e}^k$$

are rather special since they are covariantly constant. f^i define three anticommuting complex structures of the Taub–NUT manifold, their components realizing the quaternion algebra

$$f^i f^j + f^j f^i = -2\delta_{ij}, \qquad f^i f^j - f^j f^i = -2\varepsilon_{ijk}f^k.$$

Existence of these Killing–Yano tensors is linked to the hyper-Kähler geometry of the manifold and shows directly the relation between the geometry and the $N = 4$ supersymmetric extension

of the theory [2, 18]. Moreover, we can give a *physical* interpretation of these Killing–Yano tensors defining the *spin-like* operators,

$$\Sigma_i = -\frac{i}{4} f^i_{\hat{\alpha}\hat{\beta}} \gamma^{\hat{\alpha}} \gamma^{\hat{\beta}} = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

that have similar properties to those of the Pauli matrices. In the pseudo-classical description of a Dirac particle [2,18] the covariantly constant Killing–Yano tensors correspond to components of the spin which are separately conserved.

Here, since the Pauli matrices commute with the Klein–Gordon operator, the spin-like operators (5) commute with H^2. Remarkable existence of the Killing–Yano tensors allows one to construct *Dirac-type* operators [3]

$$Q_i = -i f^i_{\hat{\alpha}\hat{\beta}} \gamma^{\hat{\alpha}} \hat{\nabla}^{\hat{\beta}} = \{H, \Sigma_i\} = \begin{pmatrix} 0 & \sigma_i \alpha^* \\ \alpha \sigma_i & 0 \end{pmatrix} \tag{6}$$

that anticommute with D_s and γ^5 and commute with H [10]. Another Dirac operator can be defined using the fourth Killing–Yano tensor but this will be discussed separately in Section 6.

4 Equivalent representations

In [12] we have shown that in the massless case the operators Q_i $(i = 1, 2, 3)$ and the new supercharge $Q_0 = iD_s = i\gamma^5 H$ form the basis of a $N = 4$ superalgebra obeying the anticommutation relations

$$\{Q_A, Q_B\} = 2\delta_{AB} H^2, \qquad A, B, \ldots = 0, 1, 2, 3 \tag{7}$$

linked to the hyper-Kähler geometric structure of the Taub–NUT space. In addition, we associate to each Dirac operator Q_A its own Hamiltonian operator $\tilde{Q}_A = -i\gamma^5 Q_A$ obtaining thus another set of supercharges,

$$\tilde{Q}_0 = H, \qquad \tilde{Q}_i = i[H, \Sigma_i] \tag{8}$$

that obey the same anticommutation relations as (7). Thus we find that there are two similar superalgebras of operators with precise physical meaning. Obviously, since all of these operators must be self-adjoint we have to work only with *unitary* representations of these superalgebras, up to an equivalence.

The concrete form of these supercharges depends on the representation of the Dirac matrices which can be changed at any time with the help of a non-singular operator T such that all of the 4×4 matrix operators of the Dirac theory transform as $X \to X' = TXT^{-1}$. In this way one obtains an *equivalent* representation which preserves the commutation and the anticommutation relations. In [12] we have used such transformations for pointing out that the convenient representations where we work are equivalent to unitary one. We note that some properties of the transformations changing representations in theories with two Dirac operators and their possible new applications are discussed in [19].

For example, simple and convenient transformations of the form can be chosen:

$$U(\beta, \vec{\xi}) = \begin{pmatrix} \hat{U}(\beta, \vec{\xi}) & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix},$$

where $\hat{U}(\beta, \vec{\xi}) = e^{-i\beta} \hat{U}(\vec{\xi}) \in U(2) = U(1) \otimes SU(2)$ with $\hat{U}(\vec{\xi}) \in SU(2)$. This is because among these transformations one could find those linking equivalent Dirac operators.

It is interesting to observe that the $SU(2)$ transformations are generated just by the above defined spin-like operators as

$$U(\vec{\xi}) = U(0,\vec{\xi}) = e^{-i\vec{\xi}\cdot\vec{\Sigma}/2} = \begin{pmatrix} \hat{U}(\vec{\xi}) & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}. \tag{9}$$

If we take now $\vec{\xi} = 2\varphi\,\vec{n}$ with $|\vec{n}| = 1$ and $\varphi \in [0,\pi]$, we find that

$$\hat{U}(\vec{\xi}) = e^{-i\vec{\xi}\cdot\vec{\sigma}/2} = \mathbf{1}_2\cos\varphi - i\vec{n}\cdot\vec{\sigma}\sin\varphi$$

and after a little calculation we can write the concrete action of (9) as

$$Q_0' = U(\vec{\xi})Q_0U^+(\vec{\xi}) = Q_0\cos\varphi + n_iQ_i\sin\varphi, \tag{10}$$

$$Q_i' = U(\vec{\xi})Q_iU^+(\vec{\xi}) = Q_i\cos\varphi - (n_iQ_0 + \varepsilon_{ijk}n_jQ_k)\sin\varphi. \tag{11}$$

Hereby we see that the supercharges are mixed among themselves in linear combinations involving only *real* coefficients. In addition, we observe that these transformations correspond to an irreducible representation since the supercharges transform like the real components of a Pauli spinor. In other words, the usual $SU(2)$ transformations $\psi_Q \to \psi_Q' = \hat{U}^+(\vec{\xi})\psi_Q$ of the spinor-operator

$$\psi_Q = \begin{pmatrix} Q_0 - iQ_3 \\ Q_2 - iQ_1 \end{pmatrix}$$

give just the transformations (10) and (11).

5 Discrete transformations

Let us focus now only on the transformations which transform the supercharges Q_A among themselves without affecting their form. From (10) and (11) we see that there exist particular transformations,

$$Q_k = U_kQ_0U_k^+, \qquad k = 1,2,3,$$

where the matrix $U_k = \mathrm{diag}(-i\sigma_k, \mathbf{1}_2)$ is given by $-i\sigma_k \in SU(2)$. In addition, we consider the *parity* operator $P = P^{-1} = -\gamma^5$ that changes the sign of supercharges,

$$PQ_AP = -Q_A, \qquad A = 0,1,2,3. \tag{12}$$

Then it is not hard to verify that the identity $I = \mathbf{1}_4$, P and the sets of matrices U_k and PU_k $(k = 1,2,3)$ form a discrete group of order eight, the multiplication table of which is determined by the following rules

$$\begin{aligned} &P^2 = I, \qquad PU_k = U_kP, \\ &{U_1}^2 = {U_2}^2 = {U_3}^2 = P, \\ &U_1U_2 = U_3, \qquad U_2U_1 = PU_3, \quad \ldots \quad \text{etc.} \end{aligned} \tag{13}$$

We denote this group by $\mathcal{G}_Q$ since it is a realization of the quaternion group $\mathbf{Q}$ which is isomorphic with the dicyclic group $\langle 2,2,2\rangle$ [20,21]. In the representation (4) of the γ-matrices, its operators are defined by proper unitary matrices (which satisfy $G^{-1} = G^+$ and $\det G = 1$, $\forall\, G \in \mathcal{G}_Q$) constructed using the elements $\pm\mathbf{1}_2, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3$ of the natural realization of $\mathbf{Q}$ as a discrete subgroup of $SU(2)$.

The group $\mathcal{G}_Q$ is interesting because it brings together the parity that produces the transformations (12) and the operators U_k giving sequences of the form

$$Q_1 = U_3^+ Q_2 U_3 = U_2 Q_3 U_2^+ = U_1 Q_0 U_1^+, \quad \dots \quad \text{etc.}$$

that leads to the conclusion that the Dirac operators and their parity transformed $\pm Q_A$ ($A = 0,1,2,3$), are equivalent among themselves. All these operators constitute the orbit $\Omega_Q = \{Q \,|\, Q = GQ_0G^+,\ \forall G \in \mathcal{G}_Q\}$ of the group $\mathcal{G}_Q$ in the algebra of the 4×4 matrix operators. A similar orbit $\tilde{\Omega}_Q$ can be constructed for the associated Hamiltonian operators $\pm\tilde{Q}_A$ defined by (8), if we start with $\tilde{Q}_0$ instead of Q_0. It is remarkable that each of these two orbits includes *only* operators representing (up to sign) supercharges obeying superalgebras of the form (7).

In the Kaluza–Klein theory with the time trivially added [12], the time dependent term of the whole massless Dirac operator commutes with all the operators of $\mathcal{G}_Q$ so as it remains unchanged when one replaces the space parts using the discrete transformations of this group. In these conditions all the Dirac operators of Ω_Q lead to equivalent Dirac equations from the physical point of view. These can be written in Hamiltonian form as $i\partial_t \psi_A^{(\pm)} = \pm\tilde{Q}_A \psi_A^{(\pm)}$ ($A = 0,1,2,3$) and produce the same energy spectrum which coincides with that of the Klein–Gordon equation as it results from the superalgebra (7) [22, 12].

The existence of this discrete symmetry among the four supercharges of the superalgebra of the Dirac and Dirac-type operators (or the corresponding Hamiltonian operators) must be understood as a consequence of the fact that the Taub-NUT space has a hyper-Kähler structure modeled on a quaternion inner-product space [23]. In other words, the Dirac theory in this space picks up the basic quaternion character of the tangent space showing it off as the discrete symmetry due to the group $\mathcal{G}_Q \sim \mathbf{Q}$, naturally related to the specific supersymmetries of this geometry.

6 Hidden symmetries and the fifth Dirac operator

In the Taub–NUT space, in addition to the above discussed covariantly constant Killing–Yano tensors, there exists a fourth Killing–Yano tensor,

$$f^Y = -\frac{x^i}{r} f^i + \frac{2x^i}{\mu V} \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k,$$

which is not covariantly constant. The presence of f^Y is due to existence of the hidden symmetries of the Taub–NUT geometry that are encapsulated in three non-trivial Stäckel–Killing tensors. These are interpreted as the components of the so-called Runge–Lenz vector of the Taub–NUT problem and are expressed as symmetrized products of the Killing–Yano tensors f^Y and f^i ($i = 1,2,3$) [24].

As in the case of the Dirac operators (6), one can use f^Y for defining the fifth Dirac operator

$$Q_0^Y = -i\gamma^{\hat{\alpha}} \left(f^Y_{\hat{\alpha}\hat{\beta}} \hat{\nabla}^{\hat{\beta}} - \frac{1}{6} \gamma^{\hat{\beta}} \gamma^{\hat{\delta}} f^Y_{\hat{\alpha}\hat{\beta};\hat{\delta}} \right),$$

called here the *non-standard* or *hidden* Dirac operator to emphasize the connection with the hidden symmetry of the Taub–NUT problem. It is denoted by Q_0^Y instead of Q^Y as in [10] to point out its relation to the standard Dirac operator since it can be put in the form

$$Q_0^Y = i\frac{r}{\mu} \left[Q_0, \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_r V^{-1} \end{pmatrix} \right],$$

where $\sigma_r = \vec{x} \cdot \vec{\sigma}/r$. We showed that Q_0^Y commutes with $\tilde{Q}_0 = H$ and anticommutes with Q_0 and γ^5 [10]. This operator is important because it allowed us to derive the explicit form of the Runge–Lenz operator $\vec{K}$ of the Dirac field in Taub–NUT background establishing its properties [10].

We recall that the components of the conserved total angular momentum $\vec{J}$, and the operators $R_i = F^{-1}K_i$ with $F^2 = {P_4}^2 - H^2$ are just the generators of the dynamical algebra of the Dirac theory in Taub–NUT background [13, 14].

Starting with Q_0^Y we can construct a new orbit Ω^Y of $\mathcal{G}_Q$ defining

$$Q_k^Y = U_k Q_0^Y U_k^+ = i\frac{r}{\mu}\left[Q_k, \begin{pmatrix} \sigma_k\sigma_r\sigma_k & 0 \\ 0 & \sigma_r V^{-1} \end{pmatrix}\right] \tag{14}$$

(for $k = 1, 2, 3$) and observing that $PQ_A^Y P = -Q_A^Y$, $A = 0, 1, 2, 3$.

From the explicit form (14) we deduce that, in contrast with the operators of the orbits Ω_Q and $\tilde{\Omega}_Q$, those of the orbit Ω^Y have more involved algebraic properties. We can show that calculating

$$H^2(Q_0^Y)^2 = H^4 + \frac{4}{\mu^2}H^2\left(\vec{J}^2 + \frac{1}{4}\right) + 4F^2P_4^2, \tag{15}$$

and it is worth comparing it with equation (7). The Dirac-type operators Q_A are characterized by the fact that their quantal anticommutator closes on the square of the Hamiltonian of the theory. No such expectation applies to the non-standard, hidden Dirac operators Q_A^Y which close on a combination of different conserved operators. Also from equation (15) it results that $(Q_A^Y)^2 \neq (Q_B^Y)^2$ if $A \neq B$ (because $\vec{J}^2$ does not commute with U_k). Moreover, one can show that the commutators $[Q_A^Y, Q_B^Y]$ have complicated forms that can not be expressed in terms of operators Q_A^Y. Therefore, neither the commutator nor the anticommutator of the pairs of operators of this orbit do not lead to significant algebraic results as the anticommutation relations (7) of the operators Q_A $(A = 0, 1, 2, 3)$.

7 Concluding remarks

In the study of the Dirac equation in curved spaces, it has been proved that the Killing–Yano tensors play an essential role in the construction of new Dirac-type operators. The Dirac-type operators constructed with the aid of covariantly constant Killing–Yano tensors are equivalent with the standard Dirac operator. The non-covariantly constant Killing–Yano tensors generate non-standard Dirac operators that are not equivalent to the standard Dirac operator and they are associated with the hidden symmetries of the space.

The Taub–NUT space has a special geometry where the covariantly constant Killing–Yano tensors exist by virtue of the metric being self-dual and the Dirac-type operators generated by them are equivalent to the standard one. The fourth Killing–Yano tensor f^Y that is not covariantly constant exists by virtue of the metric being of type D. The corresponding non-standard or hidden Dirac operator does not close on H as it can be seen from equation (15) and is not equivalent to the Dirac-type operators. As it was mentioned, it is associated with the hidden symmetries of the space allowing the construction of the conserved vector-operator analogous to the Runge–Lenz vector of the Kepler problem.

Let us mention that in the pseudo-classical spinning particle models in curved spaces from covariantly constant K-Y tensors $f_{\mu\nu}$ conserved quantities of the type $f_{\mu\nu}\theta^\mu\theta^\nu$ depending on the Grassmann variables $\{\theta^\mu\}$ can be constructed [24]. The Grassmann variables $\{\theta^\mu\}$ transform as a tangent space vector and describe the spin of the particle. The antisymmetric tensor $S^{\mu\nu} = -i\theta^\mu\theta^\nu$ generates the internal part of the local tangent space rotations. For example, in the spinning Euclidean Taub–NUT space such operators correspond to components of the spin that are separately conserved [18].

The construction of the new supersymmetries in the context of pseudo-classical mechanics can be carried over straightforwardly to the case of quantum mechanics by the usual replacement

of phase space coordinates by operators and Poisson–Dirac brackets by anticommutators [25]. In terms of these operators the supercharges are replaced by Dirac-type operators [26]. In both cases, the correspondence principle leads to equivalent algebraic structures making obvious the relations between these approaches [18].

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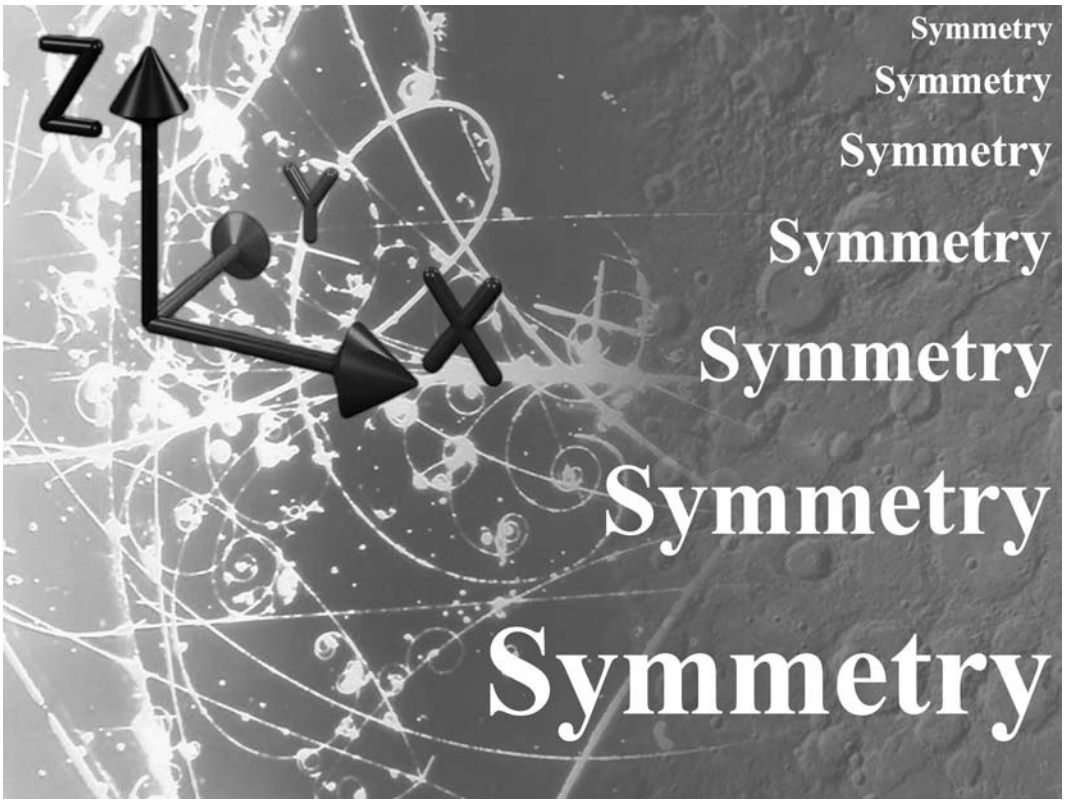
Symmetry in Physics



Symmetry in Nonlinear Mathematical Physics

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Angular Momentum and Killing–Yano Tensors

Dumitru BALEANU

Cankaya University, Faculty of Arts and Science,
Department of Mathematics and Computer Science, 06530 Balgat, Ankara, Turkey
E-mail: *dumitru@cankaya.edu.tr*

Institute of Space Sciences, P.O. Box, MG-23, R 77125 Magurele-Bucharest, Romania

Quadratic Lagrangians are obtained by adding surface terms involving the angular momentum to a free particle Lagrangian. Killing–Yano tensors corresponding to the manifold structure induced by the quadratic Lagrangians were calculated and the associated non-generic symmetries were presented.

1 Introduction

Symmetries of a Riemannian space are associated with conservation laws of geodesic motion in this space, the investigation of Killing vectors and Killing tensors are very important in classifying and understanding a space-time.

Killing tensors of rank two or higher cannot in general be interpreted as symmetries of a metric alone. But on the other hand they can be interpreted as symmetries on the phase space for single particles.

The Killing–Yano (KY) tensors of rank two or higher order [1–7] play an important role for new non-generic symmetries [8, 9] as well as the generalized KY tensors [10], which are deeply connected to generalized Runge–Lenz symmetry. Also recently, it was discovered that there is a connection between Lax tensors [11] and KY tensors of order three. KY tensors can be understood as an object generating non-generic supersymmetry, i.e. supersymmetry belonging to some specific space-times.

Within the same context of the Killing and KY tensors, the notion of geometric duality was defined and applied on Taub-NUT and Kerr–Newman space-times [12]. The direct way to construct the dual metrics is to calculate the Killing tensors [13, 14]. An alternative way is to find the KY tensor and contract two of them to find Killing tensors [15]. However, finding an explicit space-time with a physical significance admitting higher rank KY tensors is not an easy task, not only because of the complexity of the calculations, but also earlier it was found that not all metrics admit KY tensors of higher order [16].

In the last years a huge effort was devoted to analyzing of the importance of KY tensors [17–24] in several areas.

There were some attempts to find new geometries admitting KY tensors of order two or three. For example, the three- particle open Toda lattice was geometrized by a suitable canonical transformation, and it was realized as the geodesic system of a certain Riemannian geometry [25]. Adding a time-like dimension, a four-dimensional space-times admitting two Killing vector fields were found [25].

Motivated by the above results we decided to construct geometries admitting (KY) and Killing tensors by adding a surface term [26, 27] to a known free Lagrangian.

The main aim of this paper is to add, using suitable Lagrangian multipliers, the components of the angular momenta to a free Lagrangian in two or three dimensions and to study the non-generic symmetries of the induced manifolds.

2 Killing–Yano tensors and non-generic supercharges

An n^{th} rank KY tensor $f_{\nu_1\nu_2\cdots\nu_n}$, is an antisymmetric tensor fulfilling the following equations:

$$f_{\nu_1\nu_2\cdots(\nu_n;\lambda)} = 0, \tag{1}$$

where semicolon denotes the covariant derivative.

The spinning particle model was constructed to be supersymmetric [8, 28–30], therefore independent of the form of the metric there is always a conserved supercharge

$$Q_0 = \Pi_\mu \psi^\mu.$$

Here, Π_μ is the covariant momenta and ψ^μ are odd Grassmann variables. The existence of Killing–Yano tensors of valence n are related to non-generic supersymmetries described by the supercharge

$$Q_f = f_{\nu_1\nu_2\cdots\nu_r}\Pi^{\nu_1}\psi^{\nu_2}\cdots\psi^{\nu_r} \tag{2}$$

which is a superinvariant

$$\{Q_0, Q_f\} = 0.$$

The KY equation and the Jacobi identities guarantee that it is also a constant of motion

$$\{Q_f, H\} = 0 \qquad \text{with} \quad H = \frac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu$$

and with the appropriate definitions of the brackets.

3 Killing tensors and dual space-times

A Killing tensor of valence two is defined through the equation

$$K_{(\mu\nu;\alpha)} = 0. \tag{3}$$

Killing–Yano tensors of any valence can be considered as the square root of the Killing tensors of valence two in the sense that, their appropriate contractions yield

$$K_{\mu\nu} = g^{\alpha\beta}f_{\mu\alpha}f_{\beta\nu} \tag{4}$$

or for valence three it can be written as

$$K_{\mu\nu} = g^{\alpha\delta}g^{\beta\gamma}f_{\mu\alpha\beta}f_{\gamma\delta\nu}.$$

It has been shown in detail in reference [12] that $K^{\mu\nu}$ and $g^{\mu\nu}$ are reciprocally the contravariant components of the Killing tensors with respect to each other. If $K^{\mu\nu}$ is non-degenerate, then through the relation

$$K^{\mu\alpha}k_{\alpha\nu} = \delta^\mu{}_\nu,$$

the second rank non-degenerate tensor $k_{\mu\nu}$, can be viewed as the metric on the "dual" space. Furthermore, the notion of geometric duality extends to that of phase space. The constant of motion

$$K = \frac{1}{2}K^{\mu\nu}p_\mu p_\nu,$$

generates symmetry transformations on the phase space linear in momentum

$$\{x^\mu, K\} = K^{\mu\nu} p_\nu.$$

The Poisson brackets satisfy

$$\{H, K\} = 0,$$

where

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu.$$

Thus, in the phase space there is a reciprocal model with constant of motion H and the Hamiltonian K.

4 Extended Lagrangians and their corresponding geometries

Let us assume that a given free Lagrangian $L(\dot{q}^i, q^i)$ admits a set of constants of motion denoted by L_i, $i = 1, \ldots, 3$. If we add the components of the angular momentum corresponding to L, the extended Lagrangian [31]

$$L' = L + \dot{\lambda}^i L_i, \qquad i = 1, \ldots, 3$$

can be rewritten as

$$L' = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j.$$

Since a_{ij} is symmetric by construction, the issue is to find a way to construct induced manifolds. In other words we are looking to find whether a_{ij} is singular or not. If the matrix a_{ij} is singular, L' corresponds to a singular system [31]. Assuming that a_{ij} is a singular $n \times n$ matrix of rank $n-1$, we obtain non-singular symmetric matrices of order $(n-1) \times (n-1)$, where n will be 3, 5 and 6. The final step is to consider the obtained matrices as metrics on the extended space and to investigate their generic and non-generic symmetries.

4.1 The nonsingular case

As a starting point let us consider the following Lagrangian

$$L' = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 \right) + \dot{\lambda_3} (x\dot{y} - y\dot{x}). \tag{5}$$

From (5) we obtain $L^{'} = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j$, where a_{ij} is given by

$$a_{ij} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{pmatrix}. \tag{6}$$

If a_{ij} is considered as a metric of a manifold, by direct calculations we obtain that the corresponding Killing vector [2] of (6) is given by

$$V = (y, -x, 0).$$

From (1) a KY tensor of order two is defined as

$$D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0. \tag{7}$$

Solving (7) corresponding to (6) we obtained the following KY tensor

$$f_{12}=0, \qquad f_{23}=-Cx\sqrt{x^2+y^2}, \qquad f_{13}=Cy\sqrt{x^2+y^2}, \tag{8}$$

where C is a constant.

If a (KY) tensor exists, then a Killing tensor of order two is generated as in (4). Taking into account (4) and (8) a Killing tensor is constructed as

$$K_{ij}=\begin{pmatrix} y^2 & -xy & -y\left(y^2+x^2\right) \\ -xy & x^2 & x\left(x^2+y^2\right) \\ -y\left(y^2+x^2\right) & x\left(x^2+y^2\right) & 0 \end{pmatrix}.$$

5 The motion on a sphere and its induced geometries

It was proved in [32] that the motion on a sphere admits four constants of motion, the Hamiltonian and three components of the angular momentum. In the following using the surface term we will generate four-dimensional manifolds. In this case the Lagrangian is given by

$$\begin{aligned} L'=&\frac{1}{2}\left(1+\frac{x^2}{u}\right)\dot{x}^2+\frac{1}{2}\left(1+\frac{y^2}{u}\right)\dot{y}^2+\frac{xy}{u}\dot{x}\dot{y}-\frac{xy}{\sqrt{u}}\dot{\lambda_1}\dot{x}+\left(\frac{x^2}{\sqrt{u}}+\sqrt{u}\right)\dot{\lambda_2}\dot{x}\\ &-\left(\frac{y^2}{\sqrt{u}}+\sqrt{u}\right)\dot{\lambda_1}\dot{y}+\frac{xy}{\sqrt{u}}\dot{\lambda_2}\dot{y}+x\dot{\lambda_3}\dot{y}-y\dot{\lambda_3}\dot{x}, \end{aligned} \tag{9}$$

where $u=1-x^2-y^2$. From (9)we identify the singular matrix a_{ij} as

$$a_{ij}=\begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}}+\sqrt{u} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & \frac{xy}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & 0 & 0 & 0 \\ \frac{x^2}{\sqrt{u}}+\sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

Because (10) is a singular matrix of rank 4 we identify three symmetric minors of order four. If we treat these minors as metrics, we observe that they are not conformaly flat but their scalar curvatures are zero. The first metric is given by

$$g^{(1)}_{\mu\nu}=\begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & \sqrt{u}+\frac{x^2}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & \frac{xy}{\sqrt{u}} & x \\ \sqrt{u}+\frac{x^2}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix}. \tag{11}$$

The Killing vectors of (11) have the following forms

$$V_1=(y,-x,0,0), \qquad V_2=\left(\sqrt{1-x^2-y^2}+\frac{x^2}{1-x^2-y^2},\frac{xy}{1-x^2-y^2},0,0\right),$$
$$V_3=\left(-\frac{xy}{1-x^2-y^2},-\sqrt{1-x^2-y^2}-\frac{y^2}{1-x^2-y^2},0,0\right).$$

The next step is to investigate its KY tensors. Solving (7) we obtain the following set of solutions:

One-solution is given by

$$f_{21} = \frac{C_1}{\sqrt{1-x^2-y^2}}, \tag{12}$$

others zero.

Two-by-two solution has the form

$$f_{31} = f_{42} = C. \tag{13}$$

Finally, three by three solution is

$$f_{21} = \frac{C_1}{\sqrt{-1+x^2+y^2}}, \qquad f_{31} = f_{42} = C, \tag{14}$$

where C and C_1 are constants.

Using (2), (12), (13) and (14) we can obtain the form of the non-generic supercharge Q_0.

From (10) two more metrics can be identified as

$$g^{(2)}_{\mu\nu} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\sqrt{u}-\frac{y^2}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\sqrt{u}-\frac{y^2}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix} \tag{15}$$

and

$$g^{(3)}_{\mu\rho} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}}+\sqrt{u} \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & \frac{xy}{\sqrt{u}} \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & 0 & 0 \\ \frac{x^2}{\sqrt{u}}+\sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 \end{pmatrix} \tag{16}$$

respectively. By direct calculations we obtained that the metrics (15) and (16) admit the same Killing vector as in (11). Solving the equation (7) corresponding to (15) and (16) we find one non-zero component of KY tensor as $f_{21} = \frac{C_1}{\sqrt{1-x^2-y^2}}$.

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Complex Extension of Quantum Mechanics

Carl M. BENDER

Department of Physics, Washington University, St. Louis, MO 63130, USA
E-mail: *cmb@wustl.edu*

This talk examines Hamiltonians H that are not Hermitian but do exhibit space-time reflection ($\mathcal{PT}$) symmetry. If the ($\mathcal{PT}$) symmetry of H is not spontaneously broken, then the spectrum of H is entirely real and positive. Examples of $\mathcal{PT}$-symmetric non-Hermitian Hamiltonians are $H = p^2 + ix^3$ and $H = p^2 - x^4$. The apparent shortcoming of quantum theories arising from $\mathcal{PT}$-symmetric Hamiltonians is that the $\mathcal{PT}$ norm is not positive definite. This suggests that it may be difficult to develop a quantum theory based on such Hamiltonians. In this talk it is shown that these difficulties can be overcome by introducing a previously unnoticed underlying physical symmetry $\mathcal{C}$ of Hamiltonians having an unbroken $\mathcal{PT}$ symmetry. Using $\mathcal{C}$, it is shown how to construct an inner product whose associated norm *is* positive definite. The result is a new class of fully consistent complex quantum theories. Observables in these theories exhibit $\mathcal{CPT}$ symmetry, probabilities are positive, and the dynamics is governed by unitary time evolution.

1 Introduction

In this talk we discuss an alternative to a standard axiom of quantum mechanics; namely, that the Hamiltonian H, which incorporates the symmetries and specifies the dynamics of a quantum theory, must be Hermitian: $H = H^\dagger$. It is commonly believed that H must be Hermitian in order to ensure that the energy spectrum (the eigenvalues of H) is real and that the time evolution is unitary (probability is conserved in time). Although this axiom is sufficient to guarantee these desired properties, it is not necessary. We believe that the condition of Hermiticity is a mathematical requirement whose physical basis is obscure. We demonstrate here that there is a more physical alternative axiom, which we refer to as space-time reflection symmetry ($\mathcal{PT}$ symmetry): $H = H^{\mathcal{PT}}$. This symmetry allows for the possibility of non-Hermitian and complex Hamiltonians but still leads to a consistent theory of quantum mechanics.

We also show that because $\mathcal{PT}$ symmetry is an alternative condition to Hermiticity, it is now possible to construct infinitely many new Hamiltonians that would have been rejected in the past because they are not Hermitian. An example of such a Hamiltonian is $H = p^2 + ix^3$. It should be emphasized that we do not regard the condition of Hermiticity as wrong. Rather, the condition of $\mathcal{PT}$ symmetry offers the possibility of studying new quantum theories.

Let us recall the properties of the space reflection (parity) operator $\mathcal{P}$ and the time-reflection operator $\mathcal{T}$: $\mathcal{P}$ is *linear* and has the effect $p \to -p$ and $x \to -x$; $\mathcal{T}$ is antilinear and has the effect $p \to -p$, $x \to x$, and $i \to -i$. Note that $\mathcal{T}$ changes the sign of i because, like $\mathcal{P}$, it preserves the fundamental commutation relation of quantum mechanics, $[x, p] = i$.

It is easy to construct Hamiltonians that are not Hermitian but do possess $\mathcal{PT}$ symmetry. For example, consider the one-parameter family of Hamiltonians

$$H = p^2 + x^2(ix)^\epsilon \qquad (\epsilon \text{ real}). \tag{1}$$

While H in (1) is not symmetric under $\mathcal{P}$ or $\mathcal{T}$ separately, it is invariant under their combined operation. We say that such Hamiltonians possess space-time reflection symmetry. Other complex Hamiltonians having $\mathcal{PT}$ symmetry are $H = p^2 + x^4(ix)^\epsilon$ and $H = p^2 + x^6(ix)^\epsilon$ [1][1].

[1]These classes of Hamiltonians are all *different*. For example, the Hamiltonian obtained by continuing H in (1) along the path $\epsilon : 0 \to 8$ has a different spectrum from the Hamiltonian that is obtained by continuing

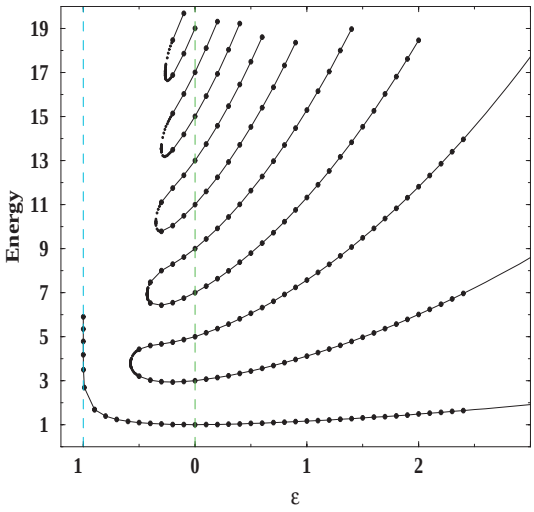

Figure 1. Energy levels of the Hamiltonian $H = p^2 + x^2(ix)^\epsilon$ as a function of the parameter ϵ. There are three regions: When $\epsilon \geq 0$, the spectrum is real and positive and the energy levels rise with increasing ϵ. The lower bound of this region, $\epsilon = 0$, corresponds to the harmonic oscillator, whose energy levels are $E_n = 2n + 1$. When $-1 < \epsilon < 0$, there is a finite number of real positive eigenvalues and an infinite number of complex conjugate pairs of eigenvalues. As ϵ decreases from 0 to -1, the number of real eigenvalues decreases; when $\epsilon \leq -0.57793$, the only real eigenvalue is the ground-state energy. As ϵ approaches -1^+, the ground-state energy becomes infinite. When $\epsilon \leq -1$ there are no real eigenvalues.

The class of $\mathcal{PT}$-symmetric Hamiltonians is larger than and includes real symmetric Hermitians because any real symmetric Hamiltonian is automatically $\mathcal{PT}$-symmetric. For example, consider the real symmetric Hamiltonian $H = p^2 + x^2 + 2x$. This Hamiltonian is time-reversal symmetric, but according to the usual definition of space reflection for which $x \to -x$, this Hamiltonian appears not to have $\mathcal{PT}$ symmetry. However, recall that the parity operator is defined only up to unitary equivalence. In this example, if we express the Hamiltonian in the form $H = p^2 + (x+1)^2 - 1$, then it is evident that H is $\mathcal{PT}$ symmetric, provided that the parity operator performs a space reflection about the point $x = -1$ rather than $x = 0$. See Ref. [2] for the construction of the relevant parity operator.

With properly defined boundary conditions the spectrum of the Hamiltonian H in (1) is *real and positive* when $\epsilon \geq 0$ [3] and the spectrum is partly real and partly complex when $\epsilon < 0$. The eigenvalues have been computed numerically to very high precision, and the real eigenvalues are plotted as functions of ϵ in Fig. 1.

We say that the $\mathcal{PT}$ symmetry of a Hamiltonian H is *unbroken* if all of the eigenfunctions of H are simultaneously eigenfunctions of $\mathcal{PT}$[2]. Here is a proof that if the $\mathcal{PT}$ symmetry of a Hamiltonian H is unbroken, then the spectrum of H is real: Assume that H possesses $\mathcal{PT}$ symmetry (that is, that H commutes with the $\mathcal{PT}$ operator), and that if ϕ is an eigenstate of H with eigenvalue E, then it is simultaneously an eigenstate of $\mathcal{PT}$ with eigenvalue λ:

$$H\phi = E\phi \qquad \text{and} \qquad \mathcal{PT}\phi = \lambda\phi. \tag{2}$$

$H = p^2 + x^6(ix)^\epsilon$ along the path $\epsilon : 0 \to 4$. This is because the boundary conditions on the eigenfunctions are different.

[2] If a system is defined by an equation that possesses a discrete symmetry, the solution to this equation need not exhibit that symmetry. For example, the differential equation $\ddot{y}(t) = y(t)$ is symmetric under time reversal $t \to -t$. The solutions $y(t) = e^t$ and $y(t) = e^{-t}$ do not exhibit time-reversal symmetry while the solution $y(t) = \cosh(t)$ is time-reversal symmetric. The same is true with a system whose Hamiltonian is $\mathcal{PT}$ symmetric. Even if the Schrödinger equation and the corresponding boundary conditions are $\mathcal{PT}$ symmetric, the wave function that solves the Schrödinger equation boundary value problem may not be symmetric under space-time reflection. When the solution exhibits $\mathcal{PT}$ symmetry, we say that the $\mathcal{PT}$ symmetry is unbroken. Conversely, if the solution does not possess $\mathcal{PT}$ symmetry, we say that the $\mathcal{PT}$ symmetry is broken.

We begin by showing that the eigenvalue λ is a pure phase. Multiply $\mathcal{PT}\phi = \lambda\phi$ on the left by $\mathcal{PT}$ and use the fact that $\mathcal{P}$ and $\mathcal{T}$ commute and that $\mathcal{P}^2 = \mathcal{T}^2 = 1$ to conclude that $\phi = \lambda^*\lambda\phi$ and thus $\lambda = e^{i\alpha}$ for some real α. Next, introduce a convention that we use throughout this talk. Without loss of generality we replace the eigenstate ϕ by $e^{-i\alpha/2}\phi$ so that its eigenvalue under the operator $\mathcal{PT}$ is unity: $\mathcal{PT}\phi = \phi$. Next, multiply the eigenvalue equation $H\phi = E\phi$ on the left by $\mathcal{PT}$ and use $[\mathcal{PT}, H] = 0$ to obtain $E\phi = E^*\phi$. Hence, $E = E^*$ and E is real.

The crucial assumption in this argument is that ϕ is simultaneously an eigenstate of H and $\mathcal{PT}$. In quantum mechanics if a linear operator X commutes with H, then the eigenstates of H are also eigenstates of X. However, the operator $\mathcal{PT}$ is not linear (it is antilinear) and thus we must make the extra assumption that the $\mathcal{PT}$ symmetry of H is unbroken; that is, that ϕ is simultaneously an eigenstate of H and $\mathcal{PT}$. This extra assumption is nontrivial because it is hard to determine *a priori* whether the $\mathcal{PT}$ symmetry of a given H is broken or unbroken. For the Hamiltonian H in (1) the $\mathcal{PT}$ symmetry is unbroken when $\epsilon \geq 0$ and broken when $\epsilon < 0$. The conventional Hermitian Hamiltonian for the quantum mechanical harmonic oscillator lies at the boundary of the unbroken and the broken regimes. Recently, Dorey *et al.* proved rigorously that the spectrum of H in (1) is real and positive [4] in the region $\epsilon \geq 0$. Many other $\mathcal{PT}$-symmetric Hamiltonians for which space-time reflection symmetry is not broken have been investigated, and the spectra of these Hamiltonians have also been shown to be real and positive [5].

While it is useful to show that a given non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian operator has a positive real spectrum, the urgent question is whether such a Hamiltonian defines a physical theory of quantum mechanics. By a *physical theory* we mean that there is a Hilbert space of state vectors and that this Hilbert space has an inner product with a positive norm. In quantum mechanics we interpret the norm of a state as a probability and this probability must be positive. Furthermore, we must show that the time evolution of the theory is unitary. This means that as a state vector evolves in time the probability does not leak away.

It is not obvious whether a Hamiltonian such as H in (1) gives a consistent quantum theory. Indeed, past investigations of this Hamiltonian have shown that while the spectrum is entirely real and positive when $\epsilon \geq 0$, one inevitably encountered the severe problem of Hilbert spaces endowed with indefinite metrics [6]. We will identify a new symmetry that all $\mathcal{PT}$-symmetric Hamiltonians having an unbroken $\mathcal{PT}$-symmetry possess. We denote the operator representing this symmetry by $\mathcal{C}$ because the properties of this operator resemble those of the charge conjugation operator in particle physics. This will allow us to introduce an inner product structure associated with $\mathcal{CPT}$ conjugation for which the norms of quantum states are positive definite. We will see that $\mathcal{CPT}$ symmetry is an alternative to Hermiticity; it introduces the new concept of a *dynamically determined* inner product (one that is defined by the Hamiltonian itself). As a consequence, we will extend the Hamiltonian and its eigenstates into the complex domain so that the associated eigenvalues are real and the underlying dynamics is unitary.

2 Construction of the $\mathcal{C}$ operator

We begin by summarizing the mathematical properties of the solution to the Sturm–Liouville differential equation eigenvalue problem

$$-\phi_n''(x) + x^2(ix)^\epsilon\phi_n(x) = E_n\phi_n(x) \tag{3}$$

associated with the Hamiltonian H in (1). This differential equation must be imposed on an infinite contour in the complex-x plane. For large $|x|$ this contour lies in wedges placed symmetrically with respect to the imaginary-x axis [3]. The boundary conditions on the eigenfunctions are that $\phi(x) \to 0$ exponentially rapidly as $|x| \to \infty$ on the contour. For $0 \leq \epsilon < 2$, the contour may lie on the real axis.

When $\epsilon \geq 0$, the Hamiltonian has an unbroken $\mathcal{PT}$ symmetry. Thus, the eigenfunctions $\phi_n(x)$ are simultaneously eigenstates of the $\mathcal{PT}$ operator: $\mathcal{PT}\phi_n(x) = \lambda_n\phi_n(x)$. As we argued above, λ_n is a pure phase and, without loss of generality, for each n this phase can be absorbed into $\phi_n(x)$ by a multiplicative rescaling so that the new eigenvalue is unity:

$$\mathcal{PT}\phi_n(x) = \phi_n^*(-x) = \phi_n(x). \tag{4}$$

There is strong evidence that, when properly normalized, the eigenfunctions $\phi_n(x)$ are complete. The coordinate-space statement of completeness reads

$$\sum_n (-1)^n \phi_n(x)\phi_n(y) = \delta(x-y) \qquad (x, y \text{ real}). \tag{5}$$

This nontrivial result has been verified numerically to extremely high accuracy (twenty decimal places) [7, 8]. There is a factor of $(-1)^n$ in the sum. This unusual factor does not appear in conventional quantum mechanics. The presence of this factor is explained in the following discussion of orthonormality [see (7)].

Here is where we encounter the problem associated with non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians. The obvious choice for the inner product of two functions $f(x)$ and $g(x)$ is

$$(f, g) \equiv \int dx\, [\mathcal{PT} f(x)]g(x), \tag{6}$$

where $\mathcal{PT} f(x) = [f(-x)]^*$ and the integral path is the above contour in the complex-x plane. The apparent advantage of this inner product is that the associated norm (f, f) is independent of the overall phase of $f(x)$ and is conserved in time. Phase independence is desired because in quantum mechanics the objective is to construct a space of rays to represent quantum mechanical states. With respect to this inner product the eigenfunctions $\phi_m(x)$ and $\phi_n(x)$ of H in (1) are orthogonal for $n \neq m$. However, when $m = n$ the norm is evidently *not positive*:

$$(\phi_m, \phi_n) = (-1)^n \delta_{mn}. \tag{7}$$

This result is apparently true for all values of ϵ in (3) and it has been verified numerically to extremely high precision. Because the norms of the eigenfunctions alternate in sign, the metric associated with the $\mathcal{PT}$ inner product $(\cdot, \cdot)$ is indefinite. This split signature (sign alternation) is a *generic* feature of the $\mathcal{PT}$ inner product. Extensive numerical calculations verify that the formula in (7) holds for all $\epsilon \geq 0$.

Despite the lack of positivity of the inner product, we proceed with the usual analysis that one would perform for any Sturm–Liouville problem of the form $H\phi_n = E_n\phi_n$. First, we use the inner product formula (7) to verify that (5) is the representation of the unity operator. That is, we verify that $\int dy\, \delta(x-y)\delta(y-z) = \delta(x-z)$.

Second, we reconstruct the parity operator $\mathcal{P}$ in terms of the eigenstates. The parity operator in position space is $\mathcal{P}(x, y) = \delta(x+y)$, so from (5) we get

$$\mathcal{P}(x, y) = \sum_n (-1)^n \phi_n(x)\phi_n(-y). \tag{8}$$

By virtue of (7) the square of the parity operator is unity: $\mathcal{P}^2 = 1$.

Third, we construct H in coordinate space: $H(x, y) = \sum_n (-1)^n E_n \phi_n(x)\phi_n(y)$. Using (5)–(7) we can see that this Hamiltonian satisfies $H\phi_n(x) = E_n\phi_n(x)$.

Fourth, we construct the Green's function $G(x, y) = \sum_n (-1)^n \frac{1}{E_n}\phi_n(x)\phi_n(y)$ in coordinate space. The Green's function is the functional inverse of H; that is, G satisfies

$$\int dy\, H(x, y)G(y, z) = \left[-\frac{d^2}{dx^2} + x^2(ix)^\epsilon\right] G(x, z) = \delta(x-z). \tag{9}$$

While the time-independent Schrödinger equation (3) cannot be solved analytically, the differential equation for $G(x,z)$ in (9) *can* be solved in closed form [8]. The technique is to consider the case $0<\epsilon<2$ so that we may treat x as real and then to decompose the x axis into two regions, $x>z$ and $x<z$. We can solve the differential equation in each region in terms of Bessel functions. From this representation of the Green's function we construct an exact closed-form expression for the *spectral zeta function* (sum of the inverses of the energy eigenvalues). To do so we set $y=x$ in $G(x,y)$ and use (7) to integrate over x. For all $\epsilon>0$ we obtain [8]

$$\sum_n \frac{1}{E_n} = \left[1+\frac{\cos\left(\frac{3\epsilon\pi}{2\epsilon+8}\right)\sin\left(\frac{\pi}{4+\epsilon}\right)}{\cos\left(\frac{\epsilon\pi}{4+2\epsilon}\right)\sin\left(\frac{3\pi}{4+\epsilon}\right)}\right]\frac{\Gamma\left(\frac{1}{4+\epsilon}\right)\Gamma\left(\frac{2}{4+\epsilon}\right)\Gamma\left(\frac{\epsilon}{4+\epsilon}\right)}{(4+\epsilon)^{\frac{4+2\epsilon}{4+\epsilon}}\Gamma\left(\frac{1+\epsilon}{4+\epsilon}\right)\Gamma\left(\frac{2+\epsilon}{4+\epsilon}\right)}. \tag{10}$$

Having presented these general Sturm–Liouville constructions, we now address the crucial question of whether a $\mathcal{PT}$-symmetric Hamiltonian defines a physically viable quantum mechanics or whether it merely provides an intriguing Sturm–Liouville eigenvalue problem. The apparent difficulty with formulating a quantum theory is that the vector space of quantum states is spanned by energy eigenstates, of which half have norm $+1$ and half have norm -1. Because the norm of the states carries a probabilistic interpretation in standard quantum theory, the existence of an indefinite metric in (7) seems to be a serious obstacle.

The situation here in which half of the energy eigenstates have positive norm and half have negative norm is analogous to the problem that Dirac encountered in formulating the spinor wave equation in relativistic quantum theory [9]. Following Dirac's approach, we attack the problem of an indefinite norm by finding a physical interpretation for the negative norm states. We claim that in *any* theory having an unbroken $\mathcal{PT}$ symmetry there exists a symmetry of the Hamiltonian connected with the fact that there are equal numbers of positive-norm and negative-norm states. To describe this symmetry we construct a linear operator denoted by $\mathcal{C}$ and represented in position space as a sum over the energy eigenstates of the Hamiltonian [10]:

$$\mathcal{C}(x,y)=\sum_n \phi_n(x)\phi_n(y). \tag{11}$$

As stated earlier, the properties of this new operator $\mathcal{C}$ resemble those of the charge conjugation operator in quantum field theory. For example, we can use (5)–(7) to verify that the square of $\mathcal{C}$ is unity ($\mathcal{C}^2=1$): $\int dy\,\mathcal{C}(x,y)\mathcal{C}(y,z)=\delta(x-z)$. Thus, the eigenvalues of $\mathcal{C}$ are ±1. Also, $\mathcal{C}$ commutes with the Hamiltonian H. Therefore, since $\mathcal{C}$ is linear, the eigenstates of H have definite values of $\mathcal{C}$. Specifically, if the energy eigenstates satisfy (7), then we have $\mathcal{C}\phi_n=(-1)^n\phi_n$ because

$$\mathcal{C}\phi_n(x)=\int dy\,\mathcal{C}(x,y)\phi_n(y)=\sum_m \phi_m(x)\int dy\,\phi_m(y)\phi_n(y).$$

We then use $\int dy\,\phi_m(y)\phi_n(y)=(\phi_m,\phi_n)$ according to our convention. Thus, $\mathcal{C}$ is the operator observable that represents the measurement of the signature of the $\mathcal{PT}$ norm of a state[3].

The operators $\mathcal{P}$ and $\mathcal{C}$ are distinct square roots of the unity operator $\delta(x-y)$. That is, $\mathcal{P}^2=\mathcal{C}^2=1$, but $\mathcal{P}\neq\mathcal{C}$. Indeed, $\mathcal{P}$ is real, while $\mathcal{C}$ is complex[4]. Furthermore, these two operators do not commute; in the position representation

$$(\mathcal{CP})(x,y)=\sum_n \phi_n(x)\phi_n(-y) \qquad \text{but} \qquad (\mathcal{PC})(x,y)=\sum_n \phi_n(-x)\phi_n(y), \tag{12}$$

which shows that $\mathcal{CP}=(\mathcal{PC})^*$. However, $\mathcal{C}$ *does* commute with $\mathcal{PT}$.

[3]The $\mathcal{PT}$ norm of a state determines its parity type. We can regard $\mathcal{C}$ as representing the operator that determines the $\mathcal{C}$ charge of the state. Quantum states having opposite $\mathcal{C}$ charge possess opposite parity type.

[4]The parity operator in coordinate space is explicitly real $\mathcal{P}(x,y)=\delta(x+y)$; the operator $\mathcal{C}(x,y)$ is complex because it is a sum of products of complex functions, as we see in (11). The complexity of the $\mathcal{C}$ operator can be seen explicitly in perturbative calculations of $\mathcal{C}(x,y)$ [11].

Finally, having obtained the operator $\mathcal{C}$ we define a new inner product structure having *positive definite* signature by

$$\langle f|g\rangle \equiv \int_{\rm C} dx\,[\mathcal{CPT}f(x)]g(x). \tag{13}$$

Like the $\mathcal{PT}$ inner product (6), this inner product is phase independent and conserved in time. This is because the time evolution operator, just as in ordinary quantum mechanics, is e^{iHt}. The fact that H commutes with the $\mathcal{PT}$ and the $\mathcal{CPT}$ operators implies that both inner products, (6) and (13), remain time independent as the states evolve in time. However, unlike (6), the inner product (13) is positive definite because $\mathcal{C}$ contributes -1 when it acts on states with negative $\mathcal{PT}$ norm. In terms of the $\mathcal{CPT}$ conjugate, the completeness condition (5) reads

$$\sum_n \phi_n(x)[\mathcal{CPT}\phi_n(y)] = \delta(x-y). \tag{14}$$

Unlike the inner product of conventional quantum mechanics, the $\mathcal{CPT}$ inner product (14) is *dynamically determined*; it depends implicitly on the Hamiltonian.

The operator $\mathcal{C}$ does not exist as a distinct entity in conventional quantum mechanics. Indeed, if we allow the parameter ϵ in (1) to tend to zero, the operator $\mathcal{C}$ in this limit becomes identical to $\mathcal{P}$. Thus, in this limit the $\mathcal{CPT}$ operator becomes $\mathcal{T}$, which is just complex conjugation. As a consequence, the inner product (13) defined with respect to the $\mathcal{CPT}$ conjugation reduces to the complex conjugate inner product of conventional quantum mechanics when $\epsilon \to 0$. Similarly, in this limit (14) reduces to the usual statement of completeness $\sum_n \phi_n(x)\phi_n^*(y) = \delta(x-y)$.

The $\mathcal{CPT}$ inner-product (13) is independent of the choice of integration contour C so long as C lies inside the asymptotic wedges associated with the boundary conditions for the Sturm–Liouville problem (2). Path independence follows from Cauchy's theorem and the analyticity of the integrand. In ordinary quantum mechanics, where the positive-definite inner product has the form $\int dx\, f^*(x)g(x)$, the integral must be taken along the real axis and the path of the integration cannot be deformed into the complex plane because the integrand is not analytic[5]. The $\mathcal{PT}$ inner product (6) shares with (13) the advantage of analyticity and path independence, but suffers from nonpositivity. We find it surprising that a positive-definite metric can be constructed using $\mathcal{CPT}$ conjugation without disturbing the path independence of the inner-product integral.

Finally, we explain why $\mathcal{PT}$-symmetric theories are unitary. Time evolution is determined by the operator e^{-iHt}, whether the theory is expressed in terms of a $\mathcal{PT}$-symmetric Hamiltonian or just an ordinary Hermitian Hamiltonian. To establish the global unitarity of a theory we must show that as a state vector evolves its norm does not change in time. If $\psi_0(x)$ is a prescribed initial wave function belonging to the Hilbert space spanned by the energy eigenstates, then it evolves into the state $\psi_t(x)$ at time t according to $\psi_t(x) = e^{-iHt}\psi_0(x)$. With respect to the $\mathcal{CPT}$ inner product defined in (13), the norm of the vector $\psi_t(x)$ does not change in time, $\langle\psi_t|\psi_t\rangle = \langle\psi_0|\psi_0\rangle$, because the Hamiltonian H commutes with the $\mathcal{CPT}$ operator.

Establishing unitarity at a local level is harder. Here, we must show that in coordinate space there exists a local probability density that satisfies a continuity equation so that the probability does not leak away. This is a subtle result because the probability current flows in the complex plane rather than along the real axis as in conventional Hermitian quantum mechanics. Preliminary numerical studies indeed indicate that the continuity equation is fulfilled [12].

[5]If a function satisfies a linear ordinary differential equation, then the function is analytic wherever the coefficient functions of the differential equation are analytic. The Schrödinger equation (3) is linear and its coefficients are analytic except for a branch cut at the origin; this branch cut can be taken to run up the imaginary axis. We choose the integration contour for the inner product (7) so that it does not cross the positive imaginary axis. Path independence occurs because the integrand of the inner product (7) is a product of analytic functions.

3 Illustrative example: A 2×2 matrix Hamiltonian

We illustrate the above results concerning $\mathcal{PT}$-symmetric quantum mechanics using the finite-dimensional 2×2 matrix Hamiltonian

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \tag{15}$$

where the three parameters r, s, and θ are real. This Hamiltonian is not Hermitian in the usual sense, but it is $\mathcal{PT}$ symmetric, where the parity operator is [13]

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

and $\mathcal{T}$ performs complex conjugation.

There are two parametric regions for this Hamiltonian. When $s^2 < r^2 \sin^2\theta$, the energy eigenvalues form a complex conjugate pair. This is the region of broken $\mathcal{PT}$ symmetry. On the other hand, if $s^2 \geq r^2 \sin^2\theta$, then the eigenvalues $\varepsilon_\pm = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta}$ are real. This is the region of unbroken $\mathcal{PT}$ symmetry. In the unbroken region the simultaneous eigenstates of the operators H and $\mathcal{PT}$ are

$$|\varepsilon_+\rangle = \frac{1}{\sqrt{2\cos\alpha}} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} \qquad \text{and} \qquad |\varepsilon_-\rangle = \frac{i}{\sqrt{2\cos\alpha}} \begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2} \end{pmatrix}, \tag{17}$$

where we set $\sin\alpha = (r/s)\sin\theta$. It is easily verified that $(\varepsilon_\pm, \varepsilon_\pm) = \pm 1$ and that $(\varepsilon_\pm, \varepsilon_\mp) = 0$, recalling that $(u, v) = (\mathcal{PT}u)\cdot v$. Therefore, with respect to the $\mathcal{PT}$ inner product, the resulting vector space spanned by energy eigenstates has a metric of signature $(+,-)$. The condition $s^2 > r^2\sin^2\theta$ ensures that $\mathcal{PT}$ symmetry is not broken. If this condition is violated, the states (17) are no longer eigenstates of $\mathcal{PT}$ because α becomes imaginary[6].

Next, we construct the operator $\mathcal{C}$:

$$\mathcal{C} = \frac{1}{\cos\alpha} \begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}. \tag{18}$$

Note that $\mathcal{C}$ is distinct from H and $\mathcal{P}$ and has the key property that $\mathcal{C}|\varepsilon_\pm\rangle = \pm|\varepsilon_\pm\rangle$. The operator $\mathcal{C}$ commutes with H and satisfies $\mathcal{C}^2 = 1$. The eigenvalues of $\mathcal{C}$ are precisely the signs of the $\mathcal{PT}$ norms of the corresponding eigenstates. Using the operator $\mathcal{C}$ we construct the new inner product structure $\langle u|v\rangle = (\mathcal{CPT}u)\cdot v$. This inner product is positive definite because $\langle\varepsilon_\pm|\varepsilon_\pm\rangle = 1$. Thus, the two-dimensional Hilbert space spanned by $|\varepsilon_\pm\rangle$, with inner product $\langle\cdot|\cdot\rangle$, has a Hermitian structure with signature $(+,+)$.

Let us demonstrate explicitly that the $\mathcal{CPT}$ norm of any vector is positive. For the arbitrary vector $\psi = \binom{a}{b}$, where a and b are any complex numbers, we see that $\mathcal{T}\psi = \binom{a^*}{b^*}$, that $\mathcal{PT}\psi = \binom{b^*}{a^*}$, and that $\mathcal{CPT}\psi = \frac{1}{\cos\alpha}\binom{a^* + ib^*\sin\alpha}{b^* - ia^*\sin\alpha}$. Thus, $\langle\psi|\psi\rangle = (\mathcal{CPT}\psi)\cdot\psi = \frac{1}{\cos\alpha}[a^*a + b^*b + i(b^*b - a^*a)\sin\alpha]$. Now let $a = x + iy$ and $b = u + iv$, where x, y, u, and v are real. Then

$$\langle\psi|\psi\rangle = \left(x^2 + v^2 + 2xv\sin\alpha + y^2 + u^2 - 2yu\sin\alpha\right)/\cos(\alpha), \tag{19}$$

which is explicitly positive and vanishes only if $x = y = u = v = 0$.

Since $\langle u|$ denotes the $\mathcal{CPT}$-conjugate of $|u\rangle$, the completeness condition reads

$$|\varepsilon_+\rangle\langle\varepsilon_+| + |\varepsilon_-\rangle\langle\varepsilon_-| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{20}$$

[6]When $\mathcal{PT}$ symmetry is broken, we find that the $\mathcal{PT}$ norm of the energy eigenstate vanishes.

Furthermore, using the $\mathcal{CPT}$ conjugate $\langle\varepsilon_\pm|$, we have $\mathcal{C}$ as $\mathcal{C} = |\varepsilon_+\rangle\langle\varepsilon_+| - |\varepsilon_-\rangle\langle\varepsilon_-|$, instead of the representation in (11), which uses the $\mathcal{PT}$ conjugate.

We represent an observable in this theory by a $\mathcal{CPT}$-invariant operator; that is, one that commutes with $\mathcal{CPT}$. Thus, if $\mathcal{CPT}$ symmetry is not broken, the eigenvalues of the observable are real. The operator $\mathcal{C}$ satisfies this requirement, so it is an observable. For the two-state system, if we set $\theta = 0$, then the Hamiltonian (15) becomes Hermitian. However, $\mathcal{C}$ then reduces to the parity operator $\mathcal{P}$. As a consequence, the requirement of $\mathcal{CPT}$ invariance reduces to the standard condition of Hermiticity for a symmetric matrix, namely, that $H = H^*$. This is why the hidden symmetry $\mathcal{C}$ was not noticed previously. The operator $\mathcal{C}$ emerges only when we extend a real symmetric Hamiltonian into the complex domain.

We have also calculated the $\mathcal{C}$ operator in infinite-dimensional quantum mechanical models. For an $x^2 + ix^3$ potential $\mathcal{C}$ can be obtained from the summation in (11) using perturbative methods and for an $x^2 - x^4$ potential $\mathcal{C}$ can be calculated using nonperturbative methods [11].

4 Applications and possible observable consequences

Could non-Hermitian, $\mathcal{PT}$-symmetric Hamiltonians be used to describe experimentally observable phenomena? Non-Hermitian Hamiltonians have *already* been used to describe interacting systems. Wu showed that the ground state of a Bose system of hard spheres is described by a non-Hermitian Hamiltonian [14]. Wu found that the ground-state energy of this system is real and conjectured that all energy levels were real. Hollowood showed that even though the Hamiltonian of a complex Toda lattice is non-Hermitian, the energy levels are real [15]. Non-Hermitian Hamiltonians of the form $H = p^2 + ix^3$ also arise in various Reggeon field theory models that exhibit real positive spectra [16]. In these examples the fact that a non-Hermitian Hamiltonian had a real spectrum appeared mysterious at the time, but now we know why: In each case the non-Hermitian Hamiltonian is $\mathcal{PT}$-symmetric. In each case the Hamiltonian constructed so that the position operator x or the field operator ϕ is always multiplied by i.

An experimental signal of a complex Hamiltonian might be found in the context of condensed matter physics. Consider the complex crystal lattice whose potential is $V(x) = i\sin x$. While the Hamiltonian $H = p^2 + i\sin x$ is not Hermitian, it is $\mathcal{PT}$-symmetric, and all of the energy bands are *real*. However, at the edge of the bands the wave function of a particle in such a lattice is always bosonic (2π-periodic) and, unlike the case of ordinary crystal lattices, the wave function is never fermionic (4π-periodic) [17]. Direct observation of such a band structure would give unambiguous evidence of a $\mathcal{PT}$-symmetric Hamiltonian.

There are many opportunities for the use of non-Hermitian Hamiltonians in the study of quantum field theory. For example, a scalar quantum field theory with a cubic self-interaction described by the Lagrangian $\mathcal{L} = \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + g\varphi^3$ is physically unacceptable because the energy spectrum is not bounded below. However, the cubic scalar quantum field theory that corresponds to H in (1) with $\epsilon = 1$ is given by the Lagrangian density $\mathcal{L} = \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 + ig\varphi^3$. This is a new, physically acceptable quantum field theory. Moreover, the theory that corresponds to H in (1) with $\epsilon = 2$ is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 - \frac{1}{4}g\varphi^4. \tag{21}$$

This "wrong-sign" field theory is remarkable because, in addition to the energy spectrum being real and positive, the one-point Green's function (the vacuum expectation value of the field φ) is *nonzero* [18]. Also, the field theory is renormalizable, and in four dimensions is asymptotically free (and thus nontrivial) [19]. Based on these features, we believe that the theory may provide a setting to describe the dynamics of the Higgs sector in the standard model.

Other field theory models whose Hamiltonians are non-Hermitian and $\mathcal{PT}$-symmetric have also been studied. For example, $\mathcal{PT}$-symmetric electrodynamics is particularly interesting because it is asymptotically free (unlike ordinary electrodynamics) and because the direction of the Casimir force is the negative of that in ordinary electrodynamics [20]. This theory is remarkable because it can determine its own coupling constant. Supersymmetric $\mathcal{PT}$-symmetric quantum field theories have also been studied [21].

We have found that $\mathcal{PT}$-symmetric quantum theories exhibit surprising and new phenomena. For example, for g sufficiently small the $-g\varphi^4$ theory described by the Lagrangian (21) possesses bound states (the conventional $g\varphi^4$ theory does not because the potential is repulsive). The bound states occur for all dimensions $0 \leq D < 3$ [22], but for purposes of illustration we describe the bound states in the context of one-dimensional quantum field theory (quantum mechanics). For the conventional anharmonic oscillator, which is described by the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + \frac{1}{4}gx^4 \qquad (g > 0), \tag{22}$$

the small-g Rayleigh-Schrödinger perturbation series for the kth energy level E_k is

$$E_k \sim m\left[k + \frac{1}{2} + \frac{3}{4}(2k^2 + 2k + 1)\nu + \mathrm{O}\left(\nu^2\right)\right] \qquad (\nu \to 0^+), \tag{23}$$

where $\nu = g/(4m^3)$. The *renormalized mass* M is defined as the first excitation above the ground state: $M \equiv E_1 - E_0 \sim m\big[1 + 3\nu + \mathrm{O}\big(\nu^2\big)\big]$ as $\nu \to 0^+$.

To determine if the two-particle state is bound, we examine the second excitation above the ground state using (23). We define

$$B_2 \equiv E_2 - E_0 \sim m\left[2 + 9\nu + \mathrm{O}\left(\nu^2\right)\right] \qquad (\nu \to 0^+). \tag{24}$$

If $B_2 < 2M$, then a two-particle bound state exists and the (negative) binding energy is $B_2 - 2M$. If $B_2 > 2M$, then the second excitation above the vacuum is interpreted as an unbound two-particle state. From (24) we see that in the small-coupling region, where perturbation theory is valid, the conventional anharmonic oscillator does not possess a bound state. Indeed, using WKB, variational methods, or numerical calculations, one can show that there is no two-particle bound state for any value of $g > 0$. Because there is no bound state the gx^4 interaction may be considered to represent a repulsive force[7].

The perturbation series for the non-Hermitian, $\mathcal{PT}$-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 - \frac{1}{4}gx^4 \qquad (g > 0), \tag{25}$$

is obtained from the perturbation series for the conventional anharmonic oscillator by replacing $\nu \to -\nu$. Thus, while the conventional anharmonic oscillator does not possess a two-particle bound state, the $\mathcal{PT}$-symmetric oscillator does indeed possess such a state. We measure the binding energy of this state in units of the renormalized mass M and we define the *dimensionless* binding energy Δ_2 by

$$\Delta_2 \equiv (B_2 - 2M)/M \sim -3\nu + \mathrm{O}\left(\nu^2\right) \qquad (\nu \to 0^+). \tag{26}$$

This bound state disappears when ν increases beyond $\nu = 0.0465$. As ν continues to increase, Δ_2 reaches a maximum value of 0.427 at $\nu = 0.13$ and then approaches 0.28 as $\nu \to \infty$.

[7] In general, a repulsive force in a quantum field theory is represented by an energy dependence in which the energy of a two-particle state decreases with separation. The conventional anharmonic oscillator Hamiltonian corresponds to a field theory in one space-time dimension, where there cannot be any spatial dependence. In this case the repulsive nature of the force is understood to mean that the energy B_2 needed to create two particles at a given time is more than twice the energy M needed to create one particle.

In the $\mathcal{PT}$-symmetric anharmonic oscillator, there are not only two-particle bound states but also k-particle bound states for all $k \geq 2$. The dimensionless binding energies are

$$\Delta_k \equiv (B_k - kM)/M \sim -3k(k-1)\nu/2 + \mathrm{O}\left(\nu^2\right) \qquad (\nu \to 0+). \tag{27}$$

The coefficient of ν is negative. Since the dimensionless binding energy becomes negative as ν increases from 0, there is a k-particle bound state. The higher multiparticle bound states cease to be bound for smaller values of ν; starting with the three-particle bound state, the binding energy of these states becomes positive as ν increases past 0.039, 0.034, 0.030, and 0.027.

Thus, for any value of ν there are always a finite number of bound states and an infinite number of unbound states. The number of bound states decreases with increasing ν until there are no bound states at all. There is a range of ν for which there are only two- and three-particle bound states, just like the physical world in which one observes only states of two and three bound quarks. In this range of ν if one has an initial state containing a number of particles (renormalized masses), these particles will clump together into bound states, releasing energy in the process. Depending on the value of ν, the final state will consist either of two- or of three-particle bound states, whichever is energetically favored. There is a special value of ν for which two- and three-particle bound states can exist in thermodynamic equilibrium.

How does a $g\varphi^3$ theory compare with a $g\varphi^4$ theory? A $g\varphi^3$ theory has an attractive force. Bound states arising as a consequence of this force can be found by using the Bethe–Salpeter equation. However, the $g\varphi^3$ field theory is unacceptable because the spectrum is not bounded below. If we replace g by ig, the spectrum becomes real and positive, but now the force becomes repulsive and there are no bound states. The same is true for a two-scalar theory with interaction of the form $ig\varphi^2\chi$, which is a model of scalar electrodynamics that has no analog of positronium.

Another feature of $\mathcal{PT}$-symmetric quantum field theory that distinguishes it from conventional quantum field theory is the commutation relation between the $\mathcal{P}$ and $\mathcal{C}$ operators. If we write $\mathcal{C} = \mathcal{C}_{\mathrm{R}} + i\mathcal{C}_{\mathrm{I}}$, where $\mathcal{C}_{\mathrm{R}}$ and $\mathcal{C}_{\mathrm{I}}$ are real, then $\mathcal{C}_{\mathrm{R}}\mathcal{P} = \mathcal{P}\mathcal{C}_{\mathrm{R}}$ and $\mathcal{C}_{\mathrm{I}}\mathcal{P} = -\mathcal{P}\mathcal{C}_{\mathrm{I}}$. These commutation and anticommutation relations suggest the possibility of interpreting $\mathcal{PT}$-symmetric quantum field theory as describing both bosonic and fermionic degrees of freedom, an idea analogous to the supersymmetric quantum theories. The distinction here, however, is that the supersymmetry can be broken; that is, bosonic and fermionic counterparts can have different masses without breaking the $\mathcal{PT}$ symmetry. Therefore, another possible observable experimental consequence might be the breaking of the supersymmetry.

5 Concluding remarks

We have described an alternative to the axiom of standard quantum mechanics that the Hamiltonian must be Hermitian. We have shown that Hermiticity may be replaced by the more physical condition of $\mathcal{PT}$ (space-time reflection) symmetry. Space-time reflection symmetry is distinct from the condition of Hermiticity, so it is possible to consider new quantum theories, such as quantum field theories whose self-interaction potentials are $ig\varphi^3$ or $-g\varphi^4$. Such theories have previously been thought to be mathematically and physically unacceptable because the spectrum might not be real and because the time evolution might not be unitary.

These new theories are complex extensions of ordinary quantum mechanics; they are continuations of real symmetric Hamiltonians to complex Hamiltonians. The idea of analytically continuing a Hamiltonian was first discussed by Dyson, who argued heuristically that perturbation theory for quantum electrodynamics diverges [23]. Dyson's argument involves rotating the electric charge e into the complex plane $e \to ie$. Applied to the anharmonic oscillator (22), Dyson's argument goes: If the coupling constant g is continued in the complex-g plane to $-g$, then the potential is no longer bounded below, so the resulting theory has no ground state. Thus, the ground-state energy $E_0(g)$ has an abrupt transition at $g = 0$. As a series in powers

of g, $E_0(g)$ must have a zero radius of convergence because $E_0(g)$ is singular at $g = 0$. Hence, the perturbation series must diverge for all $g \neq 0$. The perturbation series does indeed diverge, but this argument is flawed because the spectrum of the Hamiltonian (25) remains ambiguous until the boundary conditions that the wave functions must satisfy are specified. The spectrum depends crucially on how this Hamiltonian with a negative coupling constant is obtained.

There are two ways to obtain H in (25). First, one can substitute $g = |g|e^{i\theta}$ into (22) and rotate from $\theta = 0$ to $\theta = \pi$. Under this rotation, the ground-state energy $E_0(g)$ becomes complex. Evidently, $E_0(g)$ is real and positive when $g > 0$ and complex when $g < 0$[8]. Second, one can obtain (25) as a limit of the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}m^2x^2 + \frac{1}{4}gx^2(ix)^\epsilon \qquad (g > 0) \tag{28}$$

as $\epsilon : 0 \to 2$. The spectrum of this Hamiltonian is real, positive, and discrete. The spectrum of the limiting H in (25) obtained in this manner is similar in structure to that of H in (22).

How can H in (25) possess two such astonishingly different spectra? The answer lies in the boundary conditions on the wave functions $\phi_n(x)$. In the first case, in which $\theta = \arg g$ is rotated in the complex-g plane from 0 to π, $\psi_n(x)$ vanishes in the complex-x plane as $|x| \to \infty$ inside the wedges $-\pi/3 < \arg x < 0$ and $-4\pi/3 < \arg x < -\pi$. In the second case, in which the exponent ϵ ranges from 0 to 2, $\phi_n(x)$ vanishes in the complex-x plane as $|x| \to \infty$ inside the wedges $-\pi/3 < \arg x < 0$ and $-\pi < \arg x < -2\pi/3$. In this second case the boundary conditions hold in wedges that are symmetric with respect to the imaginary axis; these boundary conditions enforce the $\mathcal{PT}$ symmetry of H and are responsible for the reality of the energy spectrum.

Apart from the spectra, there is another striking difference between the two theories corresponding to H in (25). The one-point Green's function $G_1(g)$ is defined as the expectation value of the operator x in the ground-state wave function $\phi_0(x)$,

$$G_1(g) = \langle 0|x|0\rangle / \langle 0|0\rangle \equiv \int_C dx\, x\psi_0^2(x) \Big/ \int_C dx\, \psi_0^2(x), \tag{29}$$

where C is a contour that lies in the asymptotic wedges described above. The value of $G_1(g)$ for H in (25) depends on the limiting process by which we obtain H. If we substitute $g = g_0e^{i\theta}$ into the Hamiltonian (22) and rotate from $\theta = 0$ to $\theta = \pi$, we find that $G_1(g) = 0$ for all g on the semicircle in the complex-g plane. Thus, this rotation in the g plane preserves parity symmetry ($x \to -x$). However, if we define H in (25) by using the Hamiltonian in (28) and by allowing ϵ to range from 0 to 2, we find that $G_1(g) \neq 0$. Indeed, $G_1(g) \neq 0$ for *all* values of $\epsilon > 0$. Thus, in this theory $\mathcal{PT}$ symmetry (reflection about the imaginary axis, $x \to -x^*$) is preserved, but parity symmetry is permanently broken. Hence, one might be able to describe the dynamics of the Higgs sector by using a $-g\varphi^4$ quantum field theory.

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The Multiple Point Principle: Realized Vacuum in Nature is Maximally Degenerate

D.L. BENNETT † *and H.B. NIELSEN* ‡

† *Brookes Institute for Advanced Studies, Bøgevej 6, 2900 Hellerup, Denmark*
E-mail: *bennett@alf.nbi.dk*

‡ *The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*
E-mail: *hbech@alf.nbi.dk*

We put forward the multiple point principle as a fine-tuning mechanism that can explain why the parameters of the standard model have the values observed experimentally. The principle states that the parameter values realized in Nature coincide with the surface (e.g. the point) in the action parameter space that lies in the boundary that separates the maximum number of regulator-induced phases (e.g., the lattice artifact phases of a lattice gauge theory). We argue that a mild form of non-locality – namely that embodied in allowing diffeomorphism invariant contributions to the action – seems to appear in any fine-tuning problem. We demonstrate that the multiple point principle solution to fine-tuning has the very special property of avoiding the paradoxes that can arise in the presence of non-locality.

1 Introduction

The Multiple Point Principle (MPP) states that fundamental physical parameters assume values that correspond to having a maximal number of different coexisting "phases" for the physically realized vacuum. There is phenomenological evidence suggesting that some or all of the about 20 parameters in the Standard Model (SM) that are not predicted within the framework of the SM correspond to the MPP values of these parameters [1–3]. That these parameters take on special values (i.e., the multiple point values) poses from one viewpoint a fine tuning problem (why do constants of Nature take the MPP values). From another viewpoint, assuming the MPP as a law of Nature leads to a mechanism for fine-tuning. It is the latter viewpoint that is developed here. Moreover, we shall argue that a mild form of non locality is inherent to fine-tuning problems in general. We therefore develop a model for the relationship between fine-tuning, non-locality and the MPP.

2 Fine-tuning seems to require some form of non-locality

Explaining the (dressed) value of the cosmological constant is an example of a fine-tuning problem that would seem to require the breakdown of locality at least in a mild sense. As with any fine-tuning problem, the cosmological constant problem calls for a way to make the coupling dynamical in such a way that the values of such couplings are maintained at constant values (required for translational invariance). But this leads to a problem: if a coupling (e.g., the cosmological constant) is dynamical, the demands of a strictly local theory would be that the bare coupling can only depend on the space-time point in question and indirectly on the past but certainly not on the future. However, if the bare cosmological constant (that is to be dynamically maintained at a constant value) immediately following the big bang is to already have its value fine-tuned once and for all – to say 120 decimal places – to the value that makes the dressed cosmological constant so small as suggested phenomenologically, we definitely have a problem with locality.

The problem is that the bare cosmological constant is relatable to the value of the dressed cosmological constant only if the details of the dressed cosmological constant (that did not exist when the bare value was already tuned to the valued required for the dressed vacuum) that will evolve in the future are known at the time of big bang [4]. We are forced to conclude that a strict principle of locality is not allowed if we want to have a dynamically maintained bare coupling and renormalization group corrections of a quantum field theory with a well-defined vacuum.

This suggests models with a mild form of non-locality consisting of an interaction that is the same between any pair of points in space-time independent of the distance between these points. Assuming that this sort of non-locality is manifested through a non-local action $\hat{S}_{\rm nl}$, this symmetry between any pair of space-time points (i.e., identical interaction regardless of separation) is insured by requiring the invariance of $\hat{S}_{\rm nl}$ under diffeomorphisms (reparameterization invariance). The non-local action $\hat{S}_{nl}$ is a function of functionals $I_{f_j}[\phi(x)]$: $\hat{S}_{\rm nl} = \hat{S}_{\rm nl}(\{I_{f_j}[\phi(x)\})$ where $I_{f_j}[\phi(x)] \stackrel{\rm def}{=} \int dx^4\sqrt{g(x)}f_j(\phi(x))$ and $f_j(\phi)$ might typically be a Lagrange density e.g., $f_j(\phi) = \mathcal{L}_j(\phi(x), \partial_\mu\phi(x))$. The symbol $\phi(x)$ stands for all the fields (and derivatives of same) of the theory.

An example of a nonlocal action would be any nonlinear function of the (reparameterization invariant) functionals $I_{f_i}, I_{f_j}, \ldots$; e.g., a term $\int d^4x \int d^4y\sqrt{g(x)g(y)}\phi^2(x)\phi^4(y)$. Another example of a non-local (and nonlinear) action term more relevant to this paper is associated with having fixed values $I_{\text{fixed } f_j}$ (fixed in the sense of being a law of Nature) of some extensive quantities $I_{f_j}[\phi]$. This amounts to having a δ-function term $\exp(S_{\rm nl}(\{I_{f_j}\}) = \prod_j \delta(I_{f_j}[\phi] - I_{\text{fixed } f_j})$ in the functional integration measure and results in the non-locality that, strictly speaking, is inherent to any microcanonical ensemble (but which often is "approximated away" by using a canonical ensemble when phase space volume (or functional integration measure) is a sufficiently rapidly varying function of the extensive quantities).

An extensive quantity $I_{f_j}[\phi(x)]$ has a value for each imaginable Feynman path integral history of the Universe as it evolves from Big Bang to Big Crunch. The value $I_{\text{fixed } f_j}$ is by assumption "frozen in" and cannot change during the lifetime of the Universe. This unchangeable "choice" $I_{\text{fixed } f_j}$ then singles out a subset of all possible Feynman path integral histories that is consistent with the space-time evolution of our actually realized Universe having $I_{f_j}[\phi] = I_{\text{fixed } f_j}$.

An interaction that is the same between the fields at *any* pair of space-time points – regardless of separation – would not likely be perceived as a non-local interaction. Rather such space-time omnipresent fields – a sort of background that is forever everywhere the same – would likely be interpreted as simply constants of Nature. This feature is reminiscent of baby universe theory the essence of which is that a physical constant can depend on something and still be a constant as a function of space-time.

3 The Multiple Point Principle (MPP)

The MPP was originally put forward in connection with theoretical predictions for the values of the three gauge coupling constants [1, 2]. In addition to the assumption of the MPP, we also assumed in this first application of MPP our so-called Anti-GUT gauge group $G_{\rm Anti-GUT}$ which consists of the 3-fold replication of the Standard Model Group (SMG): $G_{\rm Anti-GUT} = {\rm SMG}\otimes{\rm SMG}\otimes{\rm SMG} \stackrel{\rm def}{=} {\rm SMG}^3$ (in the extended version: $({\rm SMG}\times U(1))^3$) having one SMG factor for each generation of fermions and gauge bosons. We postulate that $G_{\rm Anti-GUT}$ is broken to the diagonal subgroup (i.e., the usual SMG) at roughly the Planck scale.

In the original context of predicting the standard model gauge couplings using MPP (originally referred to as the principle of multiple point criticality), the principle asserts that the Planck scale values of the standard model gauge group couplings coincide with the multiple point, i.e., the point that lies in the boundary separating the maximum number of phases in

the action parameter space corresponding to the gauge group $G_{\mathrm{Anti-GUT}}$. The (Planck scale) predictions for the gauge couplings are subsequently identified with the parameter values at the point in the action parameter space for the diagonal subgroup of $G_{\mathrm{Anti-GUT}}$ that is inherited from the multiple point for $G_{\mathrm{Anti-GUT}}$ after the Planck scale breakdown of the latter.

The idea was developed in the context of lattice gauge theory and the phases to which we refer are usually dismissed as lattice artifacts (e.g., a Higgsed phase, a confined or Coulomb-like phase). Such phases have been studied extensively in the literature for simple gauge groups and semi simple gauge groups with discrete subgroups (e.g. $SU(2)$ and $SU(3)$). One typically finds first order phase transitions between confined and Coulomb-like phases at critical values of the action parameters.

Taking such lattice artifact phases as physical reflects our suspicion that such phases are inherent to having a regulator. As a regulator in some form (be it a lattice, strings or whatever) is always needed for the consistency of any quantum field theory, it is consistent to assume the existence of a fundamental regulator. The "artifact" phases that arise in a theory with such a fundamental regulator (that we have chosen to implement as a fundamental lattice) are accordingly taken as ontological phases that have physical significance at the scale of the fundamental regulator (e.g., lattice). The assumption of an ontological fundamental regulator implies the existence of monopoles in terms of which the regulator induced phase can also be studied [5].

Finding the multiple point in an action parameter space corresponding to the gauge group $G_{\mathrm{Anti-GUT}}$ is more complicated than for a single $SU(2)$ or $SU(3)$ say. The boundaries between phases in the action parameter space (i.e., the phase diagram) must be sought in a high dimensional parameter space essentially because $G_{\mathrm{Anti-GUT}}$ being a non-simple group has many subgroups and invariant subgroups.

In fact, there is a distinct phase for each subgroup pair (K, H), where K is a subgroup and H is an invariant subgroup such that $H \lhd K \subseteq G_{\mathrm{Anti-GUT}}$. An element $U \in G_{\mathrm{Anti-GUT}}$ can be parameterized as $U = U(g, k, h)$ where the Higgsed (gauge) degrees of freedom are elements g of the homogeneous space $G_{\mathrm{Anti-GUT}}/K$. The (un-Higgsed) Coulomb-like and confined degrees of freedom are respectively the elements k of the factor group K/H and the elements $h \in H$.

4 A familiar analogy to the MPP as a fine-tuning mechanism

Some important features of the MPP as a fine-tuning mechanism can be illustrated using an analogy to the familiar system in which the solid, liquid and vapor phases of water coexist. This occurs at the "triple point" of water, i.e., at the "triple point" values of temperature and pressure. Because the transitions between these three phases are all first order, there is a whole range of combinations of the extensive variables energy and volume for which the system can only be realized by having the coexistence of the ice, liquid and vapor phases. But these three phases coexist only for the triple point values of temperature and pressure, so there is a whole range of combinations of energy and volume that map onto the triple point values of the conjugate intensive variables temperature and pressure with the result that these variables are fine-tuned to the triple point values. In this illustrative analogy, the triple point of water in the phase diagram spanned by the intensive parameters temperature and pressure is analogous to the multiple point. As already stated, the multiple point is the (or a) point in the phase diagram that "touches" the maximum number of phases. In a phase diagram spanned by D intensive parameters (couplings), a generic multiple point can be in contact with up to $D + 1$ phases (in the illustrative example, $D = 2$ and the triple point is in contact with the $D + 1 = 3$ phases ice, liquid and vapour). In a non-generic situation, the multiple point can be in contact with more than $D + 1$ phases (e.g., accidently or due to symmetries).

For ease of illustration, consider now an even simpler system consisting of n_{H_2O} moles of H_2O in which just the ice and liquid phases coexist (at constant pressure). Such a system is unavoidably realized (and the temperature fine-tuned to 0° C $= 273.15^\circ$ K) for *any* value of the energy density $\rho_E = E/V_{n_{H_2O}}$ (E and $V_{n_{H_2O}}$ are respectively the energy and volume of the n_{H_2O} moles of H_2O) in the *finite* interval

$$\frac{n_{H_2O}}{V_{n_{H_2O}}} \int_{0^\circ \text{ K}}^{273^\circ \text{ K}} C_{p,\text{ice}}(T)dT < \rho_E < \frac{n_{H_2O}}{V_{n_{H_2O}}} \left(\int_{0^\circ \text{ K}}^{273^\circ \text{ K}} C_{p,\text{ice}}(T)dT + (\text{molar heat of melting}) \right)$$
($C_{p,\text{ice}}$ is the molar heat capacity of ice at constant pressure (e.g., 1 atm.)). (1)

For any ρ_E in this interval, the system cannot be realized as a single phase but rather only as an equilibriated mixture of ice and liquid water. Even choosing ρ_E at random there is a finite chance of landing in this interval in which case the temperature will be fine-tuned to 273.15° K.

5 The history of our universe as a fine tuner

Consider an analogy between the (3-dimensional) ice-water system with ρ_E in the interval of equation (1) and our 4-dimensional universe with the value of an extensive variable $I_{f_j}[\phi(x)] \stackrel{\text{def}}{=} \int dx^4 \sqrt{g(x)} f_j(\phi(x))$ (with f_j any function of ϕ – see also Section 2 for notation) primordially fixed at a value $I_{\text{fixed } f_j}$ that can only be realized as a combination of two (for the sake of example – really there could be more than two) coexisting phases i.e., two degenerate vacuum states at field values that we denote as ϕ_{us} and ϕ_{other} where we take $\phi_{\text{us}} < \phi_{\text{other}}$. Here we are anticipating the introduction of an effective potential V_{eff} that has relative minima at the field values ϕ_{us} and ϕ_{other}. In 4-space, one generic possibility for having coexistent phases would be to have a phase with ϕ_{us} in an early epoch including, say, the universe as we know it and a phase with ϕ_{other} in a later epoch:

$$I_{\text{fixed } f_j} = f_j(\phi_{\text{us}})(t_{\text{ignit}} - t_{BB})V_3 + f_j(\phi_{\text{other}})(t_{BC} - t_{\text{ignit}})V_3, \tag{2}$$

where t_{ignit} is the "ignition" time (in the future) at which there is a first-order phase transition from the vacuum at ϕ_{us} to the later vacuum at ϕ_{other}. V_3 is the 3-volume of the universe. The value of the "coupling constant" conjugate to $I_{\text{fixed } f_j}$ gets fine-tuned (unavoidably by assumption of the coexistence of the two phases separated by a first order transition) by a mechanism that also depends on a phase that will first be realized in the future (at t_{ignit}). Such a mechanism is non-local. Note, in particular, that the right-hand side of equation (2) depends on t_{ignit}.

In order to formally define a "coupling constant" (intensive quantity) conjugate to some extensive quantity (e.g., $I_{\text{fixed } f_j}$), we introduce non-locality more abstractly. Let us restrict the non-local action $\hat{S}_{\text{nl}} = \hat{S}_{\text{nl}}(\{I_j[\phi(x)]\})$ to being a (also reparameterization invariant) non-local potential V_{nl} that is a function of (not necessarily independent) functionals

$$V_{\text{nl}} \stackrel{\text{def}}{=} V_{\text{nl}}(I_{f_i}[\phi], I_{f_j}[\phi], \ldots).$$

Define now an effective potential V_{eff} such that

$$\frac{\partial V_{\text{eff}}(\phi(x))}{\partial \phi(x)} \stackrel{\text{def}}{=} \frac{\delta V_{\text{nl}}(\{I_{f_j}[\phi]\})}{\delta \phi(x)}\Big|_{\text{near min}} = \sum_i \left(\frac{\partial V_{\text{nl}}(\{I_{f_j}\})}{\partial I_{f_i}} \frac{\delta I_{f_i}[\phi]}{\delta \phi(x)} \right) \Big|_{\text{near min}}$$
$$= \sum_i \frac{\partial V_{nl}(\{I_{f_j}\})}{\partial I_{f_i}}\Big|_{\text{near min}} f_i'(\phi(x)). \tag{3}$$

The subscript "near min" denotes the approximate ground state of the whole universe, up to deviations of $\phi(x)$ from its vacuum value (or vacuum values for a multi-phase vacuum) by any amount in relatively small space-time regions. The solution to equation (3) is

$$V_{\rm eff}(\phi) = \sum_i \frac{\partial V_{\rm nl}(\{I_{f_j}\})}{\partial I_{f_i}} f_i(\phi). \tag{4}$$

We can identify the $\frac{\partial V_{\rm nl}(\{I_{f_j}\})}{\partial I_{f_i}}$ as intensive quantities conjugate to the I_{f_i}.

Consider now the effective potential (4) in the special case that $V_{\rm nl}(\{I_{f_j}\}) = V_{\rm nl}(I_2, I_4) \stackrel{\rm def}{=} V_{\rm nl}(\int d^4x\sqrt{g(x)}\phi^2(x), \int d^4y\sqrt{g(y)}\phi^4(y))$ in which case, (4) becomes

$$V_{\rm eff} = \frac{\partial V_{\rm nl}(I_2, I_4)}{\partial I_2}\phi^2(x) + \frac{\partial V_{\rm nl}(I_2, I_4)}{\partial I_4}\phi^4(x) \stackrel{\rm def}{=} \frac{1}{2}m^2_{\rm Higgs}\phi^2(x) + \frac{1}{4}\lambda\phi^4(x), \tag{5}$$

where the right-hand side of this equation, which also defines the (intensive) couplings $m^2_{\rm Higgs}$ and λ, is recognised as a prototype scalar potential at the tree level. Of course, the form of $V_{\rm nl}$ is, at least *a priori*, completely unknown to us, so – for example – the coupling constant $m^2_{\rm Higgs}$ cannot be calculated from equation (5). The potential of equation (5) with $m^2_{\rm Higgs} < 0$ has an asymmetric minimum – at, say, the value $\phi_{\rm us}$ resulting in spontaneous symmetry breakdown in the familiar way. This is just standard physics (without non-locality).

Actually, we want to consider the potential $V_{\rm eff}$ having the two relative minima $\phi_{\rm us}$ and $\phi_{\rm other}$ – both at nonvanishing values of ϕ – alluded to at the beginning of this section. The second minimum comes about at a value $\phi_{\rm other} > \phi_{\rm us}$ when radiative corrections to (5) are taken into account and the top quark mass is not too large [6, 7, 3]. Which of these vacua – the one at $\phi_{\rm us}$ or $\phi_{\rm other}$ – would be the stable one in this two-minima Standard Model effective Higgs field potential, depends on the value of $m^2_{\rm Higgs}$. Since I_2 and I_4 are functions of $t_{\rm ignit}$ (as seen from equation (2) with $f_j = \phi^2$ or ϕ^4), $m^2_{\rm Higgs} \stackrel{\rm def}{=} \frac{\partial V_{\rm nl}(\{I_2, I_4\})}{\partial I_2}$ is also a function of $t_{\rm ignit}$.

Let us first use "normal physics" to see how the relative depths of the two minima of the double well are related to $m^2_{\rm Higgs}$ and to $t_{\rm ignit}$. It can be deduced from [7] that a large negative value of $m^2_{\rm Higgs}$ corresponds to the relative minimum $V_{\rm eff}(\phi_{\rm other})$ being deeper than $V_{\rm eff}(\phi_{\rm us})$ (in which by assumption the Universe starts off following Big Bang) than for smaller negative values of $m^2_{\rm Higgs}$ (see Fig. 1). It can also be argued quite plausibly that a minimum in $V_{\rm eff}$ at $\phi_{\rm other}$ much deeper than that at $\phi_{\rm us}$ would correspond to an early (small) $t_{\rm ignit}$ inasmuch as the "false" vacuum at ϕ_{us} would be very unstable. However, as the value of the potential at $\phi_{\rm other}$ approaches that at $\phi_{\rm us}$, $t_{\rm ignit}$ becomes longer and longer, and approaches infinity as the values of $V_{\rm eff}$ at $\phi_{\rm us}$ and $\phi_{\rm other}$ become the same. The development of the double well potential and $m^2_{\rm Higgs}$ as a function of $t_{\rm ignit}$ is illustrated in Fig. 1. Note that the larger the difference $|V_{\rm eff}(\phi_{\rm other}) - V_{\rm eff}(\phi_{\rm us})|$ the more the realization of, say, $I_{\rm fixed\,2}$ will in general depend on $t_{\rm ignit}$. If $V_{\rm eff}(\phi_{\rm us}) = V_{\rm eff}(\phi_{\rm other})$, $t_{\rm ignit}$ plays no role in realizing e.g. $I_{\rm fixed\,2}$, and the value of $m^2_{\rm Higgs}$ becomes independent of $t_{\rm ignit}$.

6 Avoiding paradoxes arising from non-locality

In general, the presence of non-locality leads to paradoxes. While the form of the non-local action (or potential $V_{\rm nl}$ in this discussion) is unknown to us, we make the 4 generically representative guesses portrayed as the 4 non-locality curves in Fig. 1. In particular, non-locality curves having a negative slope as a function of $t_{\rm ignit}$ lead to paradoxes in the following manner. Consider the non-locality curve in Fig. 1 drawn with bold line that is redrawn in a rotated position in Fig. 2. Let us make the assumption that $t_{\rm ignit}$ is large and see that this leads to a contradiction. Assuming that $t_{\rm ignit}$ is large, it is seen from the non-locality function in Fig. 2 (call it

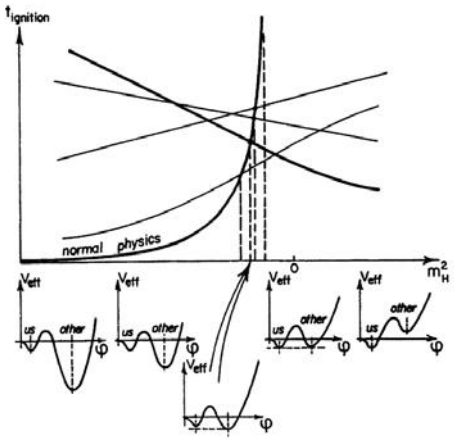

Figure 1. The development of the double well potential and $m_{\rm Higgs}$ as a function of $t_{\rm ignit}$. Note that all the more or less randomly drawn non-locality curves intersect the "normal physics" curve near where the vacua are degenerate (i.e., the MPP solution).

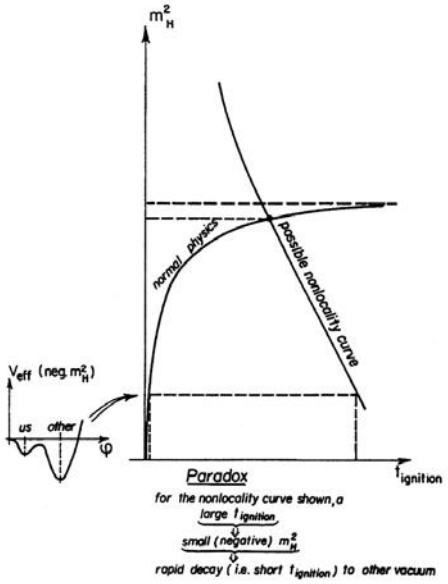

Figure 2. Many non-locality curves could lead to paradoxes similar to the "matricide" paradox. Such paradoxes are avoided if the value of $m_{\rm Higgs}$ is fine-tuned to the multiple point critical value. This corresponds to the intersection of the "normal physics" curve with the "possible non-locality" curve.

$m^2_{\rm Higgs\ nl}(t_{\rm ignit})$ to distinguish it from the "normal physics" $m^2_{\rm Higgs}(t_{\rm ignit})$) that this implies that the "normal physics" $m^2_{\rm Higgs}$ has a large negative value. But a large negative value of $m^2_{\rm Higgs}$ corresponds in "normal physics" to a (false) vacuum at $\phi_{\rm us}$ that is very unstable and therefore to a very short $t_{\rm ignit}$ corresponding to a rapid decay to the stable vacuum at $\phi_{\rm other}$. So the paradox appears: the assumption of a *large* $t_{\rm ignit}$ implies a *small* $t_{\rm ignit}$. This happens because in general $m^2_{\rm Higgs}(t_{\rm ignit}) \neq m^2_{\rm Higgs\ nl}(t_{\rm ignit})$ and is akin to the "matricide" paradox encountered for example when dealing with "time machines". It is well-known [8–10] that Nature avoids such paradoxes by choosing a very clever solution in situations where these paradoxes lure.

In the case of the paradoxes that can come about due to non-locality of the type considered here, a clever solution that avoids paradoxes is available to Nature in the form of the Multiple Point Principle (MPP). The MPP solution corresponds to the intersection of the "normal physics" curve and the "non-locality curve" in Fig. 2. because here the vacua at $\phi_{\rm us}$ and $\phi_{\rm other}$ are (essentially) degenerate. But at this intersection point, $m^2_{\rm Higgs}(t_{\rm ignit}) = m^2_{\rm Higgs\ nl}(t_{\rm ignit})$ so the paradox is avoided. So the paradox is avoided at the multiple point. But at the multiple point, an intensive parameter has its value fine-tuned for a wide range of values of the conjugate extensive quantity. Fine-tuning can therefore be understood as a consequence of Nature's way of avoiding paradoxes that can come about due to non-locality.

It should be pointed out that the paradox-free solution corresponding to the intersection of the two curves in Fig. 2 occurs for a value of $m^2_{\rm Higgs}$ corresponding to "our" vacuum at $\phi_{\rm us}$ being very slightly unstable. The value of $m^2_{\rm Higgs}$ corresponding to the vacua at $\phi_{\rm us}$ and $\phi_{\rm other}$ being (precisely) degenerate is slightly less negative than that corresponding to the multiple point value of $m^2_{\rm Higgs}$ at the intersection of the curves. Note that the multiple point value of $m^2_{\rm Higgs}$ is very insensitive to which "guess" we use for the non-local action. Indeed all the "non-locality" curves in Fig. 1 intersect the "normal physics" curve at values of $m^2_{\rm Higgs}$ that are tightly nested together. The reason for this is that $m^2_{\rm Higgs}$ is a very slowly varying function of $t_{\rm ignit}$ as

$m^2_{\rm Higgs}(t_{\rm ignit})$ approaches the value corresponding to degenerate vacua. The more nearly parallel the "normal physics" and the "non-locality" curves at the point of intersection, the smaller are the (paradoxical) effects of non-locality. For a point of intersection at values of $t_{\rm ignit}$ sufficiently large that (the "normal physics") $m^2_{\rm Higgs}(t_{\rm ignit}) \approx m^2_{\rm Higgs}(\infty)$, the non-locality effects disappear as the curves become parallel since both curves become independent of $t_{\rm ignit}$. If the curves were parallel, there would also be the possibility that these do not intersect, in which case there would be no "miraculous solution" that could avoid the paradoxes imbued in having non-locality.

If the interval $|\phi^2_{\rm other}–\phi^2_{\rm us}|$ is large (e.g. of the order of the largest physically conceivable scale (Planck?) if tuning is to be maximally effective) and if $I_{\rm fixed\,2}$ falls not too close to the ends of this interval, then $t_{\rm ignit}$ will be something of the order of half the life of the universe. Actually, the approximate degeneracy of the vacua $V_{\rm eff}(\phi_{\rm us}) \approx V_{\rm eff}(\phi_{\rm other})$ may be characteristic of the temperature of the post-Big Bang universe in the present epoch and *not* characteristic of the high temperature that prevailed immediately following the Big Bang. At high temperatures, the free energy is considerably smaller than the total energy if the entropy is large enough. A phase with a large number of light particles – for example a Coulomb-like vacuum such as the "us" phase in which we live – could very plausibly be so strongly favoured at high temperatures that other phases – for example the "other" vacuum – simply disappeared at the high temperature of the universe immediately following the Big Bang.

If this were to have depleted the universe of the phase having $\phi_{\rm other}$ at high temperatures, it would indeed be difficult to re-establish it in a lower temperature universe even if the vacuum at $\phi_{\rm us}$ were to be only meta-stable and the vacuum at $\phi_{\rm other}$ were the true vacuum at the lower temperature. Such an exchange of the true vacuum is indeed a possibility in going to lower temperatures inasmuch as the difference between the total energy and the free energy decreases in going to lower temperatures. Accordingly, this difference becomes less effective in favoring a Coulomb-like phase at the expense of a phase with heavier particles.

At this point we point out that when we say the "vacuum at $\phi_{\rm us}$" and "vacuum at $\phi_{\rm other}$" we are thinking of the approximation $\phi = \phi_{\rm us}$ and $\phi = \phi_{\rm other}$ almost everywhere, and at all times in respectively the early and late epochs of the universe in our discussion. More correctly, we should talk about vacuum densities $\langle\phi(x^0,\vec{x})\rangle_{\rm us}$ and $\langle\phi(x^0,\vec{x})\rangle_{\rm other}$ where

$$\langle\phi(x^0,\vec{x})\rangle_{\rm us} \stackrel{\rm def}{=} \frac{1}{V_{\rm us}}\int_{t_{BB}}^{t_{\rm ignit}} dx^0 \int d^3\vec{x}\sqrt{g(x)}\phi(x^0,\vec{x}),$$

where $V_{\rm us}$ denotes the the 4-volume of the universe in the first epoch. $\langle\phi(x^0,\vec{x})\rangle_{\rm other}$ is defined analogously. The more correct $\langle\phi(x^0,\vec{x})\rangle_{\rm us}$ is mentioned here so as not to confuse the reader when we talk about changes in $V_{\rm eff}(\phi_{\rm us}(x^0,\vec{x}))$ as the universe cools following Big Bang.

Recall now that the value of say of $I_{\rm fixed\,2}$ can easily (i.e. as a generic possibility) assume a value that requires that the universe to be in the "phase" with $\langle\phi(x^0,\vec{x})\rangle_{\rm other}$ during a sizeable part of its life, if the universe is to have multiple point parameters in the course of its evolution (as required for avoiding the paradoxes that accompany non-locality). How can Nature overcome the energy barrier that must be surmounted in order to bring about the decay of the slightly unstable (false) vacuum with $\langle\phi(x^0,\vec{x})\rangle_{\rm us}$ to the vacuum with $\langle\phi(x^0,\vec{x})\rangle_{\rm other}$? Even producing just a tiny "seed" of the "true" vacuum having $\langle\phi(x^0,\vec{x})\rangle_{\rm other}$ would be very difficult. What miraculously clever means can Nature devise so as to avoid deviations from a multiply critical evolution of the universe? One ingenious master plan that Nature may have implemented is the creation of life with the express "purpose" of evolving some (super intelligent?) physicists that could ignite a "vacuum bomb" by first creating in some very expensive accelerator the required "seed" of the "correct" vacuum having $\langle\phi(x^0,\vec{x})\rangle_{\rm other}$ that subsequently would engulf the universe in a (for us) cataclysmic transition to the "other" phase thereby permitting the continued evolution of a "paradox-free" universe!

7 Conclusion

We attempt to justify the assertion that fine-tuning in Nature seems to imply a fundamental form of non-local interaction. This could be manifested in a phenomenologically acceptable form as everywhere in space-time identical interactions between any pair of space-time points. This would be implemented by requiring the non-local action to be diffeomorphism invariant.

Next we put forth our multiple point principle which states that coupling parameters in the Standard Model tend to assume values that correspond to the values of action parameters lying at the junction of a maximum number of regulator induced phases (e.g., so-called "lattice artifact phases") separated from one another in action parameter space by first order transitions. The action which of course is defined on a gauge group (e.g., the non-simple SM gauge group) governs fluctuation patterns along the various subgroup combinations (K, H) with $H \lhd K \subseteq G$ that characterize the phases that come together at the multiple point. We then consider extensive quantities that are functions of functionals $I_{f_j}[\phi(x)]$ that are essentially Feymann path histories of the Universe for functions $f_j(\phi)$ of the fields $\phi(x)$ and derivatives of these fields. We then think of the generic situation in which these extensive quantities can happen to be fixed at values that require the universe to be realized as two or more coexisting phases. We draw on the analogy to the forced coexistence of ice and liquid water that occurs for a whole range of possible total energies because of the finite heat of melting (first order phase transition). With our multiple point principle, the intensive quantities (couplings) conjugate to extensive quantities fixed in this way become fine-tuned in a manner analogous to the fine tuning of temperature to 0° C when the total energy of a system of H_2O can only be realized as coexisting ice and liquid phases.

One generic way of having coexisting phases in a quantum field theory in 3+1 dimensions would be to have different phases in different epochs of the lifetime of the Universe with phase transitions occurring at various times in the course of the lifetime of the Universe. If the transitions were first order, one would have fine-tuning of (intensive) couplings conjugate to extensive quantity values that can only be realized by having coexisting (i.e., more than one) phases. But such a fine-tuning would involve non-locality: the fine-tuned values of coupling constants would depend on future phase transitions into phases that do not even exist at the time such couplings are fine-tuned.

Even non-locality of this sort (i.e., non-locally manifested as a diffeomorphism invariant non-local action) can lead to paradoxes of the "matricide paradox" type. We argue that such paradoxes are avoided when Nature chooses the multiple point principle solution to the problem of fine-tuning.

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Generalized Lorentz Symmetry and Nonlinear Spinor Fields in a Flat Finslerian Space-Time

George BOGOSLOVSKY † *and Hubert GOENNER* ‡

† *Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow, Russia*
E-mail: *bogoslov@theory.sinp.msu.ru*

‡ *Institute for Theoretical Physics, University of Göttingen, Göttingen, Germany*
E-mail: *goenner@theorie.physik.uni-goettingen.de*

The work is devoted to the generalization of the Dirac equation for a flat locally anisotropic, i.e. Finslerian space-time. At first we reproduce the corresponding metric and a group of the generalized Lorentz transformations, which has the meaning of the relativistic symmetry group of such event space. Next, proceeding from the requirement of the generalized Lorentz invariance we find a generalized Dirac equation in its explicit form. An exact solution of the nonlinear generalized Dirac equation is also presented.

1 Introduction

In spite of the impressive successes of the unified gauge theory of strong, weak and electromagnetic interactions, known as the Standard Model, one cannot a priori rule out the possibility that Lorentz symmetry underlying the theory is an approximate symmetry of nature. This implies that at the energies already attainable today empirical evidence may be obtained in favor of violation of Lorentz symmetry. At the same time it is obvious that such effects might manifest themselves only as strongly suppressed effects of Planck-scale physics.

Theoretical speculations about a possible violation of Lorentz symmetry continue for more than forty years and they are briefly outlined in [1]. Nevertheless we note here that, along with the spontaneous breaking [2], one of the first and, as it appeared subsequently, fruitful ideas relating to a possible violation of Lorentz symmetry was the idea [3] according to which the metric of event space differs from Minkowski metric and the physically equivalent inertial reference frames are linked by some transformations which differ from Lorentz ones. In [4] such transformations were called generalized Lorentz transformations. Note also that the idea about the existence of generalized Lorentz transformations was suggested in connection with the situation in the physics of ultra-high energy cosmic rays, namely, with the absence of the Greisen–Zatsepin–Kuz'min effect (the so-called GZK cutoff) predicted [5, 6] on the basis of conventional relativistic theory. The absence of the GZK cutoff has yet not been explained convincingly and still remains the main empirical fact which indirectly speaks in favor of violation of Lorentz symmetry.

Interest in the problem of violation of Lorentz and CPT symmetries has revived in recent years [7] in connection with the construction of a phenomenological theory reffered to as the Standard-Model Extension [8].

In the present work, which is in essence devoted to the same problem, we proceed from the assumption [9] that phase transitions with breaking of gauge symmetries should be accompanied by phase transitions in the geometric structure of space-time.

Our study is based on the fact [10] that the Lorentz symmetry is not the only possible realization of the relativistic symmetry. Another admissible realization of the relativistic symmetry is obtained with the aid of nonunimodular matrices belonging to a group of the generalized Lorentz

transformations. In contrast to the conventional Lorentz transformations, the generalized ones conformally modify Minkowski metric but leave invariant the corresponding Finslerian metric which describes a flat locally anisotropic space-time. Thus, from the formal point of view the locally anisotropic space-time appeares as the necessary consequence of the existence of a group of the generalized Lorentz transformations. As for the physical nature of the anisotropy, there are some reasons to suppose that a fermion-antifermion condensate, which may arise [11] (instead of elementary Higgs condensate) in the spontaneous breaking of initial gauge symmetries, turns out to be anisotropic and its anisotropy determines the local anisotropy of event space. Obviously, verification of this hypothesis is far from being trivial. Therefore the opening investigations in this direction, as presented here, are aimed at the most fundamental problem, namely, at the generalization of the Dirac equation for the locally anisotropic space-time.

2 Flat locally anisotropic space-time as a geometric invariant of a group of the generalized Lorentz transformations

Consider the metric [4] of a flat locally anisotropic space-time

$$ds^2 = \left[\frac{(dx_0 - \boldsymbol{\nu} d\boldsymbol{x})^2}{dx_0^2 - d\boldsymbol{x}^2}\right]^r \left(dx_0^2 - d\boldsymbol{x}^2\right). \tag{1}$$

Being not a quadratic form but a homogeneous function of the coordinate differentials of degree two, this metric falls within the category of Finsler metrics [12]. It depends on two constant parameters r and $\boldsymbol{\nu}$, in which case the unit vector $\boldsymbol{\nu}$ indicates a preferred direction in 3D space while r determines the magnitude of space anisotropy, characterizing the degree of deviation of the metric (1) from the Minkowski metric. Thus the anisotropic event space (1) is the generalization of the isotropic Minkowski space of conventional special relativity theory.

The 3-parameter noncompact group of the generalized Lorentz transformations, which leave the metric (1) invariant, appears as

$$x'^i = D(\boldsymbol{v}, \boldsymbol{\nu}) R^i_j(\boldsymbol{v}, \boldsymbol{\nu}) L^j_k(\boldsymbol{v}) x^k, \tag{2}$$

where $\boldsymbol{v}$ denotes the velocities of moving (primed) inertial reference frames; the matrices $L^j_k(\boldsymbol{v})$ represent the ordinary Lorentz boosts; the matrices $R^i_j(\boldsymbol{v}, \boldsymbol{\nu})$ represent additional rotations of the spatial axes of the moving frames around the vectors $[\boldsymbol{v}\,\boldsymbol{\nu}]$ through the angles

$$\varphi = \arccos\left\{1 - \frac{\left(1 - \sqrt{1 - \boldsymbol{v}^2/c^2}\right)[\boldsymbol{v}\boldsymbol{\nu}]^2}{(1 - \boldsymbol{v}\boldsymbol{\nu}/c)\boldsymbol{v}^2}\right\}$$

of relativistic aberration of $\boldsymbol{\nu}$; and the diagonal matrices

$$D(\boldsymbol{v}, \boldsymbol{\nu}) = \left(\frac{1 - \boldsymbol{v}\boldsymbol{\nu}/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}}\right)^r I,$$

the additional dilatational transformations of the event coordinates. The structure of the transformations (2) ensures the fact that in spite of a new geometry of event space the 3-velocity space remains Lobachevski space.

With the inclusion of the 1-parameter group of rotations about $\boldsymbol{\nu}$ and of the 4-parameter group of translations the inhomogeneous isometry group of the Finslerian event space (1) turns out to have eight parameters. If the third spatial axis is chosen along $\boldsymbol{\nu}$, then its generators can

be written as

$$
\begin{aligned}
X_1 &= -\left(x^1 p_0 + x^0 p_1\right) - \left(x^1 p_3 - x^3 p_1\right), \\
X_2 &= -\left(x^2 p_0 + x^0 p_2\right) + \left(x^3 p_2 - x^2 p_3\right), \\
X_3 &= -r x^i p_i - \left(x^3 p_0 + x^0 p_3\right), \\
R_3 &= x^2 p_1 - x^1 p_2; \qquad p_i = \partial/\partial x^i.
\end{aligned}
$$

These generators satisfy the commutation relations

$$
\begin{aligned}
&[X_1 X_2] = 0, && [R_3 X_3] = 0, \\
&[X_3 X_1] = X_1, && [R_3 X_1] = X_2, \\
&[X_3 X_2] = X_2, && [R_3 X_2] = -X_1; \\
&[p_i p_j] = 0; \\
&[X_1 p_0] = p_1, && [X_2 p_0] = p_2, && [X_3 p_0] = r p_0 + p_3, && [R_3 p_0] = 0, \\
&[X_1 p_1] = p_0 + p_3, && [X_2 p_1] = 0, && [X_3 p_1] = r p_1, && [R_3 p_1] = p_2, \\
&[X_1 p_2] = 0, && [X_2 p_2] = p_0 + p_3, && [X_3 p_2] = r p_2, && [R_3 p_2] = -p_1, \\
&[X_1 p_3] = -p_1, && [X_2 p_3] = -p_2, && [X_3 p_3] = r p_3 + p_0, && [R_3 p_3] = 0.
\end{aligned}
$$

As one can see the 8-parameter inhomogeneous isometry group of the space-time (1) is a subgroup of the 11-parameter extended Poincaré (similitude) group [13] whereas the homogeneous one is isomorphic to the corresponding 4-parameter subgroup (with the generators X_1, X_2, $X_3|_{r=0}$, R_3) of the homogeneous Lorentz group. It is shown in [14] that the 6-parameter homogeneous Lorentz group has no 5-parameter subgroup while the 4-parameter subgroup is unique (up to isomorphisms). Thus, the transition from Minkowski space to the Finslerian event space (1) implies a minimum of symmetry-breaking of the Lorentz symmetry. However the relativistic symmetry is maintained in the form of the generalized Lorentz symmetry.

3 Covariant formulation of the theory. Generalized relativistic point mechanics

Because of nonunimodularity of the matrices $\mathcal{L}^i_k = DR^i_j L^j_k$ representing the generalized Lorentz transformations (2), the transformational properties of some geometric entities turn out to be changed as compared with conventional special relativity theory. For instance, a 4-volume element $dx^0 d^3x$ is no longer invariant but is a scalar density of weight -1, i.e., it transforms as follows: $dx'^0 d^3x' = J^{-1} dx^0 d^3x$, where J is the Jacobian, $J = |\partial x^k/\partial x'^j| = |\mathcal{L}^{-1}{}^k_j| = D^{-4}$. Similarly, matrices η_{ik} and η^{ik} having the identical forms $\eta_{ik} = \mathrm{diag}\,(1,-1,-1,-1)$ and $\eta^{ik} = \mathrm{diag}\,(1,-1,-1,-1)$ in all frames of reference related by the transformations (2) are no longer invariant tensors but are, respectively, a covariant tensor density of weight $-1/2$ and a contravariant tensor density of weight $1/2$. This statement signifies that $\eta'_{ik} = J^{-1/2}\mathcal{L}^{-1}{}^l_i \mathcal{L}^{-1}{}^m_k \eta_{lm} = \eta_{ik}$ and $\eta'^{ik} = J^{1/2}\mathcal{L}^i_l \mathcal{L}^k_m \eta^{lm} = \eta^{ik}$. Then it is clear that $\eta_{ik}\eta^{kl} = \delta^l_i$ is a unit tensor. Later on we shall be using η_{ik} and η^{ik} to lower and raise indices. The process, however, will be accompanied by a change in weight. We shall be also in need of an entity ν^i which indicates a preferred direction in 4D space-time and whose components have the same values, $\{\nu^0 = 1, \boldsymbol{\nu}\}$, in all frames of reference related by the transformations (2). It is easy to verify that $\nu'^i = J^{(1+r)/(4r)}\mathcal{L}^i_k \nu^k = \nu^i$, i.e., that ν^i is a contravariant vector density of weight $(1+r)/(4r)$, in which case $\nu_i \nu^i = 0$ is an invariant equation.

Using the ν^i and η_{ik} one can represent the metric (1) as an explicit invariant of the transformations (2): $ds^2 = \left[(\nu_i dx^i)^2/dx_j dx^j\right]^r dx_k dx^k$. With the aid of this expression we arrive at the relativistically invariant action for a free particle in the flat anisotropic space. The action and

its variation appear as

$$S = -m\int_a^b ds, \qquad \delta S = -\int_a^b p_i\, d\delta x^i.$$

Hereafter we put $c = \hbar = 1$. The principle of least action under the condition $(\delta x^i)|_a = (\delta x^i)|_b = 0$ leads to $p_i = \text{const}$, i.e., to rectilinear inertial motion. And if one varies the coordinates of point b under the condition $p_i = \text{const}$, then $p_i = -\partial S/\partial x^i$, i.e., p_i is a canonical 4-momentum. Since η^{ik} is a contravariant tensor density of weight 1/2 and since p_k is a covariant vector it is clear that the 4-momentum $p^i = (p^0, \boldsymbol{p}) = \eta^{ik}p_k$ is transformed as a contravariant vector density of 1/2, i.e.,

$$p'^i = J^{1/2}\mathcal{L}^i_k p^k = D^{-1}R^i_l L^l_k p^k. \tag{3}$$

Thus we have arrived at the generalized Lorentz transformations for 4-momenta. Note that the dilatational transformations of p^i are inverse to those of x^i (cf. (2)). In an explicit form

$$p^i = m\left(\frac{\nu_l dx^l}{\sqrt{dx_j dx^j}}\right)^r \left((1-r)\frac{dx^i}{\sqrt{dx_a dx^a}} + r\frac{\nu^i\sqrt{dx_b dx^b}}{\nu_n dx^n}\right). \tag{4}$$

Since the direction of p^i is not aligned with the direction of dx^i we introduce (apart from p^i) the so-called kinematic 4-momentum k^i which has the same transformational properties as p^i.

$$k^i = m(1-r)\frac{dx^i}{\sqrt{dx_n dx^n}}\left(\frac{\nu_l dx^l}{\sqrt{dx_j dx^j}}\right)^r. \tag{5}$$

Taking into account the equation $\nu_i\nu^i = 0$ we obtain the following relations

$$p^i = k^i + \frac{rk_l k^l}{(1-r)k_n\nu^n}\nu^i, \qquad k^i = p^i - \frac{rp_l p^l}{(1+r)p_n\nu^n}\nu^i. \tag{6}$$

As for the 3-velocity of a particle, it is determined by the formula $\boldsymbol{v} = \boldsymbol{k}/k^0$.

The components of canonical 4-momenta satisfy the mass shell equation

$$\left[(\nu_i p^i)^2/p_j p^j\right]^{-r} p_a p^a = m^2(1-r)^{(1-r)}(1+r)^{(1+r)}. \tag{7}$$

This equation is an invariant of the transformations (3). The same mass shell but in a space of kinematic 4-momenta is described by the equation

$$\left[(\nu_i k^i)^2/k_j k^j\right]^{-r} k_a k^a = m^2(1-r)^2. \tag{8}$$

The latter two equations lead us to the important conclusion, namely, that the motion of free massless particles in anisotropic space is similar to their motion in isotropic space, i.e., massless particles do not perceive the space anisotropy whereas the motion of massive particles is analogous to that of quasiparticles in a crystalline medium.

According to (8), the mass shell for $r \neq 0$ is a deformed two-sheeted hyperboloid inscribed into a light cone. In order to show how its deformation changes with the magnitude r of space anisotropy it is reasonable to proceed from the relations (5) which determine four-dimensional coordinates of points belonging to the upper sheet of the deformed hyperboloid as explicit functions of 3-velocities $\boldsymbol{v} = d\boldsymbol{x}/dx^0$. The results of calculations, presented in Fig. 1, clearly demonstrate the fact that, if $r \to 1$, the mass shell in a space of kinematic 4-momenta converges (nonuniformly) to a light cone. As for the canonical momenta p^i, there is nonuniform convergence: $p^i \to k^i + m\nu^i$, where $k_i k^i = 0$. Physically this means that the effective inertial mass of a particle present in anisotropic (Finslerian) space depends on the magnitude r of a constant anisotropy field and disappears at all if r reaches the value equal to unity.

Thus, with a view to generalizing the Dirac Lagrangian for the Finslerian space-time, we have arrived at the following guiding principle: a generalized Lagrangian, in the limit $r = 1$, must be reducible (up to a 4-divergence) to the standard massless Dirac Lagrangian.

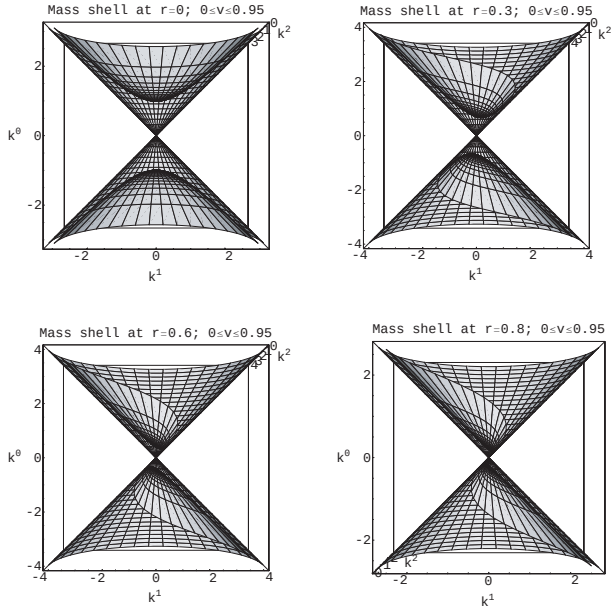

Figure 1. Parametric 3D plots of the mass shells in a space of kinematic 4-momenta.

4 Transformational properties of fermion-antifermion fields in the flat Finslerian space-time. Generalized Dirac equation and its planewave solutions

Starting to generalize the Dirac Lagrangian for the anisotropic Finslerian space-time (1), first consider the standard massless one $(i/2)\left(\bar{\psi}\gamma^n\partial_n\psi - \partial_n\bar{\psi}\gamma^n\psi\right)$. In the preceding Section we have drawn the conclusion that massless particles do not perceive the space anisotropy. This means that the Lagrangian considered need not be modified and it can be used as the kinetic term of a massive generalized Dirac Lagrangian. Since under the generalized Lorentz transformations (2) the 4-volume dx^0d^3x behaves as a scalar density of weight -1 and the action must remain invariant, it follows that the kinetic term (just as the entire Lagrangian) must be a scalar density of weight 1. This condition is fulfilled in the case where the generalized Lorentz transformations (2) of the coordinates are accompanied by the following transformations of the fields ψ and $\bar{\psi}$:

$$\psi'(x') = D^{-3/2}S_RS_L\psi(x) = J^{3/8}S\psi(x), \tag{9}$$

$$\bar{\psi}'(x') = \bar{\psi}(x)J^{3/8}S^{-1}, \tag{10}$$

where the matrices $S = S_RS_L$ satisfy the standard condition $S^{-1}\gamma^iS = \Lambda^i_k\gamma^k$, in which case $\Lambda^i_k = R^i_jL^j_k$; the matrices S_L and S_R represent, respectively, the Lorenz boosts and additional rotations of bispinors. In an explicit form

$$S_L = \sqrt{\frac{1+\sqrt{1-\boldsymbol{v}^2}}{2\sqrt{1-\boldsymbol{v}^2}}}\,I - \sqrt{\frac{1-\sqrt{1-\boldsymbol{v}^2}}{2\sqrt{1-\boldsymbol{v}^2}}}\,\frac{\gamma^0\boldsymbol{v}\boldsymbol{\gamma}}{|\boldsymbol{v}|}, \tag{11}$$

$$S_R = \sqrt{1-\frac{\left(1-\sqrt{1-\boldsymbol{v}^2}\right)[\boldsymbol{v}\boldsymbol{\nu}]^2}{2(1-\boldsymbol{v}\boldsymbol{\nu})\,\boldsymbol{v}^2}}\,I + i\sqrt{\frac{1-\sqrt{1-\boldsymbol{v}^2}}{2(1-\boldsymbol{v}\boldsymbol{\nu})}}\,\frac{[\boldsymbol{v}\boldsymbol{\nu}]}{|\boldsymbol{v}|}\,\boldsymbol{\Sigma}, \tag{12}$$

where $\boldsymbol{v}$ denotes the velocities of moving (primed) reference frames, γ^n are the Dirac matrices, $\boldsymbol{\Sigma} = \mathrm{diag}\,(\boldsymbol{\sigma},\boldsymbol{\sigma})$ and $\boldsymbol{\sigma}$ are the Pauli matrices. Thus in the flat Finslerian space-time (1) the fields ψ and $\bar{\psi}$ are, according to (9) and (10), bispinor density fields of weight 3/8.

In order to generalize the massive term $-m\bar\psi\psi$ of the Dirac Lagrangian we remind that a generalized massive term, like the kinetic one, must be a scalar density of weight 1. It can be verified that for the bispinor density fields: $\bar\psi\psi$ is a scalar density of weight 3/4, $\bar\psi\gamma^n\psi$ is a contravariant vector density of weight 1, $[(\nu_n\bar\psi\gamma^n\psi/\bar\psi\psi)^2]^{r/2}\bar\psi\psi$ is a scalar density of weight 1 and $[(\nu_n\bar\psi\gamma^n\psi/\bar\psi\psi)^2]^{-3r/2}\bar\psi\psi$ is a scalar, in which case the latter Finslerian form generalizes the scalar bilinear form $\bar\psi\psi$ of conventional theory.

Now we are able to write down a Lagrangian for the bispinor density fields representing such a generalization of the standard Dirac Lagrangian that the corresponding field equations turn out to be invariant under the group of generalized Lorentz transformations. It appears as

$$\mathcal{L}=\frac{i}{2}\left(\bar\psi\gamma^n\partial_n\psi-\partial_n\bar\psi\gamma^n\psi\right)-m\left[\left(\frac{\nu_n\bar\psi\gamma^n\psi}{\bar\psi\psi}\right)^2\right]^{r/2}\bar\psi\psi.$$

This Lagrangian leads to the following generalized Dirac equations:

$$i\gamma^a\partial_a\psi-m\left[\left(\frac{\nu_nj^n}{\bar\psi\psi}\right)^2\right]^{\frac{r}{2}}\left\{(1-r)I+r\left(\frac{\bar\psi\psi}{\nu_nj^n}\right)\nu_a\gamma^a\right\}\psi=0, \tag{13}$$

$$i\partial_a\bar\psi\gamma^a+m\left[\left(\frac{\nu_nj^n}{\bar\psi\psi}\right)^2\right]^{\frac{r}{2}}\bar\psi\left\{(1-r)I+r\left(\frac{\bar\psi\psi}{\nu_nj^n}\right)\nu_a\gamma^a\right\}=0, \tag{14}$$

where $j^n=\bar\psi\gamma^n\psi$. The operation: $\bar\psi$ (13) + (14) ψ provides the equation $\partial_nj^n=0$. Thus j^n is a preserved current. And at last, owing to the operation: $\bar\psi$ (13)–(14) ψ, we conclude that $\mathcal{L}=0$ on the solutions of equations (13) and (14).

Due to translational invariance, the generalized Dirac equations (13), (14) must admit solutions in the form of plane waves $\psi(x)=u(p)\exp(-ip_ax^a)$. This means that the amplitude $u(p)$ must satisfy the equations:

$$p_a\left[\gamma^a-\left(\frac{\bar u\gamma^au}{\bar uu}\right)\left\{(1-r)I+r\left(\frac{\bar uu}{\nu_l\bar u\gamma^lu}\right)\nu_n\gamma^n\right\}\right]u=0, \tag{15}$$

$$\bar up_a\left[\gamma^a-\left(\frac{\bar u\gamma^au}{\bar uu}\right)\left\{(1-r)I+r\left(\frac{\bar uu}{\nu_l\bar u\gamma^lu}\right)\nu_n\gamma^n\right\}\right]=0, \tag{16}$$

$$p_a\bar u\gamma^au=m\left[\left(\frac{\nu_l\bar u\gamma^lu}{\bar uu}\right)^2\right]^{r/2}\bar uu. \tag{17}$$

Equations (15)–(17) lead to the invariant dispersion relation

$$\sqrt{p_ap^a/(1-r^2)}=\pm m\left[(1+r)(\nu_ip^i)^2/((1-r)p_jp^j)\right]^{r/2}, \tag{18}$$

where the sign + corresponds to positive frequency states whereas the sign − corresponds to negative ones. It is worth mentioning that the mass shell equation (7) can be obtained from (18). In order to find the planewave solutions in a general form, i.e. at arbitrary momentum p^a we, for a start, confine ourselves to the rest frame and try the following ansatz:

$$\psi_+(x)=\sqrt{2m(1-r)}\begin{pmatrix}\varphi\\0\end{pmatrix}e^{-i(mx^0-mr\boldsymbol{\nu x})},\quad \psi_-(x)=\sqrt{2m(1-r)}\begin{pmatrix}0\\\chi\end{pmatrix}e^{i(mx^0-mr\boldsymbol{\nu x})},$$

where $\sqrt{2m(1-r)}$ is a normalizing multiplier and φ, χ are arbitrary 3-spinors normalized by means of $\varphi^\dagger\varphi=1$, $\chi^\dagger\chi=1$. It is easy to verify that the corresponding positive and negative frequency bispinor density amplitudes satisfy equations (15)–(17). Note once more that these

solutions are found in the rest frame, in which $p^a = \{m, rm\boldsymbol{\nu}\}$, whereas kinematic 4-momentum $k^a = \{m(1-r), \mathbf{0}\}$ and, respectively, $\boldsymbol{v} = \boldsymbol{k}/k^0 = 0$. Taking into account the transformational properties (9)–(12) we find planewave solutions of equation (13) in the final form:

$$\psi_+(x) = \frac{\sqrt{k_0^2 - \boldsymbol{k}^2}}{m(1-r)} \begin{pmatrix} \sqrt{k_0 + \sqrt{k_0^2 - \boldsymbol{k}^2}}\ \varphi \\ \sqrt{k_0 - \sqrt{k_0^2 - \boldsymbol{k}^2}}\ (\boldsymbol{n}\boldsymbol{\sigma})\varphi \end{pmatrix} e^{-ip_a x^a},$$

$$\psi_-(x) = \frac{\sqrt{k_0^2 - \boldsymbol{k}^2}}{m(1-r)} \begin{pmatrix} \sqrt{k_0 - \sqrt{k_0^2 - \boldsymbol{k}^2}}\ (\boldsymbol{n}\boldsymbol{\sigma})\chi \\ \sqrt{k_0 + \sqrt{k_0^2 - \boldsymbol{k}^2}}\ \chi \end{pmatrix} e^{ip_a x^a},$$

where the unit vector $\boldsymbol{n}$ indicates the direction of $\boldsymbol{k}$, in which case k^a and p^a are related by (6). The bispinor density fields $\psi_\pm$ are normalized with the help of the invariant conditions: $[(\nu_n \bar{\psi}_\pm \gamma^n \psi_\pm / \bar{\psi}_\pm \psi_\pm)^2]^{-3r/2} \bar{\psi}_\pm \psi_\pm = \pm 2m(1-r)$. As for the dispersion relation (18), in terms of k^a it takes the form $\sqrt{k_a k^a} = \pm m(1-r)\left[(\nu_i k^i)^2/k_j k^j\right]^{r/2}$. One of its solutions corresponds to massive fermions and, according to (5), admits the parametric representation by means of 3-velocities $\boldsymbol{v}$. Another solution corresponds to massless fermions and has the form $k^a \propto \nu^a$. Note at last that, in the limit $r = 1$, equation (13) takes the form $i\gamma^a \partial_a \psi - m\nu_a \gamma^a \psi = 0$ and, after the local gauge transformation $\psi \to \exp(-im\nu_a x^a)\psi$, reduces to the massless Dirac equation.

5 Conclusion

Summing up the results of the present work, we would like to emphasize that the spontaneous breaking of Lorentz symmetry does not necessarily signify the breaking of relativistic symmetry and may turn out to be a secondary effect induced by the spontaneous breaking of gauge symmetry. Here, the 10-parameter Poincaré group of an initial massless gauge-invariant theory is reduced to the 8-parameter inhomogeneous group of the generalized Lorentz transformations, which assumes in this case the role of the relativistic symmetry group of the corresponding vacuum solution of the theory. And vacuum itself, if it is regarded as space-time filled with a fermion-antifermion condensate, assumes anisotropic Finslerian geometry instead of Minkowski geometry. Within the framework of this picture the rearrangement of initial vacuum and the appearance of masses in the initial massless fields are not due to the standard Higgs mechanism but result from collective quantum effects peculiar to nonlinear dynamic systems.

Reverting to the translationally invariant generalized Dirac equation (13), which was obtained from the requirement of the generalized Lorentz symmetry, we see that it is essentially nonlinear. However this nonlinearity disappears in two cases: firstly, if the anisotropy field, constant over the whole space (and more exactly, its magnitude r) tends to zero (in this case (13) changes to the standard massive Dirac equation), and secondly, if r tends to its maximally attainable value equal to unity. In the latter case the anisotropy field turns out to be purely gauge while the massive fermion-antifermion field proves itself as the corresponding massless one. This means that the equation (13) describes the dynamics of the massive fermion-antifermion field in an anisotropic medium (in a relativistically invariant anisotropic condensate), in which case the effective inertial mass of the fermion-antifermion field has the dynamic origin and depends on the degree of order of the condensate, which should be a function of temperature.

Concluding the discussion of the nonlinear generalization of the Dirac equation, we note in addition that so far we have succeeded in constructing only simple, namely, planewave solutions of this equation. However, efficient algebraic-theoretical methods of constructing exact solutions

for a wide class of nonlinear spinor equations have already been evolved [15]. Using these methods, one can in principle obtain, also, other and, which is especially important, nongenerable families of exact solutions of the equation (13). As for the general conceptual problems relating to nonlinear generalizations of the Dirac equation [16], we hope to give more attention to them in our subsequent publications.

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Axially Symmetrical Solutions of the Klein–Gordon–Fock Equation

A.A. BORGHARDT, D.Ya. KARPENKO, T.A. KHACHATUROVA

Donetsk Physical and Technical Institute of NAS of Ukraine, 83114 Donetsk, Ukraine
E-mail: *ph_d@ukr.net*

It is shown that axially symmetrical solutions of the Klein–Gordon–Fock equation are classified by eigenfunctions of the wave operator that change the sign of eigenfunctions, i.e., the energy spectrum of the wave operator contains a point with a zero energy (frequency). The thermodynamics of the Bose–Einstein condensation is discussed on the basis of the energy spectrum under consideration. There is a phase transition in this mathematically integrable model.

In this note we discuss the thermodynamics of free relativistic spinless particles which are described by the Klein–Fock–Gordon equation (KFG)

$$(\hat{L} + \Delta_\perp - k_0^2)\Psi(x, y, z, t) = 0, \tag{1}$$

where $\hat{L} = (-1/c^2)\partial^2/\partial t^2 + \partial^2/\partial z^2$ is a wave (hyperbolic) operator, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is an elliptical operator, and $k_0 = (mc/h)$ is the inverse Compton wavelength.

As the eigenfunctions of equation (1) we put those that change the sign of operator $\hat{L}$ [1]

$$\Psi_{(+)} = \phi_{(+)}(x, y) \exp i\left(-ct\sqrt{k_z^2 + Q^2} + zk_z\right)$$

and

$$\Psi_{(-)} = \phi_{(-)}(x, y) \exp i\left(-\omega t + z\sqrt{(\omega^2/c^2) + \tilde{Q}^2}\right).$$

It is easy to verify that $\hat{L}\Psi_{(+)} = Q^2\Psi_{(+)}$, $\hat{L}\Psi_{(-)} = -\tilde{Q}^2\Psi_{(-)}$. Substitution of the eigenfunctions $\Psi_{(\pm)}$ in equation (1) gives equations for a definition of functions $\phi_{(\pm)}(x, y)$ as eigenfunctions of the operator $\Delta_\perp$:

$$\left(\Delta_\perp^2 - k_0^2 + Q^2\right)\phi_{(+)}(x, y) = 0, \qquad \left(\Delta_\perp^2 - k_0^2 - \tilde{Q}^2\right)\phi_{(-)}(x, y) = 0,$$

where functions $\phi_{(\pm)}(x, y)$ have the form

$$\phi_{(+)}(x, y) = \exp\left(i\left(k_x x + k_y y\right)\right), \qquad k_x^2 + k_y^2 = Q^2 - k_0^2 \geqslant 0,$$
$$\phi_{(-)}(x, y) = K\left(k_\perp x_\perp\right), \qquad x_\perp = \sqrt{x^2 + y^2}$$

here $k_\perp = \sqrt{k_x^2 + k_y^2}$, $k_\perp^2 = k_0^2 + \tilde{Q}^2 > 0$. The function K_0 is expressed by the modified Bessel function or McDonald function. The wave numbers and angular frequency k_z, $k_\perp$, ω satisfy the dispersion relations for the function $\Psi_{(+)}$:

$$\omega/c = \sqrt{k_z^2 + k_\perp^2 + k_0^2} > 0, \tag{2}$$

for the functions $\Psi_{(-)}$:

$$\omega/c = \sqrt{k_z^2 - k_\perp^2 + k_0^2} \geqslant 0. \tag{3}$$

Dispersion relations of equation (2) and equation (3) are represented by the hyperbolic branches in $(\omega/c, k_z)$ plane at a fixed transversal momentum. It should be noted that the dispersion relation of equation (3) contains the point $\omega = 0$ at $|k_z| = \sqrt{k_\perp^2 - k_0^2} \geqslant 0$, $k_\perp \geqslant k_0$.

The group velocity $v_z = \partial\omega/\partial k_z = ck_z/\omega$ tends to infinity at $\omega = 0$. The dispersion expression of the type of equation (3) was studied by Migdal [2], where the relativistic limit was considered in the nucleon environment (in the gas approximation). The nucleon environment is a thermostat. In the paper [2] was pointed that there exists such k at which $\omega = 0$ and therefore $\partial\omega/\partial k_z \to \infty$ [2]. Migdal called this phenomenon the effect of a disappearing rest mass and a π-condensation [2].

The aim of this note is to consider the thermodynamics of the model described by dispersion expressions of equation (2) and equation (3) and to study the Bose–Einstein condensation within it.

Let us write the full number of free particles N in the volume V in the Bose–Einstein statistics (BES) [3]

$$N = \frac{Vk_0^3}{2\pi^2}\left(\int g_1(\xi)\, f(\xi, T, \mu)\, d\xi + \frac{1}{2}\int g_2(\xi)\, f(\xi, T, \mu)\, d\xi\right). \tag{4}$$

The following notations have been used: $g_1(\xi) = \xi\sqrt{\xi^2 - 1}$, $\xi = h\omega/mc^2 = \omega/k_0 c$, $g_2(\xi) = \xi\sqrt{\xi^2 + \alpha^2} - \xi^2$, $\alpha = m^*/m$, $m^* = (h/c)\sqrt{Q^2 - k_0^2}$, where m^* is the value depending on the cutting parameter Q. The distribution function f

$$f = \left(\exp\left((\xi\theta/T) + (\mu/k_B T)\right) - 1\right)^{-1} = \sum_1^\infty \exp\left(-n\left((\xi\theta/T) + (\mu/k_B T)\right)\right),$$

here $\theta = mc^2/k_B$ is the characteristic temperature of the relativistic thermodynamics. For electrons with mass $m \approx 10^{-27}$ g and temperature $\theta \approx 10^{10^0}$ K, k_B is the Boltzmann constant and μ is the chemical potential. An integral of equation (4) with the density $g_1(\xi)$ was calculated by Pauli [4]. The second integral of equation (4) is calculated by the substitution $\xi = \alpha \operatorname{sh}(\theta)$ and using the table integral [5]

$$\int_0^\infty d\theta \exp(-q \operatorname{sh}\theta) = (\pi/2)\left(\mathbf{H}_0(q) - N_0(q)\right) = (\pi/2)\, A_0, \tag{5}$$

where $\mathbf{H}_0$ is the Struve function of zero-th order and N_0 is the Bessel function of the second kind.

With the table integral of equation (5) we represent the full number of particles N as the series:

$$N = \frac{Vk_0^3}{2\pi^2}\sum_{n=1}^\infty (K_2(n\theta/T)/(n\theta/T) + \left(\alpha^3/2\right) B(n\theta^*/T)) \exp(-\mu n/k_B T), \tag{6}$$

where $B(q) = (\pi/2)\left(\left(2A_1(q)/q^2 - A_0(q)/q\right)\right)$, $\theta^* = \left(m^* c^2/k_B\right)$. The series of equation (6) can be summarized in the asymptotic at low temperature $T \ll \theta$, therefore we have

$$N \approx \frac{Vk_0^3}{2\pi^2}\left(\sqrt{\frac{\pi}{2}}\,(T/\theta)^{3/2} F_{3/2}((\theta/T) + \mu/k_B T) + \frac{\alpha}{2}\left(\frac{T}{\theta}\right)^2 F_2\left(\frac{\mu}{k_B T}\right)\right), \tag{7}$$

where $F_p(x) = \sum\limits_1^\infty (\exp(-nx))/n^p$, $F_p(0) = \sum\limits_1^\infty 1/n^p = \zeta(p)$, and $\zeta(p)$ is the zeta Riemannian function. Taking into account that $F_p(x) \to 0$ at $x \gg 1$, we can see that the first term in

equation (7) is negligibly small compared with the second one. Therefore in the asymptotic limit at low temperatures $T \ll \theta$ we get an approximate relation

$$N \approx \frac{Vk_0^3}{2\pi^2}\alpha\left(\frac{T}{\theta}\right)^2 F\left(\mu/k_BT\right) = \frac{Vk^*}{4\pi^2}\left(\frac{k_BT}{hc^2}\right)^2 F_2\left(\mu/k_BT\right) \tag{8}$$

with $k^* = (m^*c/h) = \sqrt{Q^2 - k_0^2}$.

The critical temperature of condensation $T = T_c$ is determined from equation (8) on condition that the chemical potential μ is zero and the full number of particles N is a constant

$$N = \frac{Vk^*}{4\pi^2}\left(\frac{k_BT}{hc^2}\right)^2 \zeta\left(2\right) = \text{const}, \qquad T_c \sim \left(\frac{N}{V}\right)^{1/2}, \tag{9}$$

where $\zeta\left(2\right) = \sum\limits_1^\infty 1/n^2 = 1.64$

Define as N_0 a number of particles (condensate) with the zero-th energy and the temperature $T > T_c$: $N = N_0 + N_1$, where

$$N_1 = N\left(T/T_c\right)^2, \qquad T \geqslant T_c,$$
$$N_0 = N\left(1 - T^2/T_c^2\right), \qquad T \leqslant T_c.$$

In the nonrelativistic thermodynamics the condensation depends on the temperature according to the following law: $(N_1/N) = (T/T_c)^{3/2}$.

Let us calculate the internal energy of condensate U, at $T \ll \theta$

$$U = \frac{Vk_0^3}{4\pi^2}mc^2 \int \xi g_2(\xi) f\left(\xi, \theta, \mu = 0\right) d\xi \simeq 1.5Nk_BT\left(T/T_c\right)^2. \tag{10}$$

In equation (10) we have used an approximate equality $2\zeta(3)/\zeta(2) \approx (2 \cdot 1.2)/1.6 = 1.5$, respectively, the capacity C_V will be: $C_V = 4.5Nk_B\left(T/T_c\right)^2$. Expressions of equation (8) and (9) enable us to calculate the chemical potential μ:

$$\left(T/T_c\right)^2 \zeta\left(2\right) = \sum_{n=1}^{\infty}(1/n^2)\exp\left(-\mu n/k_BT\right). \tag{11}$$

In equation (11) we can make an approximate summation up in the vicinity of the critical temperature $\left(T - T_c\right)/T_c \ll 1$ on condition that $(\mu/k_BT) \ll 1$. It gives:

$$\mu \approx 2k_BT\ln(T/T_c), \qquad \text{at} \qquad T - T_c \ll T_c.$$

Now we can calculate the internal energy U and capacity C_V of overcondensated particles in the temperature interval $T - T_c \ll T_c$; $U = 1.5Nk_BT$, $C_v = 1.5Nk_B$. The overcondensate consisting of relativistic bosons with energy $U = 1.5Nk_BT$ has the same behavior as an ideal undegenerated gas.

Using the definition of a Heaviside step we write the internal energy of the condensate of relativistic bosons in the form

$$U = 1.5Nk_BT\left(\left(T/T_c\right)^2\theta(T_c - T) + \theta(T - T_c)\right). \tag{12}$$

The internal energy U in equation (12) is continuous at temperature $T = T_C$ and has an angular point (break). The capacity C_V has the discontinuity at $T = T_C$ and the value of this discontinuity will be $\Delta C = C_V(T_C + 0) - (T_C - 0) = -3k_bN$. In the nonrelativistic thermodynamics the capacity C_V versus temperature dependence is continuous.

To write an equation of state of a condensate we calculate the thermodynamic potential Ω:

$$\Omega = k_B T \frac{V k_0^3}{4\pi^2} \int g_2(\xi) \ln\left(1 - \exp\left(-\left(\xi\theta/T\right) - \left(\mu/k_B T\right)\right)\right) \approx -U/3. \tag{13}$$

In equation (13) integration by parts was made and an asymptotic of special functions at large values of the argument (i.e., at low temperatures) used. From equation (13) it follows that relations between thermodynamic potential Ω and internal energy U are the same as for massless relativistic particles.

Entropy S is obtained by formula: $S = \int dT C_V(T)/T + \text{const}$, where the constant is determined from the condition of the continuous entropy at $T = T_C$.

$$S = \frac{3}{2} N k_B \left(\frac{3}{2}\left(T/T_C\right)^2 \theta\left(T_C - T\right) + \left((3/2) + \ln(T/T_C)\right)\theta\left(T - T_C\right)\right). \tag{14}$$

Equation (14) is true in the interval of temperatures $\left(T - T_C\right)/T_C \ll 1$. Entropy of condensate is smaller than the entropy of an overcondensate, i.e., the condensate is a more ordered state.

Now we have to consider the density of the number of particles N of equation (6) at high temperatures $\theta/T < 1$. Using the asymptotes of special functions at small values of the argument, we have

$$N \approx \frac{V k_0^3}{2\pi^2}\left(\left(T/\theta\right)^3 F_3\left(\mu/k_B T\right) + \frac{\alpha^2}{2}\left(k_B T/\hbar c\right) F_1\left(\mu/k_B T\right)\right). \tag{15}$$

There is no critical temperature in equation (15) because $F_1(0) = \sum_1^\infty \frac{1}{n} \to \infty$.

In conclusion, we have shown that axially symmetrical solution of the Klein–Gordon–Fock equation with equation (3) as the dispersion relation yields a Bose–Einstein condensation at low temperatures.

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Self-Gravitating Global Monopole and Nonsingular Cosmology

K.A. BRONNIKOV † *and B.E. MEIEROVICH* ‡

† *VNIIMS, 3-1 M. Ulyanovoy Str., 117313 Moscow, Russia;*
Institute of Gravitation and Cosmology, PFUR,
6 Miklukho-Maklaya Str., 117198 Moscow , Russia
E-mail: *kb@rgs.mccme.ru*

‡ *Kapitza Institute for Physical Problems, 2 Kosygina Str., 117334 Moscow, Russia*
E-mail: *meierovich@yahoo.com*

We review some recent results concerning the properties of a spherically symmetric global monopole in $(D = d+2)$-dimensional general relativity. Some common features of monopole solutions are found independently of the choice of the symmetry-breaking potential. Thus, the solutions show six types of qualitative behavior and can contain at most one simple horizon. For the standard Mexican hat (Higgs) potential, we analytically find the D-dependent range of the parameter γ (characterizing the gravitational field strength) in which there exist globally regular solutions with a monotonically growing Higgs field, containing a horizon and a Kantowski–Sachs (KS) cosmology outside it, with the topology of spatial sections $\mathbb{R} \times \mathbb{S}^d$. Their cosmological properties favor the idea that the standard Big Bang might be replaced with a nonsingular static core and a horizon appearing as a result of some symmetry-breaking phase transition on the Planck energy scale. We have also found families of new solutions with an oscillating Higgs field, parametrized by the number of its knots. All such solutions describe space-times of finite size, possessing a regular center, a horizon and a singularity beyond it.

According to the Standard cosmological model, the Universe has been expanding and cooling from a split second after the Big Bang to the present time and remained uniform and isotropic on the large in doing so. In the process of its evolution, the Universe has experienced a chain of phase transitions with spontaneous symmetry breaking, including Grand Unification and electroweak phase transitions, formation of neutrons and protons from quarks, recombination, and so forth. Regions with spontaneously broken symmetry which are more than the correlation length apart, are statistically independent. At interfaces between these regions, the so-called topological defects necessarily arise.

A systematic exposition of the potential role of topological defects in our Universe has been provided by Vilenkin and Shellard [1]; see there also the necessary references to the previous work. The particular types of defects: domain walls, strings, monopoles, or textures are determined by the topological properties of vacuum. If the vacuum manifold after the breakdown is not shrinkable to a point, then solutions of Polyakov–'t Hooft monopole type appear in quantum field theory.

Spontaneous symmetry breaking is well known to play a fundamental role in modern attempts to construct particle theories. In this context, one mostly deals with internal symmetries rather than with those associated with space-time transformations: examples are the Grand Unification symmetry, the electroweak and isotopic symmetries and supersymmetry, whose transformations mix bosons and fermions. Topological defects, caused by spontaneous breaking of internal symmetries independent of space-time coordinates, are called global.

A fundamental property of global symmetry violation is the Goldstone degree of freedom. In the monopole case, the term related to the Goldstone boson in the energy-momentum tensor

decreases rather slowly away from the center. As a result, the total energy of a global monopole grows linearly with distance, in other words, diverges. Without gravity such a divergence is a general property of spontaneously broken global symmetries. The self-gravity of a global monopole, if not entirely removes this difficulty, allows one to consider it from a new standpoint.

We have performed a general study of the properties of static, spherically symmetric global monopoles in general relativity [2]. Most of the results were extended [3] to spherically symmetric configurations of arbitrary dimension with the topology $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^d$, but in this presentation we shall for simplicity mostly adhere to $D = 4$ ($d = 2$). In this case, the Lagrangian is taken in the form

$$L = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - V(\phi) + \frac{R}{16\pi G}, \tag{1}$$

where R is the scalar curvature, G is the gravitational constant, ϕ^a, $a = 1, 2, 3$ is a scalar field multiplet, and $V(\phi)$ is a symmetry-breaking potential depending on $\phi = \pm\sqrt{\phi^a \phi^a}$. We assume the static, spherically symmetric metric

$$ds^2 = A(\rho)dt^2 - \frac{d\rho^2}{A(\rho)} - r^2(\rho)d\Omega^2 \tag{2}$$

($d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\varphi^2$) and a "hedgehog" scalar field configuration:

$$\phi^a = \phi(\rho)n^a, \qquad n^a = \{\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta\}\,. \tag{3}$$

Without loss of generality we take $\rho \geq 0$ and attribute $\rho = 0$ to a regular center.

Our approach was different from most of previous studies which had used the boundary condition that $V = 0$ at spatial infinity. We did not even require the existence of a spatial asymptotic. Instead, we required regularity at the center and tried to observe the properties of the whole set of global monopole solutions. Also, instead of dealing only with a particular form of the symmetry breaking potential (usually the Mexican hat potential), we found some general features of solutions valid independently of the particular shape of $V(\phi)$. The main results are as follows.

The Einstein equations lead to $r'' \leq 0$. Since at a regular center $r' > 0$, this leaves three possibilities for the function $r(\rho)$:

(a) monotonic growth with a decreasing slope, but $r \to \infty$ as $\rho \to \infty$,

(b) monotonic growth with $r \to r_{\max} < \infty$ as $\rho \to \infty$, and

(c) growth up to $r_{\max}$ at some $\rho_1 < \infty$ and further decrease, reaching $r = 0$ at some finite $\rho_2 > \rho_1$.

The other metric function, $A(\rho)$, determines the causal structure of space-time: zeros of any order of $A(\rho)$ correspond to Killing horizon of the same order.

Proposition 1. *Under the assumption that $\phi^2 < 1/(8\pi G)$ in the whole space, our system with a regular center can have either no horizon, or one simple horizon, and in the latter case its global structure is the same as that of de Sitter space-time.*

For nonnegative $V(\phi)$ there is no restriction on the magnitude of ϕ.

Proposition 2. *If $V(\phi) \geq 0$, our system with a regular center can have either no horizon, or one simple horizon, and in the latter case its global structure is the same as that of de Sitter space-time.*

Outside the horizon, in the so-called T-region, ρ becomes a temporal coordinate, and the geometry corresponds to homogeneous anisotropic cosmological models of Kantowski–Sachs (KS) type, where spatial sections have the topology $\mathbb{R} \times \mathbb{S}^2$.

Thus, depending on the behavior of $r(\rho)$ (items a–c) and $A(\rho)$ (with or without a horizon), all possible solutions can be divided into six qualitatively different classes. There are two more general results valid for any nonnegative potentials.

Proposition 3. *If $V(\phi) \geq 0$, the second center $r = 0$, if any, is singular.*

Proposition 4. *If $V(\phi) \geq 0$ and the solution is asymptotically flat, the mass M of the global monopole is negative.*

In Proposition 4, asymptotic flatness is understood up to the solid angle deficit $\Delta < 1$: at large ρ we have $r(\rho) \approx \rho$ and

$$ds^2 = \left(\frac{1-\Delta}{\alpha^2} - \frac{2GM}{r}\right)dt^2 - \left(\frac{1-\Delta}{\alpha^2} - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2, \tag{4}$$

where α is a model-dependent constant. To our knowledge, Proposition 4 had been obtained previously only numerically for the particular potential (5) (see below).

Thus, even before studying particular solutions, we have a more or less complete knowledge of what can be expected from such global monopole systems.

We further considered [2] the most frequently used Mexican hat potential

$$V(\phi) = \frac{1}{4}\lambda\left(\phi^a\phi^a - \eta^2\right)^2 = \frac{1}{4}\eta^4\lambda\left(f^2-1\right)^2, \tag{5}$$

where $\eta > 0$ characterizes the energy of symmetry breaking, λ is a dimensionless coupling constant and $f(\rho) = \phi(\rho)/\eta$ is the normalized field magnitude playing the role of an order parameter. The model has a global $SO(3)$ symmetry, which can be spontaneously broken to $SO(2)$ due to the potential wells ($V = 0$) at $f = \pm 1$.

Our analytical and numerical study for the potential (5) has confirmed previous results of other authors concerning configurations with a monotonically growing field magnitude f.

The solution properties are basically governed by the values of the single dimensionless parameter

$$\gamma = 8\pi G\eta^2, \tag{6}$$

characterizing the gravitational field strength. Thus, for $\gamma < 1$ the solutions have a spatial asymptotic with the metric (4). For $1 < \gamma < 3$, the so-called supermassive global monopole, the solutions contain a cosmological horizon and a KS model outside it.

We have obtained analytically the upper limit $\gamma_0 = 3$ for the existence of static monopole solutions, previously found numerically by Liebling [4]. To do so, we have used the fact that near a critical value of γ the field magnitude is small everywhere inside the horizon, making it possible to formulate a well-posed eigenvalue problem for the field f against the background of the de Sitter metric (which solves the Einstein equations in case $f \equiv 0$). The linear equation for f has the form

$$\frac{d}{dx}\left[x^2\left(1 - \frac{x^2}{x_h^2}\right)\frac{df}{dx}\right] - \left(2 - x^2\right) f = 0, \tag{7}$$

where x is a dimensionless variable proportional to r and $x_h = \sqrt{12/\gamma}$ is the value of x at the horizon. The boundary conditions are

$$f\Big|_{x=0} = 0, \qquad |f(x_h)| < \infty. \tag{8}$$

Nontrivial solutions of (7) with these boundary conditions exist for the sequence of eigenvalues

$$\gamma_n = \frac{3}{(n+1/2)(n+2)}, \qquad n = 0, 1, 2, \ldots, \tag{9}$$

where n is the number of nodes of the corresponding eigenfunctions $f_n(x)$. The eigenvalue $\gamma_0 = 3$ is the sought-for critical value of γ for monotonically growing f. In case $\gamma > 3$ static monopole solutions are absent.

The solutions with $n \geq 1$ form new families, which we had not met in the existing literature. In these solutions, existing for $\gamma < \gamma_n$, the field function f (which has no reason to be small when γ is far from γ_n), changes its sign n times. All such solutions turn out to have a singularity ($f \to \infty$, $r \to 0$) at some finite value of ρ beyond the horizon.

The solutions with a static nonsingular monopole core and a KS cosmological model in a T-region ($A(\rho) < 0$) outside the horizon are of particular interest. Changing the notations, $t \to y \in \mathbb{R}$, and introducing the proper time of a comoving observer $\tau = \int d\rho/\sqrt{|A(\rho)|}$, we can rewrite the metric as

$$ds^2 = d\tau^2 - |A(\tau)|dy^2 - r^2(\tau)d\Omega^2. \tag{10}$$

The model expands in one of the directions (along the y axis) from zero at the horizon (say, $\tau = 0$) to finite values at large τ in a process which, on its early stage, resembles inflation. In two other directions, corresponding to $\mathbb{S}^2$, the model expands from a finite size and finite expansion rate at $\tau = 0$ to a linear regime at large τ. Like other regular models with the de Sitter causal structure, i.e., a static core and expansion beyond a horizon, these models drastically differ from standard Big Bang models in that the expansion starts from a nonsingular surface, and cosmological comoving observers can receive information in the form of particles and light quanta from the static region, situated in the absolute past with respect to them. Moreover, in our case the static core is nonsingular, and it is thus an example of an entirely nonsingular cosmology in the spirit of the views advocated by Gliner and Dymnikova [5, 6] (see also references therein).

The nonzero symmetry-breaking potential plays the role of a time-dependent cosmological constant, a kind of hidden vacuum matter. For an observer in the T-region the potential decreases with time, and the hidden vacuum matter gradually disappears.

The present simple model cannot be directly applied to our Universe (in particular, due to lack of isotropization), it can at most pretend to describe the earliest, near-Planckian stage in an approximate, classical manner. It nevertheless may be considered as an argument in favor of the idea that the standard Big Bang might be replaced with a nonsingular static core and a horizon appearing as a result of some symmetry-breaking phase transition on the Planck energy scale.

As another cosmological application of the global monopole, one should mention the concept of topological inflation, related to the existence of a de Sitter core of the monopole, which can inflate due to its instability [7].

In Ref. [3] we have extended the above consideration to ($D = d + 2$)-dimensional general relativity, with the space-time topology $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^d$. The qualitative features of the solutions are mostly preserved, in particular, there are the same six types of behavior, and Propositions 1–4 still hold. Outside the horizon (if any), the metric again corresponds to a Kantowski–Sachs type cosmology, now with the topology of spatial sections $\mathbb{R} \times \mathbb{S}^d$.

For the Mexican hat potential (5), the strength parameter is defined as $\gamma = \kappa^2\eta^2$, where κ^2 is the gravitational constant of D-dimensional theory and η is the symmetry breaking characteristic from (5). There are again two critical values of γ for each D: for monotonic $f(\rho)$, solutions with static spatial infinity exist with $\gamma < d - 1$, while solutions with a horizon and an infinitely expanding KS exterior correspond to $d - 1 < \gamma < 2d(d + 1)/(d + 2)$. For solutions where $f(\rho)$

changes its sign n times, instead of (9), the critical values of γ are

$$\gamma_n = \frac{2d(d+1)}{(2n+1)(2n+d+2)}, \tag{11}$$

which reduces to (9) when $d = 2$.

In the important case when the horizon is far from the monopole core, the temporal evolution of the KS metric is described analytically. The Kantowski–Sachs space-time contains a subspace with a closed Friedmann–Robertson–Walker metric. Our estimates show that the 5-dimensional global monopole model is in principle able to give plausible cosmological parameters. However, within our macroscopic theory without specifying the physical nature of vacuum we cannot unequivocally explain why the fourth spatial dimension (the one that played the role of time in the static region) is not observable. Quantitative estimates certainly require a more complete model including further phase transitions, one of which should explain the unobservable nature of the extra dimension.

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Internal Symmetry from Division Algebras in Pure Spinor Geometry

P. BUDINICH

International School for Advanced Studies, Trieste, Italy
E-mail: *fit@ictp.trieste.it*

The É. Cartan's equations defining "simple" spinors (renamed "pure" by C. Chevalley) are interpreted as equations of motions for fermion multiplets in momentum spaces which, in a constructive approach based bilinearly on those spinors, result compact and Lorentzian, naturally ending up with a ten-dimension space. The equations found are most of those traditionally adopted ad hoc by theoretical physics in order to represent the observed phenomenology of elementary particles. In particular, it is shown how the known internal symmetry groups, in particular those of the standard model, might derive from the 3 complex division algebras correlated with the associated Clifford algebras. They also explain the origin of charges, the tendency of fermions to appear in charged-neutral doublets, as well as the origin of families. The adoption of the Cartan's conjecture on the non-elementary nature of Euclidean geometry (bilinearly generated by simple or pure spinors) might throw light on several problematic aspects of particle physics.

1 Foreword

It is well known that fermions and bosons are to be conceived as the quanta of spinor- and of Euclidean tensor-fields, respectively. Now it is generally agreed that bosons may be always bilinearly represented in terms of fermions which then appear as the elementary constituents of matter. The consequent physical assumption is that bosons should be bound states of fermions, having in mind the example of nuclei, say, bound states of nucleons.

However this interpretation appears some times not viable; as in the case of the photon which, notoriously, may not be conceived as a bound state of neutrinos as proposed a long time ago by L. de Broglie. Then one may shift back the attention to the classical fields, before quantization, and try to see if possibly the Euclidean tensor fields might be steadily bilinearly represented in terms of spinor fields, and, consequently, the physical, somehow naive, explanation in terms of bound states might be substituted by the purely geometrical one which, by itself, may offer aspects of great interest and of deep meanings, since it could reveal, among others, a striking parallelism between geometry and physics, as we will see. This way was attempted by several distinguished authors, starting from W. Heisenberg.

In this paper we will try to show that indeed the geometrical way might be the right one, provided the spinors adopted are those which É. Cartan named simple [1], later renamed pure by E. Chevalley [2]. In fact É. Cartan himself advocated the conjecture that Euclidean geometry might have to be conceived as not elementary insofar its elementary constituents might be represented by simple spinor.

In the following we will assume the reader familiar with spinor geometry rich of published literature [1–4] and try to concentrate on the synthetic exposition of some results.

2 Basic definitions and propositions

Given a space $W = \mathbb{C}^{2n}$ with Clifford algebra $\mathbb{C}(2n) = \operatorname{End} S$ with generators γ_a obeying:

$$[\gamma_a, \gamma_b]_+ = 2\delta_{ab}, \qquad a, b, = 1, 2, \ldots, 2n,$$

let $\psi_D \in S$ represent a 2^n-dimensional (Dirac) spinor. For $z \in W$, the Cartan's equation:

$$z_a \gamma^a \psi_D = 0, \qquad a = 1, 2, \ldots, 2n \tag{1}$$

defines a totally null plane of dimension d, indicated in the following with $T_d(\psi_D)$.

The volume element γ_{2n+1} defined by: $\gamma_{2n+1} = \gamma_1 \gamma_2 \cdots \gamma_{2n}$ anticommutes with all γ_a and $\gamma_1, \gamma_2, \ldots, \gamma_{2n}, \gamma_{2n+1}$ generate $\mathbb{C}\ell(2n+1)$ whose associated Pauli spinors will be indicated with ψ_P. We will further indicate with φ_W the Weyl spinors defined by:

$$\varphi_W^{\pm} = \frac{1}{2}(1 \pm \gamma_{2n+1})\psi_D; \tag{2}$$

they are 2^{n-1}-dimensional and associated with the even subalgebra $\mathbb{C}\ell_0(2n)$ of $\mathbb{C}\ell(2n)$. The corresponding Cartan–Weyl equations will be:

$$z_a \gamma^a (1 \pm \gamma_{2n+1})\psi_D = 0, \qquad a = 1, 2, \ldots, 2n. \tag{3}$$

Definition 1. A Weyl spinor φ_W associated with $\mathbb{C}\ell_0(2n)$ is simple or pure if the dimension d of the associated totally null plane: $T_d(\varphi_W)$ is maximal; that is equal n.

É. Cartan proved that a simple spinor φ_W is equivalent to T_n (φ_W) (up to a sign) and he stressed the importance of this equivalence; in so far it establishes the fundamental link between spinor-geometry and a specially elegant and simple sector of Euclidean geometry: the projective one; from which presumably derives the qualification "simple" for the corresponding spinors; now however substituted in the literature with the word "pure" later introduced by Chevalley.

Observe that while the dimension of $T_n(\varphi_W)$ increases linearly with n, those of φ_W increase with n as 2^{n-1}; therefore for high n, simple φ_W will have to be subject to constraint relations; and in fact all Weyl spinors are simple for $n = 1, 2, 3$ while for $n = 4, 5, 6, 7$ simple spinors are subject to $1, 10, 66, 364$ constraint relations respectively.

We will represent spinors with one column matrices, and γ_a with square ones, and then let us define the main automorphism B of $\mathbb{C}\ell(2n)$: $B\gamma_a = \gamma_a^t B$; $B\psi = \psi^t B$ where γ_a^t and ψ^t means γ_a and ψ transposed.

For $\psi, \phi \in S$ we have [5]:

$$\psi \otimes B\phi = \sum_{j=0}^{n} T_j = \sum_{j=0}^{n} {}_{[}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_j]} T^{a_1 a_2 \cdots a_j}, \tag{4}$$

where the products of γ_a matrices are antisymmetrized and where:

$$T_{a_1 a_2 \cdots a_j} = \frac{1}{2^n} \langle B\phi, {}_{[}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_j]}\psi \rangle. \tag{5}$$

Set in (4) $\phi = \psi = \varphi_W$ and we have:

Proposition 1. *φ_W is simple iff in equation* (4)*:*

$$T_0 = T_1 = \cdots = T_{n-1} = 0, \quad \text{while} \tag{6}$$

$$T_n = \frac{1}{2^n} \langle B\varphi_W, {}_{[}\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_n]}\varphi_W \rangle \neq 0. \tag{7}$$

Equations (6) represent the constraint relations while T_n represents the maximal t.n. plane bilinear in φ_W and equivalent, up to a sign to φ_W simple.

Let us now multiply equation (4) on the left by γ_a and apply it to $\gamma_a\psi$; if we sum over a and set it to zero we obtain:

$$\gamma_a\psi \otimes B\phi\ \gamma^a\psi = z_a\gamma^a\psi = 0, \qquad a = 1, 2, \ldots, 2n \tag{8}$$

that is equation (1) where:

$$z_a = \frac{1}{2^n}\langle B\phi, \gamma_a\psi\rangle. \tag{9}$$

We have now the fundamental:

Proposition 2. *Let $z \in W = \mathbb{C}^{2n}$ with components $z_a = \langle B\phi, \gamma_a\psi\rangle$; for ϕ arbitrary, $z_a z^a = 0$ iff $\psi := \varphi_W$ is simple or pure.*

The formal proof is in [5]; however there is also a geometrically visible one. In fact it is obvious that z, defined by (9), represents the intersection of the planes $T_d(\phi)$ and $T_{d'}(\varphi_W)$ now for φ_W simple $d' = n$ is maximal and then z must be null, viceversa if this has to be true for every ϕ, it has to be $d' = n$, maximal, and φ_W is simple.

Let us now consider the isomorphism of Clifford algebras:

$$\mathbb{C}\ell(2n) \simeq \mathbb{C}\ell_0(2n+1) \tag{10}$$

both central simple, and:

$$\mathbb{C}\ell(2n+1) \simeq \mathbb{C}\ell_0(2n+2) \tag{11}$$

both non-simple, from which we have the isomorphisms and subsequent embeddings of Clifford algebras:

$$\mathbb{C}\ell(2n) \simeq \mathbb{C}\ell_0(2n+1) \hookrightarrow \mathbb{C}\ell(2n+1) \simeq \mathbb{C}\ell_0(2n+2) \hookrightarrow \mathbb{C}\ell(2n+2) \tag{12}$$

and the corresponding ones for the associated spinors:

$$\psi_D \simeq \psi_P \hookrightarrow \psi_P \oplus \psi_P \simeq \psi_W \oplus \psi_W = \Psi_D \simeq \psi_D \oplus \psi_D \tag{13}$$

which implies that a Dirac or Pauli spinor is isomorphic to a doublet of Dirac or Pauli or Weyl spinors. These isomorphisms may be represented explicitly. In fact let γ_a be the generators of $\mathbb{C}\ell(2n)$ and Γ_A with $A = 1, 2, \ldots, 2n+2$, those of $\mathbb{C}\ell(2n+2)$. Then for

$$\Gamma_a^{(m)} = \sigma_m \otimes \gamma_a, \qquad m = 0, 1, 2, 3, \quad a = 1, 2, \ldots, 2n, \tag{14}$$

where $\sigma_0 = 1$ and σ_1, σ_2, σ_3 are Pauli matrices, the corresponding spinor ψ associated with $\mathbb{C}\ell(2n+2)$ is:

$$\Psi^{(m)} = \begin{pmatrix} \psi_1^{(m)} \\ \psi_2^{(m)} \end{pmatrix}, \tag{15}$$

where, for $m = 0; 1, 2$, and 3 $\psi_1^{(m)}$ and $\psi_2^{(m)}$ are Dirac, Weyl and Pauli spinors respectively. Now defining the unitary operators:

$$U_m = 1 \otimes L + \sigma_m \otimes R = U_m^{-1}, \qquad m = 0, 1, 2, 3, \tag{16}$$

where

$$L = \frac{1}{2}(1 + \gamma_{2n+1}), \qquad R = \frac{1}{2}(1 - \gamma_{2n+1}) \tag{17}$$

we have:

$$U_j \Gamma_A^{(0)} U_j^{-1} = \Gamma_A^{(j)}, \qquad U_j \Psi^{(0)} = \Psi^{(j)}, \tag{18}$$

as easily verified [6], from which

Proposition 3. *Dirac and Pauli spinors may be isomorphically represented by Dirac, Pauli or Weyl spinor doublets.*

Propositions 2 and 3 represent the basic geometrical tools for our job. All we have to do is to restrict the above to real spaces, which will result unambiguously Lorentzian, and then to read the Cartan–Weyl equations we obtain in physical terms. We will see that indeed the Euclidean tensor fields will be bilinear in spinor fields as already anticipated by Proposition 2; and then their quanta: the bosons, bilinear in fermions, even in the case they can not be bound states. In fact we will obtain also Maxwell's equations from neutrino Weyl equations. Obviously the geometrical way derives from the basic properties Euclidean spaces as generated by simple spinors in the frame of Cartan's conjecture. Physical fermions may well be represented by non simple spinors, for which the bound state approach will be relevant, as in the case of nuclei.

We will start from the simplest and transparent case of $n = 1$ and then find out the rule for going from n to $n + 1$.

3 The elementary case of $n = 1$. The signature

Start from $\mathbb{C}\ell(2)$ and let $\varphi = \begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}$ and $\psi = \begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}$ represent two of its associated Dirac spinors, or Pauli spinors of the isomorphic $\mathbb{C}\ell_0(3)$, generated by the Pauli matrices σ_1, σ_2, σ_3. Insert them in equation (4), where now $B = -i\sigma_2 = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} := \epsilon$, which becomes:

$$\begin{pmatrix}\varphi_0\psi_1 & -\varphi_0\psi_0\\ \varphi_1\psi_1 & -\varphi_1\psi_0\end{pmatrix} \equiv \varphi \otimes B\psi = z_0 + z_j\sigma^j \equiv \begin{pmatrix}z_0 + z_3 & z_1 - iz_2\\ z_1 + iz_2 & z_0 - z_3\end{pmatrix} \tag{19}$$

from which we easily get both the z-vector components bilinear in the spinors ψ and $\varphi : z_\mu = \frac{1}{2}\psi^t\epsilon\sigma_\mu\varphi$ (compare the matrices) and the nullness of the vector $z : z_\mu z^\mu = z_0^2 - z_1^2 - z_2^2 - z_3^2 \equiv 0$ (compute the determinants of the matrices) in agreement with Proposition 2.

In order to restrict to the real z_0 and z_j, of interest for physics, we need to introduce the conjugation operator C defined by: $C\gamma_a = \bar{\gamma}_a C$, $C\varphi = \bar{\varphi}C$, where $\bar{\gamma}_a$ and $\bar{\varphi}$ mean γ_a and φ complex conjugate. Then equation (19) may be expressed, and uniquely, [6] in the form:

$$\begin{pmatrix}\varphi_0\bar{\varphi}_0 & \varphi_0\bar{\varphi}_1\\ \varphi_1\bar{\varphi}_0 & \varphi_1\bar{\varphi}_1\end{pmatrix} = p_0 + p_j\sigma^j = \begin{pmatrix}p_0 + p_3 & p_1 - ip_2\\ p_1 + ip_2 & p_0 - p_3\end{pmatrix} \tag{19$'$}$$

and now

$$p_\mu = \varphi^\dagger\sigma_\mu\varphi, \qquad \mu = 0, 1, 2, 3, \tag{20}$$

where $\varphi^\dagger$ means φ Hermitian conjugate. Then we have, again identically:

$$p_\mu p^\mu = p_0^2 - p_1^2 - p_2^2 - p_3^2 \equiv 0 \tag{21}$$

which shows how p_μ are the components of a null or optical vector of a momentum space with Minkowski signature. This is a particular case of application of Proposition 2. In fact imbed $\mathbb{C}\ell_0(3)$ in the non simple $\mathbb{C}\ell(3)$ isomorphic to $\mathbb{C}\ell_0(1, 3)$ with generators $\gamma_\mu = \{\sigma_1 \otimes 1, -i\sigma_2 \otimes \sigma_j\}$ and $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_3 \otimes 1$. Then we may identify the above Pauli spinor with one of the two Weyl spinor defined by

$$\varphi_\pm = \frac{1}{2}(1 \pm \gamma_5)\,\psi, \tag{22}$$

where ψ is a Dirac spinor associated with $\mathbb{C}\ell(1,3)$. Then equation (20) identifies with one of the two:

$$p_\mu^{(\pm)} = \tilde{\psi}\gamma_\mu(1 \pm \gamma_5)\psi, \qquad \mu = 0,1,2,3, \tag{23}$$

where $\tilde{\psi} = \psi^\dagger\gamma_0$.

Now the vectors $p^\pm$ are null or optical because of Proposition 2, since the Weyl spinors $\varphi_\pm$ are simple or pure. The corresponding Cartan–Weyl equations will be:

$$p_\mu\gamma^\mu(1 \pm \gamma_5)\psi = 0 \tag{24}$$

which may be expressed in Minkowski space-time if p_μ are interpreted as generators of Poincaré translations: $p_\mu \to i\frac{\partial}{\partial x_\mu}$. They identify, after introduction of the Planck's constant, with the known wave equation of motion for massless neutrinos.

Observe that in this unique derivation, obtained by merely imposing the reality of the p_μ components, Minkowski signature derives from quaternions, as may be seen already from equations (19) and (19′) and from their correlation with Clifford algebras, in fact notoriously $\mathbb{C}\ell(1,3) = H(2)$, where H stands for quaternions. One might then affirm that Minkowski signature is the image in nature of quaternions.

It is interesting to observe that if we define the electromagnetic tensors F with components

$$F_{\mu\nu}^{(\pm)} = \tilde{\psi}[\gamma_\mu, \gamma_\nu](1 \pm \gamma_5)\psi \tag{25}$$

we obtain, from Cartan–Weyl equation (24) the Maxwell's equations in empty space[1] [6]:

$$p_\mu F_+^{\mu\nu} = 0, \qquad \epsilon_{\lambda\rho\mu\nu}p^\rho F_-^{\mu\nu} = 0. \tag{26}$$

It is easy to see that also the electromagnetic potential may be bilinearly expressed in terms of Weyl spinors and then, in the quantised theory, the photon will result bilinear in neutrinos. However, now the bound-state assumption will not be necessary, and furthermore, as well known, it would not work.

In the last part of this section we naturally operated the transition from $n = 1$ to $n = 2$, we have now only to generalize it. To this end it is easy to show that the same results may be obtained starting from the neutral Clifford algebra $\mathbb{C}\ell(1,1)$ which also brings to $\mathbb{C}\ell(1,3)$ as above.

4 The rule: from n to $n+1$

Let $\psi_D \in S$ for $\mathbb{C}\ell(1,2n-1) = \text{End } S$ with generators γ_a. Define the Weyl spinors $\varphi_W^\pm$ as in equation (2) and let them be simple or pure, then because of Proposition 2:

$$p_a^\pm = \tilde{\psi}_D\gamma_a(1 \pm \gamma_{2n+1})\psi_D, \qquad a = 1,2,\ldots,2n \tag{27}$$

define null vectors in $\mathbb{R}^{1,2n-1}$.

Now we have $\varphi_W^+ \oplus \varphi_W^- = \psi_D$ and correspondingly: $p_a^+ + p_a^- = p_a = \tilde{\psi}_D\gamma_a\psi_D$ which are the components of a non-null vector, which however is the projection in $\mathbb{R}^{1,2n-1}$ of a null vector of $\mathbb{R}^{1,2n+1}$ with real components:

$$P_A = \tilde{\Psi}\Gamma_A(1 + \Gamma_{2n+3})\Psi, \qquad A = 1,2,\ldots,2n+2 \tag{28}$$

[1] Also the inhomogeneous Maxwell's equations in the presence of external electromagnetic sources may be obtained from spinor geometry [7].

with $\Psi \in S$, $\mathbb{C}\ell(1,2n+1) = \mathrm{End}\, S$, generated by Γ_A, which defines the Weyl (simple) spinors

$$\psi_W^{\pm} = \frac{1}{2}(1 \pm \Gamma_{2n+1})\Psi_D. \tag{29}$$

It may be shown [6] that the real components P_A may be written in the form: $P_a = p_a = \tilde{\psi}_D \gamma_a \psi_D$; $P_{2n+1} = i\tilde{\psi}_D \gamma_{2n+1}\psi_D$; $P_{2n+2} = \tilde{\psi}_D \psi_D$.

Therefore the rule is:

$$\varphi_W^+ \oplus \varphi_W^- = \psi_D \tag{30}$$

and

$$p_a^+ \oplus p_a^- \hookrightarrow P_A = \{p_a, P_{2n+1}, P_{2n+2}\}. \tag{31}$$

The corresponding Cartan–Weyl equation is:

$$P_A \Gamma^A (1 \pm \Gamma_{2n+3})\Psi_D = 0, \qquad A = 1, 2, \ldots, 2n+2 \tag{32}$$

which, because of Proposition 3, may be set in the form [6]:

$$(P^a \gamma_a^{(m)} + iP_{2n+1}\gamma_{2n+1}^{(m)} \pm P_{2n+2})\psi^{(m)} = 0, \qquad m = 0, 1, 2, 3 \tag{33}$$

and a similar one for the signature $(2n+1, 1)$:

$$(P^a \gamma_a^{(m)} + P_{2n+1}\gamma_{2n+1}^{(m)} \pm iP_{2n+2})\psi^{(m)} = 0, \qquad m = 0, 1, 2, 3. \tag{33'}$$

To interpret physically equations (33) and (33′) we have only to interpret the first four P_μ as generators of Poincaré translations: $i\frac{\partial}{\partial x_\mu}$ by which Minkowski space-time is generated as a homogeneous space, and then consider both the spinor $\psi^{(m)}$ and P_j, with $j > 5$ as functions of x_μ; the latter representing external fields (bilinear in spinors).

We see then that the rule which derives from the generalization of the natural first step of Section 3, and coherent with Cartan's conjecture, foresees steadily the appearance of Lorentzian signatures for all values of n.

Remark 1. Observe that in equations (33) or (33′) the P_A components, have to define a null vector of $\mathbb{R}^{1,2n+1}$ or $\mathbb{R}^{2n+1,1}$, in order to have for $\psi^{(m)}$ non-null solutions. This condition is geometrically implied by Proposition 2 if we adopt for our momentum space Cartan's conjecture on the fundamental role of simple spinors. The corresponding momentum space will then result compact and equivalent to the Poincaré invariant mass-sphere:

$$\pm P_\mu P^\mu = M_n^2 = P_5^2 + P_6^2 + \cdots + P_{2n+2}^2 \tag{34}$$

whose radius increases with n; that is, with the dimension 2^{n-2} of the fermion multiplet we are dealing with. In such space, in the quantized field theory there will not be ultraviolet divergences.

Observe that in Section 3 we naturally operated the transition from $n = 1$ to $n = 2$ which is a particular case: for $n = 1$, of the general rule defined in this section. In that case in terms of Clifford algebras we went, de facto, from $\mathbb{C}\ell(1,1)$ to $\mathbb{C}\ell(1,3)$.

Now let us recall the Bott periodicity theorem on Clifford algebras, stating that: $\mathbb{C}\ell(n+8, m) = \mathbb{C}\ell(n, m+8) = \mathbb{C}\ell(n,m) \otimes \mathbb{R}(16)$.

If we apply it to our case we should increase n, step by step from $n = 1$ up to $n = 5$ arriving at

$$\mathbb{C}\ell(1,1) \otimes \mathbb{R}(16) = \mathbb{C}\ell(1,9) = \mathbb{C}\ell(9,1) = \mathbb{R}(32) \tag{35}$$

since $\mathbb{C}\ell(1,1) = \mathbb{R}(2)$, after which the cycle, because of the periodicity theorem, will be repeated. Now it happens that it is precisely $\mathbb{R}^{1,9}$ the higher dimensional space which is generally adopted to explain the main features of elementary particle physics. This coincidence might not be accidental.

We will now concisely list the features which are naturally and uniquely emerging at the various steps of our construction and, somehow surprisingly, we will see that they reproduce most of the known elementary particle properties. Several of these appear as due to the known correlations of Clifford algebras with division algebras.

5 The steps from $n = 2$ to $n = 5$

We will now study equations (33) or (33′) for increasing n: from 2 to 5, and try to interpret them as equations of motions (for $p_\mu \to i\frac{\partial}{\partial x_\mu}$) for fermions or fermions multiplets. In this journey we will naturally find the equations, traditionally postulated ad hoc, in order to represent elementary phenomena of fermions. We will find them more or less in the same order as they were historically postulated: from the isospin symmetry for nuclear forces up to the $SU(3)$ ones for quarks; however now they derive from pure spinor geometry and, in particular, internal symmetries appear to originate from the division algebras correlated with the corresponding Clifford algebras, but we will also obtain indications on the possible geometrical origin of charges, families and some other features.

$n = 2$. Equation (33′) for $\mathbb{C}\ell(3,1) = R(4)$ may represent Majorana spinors [6].

$n = 3$. Equation (33′) for $m = 0$ is: $(p_a\Gamma^a + p_7\Gamma_7 + ip_8)\ N = 0$, where Γ_a $(a = 1, 2, \ldots, 6)$ are the generators of $\mathbb{C}\ell(5,1)$, $N = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$, since $m = 0$, is a doublet of $\mathbb{C}\ell(3,1)$ Dirac spinors. It may be written in the form:

$$(p_\mu 1 \otimes \gamma^\mu + \vec{\pi}\cdot\vec{\sigma}\otimes\gamma_5 + M)N = 0, \tag{36}$$

where the pion field $\vec{\pi} = \frac{1}{8}\tilde{N}\vec{\sigma}\otimes\gamma_5 N$ is bilinear in N. Equation (36) well represents (for $p_\mu \to i\frac{\partial}{\partial x_\mu}$) the pion nucleon equation of motion with isospin symmetry $SU(2)$ clearly of quaternionic origin. However now $SU(2)$ is not the covering of $SU(3)$; it is generated by the reflection operators Γ_5, Γ_6, Γ_7, which are the same as those of the conformal group [6]. Obviously one might also obtain from equation (36) the equation of motion for the pion field in a similar way as Maxwell's equations were derived from massless neutrino equation (24) in Section 3.

From equation (36) one may also derive the role of complex numbers at the origin of the electric charge. In fact, let us write (36) explicitly for ψ_1 and ψ_2:

$$\begin{aligned}&(p_\mu\gamma^\mu + p_7\gamma_5 + ip_8)\,\psi_1 + \gamma_5\,(p_5 - ip_6)\,\psi_2 = 0,\\ &(p_\mu\gamma^\mu - p_7\gamma_5 + ip_8)\,\psi_2 + \gamma_5\,(p_5 + ip_6)\,\psi_1 = 0\,.\end{aligned} \tag{37}$$

after defining: $p_5 \pm ip_6 = \rho e^{\pm\frac{i\omega}{2}}$, where $\rho^2 = p_5^2 + p_6^2$, we obtain:

$$\begin{aligned}&(p_\mu\gamma^\mu + p_7\gamma_5 + ip_8)\,e^{i\frac{\omega}{2}}\psi_1 + \gamma_5\rho\psi_2 = 0,\\ &(p_\mu\gamma^\mu - p_7\gamma_5 + ip_8)\,\psi_2 + \gamma_5\rho e^{i\frac{\omega}{2}}\psi_1 = 0\end{aligned} \tag{37′}$$

which presents a $U(1)$ phase invariance of ψ_1 generated by $J_{56} = \frac{1}{2}[\Gamma_5, \Gamma_6]$ which being local induces a covariant derivative and (36) may be written in the form:

$$\left\{\gamma_\mu\left[i\frac{\partial}{\partial x_\mu} + \frac{e}{2}\,(1 - i\Gamma_5\Gamma_6)\,A_\mu\right] + \vec{\pi}\cdot\vec{\sigma}\otimes\gamma_5 + M\right\}\begin{pmatrix}p\\ n\end{pmatrix} = 0, \tag{38}$$

where we set $\psi_1 = p$, $\psi_2 = n$, well representing the proton-neutron doublet interacting with the pion and with the electromagnetic potential A_μ.

$\boldsymbol{n = 4}$. Equation (33′) for $m = 0$ is:

$$(p_A G^A + p_9 G_9 + i p_{10})\Theta = 0, \qquad A = 1, 2, \ldots, 8, \tag{39}$$

where G_A are generators of $\mathbb{C}\ell(7,1)$ and $\Theta = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ is a doublet of $\mathbb{C}\ell(5,1)$ Dirac spinors.

It is easy to see [6] that N_1 (or N_2) presents a $U(1)$ covariance generated by $J_{78} = \frac{1}{2}[G_7, G_8]$ at the origin of a charge for N_1 from which N_2 (or N_1) is free. If interpreted as the strong charge, the N_1 may represent a baryon doublet and N_2 a lepton one (from which the similarity of lepton-baryon families, discussed below).

But we may also obtain a non-Abelian gauge field. We will illustrate it in the case of the possible geometrical origin of the electroweak model. In fact supposing $\Theta = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$ to represent leptons (see next step: $n = 5$) and, starting from equation (33), assume the generators G_A of the form (we selected the signature $(1,7)$):

$$\begin{aligned} &G_\mu = 1_2 \otimes \sigma_3 \otimes \gamma_\mu, \qquad G_{5,6} = i\sigma_1 \otimes \sigma_{1,2} \otimes 1_4, \\ &G_7 = i\sigma_1 \otimes \sigma_3 \otimes \gamma_5, \qquad G_{8,9} = i\sigma_{2,3} \otimes 1_2 \otimes \gamma_5. \end{aligned} \tag{40}$$

Then from equation (39) we may easily derive:

$$\begin{aligned} &p_\mu \sigma_3 \otimes \gamma^\mu L_{1R} + (p_9 - p_{10}) L_{1L} + \left(p_8 + \frac{i}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma} \right) L_{2L} = 0, \\ &p_\mu \sigma_3 \otimes \gamma^\mu L_{2R} + (p_9 + p_{10}) L_{2L} + \left(p_8 - \frac{i}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma} \right) L_{1L} = 0, \end{aligned} \tag{41}$$

where $\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma} = p_5\sigma_1 + p_6\sigma_2 + p_7\sigma_3$ and L_{jL}, L_{jR}; represent, for $j = 1, 2$, left-handed and right-handed projections of L_1 and L_2 respectively. Now defining

$$p_8 \pm \frac{i}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma} = \rho e^{\pm\frac{i}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma}} = \rho e^{\pm\frac{q}{2}}, \tag{42}$$

where $q = i\boldsymbol{\omega}\cdot\boldsymbol{\sigma}$ represents an imaginary quaternion, it is easy to see that L_{1L} presents a phase invariance $e^{i\frac{q}{2}} \to e^{i\frac{q'}{2}}$ from which L_{2L} is free. Since q is local this gives rise to a non-Abelian covariant derivative

$$D_\mu = \frac{\partial}{\partial x_\mu} + i\boldsymbol{\sigma}\cdot\boldsymbol{w}_\mu \tag{43}$$

which if, applied to the electron e and left-handed neutrino ν_L doublet: $L_1 = \begin{pmatrix} e \\ \nu_L \end{pmatrix}$ gives origin to the equation:

$$\left(i\tfrac{\partial}{\partial x_\mu} + m \right) \begin{pmatrix} e \\ \nu_L \end{pmatrix} + \boldsymbol{\sigma}\cdot\boldsymbol{w}_\mu \gamma^\mu \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} + B_\mu \gamma^\mu e_R = 0, \tag{44}$$

which is the strating equation for the electroweak model (equation (44) may also be obtain directly from Proposition 3 [6]).

$\boldsymbol{n = 5}$. Equation (33′) for $m = 0$ is:

$$(p_\alpha \mathcal{G}^\alpha + P_{11}\mathcal{G}_{11} + iP_{12})\Phi = 0, \qquad \alpha = 1, 2, \ldots, 10. \tag{45}$$

With $\mathcal{G}_\alpha$ generators of $\mathbb{C}\ell(9,1)$ and $\Phi = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}$ a doublet of $\mathbb{C}\ell(7,1)$ Dirac spinors. Again, Θ_1 has a $U(1)$ phase invariance generated by $J_{9,10} = \frac{1}{2}[\mathcal{G}_9, \mathcal{G}_{10}]$ not presented by Θ_2; we will then assume $\Theta_1 := \Theta_B$ to represent a quadruplet of baryons and $\Theta_2 := \Theta_\mathcal{L}$ a quadruplet of leptons.

6 The baryon quadruplet Θ_B

Θ_B obey equation (39), therefore it defines the invariant mass of equation (34):

$$-p_\mu p^\mu = M_4^2 = p_5^2 + p_6^2 + p_7^2 + p_8^2 + p_9^2 + p_{10}^2 \tag{46}$$

which defines a sphere S_5 presenting a symmetry $SO(6)$ orthogonal to the Poincaré group. Therefore the maximal internal symmetry for the baryon quadruplet Θ_B could be $SU(4)$. However before this there is an internal symmetry originating from the maximal complex division algebra: that of octonions. In fact it was shown [8] that $\mathrm{Spin}\,(1,9) = \mathrm{Spin}\,(9,1) \cong SL(2,\mathbf{o})$ where $\mathbf{o}$ stands for octonion (even if the isomorphism is restricted to the infinitesimal groups).

In order to set it in evidence let us first remind that equation (39) for Θ_B derives from the Cartan–Weyl equation (32) which in our case is:

$$p_\alpha \mathcal{G}^\alpha (1 + \mathcal{G}_{11})\Phi = 0, \qquad \alpha = 1, 2, \ldots, 10 \tag{47}$$

and, since Θ_B presents an $U(1)$ local covariance it will determine a covariant derivative in space-times such that equation (47) may be written in the form:

$$\left[i \left(\frac{\partial}{\partial x_\mu} - igA^\mu_{(m)} \right) \mathcal{G}^{(m)}_\mu + \sum_{j=5}^{9} p_j \mathcal{G}^{(m)}_j + ip_{10} \right] \Phi^{(m)} = 0, \tag{48}$$

where the indices m derive from Proposition 3 as in equation (33′).

It may be shown [6] that the isomorphism with the octonion algebra may be represented through the $\mathcal{G}_\alpha$ matrices. Precisely $\mathcal{G}^{(n)}_\mu$ and $\mathcal{G}_{6+n}$ for $n = 1, 2, 3$ may represent the first 3 octonion imaginary unit e_1, e_2, e_3 respectively while $i\mathcal{G}_{11}$ represent the seventh one e_7, such that the projector $\frac{1}{2}(1 + \mathcal{G}_{11})$ selects a particular direction in octonion space. This then reduces the automorphism group G_2 of octonions to $SU(3)$. In this way the so-called complex octonions may be defined:

$$U_\pm = (1 \pm \mathcal{G}_{11}), \qquad V^{(n)}_{\mu\pm} = \mathcal{G}^{(n)}_\mu U_\pm, \qquad V^{(n)}_\pm = \mathcal{G}_{6+n} U_\pm. \tag{49}$$

They represent an $SU(3)$ invariant algebra [9]. Precisely $U_\pm$ transform as singlets and $V^{(n)}_{\mu+}$ and $V^{(n)}_+$ transform as the (3) representation of $SU(3)$ while $V^{(n)}_{\mu-}$ and $V^{(n)}_-$ as the $(\bar{3})$ one. In this way equation (48) may be set in the form:

$$\left[i \left(\frac{\partial}{\partial x_\mu} - igA^\mu_{(n)} \right) V^{(n)}_{\mu+} + p_5 \mathcal{G}_5 + p_6 \mathcal{G}_6 + \sum_{n=}^{3} p_{6+n} V^{(n)}_+ ip_{10} \right] U_+ \Phi = 0, \tag{50}$$

where the term with $V^{(n)}_{\mu+}$ may be interpreted to represent $SU(3)$ color and the one with $V^{(n)}_+$ represents $SU(3)$ flavors. Equation (50) has no direct physical interpretation since the vector component like p_{6+n} are bilinear in spinors. Formally we could obtain $SU(3)$ flavor covariance if we would define $p_{6+n} = \Phi^\dagger \mathcal{G}_0 V^{(n)}_- \Phi$.

But it should also be possible to obtain the known Gell–Mann 3×3 representation of the pseudo-octonion algebra [10] by acting with the 3 operators corresponding to $V^{(n)}_+$ on Cartan-standard spinors, or equivalently, on the vacuum of a Fock representation of spinor space as in Ref. [5] to obtain, as minimal left ideals, 3 spinors representing quarks, as will be discussed elsewhere. In this way the $SU(3)$ symmetry both of flavor and color might be obtained in the framework of the algebraic theory of spinors, as already obtained by other authors [11, 12].

According to the present geometrical scheme a fourth quark should exist presenting with the other 3 on $SU(4)$ symmetry. It could be discovered at higher energies.

7 The lepton quadruplet $\Theta_{\mathcal{L}}$

We have seen that $n \to n+1$ means doubling the dimension of spinor space and adding two more terms to the equations of motion. This means that in our geometrical scheme dimensional reduction means $n \to n-1$ and is equivalent to reducing to one half the dimension of spinor space and decoupling of the equations of motions. Therefore the quadruplet $\Theta_{\mathcal{L}}$, missing strong charge, will have to be reduced to a doublet; as in fact it appears in nature where lepton always appear as charged-neutral doublets. Now it may be easily seen that this dimensional reduction, because of Proposition 3, may be operated in 3 non-equivalent ways giving origin to 3 lepton-neutrino families differing in the values of the invariant masses [6]. And in nature electron, muon and τ lepton seem to differ mainly in masses. It may be shown that the 3 families may be correlated with the 3 imaginary units of quaternions [13].

Observe that, if $\Theta_{\mathcal{L}} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$ where L_1 and L_2 are doublets of leptons, as we have seen in Section 4, for $n = 4$, it is foreseen that if L_1 presents electroweak interactions, L_2 do not. Therefore L_2 needs a further dimensional reduction and decoupling of the equations of motion; that is a reduction from $n = 3$ to $n = 2$ and $n = 1$ obtaining the equations for Majorana fermions and neutrinos. They could be the candidates for the explanation of the origin of black matter [6].

8 Concluding remarks and outlook

We have seen how Cartan's simple spinors are appropriate to represent most of the observed properties of fermions. They plainly explain internal symmetries as due to their correlation with Clifford algebras; they explain the origin of charges and the tendency of fermions to appear in charged-neutral pairs or pairs of multiplets, as the baryon-leptons ones, with similar properties; in particular the origin of families.

In this framework the main role of simplicity is to render compact (due to Proposition 2) the momentum spaces where the equations of motion are naturally formulated. This implies that the corresponding quantized theory (in second quantization) will be free from one of the most severe difficulties of quantum field theories; that of ultraviolet divergences.

In this preliminary approach there appear several aspects which deserve further study. One is the physical meaning of equation (34) defining invariant masses (which implicitly contains also the charges) and the possibility it may offer to compute their values. Another is the constraint relations which simple or pure spinors must obey. In our case they are in number 66 for the spinor appearing for $n = 5$ in Section 5 (and in our interpretation representing 4 baryons and 4 leptons), and in number 10 for the $\mathbb{C}\ell(1,9)$-pure spinors considered in Section 6 (in our interpretation representing 4 baryons). It is interesting to observe that this case has already been studied with unexpected positive results in superstring theory [14].

However, this approach might present further interesting aspects. In fact let us assume as a postulate what it indicates: that for the appropriate description of the wave mechanics of fermions we need the geometry of Cartan's simple spinors. Now this mechanics is generally conceived as the constituent of classical mechanics of macroscopic bodies of which fermions are in turn the elementary constituents. Let us now remind that, classical mechanics is well representable with Euclidean geometry (or with its Riemanian generalizations) in space-time and, in this framework, the physical interpretation of quantum mechanics presents known and widely discussed difficulties. But now there could be another possibility, once again obtained by shifting the attention from physics to geometry. In fact if the geometry to be adopted for dealing with fermions is that of Cartan's simple spinors, then, following Cartan, those spinors might represent the elementary constituents of Euclidean geometry and the problem may be

shifted on how, from simple spinors, one may construct the elements of Euclidean geometry. All this may be synthetically represented in the following table, where with Perceptible World we mean that which can be directly perceived with out senses and is then accessible to our ordinary intuition. The arrows indicate embeddings.

	PHYSICAL OBJECTS	MECHANICS	GEOMETRICAL INSTRUMENTS
PERCEPTIBLE WORLD	Macroscopic bodies	Classical mechanics	Euclidean geometry
	↑ 1	↑ 2	↑ 3
ELEMENTARY CONSTITUENTS	Fermions	Quantum mechanics	Cartan's simple spinors

For the embedding 1 there is no problem, (bosons are bilinears of fermions). The embedding 2 is problematic; it could be substituted with the embedding 3 following Cartan's conjecture[2].

In other words the problem of how quantum mechanics may be embedded in classical mechanics, and then understood in its frame, would be shifted from physics to geometry.

Now notoriously, this problem when dealt with in the frame of the embedding 2, that is in the frame of physics, gives rise to known paradoxes; that is to propositions contradicting our ordinary intuition. If we shift the problem to the embedding 3 that is to how one may construct Euclidean geometry from that of Cartan's simple spinors we will have to deal only with abstract mathematical and geometrical objects for which we do not need the steady control of our ordinary intuition[3].

Furthermore, when the progress of theoretical physics is guided by geometry or mathematics it may well happen that apparent paradoxes may arise, as it happened in relativity with the apparent paradoxes of dilatation of proper time[4], and then they are accepted as a appropriate corrections to the errors and limitations of our ordinary intuition.

Some tentative consequence of such a geometrization of the problem may be set in evidence already at this preliminary stage. In fact we have seen that with simple or pure spinors we may only obtain, bilinearly (according to Proposition 2) Euclidean null vectors; the ordinary vectors may then be obtained only as sums or integrals of them; and the latter may represent strings [15]. This might explain the motivation of the necessity of strings, in quantum mechanics. In fact coherently with this approach we cannot introduce in our quantum mechanics the concept of point-event which is a concept of Euclidean geometry in space-time valid only for classical mechanics. Neither we can construct, via Fourier transforms, this concept, since for that we would need an infinite momentum space of which we do not dispose, as seen above. Therefore the resulting quantum mechanics in space-time will have to be fundamentally non local. The rigorous way to represent this non locality should be studied in dealing with the geometrical problem of embedding Cartan's simple spinors in Euclidean geometry and will be dealt with elsewhere.

[2]This, among others, would set in evidence a striking parallelism between physics and geometry.

[3]As an example, mentioned above, a simple spinor associated with a complex Euclidean space of dimension $2n$ is equivalent (up to a sign) to a totally null plane of dimension n whose vectors are all null and mutually orthogonal, which is mathematically correct but no easily accessible to our ordinary intuition.

[4]Like the one: it takes about 3 years for light to arrive here from α centauri. I could arrive there tomorrow if I could dispose of a vehicle travelling with a velocity slightly smaller than that of light.

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Quantization of Nonlinear Fields on the Classical Background by Means of Bogoliubov Group Variables

Margarita V. CHICHIKINA and Oleg A. KHRUSTALEV

Physical Department, Moscow State University, 119992 Moscow, Russia
E-mail: *chich@goa.bog.msu.ru*

The scheme of quantization of gravitational field in the neighborhood of the classical component by means of the Bogoliubov group variables is presented. Nontrivial exact solutions of Einstein equation are chosen to be the classical background. The system is assumed to be invariant with respect to arbitrary group transformation of the variable x.

1 Introduction

In the present article the theory of quantization of nonlinear boson fields in a vicinity of a nontrivial classical component by means of the Bogoliubov group variables is presented. It is possible to show that quantization of any nonlinear physical system should include quantization of a boson field in a vicinity of a classical background. If it is impossible to consider a classical field from the very beginning to be small (gravitation and extended particles), there is a problem of the description of properties of the physical system, in which main effect is the separation of a classical boson field. The known theories of quantization on a classical background meet two basic difficulties. The first and the main one is a problem of the conservation laws, and the second is the zero-mode problem. To bypass these problems we propose to use an idea of Bogoliubov. In this work we propose new quantizations that will allow to use the Bogoliubov group variables for systems having arbitrary symmetry group (provided that we know representation of this group), including ones for non-stationary systems.

2 The basic ideas of a new method of quantization

In this section we present schematically the basic ideas of quantization of boson fields by means of method of the Bogoliubov group variables. In subsequent sections we shall show how this approach works in the case of gravitational field.

Development of a scheme of quantization is done in three basic stages.

I. At the first stage we pass to new variables, i.e., Bogoliubov group variables.

Let variable x' be related to x by the group transformation:

$$x' = F(x,a), \qquad x'' = F\left(F(x,a),b\right) = F(x,c), \qquad c = \varphi\left(a,b\right).$$

Variations of coordinates at a variation of parameters of group a are:

$$\left(\delta x'\right)^i = \xi^i_s(x')B^s_p(a)\delta a^p,$$

where $i = 0,1,2,3$ is the number of the coordinate, $p = 1,\ldots,r$, and k is the number of generators of group. The group properties of the transformation are defined by the tensor $B^s_p(a)$.

Let us define Bogoliubov transformation as follows:

$$f(x) = gv(x') + u(x'),$$

the dimensionless parameter g is supposed to be large, and $(u(x'), a)$ are independent new variables (Bogoliubov group variables). The explicitly separated large component depends on x', as well as a quantum part. Thus, we restore invariance with respect to the transformation group, which was broken by explicit separation of classical component, as it was pointed out in Introduction. However, consideration of the variable τ as independent leads to the fact that the right-hand side of our equation contains now k variables more than the left one. In order to equalize the number of variables we shall impose some invariant conditions on $u(x')$.

Let us make it as follows: we choose some functions $N^a(x')$ (the number of functions is equal to the number of group parameters) and we shall impose the following conditions on the function $u(x)$:

$$\omega(N^a, u) = \int_\Sigma d\sigma \big(N^a_n u - N^a u_n\big) = 0.$$

This form is invariant with respect to group transformations.

Use of this condition allows to obtain the equations, which determine group variables as functionals of initial $f(x)$ and $f_n(x)$ on D in the differential form:

$$\frac{\delta\tau^a}{\delta f(x)} = -\frac{1}{g} Q^a_b \tilde{N}^b_n(x'), \qquad \frac{\delta\tau^a}{\delta f_n(x)} = \frac{1}{g} Q^a_b \tilde{N}^b(x'),$$

where Q^a_b is a solution of the equation $Q^a_b = \delta^a_b - \frac{1}{g} R^a_c Q^c_b$. $\tilde{N}^a$ is a linear combination of N^a, such that parities $\omega(\tilde{N}^a, v_b) = 0$; and R^a_c is a complex number obtained with the help of $v(x')$ and $u(x')$.

II. At the second stage we develop the perturbation theory. All integrals of motions are expressed in new variables. Analyzing the obtained expressions, we come to conditions at which application of perturbation theory is correct, namely: the classical part has to satisfy some equations. These equations appear to coincide with Euler–Lagrange equations. However, this result is not trivial: it was received not as a consequence of variational principle, but as a condition for development of a perturbative scheme for our system in terms of new variables, while for obtaining of these equations we did not need to know the Hamiltonian structure as the generator of translations in time.

So, at the second stage we obtain equations of motion for classical components.

It is done as follows: Having in explicit form expressions for new variable as functionals from old variable on D it is possible to define the operators $\hat{q}(x)$ and $\hat{p}(x)$:

$$\hat{p}(x) = \frac{1}{\sqrt{2}} \left(f_n(x) + i \frac{\delta}{\delta f(x)} \right), \qquad \hat{q}(x) = \frac{1}{\sqrt{2}} \left(f(x) - i \frac{\delta}{\delta f_n(x)} \right).$$

They are Hermitian in space, where the scalar product looks like:

$$\langle F_1 \,|\, F_2 \rangle = \int Df Df_n F_1[f, f_n] F_2[f, f_n],$$

and also satisfy to formal commutation relations:

$$[\hat{q}(x), \hat{p}(x')] = i\delta(x - x').$$

Therefore, we can treat $\hat{q}(x)$ and $\hat{p}(x)$ as operators of coordinate and momentum of field oscillators, and to develop a scheme of secondary quantization. However, there is another pair of self-adjoint operators, constructed from $f_n(x)$, $f(x)$, $\frac{\delta}{\delta f(x)}$ and $\frac{\delta}{\delta f_n(x)}$, which satisfy the same commutation relation. Thus, the number of possible field states appears to be doubled and requires a reduction.

Therefore, we shall act as follows: We shall develop a scheme of the perturbation theory in terms of Bogoliubov variables despite appearance of extra states. After that we make a reduction of states number, and this reduction will be, as a matter of fact, a dynamical scheme.

Now we have quantum operators of coordinate and momentum (and they are are expressed in terms of new variable). Now we can perform quantization of the system by the following substitution:

$$f(x') \longrightarrow \hat{q}(x), \qquad f_n(x') \longrightarrow \hat{p}(x).$$

In terms of the new variable $\hat{q}(x)$ and $\hat{p}(x)$ are series in inverted powers of the coupling constant. Hence, and the integrals of a motion of system are series:

$$O = g^2 O_{-2} + gO_{-1} + O_0 + g^{-1}O_1 + \cdots .$$

Here O_{-2} are complex numbers, the operators O_{-1} are linear with respect to $u(x')$, $u_n(x')$, $\frac{\partial}{\partial u(x')}$, $\frac{\partial}{\partial u_n(x')}$. Since there are no normalizable eigenvectors of such operators, for correct development of perturbation theory, it is necessary that operators are identically equal to zero. This requirement is satisfied if the classical component satisfies equations of motion on D (at certain boundary conditions). Then the operators O_{-1} are equal to zero.

Therefore, further we shall consider the classical part to be the solution of Cauchy problem with boundary conditions on D.

III. The third stage is construction of the system state space.

Above we have mentioned that the number of possible field states is double in comparison with a real situation, and consequently reduction of number of states is necessary.

First of all, we shall analyze the number of independent variables. The initial number of independent variables was ∞. After definitions of the Bogoliubov group variables (they, we remind, are considered as independent) this number became $\infty + k$. This number was doubled because of the way of determination of the creation-annihilation operators: $(\infty + k) * 2 = 2 * \infty + 2 * k$. Constraints, which are introduced together with definition of Bogoliubov transformation, reduce number of independent variable by k (the number of group parameters). And at the present moment the number of possible field states is $2 * \infty + k$.

Let us separate from field variables $u(x')$ k some variables r_a, which have no physical sense and are related to the method of realization of the perturbation theory. Then the number of states is $2 * \infty$, and it is possible to describe the field by means of the variable $w(x')$, that is defined as follows:

$$u(x') = w(x') + \tilde{N}^a(x')r_a, \qquad u_n(x') = w_n(x') + \tilde{N}^a_n(x')r_a.$$

The necessary reduction of the state number can be made as follows: let us assume that a state of a field is defined by functionals $w(x')$ and $w_n(x')$, on which $\frac{\delta}{\delta w(x')}$ and $\frac{\delta}{\delta w_n(x')}$ act by a following way:

$$\frac{\delta}{\delta w(x')} \longrightarrow \frac{\delta}{\delta w(x')} - iw_n(x'), \qquad \frac{\delta}{\delta w_n(x')} \longrightarrow -iw(x').$$

(It can be achieved, for example, by introduction of holomorphic representation in a certain isomorphically connected space.) After reductions we have the following independent variables:

$$(k \text{ of group parameters}) + (k \text{ variable } r_a) + (\infty - k\text{-dimensional space of functions } w).$$

Variables r_a have no physical sense. They have appeared as the remainder of reduction of states space in the terms of Bogoliubov group variables. Separation of these variable is related to structure of integrals of a motion in zero order and also dynamical by nature.

At this stage of construction of our scheme we obtained the correct number of degrees of freedom that will be equal to their real number. During realization of the reduction we obtain equations of motion for quantum correction for our field (the equations appear to be nonlinear). Also at this stage the explicit expressions for creation-annihilation operators are obtained, as well as the spectrum of frequencies, that is actually the energy spectrum. Besides, we obtain expressions for integrals of motion as generators of corresponding symmetry transformations:

$$O_0 = i\Im^\alpha \frac{\partial}{\partial \tau^\alpha}.$$

In particular, the Hamiltonian is the generator of shift on time. Let us note that equations of motion of a classical part and explicit form of the Hamiltonian are received independently, that is to overcome basic difficulty of the description of non-stationary systems, which was described in Introduction.

Also in the zero-th order we shall obtain Heizenberg equation in terms of new variables.

The summary of the received results:

– equation of motion of a classical part;
– equation of motion of the quantum addendum;
– energy spectrum;
– expressions for integrals of motion;
– Heisenberg equations;
– explicit form of the field operator in the terms of new variables,

allows to assert that our theory gives the complete description of a system of boson fields, quantized on classical background. At the same time, the theory guarantees exact fulfillment of the conservation laws in any order of perturbation theory and also allows to avoid a zero-mode problem.

3 Gravity quantization as example of application of Bogoliubov group variables

Now we apply an approach presented above to description of a gravitational field from the point of view of quantum field theory. We consider quantization in vicinity of a solution of Einstein equation and use the method of Bogoliubov transformations for this purpose.

Let the variables x' be connected with x by an arbitrary group transformation:

$$x' = f(x,a), \qquad x'' = f\left(f(x,a),b\right) = f(x,c), \qquad c = \varphi\left(a,b\right).$$

Let us consider pairs of functions $f_{st}(x)$, $f_n^{st}(x)$ and define the Bogoliubov transformation as follows:

$$f_{st}(x) = gv_{st}(x') + u_{st}(x'), \qquad f_n^{st}(x) = gv_n^{st}(x') + u_n^{st}(x'),$$

the dimensionless parameter g is assumed to be large, and group parameters a^p are new independent variables.

For quantization we use operators $\hat{q}_{st}(x)$ and $\hat{p}^{st}(x')$:

$$u_{st}(x') \mapsto \hat{q}^{st}(x), \qquad u_n^{st}(x') \mapsto \hat{p}^{st}(x).$$

In the expansion of action into series with respect to inverted powers of coupling constant operators S_0 are complex numbers. There are no normalizable eigenvectors of S_1, so it is necessary to set them to zero for perturbation theory construction. It is possible if the following equations hold true on Σ:

$$F_{stn} = \frac{2a}{\sqrt{F}}\left(F_{stn} - \frac{1}{2}F_nF_{st}\right),$$

$$F^{st}_{nn} = \frac{a}{2\sqrt{F}}\left(F_{lkn}F^{lk}_{n} - \frac{1}{2}F^{2}_{n}\right)F^{st} - \frac{2a}{\sqrt{F}}\left(F^{st}_{n}F^{kl}_{n}F_{stn} - \frac{1}{2}F_{n}F^{st}_{n}\right)$$
$$- a\sqrt{F}\left(R^{st} - \frac{1}{2}F^{st}R\right) - \sqrt{F}\left(F^{sl}c^{t}_{;l} - F^{st}c^{l}_{;l}\right).$$

here $F_{st}(x')$ is linearly connected with $v(x')$. These equations could be treated as evolution equations.

Hereinafter, we assume $F_{st}(x)$ to be a solution of those equations, and $F_{st}(x')$ and $F^{st}_{n}(x')$ are the solution of the Cauchy problem on Σ, so we can state that on the 3D manifold the evolution equation holds true. Also it is possible to obtain conditions on the choice of Σ surface. These equations could be treated as constraint equations

$$\frac{1}{\sqrt{F}}\left(F^{st}_{n}F_{stn} - \frac{1}{2}F^{2}_{n}\right) - \sqrt{F}R(F) = 0, \qquad F^{sl}_{;l} = 0.$$

According to (3+1) ADM theory if evolution equations and constraint equations are satisfied in 3D space it means that in 4D manifold Einstein equations are fulfilled.

So we can state that Einstein equations is a necessary condition for the perturbation theory to be applicable. We would like to underline that Einstein equations were obtained in the process of perturbation theory construction as a condition of validity, not as a consequence of the variational principle.

Quantum addendum to the classical gravitational field is obtained in the process of construction of the system state space. It should be a solution of the equations

$$\phi_{stm}(\vec{x}) = \frac{2a}{\sqrt{F}}\left(\phi_{stm}(\vec{x}) - \frac{1}{2}F_{st}\phi_{m}(\vec{x})\right),$$
$$\omega^{2}_{m}\phi^{st}_{m}(\vec{x}) = a\sqrt{F}\left(R\phi^{st}_{m}(\vec{x}) - R^{st}(\phi_{m}(\vec{x}))\right).$$

Creation-annihilation operators also may be constructed in the process of system state space construction, as well as Hamiltonian as generator of time transformation:

$$H = -iB^{-1}{}^{p}_{0}(a)\frac{\partial}{\partial a^{p}}.$$

We applied Bogoliubov transformation to the quantization of gravitational field in the neighborhood of a nontrivial classical component that permitted us to avoid zero-mode problem.

Einstein equations for the classical component were obtained as a necessary condition for the perturbation theory to be applicable, not as a consequence of the variational principle.

We obtained expressions for quantum corrections of the field operator and explicit form of state vector that allows us to calculate quantum corrections to the observables such as effective mass, energy spectrum and so on.

Such calculations for the physically interesting cases like Kerr, Schwartzshild and other exact solutions of Einstein equation are planned for research in the nearest future.

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Symmetry Conditions on Dirac Observables

Heinz Otto CORDES

Department of Mathematics, University of California, Berkeley, CA 94720 USA
E-mail: *cordes@math.berkeley.edu*

Using our Dirac invariant algebra we attempt a mathematically and philosophically clean theory of Dirac observables, within the environment of the 4 books [10,9,11,4]. All classical objections to one-particle Dirac theory seem removed, while also some principal objections to von Neumann's observable theory may be cured. Heisenberg's uncertainty principle appears improved. There is a clean and mathematically precise pseudodifferential Foldy–Wouthuysen transform, not only for the supersymmeytric case, but also for general (C^∞-) potentials.

1 Introduction

The free Dirac Hamiltonian $H_0 = \alpha D + \beta$ is a self-adjoint square root of $1 - \Delta$, with the Laplace operator Δ. The free Schrödinger Hamiltonian $H_0^S = 1 - \frac{1}{2}\Delta$ (where "1" is the "rest energy" mc^2) seems related to H_0 like $\sqrt{1+x}$ its approximation $1 + \frac{1}{2}x$ – second partial sum of its Taylor expansion.

From that aspect the early discoverers of Quantum Mechanics were lucky that the energies of the bound states of hydrogen (a few eV) are small, compared to the mass energy of an electron (approx. 500 000 eV), because this should make "Schrödinger" a good approximation of "Dirac". But what about the continuous spectrum of both operators – governing scattering theory. Note that today big machines scatter with energies of about 10,000 – compared to the above $1 = mc^2$. So, from that aspect the precise scattering theory of both Hamiltonians (with potentials added) should give completely different results, and one may have to put ones trust into one of them, because at most one of them can be applicable.

We decide for the Dirac Hamiltonian which can explain the fine structure of spectral lines, Schrödinger cannot. This afortiori, because we believe to have a cure for the classical deficiencies of Dirac's theory – a different one as offered by quantum field (multi-particle) theory. Moreover, the resulting observable theory appears more consistent and offers insights, the Schrödinger Hamiltonian cannot supply.

Our main point is that the Dirac equation (with time-independent electromagnetic potentials) possesses an *invariant algebra of pseudodifferential operators* we want to exhibit as the algebra of "precisely predictable observables"[1]. The Dirac Hamiltonian itself

$$H = \alpha \cdot (D - \boldsymbol{A}(x)) + \beta + \boldsymbol{V}(x) \tag{1}$$

is precisely predictable. But other dynamical observables, like location, are not. However, location possesses "precisely predictable approximations", with an "error" of the order of magnitude of the Compton wave length – that is what Physicists assume anyway: The electron also may be observed as a wave, one should not expect a prediction of its location more accurate than its wave length.

Furthermore, no transition from "electron states" to "positron states" is ever precisely predictable – and no approximation to it exists. That should save us from the "electron see", and make one-particle Dirac theory consistent. A restriction to anti-symmetric states for multi-particle theory may be required for the Pauli principle, but is superfluous for Dirac's theory.

[1]Similar algebras exist for the "Dirac-type" equations of Yang–Mills theory.

There exist precisely predictable approximations to observables like spin (or current), but *only* if we already know for sure that the particle is an electron (a positron).

We might assume that this algebra was not known by Dirac or his contemporaries.

Some details: First an explanation why it might be reasonable to insist on observables which are pseudodifferential operators (abbrev.: ψdo). An observable interests us only if we can hope that our theory is able to *predict* it. We must expect our observable does not change too fast if we translate – rotate – dilate our coordinates, and both so, in configuration space and momentum space, because there always are errors of such kind in our measurements.

In above statement, if we replace "*does not change too fast*" by "*changes smoothly*", then we arrive at a mathematical concept we might start working with. Namely, translation, rotation and dilation describe 3 Lie groups under the action of which our observable should be "smooth" – both in configuration space and momentum space. Well, there are (precise mathematical) theorems stating that such a (Hilbert space) operator *must* be a ψdo (cf. [1] or, [4, Ch. 8]). Moreover, then, a "symbol calculus" – giving Leibniz-type formulas – is correct.

This is why we think it natural to assume observables as ψdo's.

Next, *time propagation* of an observable A. is given by the *Heisenberg transform*

$$A_t = e^{iHt} A e^{-iHt}. \tag{2}$$

The class of (*strictly classical*) ψdo-s A with the property that A_t is a *smooth* function of t forms an algebra $\mathcal{P}$. Here we are using a somewhat special concept of *smoothness* with derivatives $\partial_t^k A_t$ required to exist and be continuous as maps between certain polynomially weighted L^2-Sobolev spaces.

In our earlier theory (first published in 1983) we suggested using the self-adjoint operators of $\mathcal{P}$ as "precisely predictable observables".

In that respect we remind of the concept formulated precisely by J. von Neumann (in his book "Principles of Quantum Mechanics" of 1932): *To predict an observable A (with discrete spectrum) in the state ψ (a unit vector of the Hilbert space $\mathcal{H}$) we expand ψ into a series, after the eigenfunctions φ_j of A:*

$$\psi = \sum_{j=1}^{\infty} a_j \varphi_j, \qquad a_j = (\varphi_j, \psi), \tag{3}$$

with an orthonormal base $\{\varphi_j,\ j = 1, 2, \ldots\}$ of eigenvectors of A. One predicts the value λ for A with probability $\pi = \sum |a_j|^2$, with the sum over all j with $A\varphi_j = \lambda\varphi_j$. Only an eigenvalue of A can be measured, and the expectation value of the measurement is (predicted as) $\sum\limits_{j=1}^{\infty} \lambda_j |a_j|^2 = (\psi, A\psi)$ (Extension for A with general spectrum well known but omitted).

Now, we propose to apply above prediction rules **only** to observables (i.e., self-adjoint operators) within our above algebra $\mathcal{P}$ which then are called "precisely predictable". For other observables (not contained in $\mathcal{P}$) we devised an *approximation scheme*. Some observables – like location, momentum, may be approximated by observables in $\mathcal{P}$, and then there is an infinite series of (closer and closer) approximations, asymptotically convergent. For others, like spin and current, an approximation is possible only for specific physical states.

Now, before entering further physical discussions, we must explain a series of (precise mathematical) theorems. The first is related to the well known Foldy–Wouthuysen transform: That transform was introduced as an *approximate decoupling* (of electronic and positronic states) with an error of the order $\frac{1}{c^2}, \frac{1}{c^4}, \ldots$, (with c = speed of light). For a supersymmetric Hamiltonian this decoupling could be made precise. In 1983 we proved (without supersymmetry):

Theorem 1. *Under the general assumptions (X), below, there exists a unitary operator U of $\mathcal{H}$ which also is a strictly classical ψdo of order $0 = (0, 0)$ such that the Dirac operator* (1) *is*

decoupled modulo a reminder of order $-\infty$. That is,

$$U^*HU - \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \mathcal{O}(-\infty), \tag{4}$$

with certain (2×2-blocks of) ψdo-s X, Y having well described asymptotic expansions, and with $\mathcal{O}(-\infty)$ denoting the class of (globally defined) integral operators with kernel $k(x, y)$ in the Schwartz-space $\mathcal{S}(\mathbb{R}^6)$ of rapidly decreasing functions in all 6 variables x, y.

Theorem 2. *A strictly classical ψdo A belongs to the algebra $\mathcal{P}$ if and only if it decouples exactly as H above, under the unitary transform U of Theorem 1. That is, we have*

$$U^*AU - \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \in \mathcal{O}(-\infty), \tag{5}$$

with U of Theorem 1 and ψdo-s M, N of the same order than A.

Theorem 2 also requires the general assumptions (X), as follows:

The potentials $\mathbf{A}$, $\mathbf{V}$ are C^∞ and their partial derivatives of order k decay like $(1 + |x|)^{-k-1}$ at infinity.

Note that the Coulomb potential will need a "cap" near 0 to make it C^∞, but then will satisfy assumptions (X).

A physical reflection will cause us now to look for a removal of the reminder in $\mathcal{O}(-\infty)$.

Question. Does there exist another unitary map $V = 1 + \mathcal{O}(-\infty)$ such that, with notations of Theorem 1, we have

$$V^*U^*HUV = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \tag{6}$$

with slightly changed X, Y – just with some term of $\mathcal{O}(-\infty)$ to be added?

The answer to this question is a "**YES**", but **only** after correcting the underlying direct decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, with $\mathcal{H}_\pm$ denoting the subspaces with vanishing last to (first two) components by subtracting a *finite dimensional* subspace $\mathcal{Z}$ (with components in $\mathcal{S}(\mathbb{R}^3)$) from one of the spaces and adding it to the other one (cf. [7]).

This "obstruction" may mean that the potentials are so strong that some electron eigenvalues have entered into the "positron's continuous spectrum" (or, vice versa). Note, however, that our techniques of proof do not allow to decide if, for example, the (capped) Coulomb potential (with fine structure constant coefficient) needs this correction.

Why do we find this result interesting?

Note that this really will amount to a clean split of $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ into a direct sum of spaces of electron and positron physical states. We will correct our older notion of precisely predictable observables by introducing the smaller algebra $\mathcal{Q}$ of strictly classical ψdo-s which decouple strictly under the unitary transform $W = UV$. This means that a precisely predictable observable leaves the two subspaces of electronic and positronic physical states invariant.

Then, in view of prediction of measurements, those two spaces are strictly separated by a "fire-wall" Nothing can move across.

Note, by the way, even an observable of finite rank (and of order $-\infty$) which sends $\mathcal{H}_+$ into $\mathcal{H}_-$ would oscillate in time with more than twice the frequency of the Compton wave length. So, how could anybody hope to predict its value?

We will offer more (mathematical) details in the section below, but here will close with the following final remarks:

Remark 1. In our opinion, our modification of von Neumann's observable concept will remove *all* the old objections to Dirac's theory – there will be no need for the Pauli principle and an electron sea, and no "Zitterbewegung", nor a Klein Paradox.

Remark 2. On the other hand, we have become very curious about explicit spectral theory of some precisely predictable observables. For example – assuming $\boldsymbol{A} = 0$ – the energy observable H of (1) splits into kinetic and potential energy, the potential energy represented by $\boldsymbol{V}$. Now, while H is precisely predictable, $\boldsymbol{V}$ and $H - \boldsymbol{V}$ are not. But both have precisely predictable approximations. However these approximations seem to get very bad near $x = 0$. We have been curious about their spectral theory, reflecting on the possibility of getting some hints about special forces acting on an electron very close to the nucleus.

2 Some mathematical details

In this section we shall review some details, augmenting the statements of Section 1.

A (zero order) strictly classical ψdo A is of the form

$$Au(x) = (2\pi)^{-3/2} \int a^{\vee^2}(x, x-y)u(y)dy, \qquad u \in \mathcal{S}, \tag{7}$$

where "$(\cdot)^{\vee^2}$" denotes the inverse Fourier transform with respect to the second (set of) variable(s), and where the integral is a distribution integral – value of the distribution at $u \in \mathcal{S}$. The "symbol" $a(x, \xi)$ is a C^∞-function with this property: x-differentiation of order k followed by ξ-differentiation of order l gives a function bounded by $c(1 + |x|)^{-k}(1 + |\xi|)^{-l}$ with constant c. This class of symbols is called ψc_0; the class of all such operators $Op\psi c_0$. Note, $Op\psi c_0$ forms an adjoint invariant subalgebra of $L(\mathcal{H})$ with symbol of product and adjoint given by Leibniz-type rules as asymptotically convergent infinite series.

For strictly classical ψdo-s of arbitrary order the symbol $a(x, \xi)$ is a polynomial in x, ξ with coefficients in ψc_0 The classes are called ψc and $Op\psi c$. Operators may still be written as (7), or also,

$$a(x, D) = \sum x^\alpha a_{\alpha\beta}(x, D)D^\beta, \qquad a_{\alpha\beta} \in \psi c_0. \tag{8}$$

Again, $Op\psi c$ is an algebra (with Leibniz formulas), $\mathcal{O}(-\infty)$ a 2-sided ideal of $Op\psi c_0$ and $Op\psi c$.

Note, the distribution $a^{\vee^2}(x, x-\cdot)$ is singular only at the point x, and behaves like a function in $\mathcal{S}$ otherwise. At $x = y$ the integral (7) may be a Cauchy-type singular integral, or, generally, an "Hadamard finite part". Away from 0 (8) is a Lebesgue integral.

In (8), if all functions $a_{\alpha\beta}$ are independent of ξ, then $a(x, D)$ is a differential operator. Especially, **all of the common dynamical observables of Dirac theory** belong to our algebra $Op\psi c$ of strictly classical ψdo-s. Here we may remind of the fact that Dirac observables are 4×4-matrices, so we are really working here with $Op\psi c$ consisting of 4×4-matrices of operators of the form (8).

While $Op\psi c_0$ is a subalgebra of $L(\mathcal{H})$, $\mathcal{H} = L^2(\mathbb{R}^3)$, a general $\psi do \in Op\psi c$ is continuous $\mathcal{H}_s \to \mathcal{H}_{s-m}$ (for all s) with the "polynomially weighted" L^2-Sobolev spaces $\mathcal{H}_s = \langle D\rangle^{-s_1}\langle x\rangle^{-s_2}\mathcal{H}$ with $\langle x\rangle = \sqrt{1+x^2}$, $\langle D\rangle = \sqrt{1+D^2} = \sqrt{1-\Delta}$, and with $m = (m_1, m_2) = (D, x)$-degree of the polynomial (8). Such m is called an *order* of the ψdo A. There are other (perhaps better) orders, and m may be a pair of reals (of arbitrary sign). "Order $= -\infty$" means that $A : \mathcal{H}_s \to \mathcal{H}_t$ is continuous for all s, t. Such operators form the ideal $\mathcal{O}(-\infty)$.

With this preparation we can give a better description of above facts:

Essentially, the algebra $\mathcal{P}$ may be built from (4×4-matrices of) ψdo-s $A = a(x, D)$ with the property that $a(x, \xi)$ commutes with the symbol $h(x, \xi) = \alpha(\xi - \boldsymbol{A}) + \boldsymbol{V}$ of the Hamiltonian H.

If $[a,h]=0$ for some a of order m then a z of order $m-e=(m_1-1,m_2-1)$ may be found such that $A+Z=a(x,D)+z(x,D)$ belongs to $\mathcal{P}$. Moreover, every $P\in\mathcal{P}$ of order m splits: $P=A+Z$, where $[a,h]=0$, z of order $m-e$. Given a ψdo A – such as the location operator x, for example – with the property that its symbol commutes with h one may explicitly construct an infinite series of "corrections" Z_j such that $A+\sum Z_j$ is asymptotically convergent and represents an operator of $\mathcal{P}$. For the location coordinate x_j one gets the symbol[2]

$$x_j^{\mathrm{corr}}=x_j+\frac{1}{2i\sqrt{1+(\xi-\boldsymbol{A})^2}}(\alpha_j h_0-\xi_j)+\cdots\,,\qquad h_0=\alpha\xi+\beta, \tag{9}$$

and we will propose this operator (still a bit more corrected, adding an $\mathcal{O}(-\infty)$ to get into $\mathcal{Q}$) as the precisely predictable approximation of the location.

For other observables – spin or current, for example – the symbol does not commute with h. But then we may find a "$\mathcal{Q}$-approximation" for the operator with symbol *pap* – with the $(4\times 4$-) projection matrix $p(x,\xi)$ onto one of the two eigenspaces of $h(x,\xi)$. One finds that *pap* is a symbol in ψc commuting with h. Physically this means that *spin or current can be approximately predicted, but only if it is known with certainty that the particle is an electron (a positron).*

Regarding Theorem 1. The construction of the symbol u of the unitary operator U starts with choosing a (unitary 4×4-) matrix $u_0(x,\xi)$ diagonalizing the symbol $h(x,\xi)=\alpha(\xi-\boldsymbol{A})+\boldsymbol{V}$ as zero-th approximation (of order 0). Then an infinite series of correction symbols u_j of order $-je$ (resembling those of the Foldy–Wouthuysen transform) is constructed by an iteration, such that the asymptotic sum $u=\sum u_j$ satisfies $u(x,D)^*u(x,D)=1+\mathcal{O}(-\infty)$, $u(x,D)^*Hu(x,D)=\begin{pmatrix}X&0\\0&Y\end{pmatrix}+\mathcal{O}(-\infty)$. Then the removal of $\mathcal{O}(-\infty)$ only requires work within the algebra $\mathbb{C}+\mathcal{O}(-\infty)$ of Fredholm type integral operators with very regular kernel – no singular integrals anymore. For more details cf. [3].

For the discussion of Theorem 2 we refer to [6, Theorem 1.3].

It is clear that the relations

$$U^*U=1,\qquad U^*HU=\begin{pmatrix}X&0\\0&Y\end{pmatrix} \tag{10}$$

indicate a spectral split of H which we are tempted to identify with a split of the form

$$1=(1-E_\lambda)+E_\lambda, \tag{11}$$

with the spectral family E_λ of H. We then will expect that $|\lambda|<1$, so that one may hope for a split within the gap between continuous spectra of electron and positron. Interestingly, there seems to exist an obstruction: One may attempt to *build* the decoupling (10) (by a ψdo U) to correspond to (11) for some specific $\lambda=\lambda_0$, but then it will be necessary to first modify the two spaces $\mathcal{H}_-=E_\lambda\mathcal{H}$, $\mathcal{H}_+=(1-E_\lambda)\mathcal{H}$ by shifting a finite dimensional subspace $\mathcal{Z}$ from one of the spaces to the other one, according to a "deficiency index" $\iota_\lambda=\dim\mathcal{Z}$. This index changes monotonely as λ increases from -1 to $+1$. If it goes through zero, then we have found a split (10). If not then we first must make a finite dimensional "residual" correction of the spaces $\mathcal{H}_\pm$ with first two (last two) components zero, before we can reach (10). For more details cf. [7].

3 Final comments

It should be noted, that our above clean separation between electron and positron states still needs some "physical input", insofar as it is *not* unique: It will insist on the total dimension of

[2]Note our unit of length here is the Compton wave length.

either space, in the sense that we may not arbitrarily cast a finite dimensional subspace from $\mathcal{H}_\pm$ to $\mathcal{H}_\mp$. However, for example, we might exchange a (finite number of) positronic eigenvector(s) for an (equal number of) electronic one(s), while getting an equivalent mathematical theory.

It is, of course, evident that an abstract decoupling (by just a unitary U, not necessarily in $Op\psi c_0$) exists for just any spectral split (11), arbitrary λ. Our unitary $U \in Op\psi c_0$ must also be continuous $\mathcal{H}_s \to \mathcal{H}_s$, for every $s = (s_1, s_2)$, not only for $s = 0$, while that property does not alone make $U \in Op\psi c_0$. Note also, our construction of U (in [3] or [4]) leaves lots of ambiguity.

It may be important to observe that our "induced Hamiltonians" X, Y for the electronic (positronic) states (while not differential operators anymore) are strictly classical ψdo-s, of order (1,0), with known asymptotic expansions. They are semi-bounded from below (above), as proper Hamiltonians should be. The "residual" (finite dimensional) space $\mathcal{Z}$, if it occurs, might indicate yet that *not all electronic (positronic) states obey their corresponding Hamiltonian*. In principle, for example, there could be some very negative energy states still be possible for electrons under a Coulomb type potential, assuming that there indeed would exist a residual $\mathcal{Z}$ of electronic states governed by the operator Y.

In conclusion we note that we are aware of a large number of older attempts to deal with deficiencies of the Dirac theory. For a survey of literature we refer to the book [11], especially its notes.

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Some Applications of a Lorentz-Like Formulation of Galilean Invariance

Marc DE MONTIGNY [†], *Faqir KHANNA* [‡] *and Ademir SANTANA* [§]

[†] *Faculté Saint-Jean, University of Alberta, Edmonton, Alberta, T6C 4G9, Canada*
E-mail: *montigny@phys.ualberta.ca*

[‡] *Department of Physics, University of Alberta, Edmonton, Alberta, T6J 2J1, Canada*
E-mail: *khanna@phys.ualberta.ca*

[§] *Instituto de Fisica, Universidade Federal da Bahia, Salvador, Bahia, 40210-340, Brazil*
E-mail: *santana@ufba.br*

We describe a metric formulation of Galilean covariance in $4+1$ dimensions. As a first example, we recover the two Galilean limits of electromagnetism investigated previously by Le Bellac and Lévy-Leblond. Then we describe the field theoretical formulation of some fluid and superfluid models. Finally the non-relativistic Bhabha equations for spin 0 and 1 particles, and the Dirac equation for spin 1/2 are considered.

1 Introduction

Almost one century has elapsed since Galilei relativity was superseded by Einstein's theory as a realistic framework for describing high velocity phenomena. Yet there exists a wealth of systems at low-energy, particularly in condensed matter physics and nuclear physics, where any new method involving Galilean invariance is likely to be useful. In fact, in most many-body theories, Galilean invariance simply cannot be ignored. Moreover, contrary to popular belief, the mathematical structure of the Galilei group is more intricate than that of the Lorentz group. A case in point is that the representation theory of the Galilei group was thoroughly investigated nearly twenty years after its relativistic counterpart. The general program discussed here consists of a metric formulation of Galilei-invariance, so that one can use Galilean covariance, tensor analysis, etc. as a guiding principle to devise many-body models. Essentially, we exploit the well-known fact that the central extension of the Galilei algebra in 3+1 dimensions is a subalgebra of the Poincaré algebra in $4+1$ dimensions. Hereafter we summarize the articles [1–5], where further details can be found. Our geometrical approach follows the articles of Takahashi and his collaborators [6]. Other five-dimensional formalisms can be found in [7, 8].

We define a *Galilei-vector* $(\boldsymbol{x}, t, s)$ such that a boost acts on it as

$$\begin{aligned} \boldsymbol{x}' &= \boldsymbol{x} - \boldsymbol{V}t, \qquad t' = t, \\ s' &= s - \boldsymbol{V}\cdot\boldsymbol{x} + \frac{1}{2}\boldsymbol{V}^2 t, \end{aligned} \tag{1}$$

with relative velocity $\boldsymbol{V}$. Note that the units of s are $\frac{L^2}{T}$. The scalar product,

$$(A|B) = A^\mu B_\mu \equiv \boldsymbol{A}\cdot\boldsymbol{B} - A_4 B_5 - A_5 B_4, \tag{2}$$

of two Galilei-vectors A and B is invariant under transformation (1). This amounts to saying that we work on the light front in $4+1$ dimensions. The need for an additional coordinate may be explained in different ways: (1) as the phase required by the quantum wave function in order to keep the Schrödinger equation Galilei-invariant; (2) as a term added to the classical

free Lagrangian so that it becomes Galilei-invariant rather than quasi-invariant; (3) as noted in [8], it may be understood as a control parameter that makes up for lack of a signal with a universal velocity. We expect this approach to be useful in field theories. However, we do not claim that any Galilei-invariant theory can be expressed in this way. Neither do we suggest that this formalism will lead to the respective non-relativistic limits (for instance, the Chaplygin gas model is obtained from the Nambu–Goto action [9]).

Equation (1) can be written as

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}, \tag{3}$$

where $\Lambda^{\mu'}{}_{\nu}$ is the $(\mu'\nu)$-entry, or

$$\begin{pmatrix} x^{1'} \\ x^{2'} \\ x^{3'} \\ x^{4'} \\ x^{5'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -V_1 & 0 \\ 0 & 1 & 0 & -V_2 & 0 \\ 0 & 0 & 1 & -V_3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -V_1 & -V_2 & -V_3 & \frac{1}{2}\boldsymbol{V}^2 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{pmatrix}. \tag{4}$$

For a Galilei-oneform we have:

$$x_{\mu'} = \Lambda^{\nu}{}_{\mu'} x_{\nu}, \tag{5}$$

where now $\Lambda^{\nu}{}_{\mu'}$ is the $(\nu\mu')$-entry, with $\Lambda^{\nu}{}_{\mu'} x_{\nu}$ as in equation (4) with the change $V_j \to -V_j$.

Throughout this paper, except in Section 2, we utilize the Galilei-vectors $(x^1, \ldots, x^5)$ with each component having units of length:

$$(x^1, \ldots, x^5) = \left(\boldsymbol{x}, vt, \frac{s}{v}\right), \tag{6}$$

where v has units of velocity. For a real field $\tilde{\phi}$, the projection is defined as

$$\tilde{\phi}(x) \equiv \phi(\boldsymbol{x}, t) + a_0 s, \tag{7}$$

with a_0 a dimensionless constant. For a complex field $\tilde{\psi}$ we use the definition:

$$\tilde{\psi}(x) \equiv e^{i a_0 m s} \psi(\boldsymbol{x}, t), \tag{8}$$

with natural units, such that $\hbar = 1$. We use $a_0 = +1$ or -1.

If we use $(\boldsymbol{x}, t) \to x^{\mu} = (\boldsymbol{x}, t, s)$, then using the five-momentum $p_{\mu} \equiv -i\partial_{\mu} = (-i\boldsymbol{\nabla}, -i\partial_t, -i\partial_s)$ with $E = i\partial_t$ and $m = i\partial_s$, we obtain $p_{\mu} = (\boldsymbol{p}, -E, -m)$ and $p^{\mu} = g^{\mu\nu} p_{\nu} = (\boldsymbol{p}, m, E)$. Thereupon the mass does not enter as an external parameter, but rather as a remnant of the fifth component of the particle's momentum, starting from an apparently massless theory in $4+1$ dimensions!

2 Galilean electromagnetism

Here we recover the two 'Galilean limits' of electromagnetism obtained thirty years ago by Le Bellac and Lévy-Leblond [10].

The Lorentz transformations of a four-vector $(u^0, \boldsymbol{u})$,

$$\begin{aligned} u^{0\prime} &= \gamma\left(u^0 - \frac{1}{c}\boldsymbol{V}\cdot\boldsymbol{u}\right), \\ \boldsymbol{u}' &= \boldsymbol{u} - \gamma\frac{\boldsymbol{V}}{c}u^0 + \frac{\boldsymbol{V}}{\boldsymbol{V}^2}(\gamma - 1)\boldsymbol{V}\cdot\boldsymbol{u}, \end{aligned} \tag{9}$$

where $\gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{V}^2/c^2}}$, admits two well defined Galilean limits [10]. One is related to largely timelike vectors, with $u^{0\prime} = u^0$ and $\boldsymbol{u}' = \boldsymbol{u} - \frac{1}{c}\boldsymbol{V}u^0$, and corresponds to the 'electric' limit. The second limit is for largely spacelike vectors, with $u^{0\prime} = u^0 - \frac{1}{c}\boldsymbol{V}\cdot\boldsymbol{u}$ and $\boldsymbol{u}' = \boldsymbol{u}$, and is associated with the 'magnetic' limit. Throughout this section, we define the embedding of the Newtonian space-time into the de Sitter space by

$$(\boldsymbol{x}, t) \hookrightarrow x = (\boldsymbol{x}, t, 0)\,, \tag{10}$$

so that $\partial_k = \boldsymbol{\nabla}_k$, $\partial_4 = \partial_t$ and $\partial_5 = 0$. The electric and magnetic limits will be obtained by considering two particular embeddings of the five-potential.

From equation (5), we find that

$$\begin{aligned} &\boldsymbol{A}' = \boldsymbol{A} + \boldsymbol{V}A_5, \\ &A_{4'} = A_4 + \boldsymbol{V}\cdot\boldsymbol{A} + \frac{1}{2}\boldsymbol{V}^2A_5, \qquad A_{5'} = A_5. \end{aligned} \tag{11}$$

Let us denote the components of the five-dimensional electromagnetic antisymmetric tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ as

$$F_{\mu\nu} = \begin{pmatrix} 0 & b_3 & -b_2 & c_1 & d_1 \\ -b_3 & 0 & b_1 & c_2 & d_2 \\ b_2 & -b_1 & 0 & c_3 & d_3 \\ -c_1 & -c_2 & -c_3 & 0 & a \\ -d_1 & -d_2 & -d_3 & -a & 0 \end{pmatrix}. \tag{12}$$

They are expressed in terms of the five-potential A as

$$\begin{aligned} &\boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{A}, \qquad \boldsymbol{c} = \boldsymbol{\nabla}A_4 - \partial_4\boldsymbol{A}, \\ &\boldsymbol{d} = \boldsymbol{\nabla}A_5 - \partial_5\boldsymbol{A}, \qquad a = \partial_4A_5 - \partial_5A_4. \end{aligned} \tag{13}$$

The external five-current, $j_\mu = (\boldsymbol{j}, j_4, j_5)$, also transforms as a five-vector and one writes the continuity equation as

$$\partial^\mu j_\mu = \boldsymbol{\nabla}\cdot\boldsymbol{j} - \partial_4 j_5 - \partial_5 j_4 = 0. \tag{14}$$

In terms of the components in equation (12), the Maxwell equations,

$$\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} = 0 \tag{15}$$

and

$$\partial_\nu F^{\mu\nu} = j^\mu \tag{16}$$

become

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\boldsymbol{b} = 0, \qquad \boldsymbol{\nabla}\times\boldsymbol{c} + \partial_4\boldsymbol{b} = \boldsymbol{0}, \\ &\boldsymbol{\nabla}\times\boldsymbol{d} + \partial_5\boldsymbol{b} = \boldsymbol{0}, \qquad \boldsymbol{\nabla}a - \partial_4\boldsymbol{d} + \partial_5\boldsymbol{c} = \boldsymbol{0}, \end{aligned} \tag{17}$$

and

$$\boldsymbol{\nabla}\times\boldsymbol{b} - \partial_5\boldsymbol{c} - \partial_4\boldsymbol{d} = \boldsymbol{j}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{c} - \partial_4 a = -j_4, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{d} + \partial_5 a = -j_5, \tag{18}$$

respectively. Finally, the electromagnetic tensor transforms like $F_{\mu'\nu'} = \Lambda_{\mu'}{}^{\alpha}\Lambda_{\nu'}{}^{\beta}F_{\alpha\beta}$, so that its components transform as

$$\begin{aligned} &a' = a + \boldsymbol{V}\cdot\boldsymbol{d}, \qquad \boldsymbol{b}' = \boldsymbol{b} - \boldsymbol{V}\times\boldsymbol{d}, \\ &\boldsymbol{c}' = \boldsymbol{c} + \boldsymbol{V}\times\boldsymbol{b} + \frac{1}{2}\boldsymbol{V}^2\boldsymbol{d} - a\boldsymbol{V} - \boldsymbol{V}(\boldsymbol{V}\cdot\boldsymbol{d}), \qquad \boldsymbol{d}' = \boldsymbol{d}. \end{aligned} \tag{19}$$

2.1 Electric limit

The electric limit corresponds to the embedding

$$(\boldsymbol{A}_e, \phi_e) \hookrightarrow A_e = \left(\boldsymbol{A}_e, 0, -\frac{1}{k_1}\phi_e\right), \tag{20}$$

and

$$(\boldsymbol{j}_e, \rho_e) \hookrightarrow j_e = (k_2\boldsymbol{j}_e, 0, -k_2\rho_e). \tag{21}$$

If we define $\boldsymbol{B}_e \equiv \boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{A}_e$ and take $\boldsymbol{E}_e \equiv k_1\boldsymbol{d} = \frac{1}{\mu_0\epsilon_0}\boldsymbol{d} = -\boldsymbol{\nabla}\phi_e$, then from equations (13) and (19), we find that the field components transform like

$$\boldsymbol{E}'_e = \boldsymbol{E}_e, \qquad \boldsymbol{B}'_e = \boldsymbol{B}_e - \mu_0\epsilon_0\boldsymbol{V}\times\boldsymbol{E}_e, \tag{22}$$

as in [10]. From equations (17) and (18), with $k_2 \equiv \mu_0$, we find the wave equations

$$\begin{aligned} &\boldsymbol{\nabla}\times\boldsymbol{E}_e = \boldsymbol{0}, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}_e = 0, \\ &\boldsymbol{\nabla}\times\boldsymbol{B}_e - \mu_0\epsilon_0\partial_t\boldsymbol{E}_e = \mu_0\boldsymbol{j}_e, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E}_e = \frac{1}{\epsilon_0}\rho_e, \end{aligned} \tag{23}$$

as in equation (2.8) of [10]. Note that the Faraday term is missing in the first equation.

2.2 Magnetic limit

The magnetic limit corresponds to the embedding

$$(\boldsymbol{A}_m, \phi_m) \hookrightarrow A_m = (\boldsymbol{A}_m, \phi_m, 0), \tag{24}$$

and

$$(\boldsymbol{j}_m, \rho_m) \hookrightarrow j_m = (k_3\boldsymbol{j}_m, -k_4\rho_m, 0). \tag{25}$$

From equation (14), we find $\boldsymbol{\nabla}\cdot\boldsymbol{j} - \partial_4 j_5 - \partial_5 j_4 = \boldsymbol{\nabla}\cdot\boldsymbol{j}_m = 0$, which shows that the current $\boldsymbol{j}_m$ cannot be related to a transport of charge [10].

By defining $\boldsymbol{B}_m \equiv \boldsymbol{b} = \boldsymbol{\nabla}\times\boldsymbol{A}_m$ and taking $\boldsymbol{E}_m \equiv \boldsymbol{c} = -\boldsymbol{\nabla}\phi_m - \partial_t\boldsymbol{A}_m$, then from equation (19) we get

$$\boldsymbol{E}'_m = \boldsymbol{E}_m + \boldsymbol{V}\times\boldsymbol{B}_m, \qquad \boldsymbol{B}'_m = \boldsymbol{B}_m. \tag{26}$$

Finally, equations (17) and (18) show that the Maxwell equations reduce to

$$\begin{aligned} &\boldsymbol{\nabla}\times\boldsymbol{E}_m = -\partial_t\boldsymbol{B}_m, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{B}_m = 0, \\ &\boldsymbol{\nabla}\times\boldsymbol{B}_m = \mu_0\boldsymbol{j}_m, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{E}_m = \frac{1}{\epsilon_0}\rho_m \end{aligned} \tag{27}$$

in agreement with [10]. The displacement current term is missing in the third equation.

3 Fluid and superfluid equations

3.1 Euler equation for fluids

Define the functional Lagrangian as

$$\tilde{\mathcal{L}}[\tilde{\rho}, \tilde{\phi}] = -\frac{1}{2}\tilde{\rho}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - V(\tilde{\rho}). \tag{28}$$

The Euler–Lagrange equation for $\tilde{\rho}$ leads to $\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} + V'(\tilde{\rho}) = 0$. By defining the embedding as in equations (6) and (7), with $a_0 = -1$ and $\tilde{\rho}(x) \equiv \rho(\boldsymbol{x}, t)$, we find

$$\frac{1}{2}\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi + \partial_t\phi = -V'. \tag{29}$$

The gradient of this expression gives

$$(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\phi + \partial_t(\boldsymbol{\nabla}\phi) = -\boldsymbol{\nabla}(V'). \tag{30}$$

With $\boldsymbol{v} = \boldsymbol{\nabla}\phi$ (so that ϕ is a velocity potential) and $\boldsymbol{\nabla}(V') = \frac{1}{\rho}\boldsymbol{\nabla}p$ (where p is the pressure) we find the Euler equation,

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}p, \tag{31}$$

which is a particular case of the Navier–Stokes equation, with viscosity and body force both equal to zero. The Lagrangian of equation (28) can be deduced from

$$\tilde{\mathcal{L}}[\tilde{\psi}, \tilde{\psi}^*] \propto \partial_\mu\tilde{\psi}\partial^\mu\tilde{\psi}^* - V(|\tilde{\psi}|), \tag{32}$$

with complex field $\tilde{\psi}$, by defining the real fields $\tilde{\rho}$ and $\tilde{\phi}$ with the Madelung substitution $\tilde{\psi} \equiv \sqrt{\tilde{\rho}}e^{i\tilde{\phi}}$.

3.2 Generalized models for non-barotropic fluids

In [3] we noticed that the Takahashi model for compressible irrotational barotropic fluids with pressure proportional to the square of the mass density [6] can be expressed in a Galilean covariant form as

$$\tilde{\mathcal{L}} = \frac{\rho_0}{8v_0^2}\left(\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} - 2v_0^2\right)^2. \tag{33}$$

In this section, we generalize equation (33) by relaxing $p \propto \rho^2$ (p: pressure, ρ: density of the fluid) to $p \propto \rho^\gamma$ ($\gamma \geq 1$). For $\gamma \neq 1$ we consider

$$\tilde{\mathcal{L}} \propto (\partial\tilde{\phi}\partial\tilde{\phi} - v_0^2)^\gamma, \tag{34}$$

so that variation of the field $\tilde{\phi}$ gives

$$\left(\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - v_0^2\right)\partial_\nu\partial^\nu\tilde{\phi} + (\gamma-1)\partial_{\mu\nu}\tilde{\phi}\partial^\mu\tilde{\phi}\partial^\nu\tilde{\phi} = 0. \tag{35}$$

Using equation (6) with $a_0 = -1$, it becomes

$$v_0^2\nabla^2\phi - (\gamma-1)\partial_t^2\phi = \nabla^2\phi\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi\right) + (\gamma-1)\nabla\phi\cdot\nabla\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi + 2\partial_t\phi\right). \tag{36}$$

If $\gamma = 1$, it reduces further:

$$v_0^2\nabla^2\phi = \nabla^2\phi\left(\frac{1}{2}\nabla\phi\cdot\nabla\phi + \partial_t\phi\right). \tag{37}$$

When $\gamma \neq 1$, we recover the Takahashi model [6].

Other equations relevant in condensed matter physics are obtained by generalizing equation (32). For instance, consider

$$\tilde{\mathcal{L}}[\tilde{\psi}, \tilde{\psi}^*] \propto (\partial\tilde{\psi}\partial\tilde{\psi}^* - V(|\tilde{\psi}|))^p, \tag{38}$$

with a complex field $\tilde{\psi}$. The choices $p = 1$ and $V = \lambda|\tilde{\psi}|^4$, together with the embedding in equations (6) and (8), give us

$$\mathcal{L} \propto \nabla\psi \cdot \nabla\psi^* - im(\psi^*\partial_t\psi - \psi\partial_t\psi^*) - \lambda|\psi|^4. \tag{39}$$

The Euler–Lagrange equation, with $a_0 = -1$, leads to the non-linear Schrödinger, or Gross–Pitaevski, equation:

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + \frac{\lambda}{m}|\psi|^2\psi. \tag{40}$$

3.3 Model of non-viscous fluids and liquid helium

As a last example, let us consider equation (28) with a five-dimensional Clebsch transformation $\partial\tilde{\phi} \to \partial\tilde{\phi} + \tilde{\alpha}\partial\tilde{\beta}$:

$$\tilde{\mathcal{L}} = -\frac{\tilde{\rho}}{2v_0^2}(\partial_\mu\tilde{\phi} + \tilde{\alpha}\partial_\mu\tilde{\beta})(\partial^\mu\tilde{\phi} + \tilde{\alpha}\partial^\mu\tilde{\beta}) - V(\tilde{\rho}). \tag{41}$$

Next we define $\tilde{\alpha}(x) = \alpha(\boldsymbol{x}, t)$, $\tilde{\beta}(x) = \beta(\boldsymbol{x}, t)$, and $\tilde{\rho}(x) = \rho(\boldsymbol{x}, t)$, with equation (7) for $\tilde{\phi}(x)$ and equation (6) for the coordinates. Here we take $a_0 = +1$. Then the Lagrangian in equation (41) becomes

$$\mathcal{L} = \frac{\rho}{v_0^2}\left(\partial_t\phi - \frac{1}{2}\nabla\phi\cdot\nabla\phi + \alpha\left(\partial_t\beta - \frac{1}{2}\alpha\nabla\beta\cdot\nabla\beta\right) - \alpha\nabla\phi\cdot\nabla\beta\right) - V(\rho). \tag{42}$$

This may be expressed as

$$\mathcal{L} = \frac{\rho}{v_0^2}\left(\partial_t\phi + \alpha\partial_t\beta - \frac{1}{2}\boldsymbol{v}^2\right) - V(\rho), \tag{43}$$

where $\boldsymbol{v} = -\boldsymbol{\nabla}\phi - \alpha\boldsymbol{\nabla}\beta$. This Lagrangian was employed by Thellung and Ziman to describe the rotational components of liquid helium (see section 4.3 of [2]).

4 Bhabha and Duffin–Kemmer–Petiau equations: spin zero and spin one

In this section, we briefly summarize the references [4] and [5]. The Duffin–Kemmer–Petiau (DKP) equation is

$$(\beta^\mu\partial_\mu + k)\Psi = 0, \tag{44}$$

with matrices β satisfying the DKP algebra:

$$\beta^\mu\beta^\lambda\beta^\nu + \beta^\nu\beta^\lambda\beta^\mu = g^{\mu\lambda}\beta^\nu + g^{\nu\lambda}\beta^\mu, \tag{45}$$

where $g^{\mu\nu}$ is the Galilean metric. The adjoint of Ψ is defined as $\overline{\Psi} \equiv \Psi^\dagger\eta$, where $\eta = (\beta^4+\beta^5)^2+\mathbf{1}$. In the following we use the momentum version of equation (44):

$$(\beta^\mu p_\mu - ik)\Psi = 0. \tag{46}$$

4.1 DKP equation for spin zero and spin one

For spinless particles, the β's can be taken as in reference [5]. The DKP oscillator is described by performing the non-minimal substitution

$$\boldsymbol{p} \to \boldsymbol{p} + i\omega\eta\boldsymbol{r}. \tag{47}$$

Then equation (46) results in the following equation [5]:

$$E\phi = \left(\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega\right)\phi. \tag{48}$$

This equation has been obtained as the low-velocity limit of the energy of the DKP oscillator [11].

For spin one, we use a fifteen-dimensional representation of the DKP algebra [4]. We consider the DKP harmonic oscillator by first performing the non-minimal substitution, equation (47). Then equation (46) can be cast into the form [4]

$$E\boldsymbol{A} = \left[\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega - \frac{\omega}{\hbar}\boldsymbol{L}\cdot\boldsymbol{S}\right]\boldsymbol{A}. \tag{49}$$

This is the non-relativistic energy obtained in [11].

5 Dirac equation: spin 1/2

The details for this section are in [4]. Some recent developments, including the interaction with an external gauge field, are described in [1]. The non-relativistic Dirac equation is

$$(\gamma^\mu\partial_\mu + k)\,\Psi = 0, \qquad \mu = 1,\ldots,5, \tag{50}$$

written in momentum space as equation (46) with the β's replaced by γ's. The gamma matrices satisfy

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{51}$$

and can be chosen as

$$\gamma^n = \begin{pmatrix} \sigma_n & 0 \\ 0 & -\sigma_n \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \tag{52}$$

where each entry is a two-by-two matrix and the σ_n are the spin Pauli matrices. The adjoint spinor is defined as $\overline{\Psi} = \Psi^\dagger\zeta$, where $\zeta = \frac{-i}{\sqrt{2}}\left(\gamma^4+\gamma^5\right) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

Now let us consider the harmonic oscillator. If we perform the non-minimal substitution, equation (47), with η now replaced by ζ, for a spinor $\Psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ we find the Lévy-Leblond equation [12]:

$$\begin{aligned}
&(\boldsymbol{\sigma}\cdot\boldsymbol{p} - ik)\,\varphi + \left(\omega\boldsymbol{\sigma}\cdot\boldsymbol{r} + \sqrt{2}p_5\right)\chi = 0, \\
&(\boldsymbol{\sigma}\cdot\boldsymbol{p} + ik)\,\chi + \left(\sqrt{2}p_4 - \omega\boldsymbol{\sigma}\cdot\boldsymbol{r}\right)\varphi = 0.
\end{aligned} \tag{53}$$

Defining $p_4 = p_5$ and $\chi = -i\varphi$ we find [4]

$$E\varphi = \left(\frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2\boldsymbol{r}^2 - \frac{3}{2}\hbar\omega - \frac{2}{\hbar}\omega\boldsymbol{L}\cdot\boldsymbol{S}\right)\varphi, \tag{54}$$

where $\mathbf{S} \equiv \frac{1}{2}\hbar\boldsymbol{\sigma}$. This is in agreement with the low-velocity limit of the Dirac oscillator investigated in [13]. We plan to quantize the systems discussed in this paper following the same lines as the scalar field [14].

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Supplementary Solutions for Quasi-Exactly Solvable One-Dimensional Equations

Nathalie DEBERGH † *and Bruno VAN DEN BOSSCHE* ‡

† *Techniques du Son et de l'Image, Montéfiore Institute (B28), University of Liège, B4000 Liège, Belgique*
E-mail: *Nathalie.Debergh@ulg.ac.be*

‡ *B4000 Liège, Belgique*

Quasi-exactly solvable one-dimensional Schrödinger equations can be specified in order to exhibit supplementary analytic eigenstates. While the usual solutions are preserved by the $sl(2,\mathbb{R})$ generators, the additional ones are stabilized at the level of the universal enveloping algebra of this Lie structure. We discuss the square-integrability, the orthogonality of these supplementary solutions as well as the reality of the corresponding energies.

1 Introduction

Since Schrödinger wrote his famous equation in 1926, there has been a constant effort to find different methods for solving it. The present stage of the problem is that one-dimensional time-independent Schrödinger equations can be divided into three categories according to their degree of solvability. There are the *exactly solvable* ones for which the whole set of eigenvalues and eigenstates can be analytically determined, the *non solvable* ones which require a numerical or a perturbative treatment and the intermediate category, the *quasi-exactly solvable* (Q.E.S.) ones [1], for which a finite number of analytic solutions are known. The first explicit example of Q.E.S. potentials was found by Razavy [2].

Exactly solvable as well as most of the Q.E.S. Schrödinger equations are built up from linear and quadratic combinations of the $sl(2,\mathbb{R})$ generators [3]. For example, the Razavy potential illustrates this statement. What we want to emphasize here is that we can constrain in a specific way these combinations [4, 5] so that supplementary analytic eigenstates (and energies) arise. More precisely, the usual solutions are preserved by each of the $sl(2,\mathbb{R})$ generators while the supplementary ones are stabilized by adequate elements of the universal enveloping algebra of $sl(2,\mathbb{R})$. Moreover, the whole set of eigenstates exhibit interesting physical properties and call in question the self-adjointness of the corresponding Hamiltonians.

In Section 2, we prove how to construct a Q.E.S. operator such that it admits supplementary eigenstates with respect to the standard ones. We illustrate these developments on a Q.E.S. interaction generalizing the Razavy potential in Section 3. Finally, we discuss in Section 4 the square-integrability, the orthogonality of these supplementary solutions, as well as the reality of the corresponding energies.

2 Supplementary solutions to Q.E.S. equations

It is well known [3] that the $(n+1)$-dimensional space of monomials

$$V_1 = \{1, y, \ldots, y^n\}, \tag{1}$$

where n is a positive integer, is preserved by the differential operators

$$j_+ = -y^2\frac{d}{dy} + ny, \tag{2}$$

$$j_0 = y\frac{d}{dy} - \frac{n}{2}, \tag{3}$$

$$j_- = \frac{d}{dy}, \tag{4}$$

in the sense that

$$j_+ y^k = (n-k)\, y^{k+1}, \qquad j_0 y^k = \left(k - \frac{n}{2}\right) y^k, \qquad j_- y^k = k\, y^{k-1}, \qquad k = 0, 1, \ldots, n.$$

The operators (2)–(4) generate the $sl(2,\mathbb{R})$ Lie algebra characterized by the commutation relations

$$[j_0, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = 2j_0,$$

as it can be checked directly.

Moreover, any function of the $sl(2,\mathbb{R})$ generators will still keep the space V_1 invariant. In particular, being interested in the Schrödinger second-order differential operator, we consider the most general operator preserving V_1 i.e.

$$T = C_{+0} j_+ j_0 + C_{00} j_0^2 + C_{0-} j_0 j_- + C_+ j_+ + C_0 j_0 + C_- j_- + C_*. \tag{5}$$

In this expression, we have suppressed, without losing generality, the term $C_{+-} j_+ j_-$ due to the fact that the $sl(2,\mathbb{R})$ Casimir operator is given by

$$C = j_+ j_- + j_0^2 - j_0,$$

as well as the two terms $C_{++} j_+^2$ and $C_{--} j_-^2$, since it has been proved in [6] that these terms are redundant with respect to the canonical forms.

The operator (5) is constructed from three operators of different gradings

$$J_+ \equiv j_+(C_{+0} j_0 + C_+), \qquad J_- \equiv (C_{0-} j_0 + C_-) j_-$$

of respective gradings $+1$ and -1 and the diagonal (i.e. of grading 0)

$$J_0 \equiv C_{00} j_0^2 + C_0 j_0 + C_*.$$

Let us now ask for supplementary elements, say $\lambda_0(y), \lambda_1(y), \ldots, \lambda_{N-n-1}(y)$ with N a positive integer such that $N \geq n+1$, to be preserved by T. In other words, we do not limit ourselves to the standard Q.E.S. space V_1 but we add a $(N-n)$-dimensional space which we require in direct sum with V_1 without loss of generality (the dimension being adaptable). Following the gradings in (5), we thus ask for

$$J_+ \lambda_r(y) \sim \lambda_{r+1}(y), \qquad r = 0, 1, \ldots, N-n-2; \qquad J_+ \lambda_{N-n-1}(y) = 0,$$
$$J_- \lambda_r(y) \sim \lambda_{r-1}(y), \qquad r = 1, 2, \ldots, N-n-1; \qquad J_- \lambda_0(y) = 0.$$

These relations constrain the reals C_+ and C_- as well as the elements $\lambda_r(y)$ $(r = 0, 1, \ldots, N-n-1)$ in the following way

$$C_+ = \left(1 + \frac{3n}{2} - N - a\right) C_{+0}, \qquad C_- = \left(1 + \frac{n}{2} - a\right) C_{0-},$$
$$\lambda_r(y) = y^{a+r}, \qquad r = 0, 1, \ldots, N-n-1,$$

where a is an arbitrary real number. Consequently, the operator

$$T = C_{+0} j_+ \left(j_0 + 1 + \frac{3n}{2} - N - a\right) + C_{0-}(j_0 + 1 + \frac{n}{2} - a) j_- + C_{00} j_0^2 + C_0 j_0 + C_* \tag{6}$$

does preserve the $(N+1)$-dimensional space

$$V = V_1 \oplus V_2,$$

where V_1 is given in equation (1) while V_2 is

$$V_2 = \{y^a, y^{a+1}, \ldots, y^{a+N-n-1}\}.$$

With account of equations (2)–(4), the operator (6) simply reads

$$T = F(y)\frac{d^2}{dy^2} + G(y)\frac{d}{dy} + H(y) \tag{7}$$

with

$$\begin{aligned} F(y) &= -C_{+0}y^3 + C_{00}y^2 + C_{0-}y, \\ G(y) &= C_{+0}(N+a-2)y^2 + (C_{00}(1-n) + C_0)y + C_{0-}(1-a), \\ H(y) &= C_{+0}(1+n-N-a)n\, y + \frac{n^2}{4}C_{00} - \frac{n}{2}C_0 + C_*. \end{aligned} \tag{8}$$

The usual way [3] for transforming the operator (7) into a Schrödinger one

$$T \to H = -\frac{d^2}{dx^2} + V(x)$$

is to perform the change of variables

$$x \equiv f(y) \equiv \int \sqrt{\frac{-1}{F(y)}}\, dy \tag{9}$$

as well as the "gauge" transformation

$$\Lambda(x) \equiv \exp\left\{\frac{1}{2}\int^{y(x)} f'(y)\left[F(y)f''(y) + G(y)f'(y)\right] dy\right\}.$$

The potential $V(x)$ is then given by

$$V(x) = \left\{H(y) + \frac{1}{\Lambda(x)}\frac{d\Lambda}{dx}\left[F(y)f''(y) + G(y)f'(y)\right] - \frac{1}{\Lambda(x)}\frac{d^2\Lambda}{dx^2}\right\}_{y\to y(x)}.$$

3 An example

The change of variables (9) together with (8) leads to an elliptic function technically difficult to handle. We avoid this difficulty by choosing

$$C_{+0} = -1, \qquad C_{00} = 2, \qquad C_{0-} = 1$$

which implies that

$$x = -\left(\tanh\frac{y}{2}\right)^2.$$

We also introduce the new parameter a_1

$$a_1 \equiv 2(N-n) + C_0$$

and fix C_* as

$$C_* = \frac{1}{2}(n-N)(a_1+n-N)$$

for esthetical reasons. In this case, we have

$$\Lambda(x) = \exp\left(\frac{1}{2}(2a+a_1-1-N)\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^{-N}\left(\tanh\frac{x}{2}\right)^{a-\frac{1}{2}}$$

and

$$\begin{aligned} V(x) = \frac{1}{32}\Bigg\{ & 7+28a^2-a_1(6+a_1)+32n(n+1)-2N-2N(3a_1+16n)+15N^2 \\ & +4a(-7+3a_1-8n+N)-8(N+1)(2a+a_1-1-N)\cosh x \\ & +(2a+a_1-1-N)^2\cosh 2x-(8(N-2n+a-2)(N-2n+a)+6)\frac{1}{(\cosh\frac{x}{2})^2} \\ & +2(4a^2-1)\frac{1}{(\sinh\frac{x}{2})^2}\Bigg\}. \end{aligned} \tag{10}$$

Let us first point out some characteristics of this Q.E.S. potential.

The behavior near the origin is

$$V(x\to 0) \sim \left(a^2-\frac{1}{4}\right)\frac{1}{x^2}. \tag{11}$$

The coefficient in front of $\frac{1}{x^2}$ implies that the potential is attractive and unbounded at the origin if $a \in]-\frac{1}{2},\frac{1}{2}[$ and repulsive otherwise. The cases $a=\pm\frac{1}{2}$ are particular since the potential is not singular anymore. We recover here the Razavy interaction [2]. Up to these values, the problem is thus restricted to a half-line. Moreover equation (11) is such that the particle does not fall on the center [7]. Let us also notice that

$$\lim_{x\to\pm\infty} V(x) = +\infty,$$

a physical eigenstate will thus have to vanish at infinity.

Second we turn to the eigenstates which are given by

$$\begin{aligned} \psi_n(x) &= \exp\left(-\frac{1}{2}(2a+a_1-1-N)\left(\cosh\frac{x}{2}\right)^2\right) \\ &\quad\times\left(\cosh\frac{x}{2}\right)^N\left(\tanh\frac{x}{2}\right)^{\frac{1}{2}-a}P_n\left(\left(\tanh\frac{x}{2}\right)^2\right), \\ \tilde{\psi}_{N-n-1}(x) &= \exp\left(-\frac{1}{2}(2a+a_1-1-N)\left(\cosh\frac{x}{2}\right)^2\right) \\ &\quad\times\left(\cosh\frac{x}{2}\right)^N\left(\tanh\frac{x}{2}\right)^{\frac{1}{2}+a}\tilde{P}_{N-n-1}\left(\left(\tanh\frac{x}{2}\right)^2\right), \end{aligned}$$

where $P_n(u)$ and $\tilde{P}_{N-n-1}(u)$ are polynomials of respective orders n and $(N-n-1)$ of the real variable u. The untilded solutions are those belonging to the standard Q.E.S. space V_1 while the tilded ones refer to the additional space V_2.

In what concerns the energies, they can be obtained as solutions of algebraic equations of degree $N+1$. For example, if $N=3$ and $n=1$, they are

$$E_\pm = \frac{5}{2}-a_1\pm\frac{1}{2}\sqrt{a_1^2-8a_1-4a^2+20} \tag{12}$$

and

$$\tilde{E}_{\pm} = \frac{5}{2} - a_1 - 4a + 2a^2 + aa_1 \pm \frac{1}{2}\sqrt{a_1^2 - 8a_1 + 8aa_1 + 12a^2 - 32a + 20} \tag{13}$$

corresponding to the respective eigenstates

$$\psi_{\pm}(x) \sim \exp\left(-\frac{1}{2}(2a + a_1 - 4)\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(\tanh\frac{x}{2}\right)^{\frac{1}{2}-a} \times \left[(a-1) + \left(E_{\pm} + \frac{3}{2}(a_1 - 3)\right)\left(\tanh\frac{x}{2}\right)^2\right], \tag{14}$$

and

$$\tilde{\psi}_{\pm}(x) \sim \exp\left(-\frac{1}{2}(2a + a_1 - 4)\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(\tanh\frac{x}{2}\right)^{\frac{1}{2}+a} \times \left[(a+1) - \left(\tilde{E}_{\pm} + \frac{1}{2}(3 - 2a)(2a + a_1 - 3)\right)\left(\tanh\frac{x}{2}\right)^2\right]. \tag{15}$$

4 Real energies and self-adjointness

A look at equations (12), (13) shows that a particular choice of a and a_1 can lead to complex energies despite of the fact that the potential (10) is always real. This situation is the opposite of the one encountered in [8] where complex potentials lead to real spectra.

We thus have to analyze the self-adjointness of the kinetic term. This point is usually left aside because the radial eigenstates traditionally vanish at the origin. This is no longer true here (see equations (14), (15) for an arbitrary a) and we thus have to come back to the behavior of the functions (14), (15) at the origin.

Let us start with

$$H\psi = E_{\psi}\psi, \qquad H\phi = E_{\phi}\phi.$$

Multiplying both sides of these equations by $\phi^{\dagger}$ and $\psi^{\dagger}$, respectively, taking the conjugate of the resulting second equation and subtracting them, we obtain after integration and for a real potential, the standard relationship

$$\left(\psi\frac{d}{dx}\phi^{\dagger} - \phi^{\dagger}\frac{d}{dx}\psi\right)\Big|_{x_1}^{x_2} = (E_{\psi} - \bar{E}_{\phi})\int_{x_1}^{x_2}\phi^{\dagger}\psi\, dx. \tag{16}$$

The behavior of the functions (14), (15) at the origin is

$$\psi(x \to 0) \sim A_{\psi}x^{-\delta} + B_{\psi}x^{2-\delta} + \cdots, \tag{17}$$

where A_{ψ}, B_{ψ}, ... are expansion coefficients while

$$\delta = -\frac{1}{2} + a \qquad \text{or} \qquad \delta = -\frac{1}{2} - a$$

if the spaces V_1 or V_2 are under study, respectively. Inserting (17) in (16), we obtain

$$2(A_{\psi}\bar{B}_{\phi} - B_{\psi}\bar{A}_{\phi})\left[x^{1-2\delta} + O(x^{3-2\delta})\right]\,|_{x\to 0} = (E_{\psi} - \bar{E}_{\phi})\int_0^{+\infty}\phi^{\dagger}\psi\, dx. \tag{18}$$

If the functions ψ and ϕ belong to the same space, either V_1 or V_2, equation (18) proves that these functions are orthogonal if

$$1 - 2\delta > 0,$$

i.e.

$$a < 1 \ \text{ for } V_1 \quad \text{and} \quad a > -1 \ \text{ for } V_2. \tag{19}$$

Moreover, taking $\psi = \phi$, we deduce from equation (18) that the energies are real. On the other hand, if $1 - 2\delta \leq 0$, neither the orthogonality nor the reality of the energies can be guaranteed. Note that the condition $1-2\delta > 0$ is also the one for the eigenstates (14), (15) (see equation (17)) to be square-integrable at the origin. Besides these eigenstates are square-integrable at infinity if

$$a_1 + 2a - 4 > 0. \tag{20}$$

The constraints (19), (20) thus fix the domains of a and a_1 in order to ensure the reality of the energies as well as the physical relevance of the eigenstates. These constraints are for example met with $a = \frac{1}{4}$, $a_1 = 4$. In this case, we have

$$E_\pm = -\frac{3}{2} \pm \frac{1}{4}\sqrt{15}, \qquad \tilde{E}_\pm = -\frac{11}{8} \pm \frac{1}{4}\sqrt{19}$$

while

$$\psi_\pm = \exp\left(-\frac{1}{4}\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(\tanh\frac{x}{2}\right)^{\frac{1}{4}}\left(3 \pm \sqrt{15}\left(\tanh\frac{x}{2}\right)^2\right), \tag{21}$$

$$\tilde{\psi}_\pm = \exp\left(-\frac{1}{4}\left(\cosh\frac{x}{2}\right)^2\right)\cosh\frac{x}{2}\left(\tanh\frac{x}{2}\right)^{\frac{3}{4}}\left(7 \pm \sqrt{19} + \left(3 \mp \sqrt{19}\right)\cosh x\right). \tag{22}$$

Among these eigenstates, all vanishing at the origin, two (ψ_+ and $\tilde{\psi}_+$) have one node while the two others do not vanish on R_0^+. Another particular case is $a = \frac{1}{2}$, $a_1 = 4$. This implies

$$E_\pm = -\frac{3}{2} \pm \frac{1}{2}\sqrt{3}, \qquad \tilde{E}_\pm = -1 \pm \frac{1}{2}\sqrt{7}$$

and

$$\psi_\pm = \exp\left(-\frac{1}{2}\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(1 \pm \sqrt{3}\left(\tanh\frac{x}{2}\right)^2\right), \tag{23}$$

$$\tilde{\psi}_\pm = \exp\left(-\frac{1}{2}\left(\cosh\frac{x}{2}\right)^2\right)\sinh\frac{x}{2}\left(5 \pm \sqrt{7} + \left(1 \mp \sqrt{7}\right)\cosh x\right). \tag{24}$$

As already noticed, there is no more singularity so that here x belongs to the whole line. The number of nodes is 0, 1, 2 and 3 corresponding to the eigenstates ψ_-, $\tilde{\psi}_-$, ψ_+ and $\tilde{\psi}_+$, respectively. Finally, we choose $a = \frac{3}{4}$, $a_1 = 4$. We obtain

$$E_\pm = -\frac{3}{2} \pm \frac{1}{4}\sqrt{7}, \qquad \tilde{E}_\pm = -\frac{3}{8} \pm \frac{1}{4}\sqrt{43}$$

and

$$\psi_\pm = \exp\left(-\frac{3}{4}\left(\cosh\frac{x}{2}\right)^2\right)\left(\cosh\frac{x}{2}\right)^3\left(\tanh\frac{x}{2}\right)^{-\frac{1}{4}}\left(1 \mp \sqrt{7}\left(\tanh\frac{x}{2}\right)^2\right), \tag{25}$$

$$\tilde{\psi}_\pm = \exp\left(-\frac{3}{4}\left(\cosh\frac{x}{2}\right)^2\right)\cosh\frac{x}{2}\left(\tanh\frac{x}{2}\right)^{\frac{5}{4}}\left(13 \pm \sqrt{43} + \left(1 \mp \sqrt{43}\right)\cosh x\right). \tag{26}$$

The number of respective nodes is kept with respect to the context $a = \frac{1}{4}$, while the eigenstates have a different behavior at the origin: If the tilded functions still vanish when $x \to 0$, $\psi_\pm$ goes to $\mp\infty$ at the origin.

In each of these three cases, beside the reality of the energies, we confirm the square-integrability of each eigenstate (on R_0^+ or R) as well as their orthogonality within the same subspace ($\psi_\pm$ are orthogonal, $\tilde{\psi}_\pm$, too).

Let us now take a look at equation (18) when ψ and ϕ belong to different subspaces, say ψ to V_1 and ϕ to V_2. We have

$$2aA_\psi \bar{A}_\phi \left[1 + O\left(x^2\right)\right] |_{x\to 0} = (E_\psi - \bar{E}_\phi) \int_0^{+\infty} \phi^\dagger \psi \, dx \tag{27}$$

It is clear that the left-hand side of the relations (27) is not vanishing. Hence the eigenstates of two different subspaces are not orthogonal in general. This can be checked directly for the states (21), (22) as well as for the ones in equations (25), (26). However, the situation is special when $a = \pm\frac{1}{2}$. We have $\delta = 0$ either for the eigenstates lying in V_1 or for the ones of V_2. This means that the eigenstate at the origin goes to a constant in one case and is vanishing in the other case. One of the subspaces contains the even eigenstates and the second subspace contains the odd ones. These subspaces are now orthogonal to each other. All these statements are confirmed through equations (23), (24).

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Jordan–Wigner Fermions and Dynamic Probes of Quantum Spin Chains

Oleg DERZHKO and Taras VERKHOLYAK

Institute for Condensed Matter Physics NASU, 1 Svientsitskii Str., 79011 L'viv-11, Ukraine
E-mail: *derzhko@icmp.lviv.ua, werch@icmp.lviv.ua*

With the help of the Jordan–Wigner transformation the spin-$\frac{1}{2}$ XY chains can be reformulated in terms of noninteracting spinless fermions and, as a result, many statistical-mechanics calculations can be performed rigorously, i.e. without making any simplifying approximations. We are interested in dynamic properties of such chains (time-dependent spin correlation functions, dynamic structure factors, dynamic susceptibilities) which are of great importance for interpretation of experimental data. We have worked out a number of dynamic quantities explicitly as well as have performed a general analysis of the two-fermion continua which are relevant for different dynamic quantities.

1 Introductory remarks

The spin-$\frac{1}{2}$ XY chains are known as a simplest quantum interacting system for which a lot of statistical-mechanics calculations can be performed exactly [1]. The properties of such quantum spin chains were studied intensively during last more than forty years and the interest in such models may be renewed owing to a discovery of almost spin-$\frac{1}{2}$ XY chain compounds (see e.g. [2, 3]). On the other hand, such studies are interesting in their own rights since they provide a set of reference results which may be useful for understanding the much more common Heisenberg chains.

In what follows we consider the spin-$\frac{1}{2}$ anisotropic XY chain in a transverse field with an additional Dzyaloshinskii–Moriya interaction directed along z-axis in spin space to elucidate the effects of the Dzyaloshinskii–Moriya interaction on the dynamic properties of quantum spin chains. The Dzyaloshinskii–Moriya interaction plays important role in a number of quasi-one-dimensional materials and although it is generally small, its effects could be very important [4–6]. The Dzyaloshinskii–Moriya interaction also arises in description of the nonequlibrium steady states of spin chains [7]. Let us recall what is known about the dynamics of the considered quantum spin chains. The spin-$\frac{1}{2}$ XY chain with the Dzyaloshinskii–Moriya interaction was introduced in Ref. [8] (see also Ref. [9]) and the effects of this interaction on the zz dynamics were analyzed in Refs. [10, 11]. In particular, the zz dynamic susceptibility $\chi_{zz}(\kappa,\omega)$ of the spin-$\frac{1}{2}$ anisotropic XY chain with the Dzyaloshinskii–Moriya interaction was derived explicitly for $\kappa = 0$ [10] and $\kappa \neq 0$ [11]. Nevertheless, the effects of the Dzyaloshinskii–Moriya interaction on the two-fermion excitation continuum which governs zz dynamics [12, 13] have not been examined yet. There are notorious difficulties in calculations of the xx, xy, yx, yy dynamic quantities (for references see [14]), and to our best knowledge the effects of the Dzyaloshinskii–Moriya interaction on such quantities have not been reported until now. On the other hand, we should mentioned here the recent papers on the Heisenberg chains with the Dzyaloshinskii–Moriya interaction [15, 16]. Thus, using the symmetry arguments for the antiferromagnetic isotropic Heisenberg (XXX) chain with the Dzyaloshinskii–Moriya term directed along z-axis in spin space it was shown that although the Dzyaloshinskii–Moriya interaction may leave the spectrum of the problem unchanged, it can essentially influence the spin correlations / dynamic susceptibilities. In what follows we also derive such a conclusion in the case of the isotropic XY (i.e. $XX0$) chain, however, calculated different dynamic structure factors explicitly.

In the present paper we report the first results for the Dzyaloshinskii–Moriya effects on the two-fermion excitation continuum, which governs the zz dynamics of spin-$\frac{1}{2}$ XY chains and on the xx, xy, yx, yy dynamics of such chains. We start with presenting the basic formulas of the Jordan–Wigner fermionization approach (Section 2). Then we consider the case of isotropic XY exchange interaction. This case is essentially simpler than the case of anisotropic XY exchange interaction since the Dzyaloshinskii–Moriya interaction can be eliminated by a spin axes rotation. As a result, we can examine the dynamic structure factors of the chain with the Dzyaloshinskii–Moriya interaction using the obtained earlier results for such a chain without the Dzyaloshinskii–Moriya interaction (Section 3). Finally, we summarize our results (Section 4).

2 Dzyaloshinskii–Moriya interaction and Jordan–Wigner fermions

In what follows we consider $N \to \infty$ spins $\frac{1}{2}$ arranged in a circle and governed by the Hamiltonian

$$\begin{aligned} H &= \sum_{n=1}^{N} \Omega s_n^z + \sum_{n=1}^{N} \left(J^x s_n^x s_{n+1}^x + J^y s_n^y s_{n+1}^y \right) + \sum_{n=1}^{N} D \left(s_n^x s_{n+1}^y - s_n^y s_{n+1}^x \right) \\ &= \sum_{n=1}^{N} \Omega \left(s_n^+ s_n^- - \frac{1}{2} \right) + \frac{1}{2} \sum_{n=1}^{N} \left(\left(I^+ + \mathrm{i}D \right) s_n^+ s_{n+1}^- \right. \\ &\quad \left. + \left(I^+ - \mathrm{i}D \right) s_n^- s_{n+1}^+ + I^- \left(s_n^+ s_{n+1}^+ + s_n^- s_{n+1}^- \right) \right). \end{aligned} \tag{1}$$

Here $J^\alpha = 2I^\alpha$, $\alpha = x, y$ is the anisotropic XY exchange interaction, $I^\pm = I^x \pm I^y$, D is the Dzyaloshinskii–Moriya interaction and Ω is the transverse field. It is worthwhile to note that making use of the transformation $\tilde{s}_n^x = s_n^x$, $\tilde{s}_n^y = -s_n^y$, $\tilde{s}_n^z = -s_n^z$ (a π rotation of all spins about the x-axis) one gets again (1) with the parameters $-\Omega$, J^x, J^y, $-D$, whereas a similar transformation, $\tilde{s}_n^x = (-1)^n s_n^x$, $\tilde{s}_n^y = (-1)^n s_n^y$, $\tilde{s}_n^z = s_n^z$, yields (1) with the parameters Ω, $-J^x$, $-J^y$, $-D$. The renumbering of sites $j \to N - j + 1$, $j = 1, 2, \ldots, N$ in (1) yields again (1) with the parameters Ω, J^x, J^y, $-D$. These symmetry remarks permit to reduce the range of parameters for a study of the properties of the model.

We are interested in dynamics of quantum spin chain (1). For this purpose we need the two-spin time-dependent correlation functions, $\left\langle s_n^\alpha(t) s_{n+m}^\beta(0) \right\rangle$, α, $\beta = x$, y, z, the angular brackets denote the canonical thermodynamic averaging, which yield the dynamic structure factors

$$S_{\alpha\beta}(\kappa, \omega) = \sum_{m=1}^{N} \exp\left(\mathrm{i}\kappa m\right) \int_{-\infty}^{\infty} \mathrm{d}t \exp\left(\mathrm{i}\omega t\right) \left\langle s_n^\alpha(t) s_{n+m}^\beta(0) \right\rangle. \tag{2}$$

Another dynamic quantity, the dynamic susceptibility $\chi_{\alpha\beta}(\kappa, \omega)$, can be obtained from the dynamic structure factor (2) using the fluctuation-dissipation theorem and the Kramers–Kronig transformation.

To derive the statistical-mechanics quantities of the spin model (1) we first use the Jordan–Wigner transformation

$$\begin{aligned} &c_1^+ = s_1^+, \quad c_1 = s_1^-, \quad c_n^+ = \left(-2s_1^z\right) \ldots \left(-2s_{n-1}^z\right) s_n^+, \\ &c_n = \left(-2s_1^z\right) \ldots \left(-2s_{n-1}^z\right) s_n^-, \quad n = 2, \ldots, N \end{aligned} \tag{3}$$

to express the spin Hamiltonian in fermionic language

$$H=\sum_{n=1}^{N}\Omega\left(c_n^+c_n-\frac{1}{2}\right)$$
$$+\frac{1}{2}\sum_{n=1}^{N}\left(\left(I^++\mathrm{i}D\right)c_n^+c_{n+1}-\left(I^+-\mathrm{i}D\right)c_nc_{n+1}^++I^-\left(c_n^+c_{n+1}^+-c_nc_{n+1}\right)\right). \tag{4}$$

Here the periodic boundary conditions are implied (the boundary term not important for further calculations when $N\to\infty$ has been omitted). Then we perform the Fourier transformation

$$c_n^+=\frac{1}{\sqrt{N}}\sum_{\kappa}\exp\left(\mathrm{i}\kappa n\right)c_\kappa^+,\qquad c_n=\frac{1}{\sqrt{N}}\sum_{\kappa}\exp\left(-\mathrm{i}\kappa n\right)c_\kappa,$$
$$c_\kappa^+=\frac{1}{\sqrt{N}}\sum_{n}\exp\left(-\mathrm{i}\kappa n\right)c_n^+,\qquad c_\kappa=\frac{1}{\sqrt{N}}\sum_{n}\exp\left(\mathrm{i}\kappa n\right)c_n \tag{5}$$

with $\kappa=\frac{2\pi}{N}n$ and $n=-\frac{N}{2},-\frac{N}{2}+1,\ldots,\frac{N}{2}-1$ (if N is even) or $n=-\frac{N-1}{2},-\frac{N-1}{2}+1,\ldots,\frac{N-1}{2}$ (if N is odd) and the Bogolyubov transformation

$$\begin{pmatrix}\beta_\kappa\\ \beta_{-\kappa}^+\end{pmatrix}=\begin{pmatrix}\mathrm{i}u_\kappa & v_\kappa\\ v_\kappa & \mathrm{i}u_\kappa\end{pmatrix}\begin{pmatrix}c_\kappa\\ c_{-\kappa}^+\end{pmatrix},\qquad u_\kappa=\mathrm{sgn}\left(I^-\sin\kappa\right)\frac{1}{\sqrt{2}}\sqrt{1+\frac{\Omega+I^+\cos\kappa}{\lambda_\kappa}},$$
$$v_\kappa=\frac{1}{\sqrt{2}}\sqrt{1-\frac{\Omega+I^+\cos\kappa}{\lambda_\kappa}},\qquad \lambda_\kappa=\sqrt{\left(\Omega+I^+\cos\kappa\right)^2+\left(I^-\sin\kappa\right)^2} \tag{6}$$

to get instead of (4) the final fermionic Hamiltonian

$$H=\sum_{\kappa}\left(-\frac{\Omega}{2}+\left(\Omega+I^+\cos\kappa+D\sin\kappa\right)c_\kappa^+c_\kappa-\mathrm{i}\frac{1}{2}I^-\sin\kappa\left(c_\kappa^+c_{-\kappa}^++c_\kappa c_{-\kappa}\right)\right)$$
$$=\sum_{\kappa}{}'\left(\begin{pmatrix}c_\kappa^+ & c_{-\kappa}\end{pmatrix}\begin{pmatrix}\Omega+I^+\cos\kappa+D\sin\kappa & -\mathrm{i}I^-\sin\kappa\\ \mathrm{i}I^-\sin\kappa & -\Omega-I^+\cos\kappa+D\sin\kappa\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix}c_\kappa\\ c_{-\kappa}^+\end{pmatrix}-D\sin\kappa\right)$$
$$=\sum_{\kappa}{}'\left(\begin{pmatrix}\beta_\kappa^+ & \beta_{-\kappa}\end{pmatrix}\begin{pmatrix}D\sin\kappa+\lambda_\kappa & 0\\ 0 & D\sin\kappa-\lambda_\kappa\end{pmatrix}\begin{pmatrix}\beta_\kappa\\ \beta_{-\kappa}^+\end{pmatrix}-D\sin\kappa\right)$$
$$=\sum_{\kappa}{}'\left(D\sin\kappa\left(\beta_\kappa^+\beta_\kappa+\beta_{-\kappa}\beta_{-\kappa}^+-1\right)+\lambda_\kappa\left(\beta_\kappa^+\beta_\kappa-\beta_{-\kappa}\beta_{-\kappa}^+\right)\right)$$
$$=\sum_{\kappa}\left(D\sin\kappa+\lambda_\kappa\right)\left(\beta_\kappa^+\beta_\kappa-\frac{1}{2}\right)=\sum_{\kappa}\Lambda_\kappa\left(\beta_\kappa^+\beta_\kappa-\frac{1}{2}\right). \tag{7}$$

Here the prime denotes that κ varies (when $N\to\infty$) from 0 to π and the elementary excitations energies are given by

$$\Lambda_\kappa=D\sin\kappa+\lambda_\kappa=D\sin\kappa+\sqrt{\left(\Omega+I^+\cos\kappa\right)^2+\left(I^-\sin\kappa\right)^2}\neq\Lambda_{-\kappa}. \tag{8}$$

It should be noted here that the Bogolyubov transformation (6) does not depend on the Dzyaloshinskii–Moriya interaction D.

3 Isotropic XY chain

The isotropic case $J^x = J^y = J$ is essentially simpler than a general anisotropic case $J^x \neq J^y$ since the Dzyaloshinskii–Moriya interaction can be eliminated from the Hamiltonian (1) by a simple spin axes rotation (see, e.g. [4]). Really, introducing new spin variables

$$\tilde{s}_n^x = s_n^x \cos\phi_n + s_n^y \sin\phi_n, \qquad \tilde{s}_n^y = -s_n^x \sin\phi_n + s_n^y \cos\phi_n, \qquad \tilde{s}_n^z = s_n^z,$$
$$\phi_n = (n-1)\varphi, \qquad \tan\varphi = \frac{D}{J} \tag{9}$$

one finds that

$$\begin{aligned} H &= \sum_{n=1}^{N} \Omega s_n^z + \sum_{n=1}^{N} J\left(s_n^x s_{n+1}^x + s_n^y s_{n+1}^y\right) + \sum_{n=1}^{N} D\left(s_n^x s_{n+1}^y - s_n^y s_{n+1}^x\right) \\ &= \sum_{n=1}^{N} \Omega \tilde{s}_n^z + \sum_{n=1}^{N} \operatorname{sgn}(J)\sqrt{J^2+D^2}\left(\tilde{s}_n^x \tilde{s}_{n+1}^x + \tilde{s}_n^y \tilde{s}_{n+1}^y\right). \end{aligned} \tag{10}$$

Obviously, the thermodynamic properties of the model with Dzyaloshinskii–Moriya interaction are the same as of the model without such interaction, but with renormalized isotropic exchange interaction $|J| \to \sqrt{J^2+D^2}$. As a result, the Dzyaloshinskii–Moriya interaction cannot be revealed from the measurements of thermodynamic quantities. Let us pass to the dynamic quantities.

3.1 zz dynamics and two-fermion excitation continuum

The spin rotations (9) do not effect the z spin component and therefore

$$\left\langle s_n^z(t) s_{n+m}^z(0)\right\rangle\big|_{J,D} = \left\langle \tilde{s}_n^z(t) \tilde{s}_{n+m}^z(0)\right\rangle\big|_{\operatorname{sgn}(J)\sqrt{J^2+D^2},0}. \tag{11}$$

As a result we may use the long-known results for the zz dynamics (see, e.g. [12, 13])

$$\begin{aligned} S_{zz}(\kappa,\omega)|_{J,D} &= S_{zz}(\kappa,\omega)|_{\operatorname{sgn}(J)\sqrt{J^2+D^2},0} \\ &= \int_{-\pi}^{\pi} \mathrm{d}\kappa_1 n_{\kappa_1}\left(1-n_{\kappa_1-\kappa}\right)\delta\left(\omega+\Lambda_{\kappa_1}-\Lambda_{\kappa_1-\kappa}\right). \end{aligned} \tag{12}$$

Here $\Lambda_\kappa = \Omega + \operatorname{sgn}(J)\sqrt{J^2+D^2}\cos\kappa$ is the elementary excitation energy and $n_\kappa = \frac{1}{1+\exp(\beta\Lambda_\kappa)}$ is the Fermi function. From Refs. [12, 13] we know that the zz dynamic structure factor (12) is governed by the two-fermion excitation continuum. The lower, middle, and upper boundaries of the continuum in the plane wavevector κ – frequency ω are given by

$$\frac{\omega_l}{\sqrt{J^2+D^2}} = 2\sin\frac{|\kappa|}{2}\left|\sin\left(\frac{|\kappa|}{2}-\alpha\right)\right|, \tag{13}$$

$$\frac{\omega_m}{\sqrt{J^2+D^2}} = 2\sin\frac{|\kappa|}{2}\sin\left(\frac{|\kappa|}{2}+\alpha\right), \tag{14}$$

and

$$\frac{\omega_u}{\sqrt{J^2+D^2}} = \begin{cases} 2\sin\dfrac{|\kappa|}{2}\sin\left(\dfrac{|\kappa|}{2}+\alpha\right), & \text{if} \quad 0 \le |\kappa| \le \pi - 2\alpha, \\ 2\sin\dfrac{|\kappa|}{2}, & \text{if} \quad \pi - 2\alpha \le |\kappa| \le \pi, \end{cases} \tag{15}$$

respectively; here $\alpha = \arccos\frac{\Omega}{\sqrt{J^2+D^2}}$. The soft modes (i.e. the values of κ at which $\omega_l = 0$) occur at $|\kappa_0| = 0,\ 2\alpha$. The zz dynamic structure factor may exhibit a one-dimensional Van Hove's singularity (i.e. $S_{zz}(\kappa, \omega = \omega_s - 0) \sim \frac{1}{\sqrt{\omega_s - \omega}}$) while approaching the curve

$$\frac{\omega_s}{\sqrt{J^2+D^2}} = 2\sin\frac{|\kappa|}{2}. \tag{16}$$

As temperature increases, the lower boundary of the continuum smears out, i.e. $\frac{\omega_l}{\sqrt{J^2+D^2}} = 0$, and the upper boundary becomes $\frac{\omega_u}{\sqrt{J^2+D^2}} = 2\sin\frac{|\kappa|}{2}$. To conclude, the zz dynamics in the presence of the Dzyaloshinskii–Moriya interaction remains as for the chain without such interaction but with renormalized isotropic exchange interaction $|J| \to \sqrt{J^2+D^2}$. As a result the Dzyaloshinskii–Moriya interaction does not manifest itself in the zz dynamic quantities.

3.2 xx and xy dynamics

Let us pass to the remaining spin correlation functions. Employing equation (9) one finds

$$\begin{aligned}\left\langle s_n^x(t)s_{n+m}^x\right\rangle\big|_{J,D} &= \cos\phi_n\cos\phi_{n+m}\left\langle \tilde{s}_n^x(t)\tilde{s}_{n+m}^x\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\\ &- \cos\phi_n\sin\phi_{n+m}\left\langle \tilde{s}_n^x(t)\tilde{s}_{n+m}^y\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\\ &- \sin\phi_n\cos\phi_{n+m}\left\langle \tilde{s}_n^y(t)\tilde{s}_{n+m}^x\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\\ &+ \sin\phi_n\sin\phi_{n+m}\left\langle \tilde{s}_n^y(t)\tilde{s}_{n+m}^y\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\end{aligned} \tag{17}$$

and similar formulas for $\left\langle s_n^x(t)s_{n+m}^y\right\rangle\big|_{J,D}$, $\left\langle s_n^y(t)s_{n+m}^x\right\rangle\big|_{J,D}$ and $\left\langle s_n^y(t)s_{n+m}^y\right\rangle\big|_{J,D}$. Using further the relations $\left\langle \tilde{s}_n^x(t)\tilde{s}_{n+m}^x\right\rangle\big|_{J,0} = \left\langle \tilde{s}_n^y(t)\tilde{s}_{n+m}^y\right\rangle\big|_{J,0}$, $\left\langle \tilde{s}_n^x(t)\tilde{s}_{n+m}^y\right\rangle\big|_{J,0} = -\left\langle \tilde{s}_n^y(t)\tilde{s}_{n+m}^x\right\rangle\big|_{J,0}$ one gets

$$\begin{aligned}S_{xx}(\kappa,\omega)|_{J,D} &= \frac{1}{2}\Big(S_{xx}(\kappa+\varphi,\omega)|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0} + S_{xx}(\kappa-\varphi,\omega)|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\\ &+ \mathrm{i}\Big(S_{xy}(\kappa+\varphi,\omega)|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0} - S_{xy}(\kappa-\varphi,\omega)|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\Big)\Big) = S_{yy}(\kappa,\omega)|_{J,D}\,,\end{aligned} \tag{18}$$

$$\begin{aligned}S_{xy}(\kappa,\omega)|_{J,D} &= \frac{1}{2}\Big(S_{xy}(\kappa+\varphi,\omega)|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0} + S_{xy}(\kappa-\varphi,\omega)|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\\ &- \mathrm{i}\Big(S_{xx}(\kappa+\varphi,\omega)|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0} - S_{xx}(\kappa-\varphi,\omega)|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\Big)\Big) = -S_{yx}(\kappa,\omega)|_{J,D}\,.\end{aligned} \tag{19}$$

We may use now the results available for the isotropic XY chain without the Dzyaloshinskii–Moriya interaction to follow the effects of the latter interaction on xx (yy) and xy (yx) dynamics. We start from the exact analytical result for zero temperature $\beta = \infty$ and strong fields $\Omega > \sqrt{J^2+D^2}$ [17, 14]. The ground-state is completely polarized $|\mathrm{GS}_s\rangle = \prod_n |\downarrow_n\rangle$ (in spin language) or completely empty $c_\kappa\,|\mathrm{GS}_c\rangle = 0$ (in fermionic language). Therefore, a crucial simplification occurs

$$s_m^+\,|\mathrm{GS}_s\rangle = c_m^+\,|\mathrm{GS}_c\rangle = \frac{1}{\sqrt{N}}\sum_\kappa \exp(\mathrm{i}\kappa m)\,c_\kappa^+\,|\mathrm{GS}_c\rangle\,, \qquad s_m^-\,|\mathrm{GS}_s\rangle = 0. \tag{20}$$

As a result

$$\begin{aligned}\left\langle s_n^x(t)s_{n+m}^x\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0} &= \frac{1}{4}\left\langle \mathrm{GS}_s|s_n^-(t)s_{n+m}^+|\mathrm{GS}_s\right\rangle\\ &= \frac{1}{4N}\sum_\kappa \exp\Big(\mathrm{i}\kappa m - \mathrm{i}\Big(\Omega + \mathrm{sgn}(J)\sqrt{J^2+D^2}\cos\kappa\Big)t\Big)\,,\end{aligned} \tag{21}$$

$$\begin{aligned}\left\langle s_n^x(t)s_{n+m}^y\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0} &= \frac{1}{4\mathrm{i}}\left\langle \mathrm{GS}_s|s_n^-(t)s_{n+m}^+|\mathrm{GS}_s\right\rangle\\ &= -\mathrm{i}\left\langle s_n^x(t)s_{n+m}^x\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0}\,,\end{aligned} \tag{22}$$

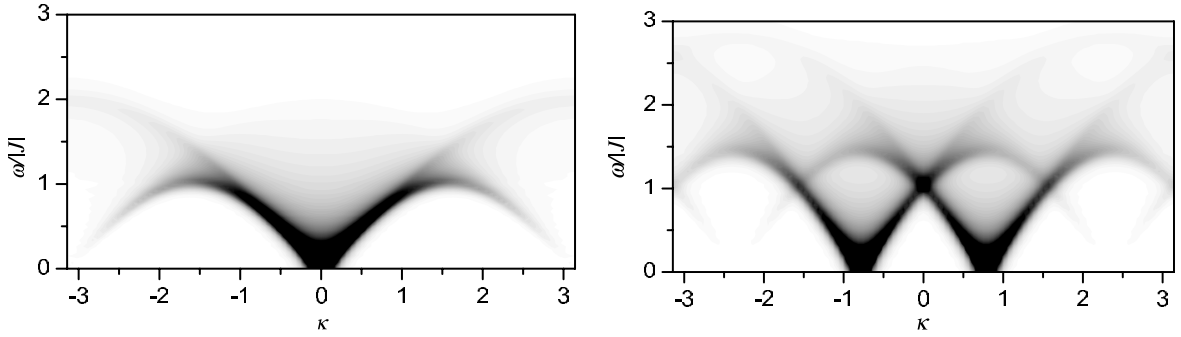

Figure 1. $S_{xx}(\kappa,\omega)$ for $\Omega = 0.001$, $J = -1$, $D = 0$ (left panel), $D = 1$ (right panel), $\beta = 20$.

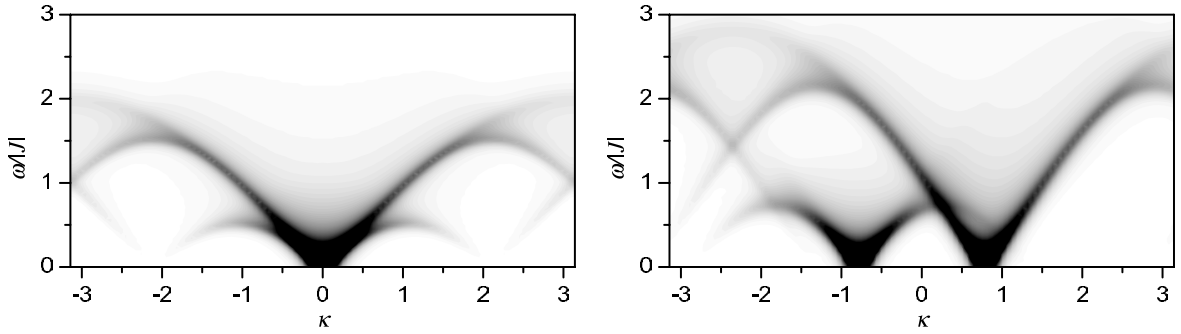

Figure 2. $S_{xx}(\kappa,\omega)$ for $\Omega = 0.5$, $J = -1$, $D = 0$ (left panel), $D = 1$ (right panel), $\beta = 20$.

and therefore

$$S_{xx}(\kappa,\omega)|_{J,D} = \mathrm{i}\, S_{xy}(\kappa,\omega)|_{J,D} = \frac{\pi}{2}\delta\left(\omega - \Omega - \mathrm{sgn}(J)\sqrt{J^2+D^2}\cos\left(\kappa+\varphi\right)\right). \tag{23}$$

If $\Omega < -\sqrt{J^2+D^2}$ instead of (20) one has

$$s_m^+ \left|\mathrm{GS}_s\right\rangle = 0, \qquad s_m^- \left|\mathrm{GS}_s\right\rangle = (-1)^m c_m \left|\mathrm{GS}_c\right\rangle = \frac{1}{\sqrt{N}}\sum_\kappa \exp\left(-\mathrm{i}\left(\kappa+\pi\right)m\right) c_\kappa \left|\mathrm{GS}_c\right\rangle. \tag{24}$$

As a result

$$\begin{aligned}\left\langle s_n^x(t) s_{n+m}^x\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0} &= -\mathrm{i}\left\langle s_n^x(t) s_{n+m}^y\right\rangle\big|_{\mathrm{sgn}(J)\sqrt{J^2+D^2},0} = \frac{1}{4}\left\langle \mathrm{GS}_s\right| s_n^+(t) s_{n+m}^-\left|\mathrm{GS}_s\right\rangle \\ &= \frac{1}{4N}\sum_\kappa \exp\left(-\mathrm{i}\left(\kappa+\pi\right)m + \mathrm{i}\left(\Omega + \mathrm{sgn}(J)\sqrt{J^2+D^2}\cos\kappa\right)t\right),\end{aligned} \tag{25}$$

and therefore

$$S_{xx}(\kappa,\omega)|_{J,D} = -\mathrm{i}\, S_{xy}(\kappa,\omega)|_{J,D} = \frac{\pi}{2}\delta\left(\omega + \Omega - \mathrm{sgn}(J)\sqrt{J^2+D^2}\cos\left(\kappa-\varphi\right)\right). \tag{26}$$

Another exact analytical results may be obtained for infinite temperature $\beta = 0$.

Let us pass to the case of finite temperatures $0 < \beta < \infty$ (and $0 \le \Omega < \sqrt{J^2+D^2}$). Unfortunately, in this case we can calculate the dynamic structure factors $S_{xx}(\kappa,\omega)$, $S_{xy}(\kappa,\omega)$ of the isotropic XY chain which enter (18), (19) only numerically. The xx dynamic structure factor (grey-scale plots) of the finite-size ($N = 400$) isotropic XY chain ($J = -1$) with the Dzyaloshinskii–Moriya interaction at low temperature ($\beta = 20$) for different strengths of the transverse field $\Omega = 0.001$ and $\Omega = 0.5$ can be seen in Fig. 1 and Fig. 2, respectively.

From the earlier studies for isotropic XY chains [14] we know that the xx dynamic structure factor at low temperatures is concentrated in the plane wavevector κ – frequency ω roughly along

the boundaries of the two-fermion continuum (13), (14), (15) (left panels in Figs. 1 and 2). If the Dzyaloshinskii–Moriya interaction is present it is not true any more (right panels in Figs. 1 and 2). The xx dynamic structure factor is concentrated mostly along the curves which correspond to the boundaries of two two-fermion continuum (13), (14), (15) which are shifted by $\pm\varphi$ along the wavevector axis κ and are renormalized $|J| \to \sqrt{J^2+D^2}$ along the frequency axis ω. Thus, the correspondence between the xx and zz dynamic quantities becomes violated. From Figs. 1, 2 we can also see arising of the asymmetry in $S_{xx}(\kappa,\omega)$ with respect to the change $\kappa \to -\kappa$ owing to the Dzyaloshinskii–Moriya interaction as the transverse field deviates from zero. As Ω increases (at fixed D), a redistribution of a weight of $S_{xx}(\kappa,\omega)$ between "left" and "right" two-fermion continua takes place until the "left" one completely disappears as Ω exceeds $\sqrt{J^2+D^2}$. (Obviously, we should change "left" to "right" and vice versa when Ω increases its value being negative.) It is important to note that the soft modes of "left" and "right" continua originating from the soft mode of the original continuum at $\kappa = 0$ occur at $\kappa = \pm\varphi$ and they are field independent; they can be used for determining of the value of the Dzyaloshinskii–Moriya interaction in the corresponding compounds. For electron-spin resonance experiments the frequency profiles of $S_{xx}(\kappa,\omega)$ at $\kappa = 0$ and $\kappa = \pi$ are relevant [18]. We see that both profiles change drastically if the Dzyaloshinskii–Moriya interaction is present. For example, the zero-frequency peak at $\kappa = 0$, which is relevant for the ferromagnetic case, moves towards higher frequency and its position is determined by the value of the Dzyaloshinskii–Moriya interaction. Similarly, the Dzyaloshinskii–Moriya interaction spectacularly changes the frequency profile at $\kappa = \pi$, which is relevant for the antiferromagnetic case. Obviously, these effects can be used for experimental determining of the value of the Dzyaloshinskii–Moriya interaction in the corresponding compounds.

4 Concluding remarks

To summarize, we have studied the xx (yy) and xy (yx) dynamic structure factors of the spin-$\frac{1}{2}$ isotropic XY chain with the Dzyaloshinskii–Moriya interaction and have demonstrated how the relation between these quantities and the zz dynamic structure factor is modified due to the Dzyaloshinskii–Moriya interaction. We have discussed how the Dzyaloshinskii–Moriya interaction may manifest itself in the dynamic experiments.

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Extensions of Physical Theories: Quantum Mechanics with Internal Degrees of Freedom and Nonlinear Transformations

H.D. DOEBNER †, *A. KOPP* † *and R. ZHDANOV* ‡

† *Department of Physics and Material Sciences, Technical University of Clausthal, Germany*
E-mail: *asi@pt.tu-clausthal.de*

‡ *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *renat@imath.kiev.ua*

We review several nonlinear mathematical models for physical systems which are based on a gauge-like structure. For systems with internal degrees of freedom like spin, a nonlinearity leads to coupling of the internal coordinates; this kind of interaction appears as an addition to the coupling through Pauli or Dirac matrices. We briefly discuss a family of nonlinear and relativistically invariant extensions of the Dirac equation.

1 Introduction

Construction of mathematical models for physical systems is often connected with the problem of extending a partly successful theory. The motivations are different: e.g. the parameter regions and the space time domains of a model are too narrow or the chosen mathematical framework is not suitable for a realistic model. Such extensions are expected to explain new observable effects and to provide corrections to the "old" theory. For their development different options are at hand: generalisations of the theory, changes of the present mathematical formalism or inventions of new mathematical methods.

2 An example

2.1 Gauge like models

We illustrate this problem for an example in the following (gauge theoretic) design: A system is modelled through linear partial differential operator (PDO) $\mathbb{D}$ on a suitable k-vector valued function space $\mathcal{G}$ over space-time $\mathbb{R}^3_x \times \mathbb{R}^1_t$. The solution variety

$$\mathcal{B} = \{f(x,t)/\mathbb{D}f(x,t) = 0,\ f(x,t) \in \mathcal{G}\}$$

serves as a *basic* background structure for the set $\mathcal{O} = \{o\}$ of (all or a part of the) physical observables o. This means that the elements of $\mathcal{O}$ are modelled through functions F_α, $\alpha = 1, \ldots, s$, of the solution variety. These *utility* functions form a set $B = \{F_\alpha(f),\ \alpha = 1, \ldots, s, f \in \mathcal{B}\}$ with a map

$$\mathcal{O} \ni o \mapsto F_\alpha(f),$$

where α depends on o. Hence the system is modelled through $\{\mathcal{G}, \mathbb{D}, B\}$. In general this description is not complete. Further properties have to be included and the physical interpretation is essential. The characterisation of the models as 'gauge like' stems from the example of electrodynamics and quantum mechanics [1].

2.2 Extensions of gauge like models

There are only few plausible and manageable possibilities to extend a gauge like theory based on $\{\mathcal{G}, \mathbb{D}, B\}$. We mention here two lines: the *deformation* of possible symmetries of $\mathbb{D}$ and the construction of nonlinear $\mathbb{D}$ through extensions with nonlinear transformations in $\mathcal{G}$.

A. Deformations of symmetries. If $\mathbb{D}$ has a group theoretical or an algebraic symmetry S, one can extend this symmetry through deformations S_{def} of S. Use the function space $\mathcal{G}'$ which carries a realisation of S_{def}, construct from $\mathbb{D}$ the corresponding deformed operator $\mathbb{D}_{\text{def}}$ and take the same utility functions as before. The result is a deformed extension $\{\mathcal{G}', \mathbb{D}_{\text{def}}, B\}$ of $\{\mathcal{G}, \mathbb{D}, B\}$.

B. Nonlinear extensions. Another method is based on an interplay between the linear operator $\mathbb{D}$ and B; it leads to nonlinear extensions. Consider a group $\mathfrak{N}$ of linear or nonlinear and invertible transformations in $\mathcal{B}$

$$\mathfrak{N} \ni N : \mathcal{B} \ni f \mapsto N[f] = N(f)f \in \mathcal{B}$$

The partial differential equation (PDE) $\mathbb{D}f = 0$ implies

$$\mathbb{D}_N = \mathbb{D}N(f) \quad \text{with} \quad \mathbb{D}_N f' = 0, \quad f' = N(f)f.$$

The operator $\mathbb{D}_N$ is nonlinear (for nonlinear N) and leads together with the utility functions to the same description as before *if* N is chosen such that the utility functions are invariant under N, namely

$$F_\alpha(N[f]) = F_\alpha(f), \qquad F_\alpha \in B.$$

Transformations with this *Equivalence Condition* [1] also generate a transformation group $\mathfrak{N}^e \subset \mathfrak{N}$ (e denotes equivalence). The theory based on $\{\mathcal{G}, \mathbb{D}, B\}$ and $\{\mathcal{G}, \mathbb{D}_N, B\}$ describes the same physical system; both are *physically equivalent*. However, through the description with $\mathbb{D}_N$ some *additional properties* A of the linear system which are not encoded in $\{\mathcal{G}, \mathbb{D}, B\}$, could be lost, e.g. the space-time symmetry or the separability of $\mathbb{D}$. To restore such properties, restrict $\mathfrak{N}$ to subgroups $\mathfrak{N}^A$ such that they are invariant and use $\mathfrak{N}^A \cap \mathfrak{N}^e = \mathfrak{N}^{A,e}$. Nonlinear extensions through $\mathfrak{N}^e \in \mathfrak{N}$ are trivial reformulations of the linear system. To get something "new", i.e. a non trivial extension, change $\mathbb{D}_N$ to $\mathbb{D}_{\text{ext}} = (\mathbb{D}_N)_{\text{ext}}$ if there is an obvious and plausible method to do this [1]. This is the case if the coefficients (functions, numbers) in $\mathbb{D}_N$ are related among each other: Break this relations, i.e. consider the coefficients as independent or free, get $\mathbb{D}_{\text{ext}}$, use the same $\mathcal{G}$ and B and find a nonequivalent description $\{\mathcal{G}, \mathbb{D}_{\text{ext}}, B\}$; hence a "new" system appears which contains the "old" system.

2.3 Quantum mechanics as a gauge like model

We specialise this design to the time evolution of a quantum mechanical observable. Here the function space $\mathcal{G}$ is a suitable Hilbert space $\mathcal{H}$, its elements are k-vector valued wave functions $\psi(x,t) = (\psi_1(x,t), \ldots, \psi_k(x,t))$, $\mathbb{D}_{\mathsf{Q}}$ is the quantum mechanical time evolution, e.g. the Schrödinger operator $\mathbb{D}_S$, and the (only, i.e. $s = 1$) utility function is the positional density for all x and t

$$F_\varrho \overline{\psi}\psi = \sum_{j=1}^{k} \psi_j^* \psi_j$$

with $\overline{\psi} = (\psi_1^*, \ldots, \psi_k^*)^T$, T denotes the transpose. All properties of the system are encoded in one utility function F_ϱ and the evolution operator (see e.g. [3]). The system is characterised through $(\mathcal{H}, \mathbb{D}_{\mathsf{Q}}, F_q)$. A similar design appears in gauge theories like electrodynamics; the situation in relativistic quantum field theory is different.

2.4 Extensions of quantum mechanical evolutions

We sketch some application of the lines A and B (Section 2.2) and present in Sections 3 and 4 the detailed version of line B.

A. Deformations of symmetries. A symmetry of a quantum system is often connected with algebraic symmetries through a linear (possibly integrable) representation of a finite or infinite dimensional (space time or kinematical) Lie algebra; examples in $\mathbb{R}^3_x \times \mathbb{R}^1_t$ are: the central extension of the Galilei algebra, the Poincaré algebra or the Lie algebra of a subgroup of the inhomogeneous diffeomorphism group of the configuration space (Borel quantisation [4,5]); inhomogeneous current algebras [6,7]. There are physical and mathematical reasons to deform (extend or contract) this algebra (in the sense of Gerstenhaber) in the category of Lie algebras or, more general, in the category of Hopf algebras. In the first case one gets a relation between Poincaré and Galilei algebras. In the latter case one finds a q-algebra or q symmetry, the evolution operator appears as a difference operator, the space-time has discrete properties with a corresponding behaviour of the wave functions and the utility function. Another related extension is choice of a noncommutative space-time. Algebraic symmetries can be formulated in the usual (linear) quantum mechanical framework. Properties and behaviour of the corresponding systems are available and a comparison with undeformed symmetries is possible. The interest in this mathematical and physical field is strong. However, there is up to now only "weak" experimental evidence for extended symmetries.

B. Nonlinear extensions. Quantum mechanics is based deeply on linear structures (the above mentioned gauge like design is e.g. a linear construction). A physically acceptable nonlinear extension of quantum mechanics – e.g. with nonlinear evolutions equations – such that the linear theory appears as a linear approximation is not known. The reason is that the linear framework does not allow for nonlinear operators (Hamiltonians) [8]. It is in principle difficult to develop such extensions; also the usual physical interpretation breaks down. On the other hand a deeper reason for a linear structure of quantum mechanics is unknown. Therefore a reasonable first ansatz for a nonlinear extension is of interest, especially a derivation of nonlinear evolutions equations from first (physical) principles. The known methods for calculation of physical properties can be applied in an extended theory as approximations for deviations from the linear theory. In recent quantum mechanical precision experiments such nonlinear deviations were not found [9]. But future experiments and a new experimental design could reveal quantum mechanical nonlinearities.

3 Nonlinear Schrödinger, Pauli and Dirac equations

3.1 Linear evolution equation and a locality condition

We present and partly review nonlinear extensions for the evolution of a quantum mechanical observable in the $(\mathcal{H}, \mathbb{D}_{\mathrm{Q}}, F_\varrho)$ design (see Section 2.3). For $k = 1$, $\mathbb{D}_{\mathrm{Q}}$ is the Schrödinger operator $\left(\nu_1 = -\frac{\hbar}{2m},\ \mu_0 = \frac{1}{\hbar}\right)$

$$\mathbb{D}_S = \left(-i\partial_t + \nu_1 \Delta + \mu_0 V(x) \right);$$

for $k = 2$, $\mathbb{D}_{\mathrm{Q}}$ is the Pauli operator

$$\mathbb{D}_P = \left(-i\partial_t + V_1 \Delta + \mu_0 V(x) + \alpha \vec{B}(x,t) \times \vec{\sigma} + \beta \vec{L} \times \vec{\sigma} \right),$$

with $\vec{\sigma}$ as Pauli matrices, $\vec{B}(x,t)$ as external field and $\vec{L}$ as angular momentum; for $k = 4$, $\mathbb{D}_{\mathrm{Q}}$ is the Dirac operator ($\hbar = 1$)

$$\mathbb{D}_D = \left(j_\mu p^\mu - m \right), \qquad p^\mu = i \frac{\partial}{\partial x^\mu};$$

we use the notation of [13]; the component index k runs from 0 to 3, the time component is labelled by 0; the solutions depend on x_0, x_1, x_2, x_3.

As a first *additional simplifying property A* for $\mathbb{D}$ (to avoid e.g. boundary value discussions) we want that the highest order of $\mathbb{D}_{\mathrm{Q}}$ and of $\mathbb{D}_{\mathrm{Q},N}$ is the same. This implies a locality condition for N

$$N(\psi) = (N_1[\psi], \ldots, N_k[\psi]) = (N_1(\psi)\psi_1, \ldots, N_k(\psi)\psi_k).$$

The nonlinear transformations $N_i(\psi)$, $i = 1, \ldots, k$ act on $\mathcal{H}$ as multiplicative operators and depend on ψ only. They form a group $\mathfrak{N}^l \subset \mathfrak{N}$ (l stands for locality).

3.2 A separability condition

A next additional property, a *separability condition*, stems from the fact that quantum mechanical systems are composed of n one-particle systems with Hilbert spaces $\mathcal{H}_j$ and self adjoint operators $\mathbb{A}_j$ for the same observable o' in each system $j = 1, \ldots, n$. The Hilbert space for the n-particle system is chosen as

$$\mathcal{H}^{(n)} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$$

with a dense set of product states

$$P^{(n)} = \left\{\psi^1 \otimes \cdots \otimes \psi^n,\ \psi^j \in \mathcal{H}_j,\ j = 1, \ldots, n\right\}.$$

The corresponding observable o' of the n-particle system is represented on the product states as

$$\mathbb{A}^{(n)} = \left(\mathbb{A}_1\psi^1 \otimes \cdots \otimes \mathbb{A}_n\psi^n\right)$$

and extend to $\mathcal{H}^{(n)}$ through linear completion.

For a nonlinear operator, like N, this procedure is not possible. As in the linear case $N^{(n)}$ can be defined on product states $\mathcal{P}^{(n)}$ but an extension to $\mathcal{H}^{(n)}$ is not defined; some further guideline is needed. Here one of the difficulties for nonlinear extensions shows up. It is plausible to require as a minimal assumption that for a 2-particle system the relation

$$N^{(2)}\left[\psi^1 \otimes \psi^2\right] = N\left[\psi^1\right] \otimes N\left[\psi^2\right]$$

holds. This is a *separability condition* [1], it implies that a k -component function $N(\psi)$ on $P^{(2)}$ exists such that for $N \ni \mathfrak{N}^l$ and any two ψ^1, ψ^2, $i = 1, 2$ the condition holds. Such N form a group $\mathfrak{N}^{l,s}$ (s for separability). Its elements are calculated in [10]. We quote the results for $\mathcal{G}$ (which may be specified for the corresponding Hilbert space) and use the (not unique) polar decomposition for $\psi_j(x,t) = R_j(x,t)\exp iS_j(x,t)$.

The transformations

$$N : \mathcal{G} \mapsto \mathcal{G}$$

which satisfy the locality and separability condition are given through

$$N_j(\psi)_{[a,b,F]} = (R_j)^{a-1}\exp\left((b-i)S_j\right) \cdot F_j\left(\frac{R_2}{R_1}, \ldots, \frac{R_n}{R_1}, S_2 - S_1, \ldots, S_n - S_1\right)$$

The N are labelled by two complex parameter a, b and k functions

$$F(\psi) = (F_1(\psi), \ldots, F_k(\psi)).$$

In order to derive the group multiplication law, arrange the parameter $a = \widetilde{a} + i\widehat{a}$, $b = \widetilde{b} + i\widehat{b}$ as

$$K = \begin{pmatrix} \widetilde{a} & \widehat{a} \\ \widetilde{b} & \widehat{b} \end{pmatrix}$$

and find for the product

$$N_{[K_2,F_2]}\left(N_{[K_1,F_1]}(\psi)\,\psi\right) N_{[K_1,F_1]}(\psi)\,\psi = N_{[K_3,F_3]}\psi$$

with

$$K_3 = K_2 K_1,$$
$$F_{31} = |F_{1j}|^{a_2} \exp\left(b_2 \arg F_{1j}\right) F_{2j}\left(u_2, \ldots, u_n, v_2, \ldots, v_n\right).$$

The variables in F_{1j} and in F_{2j} are as before resp. given through $(i = 2, \ldots, k)$

$$u_i = \left(\frac{R_i}{R_1}\right)^{\widetilde{a}_i} \exp \widetilde{b}_1 \left(S_1 - S_i\right) |F_{1i}||F_{11}|^{-1},$$
$$v_i = \widehat{a}_1 \ln\left(\frac{R_i}{R_1}\right) + \widehat{b}_1 \left(S_1 - S_i\right) \arg\left(F_{1i} F_{11}\right).$$

respectively. The group $\mathfrak{N}^{l,s}$ is a local (infinite) parameter group; the element N_{11} is the identity, $N_{[K,F]}$ is locally invertible in a neighbourhood of the identity; the associativity is respected. For $n > 2$ the corresponding result holds.

3.3 Quantum mechanical equivalence condition

We combine the essential equivalence condition with the locality condition (more technical) and the separability condition (necessary to build n-particle states) and determine

$$\mathfrak{N}^{l,s} \cap \mathfrak{N}^{l,e} = \mathfrak{N}^{l,s,e}.$$

We quote the result of [10]: The transformations $N \in \mathfrak{N}^{l,s,e}$ which satisfy the locality, separability and equivalence condition are given through

$$N_j(\psi)_{[a,b,F^e]} = R_j^{i\widehat{a}} \exp i\left(\widehat{b} - 1\right) S_j \cdot F_j^e\left(\frac{R_2}{R_1}, \ldots, \frac{R_n}{R_1}, S_2 - S_1, \ldots, S_n - S_1\right).$$

with the condition for the k components of F^e:

$$\sum_{j=1}^{k} R_j^2 \left(|F_j^e|^2 - 1\right) = 0.$$

The $\mathfrak{N}^{l,s,e}$ form a group.

3.4 Application for $k = 1$: Schrödinger equation

In the Schrödinger case we have in the configuration space $\mathbb{R}^3_x$ with wave function $\psi = R \exp is$ for

$$\mathfrak{N}^{l,s,e} \ni N[\psi]_{[\widehat{a},\widehat{b}]} = R^{1+i\widehat{a}} \exp i\widehat{b} s \cdot F^e.$$

F^e is a constant phase factor; we choose $F^e = 1$. This transformation is written as $(\widehat{a} = \gamma,\ \widehat{b} = \Lambda)$

$$N[\psi]_{[\gamma,\Lambda]} = \exp i\left(\gamma \ln R + (\Lambda - 1)\, s\right) \psi.$$

Following our construction we calculate [1] a nonlinear Schrödinger equation $\mathbb{D}_{S,N}\psi' = 0$ with $\psi' = N[\psi]_{[\gamma,\Lambda]}$ and find

$$
\begin{aligned}
&i\partial_t\psi = \left(\nu'\Delta + \mu_0' V + F_{DG}^{(0)}[\psi]\right)\psi,\\
&F_{DG}^{(0)}[\psi] = \mu_1'\left(R_1[\psi] - R_4[\psi]\right) - \frac{1}{2}\mu_1' R_2[\psi] + \kappa'\left(R_2[\psi] - \frac{1}{2}R_5[\psi]\right),\\
&R_1[\psi] = \frac{\nabla\cdot\mathbf{J}}{\rho}, \qquad R_2[\psi] = \frac{\Delta\rho}{\rho},\\
&R_3[\psi] = \frac{\mathbf{J}^2}{\rho^2}, \qquad R_4[\psi] = \frac{\mathbf{J}\cdot\nabla\rho}{\rho}, \qquad R_5[\psi] = \frac{(\nabla\rho)^2}{\rho^2},
\end{aligned}
$$

where $\rho = \overline{\psi}\psi$ and $\mathbf{J} = \frac{1}{2}\left(\overline{\psi}\nabla\psi - \left(\nabla\overline{\psi}\right)\psi\right)$.

The coefficients in this PDO are not independent; the constraint relations are

$$
\nu_1' = \frac{1}{\Lambda}\nu_1, \qquad \mu_0' = \Lambda\mu_0, \qquad \mu_1' = -\frac{\gamma}{\Lambda}\nu_1, \qquad \kappa' = \frac{\gamma^2 + \Lambda^2 - 1}{2\Lambda}\nu_1.
$$

Following the method mentioned in Section 2.2 we extend $\mathbb{D}_{S,N}$ through a breaking of this constraints. An extended evolution operator $(\mathbb{D}_{S,N})_{\text{ext}}^{(1)}$ appears. One can continue this procedure: Apply $N \in \mathfrak{N}^{l,s,e}$ to $\mathbb{D}_{\text{ext}}^{(1)}$ one gets an evolution operator with constraints and different from $\mathbb{D}_{\text{ext}}^{(1)}$. Break this constraints and continue. The process stops after 4 steps and yields a 8-parameter nonlinear Schrödinger equation [1] (DG equation)

$$
\begin{aligned}
i\partial_t\psi = {} & \left(\nu_1\Delta + \mu_0 V\right)\psi + i\nu_2 R_2[\psi]\psi + \mu_1 R_1[\psi]\psi + \left(\mu_2 - \frac{1}{2}\nu_1\right)R_2[\psi]\psi\\
& + \left(\mu_3 + \nu_1\right)R_3[\psi]\psi + \mu_4 R_4[\psi]\psi + \left(\mu_5 + \frac{1}{4}\nu_1\right)R_5[\psi]\psi.
\end{aligned}
$$

The DG equation was first derived [11] with a mathematically completely different method (representation theory of infinite dimensional Lie algebras) and motivated physically through an analysis of geometrical details of the kinematics of the system.

3.5 Application for $k = 2$: Pauli equation

Here, already the special complications connected with vector-valued wave functions appear. This is because a two component function $F = (F_1, F_2)$ reduce to two equivalent constant phase factors only for a special type of solutions (type 0); they have the property that different (spin-) components of the wave function are not coupled through nonlinearities in the evolution equation; only the spin matrices yields a coupling. For all other types a coupling through the nonlinear term and the spin matrices occurs. This remains valid also for extension of the non linear Pauli operator $\mathbb{D}_P$. The corresponding solutions are involved and clumsy. Even if $\alpha, \beta = 0$, i.e. if the term with the Pauli matrices is absent, the nonlinearities yield a coupling of the (spin-) components which resembles in some sense an analogue for a spin-like degree of freedom. A more detailed analysis of extensions of nonlinear Pauli equations is in preparation [12].

4 Applications for $k = 4$: Dirac equation

4.1 Poincaré invariance

We mentioned already that a description through $(\mathcal{G}, \mathbb{D}, B)$ is in general not complete. Additional properties of D could be physically significant for the interpretation and for an extension, e.g. the

behaviour under space-time translations. For the Dirac operator $\mathbb{D}_D$ the spin $\frac{1}{2}$ representation U of the Poincaré group is such an essential property. Hence, it is reasonable to choose N from $\mathfrak{N}^l$ such that $\mathbb{D}_{D,N}$ has the same transformation property as $\mathbb{D}_D$. This is guaranteed if (local) N transform under $\mathcal{U}$ as

$$N[U\psi] = UN[\psi]$$

The general form of N with the above property follows from a theorem of Fushchych and Zhdanov [13, 14] (in connection with nonlinear Dirac equation with local nonlinearities)

$$N(\psi)\psi = (f_1(X,Y) + f_2(X,Y)\gamma_5)\psi,$$

f_1, f_2 are independent complex functions depending on the invariant quantities $X = \overline{\psi}\psi$ and $Y = \overline{\psi}\gamma_5\psi$. Such N form a group $\mathfrak{N}^{l,P}$ (P for Poincaré invariance).

4.2 Consideration of separability and equivalence conditions

In this section (see also [10]) we are interested in the *quantum mechanical* Dirac equation. Therefore we assume for $N \in \mathfrak{N}^{l,P}$ also the separability and the equivalence condition. The situation in quantum field theory needs another approach and is not related to the later constructions of nonlinear extensions. With the previous result it is straightforward to calculate the corresponding transformation groups through

$$\mathfrak{N}_2 \equiv \mathfrak{N}^{l,s,P,e} = \mathfrak{N}^{l,s,e} \cap \mathfrak{N}^{l,P}.$$

We quote the result: Elements of $\mathfrak{N}_2$ are labelled through a real parameter a and two real functions ϕ and ϱ depending on the invariant Z. The transformation is given through

$$N(\psi)_{[\widehat{a},\phi,\varrho]}\psi = \exp\left(i\frac{\widehat{a}}{2}\ln\left(\overline{\psi}\psi\right) + i\phi(z) + \varrho(z)\gamma_5\right)\psi.$$

The corresponding group relations are involved.

4.3 Some extended nonlinear Dirac equations

The gauge like model yields with $\mathfrak{N}_2$ and a family $\mathbf{F}_2$ of nonlinear Dirac equations depending on m (mass) and the parameter and functions which label the transformations. We calculate

$$(\gamma_\mu p^\mu - m)N(\psi)\psi = (\gamma_\mu p^\mu + H_2(\psi))\psi.$$

For the nonlinear term $H_2(\psi)$ we find ($'$ denotes the derivative; $Z = \frac{X}{Y}$):

$$H_2(\psi) = (\gamma_\mu p^\mu X)\, i\frac{\widehat{a}}{2X} + (\gamma_\mu p^\mu Z)\left(i\phi'(Z) + \varrho'(Z)\gamma_5\right) - m\exp 2\varrho(Z)\gamma_5.$$

As explained before the families are *reformulations of the linear theory* with certain additional properties (Poincaré invariance, separability, equivalence). To get new evolutions, which are related to the linear one, we applied in Section 3.4 for $k = 1$ the method of gauge generalisation. This method can be applied also to $\mathbf{F}_2$: The coefficient function $\varrho(Z)$ and $\varrho'(Z)$ are obviously not independent. Break the relation between them, generalise the term proportional to $\frac{\widehat{\alpha}}{2X}$ and get an extension of $\mathbf{F}_2$

$$H_{2,\mathrm{ex}} = (\gamma_\mu p^\mu X)\, ig(X) + (\gamma_\mu p^\mu Z)\cdot(ik(Z) + l(Z)\gamma_5) - m\exp 2n(Z)\gamma_5.$$

This family with functions $g(X)$, $k(Z)$, $l(Z)$ and $n(Z)$ is Poincaré invariant, local and separable; the equivalence condition does not hold.

5 Conclusions

We presented and reviewed mathematical models for physical systems which are based on a gauge-like structure, i.e. with a description of observable through (utility) functions on the solution manifold of a family of PDO's. We explained some methods to extend this models, especially on extensions with nonlinear transformations on the solution manifold and a subsequent "gauge generalisation". Quantum mechanical evolutions of observable and states in a Hilbert space spanned through k-vector valued wave functions have a gauge-like structure. We applied the extension method through physically motivated nonlinear transformations which behave like nonlinear gauge transformations. In the Schrödinger case $k = 1$ some mathematical and physical details of an extended nonlinear theory and its possible relevance are known [5, 8]. For systems with $k > 1$, i.e. with internal degrees of freedom like spin, a nonlinearity leads to a coupling of the internal coordinates; this kind of interaction appears as an addition to the coupling through Pauli or Dirac matrices. For $k = 2$ the Pauli equation was mentioned [12]; for $k = 4$ quantum mechanical Dirac equations were discussed and we constructed a family of nonlinear and relativistic invariant extensions [10]. This extensions are not relevant in connection with the relativistic quantum field theory for spin $\frac{1}{2}$ particles.

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Irreducible Killing Tensors from Conformal Killing Vectors

S. Brian EDGAR [†], *Rafaelle RANI* [‡] *and Alan BARNES* [§]

[†] *Department of Mathematics, Linköping University, Linköping, Sweden S581-83*
E-mail: *bredg@mai.liu.se*

[‡] *Institute of Astronomy and Astrophysics, Department of Theoretical Astrophysics, University of Tübingen, D-72076 Tübingen, Germany*
E-mail: *rani@tat.physik.uni-tuebingen.de*

[§] *School of Engineering & Applied Science, Aston University, Birmingham B4 7ET, UK*
E-mail: *barnesa@aston.ac.uk*

Koutras has proposed some methods to construct reducible proper conformal Killing tensors and Killing tensors (which are, in general, irreducible) when a pair of orthogonal conformal Killing vectors exist in a given space. We give the completely general result demonstrating that this severe restriction of orthogonality is unnecessary. In addition we correct and extend some results concerning Killing tensors constructed from a single conformal Killing vector. We give as an example a Kimura metric and demonstrate how it is possible to construct a larger class of reducible proper conformal Killing tensors and Killing tensors than permitted by the Koutras algorithms. In addition, from our new result that all conformal Killing tensors are reducible in conformally flat spaces, we have a method of constructing all conformal Killing tensors and hence all the Killing tensors (which will in general be irreducible) of conformally flat spaces using their conformal Killing vectors.

1 Introduction

In this paper we shall consider an indirect method of constructing irreducible Killing tensors via conformal Killing vectors which has been proposed by Koutras [5], and also used recently by Amery and Maharaj [1]. However, in these two papers the underlying principle is not completely transparent nor are the algorithms obtained the most general; this is partly due to a distraction caused by the trace-free requirement in the definitions of conformal Killing tensors which is used in these two papers. Also in a paper by O'Connor and Prince [6] there has been an independent related discussion, but in the narrower context of a particular metric. We shall show that the arguments in these papers can be made more general than in the original presentations; in particular, we shall show that our more general approach enables us to obtain more conformal Killing tensors and hence more irreducible Killing tensors than those which can be obtained by the algorithms in [5, 1]. We extend a result of Weir [8] for flat spaces to conformally flat spaces and obtain the maximum number of conformal Killing tensors, which shows that they are all reducible in conformally flat spaces.

We begin with some notation and known results for Killing vectors and Killing tensors of order 2.

A *Killing vector* ξ satisfies $\xi_{(a;b)} = 0$.

A *Killing tensor of order 2* is a symmetric tensor K_{ab} such that $K_{(ab;c)} = 0$.

Reducible Killing tensors are built from Killing vectors ξ_J and the metric g_{ab}

$$K_{ab} = a_0 g_{ab} + \sum_{I=1}^{N} \sum_{J=I}^{N} a_{IJ} \xi_{I(a} \xi_{|J|b)},$$

where a_0 and a_{IJ} for $J \geq I$ are constants.

All other Killing tensors are called *irreducible (degenerate, trivial).*

- Physically the interest in Killing tensors is due to their connection with quadratic first integrals of geodesic motion and separability of classical partial differential equations.
- Irreducible Killing tensors yield quadratic first integrals which are not simply linear combinations of products of the linear first integrals associated with the Killing vectors.
- The maximum number of linearly independent Killing tensors in an n-dimensional Riemannian space is $n(n+1)^2(n+2)/12$.
- The existence of this maximum number is a necessary and sufficient condition for spaces of constant curvature.
- In n-dimensional spaces of constant curvature all Killing tensors are reducible (built from Killing vectors and the metric).

We have analogous notation and results for conformal Killing vectors and conformal Killing tensors of order 2.

A *conformal Killing vector* χ satisfies $\chi_{(a;b)} = \vartheta g_{ab}$.

A *conformal Killing tensor of order 2* is a symmetric tensor Q_{ab} such that $Q_{(ab;c)} = q_{(a}g_{bc)}$.

Reducible conformal Killing tensors are built from conformal Killing vectors χ_J and a scalar times the metric g_{ab}

$$Q_{ab} = \sum_{I=1}^{M}\sum_{J=I}^{M} a_{IJ}(\chi_{I(a}\chi_{|J|b)} + \lambda g_{ab}.$$

All other conformal Killing tensors are called *irreducible.*

A *trace-free conformal Killing tensor of order 2* is a symmetric trace-free tensor P_{ab} such that $P^i{}_i = 0$ and $P_{(ab;c)} = p_{(a}g_{bc)}$

Reducible trace-free conformal Killing tensors are built from conformal Killing vectors χ_J and a scalar times the metric g_{ab}

$$P_{ab} = \sum_{I=1}^{M}\sum_{J=I}^{M} a_{IJ}\left(\chi_{I(a}\chi_{|J|b)} - \frac{1}{n}\chi_I{}^c\chi_{Jc}g_{ab}\right).$$

All other trace-free conformal Killing tensors are called *irreducible.*

- Proper conformal Killing tensors do not generate quadratic first integrals for geodesic motion in general, but they do so for null geodesics.
- Irreducible proper conformal Killing tensors yield quadratic first integrals which are not simply linear combinations of products of the linear first integrals associated with the proper conformal Killing vectors.
- The maximum number of trace-free conformal Killing tensors in an n-dimensional Riemannian space is $(n-1)(n+2)(n+3)(n+4)/12$, [8].

2 To find all Killing tensors in conformally flat spaces

It is well known that, if χ^a is a conformal Killing vector of the metric g_{ab} with conformal factor ϑ, then it is also a conformal Killing vector of the conformally related metric $\tilde{g}_{ab} = e^{2\Omega}g_{ab}$ with conformal factor $\tilde{\vartheta} = \vartheta + \Omega_{,c}\chi^c$. We now obtain the analogous result for conformal Killing tensors:

Theorem 1. *If Q^{ab} is a conformal Killing tensor satisfying $\nabla^{(a}Q^{bc)} = q^{(a}g^{bc)}$, then Q^{ab} is also a conformal Killing tensor of the conformally related metric $\tilde{g}_{ab} = e^{2\Omega}g_{ab}$. Q^{ab} satisfies $\tilde{\nabla}^{(a}Q^{bc)} = \tilde{q}^{(a}\tilde{g}^{bc)}$, where $\tilde{q}^a = q^a + 2\Omega_{,d}Q^{da}$.*

Proof. The proof is straightforward involving an evaluation of $\tilde{\nabla}^{(a}Q^{bc)}$ using the result that

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + \delta^a_b\Omega_{,c} + \delta^a_c\Omega_{,b} - \Omega^{,a}g_{bc}.$$ ■

We cannot determine the number of linearly independent conformal Killing tensors because of the freedom in their trace; but we can consider the number of linearly independent trace-free conformal Killing tensors. From the above theorem and the analogous result for conformal Killing vectors we have,

Corollary 1. *The number of linearly independent trace-free conformal Killing tensors is invariant under conformal change of the metric. The number of linearly independent reducible trace-free conformal Killing tensors is similarly invariant.*

The maximum number of trace-free conformal Killing tensors in an $n\,(>2)$-dimensional Riemannian space has been found by Weir [8] to be $(n-1)(n+2)(n+3)(n+4)/12$, and he has shown that this number is attained in flat space.

For M conformal Killing vectors there are in general $M(M+1)/2$ symmetrised products of pairs of conformal Killing vectors; hence, in conformally flat spaces, we can construct $M(M+1)/2$ reducible trace-free conformal Killing tensors. In an n-dimensional Riemannian space there exist at most $(n+1)(n+2)/2$ linearly independent conformal Killing vectors, and the maximum number can be attained only in conformally flat spaces. Hence by substituting $M = (n+1)(n+2)/2$ we can obtain the maximum possible number of reducible conformal Killing tensors in an n-dimensional Riemannian space; but of course these need not all be linearly independent. (For example, in 4 dimensions there are 120 reducible trace-free conformal Killing tensors which can be constructed from the metric and the conformal Killing vectors, while the theoretical upper limit of linearly independent trace-free conformal Killing tensors is only 84.) However, Weir [8] has shown explicitly, in $n\,(>2)$-dimensional flat spaces, that of the $M(M+1)/2$ possible trace-free conformal Killing tensors constructed as above, only $(n-1)(n+2)(n+3)(n+4)/12$ are linearly independent. So, in $n\,(>2)$-dimensional flat spaces all $(n-1)(n+2)(n+3)(n+4)/12$ trace-free conformal Killing tensors are reducible [8].

Applying the above corollary we can extend Weir's results to conformally flat spaces:

Corollary 2. *The maximum number of linearly independent trace-free conformal Killing tensors in $n\,(>2)$ dimensions is $(n-1)(n+2)(n+3)(n+4)/12$ and is attained in conformally flat spaces. In this case all the trace-free conformal Killing tensors are reducible.*

So to find all the Killing tensors for conformally flat spaces, we simply investigate all the reducible conformal Killing tensors; these we can build up from all the conformal Killing vectors.

In four dimensions, conformally flat space-times necessarily admit 15 independent conformal Killing vectors from which 84 independent reducible conformal Killing tensors can be constructed. Hence in such space-times there is a rich supply of 'candidate' conformal Killing tensors which may satisfy the gradient condition and so be associated with possibly irreducible Killing tensors. The large number of candidate tensors means that a direct approach by hand calculation would be lengthy and error-prone. However the calculations involved, though lengthy, are routine and this enables them to be automated by using of a computer algebra package such as Reduce. Work is in progress on investigating a number of conformally flat space-times including the perfect fluid solutions [7] and the pure radiation solutions [2], the Robertson–Walker metrics and the interior Schwarzschild solution; these results will be presented elsewhere. In this paper we will restrict ourselves to a few preliminary remarks.

The generic perfect fluid solutions [7] and the pure radiation solutions [2] admit no Killing vectors and so if any gradient conformal Killing tensors are found then the associated Killing tensors will necessarily be irreducible (unless they are simply constant multiples of the metric).

Amery and Maharaj [1] found a number of conformal Killing tensors and Killing tensors in Robertson–Walker space-times using Koutras' algorithms, but because they used only mutually orthogonal conformal Killing vectors in their construction, they were only able to construct 39 'candidate' conformal Killing tensors. However, the Robertson-Walker metrics, being conformally flat, admit the maximal number, namely 84, of reducible conformal Killing tensors and so Amery and Maharaj's results are incomplete.

A generic Robertson–Walker metric admits 6 independent Killing vectors and so 22 ($= 1 + 6.7/2$) reducible Killing tensors can be constructed from the metric and the Killing vectors – of which 21 are linearly independent. Similarly for the special case of the static Einstein universe which admits a seventh Killing vector, we can construct 30 ($= 1 + 7.8/2$) reducible Killing tensors from the metric and the Killing vectors – of which 27 are linearly independent. Hence, after finding the gradient conformal Killing tensors and their associated Killing tensors of the generic Robertson–Walker metric (or of the Einstein universe), we need to determine whether they are irreducible by checking if they are independent of these 21 (or 27) reducible Killing tensors. Again the high dimension of these linear subspaces involved and the routine nature of the calculations means that the computations can be automated by use of the computer algebra system Reduce.

3 Finding irreducible Killing tensors in arbitrary spaces

Consider the most general reducible conformal Killing tensor Q_{ab}

$$Q_{ab} = \sum_{I=1}^{M} \sum_{J=I}^{M} a_{IJ}(\chi_{I(a}\chi_{|J|b)} + \lambda g_{ab})$$

satisfying

$$Q_{(ab:c)} = q_{(a}g_{bc)}.$$

When the vector q_a is a gradient vector $q_{,a}$ then Q_{ab} is said to be *of gradient type*, and has an *associated Killing tensor* K_{ab} where

$$K_{ab} = Q_{ab} - qg_{ab}.$$

Hence, we have an indirect method – using conformal Killing tensors – to find examples of Killing tensors, most of which we expect to be irreducible:

(i) From the conformal Killing vectors of a given metric, we can find all the reducible conformal Killing tensors.

If the metric admits N independent Killing vectors $\xi_1, \ldots, \xi_N$ then the linear space of all reducible conformal Killing tensor contains a linear subspace of reducible Killing tensors. We can exclude these from consideration if we choose the basis of the conformal Killing vectors $\xi_1, \ldots, \xi_N$, $\chi_{N+1}, \ldots, \chi_M$, where the ξ_I's are Killing vectors and we consider only reducible conformal Killing tensors of the form

$$Q_{ab} = \sum_{I=1}^{N} \sum_{J=N+1}^{M} a_{IJ}\xi_{I(a}\chi_{|J|b)} + \sum_{I=N+1}^{M} \sum_{J=I}^{M} a_{IJ}\chi_{I(a}\chi_{|J|b)}.$$

(ii) Next test to see if any of these conformal Killing tensors are of gradient type (including $q_a = 0$) and if so, construct the associated Killing tensors.

(iii) Check directly which of these Killing tensors are irreducible by comparison with the definition in Section 1.

4 Some sample theorems

For individual spaces we can always work from 'first principles' by systematically examining all reducible conformal Killing tensors. However, there are some general theorems which we can easily deduce, and then exploit:

Theorem 2. *Any space which admits a proper homothetic Killing vector χ_a with homothetic constant h as well as a gradient Killing vector $\xi_{,a}$ also admits a Killing tensor $K_{ab} = \chi_{(a}\xi_{,b)} - \xi g_{ab}/h$.*

Theorem 3. *Any space which admits a conformal Killing vector field χ_a which is a gradient also admits the Killing tensor $K_{ab} = \chi_a\chi_b - \chi^2 g_{ab}$, where $\chi^2 = \chi_a\chi^a$.*

Theorem 4. *Any space which admits a proper non-null conformal Killing vector field χ_a which is geodesic (that is $\chi_{a;b}\chi^b = \lambda\chi_a$) also admits the Killing tensor $K_{ab} = \chi_a\chi_b - \chi^2 g_{ab}$.*

These theorems (and others in [3]) generalise the results in [5] and [1]. It is important to note that, in all such cases the Killing tensors have to be investigated directly to determine whether they are irreducible Killing tensors, i.e. not constructed from Killing vectors and metric.

5 Example: A Kimura metric

A Kimura metric (type I) [4] given by

$$ds^2 = \frac{r^2}{b}dt^2 - \frac{1}{r^2b^2}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,$$

is of Petrov type D with a non-zero energy momentum tensor.

Four Killing vectors

$$\xi_1{}^a = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}, \qquad \xi_2{}^a = \frac{\partial}{\partial\phi},$$
$$\xi_3{}^a = -\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}, \qquad \xi_4{}^a = \frac{\partial}{\partial t},$$

which in covariant form are

$$\xi_{1a} = -r^2\sin\phi\theta_{,a} - r^2\sin\theta\cos\theta\cos\phi\,\phi_{,a}, \qquad \xi_{2a} = -r^2\sin^2\theta\phi_{,a},$$
$$\xi_{3a} = r^2\cos\phi\theta_{,a} - r^2\sin\theta\cos\theta\sin\phi\phi_{,a}, \qquad \xi_{4a} = \frac{r^2}{b}t_{,a}.$$

None are gradient vectors, but ξ_{2a} and ξ_{4a} are hypersurface orthogonal.

Two proper conformal Killing vectors

$$\chi_1{}^a = r^2\frac{\partial}{\partial r} \qquad \text{and} \qquad \chi_2{}^a = r^2t\frac{\partial}{\partial r} - \frac{1}{br}\frac{\partial}{\partial t}$$

with conformal factors r and rt respectively.

Both are gradient vectors $\chi_{1a} = -\left(r/b^2\right)_{,a}$ and $\chi_{2a} = -\left(rt/b^2\right)_{,a}$.

Eleven reducible proper conformal Killing tensors.

We can immediately write down 11 reducible proper conformal Killing tensors from the symmetrised products of each proper conformal Killing vector with each Killing vector, together with the symmetrised products of the proper conformal Killing vectors

$$\xi_{1(a}\chi_{1b)}, \quad \xi_{1(a}\chi_{2b)}, \quad \xi_{2(a}\chi_{1b)}, \quad \xi_{2(a}\chi_{2b)}, \quad \xi_{3(a}\chi_{1b)}, \quad \xi_{3(a}\chi_{2b)}, \quad \xi_{4(a}\chi_{1b)}, \quad \xi_{4(a}\chi_{2b)},$$
$$\chi_{1a}\chi_{1b}, \quad \chi_{2a}\chi_{2b}, \quad \chi_{1(a}\chi_{2b)}.$$

Killing tensors.

χ_{1a} and χ_{2a} are gradient vectors and so by Theorem 3 we obtain respectively two Killing tensors with non-zero components,

$$K_1^{tt} = \frac{1}{b}, \qquad K_1^{\theta\theta} = -\frac{1}{b^2}, \qquad K_1^{\phi\phi} = -\frac{1}{b^2 \sin^2\theta}$$

and

$$K_2^{tt} = b^2 + \frac{1}{r^2}, \qquad K_2^{tr} = -btr, \qquad K_2^{\theta\theta} = -t^2, \qquad K_2^{\phi\phi} = -\frac{t^2}{\sin^2\theta}.$$

Noting that for the case $Q_{ab} = \chi_{1(a}\chi_{2b)} + \lambda g_{ab}$,

$$q_a = r\chi_{2a} + rt\chi_{1a} = r\Big(\frac{rt}{b^2}\Big)_{,a} + rt\Big(\frac{r}{b^2}\Big)_{,a} = \Big(\frac{r^2 t}{b^2}\Big)_{,a}$$

is a gradient vector, we find the associated Killing tensor

$$K_3^{tt} = 2\frac{t}{b}, \qquad K_3^{tr} = -\frac{r}{b}, \qquad K_3^{\theta\theta} = -2\frac{t}{b^2}, \qquad K_3^{\phi\phi} = -\frac{t^2}{\sin^2\theta} b^2.$$

These are the only Killing tensors which can be found by this method.

A comparison of the Killing tensor $\boldsymbol{K}_1$ with the Killing vectors shows that it is in fact reducible since,

$$K_{1ab} = \frac{1}{b}\xi_{4a}\xi_{4b} - \frac{1}{b^2}(\xi_{1a}\xi_{2b} + \xi_{2a}\xi_{2b} + \xi_{3a}\xi_{3b}).$$

$\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are irreducible Killing tensors since it is clearly impossible to obtain, using the Killing vectors and metric, those terms in which are explicit functions of t. It is easy to confirm from observation that these three tensors are linearly independent of each other and of the metric.

In Kimura's original work [4] he sought directly for irreducible Killing tensors, and found the two tensors $\boldsymbol{K}_2$ and $\boldsymbol{K}_3$. Koutras [5] only found 8 reducible trace-free conformal Killing tensors and only the 2 Killing tensors $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, because he used his less general algorithms. O'Connor and Prince [6] obtained all three Killing tensors since they used the same more general argument as we have done.

6 Summary

We have clarified the concept and definition of *reducible* conformal Killing tensors of order 2 and their trace-free counterparts; this enables us to write down immediately all the reducible conformal Killing tensors in a space where the conformal Killing vectors are known. By identifying those reducible conformal Killing tensors of gradient type we are able to construct associated Killing tensors, most of which we expect to be irreducible. For conformally flat spaces we have shown that all conformal Killing vectors are reducible and so they can all (including both reducible and irreducible Killing tensors) be found by this indirect method.

The full details of the results summarised here are given in [3].

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The Generalized Leray Transformation for Finite Time Singular Vortices in Fluids

Homayoon ESHRAGHI [†‡] *and Yoosofali ABEDINI* [‡]

[†] *Institute for Studies in Theor. Physics and Mathematics, P.O.Box 19395-5531, Tehran, Iran*
E-mail: *eshraghi@theory.ipm.ac.ir*

[‡] *Physics Department of Zanjan University, P.O.Box 313, Zanjan, Iran*

The finite time singularity solution for a single vortex field in both viscous and non-viscous fluids is discussed. The Leray transformation, which gives self similar solutions for (local) inner region of incompressible fluids, is generalized to a dynamical time dependent case. A new generalized time T is introduced to modify the Leray equation. Two important examples are generalized to produce both decreasing and constant areas of singularity instead of an exact line of singularity in the self-similar solutions. This is done by assuming a "line source" of the matter in the core of the singularity.

1 Introduction

The problem of singularity is generally of importance in many domains. One of the interesting cases is finite time singularity in the Euler and/or Navier–Stokes equation. General mathematical investigations about the formation and creation of singularities can be found in [1–3]. The blow up solution for vortices is a fundamental problem in fluid dynamics which firstly was noticed by Leray [4]. In this solution one finds a self-similar collapsing behavior towards a singularity at a finite time t^* while the length scale is decreasing like $(t^*-t)^{1/2}$ and velocity is diverging as $(t^*-t)^{-1/2}$. The above singular solution however does not allow the fluid physical parameters like energy or momentum to be finite [5, 6]. Hence, the singular solution must be considered as an inner solution which must be appropriately matched with a non-singular outer solution [6]. There exist significant numerical results about the singular vortices and their interaction (see for instance [7–15]).

The existence of a smooth and bounded solution for the Leray equation is a rather difficult problem which is still under consideration [16]. Usually (as appears below) a singular strain field is considered to yield a (local) collapsing behavior. Although this strain field gives an infinite energy but it is a useful phenomenological model for this singularity. Following this strategy in this paper, a "line source" is assumed to exist in the core of the singularity. This line source can spread out the singularity to a non-zero area. So, it will be possible to have an area of singularity instead of a line. Therefore, our solution does not satisfy the finite energy condition and enters a phenomenological model showing the tendency of the singularity. This is due to the line source which is injecting matter and energy to the singular area. Again because of this source, it is difficult to talk about any matching between the inner and outer regions since these regions have no significant meaning here. Hence, the present solution is only a new type of singularity affected by an external line source. In doing this, we generalize the Leray transformation leading to a time dependent version. We consider two original solutions of the old Leary equation and generalize them to obtain their dynamical versions.

To introduce the Leray problem we must start from the Navier–Stokes equation for an incompressible ($\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$) fluid

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega})+\nu\nabla^2\boldsymbol{\omega}, \tag{1}$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ is the fluid velocity, $\boldsymbol{\omega}=\boldsymbol{\nabla}\times\boldsymbol{u}$ is the vorticity and ν is a (dimensional) constant representing the viscosity effect. Although the singularity has probably a close connection with the turbulence but it is convenient to consider a local idealized unsteady "strain" velocity field $\boldsymbol{u}'(\boldsymbol{x},t)$ which satisfies

$$\boldsymbol{\nabla}\times\boldsymbol{u}'=0,\qquad \boldsymbol{\nabla}\cdot\boldsymbol{u}'=0. \tag{2}$$

So the total local fluid velocity is

$$\boldsymbol{u}=\boldsymbol{u}_0+\boldsymbol{u}', \tag{3}$$

where

$$\boldsymbol{\nabla}\times\boldsymbol{u}_0=\boldsymbol{\omega}=\boldsymbol{\nabla}\times\boldsymbol{u},\qquad \boldsymbol{\nabla}\cdot\boldsymbol{u}_0=0. \tag{4}$$

Moreover, $\boldsymbol{u}_0$ is the solution of the equilibrium (non-viscous) Euler equation

$$\boldsymbol{\nabla}\times(\boldsymbol{u}_0\times\boldsymbol{\omega})=0 \tag{5}$$

and has no direct influence on the creation of $\boldsymbol{\omega}$ but it can change the dynamics of $\boldsymbol{\omega}$ because

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=\boldsymbol{\nabla}\times(u'\times\boldsymbol{\omega})+\nu\nabla^2\boldsymbol{\omega}. \tag{6}$$

Equation (6) is a direct result from (1)–(5).

2 Self-similar solution

The Leray transformation changes the variables $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{\omega}$ to new dimensionless variables $\boldsymbol{X}$, $\boldsymbol{U}$, $\boldsymbol{\Omega}$. For an inviscid fluid ($\nu=0$) the Leray transformation

$$\boldsymbol{x}\longrightarrow\boldsymbol{X}\equiv\frac{\boldsymbol{x}}{\sqrt{\Gamma(t^*-t)}},\qquad \boldsymbol{u}\longrightarrow\boldsymbol{U}(\boldsymbol{X})\equiv\sqrt{\frac{t^*-t}{\Gamma}}\boldsymbol{u},\qquad \boldsymbol{\omega}\longrightarrow\boldsymbol{\Omega}(\boldsymbol{X})\equiv(t^*-t)\boldsymbol{\omega} \tag{7}$$

changes equation (1) ($\nu=0$) to

$$\boldsymbol{\nabla}\times\left[\left(\boldsymbol{U}+\frac{1}{2}\boldsymbol{X}\right)\times\boldsymbol{\Omega}\right]=0. \tag{8}$$

In (7) the constant Γ is an arbitrary scale for the circulation for example, it can be the surface integration $\int\boldsymbol{\omega}\cdot\boldsymbol{n}da$ in an area perpendicular to $\boldsymbol{\omega}$. In a viscous fluid ($\nu\neq0$) we may write

$$\boldsymbol{x}\longrightarrow\boldsymbol{X}\equiv\frac{\boldsymbol{x}}{\sqrt{\nu(t^*-t)}},\qquad \boldsymbol{u}\longrightarrow\boldsymbol{U}(\boldsymbol{X})\equiv\sqrt{\frac{t^*-t}{\nu}}\,\boldsymbol{u}, \tag{9}$$

with $\boldsymbol{\Omega}(\boldsymbol{X})$ the same as in (7) which yields

$$\boldsymbol{\nabla}\times\left[\left(\boldsymbol{U}+\frac{1}{2}\boldsymbol{X}\right)\times\boldsymbol{\Omega}\right]+\nabla^2\boldsymbol{\Omega}=0. \tag{10}$$

In equations (8) and (10) the operator $\boldsymbol{\nabla}$ denotes the derivatives with respect to $\boldsymbol{X}$. Also $\boldsymbol{\Omega}(\boldsymbol{X})=\boldsymbol{\nabla}\times\boldsymbol{U}(\boldsymbol{X})$ and $\boldsymbol{\nabla}\cdot\boldsymbol{U}(\boldsymbol{X})=0$.

Denoting the strain field by $\boldsymbol{U}'(\boldsymbol{X})$ one can similarly write

$$\begin{aligned}&\boldsymbol{U}(\boldsymbol{X})=\boldsymbol{U}_0(\boldsymbol{X})+\boldsymbol{U}'(\boldsymbol{X}),\qquad \boldsymbol{\nabla}\times\boldsymbol{U}_0=\boldsymbol{\Omega}=\boldsymbol{\nabla}\times\boldsymbol{U},\\ &\boldsymbol{\nabla}\times\boldsymbol{U}'=\boldsymbol{0},\qquad \boldsymbol{\nabla}\cdot\boldsymbol{U}'=\boldsymbol{\nabla}\cdot\boldsymbol{U}_0=0,\end{aligned} \tag{11}$$

and

$$\boldsymbol{\nabla}\times(\boldsymbol{U}_0\times\boldsymbol{\Omega})=0,\qquad \boldsymbol{\nabla}\times\left[\left(\boldsymbol{U}'+\frac{1}{2}\boldsymbol{X}\right)\times\boldsymbol{\Omega}\right]+\nabla^2\boldsymbol{\Omega}=0. \tag{12}$$

For an inviscid fluid the last term of the second equation of (12) vanishes.

It is important to stress again that $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ are only functions of $\boldsymbol{X}$, i.e. we are sitting on a collapsing reference frame with a scale proportional to $\sqrt{t^*-t}$.

Two important singular solutions of (11) and (12) will be considered here. For both of these solutions the cylindrical coordinates $\boldsymbol{x}=(r,\theta,z)$ for usual real coordinates corresponding to $X=(R,\theta,Z)$ for new Leray coordinates are used. So $\boldsymbol{U}=(U_R,U_\theta,U_Z)$ and $\boldsymbol{\Omega}=(\Omega_R,\Omega_\theta,\Omega_z)$. The strain field for both cases is assumed to be [6]

$$\boldsymbol{U}'=c\left(-\frac{1}{2}R,0,Z\right),\qquad \text{or}\qquad \boldsymbol{u}'=\frac{c}{t^*-t}\left(-\frac{1}{2}r,0,z\right)\qquad (c=\text{const}) \tag{13}$$

which clearly satisfy (11).

The first solution. Suppose $\boldsymbol{\Omega}=(0,0,\Omega(R))$ is in the Z-direction and $\boldsymbol{U}_0=(0,U_0(R),0)$ is in the $\hat{\boldsymbol{\theta}}$ direction. So

$$\boldsymbol{\Omega}(R)=\frac{1}{R}\frac{d}{dR}[RU_0(R)]\hat{\boldsymbol{Z}}. \tag{14}$$

The first equation of (12) is automatically satisfied and its second equation regarding (13) yields

$$(c-1)\Omega+\frac{1}{2}(c-1)R\frac{d\Omega}{dR}+\frac{1}{R}\frac{d}{dR}\left(R\frac{d\Omega}{dR}\right)=0. \tag{15}$$

The solution of (15) initialing from a Gaussian form is [6]

$$\Omega=\Omega_0 e^{-\frac{c-1}{4}R^2},\qquad \text{or}\qquad \omega=\frac{\Omega_0}{t^*-t}\,e^{-\frac{(c-1)r^2}{4\nu(t^*-t)}}. \tag{16}$$

In [6] a complete discussion is given for the value of c. For a viscous fluid $c>1$ and for an inviscid fluid $c=1$. Here we restrict to $\nu\neq 0$.

Another solution of the linear equation (15) can be obtained to be

$$\Omega=Ae^{-\frac{c-1}{4}R^2}\int^R\frac{e^{\frac{c-1}{4}R'^2}}{R'}\,dR'\qquad (A=\text{const}),$$

which is divergent at $R\to 0$.

The second solution. In this case we may imagine a cylindrical beam of fluid symmetric around the Z-axis and moving parallel to it. So, $\boldsymbol{U}_0=(0,0,U_0(R))$ and $\boldsymbol{\Omega}=(0,\Omega(R),0)$. For simplicity let us neglect the viscosity term in (12) and use (13) to obtain

$$-\left(1+\frac{c}{2}\right)\Omega+\frac{c-1}{2}\,R\frac{d\Omega}{dR}=0. \tag{17}$$

The solution of (17) is

$$\Omega=AR^{\frac{c+2}{c-1}},\qquad \omega=A\Gamma^{\frac{c+2}{2(1-c)}}\frac{r^{\frac{c+2}{c-1}}}{(t^*-t)^{\frac{3c}{2(c-1)}}}\qquad (A=\text{const}). \tag{18}$$

The case of $c=1$ in (17) gives $\Omega=0$. If $\frac{c+2}{c-1}<0$ $(-2<c<1)$ then ω diverges when $r\to 0$ which is not physical. If on the other hand $0<c<1$, then for a fixed point $r\neq 0$, $\omega\to 0$ when $t\to t^*$ which does not contain singularity. Hence the best physical case is when $c>1$ or $c<-2$. It is clear that the radious of the beam must be finite.

3 Generalization of the Leray transformation

As mentioned before in the Leray transformation similarity solutions were sought and hence $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ were assumed to be only functions of $\boldsymbol{X}$. Obviously this is not the only possibility for finite time singularity. For example see [17]. The point is that when t^* is long enough, $\boldsymbol{\Omega}$ and $\boldsymbol{U}$ may change significantly and deviate from the self-similar solution, i.e. $\boldsymbol{\Omega}$ and $\boldsymbol{U}$ are not necessarily only functions of $\boldsymbol{X}$. In other words we want to allow the smooth functions $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ to change arbitrarily and not to be restricted to decreasing length scale $\sqrt{t^*-t}$. So, generally we can find another dimensionless time T such as

$$t \longrightarrow T \equiv \ln \frac{t^*}{t^*-t}\,, \tag{19}$$

where $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{X}, T)$ and $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\boldsymbol{X}, T)$. If we add (19) to (7) or (9), equations (10) and (12) will be generalized to

$$\frac{\partial \boldsymbol{\Omega}}{\partial T} = \boldsymbol{\nabla} \times \left[\left(\boldsymbol{U} + \frac{1}{2}\boldsymbol{X}\right) \times \boldsymbol{\Omega}\right] + \nabla^2 \boldsymbol{\Omega}, \tag{20}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{U}_0 \times \boldsymbol{\Omega}) = 0, \qquad \frac{\partial \boldsymbol{\Omega}}{\partial T} = \boldsymbol{\nabla} \times \left[\left(\boldsymbol{U}' + \frac{1}{2}\boldsymbol{X}\right) \times \boldsymbol{\Omega}\right] + \nabla^2 \boldsymbol{\Omega}. \tag{21}$$

Clearly $T = 0$ when $t = 0$ and $T \to +\infty$ when $t \to t^*$.

To generalize the examples given in the previous section we again consider the strain field (13) in cylindrical coordinates. It can be seen that this strain field has a tendency to collect the singularity to the Z-axis ($R = 0$). This is of course the nature of a self-similar solution. Searching another symmetric strain field $\boldsymbol{U}''$ added to $\boldsymbol{U}'$ in equation (13) having the opposite effect of $\boldsymbol{U}'$, yields

$$\boldsymbol{U}'' = F(T)\left(R^{-1}, 0, 0\right), \qquad \text{or} \qquad \boldsymbol{u}'' = \nu f(t)\left(r^{-1}, 0, 0\right), \tag{22}$$

where $F(T) = f(t)$ are some functions to be determined. The θ-dependence of the strain field is not considered here. For this case see [6] and [17]. It can be seen that

$$\boldsymbol{\nabla} \times \boldsymbol{U}'' = 0, \quad \boldsymbol{\nabla} \cdot \boldsymbol{U}'' = 2\pi F(T)\delta(R\hat{\boldsymbol{R}}), \tag{23}$$

so we do not consider the Z-axis ($R = 0$). Indeed $\boldsymbol{U}''$ represents a line source exactly on the Z-axis which is adding new mass with a fixed density (equal to the fixed fluid density) to the existing large amount of the fluid. This line source then near the Z-axis is very strong and prevents the singularity to collapse on the Z-axis.

Now let us generalize the two mentioned solutions.

The first solution. Again suppose $\boldsymbol{U}_0(R, T)$ to be only in the $\hat{\boldsymbol{\theta}}$ direction and $\boldsymbol{\Omega}(R, T)$ parallel to the Z-axis. Regarding the total strain field $\boldsymbol{U}' + \boldsymbol{U}''$ in (21) results in

$$(c-1)\Omega + \frac{1}{2}(c-1)R\frac{\partial \Omega}{\partial R} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Omega}{\partial R}\right) - \frac{F(T)}{R}\frac{\partial \Omega}{\partial R} = \frac{\partial \Omega}{\partial T}\,. \tag{24}$$

Three solution of (24) can be obtained.

i) We suggest the following solution (outside the z-axis)

$$\Omega = \Omega_0 \exp\left(Ae^T - \frac{c-1}{4}R^2\right), \qquad \omega = \frac{\Omega_0}{t^*-t}\,\exp\frac{r_0^2 - r^2}{t^*-t}\,, \tag{25}$$

where

$$r_0 \equiv \sqrt{\frac{4At^*\nu}{c-1}}\,, \tag{26}$$

and A and Ω_0 are constant. In the hole domain of $0 < r \le r_0$ the vorticity is singular while for $r > r_0$, $\omega \to 0$ when $t \to t^*$. Substituting (25) into (24) one finds.

$$F(T) = \frac{2A}{c-1}e^T, \qquad f(t) = \frac{2A}{c-1}\frac{t^*}{t^*-t}, \tag{27}$$

which means that the source is also singular in harmony with other singular parameters. This harmony which is locally in the inner region may come from complicated nonlinear interactions connected to turbulence.

ii) Assuming

$$\Omega = \Omega_0 e^{G(T)}\frac{R^2}{4}e^{-\frac{c-1}{4}R^2}$$

and substituting it in (24) results in $F(T) = 2$, $G(T) = 0$, and so

$$\Omega = \Omega_0\frac{R^2}{4}\,e^{-\frac{c-1}{4}R^2}, \qquad \omega = \frac{\Omega_0}{4\nu(t^*-t)^2}\,r^2 e^{-\frac{c-1}{4\nu}\frac{r^2}{t^*-t}} \quad (r \neq 0). \tag{28}$$

For the convergence of ω at large radiuses, it is necessary to have $c > 1$. The form of (28) introduces a finite radius

$$R_{\max} \equiv \sqrt{\frac{4}{c-1}}\,, \qquad r_{\max}(t) \equiv \sqrt{\frac{4\nu}{c-1}(t^*-t)}\,, \tag{29}$$

at which ω is maximum. This radius of course goes to zero for $t \to t^*$. Hence, $\omega \to 0$ as $r \to 0$ and is maximum on a cylindrical "shell" around the Z-axis which its radius is decreasing. If we sit on the Leray (collapsing) coordinates i.e. when $R = R_{\max} = \text{const}$, then $\omega = \omega_{\max} \propto (t^*-t)^{-1}$.

iii) Let us generally assume

$$\Omega = \Omega_0 e^{H(T)}e^{-\frac{c-1}{4}(R-R_m(T))^2}, \tag{30}$$

which from (24) one can get

$$H(T) = \frac{c-1}{2}\,T, \qquad R_m(T) = R_0 e^{-\frac{c-1}{2}T} \quad (R_0 = \text{const}), \qquad F(T) = 1 = f(t). \tag{31}$$

Hence

$$\Omega = \Omega_0 e^{\frac{c-1}{2}T}e^{-\frac{c-1}{4}(R-R_m(T))^2}, \qquad \omega = \frac{\Omega_0}{t^*}\left(\frac{t^*}{t^*-t}\right)^{\frac{c+1}{2}} e^{-\frac{c-1}{4\nu(t^*-t)}[r-r_m(t)]^2} \quad (r \neq 0), \tag{32}$$

where

$$r_m(t) \equiv R_0\sqrt{\nu t^*}\left(\frac{t^*-t}{t^*}\right)^{\frac{c}{2}} \tag{33}$$

and $R_m(T)$ is defined in (31). Moving with $r_m(t)$ a singularity of the order of $1/(t^*-t)^{\frac{c+1}{2}}$ can be seen. Since $c > 1$ this singularity is stronger than $\frac{1}{t^*-t}$. Also $r_m(t)$ is decreasing as $(t^*-t)^{c/2}$ which is faster than the Leray coordinates. Finally it is interesting to note that if only the strain field $\boldsymbol{U}''$ is considered (without $\boldsymbol{U}'$) the solutions seem to be divergent as $r \to \infty$.

The second solution. We can naturally generalize this solution to $\boldsymbol{U}_0(\boldsymbol{X},T)=(0,0,U_0(R,T))$ and $\boldsymbol{\Omega}(\boldsymbol{X},T)=-\frac{\partial U_0}{\partial R}\hat{\boldsymbol{\theta}}$. Again neglection of the viscosity yields

$$-\left(1+\frac{c}{2}\right)\Omega+\frac{c-1}{2}R\frac{\partial\Omega}{\partial R}=\frac{\partial\Omega}{\partial T}, \tag{34}$$

which is the generalization of (17). The general solution of (34) has the form

$$\Omega=e^{-(1+\frac{c}{2})T}W\left(Re^{\frac{c-1}{2}T}\right),$$

where W is an arbitrary function. Choosing a power form for W one obtains

$$\Omega(R,T)=Ae^{(\frac{k}{2}-1)T}R^{\frac{c+k}{c-1}},\qquad \omega(r,t)=\frac{A}{t^*(\Gamma t^*)^{\frac{e+k}{2(c-1)}}}\left(\frac{t^*}{t^*-t}\right)^{\frac{c}{2}\frac{k+1}{c-1}}r^{\frac{c+k}{c-1}}. \tag{35}$$

The necessary condition for the existence of singularity is

$$\frac{c(k+1)}{2(c-1)}>0. \tag{36}$$

When $r\to 0$ for a fixed time $t<t^*$ we have $\boldsymbol{\omega}\to 0$ if

$$\frac{c+k}{c-1}>0. \tag{37}$$

In the case of $k=2$, (35) reduces to (18) and (36), (37) give $c>1$ or $c<-2$. It should be again mentioned that the above solution is obtained in the presence of $\boldsymbol{U}'$ (equation (13)) only and without $\boldsymbol{U}''$.

4 Time reversibility for the inviscid limit

For the inviscid limit the last term of (20) vanishes to yield

$$\frac{\partial\boldsymbol{\Omega}}{\partial T}=\boldsymbol{\nabla}\times\left[\left(\boldsymbol{U}+\frac{1}{2}\boldsymbol{X}\right)\times\boldsymbol{\Omega}\right]. \tag{38}$$

Because of the time reversibility of Euler equation, we may think about an "expanding" solution for $-t^*<t\le 0$ [6, 10]. Indeed, the change $t\to -t$, offers the new variables $\boldsymbol{X}$, $\boldsymbol{T}_1$, $\boldsymbol{U}_1$, $\boldsymbol{\Omega}_1$ as

$$\begin{aligned}&\boldsymbol{x}\longrightarrow\boldsymbol{X}_1\equiv\frac{\boldsymbol{x}}{\sqrt{\Gamma(t^*+t)}},\qquad t\longrightarrow T_1\equiv\ln\frac{t_*}{t^*+t},\\ &\boldsymbol{u}\longrightarrow\boldsymbol{U}_1\equiv\sqrt{\frac{t^*+t}{\Gamma}}\,\boldsymbol{u},\qquad \boldsymbol{\omega}\longrightarrow\boldsymbol{\Omega}_1\equiv(t^*+t)\,\boldsymbol{\omega},\end{aligned} \tag{39}$$

which convert the Euler equation to

$$-\frac{\partial\boldsymbol{\Omega}_1}{\partial T_1}=\boldsymbol{\nabla}_1\times\left[\left(\boldsymbol{U}_1-\frac{1}{2}\boldsymbol{X}_1\right)\times\boldsymbol{\Omega}_1\right], \tag{40}$$

where $\boldsymbol{\nabla}_1$ denotes the derivative with respect to $\boldsymbol{X}_1$. It can be seen that if $\boldsymbol{\Omega}_1(\boldsymbol{X}_1,T_1)$, $\boldsymbol{U}_1(\boldsymbol{X}_1,T_1)$ is a solution of (40), then $\boldsymbol{\Omega}(\boldsymbol{X},T)=-\boldsymbol{\Omega}_1(\boldsymbol{X},T)$, $\boldsymbol{U}(\boldsymbol{X},T)=-\boldsymbol{U}_1(\boldsymbol{X},T)$ will be a solution of (38). This result is also covered in the self-similar Leray variables [6] and so, our generalization does not change this symmetry. An important conclusion is that when a singular inner solution passes through the critical time t^*, then it may change to an expanding solution [6, 10].

5 Summary

The problem of finite time singularity in fluid dynamics which was firstly noticed by Leray in 1934, has a close relation with the turbulence. Instead of complicated turbulence domains, one may assume a singular unsteady strain field which can produce singularly increasing velocities and vorticities upon a decreasing length scale. This picture that obviously violates the energy and momentum conservations cannot be global so in an inner level a singular solution must match an outer regular solution.

In the Leray transformation it is possible to find only the self-similar inner solutions when the length scale is decreasing as $\sqrt{t^* - t}$ while the fluid velocity is increasing as $(t^* - t)^{-1/2}$. In the present paper the Leray transformation, was generalized to obtain dynamical transformed Navier–Stokes and Euler equations. In this picture the finite time t^* for singularity is assumed to be long enough so that the deviation from the self-similar solution is significant. Two important cylindrically symmetric self-similar solutions were both generalized to the "dynamical" finite time singular case in which the singular areas enlarged to decreasing or constant cylindrical shells. Also the time reversibility of the Euler equation turned out to recover the same results for the generalized Leray parameters as in the usual self-similar solutions.

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Physical Nature of Lobachevsky Parallel Lines and a New Inertial Frame Transformation

Nikolai G. FADEEV

Laboratory of Particle Physics, JINR, 141980 Dubna, Moscow Region, Russia
E-mail: *fadeev@sunse.jinr.ru*

Processes of the particle and light beams propagation synchronized by Hyigens principle have been found as the physical nature for the V postulate denial and for the Lobachevsky parallel lines definition in the velocity space. The physical nature of Lobachevsky parallel lines reveals a new way to solve the main difficulty in relativity – the problem of definition of the moments of time for different space points. The first obvious consequences from the established physical correspondence, including simultaneity, proper time and inertia system coordinate transformation, are presented in this paper.

1 Introduction

In modern high energy physics the Lobachevsky velocity space is widely used for investigation of particle scattering processes. In spite of that the physical nature of Lobachevsky parallel lines (LPL) is still absent. As the existence of LPL is based on denial of the Euclidean V postulate, a physical foundation for its violation is also absent. At the present time LPL have a geometrical interpretation only either as infinite lines on a pseudospherical surface, or as hordes on an Euclidean circle [1].

Exposition of the physical nature of LPL and obtaining the first obvious consequences from processes corresponding to LPL are results of further development of main ideas of [2] and presented in this paper.

It is reasonable to sketch some general remarks on our approach. Let us consider light propagation on the basis of the Huygens principle and of the light beams independence law. So, light phenomena of diffraction and interference are omitted. Also, let us accept the constant light velocity principle. Let us imply that the reader is familiar with Lobachevsky geometry [1, 3].

There is a special remark for Huygens principle: a time moment of emitting a secondary light sphere (semisphere) from any point reached by a light front can be taken as the initial (zero) moment of time counting for that point.

Let us use the same plane light fronts as used to explain the light reflection and refraction phenomena.

2 The physical nature of Lobachevsky parallel lines

Let us have two inertial reference frames K and K_s, and one (K_s) is moving relatively to the other with some velocity V. Both systems may be associated somehow with corresponding particles. As usual, all space axes of the both frames are parallel, and the motion goes along the X-axis of the K. Let us assume that when the origins O and O_s of both frames coincide, then a plane light front (side beam directed from down to up in some plane, for instance in XY) hits the point O under a parallel angle to the X-axis (see Fig. 1a):

$$\cos\theta_L \equiv \cos\Pi(\rho/k) = th(\rho/k) = V/c \equiv \beta, \qquad (k = c) \tag{1}$$

and a light sphere (semisphere to the falling front) starts to spread out from the O (here β is a velocity V in units of c, ρ/k is a rapidity in units of $k = c$, $\Pi(\rho/k) \equiv \theta_L$ is a parallel angle, k is Lobachevsky constant, c is a light velocity). The second equality $\beta = th(\rho/c)$ in (1) is known

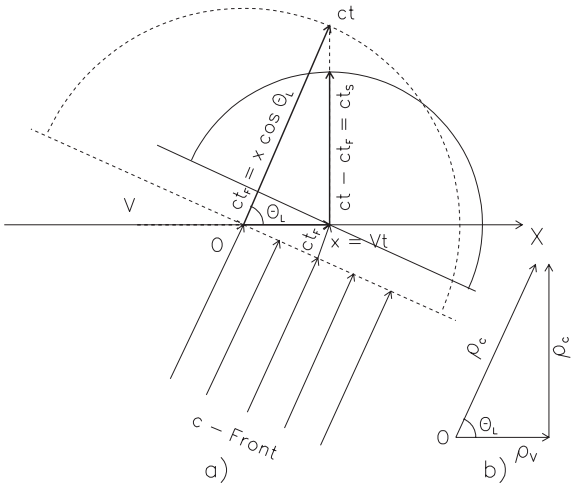

Figure 1. a) synhronization of light rays (ct and ct_s) propagation and the K_s-motion Vt by the Huygens principle due to the side light beam; b) Lobachevsky parallel lines in the velocity space plane corresponding to synchronous motions of ct, ct_s and Vt (in the Euclidean plane).

from the Beltarami model [1] and used in physics to define particle rapidity:

$$\rho/c = 1/2 \ln((1+\beta)/(1-\beta)). \tag{2}$$

The first equality in (1) can be rewritten in the form:

$$\theta_L \equiv \Pi(\rho/k) = 2\mathrm{arctg}\, e^{-\rho/c}, \tag{3}$$

known as Lobachevsky function. It is seen from (1) that for any rapidity (or/and for any velocity) there is a definite angle θ_L. For negative argument of the Lobachevsky function the parallel angle is $\pi - \theta_L$ [1]. So, this case corresponds to $(\cos(\pi - \theta_L) = -\cos\theta_L = -V/c)$ the same velocity, but in the opposite direction.

Let us consider an event $(x = Vt, t)$ in the K. Then the side beam hits a given x-point in the moment of time t_F (see Fig. 1a):

$$ct_F = x\cos\theta_L = Vt\cos\theta_L = ct\cos^2\theta_L, \tag{4}$$

i.e., later than it hits the origin O, and a new light sphere starts to spread out from a given x-point. By a given moment of time t a new sphere spreads up to the distance (or radius) ct_s:

$$ct_s = ct - ct_F = ct - x\cos\theta_L = ct - xV/c, \qquad t_s = t - xV/c^2, \tag{5}$$

and for $x = Vt$:

$$ct_s = ct - ct\cos^2\theta_L = ct\sin^2\theta_L = ct(1 - V^2/c^2), \tag{6}$$

where ct is the light sphere radius from origin O, so that $ct_s < ct$. It is obvious that the origin O_s of the K_s moves along X at the distances Vt.

Let us choose two light rays from these two spheres: one is ct from O at the angle θ_L to the X-axis on some plane, the other is ct_s from O_s (from the given x) perpendicular to the X-axis on the same plane (see Fig. 1a). These three segments ct, Vt and ct_s form some kind of

a rectangular triangle. But sides ct and ct_s have no common (intersection) point for any moment of time t, so they are parallel in any chosen Euclidean plane. As a rapidity for c is an infinity (see (2)), then this obtained triangle transforms into LPL, or more precisely, into parallel lines in one side on the Lobachevsky plane in the velocity space (see Fig. 1b).

Thus, the LPL in a velocity space corresponds to the light rays ct and ct_s emitted from different points and different times and synchronized by Huygens principle with particle motion Vt. The physical reason of absence of an intersection point is the time delay t_F (see (4)). This time is obviously a physical foundation for denial of the Vth postulate.

To find out light rays corresponding to LPL in another side, one should use analogous consideration with a side beam directed to another side (from up to down) in the same plane (see Fig. 2a and Fig. 2b). To find out light rays corresponding to the LPL for negative argument of Lobachevsky function (for $V < 0$ in the both sides), one should use side beams directed opposite to X-axis, i.e. from the right to the left (the previous ones for $V > 0$ were directed from the left to the right), see Fig. 2c and Fig. 2d. The complete picture in the Euclidean plane corresponding to the LPL on a plane in the velocity space is presented at Fig. 3.

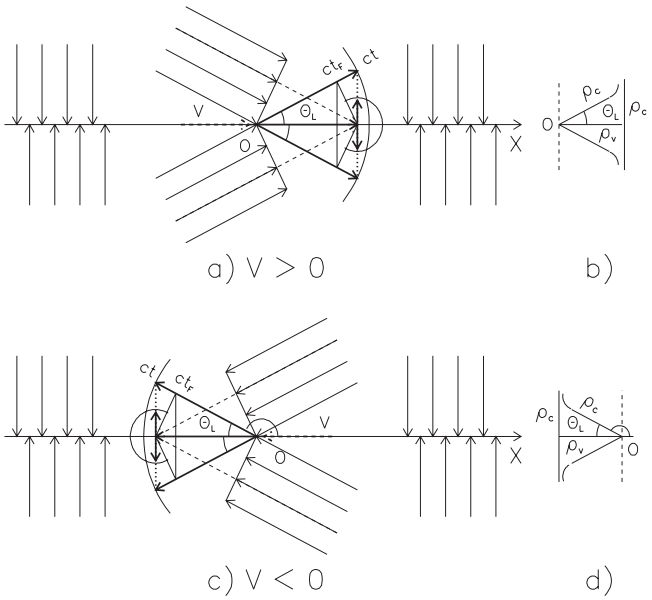

Figure 2. a) two side light beams (for $V > 0$) give two pairs of light rays ct and ct_s for both sides of the plane (up and down), synchronous with a K_s-motion Vt; b)parallel lines in both sides on Lobachevsky plane, corresponding to synchronous motions in a); c) and d) are the same one as in a) and b), but for $V < 0$.

Thus, a moving reference frame ($V > 0$ or $V < 0$) is associated with definite side light beams. For fixed frame $V = 0$ and the fixed frame is associated with straight beams as in this case $\theta_L = \pi/2$ (see (1) and Fig. 2). The physical nature of Lobachevsky parallel lines reveals a new way to solve the main difficulty in relativity – the problem of definition of moments of time for different space points.

3 x and t – coordinate transformation and light ether concept

Let us continue the previous consideration of two inertia systems K and K_s ($V > 0$). Let us assume that a straight beam hits X-axis at the same moment of time when a side beam hits

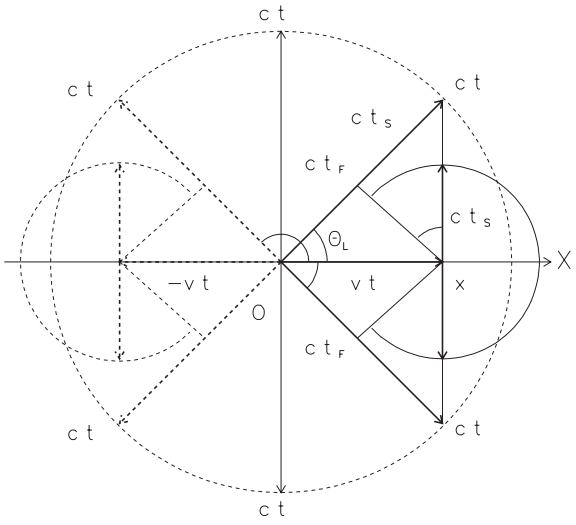

Figure 3. A summary diagram to illustrate of Huygens synchronization of corresponding light rays and two motions ($V > 0$ and $V < 0$) in K-frame.

a point where the both origins coincide. Then all x-points (including O) are "fired" simultaneously due to the straight beam, and this moment of time is usually taken as the initial one for the K frame. With respect to the side beam the initial moment of time for any x-point is shifted by the delay time t_F (see (4)). The time t_s in a given x-point (in K) by a given moment of time t (in K) is defined by (5). Thus, due to synchronization of K and K_s systems any x point has two times: t and t_s. As the velocity of K_s is known then t_s depends only on a chosen event.

Let us a measure time moment t in the fixed frame through the distance of light ray ct emitted from the point O at the parallel angle to X-axis in some plane. Then for any event (x, t) the delay time ct_F is just a projection of the given x on the chosen light ray ct (see Figs. 1–4).

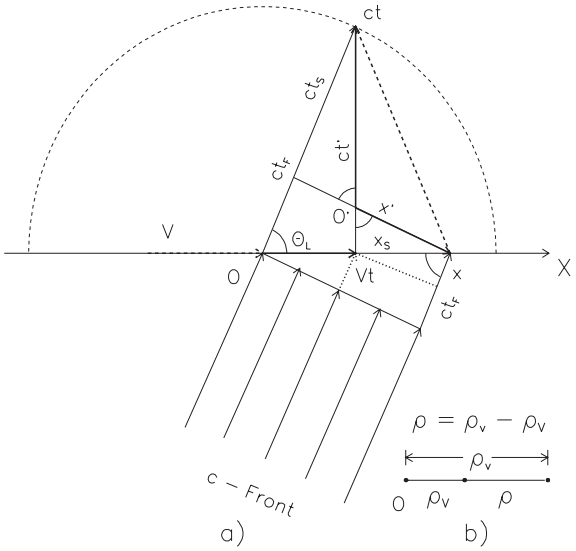

Figure 4. a) an illustration for the inertial frame x and t coordinate transformation (including Lorentz transformation); b) a velocity space diagram corresponding to x and t shifts (by the moment of time t a given x coordinate is assumed as x-position of a particle, moving with a velocity $v = x/t$ in K-frame).

It is obvious that the K_s origin displacement $Vt = ct\cos\theta_L$ is just a projection of light ray ct on the X-axis. So, a given x at a given time t has a value x_s relative to the origin O_s:

$$x_s = x - Vt = x - ct\cos\theta_L. \tag{7}$$

For any event $(x = Vt, t)$ a relative coordinate is $x_s = 0$. It means that time t_s (see (5) and (6)) is a proper time of K_s – the time "measured" by means of the "moving clock". An observer in K sees light sphere with radius ct and at the same time t a moving observer sees another light sphere with radius ct_s. Thus, for any event (x, t) in K the corresponding coordinates for K_s one can find as simple shifts (see (5) and (7)). To find out the values of shifts, one should produce the mentioned above symmetrical projections.

Let us use the established symmetry to find out Lorentz coordinates x' and t' for a moving frame. To obtain them, one should find the crossing point O' of two perpendiculars producing the mentioned projections (see Fig. 4). Then the length of the side from O' up to the x corresponds to x':

$$x' = (x - ct\cos\theta_L)/\sin\theta_L = (x - Vt)/\sqrt{1 - V^2/c^2}, \qquad x_s = x'\sin\theta_L, \tag{8}$$

and the length of the side from O' up to the ct corresponds to ct':

$$ct' = (ct - x\cos\theta_L)/\sin\theta_L = (ct - xV/c)/\sqrt{1 - V^2/c^2}, \qquad ct_s = ct'\sin\theta_L. \tag{9}$$

It is seen from (8) and (9) that primed and shifted coordinates are related as corresponding projections. But the point O', which is always accepted as the origin of the moving frame, does not coincide in space with O_s. It is also seen that $O'x'$ line is not parallel to the X-axis. So, it seems obvious that primed values cannot be accepted as coordinates of a moving frame.

Now, let us look at the length of a side between the given points x and ct (dashed line in Fig. 4). It can be obviously written through the primed and unprimed values:

$$l^2 = c^2t^2 + x^2 - 2ctx\cos\theta_L = c^2t'^2 + x'^2 + 2ct'x'\cos\theta_L, \tag{10}$$

or as a sum of two terms, either as $l^2 = s_1^2 + s_2^2$ (for that one should add $\pm x^2$ to the left part of (10) and $\pm x'^2$ to the right part of it), or as $l^2 = -s_1^2 + s_3^2$ (for that one should add $\pm c^2t^2$ to the left part of (10) and $\pm c^2t'^2$ to the right part of it), where:

$$s_1^2 = c^2t^2 - x^2 = c^2t'^2 - x'^2 = \gamma^2(c^2t_s^2 - x_s^2), \qquad \gamma = 1/\sin\theta_L = 1/\sqrt{1 - V^2/c^2}, \tag{11}$$

$$s_2^2 = 2x(x - ct\cos\theta_L) = 2x'(x' \pm ct'\cos\theta_L),$$

$$s_3^2 = 2ct(ct - x\cos\theta_L) = 2ct'(ct' \pm x'\cos\theta_L). \tag{12}$$

Term s_1^2 is known as an invariant interval. It is seen that it is only a part of full length l^2 and that this part is a result of cancelling of two equal values, either s_2^2 or s_3^2, in the possible expressions for l^2. Terms s_2^2 and s_3^2 may differ by sign: $(+)$ corresponds to the case when the point O' is inside and $(-)$ when it is outside of the angle θ_L. For an event $(x = Vt, t)$ term s_2^2 is equal to zero (as $x' = 0$) and $s_3^2 = 2s_1^2$, so $s_1^2 \equiv l^2$. Just for this case the Lorentz coordinate transformation are usually proved in manuals (see e.g. [4]).

By using second formulas in (8), (9) one can find from (3):

$$x = (x_s + ct_s\cos\theta_L)/\sin^2\theta_L = (x_s + Vt_s)/(1 - V^2/c^2), \tag{13}$$

$$ct = (ct_s + x_s\cos\theta_L)/\sin^2\theta_L = (ct_s + Vx_s/c)/(1 - V^2/c^2), \tag{14}$$

just the reverse transformation from the moving frame to the fixed one. To be sure of that, one has to solve a system of (5) and (7) with respect to x and ct. To make a geometrical meaning of the latter formulas more clear, it is useful to insert the factor $1/sin\theta_L$ into brackets (then terms in brackets are lengths of perpendiculars corresponding to the above-mentioned projection symmetry).

As seen from (5), (7) and (13), (14) that the straight and reverse transformations are different: back formulas could not be taken by changing V on $-V$. It means that one knows either that

frame moves, or it is fixed. As it was shown, changing V on $-V$ one should also choose an appropriate side light beam direction for a moving frame. So, if K_s moves in the backward to X direction ($V < 0$) one should change the sign in (5) and (7) and also in nominators of the back formulas (13), (14). Thus, choosing the corresponding (to the known velocities) straight and side light beams, any two frames may be considered in a such way that one of them can be taken as a moving frame and the other one as in the rest or vise versa.

The possible way to realize these opportunities is to make assumption that in the surrounded world there are a lot of light streams of any directions, something is like ether, but not in the rest – it is a moving light ether.

4 *y*,*z*-coordinate transformation and invariants

Let us have an event $(x, y, z = 0, t)$ in K frame and the side light beam falls onto X-axis in XY-plane as shown in Fig. 5, i.e., it falls from down to up and hits first of all the plane point (x, y) and then a point $(x, y = 0)$ on the X-axis (if y-coordinate has an opposite sign, then one can choose another side beam falling from up to down).

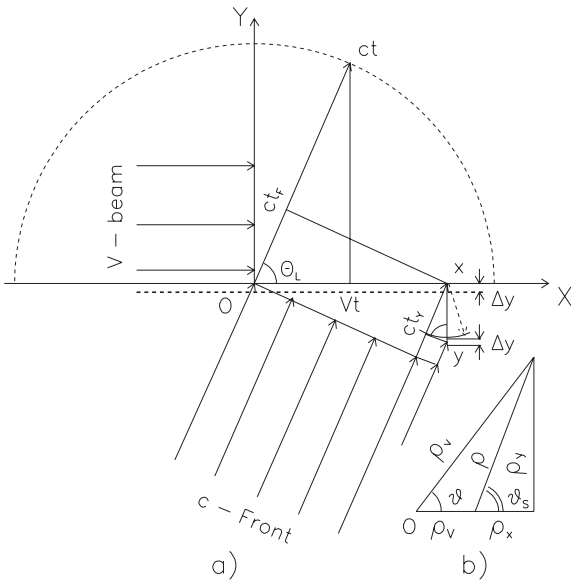

Figure 5. a) an illustration to arising of Δy-shift due to the light way difference; b) the velocity space diagram corresponding to a) (see note in Fig. 4b).

A secondary light sphere spreads out from the first point up to the X-axis (up to a point $(x, y = 0)$) for a time y/c. The side beam's ray hits this point in a moment of time $y \sin\theta_L/c$ (since the moment of time when secondary sphere starts to spread out from the first point). So, there is the light way difference:

$$c\Delta t \equiv \Delta y = y - y\sin\theta_L. \tag{15}$$

To compensate for this difference and for an y-coordinate (in a moving frame) to has the same initial moment of time as x_s, the origin of the K_s should be shifted along the Y-axis by value Δy defined by (15). Then an y-coordinate in K_s frame is:

$$y_s = y - \Delta y = y\sin\theta_L = y\sqrt{1 - V^2/c^2}. \tag{16}$$

For another transverse coordinate z_s one can get:

$$z_s = z - \Delta z = z\sin\theta_L = z\sqrt{1 - V^2/c^2} \tag{17}$$

the same way. The reverse transformation is obvious:

$$y = y_s/\sin\theta_L = y_s/\sqrt{1 - V^2/c^2}, \qquad z = z_s/\sin\theta_L = z_s/\sqrt{1 - V^2/c^2}. \tag{18}$$

For the non-invariant interval (see (11)) by using the latter formulas one can get:

$$c^2t^2 - x^2 - y^2 - z^2 = \gamma^2\left(c^2t_s^2 - x_s^2 - y_s^2 - z_s^2\right). \tag{19}$$

Thus, the obtained coordinate transformations lead to the reduced interval (i.e. to the light sphere with the reduced radius) but they do not contradict to the velocity summation law. Since the energy-momentum transformation is a direct consequence of the velocity summation law, then the Lorentz energy-momentum transformation is valid in this approach [5]. Also in [5] relativistic effects considered in detail, and the four elements complex fraction invariant and a new wave equation in framework of this approach were proposed.

5 Conclusion

- A complete correspondence has been established between Lobachevsky parallel lines in the velocity space and the processes of particle and light beams propagation in the ordinary space, synchronized by Huygens principle.
- The time delay in the emission of two light rays has been found as the physical reason for their intersection point absence and for the V postulate denial.
- New contents of the simultaneity conception, common time and proper time have been formulated.
- New inertial frames coordinate transformations (as shifts) have been obtained.
- It has been shown, that relativistic effects happen due to the coordinate and time reference point shifts. Changing the way of measuring the space or time interval lengths, one can find the way when these values are the same in fixed and moving frames.
- It has been shown that Lorentz energy-momentum transformation is a direct consequence of the relativistic velocity summation law.
- It has also been shown that Lobachevsky function is a tool to express the constant light velocity principle.
- The four elements of the complex fractional invariant and a possible wave equation have been proposed.

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Associative Algebras, Punctured Disks and the Quantization of Poisson Manifolds

Domenico FIORENZA and Riccardo LONGONI

Dipartimento di Matematica "G. Castelnuovo", Università di Roma "La Sapienza", P.le A. Moro 2, I-00185 Roma, Italy
E-mail: *fiorenza@mat.uniroma1.it, longoni@mat.uniroma1.it*

The aim of the note is to provide an introduction to the algebraic, geometric and quantum field theoretic ideas that lie behind the Kontsevich–Cattaneo–Felder formula for the quantization of Poisson structures. We show how the quantization formula itself naturally arises when one imposes the following two requirements to a Feynman integral: on the one side it has to reproduce the given Poisson structure as the first order term of its perturbative expansion; on the other side its three-point functions should describe an associative algebra. It is further shown how the Magri–Koszul brackets on 1-forms naturally fits into the theory of the Poisson sigma-model.

1 Deformation quantization as a Feynman diagrams expansion

A Poisson manifold is a differentiable manifold M endowed with a bi-vector $\alpha \in \Gamma(M; TM \wedge TM)$ such that $[\alpha, \alpha] = 0$, where $[\cdot, \cdot]$ is the Schouten–Nijenhuis bracket (see e.g. [7]). The bi-vector α defines a Poisson algebra structure on the space of smooth functions on M by

$$\{f, g\} := \langle \alpha \,|\, \mathrm{d}f \wedge \mathrm{d}g \rangle.$$

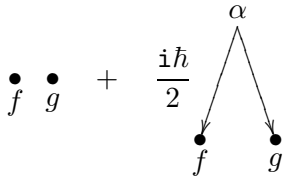
The problem of deformation quantization of the given Poisson structure is that of finding an associative $\star$-product on $C^\infty(M)[[\hbar]]$ deforming the usual pointwise product on $C^\infty(M)$ and having the Poisson bracket as the first order term in $\hbar$:

$$(f \star g)(x) = f(x)g(x) + \frac{\mathtt{i}\hbar}{2}\{f, g\}(x) + O\big(\hbar^2\big), \tag{1}$$

or, more generally,

$$(f \star g)(x) = f(x)g(x) + \frac{\mathtt{i}\hbar}{2}\big(\{f, g\} + B(f, g)\big)(x) + O\big(\hbar^2\big), \tag{2}$$

where B is a symmetric bi-differential operator. This problem has been solved by M. Kontsevich [3], and his solution was then interpreted in the language of quantum field theories by A. Cattaneo and G. Felder [2]. These notes are an attempt to explain why the Cattaneo–Felder model naturally arises when one tries to look at (1) as the perturbative expansion of a Feynman integral:

$f \star g =$  $+ O\big(\hbar^2\big).$

We see from this formula that there are two types of vertices, namely the ones labelled by the functions f, g and the ones labelled by the bi-vector α, and that the propagator is

$= \mathrm{d}x^i \otimes \partial_i,$

where ∂_i is a shorthand notation for $\partial/\partial x^i$. By the above description, we see that our fields are tangent and cotangent vectors at x; moreover, in order to look at α as to a function of the fields, we have to consider the cotangent vectors as odd fields, i.e., the coordinates η_i of a cotangent vector η are anticommuting variables. Therefore, the natural choice for the space of fields is $T_xM \oplus \Pi T^*_xM$, endowed with the natural pairing $\langle \partial_i | \mathrm{d}x^j \rangle = \delta_i^j$.

The functions f and g and the Poisson bi-vector α can be seen as functions on the space of fields, by using the Taylor expansions:

$$
\begin{aligned}
f(\xi,\eta) &:= f(x+\xi) = f(x) + \partial_i f(x)\xi^i + \frac{1}{2}\partial_i\partial_j f(x)\xi^i\xi^j + \cdots, \\
g(\xi,\eta) &:= g(x+\xi) = g(x) + \partial_i g(x)\xi^i + \frac{1}{2}\partial_i\partial_j g(x)\xi^i\xi^j + \cdots, \\
\alpha(\xi,\eta) &:= \langle \alpha(x+\xi) | \eta\wedge\eta\rangle = \alpha^{ij}(x)\eta_i\eta_j + \partial_k\alpha^{ij}(x)\eta_i\eta_j\xi^k + \cdots,
\end{aligned}
$$

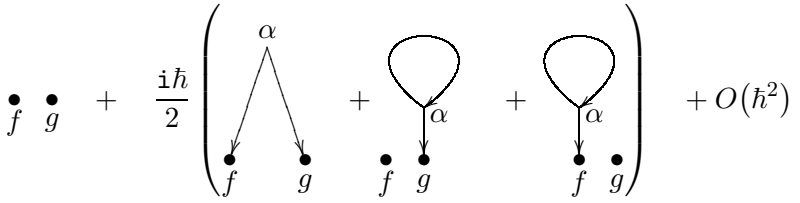
where $\xi \in T_xM$ and $\eta \in \Pi T^*_xM$. Now consider

$$
\int_{T_xM\oplus\Pi T^*_xM} \mathrm{d}\xi\mathrm{d}\eta\, f(x+\xi)\, g(x+\xi)\, e^{\frac{\mathrm{i}}{\hbar}S(\xi,\eta)} \Bigg/ \int_{T_xM\oplus\Pi T^*_xM} \mathrm{d}\xi\mathrm{d}\eta\, e^{\frac{\mathrm{i}}{\hbar}\langle\xi|\eta\rangle}, \tag{3}
$$

where the action is

$$
S(\xi,\eta) = S_{\mathrm{free}}(\xi,\eta) + S_{\mathrm{int}}(\xi,\eta) := \langle\xi|\eta\rangle + \langle\alpha(x+\xi)|\eta\wedge\eta\rangle.
$$

By the usual Feynman rules, the perturbative expansion of (3) is



which is of the form (2). Note that, if α is constant as a function of $x \in M$, then the perturbative expansion of (3) is

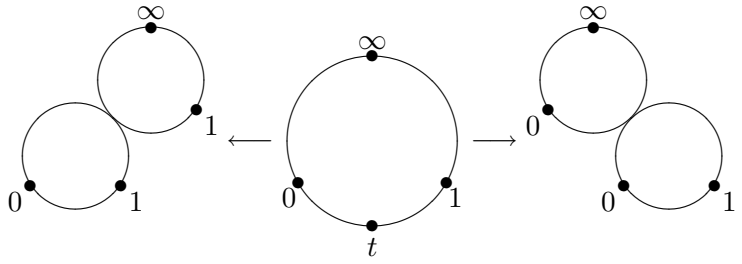
$$
(f \star g)(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\mathrm{i}\hbar}{2}\right)^n \alpha^{i_1 j_1}\cdots\alpha^{i_n j_n}\partial_{i_1}\cdots\partial_{i_n} f(x)\,\partial_{j_1}\cdots\partial_{j_n} g(x)
$$

which is precisely the Moyal $\star$-product formula. However, for general α, formula (3) does not yield an associative $\star$-product. A way to remedy this is to consider a topological space whose geometry describes the structure of associative algebras, and pull back our integral onto this space.

2 Punctured disks and associative algebras

Let D be the unit complex disk, and let B_n be the moduli space of $(n+1)$ points on the boundary of D, for $n \geq 2$. The disk D is identified with the complex upper half plane and its boundary

with $\mathbb{R}\cup\{\infty\}$. Since the group of the biholomorphisms acts 3-transitively on the set of boundary points on D, we can fix three of them to be 0, 1 and ∞, and make all the others lie in the interval $(0,1)$. Therefore B_n is just the open $(n-2)$-dimensional simplex $0<t_1<\cdots<t_{n-2}<1$. One can define a compactification $\overline{B}_n$ of B_n by adding products of $B_{n'}$, $n'<n$; these new boundary components correspond to the collapsing of two or more points in the boundary. For instance, there are two boundary components in $\overline{B}_3$ corresponding to the degenerations as $t=t_1$ goes to 0 or to 1.



Now, we look at B_2 as to an operation m_2 with two inputs (the points 0 and 1) and one output (the point ∞). Note that the two boundary components of $\overline{B}_3$ correspond to the two ways of composing m_2 with itself, namely $m_2(m_2\otimes\mathrm{id})$ and $m_2(\mathrm{id}\otimes m_2)$. So, if we find a continuous family of operations $m_3(t)$, $t\in(0,1)$, with three inputs and one output, which extends to the compactification $\overline{B}_3$ (in a way compatible with the product structure of the boundary), then the associativity of m_2 is equivalent to $m_3(0)=m_3(1)$. If moreover $m_3(t)$ is differentiable, this is equivalent to

$$m_2 \text{ associative} \quad\Leftrightarrow\quad \int_0^1 \mathrm{d}t\,\frac{\mathrm{d}m_3(t)}{\mathrm{d}t}=0.$$

Remark 1. In the language of operads, the above discussion corresponds to the well-known fact that the chain complex $C_*(\overline{B}_n)$ is the operad governing A_∞ algebras. In particular one says that m_2 is associative only up to the homotopy m_3.

Now, we want to define m_2 and m_3 on the space of smooth functions on the Poisson manifold M, in such a way that m_2 is related to equation (3). The most natural choice is to consider the "expectation value" over the maps $X\colon D\to M$ of the product $f(X(0))\,g(X(1))\,h(X(\infty))$ w.r.t. some measure to be defined, and "raise" the indices, i.e., set h to be the Dirac delta function δ_x. In other words we are looking for an operation m_2 of the form

$$m_2(f,g)(x)=\int \mathrm{d}\mu(X)\,f(X(0))\,g(X(1))\,\delta_x(X(\infty)). \tag{4}$$

As for $m_3=m_3(t)$, we set

$$m_3(f,g,h)(x)=\int \mathrm{d}\mu(X)\,f(X(0))\,g(X(t))\,h(X(1))\,\delta_x(X(\infty))$$

so that the associativity of m_2 becomes

$$\int \mathrm{d}\mu(X)\int_0^1 \mathrm{d}t\left(f(X(0))\,\frac{\mathrm{d}g(X(t))}{\mathrm{d}t}\,h(X(1))\,\delta_x(X(\infty))\right)=0. \tag{5}$$

3 The Poisson sigma-model

In this Section we want to combine equation (4), which defines an associative product, with equation (3), which has the correct first term in its perturbative expansion. First, the measure

$\mathrm{d}\mu(X)$ in equation (4) should be of the form $(1/C)\,\mathrm{d}\xi \mathrm{d}e^{\frac{\mathrm{i}}{\hbar}S(\xi,\eta)}$ as in equation (3), where C is a suitable normalization constant. In order to accomplish this, a new field, denoted by η, has to be introduced: it has to be defined on the disk and take values in $\Pi T^*_x M$. Moreover, since the new action S will be an integral over D, it is natural to take $\eta \in \Omega^1(D; X^*(\Pi T^* M))$. We are therefore led to consider the following object

$$\int \mathrm{d}X \mathrm{d}\eta\, f(X(0))\, g(X(1)) \delta_x(X(\infty))\, e^{\frac{\mathrm{i}}{\hbar}S(X,\eta)} \Bigg/ \int \mathrm{d}X \mathrm{d}\eta\, e^{\frac{\mathrm{i}}{\hbar}\int_D \langle \mathrm{d}X | \eta \rangle}, \tag{6}$$

where $S(X,\eta) = \int_D \langle \mathrm{d}X|\eta\rangle + \frac{1}{2}\int_D \langle \alpha(X)|\eta \wedge \eta\rangle$.

Notice however that in equation (3), we have a tangent vector $\xi \in T_x M$, where x is some point in M. Hence, what we should consider are infinitesimal variations of the map X around the constant map $X \equiv x$. In other terms, in equation (6) we have to replace X with $x + \xi$ where $\xi \in \Omega^0(D; X^*(TM))$.

Since the map X at the point ∞ is fixed to be equal to x by the term $\delta_x(X(\infty))$, we have to impose the boundary condition $\xi(\infty) = 0$; finally the 1-form η is required to vanish on tangent vectors to the boundary of the disk D. The action now reads

$$S(\xi,\eta) = \int_D \langle \mathrm{d}\xi | \eta \rangle + \frac{1}{2}\int_D \langle \alpha(x+\xi) | \eta \wedge \eta \rangle,$$

and we define

$$(f \star g)(x) := \frac{\displaystyle\int \mathrm{d}\xi \mathrm{d}\eta\, f(x+\xi(0))\, g(x+\xi(1))\, e^{\frac{\mathrm{i}}{\hbar}S(\xi,\eta)}}{\displaystyle\int \mathrm{d}\xi \mathrm{d}\eta\, e^{\frac{\mathrm{i}}{\hbar}\int_D \langle \mathrm{d}\xi|\eta\rangle}}. \tag{7}$$

In order to perform the perturbative expansion of (7), symmetries of the action have to be taken into account. A systematic way of doing this is via the superfield formalism, namely we consider the superdisk $D^{2|2}$ with even coordinates u^1, u^2 and Grassmann coordinates θ^1, θ^2 and set

$$\tilde{\xi}^i = \xi^i + \eta^{+\,i}_\mu \theta^\mu + \frac{1}{2}\beta^{+\,i}_{\mu\nu}\theta^\mu\theta^\nu, \qquad \tilde{\eta}_i = \beta_i + \eta_{i\,\mu}\theta^\mu + \frac{1}{2}\xi^+_{i\,\mu\nu}\theta^\mu\theta^\nu.$$

The de Rham differential now reads $\mathrm{D} = \theta_\mu \frac{\partial}{\partial u^\mu}$ and the $\star$-product becomes

$$(f \star g)(x) = \frac{\displaystyle\int_{\xi^+=\eta^+=\beta^+=0} \mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}\, f(x+\tilde{\xi}(0))\, g(x+\tilde{\xi}(1))\, e^{\frac{\mathrm{i}}{\hbar}S(\tilde{\xi},\tilde{\eta})}}{\displaystyle\int_{\xi^+=\eta^+=\beta^+=0} \mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}\, e^{\frac{\mathrm{i}}{\hbar}\int_{D^{2|2}}\langle \mathrm{D}\tilde{\xi}|\tilde{\eta}\rangle}}, \tag{8}$$

where the superaction is

$$S(\tilde{\xi},\tilde{\eta}) := \int_{D^{2|2}} \langle \mathrm{D}\tilde{\xi}|\tilde{\eta}\rangle + \frac{1}{2}\int_{D^{2|2}} \langle \alpha(x+\tilde{\xi})|\tilde{\eta}\wedge\tilde{\eta}\rangle. \tag{9}$$

Notice that besides of the original fields ξ, η (and their "antifields" ξ^+, η^+), a new field β has appeared, which can be interpreted as an infinitesimal symmetry of the original action (see Remark 4 below).

The advantage of this reformulation of the Poisson sigma-model is that we can now apply the Batalin–Vilkovisky formalism and deform the subspace $\xi^+ = \eta^+ = \beta^+ = 0$ over which the integration is performed, in such a way that the perturbative expansion is well defined.

4 Batalin–Vilkovisky formalism

We recall that for any vector space V, the space of functions on $V \oplus \Pi V^*$ is naturally endowed with a BV algebra structure [1, 6]. Using the standard terminology, we call *fields* the coordinates v^i on V and *antifields* the coordinates v_i^+ on ΠV^*. The BV bracket between two functionals $f, g\colon V \oplus \Pi V^* \to \mathbb{R}$ is given by

$$(f,g) := \frac{\overleftarrow{\partial} f}{\partial v^i}\frac{\overrightarrow{\partial} g}{\partial v_i^+} - \frac{\overleftarrow{\partial} f}{\partial v_i^+}\frac{\overrightarrow{\partial} g}{\partial v^i}$$

while the BV Laplacian is

$$\Delta f = \frac{\overrightarrow{\partial}}{\partial v_i^+}\frac{\overleftarrow{\partial}}{\partial v^i} f.$$

The BV bracket and the BV Laplacian satisfy, together with the pointwise product, the axioms of a BV algebra, namely

$$\begin{aligned}
&(f,g) = -(-1)^{(|f|-1)\,(|g|-1)}(g,f),\\
&(f,(g,h)) = ((f,g),h) + (-1)^{(|f|-1)(|g|-1)}(g,(f,h)) = 0,\\
&(f,gh) = (f,g)h + (-1)^{(|f|-1)|g|}g(f,h),\\
&(f,g) = \Delta(fg) - \Delta(f)g + (-1)^{|f|}f\Delta(g),\\
&\Delta^2 = 0.
\end{aligned}$$

In particular a Δ-cohomology is defined on the space of functional on the fields-antifields.

In our case $(\xi,\eta,\beta) \in V = \Omega^0(D, X^*(TM)) \oplus \Omega^1(D, X^*(\Pi T^*M)) \oplus \Omega^0(D, X^*(\Pi T^*M))$ and $(\xi^+,\eta^+,\beta^+) \in \Pi V^* = \Omega^2(D, X^*(\Pi T^*M)) \oplus \Omega^1(D, X^*(TM)) \oplus \Omega^2(D, X^*(TM))$. A "total degree" is then introduced by saying that a form on D with values in $X^*(TM)$ has total degree zero, while a form with values in $X^*(\Pi T^*M)$ has total degree 1. Next, we define the "ghost number" *gh* as the difference between the total degree and the degree *deg* as a differential form on D. We summarize the degrees and ghost numbers of our fields and antifields in the following table:

$gh \backslash deg$	0	1	2
-2			β^+
-1		η^+	ξ^+
0	ξ	η	
1	β		

A main feature of the BV formalism is that the integral of a Δ-closed functional $\mathcal{H}$ performed over a Lagrangian submanifold L in the space of fields-antifields, depends only on the homology class of L and that the integral of a Δ-exact functional is zero. Hence, integration defines a pairing between homology classes of Lagrangian submanifolds and Δ-cohomology classes. An easy computation shows that a functional of the form $e^{\frac{\mathrm{i}}{\hbar}S}$ is Δ-closed if and only if S satisfies the "quantum master equation"

$$(S,S) - 2i\hbar\Delta(S) = 0 \tag{10}$$

as indeed happens for the superaction (9) of the Poisson sigma-model [2] (see also Remark 3 below). More generally, if the functional $\mathcal{H}$ is of the form $\mathcal{O}\, e^{\frac{\mathrm{i}}{\hbar}S}$ for some functional $\mathcal{O}$ and some S satisfying equation (10), we have that $\Delta(\mathcal{O}\, e^{\frac{\mathrm{i}}{\hbar}S}) = 0$ if and only if $\Omega(\mathcal{O}) = 0$, where $\Omega(\mathcal{O}) :=$

$(S,\mathcal{O}) - i\hbar\Delta(\mathcal{O})$. Equation (10) immediately implies $\Omega^2 = 0$ and the relevant cohomology classes are called "observables" of the theory. Since the "expectation value" $\langle\mathcal{O}\rangle := \int_L \mathcal{O}\, e^{\frac{i}{\hbar}S}$ of an observable $\mathcal{O}$ depends only on the homology class of L, the perturbative expansion of the original path integral (7), which corresponds to integrating over the Lagrangian submanifold $\xi^+ = \eta^+ = \beta^+ = 0$ (and which is actually ill-defined due to the symmetries), can be effectively computed by choosing an appropriate submanifold where the quadratic part of the action is non-degenerate (see [2] for details).

Remark 2. For any point u in the boundary of D, one has

$$\Omega(\tilde{\xi}^i(u)) = \Omega(\tilde{\eta}_j(u)) = 0. \tag{11}$$

This gives a way to construct observables for the Poisson sigma-model from a point $u \in \partial D$ and a smooth function φ of $\tilde{\xi}$ and $\tilde{\eta}$. Indeed, the functional $\mathcal{O}_{\varphi,u}(\tilde{\xi},\tilde{\eta}) := \varphi(\tilde{\xi}(u),\tilde{\eta}(u))$ is clearly Ω-closed. In particular, $f(x+\tilde{\xi}(0))$ and $g(x+\tilde{\xi}(1))$ from equation (8) are observables.

Remark 3. Given a p-multivector field ψ, written in coordinates as $\psi(x)^{i_1,\dots,i_p}\partial_{i_1}\wedge\cdots\wedge\partial_{i_p}$, we can consider

$$S_\psi(\tilde{\xi},\tilde{\eta}) := \int_{D^{2|2}} \psi(x+\tilde{\xi})^{i_1,\dots,i_p}\tilde{\eta}_{i_1}\cdots\tilde{\eta}_{i_p}.$$

Notice that with this notation the superaction (9) becomes $S(\tilde{\xi},\tilde{\eta}) = S_{\text{free}}(\tilde{\xi},\tilde{\eta}) + S_\alpha(\tilde{\xi},\tilde{\eta})$. An explicit calculation shows that the map $\psi \mapsto S_\psi$ is a Lie algebra morphism

$$(S_{\psi_1}, S_{\psi_2}) = S_{[\psi_1,\psi_2]}$$

where we have the BV bracket on the l.h.s. and the Schouten–Nijenhuis bracket on the r.h.s. In particular, since the bi-vector α is Poisson, we have $(S_\alpha, S_\alpha) = 0$. When the "free" part of the superaction is taken into account, it is not difficult to show that $(S_{\text{free}}, S_{\text{free}}) = 0$ and $(S_{\text{free}}, S_\psi) = 0$, which in turn imply the so-called "master equation" for (9)

$$(S,S) = (S_{\text{free}} + S_\alpha, S_{\text{free}} + S_\alpha) = 0. \tag{12}$$

A consequence of this equality is that $\delta f := (S,f)$ is a coboundary operator. Finally, notice that the quantum master equation (10) descends immediately from the relations $\Delta(S_{\text{free}}) = \Delta(S_\alpha) = 0$.

Remark 4. Using the operator δ defined above, we can rewrite equation (12) as

$$\delta S = 0. \tag{13}$$

On the other hand one explicitly computes

$$\delta\tilde{\xi}^i = \mathrm{D}\tilde{\xi}^i + \alpha^{ij}(x+\tilde{\xi})\tilde{\eta}_j, \tag{14}$$

$$\delta\tilde{\eta}_i = \mathrm{D}\tilde{\eta}_i + \frac{1}{2}\partial_i\alpha^{jk}(x+\tilde{\xi})\tilde{\eta}_j\tilde{\eta}_k. \tag{15}$$

The operator $\delta|_{\xi^+=\eta^+=\beta^+=0}$ can be seen as a vector field on the space of functionals of (ξ,η) depending on the choice of β. We denote by δ_β this vector field. Now, equations (13)–(15) together imply that δ_β is an infinitesimal symmetry of the original action $S(\xi,\eta)$. Explicitly this symmetry reads

$$\delta_\beta\xi^i = \alpha^{ij}(x+\xi)\beta_j,$$
$$\delta_\beta\eta_i = -\mathrm{d}\beta_i - \partial_i\alpha^{jk}(x+\xi)\eta_j\beta_k.$$

5 Ward identities

The equation $\int_L \Delta(\mathcal{H}) = 0$ produces non-trivial identities (called "Ward identities") among the expectation values. For instance if $\phi(\tilde{\xi}, \tilde{\eta})$ is a Δ-closed functional, the following equality easily descends from the axioms of a BV algebra

$$0 = \int_L \Delta\big(e^{\frac{\mathrm{i}}{\hbar}S}\phi\big) = \int_L e^{\frac{\mathrm{i}}{\hbar}S}\delta\phi. \tag{16}$$

Now consider

$$\phi = \int_0^1 \mathrm{d}t \int \mathrm{d}\theta\, f(x+\tilde{\xi}(0))\, g(x+\tilde{\xi}(t,\theta))\, h(x+\tilde{\xi}(1)).$$

An explicit computation using equation (14) shows that

$$\delta\phi = \int_0^1 \mathrm{d}t \left(f(x+\tilde{\xi}(0))\, \frac{\mathrm{d}g(x+\tilde{\xi}(t))}{\mathrm{d}t}\, h(x+\tilde{\xi}(1)) \right).$$

Therefore equation (16) has precisely the form of equation (5) and the Ward identity for this choice of ϕ is the associativity equation

$$0 = \int_L \Delta\left(e^{\frac{\mathrm{i}}{\hbar}S}\phi\right) = ((f\star g)\star h) - (f\star(g\star h)).$$

6 The Magri–Koszul bracket

If ω is a 1-form on M we can associate to it a function on $T_xM \oplus \Pi T_x^*M$ by

$$\omega(\xi,\eta) = \omega(\xi) := \langle \omega(x+\xi)|\xi\rangle = \omega_i(x)\xi^i + \partial_j\omega_i(x)\xi^i\xi^j + \cdots .$$

Similarly, to a vector field χ we can associate the function

$$\chi(\xi,\eta) := \langle \chi(x+\xi)|\eta\rangle = \chi^i(x)\eta_i + \partial_j\chi^i(x)\eta_i\xi^j + \cdots .$$

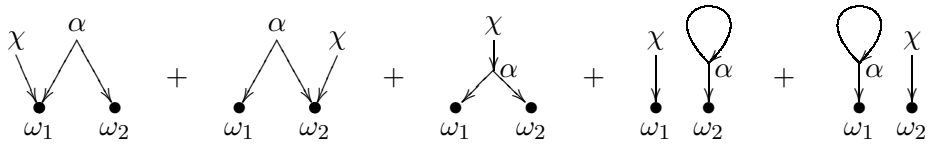
The perturbative expansion of the integral

$$\frac{\displaystyle\int_{T_xM\oplus\Pi T_x^*M} \mathrm{d}\xi\mathrm{d}\eta\, \omega_1\big(\xi\big)\, \omega_2\big(\xi\big)\, \chi\big(\xi,\eta\big)\, e^{\frac{\mathrm{i}}{\hbar}S(\xi,\eta)}}{\displaystyle\int_{T_xM\oplus\Pi T_x^*M} \mathrm{d}\xi\mathrm{d}\eta\, e^{\frac{\mathrm{i}}{\hbar}\langle\xi|\eta\rangle}}$$

is closely related to the Magri–Koszul bracket on 1-forms [4, 5]. More precisely, if we apply the Poisson sigma-model techniques to this situation, the function $\omega_1\big(\xi\big)\,\omega_2\big(\xi\big)\,\chi\big(\xi,\eta\big)$ is changed into $\omega_1\big(\tilde{\xi}(0)\big)\,\omega_2\big(\tilde{\xi}(1)\big)\,\chi\big(\tilde{\xi}(\infty),\tilde{\eta}(\infty)\big)$. Since $\xi(\infty)=0$, we have $\chi(\tilde{\xi}(\infty),\tilde{\eta}(\infty)) = \chi^i(x)\tilde{\eta}_i(\infty)$. Therefore the perturbative expansion of the path integral:

$$\frac{\displaystyle\int_{\xi^+=\eta^+=\beta^+=0} \mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}\, \omega_1(\tilde{\xi}(0))\, \omega_2(\tilde{\xi}(1))\, \chi(\tilde{\xi}(\infty),\tilde{\eta}(\infty))\, e^{\frac{\mathrm{i}}{\hbar}S(\tilde{\xi},\tilde{\eta})}}{\displaystyle\int_{\xi^+=\eta^+=\beta^+=0} \mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}\, e^{\frac{\mathrm{i}}{\hbar}\int_{D^{2|2}}\langle \mathrm{D}\tilde{\xi}|\tilde{\eta}\rangle}}, \tag{17}$$

will depend on $\chi(x)$ but not on its derivatives. The first order expansion of the integral (17) is $\frac{\mathrm{i}\hbar}{2}\langle\omega_1 \bullet \omega_2|\chi\rangle + O(\hbar^2)$ where

$\langle\omega_1 \bullet \omega_2|\chi\rangle :=$ 

If we define

$$[\omega_1, \omega_2] := \frac{\omega_1 \bullet \omega_2 - \omega_2 \bullet \omega_1}{2}$$

then

$$\begin{aligned}
[\omega_1, \omega_2] &= \alpha^{ij}(\partial_i\omega_{1\,k} + \partial_k\omega_{1,i})\omega_{2,j}\mathrm{d}x^k + \alpha^{ij}\omega_{1,i}(\partial_j\omega_{2\,k} + \partial_k\omega_{2,j})\mathrm{d}x^k + \partial_k\alpha^{ij}\omega_{1,i}\omega_{2,j}\mathrm{d}x^k \\
&= (\partial_k\alpha^{ij}\omega_{1,i}\omega_{2,j} + \alpha^{ij}\partial_k\omega_{1,i}\omega_{2,j} + \alpha^{ij}\omega_{1,i}\partial_k\omega_{2,j})\mathrm{d}x^k \\
&\quad - \alpha^{ij}\partial_j\omega_{1\,k}\omega_{2,i}\mathrm{d}x^k + \alpha^{ij}\omega_{1,i}\partial_j\omega_{2\,k}\mathrm{d}x^k \\
&= \mathrm{d}\langle\alpha|\omega_1 \wedge \omega_2\rangle + \mathcal{L}_{\alpha\lrcorner\omega_1}\omega_2 - \mathcal{L}_{\alpha\lrcorner\omega_2}\omega_1,
\end{aligned}$$

i.e., the bracket $[\omega_1, \omega_2]$ is precisely the Magri–Koszul bracket on 1-forms.

In particular one can recover the Jacobi identity for the Magri–Koszul bracket as a Ward identity (see Section 5) by choosing

$$\phi = \int_0^1 \mathrm{d}t \int \mathrm{d}\theta\, \omega_1(x + \tilde{\xi}(0))\, \omega_2(x + \tilde{\xi}(t,\theta))\, \omega_3(x + \tilde{\xi}(1))\, \chi(\tilde{\xi}(\infty), \tilde{\eta}(\infty)).$$

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Family Replicated Gauge Group Models

C.D. FROGGATT [*], *L.V. LAPERASHVILI* [†], *H.B. NIELSEN* [‡] *and Y. TAKANISHI* [§]

[*] *Department of Physics and Astronomy, Glasgow University, Glasgow G12 8QQ, Scotland*
E-mail: *c.froggatt@physics.gla.ac.uk*

[†] *ITEP, 25 B. Cheremushkinskaya Str., 117218 Moscow, Russia*
E-mail: *laper@heron.itep.ru*

[‡] *The Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*
E-mail: *hbech@alf.nbi.dk*

[§] *The Abdus Salam ICTP, Strada Costiera 11, 34100 Trieste, Italy*
E-mail: *yasutaka@ictp.trieste.it*

Family Replicated Gauge Group models of the type $SU(n)^N \times SU(m)^N$, $(SMG)^3$ and $(SMG \times U(1)_{B-L})^3$ are reviewed, where $SMG = SU(3)_c \times SU(2)_L \times U(1)_Y$ is the gauge symmetry group of the Standard Model, B is the baryon and L is the lepton numbers, respectively. It was shown that Family Replicated Gauge Group model of the latter type fits the Standard Model fermion masses and mixing angles and describes all neutrino experiment data order magnitudewise using only 5 free parameters – five vacuum expectation values of the Higgs fields which break the Family Replicated Gauge Group symmetry to the Standard Model. The possibility of $[SU(5)]^3$ or $[SO(10)]^3$ unification at the GUT-scale $\sim 10^{18}$ GeV also is briefly considered.

1 Introduction

Trying to gain insight into Nature and considering the physical processes at very small distances, physicists have made attempts to explain the well-known laws of low-energy physics as a consequence of the more fundamental laws of Nature. The contemporary low-energy physics of the electroweak and strong interactions is described by the Standard Model (SM) which unifies the Glashow–Salam–Weinberg electroweak theory with QCD – the theory of strong interactions.

The gauge symmetry group in the SM is:

$$SMG = SU(3)_c \times SU(2)_L \times U(1)_Y, \tag{1}$$

which describes elementary particle physics up to the scale ≈ 100 GeV.

Recently it was shown that the Family Replicated Gauge Groups (FRGG) of the type $SU(n)^N \times SU(m)^N$ provide new directions for research in high energy physics and quantum field theory. In the Deconstruction of space-time models [1], the authors tried to construct renormalizable asymptotically free 4-dimensional gauge theories which dynamically generate a fifth dimension (it is possible to obtain more dimensions in this way). Such theories naturally lead to electroweak symmetry breaking, relying neither on supersymmetry nor on strong dynamics at the TeV scale. The new TeV physics is perturbative and radiative corrections to the Higgs mass are finite. Thus, we see that the family replicated gauge groups provide a new way to stabilize the Higgs mass in the Standard Model.

But there exists quite different way to employ the FRGG.

2 Family Replicated Gauge Group as an extension of the Standard Model

The extension of the Standard Model with the Family Replicated Gauge Group:

$$G = (SMG)^{N_{\rm fam}} = [SU(3)_c]^{N_{\rm fam}} \times [SU(2)_L]^{N_{\rm fam}} \times [U(1)_Y]^{N_{\rm fam}} \tag{2}$$

was first suggested in the paper [2] and developed in the book [3] (see also the review [4]). Here $N_{\rm fam}$ designates the number of quark and lepton families. If $N_{\rm fam} = 3$ (as our theory predicts and experiment confirms), then the fundamental gauge group G is:

$$G = (SMG)^3 = SMG_{\rm 1st\ fam.} \times SMG_{\rm 2nd\ fam.} \times SMG_{\rm 3rd\ fam.}, \tag{3}$$

or

$$G = (SMG)^3 = [SU(3)_c]^3 \times [SU(2)_L]^3 \times [U(1)_Y]^3. \tag{4}$$

The generalized fundamental group:

$$G_f = (SMG)^3 \times U(1)_f \tag{5}$$

was suggested by fitting the SM charged fermion masses and mixing angles in paper [5]. A new generalization of our FRGG-model was suggested in papers [6], where:

$$\begin{aligned} G_{\rm ext} &= (SMG \times U(1)_{B-L})^3 \\ &\equiv [SU(3)_c]^3 \times [SU(2)_L]^3 \times [U(1)_Y]^3 \times [U(1)_{(B-L)}]^3 \end{aligned} \tag{6}$$

is the fundamental gauge group, which takes right-handed neutrinos and the see-saw mechanism into account. This extended model can describe all modern neutrino experiments, giving a reasonable fit to all the quark-lepton masses and mixing angles.

The gauge group $G = G_{\rm ext}$ contains: $3{\times}8 = 24$ gluons, $3{\times}3 = 9$ W-bosons, and $3{\times}1{+}3{\times}1 = 6$ Abelian gauge bosons.

The gauge group $G_{\rm ext} = (SMG \times U(1)_{B-L})^3$ undergoes spontaneous breakdown (at some orders of magnitude below the Planck scale) to the Standard Model Group SMG which is the diagonal subgroup of the non-Abelian sector of the group $G_{\rm ext}$. As was shown in Ref. [7], 6 different Higgs fields: ω, ρ, W, T, ϕ_{WS}, ϕ_{B-L} break our FRGG-model to the SM. The field ϕ_{WS} corresponds to the Weinberg–Salam Higgs field of Electroweak theory. Its vacuum expectation value (VEV) is fixed by the Fermi constant: $\langle\phi_{WS}\rangle = 246$ GeV, so that we have only 5 free parameters – five VEVs: $\langle\omega\rangle$, $\langle\rho\rangle$, $\langle W\rangle$, $\langle T\rangle$, $\langle\phi_{B-L}\rangle$ to fit the experiment in the framework of the SM. These five adjustable parameters were used with the aim of finding the best fit to experimental data for all fermion masses and mixing angles in the SM, and also to explain the neutrino oscillation experiments.

Experimental results on solar neutrino and atmospheric neutrino oscillations from Sudbury Neutrino Observatory (SNO Collaboration) and the Super-Kamiokande Collaboration have been used to extract the following parameters:

$$\begin{aligned} &\Delta m^2_{\rm solar} = m_2^2 - m_1^2, \qquad \Delta m^2_{\rm atm} = m_3^2 - m_2^2, \\ &\tan^2\theta_{\rm solar} = \tan^2\theta_{12}, \qquad \tan^2\theta_{\rm atm} = \tan^2\theta_{23}, \end{aligned} \tag{7}$$

where m_1, m_2, m_3 are the hierarchical left-handed neutrino effective masses for the three families. We also use the CHOOZ reactor results. It is assumed that the fundamental Yukawa couplings in our model are of order unity and so we make order of magnitude predictions. The typical fit is shown in Table 1. As we can see, the 5 parameter order of magnitude fit is encouraging.

Table 1. Best fit to conventional experimental data.
All masses are running masses at 1 GeV except the top quark mass which is the pole mass.

	Fitted	Experimental
m_u	4.4 MeV	4 MeV
m_d	4.3 MeV	9 MeV
m_e	1.6 MeV	0.5 MeV
m_c	0.64 GeV	1.4 GeV
m_s	295 MeV	200 MeV
m_μ	111 MeV	105 MeV
M_t	202 GeV	180 GeV
m_b	5.7 GeV	6.3 GeV
m_τ	1.46 GeV	1.78 GeV
V_{us}	0.11	0.22
V_{cb}	0.026	0.041
V_{ub}	0.0027	0.0035
$\Delta m^2_\odot$	9.0×10^{-5} eV^2	5.0×10^{-5} eV^2
$\Delta m^2_{\rm atm}$	1.7×10^{-3} eV^2	2.5×10^{-3} eV^2
$\tan^2\theta_\odot$	0.26	0.34
$\tan^2\theta_{\rm atm}$	0.65	1.0
$\tan^2\theta_{\rm chooz}$	2.9×10^{-2}	$< 2.6 \times 10^{-2}$

There are also 3 see-saw heavy neutrinos in this model (one right–handed neutrino in each family) with masses: M_1, M_2, M_3. The model predicts the following neutrino masses:

$$m_1 \approx 1.4 \times 10^{-3}\ \text{eV}, \qquad m_2 \approx 9.6 \times 10^{-3}\ \text{eV}, \qquad m_3 \approx 4.2 \times 10^{-2}\ \text{eV} \tag{8}$$

– for left-handed neutrinos, and

$$M_1 \approx 1.0 \times 10^{6}\ \text{GeV}, \qquad M_2 \approx 6.1 \times 10^{9}\ \text{GeV}, \qquad M_3 \approx 7.8 \times 10^{9}\ \text{GeV} \tag{9}$$

– for right-handed (heavy) neutrinos.

Finally, we conclude that our theory with the FRGG-symmetry is very successful in describing experiment. The best fit gave the following values for the VEVs:

$$\langle W\rangle \approx 0.157, \qquad \langle T\rangle \approx 0.077, \qquad \langle\omega\rangle \approx 0.244, \qquad \langle\rho\rangle \approx 0.265 \tag{10}$$

in the "fundamental units", $M_{\rm Pl} = 1$, and

$$\langle\phi_{B-L}\rangle \approx 5.25 \times 10^{15}\ \text{GeV} \tag{11}$$

which gives the see-saw scale: the scale of breakdown of the $U(1)_{B-L}$ groups ($\sim 5 \times 10^{15}$ GeV).

3 The problem of monopoles in the Standard and Family Replicated Models

The aim of the present Section is to show, following Ref. [8], that monopoles cannot be seen in the Standard Model and in its usual extensions in the literature up to the Planck scale: $M_{\rm Pl} = 1.22 \times 10^{19}$ GeV, because they have a huge magnetic charge and are completely confined or screened. Supersymmetry does not help to see monopoles.

In theories with the FRGG-symmetry the charge of monopoles is essentially diminished. Then monopoles can appear near the Planck scale and change the evolution of the fine structure

constants $\alpha_i(t)$ (here $i = 1, 2, 3$ corresponds to $U(1)$, $SU(2)$ and $SU(3)$), $t = \log(\mu^2/\mu_R^2)$, where μ is the energy variable and μ_R is the renormalization point.

Let us consider the "electric" and "magnetic" fine structure constants:

$$\alpha = \frac{g^2}{4\pi} \qquad \text{and} \qquad \tilde{\alpha} = \frac{\tilde{g}^2}{4\pi}, \tag{12}$$

where g is the coupling constant, and $\tilde{g}$ is the dual coupling constant (in QED: $g = e$ (electric charge), and $\tilde{g} = m$ (magnetic charge)).

The Renormalization Group Equation (RGE) for monopoles is:

$$\frac{d(\log\tilde{\alpha}(t))}{dt} = \beta(\tilde{\alpha}). \tag{13}$$

With the scalar monopole beta-function we have:

$$\beta(\tilde{\alpha}) = \frac{\tilde{\alpha}}{12\pi} + \left(\frac{\tilde{\alpha}}{4\pi}\right)^2 + \cdots = \frac{\tilde{\alpha}}{12\pi}\left(1 + 3\frac{\tilde{\alpha}}{4\pi} + \cdots\right). \tag{14}$$

The latter equation shows that the theory of monopoles cannot be considered perturbatively at least for

$$\tilde{\alpha} > \frac{4\pi}{3} \approx 4. \tag{15}$$

And this limit is smaller for non-Abelian monopoles.

Let us consider now the evolution of the SM running fine structure constants. The usual definition of the SM coupling constants is given in *the Modified minimal subtraction scheme* ($\overline{\text{MS}}$):

$$\alpha_1 = \frac{5}{3}\alpha_Y, \qquad \alpha_Y = \frac{\alpha}{\cos^2\theta_{\overline{\text{MS}}}}, \qquad \alpha_2 = \frac{\alpha}{\sin^2\theta_{\overline{\text{MS}}}}, \qquad \alpha_3 \equiv \alpha_s = \frac{g_s^2}{4\pi}, \tag{16}$$

where α and α_s are the electromagnetic and $SU(3)$ fine structure constants respectively, Y is the weak hypercharge, and $\theta_{\overline{\text{MS}}}$ is the Weinberg weak angle in $\overline{\text{MS}}$ scheme. Using RGEs with experimentally established parameters, it is possible to extrapolate the experimental values of the three inverse running constants $\alpha_i^{-1}(\mu)$ from the Electroweak scale to the Planck scale (see Fig. 1).

In this connection, it is very attractive to include gravity. The quantity:

$$\alpha_g = \left(\frac{\mu}{\mu_{\text{Pl}}}\right)^2 \tag{17}$$

plays the role of the running "gravitational fine structure constant" (see Ref. [8]) and the evolution of its inverse is presented in Fig. 1 together with the evolutions of $\alpha_i^{-1}(\mu)$.

Assuming the existence of the Dirac relation: $g\tilde{g} = 2\pi$ for minimal charges, we have the following expression for the renormalized charges g and $\tilde{g}$ [9]:

$$\alpha(t)\tilde{\alpha}(t) = \frac{1}{4}. \tag{18}$$

Using the Dirac relation, it is easy to estimate (in the simple SM) the Planck scale value of $\tilde{\alpha}(\mu_{\text{Pl}})$ (minimal for $U(1)_Y$ gauge group):

$$\tilde{\alpha}(\mu_{\text{Pl}}) = \frac{5}{3}\alpha_1^{-1}(\mu_{\text{Pl}})/4 \approx 55.5/4 \approx 14. \tag{19}$$

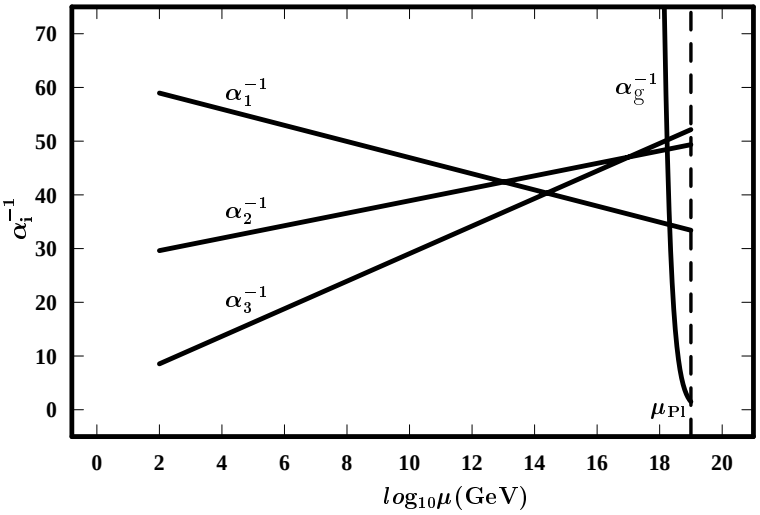

Figure 1.

This value is really very big compared with the estimate (15) and, of course, with the critical coupling $\tilde{\alpha}_{\rm crit} \approx 1$, corresponding to the confinement – deconfinement phase transition in the lattice $U(1)$ gauge theory. Clearly we cannot make a perturbation approximation with such a strong coupling $\tilde{\alpha}$. It is hard for such monopoles not to be confined.

There is an interesting way out of this problem if one wants to have the existence of monopoles, namely to extend the SM gauge group so cleverly that certain selected linear combinations of charges get bigger electric couplings than the corresponding SM couplings. That could make the monopoles which, for these certain linear combinations of charges, couple more weakly and thus have a better chance of being allowed "to exist".

An example of such an extension of the SM that can impose the possibility of allowing the existence of free monopoles is just Family Replicated Gauge Group Model (FRGGM).

FRGGs of type $[SU(N)]^{N_{\rm fam}}$ lead to the lowering of the magnetic charge of the monopole belonging to one family:

$$\tilde{\alpha}_{\text{one family}} = \frac{\tilde{\alpha}}{N_{\rm fam}}. \tag{20}$$

For $N_{\rm fam} = 3$, for $[SU(2)]^3$ and $[SU(3)]^3$, we have: $\tilde{\alpha}^{(2,3)}_{\text{one family}} = \tilde{\alpha}^{(2,3)}/3$. For the family replicated group $[U(1)]^{N_{\rm fam}}$ we obtain:

$$\tilde{\alpha}_{\text{one family}} = \frac{\tilde{\alpha}}{N^*}, \tag{21}$$

where $N^* = \frac{1}{2}N_{\rm fam}(N_{\rm fam}+1)$. For $N_{\rm fam} = 3$ and $[U(1)]^3$, we have: $\tilde{\alpha}^{(1)}_{\text{one family}} = \tilde{\alpha}^{(1)}/6$ (six times smaller!). This result was obtained previously in Ref. [10].

According to the FRGGM, at some point $\mu = \mu_G < \mu_{\rm Pl}$ (or really in a couple of steps) the fundamental group $G \equiv G_{\rm ext}$ undergoes spontaneous breakdown to its diagonal subgroup:

$$G \longrightarrow G_{\rm diag.subgr.} = \{g, g, g || g \in SMG\}, \tag{22}$$

which is identified with the usual (low-energy) group SMG.

In the Anti-GUT-model [2, 3] the FRGG breakdown was considered at $\mu_G \sim 10^{18}$ GeV. But the aim of this investigation is to show that we can see quite different consequences of the extension of the SM to FRGGM, if the G-group undergoes the breakdown to its diagonal

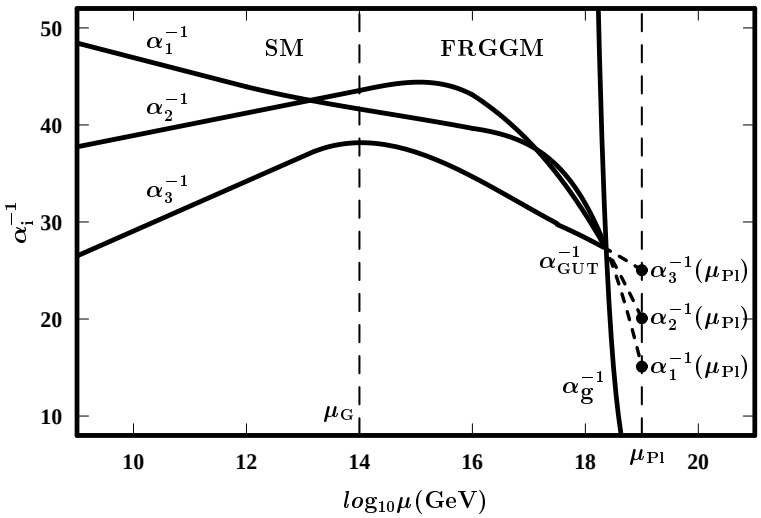

Figure 2.

subgroup (that is, SMG) not at $\mu_G \sim 10^{18}$ GeV, but at $\mu_G \sim 10^{14}$ or 10^{15} GeV, *i.e.* before the intersection of $\alpha_2^{-1}(\mu)$ with $\alpha_3^{-1}(\mu)$ at $\mu \approx 10^{16}$ GeV. In this case, in the region $\mu_G < \mu < \mu_{\rm Pl}$ there are three $SMG \times U(1)_{B-L}$ groups for the three FRGG families, and we have a lot of fermions, mass protected or not mass protected, belonging to usual families or to mirror ones. In the FRGGM the additional 5 Higgs bosons, with their large VEVs, are responsible for the mass protection of a lot of new fermions appearing in the region $\mu > \mu_G$. Here we denote the total number of fermions N_F, which is different to $N_{\rm fam}$.

Also the role of monopoles can be important in the vicinity of the Planck scale: they give contributions to the beta-functions and change the evolution of the $\alpha^{-1}(\mu)$. Finally, we obtain the following RGEs:

$$\frac{d(\alpha_i^{-1}(\mu))}{dt} = \frac{b_i}{4\pi} + \frac{N_M^{(i)}}{\alpha_i}\beta^{(m)}(\tilde{\alpha}_{U(1)}), \tag{23}$$

where b_i are given by the following values:

$$b_i = (b_1, b_2, b_3) = \left(-\frac{4N_F}{3} - \frac{1}{10}N_S,\ \frac{22}{3}N_V - \frac{4N_F}{3} - \frac{1}{6}N_S,\ 11N_V - \frac{4N_F}{3}\right). \tag{24}$$

The integers N_F, N_S, N_V, N_M are respectively the total numbers of fermions, Higgs bosons, vector gauge fields and scalar monopoles in the FRGGM considered in our theory. In our FRGG model we have $N_V = 3$, because we have 3 times more gauge fields ($N_{\rm fam} = 3$), in comparison with the SM and one Higgs scalar monopole in each family.

We have obtained the evolutions of $\alpha_i^{-1}(\mu)$ near the Planck scale by numerical calculations for: $\mu_G = 10^{14}$ GeV, $N_F = 18$, $N_S = 6$, $N_M^{(1)} = 6$, $N_M^{(2,3)} = 3$. Fig. 2 shows the existence of the unification point.

We see that in the region $\mu > \mu_G$ a lot of new fermions, and a number of monopoles near the Planck scale, change the one-loop approximation behavior of $\alpha_i^{-1}(\mu)$ which we had in the SM. In the vicinity of the Planck scale these evolutions begin to decrease, as the Planck scale $\mu = \mu_{\rm Pl}$ is approached, implying the suppression of asymptotic freedom in the non-Abelian theories. Fig. 2 gives the following Planck scale values for the α_i:

$$\alpha_1^{-1}(\mu_{\rm Pl}) \approx 13, \qquad \alpha_2^{-1}(\mu_{\rm Pl}) \approx 19, \qquad \alpha_3^{-1}(\mu_{\rm Pl}) \approx 24. \tag{25}$$

Fig. 2 demonstrates the unification of all gauge interactions, including gravity (the intersection of α_g^{-1} with α_i^{-1}), at

$$\alpha_{GUT}^{-1} \approx 27 \qquad \text{and} \qquad x_{GUT} \approx 18.4. \tag{26}$$

Here we can expect the existence of $[SU(5)]^3$ or $[SO(10)]^3$ (SUSY or not SUSY) unification.

Considering the predictions of such a theory for low-energy physics and cosmology, maybe in future we shall be able to answer the question: Does the unification $[SU(5)]^3$ or $[SO(10)]^3$ really exist near the Planck scale?

Recently F.S. Ling and P. Ramond [11] considered the group of symmetry $[SO(10)]^3$ and showed that it explains the observed hierarchies of fermion masses and mixings.

Acknowledgements

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Symmetry in Anisotropic Rotating Turbulence

José GAITE

Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, 28006 Madrid, Spain
E-mail: *jgaite@imaff.cfmac.csic.es*

In the turbulence that takes place in rotating fluids, the symmetry assumptions of the theory of fully developed turbulence may not hold, in particular, isotropy must be relaxed to axial symmetry. This symmetry reduction leads one to consider the most general axisymmetric *viscosity tensor* for a Newtonian fluid, which can be obtained by group theory methods. This tensor generates new turbulent effective forces on large scales in addition to the molecular viscous force.

1 Introduction

The turbulent state of a fluid is characterized by the presence of fluctuations of the velocity in a wide range of scales. These irregular velocity fluctuations enhance the dissipation of energy in the fluid well over the dissipation pertaining to molecular viscosity. Therefore, one speaks of an "effective" viscosity that operates in the same range of scales in which the turbulence takes place. For a general reference on modern ideas about turbulence, see the book by Uriel Frisch [1].

If we assume that the Navier–Stokes equations for viscous fluids are suitable to describe turbulence, this phenomenon appears when the *nonlinear* term becomes dominant, in particular, with respect to the dissipative term proportional to the molecular viscosity. More formally, as the nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ grows with the driving motion, the laminar flow becomes unstable, giving rise to irregular velocity fluctuations, which in turn enhance the dissipative term $\nu\nabla^2\boldsymbol{u}$, because it is very sensitive to the variations of the velocity (due to the second derivative).

The most interesting turbulent state is precisely the one with the maximal amount of nonlinearity, in which the velocity of the driving motion U (associated with a scale L) is very large in comparison with the molecular viscosity. In technical terms, this state is the solution of the equations as the Reynolds number $Re = UL/\nu \to \infty$, and it is called the state of *fully developed* turbulence. As the range of scales with velocity fluctuations grows with Re, in the $Re \to \infty$ limit it becomes formally infinite, so the fully developed turbulence is independent of the large scale dynamics associated with the driving motion and of the small scale dynamics associated with molecular dissipation and, in this sense, is "universal".

Symmetry assumptions are essential in the theory of fully developed turbulence. In fact, the classical theory of fully developed turbulence applies to maximally symmetric states, namely, invariant under time translations, on the one hand, and invariant under the space translation and rotation groups, on the other hand. So fully developed turbulence is a *stationary, homogenous and isotropic* state. Of course, these symmetries must be understood in a statistical sense, as symmetries of suitables averages of the velocity field (*correlation functions*), since the instantaneous value of this field is highly irregular.

These symmetry assumptions can be questioned; in particular, the assumption of isotropy has been questioned [2, 3, 1, 4], arguing that anisotropy can arise from a mean flow or from anisotropic driving. However, if we preserve homogeneity (space translation symmetry), the dependence of any property of the flow on its mean velocity, contradicts Galilean invariance. In contrast, an anisotropic driving is likely to produce anisotropy but contradicts the fact that the fully developed turbulence state is independent of the driving motion.

In this symmetry breaking context, we consider here the case of *rotating turbulence*. Turbulence in uniformly rotating fluids, which is called rotating turbulence, is an example of anisotropic turbulence and an area of active research [5, 6]. To be precise, in rotating turbulence the full rotation symmetry must necessarily break down to axial symmetry (the symmetry axis being the fluid rotation axis).

To study the breakdown of rotation symmetry, we shall first review the fluid equations in a rotating frame and introduce the viscosity in the standard manner [7] but *without* recourse to isotropy. This leads to the viscosity being defined by a four-rank tensor, the *viscosity tensor*. Isotropy is replaced by only symmetry around the rotation axis, given by the angular velocity $\boldsymbol{\Omega}$. Then all the components of the viscosity tensor are determined in terms of $\boldsymbol{\Omega}$ by group theory arguments [8]. Finally, from this tensor we obtain the effective anisotropic forces.

2 Dynamical equations of rotating turbulence

The hydrodynamical equations for a fluid with density $\rho(\boldsymbol{x}, t)$, velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ and pressure $P(\boldsymbol{x}, t)$ in a rotating frame are [5, 7]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P - 2\boldsymbol{\Omega} \times \boldsymbol{u} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) + \boldsymbol{f}. \tag{2}$$

The upper one, namely, the continuity equation, expresses conservation of mass while the lower one expresses conservation of momentum and includes the force from the gradient of pressure, the Coriolis and centrifugal forces, and an additional force that accounts for both friction and an external homogeneous and isotropic random force (in fact, all the forces are rather accelerations, since they have been divided by the density).

Assuming that the fluid is incompressible, with constant density, the continuity equation becomes $\nabla \cdot \boldsymbol{u} = 0$. If we further define $p = P/\rho$, every reference to the density disappears from the momentum equation. Moreover, we can also eliminate p by projecting the solenoidal (or transverse) components, obtaining

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{P}[(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}] = -\mathcal{P}(2\boldsymbol{\Omega} \times \boldsymbol{u}) + \boldsymbol{f}, \tag{3}$$

where the transverse projector is

$$\mathcal{P}_{ij} = \delta_{ij} - \partial_i \frac{1}{\nabla^2} \partial_j. \tag{4}$$

We call equation (3) the transverse rotating fluid equation. If we substitute for $\boldsymbol{f}$ an *isotropic* viscous force, it becomes the transverse rotating Navier–Stokes equation, but we do not need to be so restrictive, as we shall see. Note that the transverse rotating fluid equation (3) is *translation invariant* (assuming that $\boldsymbol{f}$ is homogeneous), in contrast to equation (2), since the centrifugal force has disappeared. Therefore, its solutions are homogeneous velocity fields and, furthermore, one can make use of the Fourier transform.

3 The effective viscosity tensor

In a fluid with no global motion, we can express the forces acting on the fluid element of volume (or mass) in terms of the stress tensor, which has the general expression

$$T_{ij} = -P\delta_{ij} + \sigma_{ij}, \tag{5}$$

where the *deviatoric* part σ_{ij} corresponds to the internal relative motion of fluid elements, producing the viscous forces $f_i = \partial_j \sigma_{ij}$. Naturally, σ_{ij} vanishes if there is no velocity gradient, $\partial_i \boldsymbol{u} = 0$; so for moderate internal relative motion we can assume that σ_{ij} is proportional to the velocity gradient, that is,

$$\sigma_{ij} = \tau_{ijmn} \partial_m u_n. \tag{6}$$

This relation characterizes the Newtonian fluids. Furthermore, one can make two assumptions [7]:

1. The components of the velocity gradient that are antisymmetric in its two indices (associated to the vorticity) are excluded; in other words, one only includes the symmetric components $u_{mn} = \partial_{(m} u_{n)}$.
2. The stress tensor can be taken to be symmetric in its two indices because of angular momentum conservation.

Under these conditions, we can write

$$\sigma_{ij} = \eta_{ijmn} u_{mn}, \tag{7}$$

defining a four-rank tensor with symmetry in the first and second pairs of indices. This tensor has already appeared in studies of anisotropic turbulence (see, e.g., Ref. [3]). Of course, the assumption of *isotropy* leads to the existence of only two proportionality constants, namely, shear and bulk viscosities, the latter playing no role in incompressible fluids.

However, in a rotating frame, neither isotropy nor the assumptions leading to the symmetry in the indices hold, so we must content ourselves with the bare proportionality provided by equation (6). In any event, we can decompose the four-rank tensor τ_{ijmn} into components with definite symmetry in the first and second pairs of indices; namely, we can make $\tau = \eta + \chi + \xi + \zeta$, where η has symmetry in the first and second pairs of indices, χ has symmetry in the first pair and antisymmetry in the second pair, ξ has antisymmetry in the first pair and symmetry in the second pair, and ζ has antisymmetry in both pairs.

4 General form of axisymmetric viscosity tensor and forces

4.1 Irreducible four-rank tensors with definite symmetry by pairs of indices

To derive the mathematical form of the decomposition $\tau = \eta + \chi + \xi + \zeta$ according to the symmetry of pairs of indices (and further decomposition still allowed), let us first work out the resolution of the general four-rank tensor T_{ijmn} into a sum of tensors of definite symmetry type given by standard Young tableaux. Young tableaux indicate certain symmetry operations performed on the indices [8]. We can consider the general four-rank tensor as a tensorial product of four vectors and, therefore, write its resolution as the Clebsch–Gordan decomposition for the linear group $GL(3)$ of the corresponding direct product:

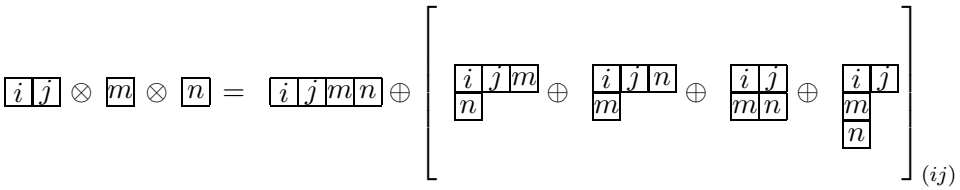
$$\begin{array}{|c|} \hline i \\ \hline \end{array} \otimes \begin{array}{|c|} \hline j \\ \hline \end{array} \otimes \begin{array}{|c|} \hline m \\ \hline \end{array} \otimes \begin{array}{|c|} \hline n \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline i & j & m & n \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline i & j & m \\ \hline n \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|c|} \hline i & j & n \\ \hline m \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|c|} \hline i & m & n \\ \hline j \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|} \hline i & j \\ \hline m & n \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline i & m \\ \hline j & n \\ \hline \end{array}$$

$$\oplus \begin{array}{|c|c|} \hline i & j \\ \hline m \\ \cline{1-1} n \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|} \hline i & m \\ \hline j \\ \cline{1-1} n \\ \cline{1-1} \end{array} \oplus \begin{array}{|c|c|} \hline i & n \\ \hline j \\ \cline{1-1} m \\ \cline{1-1} \end{array} \tag{8}$$

Now, we must project the above linear irreducible representations into irreducible representations with definite symmetry by pairs of indices. We can do it in two stages, the first one

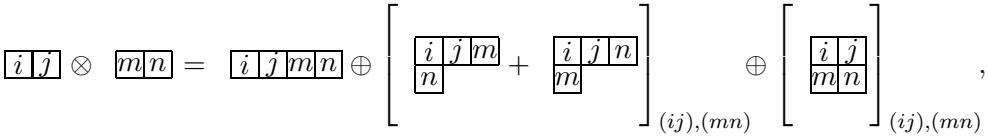
achieving symmetry in the first pair and the second one achieving symmetry in the second pair. Their diagrammatic representation is (the subindices denote either symmetrization or antisymmetrization with respect to the corresponding indices)

1. First stage:

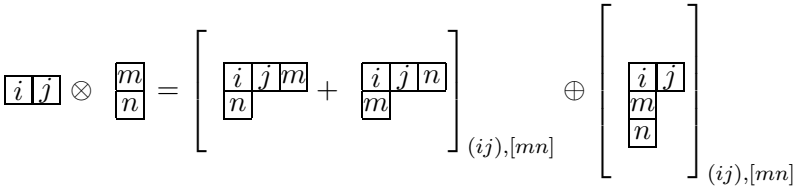


(9)

2. Second stage:



(10)



(11)

The tensor with antisymmetry in ij and symmetry in mn is analogous to the one with the reversed symmetries, given by equation (11). The tensor with antisymmetry in both pairs can be obtained straightforwardly.

4.1.1 Computation of Young tableaux

The actual derivation of the irreducible representations with definite symmetry by pairs of indices requires to compute the Young tableaux above and to perform the symmetry operations given by the subindices. It is simple linear algebra but somewhat cumbersome (for more details, see Ref. [9]). The results are (left subscripts indicate dimensions of representations):

- Part with symmetry by pairs.
  Starting from the general four-rank tensor T_{ijmn}, define

$$S_{ijmn} = T_{ijmn} + T_{jimn} + T_{ijnm} + T_{jinm}. \tag{12}$$

  Then, the computation of the Young tableaux of equation (10) yields:

  1. Pair-symmetric part:
     i) Totally symmetric part (first Young tableau):

$$_{15}S_{ijmn} = S_{ijmn} + S_{mnij} + S_{jmin} + S_{imjn} + S_{injm} + S_{jnim}. \tag{13}$$

     ii) Remaining part (last Young tableau):

$$_{6}S_{ijmn} = S_{ijmn} + S_{mnij} - \frac{1}{2}\left[S_{imjn} + S_{inmj} + S_{mjin} + S_{njmi}\right]. \tag{14}$$

  2. Pair-antisymmetric part (middle Young tableaux):

$$_{15}S'_{ijmn} = S_{ijmn} - S_{mnij}. \tag{15}$$

- Part with symmetry in the first pair and antisymmetry in the second pair.
  Now define

$$SA_{ijmn} = T_{ijmn} + T_{jimn} - T_{ijnm} - T_{jinm}\,. \tag{16}$$

  The computation of the Young tableaux of equation (11) yields:

  1. First irreducible representation:

$${}_{15}SA_{ijmn} = SA_{ijmn} + \frac{1}{2}\left[SA_{imjn} + SA_{inmj} + SA_{mjin} + SA_{njmi}\right]. \tag{17}$$

  2. Second irreducible representation:

$${}_{3}SA_{ijmn} = SA_{ijmn} - \frac{1}{2}\left[SA_{imjn} + SA_{inmj} + SA_{mjin} + SA_{njmi}\right]. \tag{18}$$

- Part with antisymmetry in the first pair and symmetry in the second pair: analogous to the preceding one.
- Part with antisymmetry by pairs.
  Define

$$A_{ijmn} = T_{ijmn} - T_{jimn} - T_{ijnm} + T_{jinm}. \tag{19}$$

  For this last part it is not really necessary to use Young tableaux. One obtains:

  1. Pair-symmetric part:

$${}_{6}A_{ijmn} = A_{ijmn} + A_{mnij}. \tag{20}$$

  2. Pair-antisymmetric part:

$${}_{3}A_{ijmn} = A_{ijmn} - A_{mnij}. \tag{21}$$

4.2 Rotation and axial symmetry

4.2.1 Reduction of the four-rank tensor under the rotation group

The preceding linearly irreducible four-rank tensors are, however, reducible under the rotation group $O(3)$ (which is a subgroup of the linear group $GL(3)$). The standard reduction of linear tensors under the rotation group $O(3)$ is performed by extracting and removing traces [8]. Indeed, the $O(3)$ irreducible representations are the symmetric traceless tensors, labelled by their rank J. Therefore, it is possible to reexpress each of the linear representations corresponding to the above four-rank tensors as a linear combination of symmetric traceless tensors of equal or lower rank. For example, from ${}_{15}S'$ we obtain the symmetric tensor $T_{ljn} = \epsilon_{lim}S'_{ijmn} + \epsilon_{jim}S'_{ilmn} + \epsilon_{nim}S'_{ijml}$, which after removing its trace (vector) leads to the $J = 3$ representation. From ${}_{15}S'$ we also obtain the symmetric rank-two tensor $T_{ij} = \delta_{mn}S'_{ijmn}$. Similarly, from ${}_{15}SA$ we obtain the symmetric tensor $T_{pij} = \epsilon_{pmn}\,SA_{ijmn} + \epsilon_{imn}\,SA_{jpmn} + \epsilon_{jmn}\,SA_{pimn}$, and we can also obtain a symmetric rank-two tensor.

The full result of the reduction under rotations of the linear representations corresponding to four-rank tensors is (in symbolic form) [8]:

- ${}_{15}S = 4 \oplus 2 \oplus 0$, while ${}_{6}S$ or ${}_{6}A = 2 \oplus 0$.
- ${}_{15}S'$, ${}_{15}SA$ or ${}_{15}AS$ yield $3 \oplus 2 \oplus 1$.

- Of course, ${}_3SA$ or ${}_3AS$, ${}_3A = 1$.

Now that we have the decomposition of the viscosity tensor in $O(3)$-irreducible components, it is easy to derive its general axisymmetric form. In order to do it, let us note that the axisymmetric form can only depend on the scalar tensors δ_{ij} and ϵ_{ijk} (the totally antisymmetric tensor), and the vector marking the symmetry axis, that is, the angular velocity Ω_i. In particular, the latter allows us to construct an infinite sequence of symmetric tensors for every rank J, namely:

$J = 1$: Ω_i

$J = 2$: $\Omega_i\Omega_j$

$J = 3$: $\Omega_i\Omega_j\Omega_k$

$\ldots \quad \ldots \quad \ldots$

These tensors must be combined with the scalar tensors according to the $O(3)$ Clebsh–Gordan decomposition above.

4.2.2 Axisymmetric viscosity tensor

The resulting axisymmetric viscosity tensor parts are:

$$\begin{aligned}
\eta^S_{ijmn} &= a_1\,(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \\
&\quad + a_2\,(\Omega_i\Omega_j\delta_{mn} + \Omega_m\Omega_n\delta_{ij} + \Omega_i\Omega_m\delta_{jn} + \Omega_j\Omega_m\delta_{in} + \Omega_i\Omega_n\delta_{jm} + \Omega_j\Omega_n\delta_{im}) \\
&\quad + a_3\,\Omega_i\Omega_j\Omega_m\Omega_n + a_4\,\delta_{ij}\delta_{mn} + a_5\,(\Omega_i\Omega_j\delta_{mn} + \Omega_m\Omega_n\delta_{ij})\,, \qquad (22)\\
\eta^A_{ijmn} &= b_1\,\Omega_q(\epsilon_{qim}\delta_{jn} + \epsilon_{qin}\delta_{jm} + \epsilon_{qjm}\delta_{in} + \epsilon_{qjn}\delta_{im}) \\
&\quad + b_2\,\Omega_q(\epsilon_{qim}\Omega_j\Omega_n + \epsilon_{qin}\Omega_j\Omega_m + \epsilon_{qjm}\Omega_i\Omega_n + \epsilon_{qjn}\Omega_i\Omega_m) \\
&\quad + b_3(\Omega_i\Omega_j\delta_{mn} - \Omega_m\Omega_n\delta_{ij}), \qquad (23)\\
\chi_{ijmn} &= (c_1\delta_{ij} + c_2\Omega_i\Omega_j)\epsilon_{lmn}\Omega_l + c_3(\Omega_i\Omega_m\delta_{jn} + \Omega_j\Omega_m\delta_{in} - \Omega_i\Omega_n\delta_{jm} - \Omega_j\Omega_n\delta_{im}) \\
&\quad + c_4(\epsilon_{imn}\Omega_j + \epsilon_{jmn}\Omega_i), \qquad (24)\\
\zeta_{ijmn} &= d_1(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) + d_2(\Omega_i\Omega_m\delta_{jn} - \Omega_j\Omega_m\delta_{in} - \Omega_i\Omega_n\delta_{jm} + \Omega_j\Omega_n\delta_{im}) \\
&\quad + d_3(\epsilon_{imn}\Omega_j - \epsilon_{jmn}\Omega_i). \qquad (25)
\end{aligned}$$

The tensor ξ with antisymmetry in the first pair and symmetry in the second pair has been suppressed, for its form is analogous to χ's, with different coefficients, say $c'_1, \ldots, c'_4$. The total number of coefficients is 19. Of course, we can let depend these scalar coefficients on Ω^2.

4.3 Effective forces

Once we have the general form of the axisymmetric viscosity tensor, we can obtain the general form of the axisymmetric effective forces, according to equation (6), by the relation

$$f_i = \partial_j\sigma_{ij} = \tau_{ijmn}\partial_{jm}u_n. \qquad (26)$$

Note that this equation implies that only the components of τ_{ijmn} symmetric in jm contribute to the force. Therefore, the number of independent axisymmetric components of the force is smaller than 19, that is, some of the 19 coefficients of the axisymmetric viscosity tensor are redundant, as we shall see.

Taking into account that $\nabla\cdot\boldsymbol{u}=0$ and suppressing gradient terms (which do not contribute to the transverse rotating fluid equation), we obtain:

$$\begin{aligned}\boldsymbol{f} &= (a_1-d_1)\nabla^2\boldsymbol{u} - b_1(\boldsymbol{\Omega}\times\nabla^2\boldsymbol{u}) - b_2(\boldsymbol{\Omega}\cdot\nabla)^2(\boldsymbol{\Omega}\times\boldsymbol{u}) - (b_2+c_2')(\boldsymbol{\Omega}\cdot\nabla)(\boldsymbol{\Omega}\times\nabla)(\boldsymbol{\Omega}\cdot\boldsymbol{u})\\ &+ (b_2+c_2)\boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\nabla)(\boldsymbol{\Omega}\cdot\boldsymbol{\omega}) + (c_4+c_4'+d_3)(\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{\omega} + (a_2-c_3+c_3'-d_2)\boldsymbol{\Omega}\nabla^2(\boldsymbol{\Omega}\cdot\boldsymbol{u})\\ &+ (a_2+c_3-c_3'-d_2)(\boldsymbol{\Omega}\cdot\nabla)^2\boldsymbol{u} + a_3\boldsymbol{\Omega}(\boldsymbol{\Omega}\cdot\nabla)^2(\boldsymbol{\Omega}\cdot\boldsymbol{u}). \qquad (27)\end{aligned}$$

Several remarks about this effective force are in order. Note that the fifth and sixth terms of the force involve the vorticity $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$ and are proportional to an odd power of $\boldsymbol{\Omega}$. The terms preceding them are also proportional to an odd power of $\boldsymbol{\Omega}$, except for the first one, which is isotropic. These terms proportional to an odd power of $\boldsymbol{\Omega}$ would not be allowed if isotropy were broken by a polar vector (instead of $\boldsymbol{\Omega}$). More importantly, 5 of the 19 coefficients are missing after making $\nabla\cdot\boldsymbol{u}=0$ and suppressing gradient terms, and some of the remaining ones are redundant: inspecting equation (27), we see that there are two redundant coefficients among c_4, c_4', d_3, two redundant coefficients among a_2, c_3, c_3', d_2, and one redundant coefficient among a_1, d_1.

After taking into account that $\nabla\cdot\boldsymbol{u}=0$ and suppressing gradient terms, there remain only the coefficients of the part of τ that is traceless in the first pair and in the second pair of indices. Gradient terms are longitudinal and the physical force must be transverse (solenoidal); but, after removing these terms, the force is still non-transverse and must be projected with the nonlocal operator $\mathcal{P}$ of equation (4). This operation brings back two supressed gradient terms, namely, $\nabla(\boldsymbol{\Omega}\cdot\boldsymbol{\omega})=0$ and $\nabla[(\boldsymbol{\Omega}\cdot\nabla)(\boldsymbol{\Omega}\cdot\boldsymbol{u})]$, in addition to producing nonlocal gradient terms.

5 Conclusions

We have studied the consequences of the axial symmetry of a uniformly rotating fluid for its homogeneous turbulent state, focusing on the four-rank tensor defining the linear relation between the stress tensor and the velocity derivatives (called the "viscosity tensor"). This study has been carried out with general methods of the theory of tensorial group representations.

The most general four-rank viscosity tensor comprises five parts:

- a tensor η^S symmetric by pairs of indices and pair symmetric, accounting for the usual proportionality relation between (anisotropic) stress and strain rate;
- a tensor η^A symmetric by pairs and pair antisymmetric embodying a new relation between stress and strain rate, typical of rotating fluids, since it does not lead to dissipation;
- a tensor χ symmetric in the first pair of indices and antisymmetric in the second, which accounts for a stress tensor coupling to vorticity;
- a tensor ξ antisymmetric in the first pair of indices and symmetric in the second, which accounts for the antisymmetric part of the stress tensor (angular momentum non-conserving) that couples to the strain rate.
- a tensor ζ antisymmetric in both pairs of indices, which accounts for the antisymmetric part of the stress tensor that couples to the vorticity. This tensor can be further decomposed into pair-symmetric and pair-antisymmetric parts, like η.

The theory of tensorial group representations allows us to find the $O(3)$ Clebsch–Gordan decomposition of these tensors and, hence, its axisymmetric form.

This variety of components of the "viscosity tensor" is reflected in the various effective forces that arise from them. However, given that the force is a vector, its number of axisymmetric independent components is smaller and, moreover, it is even smaller after suppressing gradient terms; namely, it boils down to just nine independent components.

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Equation for Doublets of Particles with Half-Integer Spin and Parasupersymmetry

Alexander GALKIN

Taras Shevchenko Kyiv National University, 60 Volodymyrs'ka Str., 01033 Kyiv-33, Ukraine
E-mail: *galkin@unicyb.kiev.ua*

In the present paper we consider the tensor-bispinor equations, which describe two particles with arbitrary half-integer spin and opposite parity. We obtain the energy levels of particle with half-integer spin s in constant magnetic field and find the parasupersymmetry which is connected with extra degeneracy of energy levels in constant magnetic field.

1 Introduction

It is well-known that equations for particles with spin more than $\frac{1}{2}$ have a lot of defects such as: acausality [1], incompatibility of the theoretical value of gyromagnetic ratio $g = \frac{1}{s}$ with experimental value $g = 2$ for any spin, complex value of energy of charged particle in constant magnetic field and others. In the papers [2, 3] tensor-bispinor equation was considered, which describes the motion of particle with half-integer spin. It was shown [2, 3], that tensor-bispinor equation remains causal after introducing the interaction with electromagnetic field. Moreover, this equation, in the case of minimal interaction, is equivalent to the equation in Dirac-like form, which was considered in the paper [4].

The aim of this paper is generalization of the result that was obtained in the papers [2,3], for the case of anomalous interaction quadratic in electromagnetic field. We note that there is a kind of anomalous interaction quadratic in electromagnetic field at which the extra degeneration of energy levels of particle appear. This degeneratior is caused by parasupersymmetry of considered equations.

In Section 2 we consider tensor-bispinor equations for free particle with arbitrary half-integer spin. We also show the equivalence of such equations and the Dirac-like equations [4]. In Section 3 the minimal and anomalous interaction with electromagnetic field is discussed. Section 4 is dedicated to the problem of motion a particle in constant magnetic field. The condition is found at which the problem of complex energy levels does not arise. Finally, in Section 5, we retrieve the conditions the constants to which should satisfy of interactions in order that the equation for particle in constant magnetic field be parasupersymmetric (It is worth to note that parasupersymmetry of equations for spin-1 and spin-0 was considered by Beckers, Debergh, Nikitin [5] and Sergeyev [6]). We also obtain corresponding parasupercharges.

2 Tensor bispinor equation for free particles with arbitrary spin

Let us consider in this section the description of motion of free particles with arbitrary half-integer spin in terms of irreducible antisymmetric tensor-bispinor $\Psi_{\gamma}^{[\mu_1,\nu_1][\mu_2,\nu_2]\cdots[\mu_n,\nu_n]}$ [2, 3] of rank n ($n = s - \frac{1}{2}$).

We suppose that the tensor-bispinor $\Psi_{\gamma}^{[\mu_1,\nu_1][\mu_2,\nu_2]\cdots[\mu_n,\nu_n]}$ satisfies the following conditions

$$\Psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_k\nu_k]\cdots[\mu_n\nu_n]} = -\Psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\nu_k\mu_k]\cdots[\mu_n\nu_n]},$$
$$\Psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_k\nu_k]\cdots[\mu_l\nu_l]\cdots[\mu_n\nu_n]} = \Psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_l\nu_l]\cdots[\mu_k\nu_k]\cdots[\mu_n\nu_n]}, \quad (1)$$

and

$$\gamma_\mu\gamma_\nu\Psi^{[\mu\nu][\mu_1\nu_1]\cdots[\mu_{n-1}\nu_{n-1}]}=0, \tag{2}$$

where γ_μ are the Dirac matrices. Moreover, $\Psi_\gamma^{[\mu_1,\nu_1][\mu_2,\nu_2]\cdots[\mu_n,\nu_n]}$ satisfy the Dirac equation

$$(\gamma_\lambda p^\lambda - m)\Psi^{[\mu_1,\nu_1]\cdots[\mu_n,\nu_n]}=0, \tag{3}$$

where $p_\mu = i\frac{\partial}{\partial x^\mu}$.

Commuting $\gamma_\mu\gamma_\nu$ and using (2), (3) with $(\gamma_\lambda p^\lambda - m)$ we come to the condition

$$p_\mu\gamma_\nu\Psi^{[\mu\nu][\mu_1\nu_1]\cdots[\mu_{n-1}\nu_{n-1}]}=0. \tag{4}$$

The wave function $\Psi_\gamma^{[\mu_1,\nu_1][\mu_2,\nu_2]\cdots[\mu_n,\nu_n]}$ belongs to the carrier space of the representation

$$\begin{aligned}&\left[D\left(s-\tfrac12,0\right)\oplus D\left(0,s-\tfrac12\right)\right]\otimes\left[D\left(\tfrac12,0\right)\oplus D\left(0,\tfrac12\right)\right]\\&\quad= D(s,0)\oplus D(0,s)\oplus D\left(s-\tfrac12,\tfrac12\right)\oplus D\left(\tfrac12,s-\tfrac12\right)\oplus D(s-1,0)\oplus D(0,s-1).\end{aligned}$$

of the Lorentz group. Thus, the wave function $\Psi_\gamma^{[\mu_1,\nu_1][\mu_2,\nu_2]\cdots[\mu_n,\nu_n]}$ has $16s$ components. Constraint (2) removed the states which correspond to the representation $D(s-1,0)\oplus D(0,s-1)$. The secondary constraint (4) nullifies the remaining part of non-physical components and, as a result, we have exactly $4(2s+1)$ independent components, i.e. twice more than it is necessary for describing particles with arbitrary spin. So equation (3) with additional conditions (1)–(2), (4) describes two particles with arbitrary spin s, but these particles have opposite parity.

To introduce the interaction with electromagnetic field in equations (1)–(4) it is preferable to write them as a single equation. For this purpose we present the wave function $\Psi_\gamma^{[\mu_1,\nu_1][\mu_2,\nu_2]\cdots[\mu_n,\nu_n]}$ in the form

$$\begin{aligned}\Psi^{[\mu_1\nu_1]\cdots[\mu_n\nu_n]}&=\Phi^{[\mu_1\nu_1]\cdots[\mu_n\nu_n]}+\frac12\Big(\gamma^{[\mu_1}A^{\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}\\&\quad+\gamma^{[\mu_2}A^{\nu_2][\mu_1\nu_1][\mu_3\nu_3]\cdots[\mu_n\nu_n]}+\cdots+\gamma^{[\mu_n}A^{\nu_n][\mu_1\nu_1]\cdots[\mu_{n-1}\nu_{n_1}]}\Big),\end{aligned} \tag{5}$$

here $\gamma^{[\mu}A^{\nu]}=\gamma^\mu A^\nu-\gamma^\nu A^\mu$. Tensors $\Phi^{[\mu_1\nu_1]\cdots[\mu_n\nu_n]}$ and $A^{\lambda[\mu_1\nu_1]\cdots[\mu_{n-1}\nu_{n-1}]}$ satisfy the following conditions

$$\begin{aligned}&\gamma_\mu\Phi^{[\mu\nu][\mu_1\nu_1]\cdots[\mu_n\nu_n]}=0,\\&\gamma_\lambda A^{\lambda[\mu_1\nu_1]\cdots[\mu_{n-1}\nu_{n-1}]}=\gamma_{\mu_1}A^{\lambda[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}=0,\\&A^{\lambda[\mu_1\nu_1]\cdots[\mu_n\nu_n]}+A^{\mu_1[\nu_1\lambda]\cdots[\mu_n\nu_n]}+A^{\nu_1[\lambda\mu_1]\cdots[\mu_n\nu_n]}=0,\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&A^{\lambda[\mu_1\nu_1]\cdots[\mu_n\nu_n]}+A^{\mu_n[\mu_1\nu_1]}+A^{\nu_n[\mu_1\nu_1]\cdots[\lambda\mu_n]}=0.\end{aligned}$$

Moreover, tensors $\Phi^{[\mu_1\nu_1]\cdots[\mu_n\nu_n]}$ and $A^{\lambda[\mu_1\nu_1]\cdots[\mu_{n-1}\nu_{n-1}]}$ satisfy conditions (1). Then we can write the single equation equivalent to (1)–(4) in the form

$$\begin{aligned}&(\gamma_\nu p^\nu-m)\Psi^{[\mu_1\nu_1][\mu_2\nu_2]\cdots[\mu_n\nu_n]}-\frac{n}{2(8n-11)}\sum_{i=1}^{n}P_1(\mu_i\nu_i)\\&\quad-\frac{n}{8n-11}\sum_{i,j=1(j>i)}^{n}P_2(\mu_i\nu_i;\mu_j\nu_j)=0,\end{aligned} \tag{6}$$

where

$$P_1(\mu_i\nu_i) = [\gamma_{\mu_i}, \gamma_{\nu_i}]p_\lambda A^{\lambda[\mu_1\nu_1]\cdots[\mu_{i-1}\nu_{i-1}][\mu_{i+1}\nu_{i+1}]\cdots[\mu_n\nu_n]},$$
$$P_2(\mu_i\nu_i;\mu_j\nu_j) = P_1(\mu_i\nu_j) - P_1(\mu_i\mu_j) + P_1(\nu_i\mu_j) - P_1(\nu_i\nu_j). \tag{7}$$

Contracting (6)–(7) with $(p_{\mu_1}\gamma_{\nu_1} - p_{\nu_1}\gamma_{\mu_1})$ we come to constraint (4).

It was shown [2,3] that equations (5)–(7) can be written in the Dirac-like form [6]

$$(\Gamma_\mu p^\mu - m)\Psi^\epsilon = 0, \qquad \epsilon = -1, 1,$$
$$(\Gamma_\mu p^\mu + m)(1 + \epsilon i\Gamma_4)[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)]\Psi^\epsilon = 24m\Psi^\epsilon. \tag{8}$$

Here Ψ^ϵ are wave functions, which have $8s$ components; matrices Γ_μ have the following form

$$\Gamma_\mu = \gamma_\mu \otimes I,$$

where γ_μ are Dirac matrices and I is $(2\tau+1)\times(2\tau+1)$ unit matrix ($\tau = s - \frac{1}{2}$). In the general case Ψ^ϵ can be expressed in terms of tensor-bispinor $\Psi_\gamma^{[\mu_1,\nu_1][\mu_2,\nu_2]\cdots[\mu_n,\nu_n]}$. For example, in the case of spin $\frac{3}{2}$ we have

$$\Phi_\alpha^{ab} = \frac{1}{2}\varepsilon_{abc}\big(\Phi_{\alpha c}^{(-1)} + \Phi_{\alpha c}^{(1)}\big), \qquad \Phi_\alpha^{oc} = \frac{i}{2}\big(\Phi_{\alpha c}^{(1)} - \Phi_{\alpha c}^{(-1)}\big), \qquad a,b,c = 1,2,3, \quad \alpha = 0,1,2,3.$$

Matrices $S_{\mu\nu}$ can be represented in the form

$$S_{\mu\nu} = j_{\mu\nu} + \tau_{\mu\nu}, \qquad j_{\mu\nu} = \frac{i}{4}[\Gamma_\mu, \Gamma_\nu]$$

and $\tau_{\mu\nu}$ satisfy the commutation relations

$$[\tau_{\mu\nu}, \tau_{\rho\sigma}] = i(g_{\mu\sigma}\tau_{\nu\rho} + g_{\nu\rho}\tau_{\mu\sigma} - g_{\mu\rho}\tau_{\nu\sigma} - g_{\nu\sigma}\tau_{\mu\rho}), \qquad [\tau_{\mu\nu}, \Gamma_\rho] = [\tau_{\mu\nu}, j_{\rho\sigma}] = 0,$$

and belong to the representation $D(\tau, 0)$ of algebra $AO(1,3)$, i.e. satisfy the relations

$$\tau_{ab} = \varepsilon_{abc}\tau_c, \qquad \tau_{0a} = i\tau_a, \qquad a,b,c = 1,2,3,$$
$$\tau_a\tau_a = \tau(\tau+1), \qquad [\tau_a, \tau_b] = i\varepsilon_{abc}\tau_c. \tag{9}$$

Taking into account (9) we can write τ_a as

$$\tau_a = 1 \otimes \hat{\tau}_a, \qquad a = 1,2,3,$$

where 1 is 4×4 unit matrix and $\hat{\tau}_a$ realize irreducible representation $D(\tau)$ of algebra $AO(3)$.

3 Interaction with electromagnetic field

The minimal interaction with electromagnetic field and anomalous interaction linear in electromagnetic field were considered in [2–4]. Using tensor-bispinor formulation we can introduce as minimal as anomalous interaction with electromagnetic field and then we come over to Dirac-like form [2]

$$\Big(\Gamma_\mu\pi^\mu - m + \frac{e}{4m}(1 - \epsilon i\Gamma_4)\Big(\frac{i}{4}(g-2)[\Gamma_\mu, \Gamma_\nu] + g\tau_{\mu\nu}\Big)F^{\mu\nu}\Big)\Psi^\epsilon = 0,$$
$$(\Gamma_\mu\pi^\mu + m)(1 + i\Gamma_4)[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)]\Psi^\epsilon = 24m\Psi^\epsilon. \tag{10}$$

Equations (10) can be generalized by introducing the anomalous interaction quadratic in electromagnetic field

$$\Big(\Gamma_\mu\pi^\mu - m + \frac{e}{4m}(1-\epsilon i\Gamma_4)\big(gS_{\mu\nu}F^{\mu\nu} + g_1(S_{\mu\nu}F^{\mu\nu})^2 + g_2F^{\mu\nu}F_{\mu\nu} - i\Gamma_\mu\Gamma_\nu F^{\mu\nu}\big)\Big)\Psi^\epsilon = 0. \tag{11}$$

Equations (11) admit the Lagrangian formulation and include coupling constants g, g_1 and g_2. We will see that the constraints g_1 and g_2 can be chosen such way that the known difficulties [2] with complex energy levels will be overcomed.

4 Particle with arbitrary half-integer spin s in constant and homogeneous magnetic field

In this section we consider the problem of interaction of a charged particle with arbitrary half-integer spin with a constant and homogeneous magnetic field. We demonstrate that for $g \neq \frac{1}{s}$ this problem leads to the known difficulties with complex energies and demonstrate that these difficulties are successfully overcomed using the generalized model (11) with bilinear in $F_{\mu\nu}$ anomalous interaction.

Equation (11) can be expressed in the form of second order equation

$$\Big(\pi_\mu\pi^\mu - m^2 + \frac{eg}{2}S_{\mu\nu}F^{\mu\nu} + \frac{eg_1}{2}(S_{\mu\nu}F^{\mu\nu})^2 + \frac{eg_2}{2}F^{\mu\nu}F_{\mu\nu}\Big)\Psi^\epsilon_+ = 0, \tag{12}$$

$$(S_{\mu\nu}S^{\mu\nu} - 15)\Psi^\epsilon_+ = 0, \qquad \Psi^\epsilon_- = \frac{1}{m}\Gamma_\mu\pi^\mu\Psi^\epsilon_+,$$

where Ψ^ϵ can be obtained, using the expression

$$\Psi^\epsilon = \Psi^\epsilon_+ + \Psi^\epsilon_-.$$

For the case of the constant and homogeneous magnetic field the vector-potential A_μ and the field tensor $F_{\mu\nu}$ are

$$A_0 = A_2 = A_3 = 0, \qquad A_1 = Hx_2, \qquad F_{0a} = F_{23} = F_{31} = 0, \qquad a = 1,2,3,$$
$$F_{12} = H_3 = H, \qquad H \geq 0,$$

H is the magnetic field strength.

The solution of equation (12) can be represented as [4]

$$\Psi^\epsilon_+ = \begin{pmatrix} \Phi^\epsilon_s \\ \hat{0} \\ \frac{1}{m}(\varepsilon + \frac{1}{s}S_a\pi_a)\Phi^\epsilon_s \\ -\frac{1}{sm}K^s_a\pi_a\Phi^\epsilon_s \end{pmatrix}.$$

Here $(K^s_3)_{mm'} = \delta_{mm'}\sqrt{s^2-m^2}$; $(K^s_1)_{mm'} \pm i(K^s_2)_{mm'} = \pm\delta_{m\pm m'}\sqrt{s(s-1)\mp m(m\mp1)\pm 2ms}$, $m, m' = -s, -s+\frac{1}{2}, \ldots, s$; $\hat{0}$ is zero matrix $1\times(2s-1)$, s is value of spin of the particle, Φ^ϵ_s is a four-component spinor which satisfy the equation

$$\big[p^2 + e^2H^2x_2^2 - eH(gS_3 + 2g_1S_3^2H + 2g_2H + 2x_2p_1)\big]\Phi^\epsilon_s = (\varepsilon^2 - m^2)\Phi^\epsilon_s. \tag{13}$$

So the problem of describing the motion of particle with spin s reduces to solving equation (13).

Using the eigenvectors Ω^s_ν ($\nu = -s, -s+\frac{1}{2}, \ldots, s$) of matrix S_3 we can represent Φ_s in the form

$$\Phi^\epsilon_s = \exp(i(p_1x_1 + p_3x_3)) \sum_{\nu=-s}^{s} f^s_\nu(x_2)\Omega^s_\nu,$$

where $f^s_\nu(x_2)$ are unknown functions, Ω^s_ν are 4 components spinor eigenvectors of S_3, $S_3\Omega^s_\nu = \nu\Omega^s_\nu$. The functions f^s_ν satisfy the equation

$$\left(\frac{d^2}{dy^2} + y^2\right) f^s_\nu(y) = \eta f^s{}_\nu(y),$$

where $\eta = \frac{\varepsilon^2 - m^2 - p_3^2}{eH} + \nu(g + 2g_1\nu H + 2g_2H)$, $x_2 = \frac{1}{eH}(p_1 + \sqrt{eH}y)$.

Requiring that $f^s{}_\nu(y) \to 0$ when $y \to \pm\infty$ we have

$$\eta = 2n + 1, \qquad n = 0, 1, 2, 3, \ldots.$$

Then the energy levels are

$$\varepsilon^2 = m^2 + p_3^2 + eH(2n + 1 - \nu(g + 2g_1\nu H + 2g_2H))$$

and eigenfunctions $f^s_\nu(y)$ take the form

$$f^s_\nu(x_2) = \exp\left(-\frac{eHx_2 - p_2}{2eH}\right) h_n\left(\frac{eHx_2 - p_2}{\sqrt{eH}}\right),$$

where $h_n(y) = \frac{H_n(y)}{||H_n(y)||}$, $H_n(y)$ are Hermitian polynomials. If $g_1 = 0$ such values of H exists for which $\varepsilon^2 \le 0$. In order that $\varepsilon^2 \ge 0$ for all H, g and g_1 must satisfy the following condition

$$|sg_1 + g_2| \ge \frac{(sg-1)^2 e}{8ms^2}.$$

5 Parasupersymmetry

In this section the parasupersymmetry of equations for particle with spin $\frac{3}{2}$ in magnetic field is discussed. Let us start with the supersymmetry of Dirac equation for particle with spin $\frac{1}{2}$ in homogeneous magnetic field. The related equation (13) takes the following form

$$\left(\varepsilon^2 - (p_1 - eHx_2)^2 - p_2^2 - p_3^2 - m^2 + 2e\sigma_3 H\right) \Phi_{\frac{1}{2}} = 0. \tag{14}$$

Here σ_3 is the Pauli matrix and we put in (13) $g_1 = g_2 = 0$ and $g = 2$. Substituting

$$\Phi_{\frac{1}{2}} = \exp(i(p_1x_1 + p_3x_3))\phi(x_2)$$

into (14) we come to the equation

$$\varepsilon'^2\phi(x_2) = H^2\phi(x_2),$$

where

$$H^2 = -\frac{d^2}{dx_2^2} + (p_1 - eHx_2)^2 - 2e\sigma_3 H$$

and $\varepsilon'^2 = \varepsilon^2 - p_3^2 - m^2$.

Following [7] we can build the supercharges, which are related to extra degeneracy of energy levels

$$Q = A\psi^\dagger, \qquad Q^\dagger = A^\dagger\psi. \tag{15}$$

A and $A^\dagger$ are bosonic creation and annihilation operators, which have the following form

$$A = \frac{1}{\sqrt{eH}}\left(\frac{d}{dx_2} - p_1 + eHx_2\right), \qquad A^\dagger = \frac{1}{\sqrt{eH}}\left(-\frac{d}{dx_2} - p_1 + eHx_2\right), \tag{16}$$

and satisfy the conditions

$$[A, A^\dagger] = 1, \qquad [N, A] = -A, \qquad [N, A^\dagger] = A^\dagger, \qquad N = A^\dagger A.$$

Operators ψ and $\psi^\dagger$ are fermionic creation and annihilation operators, which have the form

$$\psi = \sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2), \qquad \psi^\dagger = \sigma_- = \frac{1}{2}(\sigma_1 - i\sigma_2).$$

They obey the anticommutation relations

$$\{\psi, \psi^\dagger\} = 1, \qquad \{\psi^\dagger, \psi^\dagger\} = \{\psi, \psi\} = 0,$$

as well as obeying the commutation relation

$$[\psi, \psi^\dagger] = \sigma_3.$$

The operator which plays the role of SUSY Hamiltonian is

$$H^2 = QQ^\dagger + Q^\dagger Q.$$

Now let us consider the spin-$\frac{3}{2}$. In this case the equation (13) has the form

$$[-p_1^2 - p_2^2 - p_3^2 + e^2H^2x_2^2 - geHS_3 - 2eHx_2p_1]\Phi^\epsilon_{\frac{3}{2}} = (\varepsilon^2 - m^2)\Phi^\epsilon_{\frac{3}{2}},$$

where S_3 belongs to the irreducible representation $D(\frac{3}{2})$ of algebra $AO(3)$ and we put $g_1 = g_2 = 0$, $g = 2$.

Substituting

$$\Phi^\epsilon_{\frac{3}{2}} = \exp(i(p_1x_1 + p_3x_3))\phi_{\frac{3}{2}}(x_2),$$

we come to the equation

$$\varepsilon'^2\phi(x_2) = H^2_{\frac{3}{2}}\phi(x_2),$$

where $\varepsilon'^2 = \varepsilon^2 - p_3^2 - m^2$ and

$$H^2_{\frac{3}{2}} = -\frac{d^2}{dx_2^2} + (p_1 - eHx_2)^2 - 2eS_3H.$$

Here S_3 is the spin matrix belonging to the irreducible representation $D(\frac{3}{2})$ of algebra $AO(3)$.

Let us show that we obtain $N = 2$, $p = 3$ parasuperalgebra with parasupercharges (15), where bosonic creation and annihilation operators A, $A^\dagger$ (16) and parafermionic creation and annihilation operators take the following form

$$\psi = S_+ = \frac{1}{2}(S_1 + iS_2), \qquad \psi^\dagger = S_- = \frac{1}{2}(S_1 - iS_2). \tag{17}$$

Here S_1, S_2 and S_3 are spin matrices for spin $\frac{3}{2}$. It can be verified that operators ψ, $\psi^\dagger$ satisfy the double commutation relation

$$[\psi,[\psi^\dagger,\psi]]=2\psi, \qquad [\psi^\dagger,[\psi,\psi^\dagger]]=2\psi^\dagger$$

as well as satisfying the commutation relations

$$[\psi,\psi^\dagger]=S_3.$$

Operators Q, $Q^\dagger$ and $H^2_{\frac{3}{2}}$ satisfy the double commutation relations typical for $p=3$ parasupersymmetry

$$[H^2_{\frac{3}{2}},Q]=[H^2_{\frac{3}{2}},Q^\dagger]=0, \qquad [Q,[Q^\dagger,Q]]=2QH^2_{\frac{3}{2}}.$$

We notice that the mentioned parasupersymmetry appears only for $g_2\neq 0$.

It is possible to show that in the case of arbitrary half-integer spin $s=n+\frac{1}{2}$ we have the $p=n$ parasupersymmetry, where the parasuperchages have the form (15) with bosonic creation and annihilation operators A and $A^\dagger$ (16) and parafermionic creation and annihilation operators (17). The related matrices S_1, S_2 and S_3 belong to the irreducible representation $D(s)$ of algebra $AO(3)$.

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Quantum Groups and Cabibbo Mixing

Alexandre M. GAVRILIK

Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine
E-mail: *omgavr@bitp.kiev.ua*

Treating the issue of hadron masses and mass relations by the use of quantum groups $U_q(su_n)$ taken as hadron flavor symmetries suggests, at least in the case of baryons, a direct connection of the deformation parameter q with the Cabibbo angle. We discuss possible manifestations of the Cabibbo mixing implied by such connection, including unusual ones.

1 Introduction

The standard model of fundamental particles and forces incorporates an important concept of quark mixing [1] usually described by the CKM matrix, which in the Wolfenstein's form looks as

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho-{\rm i}\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-{\rm i}\eta) & -A\lambda^2 & 1 \end{pmatrix}.$$

When keeping the first two quark families, only the parameter $\lambda \approx 0.22$ (the sine of Cabibbo angle $\theta_{\rm C}$) persists. Cabibbo mixing [1] is known to play basic role in describing weak decays of mesons and baryons by means of flavor changing (e.g., strangeness changing) quark currents. Our goal is to discuss some other, unusual implications of the Cabibbo mixing which are deduced on the base of adopting instead of the flavor groups $SU(n)$ their quantum counterparts $U_q(su(n))$. Note that the idea of using [2] in place of $SU(n)$ the quantum groups (or q-deformed algebras) $U_q(su_n)$ and their representations [3] in order to treat hadronic flavor symmetries is natural, easily justifiable and very fruitful, leading to many interesting implications [2, 4]. The most important one is the possibility to link the q-parameter with the angle $\theta_{\rm C}$, and this enables to find novel connections for the concept of Cabibbo mixing. Say, the earliest modified (improved) version of the Gell-Mann–Okubo (GMO) mass relation for the $SU(3)$ octet baryons $\frac{1}{2}^+$ derived in a particular version [5] of 'diquark-quark' model (the LTK model), involves certain parameters characterizing diquark and separate quark. The alternative improved version of GMO for octet baryons is the extremely precise mass sum rule (MSR) obtained with $U_q(su_n)$. It exploits fixed value of the q-parameter linked in a simple way to $\theta_{\rm C}$, and this fact allows to find direct connection, through the fixed q, of the parameters of LTK diquark-quark model to the angle $\theta_{\rm C}$. Besides this, we discuss the connections of the $\theta_{\rm C}$ with such concepts as anyonic statistics parameter from the anyonic picture, as intercepts of the two- and three-pion (-kaon) correlation functions drawn in recent experiments on relativistic heavy ion collisions.

2 Baryon mass sum rules from q-deformation

2.1 Mass relations for octet and decuplet baryons

Consider first $\frac{1}{2}^+$ baryons from $SU(3)$ octet. Using $U_q(su_n)$, $n = 2, 3, 4$, and $U_q(u_{4,1})$ (the latter playing the role of dynamical q-algebra), the q-deformed mass relation for octet baryons

$$[2]M_N + \frac{[2]M_\Xi}{[2]-1} = [3]M_\Lambda + \Big(\frac{[2]^2}{[2]-1} - [3]\Big)M_\Sigma + \frac{A_q}{B_q}\,(M_\Xi + [2]M_N - [2]M_\Sigma - M_\Lambda) \qquad (1)$$

was obtained [6]. The pair of polynomials A_q, B_q (of the q-deuce $[2]_q \equiv q + q^{-1}$), with nonoverlapping sets of zeros, results from calculation in a particular dynamical representation (irrep) of $U_q(u_{4,1})$; different irreps lead to differing pairs A_q, B_q in (1). Each A_q possesses the mandatory factor $([2]_q - 2)$, i.e. 'classical' zero $q = 1$, and nontrivial zeros. As particular cases, the q-analog (1) yields the familiar [7] Gell-Mann–Okubo (GMO) mass relation $M_N + M_\Xi = \frac{3}{2}M_\Lambda + \frac{1}{2}M_\Sigma$ in the 'classical' limit $q = 1$, and two new mass sum rules of improved accuracy [4, 8]

$$M_N + \frac{1+\sqrt{3}}{2}M_\Xi = \frac{2M_\Lambda}{\sqrt{3}} + \frac{9-\sqrt{3}}{6}M_\Sigma \qquad (0.22\ \%), \tag{2}$$

$$M_N + \frac{M_\Xi}{[2]_{q_7}-1} = \frac{M_\Lambda}{[2]_{q_7}-1} + M_\Sigma \qquad (0.07\ \%) \tag{3}$$

which correspond to A_q and $\tilde{A}_q$ with respective zeros $q_6 = e^{\mathrm{i}\pi/6}$ and $q_7 = e^{\mathrm{i}\pi/7}$. Among the admissible irreps of $U_q(u_{4,1})$ or its 'compact' counterpart $U_q(u_5)$ there exist infinite series of irreps producing infinite set of MSRs numbered by integer m $(6 \le m < \infty)$, each given by (1) with q_m put for q, where $q_m = e^{\mathrm{i}\pi/m}$ guarantees vanishing of $\frac{A_q}{B_q}$. Each MSR in this set *agrees better with data* than the standard GMO one. Thus, a 'discrete choice' (instead of usual fitting) becomes possible; the q-polynomial A_q due to zero q_m serves as *defining* polynomial for the corresponding MSR. It is the value $q_7 = e^{\mathrm{i}\pi/7}$ and the MSR (3) that provides the best choice.

For decuplet baryons $\frac{3}{2}^+$, 1st order $SU(3)$ breaking yields well-known [7] equal spacing rule (ESR) for isoplets from **10**-plet. However, $M_{\Sigma^*} - M_\Delta$, $M_{\Xi^*} - M_{\Sigma^*}$ and $M_\Omega - M_{\Xi^*}$ significantly deviate from ESR: 152.6 MeV $\leftrightarrow$ 148.8 MeV $\leftrightarrow$ 139.0 MeV. The other known relation [9, 5]

$$(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*})/2 = M_{\Xi^*} - M_{\Sigma^*} \tag{4}$$

accounts 1st, 2nd orders in $SU(3)$-breaking and holds only slightly better than the ESR.

On the contrary, the use of q-algebras $U_q(su_n)$ gives nice improvement. Evaluation of decuplet masses with particular irreps of the dynamical $U_q(u_{4,1})$ yields the q-deformed mass relation [10]

$$(M_{\Sigma^*} - M_\Delta + M_\Omega - M_{\Xi^*})/(2\cos\theta_{\mathbf{10}}) = M_{\Xi^*} - M_{\Sigma^*}, \qquad q = \exp(\mathrm{i}\theta_{\mathbf{10}}). \tag{5}$$

This mass relation is *universal* – it results from any admissible irrep (containing $U_q(su_3)$-decuplet embedded in *20*-plet of $U_q(su_4)$) of the dynamical $U_q(u_{4,1})$. With empirical masses, equation (5) holds remarkably for $\theta_{\mathbf{10}} \simeq \frac{\pi}{14}$. It is argued that $\theta_{\mathbf{10}} = \theta_{\mathrm{C}}$, see [4] and also the next subsection.

2.2 Linking the q-parameter to the Cabibbo angle

We compare the generalization [11] $f_\pi^2 m_\pi^2 + 3 f_\eta^2 m_\eta^2 = 4 f_K^2 m_K^2$ of GMO-formula $m_\pi^2 + 3m_\eta^2 = 4m_K^2$ for pseudoscalar (PS) mesons, with $f_\pi^2 + 3f_\eta^2 = 4f_K^2$ imposed, and our q-deformed analog

$$m_\pi^2 + [3]_q^{-1}(2\ [2]_q - [3]_q)m_{\eta_8}^2 = 2\ [2]_q (2\ [2]_q - [3]_q)^{-1} m_K^2, \qquad [3]_q = [2]_q^2 - 1, \tag{6}$$

(found in [2]) of PS-mesonic GMO, with duly fixed $q = q_{\mathrm{PS}}$ and *physical* η-meson put in place of η_8 (thus there is no need for explicit mixing). We find, using $\xi_{\pi,K} \equiv (4f_K^2/f_\pi^2)^{-1}$, that

$$f_K^2/f_\pi^2 \ \leftrightarrow\ \frac{1}{2}[2]/(2[2]-[3]), \qquad [2]_\pm = 1 - \xi_{\pi,K} \pm \left(\left(1-\xi_{\pi,K}\right)^2 + 1\right)^{1/2}.$$

Since the ratio f_K/f_π is expressible through the Cabibbo angle (see e.g., [12]), we infer: the deformation parameter q_{PS} is directly connected with the Cabibbo angle.

In another way of reasoning, we use the result of [13] where the Lagrangian for quantum-group valued gauge field analog of the Weinberg–Salam (WS) model was constructed and the relation

$\tan\theta = h(q) \equiv (1-q^2)/(1+q^2)$ found. It provides proper mixing of the $U(1)$-component B_μ and the non-Abelian component A^3_μ. Physical photon $\tilde{A}_\mu$ and Z-boson of WS model appear through the Weinberg angle $\theta_{\rm W} = \theta = \arctan h(q)$. At $\theta = 0$ the potentials B_μ and A^3_μ get unmixed, but the q-deformation with $\theta \neq 0$ provides mixing inherent in the WS model. So, the mixing of (electro)weak gauge fields is adequately modelled by the q-deformation. Due to the latter, the weak angle and the q-parameter are explicitly linked. The relation [14] $\theta_{\rm W} = 2(\theta_{12}+\theta_{23}+\theta_{13})$, on the other hand, connects $\theta_{\rm W}$ with the Cabibbo angle $\theta_{12} \equiv \theta_{\rm C}$ (we neglect the 3rd family's θ_{13}, θ_{23}). Thus, the apparently different mixing angles, in the *bosonic* (interaction) and in the *fermionic* (matter) sectors of the electroweak model, are related.

We conclude that *the Cabibbo angle is linked with q-parameter* of a quantum-group (or q-algebra) based symmetry structure applied in the fermion sector. The explicit connection is remarkably simple: $\theta_{\mathbf{10}} = \theta_{\rm C}$, $\theta_{\mathbf{8}} = 2\,\theta_{\rm C}$. With $\theta_{\mathbf{8}} = \frac{\pi}{7}$ this suggests the exact value $\frac{\pi}{14}$ for $\theta_{\rm C}$.

2.3 Nonpolynomiality in $SU(3)$-breaking and Michel's statement

The universality of q-analog (5) concerns all admissible irreps of the '*compact*' dynamical $U_q(su_5)$, too. Say, calculation in the dynamical irrep $\{4000\}$ of $U_q(su_5)$ yields $M_\Delta = M_{\mathbf{10}} + \beta$, $M_{\Sigma^*} = M_{\mathbf{10}} + [2]\beta + \alpha$, $M_{\Xi^*} = M_{\mathbf{10}} + [3]\beta + [2]\alpha$, $M_\Omega = M_{\mathbf{10}} + [4]\beta + [3]\alpha$, from which (5) stems. With hypercharge Y, all the decuplet masses $M_{D_i} \equiv M(Y(D_i))$ are comprised by single formula

$$M_{D_i} = M_{\mathbf{10}} + \alpha[1 - Y(D_i)]_q + \beta[2 - Y(D_i)]_q \tag{7}$$

of explicit Y-dependence. The limit $q \to 1$ reduces it to $M_{D_i} = \tilde{M}_{\mathbf{10}} + aY(D_i)$ with $a = -\alpha - \beta$, $\tilde{M}_{\mathbf{10}} = M_{\mathbf{10}} + \alpha + 2\beta$, i.e., to linear dependence on hypercharge (or strangeness $S = Y - 1$).

Formula (7) involves *highly nonlinear dependence* of mass on hypercharge: for decuplet, Y alone causes $SU(3)$-breaking. Since for any N its q-number is $[N]_q = (q^N - q^{-N})/(q - q^{-1}) = q^{N-1} + q^{N-3} + \cdots + q^{-N+3} + q^{-N+1}$ (N terms), this shows exponential Y-dependence of masses. Such high nonlinearity makes (5) and (7) crucially different from the result (4) of traditional treatment accounting linear and quadratic effects in Y.

For octet baryon masses, *nonpolynomiality* in $SU(3)$-breaking effectively accounted [8] by the model is embodied in the expressions for isoplet masses, with explicit dependence on Y as well as isospin I, through $I(I+1)$. Matrix elements contributing to octet baryon masses contain e.g., the terms $([Y/2]_q[Y/2+1]_q - [I]_q[I+1]_q)$ or $([Y/2-1]_q[Y/2-2]_q - [I]_q[I+1]_q)$, with multipliers depending on irrep labels m_{15}, m_{55}, that show explicit dependence on hypercharge and on the q-analog $[I]_q[I+1]_q$ of $SU(2)$ Casimir. The q-bracket $[n]_q$ means $[n]_q = \frac{\sin(nh)}{\sin(h)}$, $q = \exp(ih)$, so we see that octet baryon masses depend on hypercharge Y and isospin I (hence, on $SU(3)$-breaking effects) also in highly nonlinear – *nonpolynomial* – fashion.

We note finally that the conclusion made in the preceding subsection about the ability to find, due to the link $q \leftrightarrow \theta_C$, the exact value of Cabibbo angle is in accord with Michel's statement [15] that only account of higher-order breaking effects enables gaining of such result.

3 Cabibbo angle and the anyonic statistics parameter

From N sorts of lattice fermions $c_i(\boldsymbol{x})$, $c_i^\dagger(\boldsymbol{x})$, $i = 1, \ldots, N$, with usual lattice anticommutation relations (ACRs), using the lattice angle functions [16] $\theta_\gamma(\boldsymbol{x}, \boldsymbol{y})$ and $\theta_\delta(\boldsymbol{x}, \boldsymbol{y})$ for two opposite (γ- and δ-) types of cuts, the related ordering on the lattice ($\boldsymbol{x} > \boldsymbol{y}$ or $\boldsymbol{y} > \boldsymbol{x}$), and the two types of statistical operators $K_i(\boldsymbol{x}_\gamma)$ and $K_i(\boldsymbol{x}_\delta)$,

$$K_j(\boldsymbol{x}_\gamma) = \exp\bigl(\mathrm{i}\nu \sum_{\boldsymbol{y}\neq\boldsymbol{x}} \theta_\gamma(\boldsymbol{x},\boldsymbol{y}) c_j^\dagger(\boldsymbol{y}) c_j(\boldsymbol{y})\bigr), \quad K_j(\boldsymbol{x}_\delta) = \exp\bigl(\mathrm{i}\nu \sum_{\boldsymbol{y}\neq\boldsymbol{x}} \theta_\delta(\boldsymbol{x},\boldsymbol{y}) c_j^\dagger(\boldsymbol{y}) c_j(\boldsymbol{y})\bigr), \tag{8}$$

one defines [16] the anyonic oscillators $a_i(\boldsymbol{x}_\gamma) = K_i(\boldsymbol{x}_\gamma)c_i(\boldsymbol{x})$ and $a_i(\boldsymbol{x}_\delta) = K_i(\boldsymbol{x}_\delta)c_i(\boldsymbol{x})$ involving *the anyonic statistics parameter* ν. From this definition and ACR's for lattice fermions, the relations of permutation for anyonic oscillators then follow. Some of them, e.g.

$$a_i(\boldsymbol{x}_\gamma)a_i(\boldsymbol{y}_\gamma) + q^{-\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})}a_i(\boldsymbol{y}_\gamma)a_i(\boldsymbol{x}_\gamma) = 0, \qquad a_i(\boldsymbol{x}_\gamma)a_i^\dagger(\boldsymbol{y}_\gamma) + q^{\mathrm{sgn}(\boldsymbol{x}-\boldsymbol{y})}a_i^\dagger(\boldsymbol{y}_\gamma)a_i(\boldsymbol{x}_\gamma) = 0,$$

involve the deformation parameter q connected with the statistics parameter ν of (8) as $q = \exp(\mathrm{i}\pi\nu)$). The generating elements $A_{j,j+1}$, $A_{j+1,j}$ and H_j of the quantum algebra $U_q(su_N)$ are realized bilinearly through anyonic oscillators $a_i(\boldsymbol{x}_\gamma)$, $a_i^\dagger(\boldsymbol{y}_\gamma)$ and were shown to satisfy [16] the defining relations [3] of the quantum algebra $U_q(su_N)$. Dual realization in terms of $a_i(\boldsymbol{x}_\delta)$, $a_i^\dagger(\boldsymbol{y}_\delta)$ is also valid. Then, in anyonic realization of $U_q(su_N)$, both hadron mass operator $\hat{M}$ and basis vectors for hadronic irreps are explicitly constructed [17]. Say, for the irrep $\{4000\}$ of 'dynamical' $U_q(su_5)$, in accordance with the chain of q-algebras $U_q(su_3) \subset U_q(su_4) \subset U_q(su_5)$ and respective chain of irreps $[30] \subset [300] \subset \{4000\}$, all basis state vectors $|n_1n_2n_3n_4\rangle \equiv a^\dagger_{n_1}(\boldsymbol{x}_{1\gamma})a^\dagger_{n_2}(\boldsymbol{x}_{2\gamma})a^\dagger_{n_3}(\boldsymbol{x}_{3\gamma})a^\dagger_{n_4}(\boldsymbol{x}_{4\gamma})|0\rangle$ of baryons $\frac{3}{2}^+$ are constructed by acting with lowering generators upon the highest weight vector. E.g., for isoquartet baryon $|\Delta^{++}\rangle$ we get $|\Delta^{++}\rangle = [4]^{-1/2}(|5111\rangle + q^{-1}|1511\rangle + q^{-2}|1151\rangle + q^{-3}|1115\rangle)$, and similarly for $|\Sigma^*\rangle$, $|\Xi^*\rangle$, $|\Omega^-\rangle$. The dual basis vectors $|\Delta^{++}\tilde{\rangle}$ etc., are also given. Masses M_{D_i} of baryons D_i in the dynamical $U_q(su_5)$-irrep $\{4000\}$ are calculated with mass operator $\hat{M}$ as $M_{D_i} = \tilde{\langle} D_i|\hat{M}|D_i\rangle$ to yield: $M_\Delta = M_{\mathbf{10}} + \beta$, $M_{\Sigma^*} = M_{\mathbf{10}} + [2]_q\alpha + [2]_q\beta$, $M_{\Xi^*} = M_{\mathbf{10}} + [2]_q^2\alpha + [3]_q\beta$, and $M_{\Omega^*} = M_{\mathbf{10}} + [2]_q[3]_q\alpha + [4]_q\beta$, from which the relation (5) follows. This proves applicability [17] of quantum algebras and their irreps for deriving hadron mass relations by *using their anyonic realization.*

Since $\theta_{\mathbf{10}} = \theta_C$, *we have the connection: Cabibbo angle* $\leftrightarrow$ *anyonic statistics parameter* ν.

4 Diquark-quark model parameters and the Cabibbo angle

The LTK diquark-quark model [5] uses, besides the $SU(3)$ invariant masses m_t, m_s and m_q of the $SU(3)$ diquark triplet, diquark sextet, and 3rd quark triplet (so the subscripts "t", "s", "q"), also the mass parameters δ_t, δ_s and δ_q reflecting $SU(3)$ violation in the respective multiplets.

The improved form of GMO obtained in the LTK model is

$$\tfrac{3}{2}m_\Lambda + \tfrac{1}{2}m_\Sigma - m_N - m_\Xi = \mathcal{C}_{\mathrm{LTK}} \equiv \mu_s\big(3\xi_{ts}^2 + 18\xi_{ts} - 13\big), \tag{9}$$

where $\xi_{ts} = \frac{\delta_t - \delta_q}{\delta_s - \delta_q}$, $\mu_s = V_0\gamma_s^2$, $\gamma_s = \frac{(\delta_s - \delta_q)}{6V_0}$. The μ_s must be positive being also involved [5] in the decuplet mass combination: $8\mu_s = 2m_{\Xi^*} - m_\Omega - m_{\Sigma^*} > 0$. As result, the LTK model gave the value $\xi_{ts}^{\mathrm{LTK}} = -3$, which implied: δ_t must be greater (smaller) than δ_q with the parameter δ_s being smaller (greater) than δ_q. However, the value $\xi_{ts}^{\mathrm{LTK}} = -3$ contradicts data as it renders the r.h.s. of (9) to be negative thus correcting the GMO *in wrong direction.*

On the other hand, our q-MR (3) for which $q_7 = \exp(\mathrm{i}\frac{\pi}{7})$, $\frac{\pi}{7}{=}2\theta_{\mathrm{C}}$, can be rewritten as

$$\frac{3}{2}m_\Lambda + \frac{1}{2}m_\Sigma - m_N - m_\Xi = \mathcal{C}_{q_7} \equiv \frac{2 - [2]_7}{[2]_7 - 1}(m_\Xi - m_\Lambda) - \frac{1}{2}(m_\Sigma - m_\Lambda). \tag{10}$$

Comparison of the two improved versions (9), (10) leads to the relation [18]

$$([2]_7 - 1)^{-1} = \frac{9\tilde{\xi}_{ts}^2 - 6\tilde{\xi}_{ts} + 5}{4(\tilde{\xi}_{ts}^2 - 6\tilde{\xi}_{ts} + 5)}, \qquad \tilde{\xi}_{ts} = \frac{\tilde{\delta}_t - \tilde{\delta}_q}{\tilde{\delta}_s - \tilde{\delta}_q}, \tag{11}$$

which connects the value $q_7 = \exp\big(i\frac{\pi}{7}\big)$ of q-parameter in the q-GMO (10) with the ratio ξ_{ts} of the LTK model. Remark that the values of $\tilde{\delta}_t$, $\tilde{\delta}_s$ and $\tilde{\delta}_q$ in this relation are *understood as*

the optimized ones reflecting, through the found connection, all-order account by (10) of $SU(3)$ symmetry breaking in octet baryon masses – just this fact is denoted by tildas over δ_t, δ_s, δ_q.

Since $[2]_7 = 2\cos\frac{\pi}{7} \approx 1.80194$, solving the relation (11) yields the values $\xi_{ts}^{(+)} \approx 0.741$ and $\xi_{ts}^{(-)} \approx -6.705$. By the very derivation, $\xi_{ts}^{(\pm)}$ should guarantee the validity of sum rule (9) to within 0.07 %. The both values differ essentially from the value $\xi_{ts}^{\rm LTK} = -3$ of LTK model, *providing proper positive correction to* GMO, see the above comment about $\xi_{ts}^{\rm LTK}$. Moreover, being positive, our $\xi_{ts}^{(+)}$ is not only well acceptable phenomenologically but also reflects *qualitatively* different, than $\xi_{ts}^{(-)}$ and $\xi_{ts}^{\rm LTK}$, physical situation. Namely, the mass parameter $\tilde{\delta}_t$ is greater (smaller) than $\tilde{\delta}_q$ when the mass parameter $\tilde{\delta}_s$ is greater (smaller) than $\tilde{\delta}_q$, since

$$\tilde{\delta}_t - \tilde{\delta}_q = 0.741(\tilde{\delta}_s - \tilde{\delta}_q) \qquad \text{i.e.,} \qquad \tilde{\delta}_t - \tilde{\delta}_s = 0.259(\tilde{\delta}_q - \tilde{\delta}_s).$$

Thus, the inequalities for the (optimized) parameters of LTK model should be

$$\text{either} \qquad \tilde{\delta}_q > \tilde{\delta}_t > \tilde{\delta}_s \qquad \text{or} \qquad \tilde{\delta}_s > \tilde{\delta}_t > \tilde{\delta}_q. \tag{12}$$

Since $\theta_{\mathbf{8}} = \frac{\pi}{7} = 2\ \theta_{\rm C}$, or $\theta_{\rm C} = \frac{\pi}{14}$, we finally arrive [18] at the formula yielding *direct link* of the Cabibbo angle to the (optimized) parameters of the LTK diquark-quark model:

$$\cos 2\theta_{\rm C} = \frac{13\tilde{\xi}_{ts}^2 - 30\tilde{\xi}_{ts} + 25}{2(9\tilde{\xi}_{ts}^2 - 6\tilde{\xi}_{ts} + 5)}, \qquad \tilde{\xi}_{ts} \equiv \frac{\tilde{\delta}_t - \tilde{\delta}_q}{\tilde{\delta}_s - \tilde{\delta}_q}. \tag{13}$$

This connection of the Cabibbo angle with the parameters of diquark-quark model, clearly, could not be found without equation (3) (or equation (10)) and the relation (11) deduced on its base.

5 Cabibbo mixing in multiparticle correlations?

The model of ideal gas of q-bosons based on the algebra of q-deformed oscillators of Arik–Coon (AC) or Biedenharn–Macfarlane (BM) type [19], may be used to describe [20] unusual behavior of 2-particle correlations of identical pions or kaons produced in relativistic heavy ion collisions. The approach yields explicit expressions and clear predictions [21] for the intercept λ (dependent on the temperature, particle mass, pair mean momentum, *and the deformation parameter* q).

Physical observables are evaluated as thermal averages $\langle A\rangle = {\rm Sp}(A\rho)/{\rm Sp}(\rho)$, $\rho = e^{-\beta H}$, where the Hamiltonian is $H = \sum\omega_i N_i$ and $\beta = 1/T$. With $b_i^\dagger b_i = [N_i]_q$ and $[2]_q = 2\cos\theta$, the q-deformed distribution function for the BM-type q-bosons results (see e.g. [20]) as

$$\langle b_i^\dagger b_i\rangle = (e^{\beta\omega_i} - 1)/(e^{2\beta\omega_i} - 2\cos\theta\ e^{\beta\omega_i} + 1). \tag{14}$$

At θ=0 (or q=1), it yields Bose–Einstein (BE) distribution, as q=1 recovers usual bosonic commutation relations. Deviation of q-distribution (14) from the quantum BE distribution is seen to tend towards the Boltzmann one (reduced quantum statistical effects). For kaons, whose mass $m_K > 3m_\pi$, analogous curve gets closer (than pion's one) to the BE distribution. Note that for AC-type q-bosons, the q-distribution looks more simple: $\langle b_i^\dagger b_i\rangle = (e^{\beta\omega_i} - q)^{-1}$.

5.1 Intercept λ of two-pion (two-kaon) correlations and Cabibbo angle

To obtain explicitly the intercept λ of two-particle correlations one calculates the two-particle distribution $\langle b^\dagger b^\dagger bb\rangle$ and normalizes it by $\langle b^\dagger b\rangle^2$. The result, see [20], for AC-type q-bosons reads $\lambda = q - \frac{q(1-q^2)}{e^{\omega/T} - q^2}$, $-1 \le q \le 1$, and for BM-type q-bosons, with $\mathcal{F}(\beta\omega) \equiv \cosh(\beta\omega)$, it is

$$\lambda + 1 \equiv \langle b^\dagger b^\dagger bb\rangle/(\langle b^\dagger b\rangle)^2 = \frac{2\cos\theta(\mathcal{F}(\beta\omega) - \cos\theta)^2}{(\mathcal{F}(\beta\omega) - 1)(\mathcal{F}(\beta\omega) - 2\cos^2\theta + 1)}. \tag{15}$$

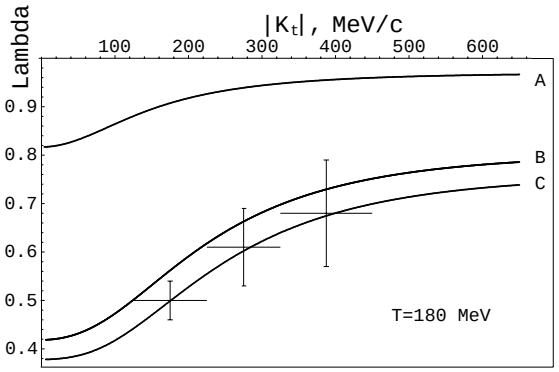

Figure 1. The intercept λ of two-pion correlation function versus transverse momentum $|\boldsymbol{K}_t|$ at fixed $T = 180$ MeV and fixed $q = \exp(i\theta)$: A) $\theta = 10^\circ$, B) $\theta = 25.7^\circ$ (i.e., $2\theta_{\rm C}$), C) $\theta = 28.5^\circ$.

With $\omega = (m^2 + \boldsymbol{K}^2)^{1/2}$, asymptotically at large mean momentum of pion (kaon) pair and fixed temperature, the intercept λ tends to the constant $\lambda^{\rm AC}_{\rm asymp} = q$ for the AC-type q-bosons (q real), and for the BM-type q-bosons (q is a phase factor) to the constant

$$\lambda^{\rm BM}_{\rm asymp} = 2\cos\theta - 1, \qquad \theta = -{\rm i}\ln q. \tag{16}$$

As suggested in [21,22], correlations of pions and kaons are characterized by the same value of q (a kind of universality). Then, experimentally there should be a tendency of merging $\lambda(\pi)$ and $\lambda(K)$ at large enough mean momenta: $\lambda_{\rm asymp}(\pi) = \lambda_{\rm asymp}(K)$. Recent RHIC/STAR experiment gives [22] the values $\lambda_1(\pi^-)$, $\lambda_2(\pi^-)$ and $\lambda_3(\pi^-)$ for π^--intercept, as averaged over three intervals of transverse momenta $\boldsymbol{K}_t$ and integrated over rapidity y, $-0.5 \le y \le 0.5$.

In Fig. 1, we show three curves for the intercept λ which correspond to fixing in (15) certain values of the deformation angle θ, along with the data $\lambda_j(\pi^-)$, j=1, 2, 3 (with error bars). The temperature is $T = 180$ MeV for all curves. We find remarkable agreement with data for the curve C (i.e., $\theta = 28.5^\circ$). Another interesting case is the curve B at $\theta = \frac{\pi}{7} = 2\theta_{\rm C} \simeq 25.7^\circ$. At $T = 180$ MeV, the curve B agrees, within error bars, with the points $\lambda_2(\pi)$ and $\lambda_3(\pi)$. However, slightly higher effective temperature $T \simeq 198$ MeV makes the curve marked by $\theta = 2\theta_{\rm C}$ respecting *all the three error bars*. Among various mixing angles known for hadrons, only the angle $2\theta_{\rm C}$ seems to be relevant to the issue of intercept $\lambda(\pi)$. Hence, it is tempting to suggest that just the angle $2\theta_{\rm C}$ embodies the assumed universality, to be seen in 2-particle correlations:

$$\lambda^{\rm BM}_{\rm asymp}(\pi)|_{\theta=\pi/7} = \lambda^{\rm BM}_{\rm asymp}(K)|_{\theta=\pi/7} = 2\cos\frac{\pi}{7} - 1 \approx 0.80194. \tag{17}$$

Insisting on the asymptotical coincidence $\lambda_{\rm asymp}(\pi) = \lambda_{\rm asymp}(K)$, we predict for kaons: *the intercept of 2-kaon correlations at any transverse momenta should obey:* $\lambda(K) \lesssim 0.80194$.

5.2 Three-particle correlations of pions (viewed as q-bosons) and $\theta_{\rm C}$

Now consider higher order (or multi-particle) correlations, first in the case of AC-type q-oscillators. It is not difficult to derive [23] the following 3-particle monomode correlation function

$$\langle a_i^\dagger a_i^\dagger a_i^\dagger a_i a_i a_i\rangle = \frac{(1+q)(1+q+q^2)}{(e^{\eta_i}-q)(e^{\eta_i}-q^2)(e^{\eta_i}-q^3)} \tag{18}$$

with $\eta_i \equiv \beta\omega_i$. From this, and 1-particle distribution $\langle a_i^\dagger a_i\rangle = \frac{1}{e^{\eta_i}-q}$, the intercept of 3-particle correlations for AC-type q-bosons in monomode case, dropping the subscript "i", reads:

$$\lambda^{(3),{\rm AC}} + 1 = \frac{\langle (a^\dagger)^3 a^3\rangle}{\langle a^\dagger a\rangle^3} = \frac{(1+q)(1+q+q^2)(e^{\eta_i}-q)^2}{(e^{\eta_i}-q^2)(e^{\eta_i}-q^3)}, \qquad -1 \le q \le 1. \tag{19}$$

The n-particle generalization of (18) is obtained in the form (see [23])

$$\langle (a_i^\dagger)^n (a_i)^n \rangle = \frac{\lfloor n \rfloor_q !}{\prod_{r=1}^n (e^{\eta_i} - q^r)}, \qquad \lfloor m \rfloor_q \equiv \frac{1-q^m}{1-q} = 1 + q + q^2 + \cdots + q^{m-1}, \tag{20}$$

where $\lfloor n \rfloor_q ! = \lfloor 1 \rfloor_q \lfloor 2 \rfloor_q \cdots \lfloor n \rfloor_q$. Moreover, this admits direct two-parameter extension [23] to the model of pq-Bose gas, based on the use of so-called pq-oscillators, in the form

$$\langle (A_i^\dagger)^n (A_i)^n \rangle = \frac{[\![n]\!]_{qp} ! \ (e^{\eta_i} - 1)}{\prod_{r=o}^n (e^{\eta_i} - q^r p^{n-r})}, \qquad [\![m]\!]_{qp} \equiv \frac{q^m - p^m}{q - p}. \tag{21}$$

With the use of one-particle distribution $\langle A^\dagger A \rangle = \frac{e^w - 1}{(e^w - q)(e^w - p)}$ of qp-bosons (see also [24]) we obtain general formula [23] for the qp-deformed n-th order intercept $\lambda_{q,p}^{(n)}$:

$$\lambda_{q,p}^{(n)} \equiv -1 + \frac{\langle A^{\dagger n} A^n \rangle}{\langle A^\dagger A \rangle^n} = -1 + [n]_{qp} ! \frac{(e^w - p)^n (e^w - q)^n}{(e^w - 1)^{n-1} \prod_{k=0}^n (e^w - q^{n-k} p^k)}. \tag{22}$$

From this, the result (19) for AC-type q-bosons follows if $p = 1$. Similarly, putting $p = q^{-1}$ reduces to the important case of BM-type q-bosons, which for the 3-particle intercept yields

$$\lambda^{(3),\mathrm{BM}} \equiv -1 + \frac{\langle a^{\dagger 3} a^3 \rangle}{\langle a^\dagger a \rangle^3} = -1 + \frac{[2]_q [3]_q (e^{2w} - 2\cos\theta e^w + 1)^2}{(e^w - 1)^2 (e^{2w} - 2\cos(3\theta) e^w + 1)} \tag{23}$$

(compare it with equation (19)). Again, like the intercept $\lambda_{\mathrm{asymp}}^{(2)}$ in (16), at large w (large momenta or low T) $\lambda_{\mathrm{asymp}}^{(3)}$ gets dependent solely on the deformation parameter $q = \exp(\mathrm{i}\theta)$, i.e.

$$\lambda_{\mathrm{asymp}}^{(3),\mathrm{BM}} = -1 + [2]_q [3]_q = -1 + 2\cos\theta (2\cos\theta - 1)(2\cos\theta + 1) \ . \tag{24}$$

To confront the results (22), (24) with experimental data, we consider the combination [25]

$$r_0 = \left(\lambda^{(3)} - 3\lambda^{(2)}\right) \left(\lambda^{(2)}\right)^{-3/2}. \tag{25}$$

Then, the r_0 that follows from (15) and (23) (or $r_{0,\mathrm{asymp}}$ stemming from (16) and (24)) is at fixed T a decreasing function of θ for $0 \leq \theta < \pi/3$. What about the Cabibbo angle? If we take the value $\theta = 2\theta_{\mathrm{C}}$ of deformation angle as we did in the preceding subsection we get $r_0|_{\theta = 2\theta_{\mathrm{C}}} \simeq 0.8955$. The universality conjecture dictates now that *all posible values of* the quantity r_0 (composed of intercepts $\lambda^{(2)}$ and $\lambda^{(3)}$) for either pions or kaons should respect the value 0.8955. The presently available data [25] extracted in Pb-Pb and Au-Au collisions seems to be yet insufficient to prove or disprove this assertion.

6 Conclusion

The use of quantum groups (quantum algebras) in the context of baryon phenomenology implies important fact that the deformation parameter q is linked very simply to the basic (fermion) Cabibbo mixing angle θ_{C}. In turn, this leads to unexpected connections for θ_{C} and for the whole concept of mixing, e.g., with the parameters of diquark-quark model of baryons, with the statistics parameter of anyonic picture, with the experimentally testable intercept parameters of multi-pion (-kaon) correlation functions, etc. Of course, it would be highly desirable to (re)obtain the considered connections of θ_{C} in a more strict way, and we hope this will be realized in a not very distant future.

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Classical Liouville Theory and the Microscopic Interpretation of Black Hole Entropy

Alex GIACOMINI

Department of Physics, University of Trento and Istituto Nazionale di Fisica Nucleare, Gruppo collegato di Trento, Italy
E-mail: *giacomin@science.unitn.it*

In this paper we will compute the entropy of a Schwarzschild black hole by finding a classical central charge of the Virasoro algebra of a Liouville theory and using the Cardy formula. This is done by performing a dimensional reduction of the Einstein–Hilbert action with the Ansatz of spherical symmetry and writing the metric in conformally flat form. We obtain two coupled field equations using the near horizon approximation the field equation for the conformal factor decouples becoming a Liouville equation. The generators of conformal transformations of a Liouville theory form a Virasoro algebra with a classical central charge. It is then possible to compute the black hole entropy via Cardy formula and so to count the microstates responsible for this entropy. This computation is independent from a specific quantum theory of gravity model.

1 Introduction

It is a well known fact that for black holes there are 4 theorems of differential geometry called the "4 laws of black hole mechanics" [1]

$$\kappa = \text{const} \quad \text{on H}, \tag{1}$$

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega\delta J + \Phi\delta e, \tag{2}$$

$$\delta A =\geq 0 \quad \text{in every phys. process}, \tag{3}$$

$$\kappa = 0 \quad \text{BH not realizable}. \tag{4}$$

One can notice the strong analogy to the laws of thermodynamics if one associates to the mass M a the energy to the surface gravity κ the temperature and to to the horizon area A the entropy. But at classical level this is only a formal analogy because a black hole being "black" has $T = 0$. The situation changes dramatically when one considers quantum effects near the horizon. Hawking [2] showed in his famous article that due to quantum effects a black hole emits a thermal radiation proportional to the surface gravity

$$T = \frac{\kappa}{2\pi}. \tag{5}$$

So the 4 laws of black hole mechanics are not only an analogy to the laws of thermodynamics but are really the laws of thermodynamics applied to black holes. So one can conclude that a black hole possesses an entropy given by the Bekenstein–Hawking formula [3]

$$S = \frac{A}{4}. \tag{6}$$

Now in thermodynamics the entropy has a statistical interpretation in terms of microstates given by the Boltzmann formula

$$S = k_B \log N \ .$$

So there arises the question which are the microstates of the black hole responsible for the entropy. One possibility would be, if we consider the black hole as a collapsed star, to take the degrees of freedom of matter as responsible for the entropy. But if we compare the entropy of a non-collapsed star of 1 solar mass that is about $S = 10^{58}k_B$ with the entropy of a black hole of 1 solar mass that is about $S = 10^{77}k_B$ we see that there are 19 magnitudes missing. One can conclude that the relevant role for the black hole entropy is played by the gravitational degrees of freedom. The problem is now that no complete quantum theory of gravity exists at the moment. There exist of course some specific models for quantum gravity like superstrings or loop quantum gravity, which can successfully compute the entropy of some classes of black holes, but it remains the fact that the black hole entropy was found by means of semiclassical calculations and already at classical level one can see the thermodynamic properties of black holes by means of the 4 laws of black hole mechanics. We will therefore show in this paper ,which is based on an article written by A. Giacomini & N. Pinamonti, [4] that it is possible to count the microstates of a black hole in a way that is completely independent of a specific model of quantum theory of gravity. This is done by finding a classical symmetry principle, in our case the 2-D conformal symmetry.
In order to understand the role of 2-D conformal symmetry in counting the black hole microstates let us recall some standard features of 2-D conformal symmetry. By introducing complex coordinates the flat metric can be written in the form

$$ds^2 = dzd\overline{z}$$

and we see that every analytic function

$$z \to f(z), \qquad \overline{z} \to \overline{f}(\overline{z})$$

defines a conformal transformation

$$ds^2 = \frac{\partial f}{\partial z}\frac{\partial \overline{f}}{\partial \overline{z}} = \rho(z\overline{z})dzd\overline{z}.$$

The infinitesimal generators of such transformations

$$G_n = z^{n+1}\partial_z$$

close a Lie algebra

$$[G_n, G_m] = (n-m)G_{n+m}. \tag{7}$$

In 2D CFT the stress energy tensor has only two components i.e. one analytic and the other antianalytic

$$T = T_{zz}, \qquad \overline{T} = \overline{T}_{\overline{z}\overline{z}}, \tag{8}$$

$$\partial_z\overline{T} = 0, \qquad \partial_{\overline{z}}T = 0. \tag{9}$$

We can therefore expand for example T in Laurent series

$$T(z) = \sum_{-\infty}^{\infty}\frac{L_n}{z^{n+2}}.$$

In the quantum case the L_n become operators and relatively to the commutator form a Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\left(m^3 - m\right)\delta_{m,-n}. \tag{10}$$

This is a central extension of the algebra (7). The central extension in the commutator algebra arises because of the normal ordering of the creation operators. It is a standard result for quantum CFT in 2-D that the asymptotic density of states for given L_0 is completely determined by the Virasoro algebra by means of the Cardy formula [5]

$$\rho(L_0) = \exp\left(2\pi\sqrt{\frac{cL_0}{6}}\right). \tag{11}$$

The entropy can therefore be calculated by using the logarithm of the Cardy formula

$$S = \ln \rho.$$

A central extension of the conformal algebra (7) can already arise at classical level in the Poisson algebra of the charges, as for example in the Liouville theory [6].

How can we now use the powerful tools of 2-D CFT for the counting of microstates of a black hole? We notice that all the relevant geometry of a black hole is encoded in the $r-t$ plane which is a 2-D manifold. Therefore if we find a classical conformal symmetry for the geometry of the $r-t$, plane which admits a Virasoro algebra with a classic central charge, we would be able to count the microstates of the black hole via Cardy formula without the use of any quantum theory of gravity. One interesting approach in this direction was made by Carlip [7] by computing the Poisson brackets of the canonical generators which preserve certain fall off conditions of the metric near the horizon. This idea is based on an article of J.D. Brown and M. Henneaux, where the central charge of the Poisson algebra of generators preserving the asymptotic structure of AdS_3 is found [8]. This approach seems to have some technical difficulties especially for the Schwarzschild black hole [9,10]. We will use in this paper a different approach: using the Ansatz of spherical symmetry (we are interested in the Schwarzschild black hole) we will perform a dimensional reduction and obtain an effective 2-D theory. Using then a near horizon approximation we will eventually find that all the dynamics is described by a Liouville theory with a computable classical central charge. We can then use this classical Virasoro algebra to compute the Schwarzschild black hole entropy via Cardy formula.

2 Dimensional reduction

Let us start with the usual 4-dimensional Einstein–Hilbert action

$$I = \frac{1}{16\pi}\int \sqrt{-g}Rd^4x. \tag{12}$$

We make the Ansatz of spherical symmetry

$$ds^2 = g^{(2)}_{ij}dx^i dx^j + \Phi^2(x_1,x_2)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{13}$$

where $g^{(2)}$ is the metric of a 2-D manifold and Φ is a radial coordinate. We write now the action (12) in function of the 2-dimensional curvature scalar $R^{(2)}$ and Φ and integrate over the angular variables. We obtain an effective action of a 2-dimensional gravity coupled to a dilaton field

$$I = \frac{1}{4}\int d^2x\sqrt{-g^{(2)}}\left(2(\nabla\Phi)^2 + \Phi^2R^{(2)} + 2\right). \tag{14}$$

One possibility would be to study the dynamics of the dilaton field and see if near horizon it becomes a conformal field. This has already been done by Solodukhin [11]. In this paper we

argue that the relevant degree of freedom is the metric of the $r-t$ plane and so in order to put this action in a more useful form we redefine

$$\Phi^2 = \eta, \qquad g^{(2)}_{ab} = \frac{1}{\sqrt{\eta}}\tilde{g}_{ab} \tag{15}$$

obtaining the action

$$I = \frac{1}{2}\int \sqrt{-\tilde{g}}\left[\frac{\eta}{2}R[\tilde{g}] + V(\eta)\right], \tag{16}$$

where the dilatonic potential $V(\eta)$ in our case is $V(\eta) = \frac{1}{\sqrt{\eta}}$.

Up to now we have a theory with 2 degrees of freedom: The field η which is related to the radius of the 2-sphere while $\tilde{g}$ contains the geometry of the $r-t$ plane. It is well known that in 2 dimensions the metric can always be written in conformally flat form

$$\tilde{g}_{ab} = e^{-2\rho}\gamma_{ab}, \tag{17}$$

where γ is the 2-dimensional Minkowski metric. Therefore all the geometry of the $r-t$ plane is described by the field ρ. The action (16) using (17) becomes

$$I = \frac{1}{2}\int d^2x \left(-\partial_a\eta\partial^a\rho + V(\eta)e^{-2\rho}\right). \tag{18}$$

We obtain so a theory of two fields propagating in flat space-time. From the last action we obtain two coupled equations of motion

$$\Box\rho + \partial_\eta V(\eta)e^{-2\rho} = 0, \qquad \Box\eta - 2V(\eta)e^{-2\rho} = 0. \tag{19}$$

As we have gauge fixed the metric to the form (17) the fields must also satisfy the constraints

$$\frac{\delta I}{\delta g^{ab}} = T_{ab} = 0. \tag{20}$$

With introducing now light coordinates $X^\pm = x^1 \pm x^2$ the equations of motion (19) take the form

$$\partial_+\partial_-\rho - \frac{\partial_\eta V(\eta)}{4}e^{-2\rho} = 0, \qquad \partial_+\partial_-\eta + \frac{V(\eta)}{2}e^{-2\rho} = 0 \tag{21}$$

and the constraints (20) can now be written as

$$T_{\pm\pm} = \partial_\pm\partial_\pm\eta + 2\partial_\pm\rho\partial_\pm\eta = 0. \tag{22}$$

Notice that the last equation of (21) plus the constraints imply the first equation of (21).

3 Near-horizon approximation

Up to now we have only made an Ansatz of spherical symmetry. If we impose the existence of a black hole the $r-t$ plane metric becomes

$$ds^2 = -N^2(r)dt^2 + \frac{1}{N^2(r)}dr^2. \tag{23}$$

Now as near horizon approximation the lapse function N can be approximated by

$$N^2(r) = 2\kappa(r - r_0). \tag{24}$$

With introducing the coordinate $r = r_0 + \kappa y^2/2$ the near horizon metric takes the Rindler form

$$ds^2 = -\kappa^2 y^2 dt^2 + dy^2 \tag{25}$$

Now introducing $x^{\pm} = \kappa t \pm \log y$ we eventually obtain

$$ds^2 = -dx^+ dx^- \exp\left(x^+ - x^-\right). \tag{26}$$

Comparing this expression with the equation (17) we can find the form of the conformal factor ρ near the horizon

$$-2\rho = x^+ - x^-. \tag{27}$$

Notice that it easy to integrate the constraint (22)

$$\partial_{\pm}\eta = \exp\left(-2\rho + C_{\mp}(x_{\mp}\right), \tag{28}$$

where $C_{\mp}$ is an arbitrary function of $x^{\mp}$ In the near horizon limit i.e. taking $x^{\pm} \to \mp\infty$ and because of (27) this means $\rho \to \infty$ we obtain

$$\partial_{\pm}\eta = 0 \Rightarrow \eta = \text{const.} \tag{29}$$

This means that in the near horizon limit the dilaton field can be considered as fixed at its value on the horizon η_0. In that case it is immediate to check that the constraints (22) and the second equation of motion in (21) are identically satisfied. As equation of motion survives, therefore only

$$\partial_+\partial_-\rho + \frac{1}{8\eta_0^{3/2}} e^{-2\rho} = 0. \tag{30}$$

This is the classical Liouville equation. Therefore we can conclude that in the near horizon limit all the dynamics of the black hole is described by the field ρ, which describes the geometry of the $r - t$ plane. The dynamics of this field is governed by the Liouville equation. It is a well known fact that the Liouville theory possesses a classical central charge.

4 Virasoro algebra

In order to find a Virasoro algebra we must introduce a set of generators. To do this let us first notice that the most general action that gives the Liouville equation is

$$I = C \int \sqrt{\hat{g}} \left(\frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \frac{\mu}{\beta^2} e^{-\beta\rho} - \frac{2}{\beta}\rho R[\hat{g}] \right). \tag{31}$$

In our case $\beta = 2$ and $\mu = 1/\eta^{3/2}$. It is important to stress that we have not derived the action (31) from the gravitational one (18) but this is the most general action that gives the Liouville equation (30). The constant C in the action is arbitrary for the moment and will be fixed later. The stress energy tensor derived from the last action is in lightcone coordinates

$$T_{\pm\pm} = C \left(\partial_\pm\rho\partial_\pm\rho + \frac{2}{\beta}\partial_\pm\partial_\pm\rho \right). \tag{32}$$

The term with the second derivatives arises from the variation of the scalar curvature term in the action (31) and remains also after choosing a flat fixed background, where R is then of

course zero. Now the constant C in the action (31) is set in such a way that the energy of the system equals the mass of the black hole M_B.

$$\int_{-l/\sqrt{2}}^{l/\sqrt{2}} T_{11} dx^2 = M_B \Rightarrow C = \frac{\kappa A}{2\pi l}. \tag{33}$$

The parameter l is a cutoff parameter that eventually must tend to infinity. Now we can define the generators

$$L_n^{\pm} = \int_{-l/2\kappa}^{l/2\kappa} \kappa dx^{\pm} \xi_n^{\pm} T_{\pm\pm}, \tag{34}$$

where the factor 2κ in the integration is used to match the Euclidean periodicity. The ξ_n are smearing fields defined as

$$\xi_n^{\pm} = \frac{l}{2\kappa\pi} \exp\left(-i\frac{2\pi\kappa}{l} n x^{\pm}\right). \tag{35}$$

We can now compute easily the Poisson brackets of the generators

$$\left\{L_n^{\pm}, L_m^{\pm}\right\}_{PB} = i(n-m)L_{m+n}^{\pm} + i\frac{c}{12} n^3 \delta_{m+n,0}, \tag{36}$$

with

$$L_0^{+} = \frac{Al}{16\pi^2}, \tag{37}$$

$$c = \frac{3A}{2l}. \tag{38}$$

Now for every fixed value of the cutoff l we are able to compute the entropy with the logarithm of the Cardy formula

$$S = 2\pi\sqrt{\frac{c^{+}L_0^{+}}{6}} = \frac{A}{4}, \tag{39}$$

which is exactly the Bekenstein–Hawking entropy. If we let the cutoff tend to infinity the L_0 generator diverges and the central charge tends to zero. But we notice in the Cardy formula we have the product $L_0 c$ and therefore the result of the computation does not depend on l. We remember that a zero central charge was also found in [12, 13]. Therefore evntually we have "counted" the microstates of a black hole without using detail of a specific model of quantum theory of gravity, but using only the classical near horizon structure of the black hole.

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Infinitesimal Affinely-Rigid Bodies in Riemann Spaces

Barbara GOŁUBOWSKA

Institute of Fundamental Technological Research PAS,
21 Swiętokrzyska Str., 00-049 Warsaw, Poland
E-mail: *bgolub@ippt.gov.pl*

In general Riemannian spaces, as rule, there are no isometries, except the identity transformation. For that reason one cannot consider extended rigid bodies in the usual sense. But one can consider approximately rigid motion of small bodies. More rigorously, they are material points with attached orthonormal bases. These bases describe internal degrees of freedom and provide the symbolic description of relative motion degrees of freedom after the limit transition, when the body diameter tend to zero. Similarly, there is no extended affinely-rigid body concept in a general Riemann space. Affine degrees of freedom may be considered merely as internal ones, represented by a general linear basis attached at the material point.

1 Introduction

We shall consider some models of an infinitesimal affinely- or metrically-rigid body. Because we cannot describe extended bodies, we use material points with attached linear bases. We have the following generalized coordinates:

1. The spatial coordinates x^i in a manifold M, $\dim M = n$.
2. The attached basis components $e^i{}_A$, where $A = 1, \ldots, n$ are co-moving indices (referring to the material spaces), and $i = 1, \ldots, n$ are spatial indices. In the metrically-rigid case e_A are orthonormal: $g_{ij}e^i{}_Ae^j{}_B = \delta_{AB}$.

E_A denotes some field of frames fixed on M once for all. In affinely-rigid case this field is not indispensable, but description based on it is more convenient and "natural". At any time instant t, the vectors $e_A(t) \in T_{x(t)}M$ of the co-moving frame are given by $e_A(t) = E_B(x(t))U^B{}_A(t)$. In the metrically-rigid case, for any t, the matrix U is orthogonal; in the affine case it is general. The choice of E strictly depends on geometry of M. In this article we discuss two cases. First is when our infinitesimal affinelly-rigid bodies will move on the sphere. The second case is pseudo-spherical, explicitly:

$$\text{in the first case}: \quad E_{(r)} = \frac{\partial}{\partial r}, \quad E_{(\varphi)} = \frac{1}{R\sin\left(\frac{r}{R}\right)}\frac{\partial}{\partial\varphi}, \quad E^r_{(\varphi)} = E^\varphi_{(r)} = 0,$$

$$\text{in the second case}: \quad E_{(r)} = \frac{\partial}{\partial r}, \quad E_{(\varphi)} = \frac{1}{R\,\mathrm{sh}\left(\frac{r}{R}\right)}\frac{\partial}{\partial\varphi}, \quad E^r_{(\varphi)} = E^\varphi_{(r)} = 0.$$

Affine velocity in the co-moving representation is given by:

$$\frac{De^i{}_B}{Dt} := e^i{}_A\Omega^A{}_B. \tag{1}$$

With the help of the standard Kronecker delta metric we can trivially shift the co-moving indices, e.g. $\Omega_{AB} := \delta_{AC}\Omega^C{}_B$. The above quantity is skew-symmetric, i.e. $\Omega_{AB} = -\Omega_{BA}$, when the motion is metrically-rigid (gyroscopic). The spatial components of affine velocity are given by: $\Omega^i{}_j = e^i{}_A\Omega^A{}_Be^B{}_j$. In the metrically-rigid motion we can shift indices with the help of g_{ij}: $\Omega_{ij} = g_{ik}\Omega^k{}_j = -\Omega_{ji}$.

2 Hamiltonian description

The previously described models of kinetic energy [2, 5–8] were invariant under the g-isometry groups acting in tangent spaces and under the orthogonal group acting in the micromaterial space $\mathbb{R}^n$. However, in spite of the affine symmetry of degrees of freedom they were not affinely invariant. It may be interesting, at least for "academic" purpose, to investigate the affinely invariant models of kinetic energy. They have the Casimir–Killing forms:

$$T_{\text{int}} = \frac{A}{2}\text{Tr}\left[\Omega^2\right] + \frac{B}{2}\left(\text{Tr}\left[\Omega\right]\right)^2, \tag{2}$$

where A, B are constants. The full kinetic energy is postulated in the form: $T = T_{\text{tr}} + T_{\text{int}}$, where T_{tr} denotes translational and T_{int} the internal part of kinetic energy, explicitly:

1. on sphere: $T_{\text{tr}} = \dfrac{m}{2}\left(\left(\dfrac{dr}{dt}\right)^2 + R^2\sin^2\left(\dfrac{r}{R}\right)\left(\dfrac{d\varphi}{dt}\right)^2\right)$, where $r \in [0, \pi R)$,
2. on pseudo-sphere: $T_{\text{tr}} = \dfrac{m}{2}\left(\left(\dfrac{dr}{dt}\right)^2 + R^2\,\text{sh}^2\left(\dfrac{r}{R}\right)\left(\dfrac{d\varphi}{dt}\right)^2\right)$, where $r \in [0, \infty]$.

Let us write ${U^B}_A$ in the two-polar decomposition $\left({U^B}_A = {R^B}_C {D^C}_K {\left(V^{-1}\right)^K}_A\right)$, explicitly:

$$R = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, \qquad V = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}, \qquad D = \begin{bmatrix} e^{\lambda-\mu} & 0 \\ 0 & e^{\lambda+\mu} \end{bmatrix}.$$

The co-moving affine velocity Ω, defined in (1), is given by: ${\Omega^A}_B = {\left(U^{-1}\right)^A}_F \Gamma^F{}_{DC} {U^D}_B {U^C}_G v^G + {\left(U^{-1}\right)^A}_C \dot{U}^C{}_B$. After some calculations we can obtain Ω in the form:

$$\begin{bmatrix} -\chi\sin(2\beta)\,\text{sh}(2\mu) + \dot\lambda + \cos(2\beta)\dot\mu & -\chi(\text{ch}(2\mu) - \cos(2\beta)\,\text{sh}(2\mu)) + \dot\beta + \sin(2\beta)\dot\mu \\ \chi(\text{ch}(2\mu) - \cos(2\beta)\,\text{sh}(2\mu)) - \dot\beta + \sin(2\beta)\dot\mu & -\chi\sin(2\beta)\,\text{sh}(2\mu) + \dot\lambda - \cos(2\beta)\dot\mu \end{bmatrix},$$

and the internal kinetic energy has the form:

$$T_{\text{int}} = -a\chi^2 + 2a\,\text{ch}(2\mu)\chi\dot\beta - a\dot\beta^2 + (a+2b)\dot\lambda^2 + a\dot\mu^2,$$

where χ are given by:

1. on the sphere: $\chi = \dot\alpha + \cos\left(\dfrac{r}{R}\right)\dot\varphi$,
2. on the pseudo-sphere: $\chi = \dot\alpha + \text{ch}\left(\dfrac{r}{R}\right)\dot\varphi$.

3 Action-angle description

3.1 Action variables

Let us write the kinetic energy in the concise form: $T = \frac{m}{2} G_{ij}\dot q^i \dot q^j$, where $\left\{q^i\right\} = \{r, \varphi, \alpha, \beta, \lambda, \mu\}$. Now we can introduce the canonical formalism: $H = T + V$, where $T = \frac{1}{2m} G^{ij} p_i p_j$. The matrix G^{jk} is reciprocal to G_{ij}, i.e. $G_{ij}G^{jk} = \delta_i{}^k$.

In the both discussed cases the stationary Hamilton–Jacobi equation

$$H\left(q^a, \frac{\partial S}{\partial q^a}\right) = \frac{1}{2m} G^{ij}\frac{\partial S}{\partial q^i}\frac{\partial S}{\partial q^j} + V(q) = E$$

is separable for potentials of the form: $V(q) = V_r(r) + V_\lambda(\lambda) + V_\mu(\mu)$. Then the action variables may be explicitly calculated, at least in principle (in quadrature sense), and the degeneracy problem may be analyzed [3]. The kinetic term in the Hamiltonian has the form:

1. sphere:

$$T = \frac{p_r^2}{2m} + \frac{\left(p_\varphi - \cos\left(\frac{r}{R}\right) p_\alpha\right)^2}{2mR^2 \sin^2\left(\frac{r}{R}\right)} + \frac{p_\alpha^2 + 2\,\mathrm{ch}\,(2\mu)\, p_\alpha p_\beta + p_\beta^2}{4A\,\mathrm{sh}\,(2\mu)} + \frac{p_\lambda^2}{4\,(A+2B)} + \frac{p_\mu^2}{4A},$$

2. pseudo-sphere:

$$T = \frac{p_r^2}{2m} + \frac{\left(p_\varphi - \mathrm{ch}\left(\frac{r}{R}\right) p_\alpha\right)^2}{2mR^2 \,\mathrm{sh}^2\left(\frac{r}{R}\right)} + \frac{p_\alpha^2 + 2\,\mathrm{ch}\,(2\mu)\, p_\alpha p_\beta + p_\beta^2}{4A\,\mathrm{sh}\,(2\mu)} + \frac{p_\lambda^2}{4\,(A+2B)} + \frac{p_\mu^2}{4A}.$$

The time-independent Hamilton–Jacobi equation has the form:

$$H\left(q, \frac{\partial S_0}{\partial q}\right) = E,$$

where

$$\begin{aligned} S_0 &= S_r\,(r) + S_\varphi(\varphi) + S_\alpha(\alpha) + S_\beta(\beta) + S_\lambda(\lambda) + S_\mu(\mu) \\ &= S_r(r) + l\varphi + C_\alpha\alpha + C_\beta\beta + S_\lambda(\lambda) + S_\mu(\mu). \end{aligned}$$

We can separate these equations if φ, α and β are cyclic variables.

Remark 1. The form of kinetic term in the Hamiltonian suggested us that perhaps λ should be cyclic too. But $\det[D] = \exp(2\lambda)$, so the size could freely expand to infinity or contract to the point (although in the infinite time). Of course, if at some initial moment the dilatations velocity $\dot{\lambda}$ had the vanishing value, λ would be constant; however, such a solution is exponentially nonstable. Because of this, for physical reasons one should use some model of $V_\lambda(\lambda)$ stabilizing the size.

In the spherical case the Hamilton–Jacobi equation has the form:

$$\begin{aligned} E = {} & \frac{1}{2m}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 + \frac{\left(l - C_\alpha \cos\left(\frac{r}{R}\right)\right)^2}{2mR^2 \sin^2\left(\frac{r}{R}\right)} + V(r) + \frac{C_\alpha^2 + 2\,\mathrm{ch}\,(2\mu)\, C_\alpha C_\beta + C_\beta^2}{4A\,\mathrm{sh}\,(2\mu)} \\ & + \frac{1}{4\,(A+2B)}\left(\frac{\partial S_\lambda(\lambda)}{\partial \lambda}\right)^2 + \frac{1}{4A}\left(\frac{\partial S_\mu\,(\mu)}{\partial \mu}\right)^2 + V_\mu(\mu) + V_\lambda(\lambda), \end{aligned} \tag{3}$$

and in the pseudo-spherical case:

$$\begin{aligned} E = {} & \frac{1}{2m}\left(\frac{\partial S_r(r)}{\partial r}\right)^2 + \frac{\left(l - C_\alpha \,\mathrm{ch}\left(\frac{r}{R}\right)\right)^2}{2mR^2 \,\mathrm{sh}^2\left(\frac{r}{R}\right)} + V(r) + \frac{C_\alpha^2 + 2\,\mathrm{ch}\,(2\mu)\, C_\alpha C_\beta + C_\beta^2}{4A\,\mathrm{sh}\,(2\mu)} \\ & + \frac{1}{4\,(A+2B)}\left(\frac{\partial S_\lambda(\lambda)}{\partial \lambda}\right)^2 + \frac{1}{4A}\left(\frac{\partial S_\mu\,(\mu)}{\partial \mu}\right)^2 + V_\mu(\mu) + V_\lambda(\lambda). \end{aligned} \tag{4}$$

The previous equations may be separated and some explicit calculations are possible. Our action variables have the form:

$$\begin{aligned} J_\varphi &= \oint p_\varphi d\varphi = \oint \frac{\partial S_\varphi(\varphi)}{\partial \varphi} d\varphi = \int_0^{2\pi} l d\varphi = 2\pi l \Rightarrow l = \frac{J_\varphi}{2\pi}, \\ J_\alpha &= \oint p_\alpha d\alpha = \oint \frac{\partial S_\alpha(\alpha)}{\partial \alpha} d\alpha = C_\alpha \int_0^{2\pi} d\alpha = 2\pi C_\alpha \Rightarrow C_\alpha = \frac{J_\alpha}{2\pi}, \\ J_\beta &= \oint p_\beta d\beta = \oint \frac{\partial S_\beta(\beta)}{\partial \beta} d\beta = C_\beta \int_0^{2\pi} d\beta = 2\pi C_\beta \Rightarrow C_\beta = \frac{J_\beta}{2\pi}. \end{aligned} \tag{5}$$

When we have J_φ, J_α and J_β, we can explicitly calculate J_r, J_λ and J_μ. Let us split the energy equation into the part depending only on r and the one depending only on λ, μ:

$$K = \frac{J_\alpha^2 + 2\,\mathrm{ch}\,(2\mu)\,J_\alpha J_\beta + J_\beta^2}{16\pi^2 A\,\mathrm{sh}^2\,(2\mu)} + N(\lambda) + \frac{1}{4A}\left(\frac{\partial S_\mu\,(\mu)}{\partial \mu}\right)^2 + V_\mu(\mu), \tag{6}$$

where

$$N(\lambda) = \frac{1}{(A+2B)}\left(\frac{\partial S_\lambda(\lambda)}{\partial \lambda}\right)^2 + V_\lambda(\lambda), \tag{7}$$

$$\text{on sphere:} \qquad E = \frac{1}{2m}\left(\frac{\partial S_r\,(r)}{\partial r}\right)^2 + \frac{\left(l - J_\alpha \cos\left(\frac{r}{R}\right)\right)^2}{8\pi^2 m R^2 \sin^2\left(\frac{r}{R}\right)} + V(r) + K, \tag{8}$$

$$\text{and on pseudo-sphere:} \qquad E = \frac{1}{2m}\left(\frac{\partial S_r\,(r)}{\partial r}\right)^2 + \frac{\left(l - J_\alpha \,\mathrm{ch}\left(\frac{r}{R}\right)\right)^2}{8\pi^2 m R^2 \,\mathrm{sh}^2\left(\frac{r}{R}\right)} + V(r) + K. \tag{9}$$

Action variables have the following form:

$$J_\mu = \oint p_\mu d\mu = \oint \sqrt{4A\,(K - N(\lambda) - V_\mu(\mu)) - \frac{J_\alpha^2 + 2\,\mathrm{ch}\,(2\mu)\,J_\alpha J_\beta + J_\beta^2}{4\pi^2\,\mathrm{sh}^2\,(2\mu)}}\,d\mu,$$
$$J_\lambda = \oint p_\lambda d\lambda = \oint \sqrt{4(A+2B)\,(N(\lambda) - V_\lambda(\lambda))}\,d\lambda,$$

for both cases;

$$\text{on sphere:} \qquad J_r = \oint p_r dr = \oint \sqrt{2m\,(E - K - V_r(r)) - \frac{\left(J_\varphi - J_\alpha \cos\left(\frac{r}{R}\right)\right)^2}{4\pi^2 R^2 \sin^2\left(\frac{r}{R}\right)}}\,dr,$$

$$\text{on pseudo-sphere:} \qquad J_r = \oint p_r dr = \oint \sqrt{2m\,(E - K - V_r(r)) - \frac{\left(J_\varphi - J_\alpha \,\mathrm{ch}\left(\frac{r}{R}\right)\right)^2}{4\pi^2 R^2 \,\mathrm{sh}^2\left(\frac{r}{R}\right)}}\,dr.$$

3.2 Some convenient models of potentials

In elastic models of internal degrees of freedom the potentials should have a local minimum at the non-deformed configuration (usually, although not necessarily, identified with $\lambda = 0$, $\mu = 0$). For example, $V_\lambda(\lambda) = \frac{\kappa}{2}\lambda^2$. In non-elastic models this is no necessary. But the potentials should prevent the body to expand to infinity or contract to a point. Thus, e.g., $V_\lambda(\lambda)$ must tend to plus infinity, when λ tends to minus infinity. Similarly, it should tend to plus infinity, or to some limit higher than its infimum, when λ tends to plus infinity. In some non-elastic problems, for certain reason connected with the theory of integrable lattices, something similar to the inverse-square rule may be convenient, e.g.

$$V_\lambda(\lambda) = \frac{\kappa}{2}\lambda^2, \quad V_\mu(\mu) = G\,\mathrm{cth}^2(2\mu), \quad \text{where} \quad \kappa,\ G \text{ are constants.} \tag{10}$$

On $\mathrm{SL}\,(2,\mathbb{R})$ the geodetic model is interesting, when V_μ vanishes at all.

Let us take into account potentials $V(\mu)$ depending only on μ like, e.g. (10). For this potential J_μ is given by:

$$J_\mu = \frac{1}{4}\left(\sqrt{(J_\alpha + J_\beta)^2 + 16AG\pi^2} + \sqrt{(J_\alpha - J_\beta)^2 + 16AG\pi^2}\right) - 2\pi\sqrt{A(G - K + N(\lambda))}.$$

Now let us consider a potential V depending only on λ. For all reasons presented in the Remark 1 we propose $V(\lambda)$ like, e.g., in (10) and for that potential J_λ is given by:

$$J_\lambda = 2\pi N(\lambda)\sqrt{\frac{2(A+2B)}{\kappa}}.$$

It is interesting to consider classical and quantum problems without V_μ at all. Then the sign of constant of separation K depends on the effective potential [4], which is given by (in a consequence of (7) and without $V(\mu)$, but not without $V(\lambda)$):

$$V_{\text{eff}}(\mu,\lambda,J_\alpha,J_\beta) = \frac{J_\alpha^2 + 2\,\text{ch}\,(2\mu)\,J_\alpha J_\beta + J_\beta^2}{4\pi^2\,\text{sh}^2\,(2\mu)} + 4AN(\lambda).$$

The orbits are bounded, when $K \leqslant 0$, an this is just the case we consider. Then J_μ is given by:

$$J_\mu = \frac{1}{2}J_\alpha - 2\pi\sqrt{A(N(\lambda)-K)} \qquad \text{and} \qquad N(\lambda) = \frac{J_\lambda}{2\pi}\sqrt{\frac{\kappa}{2(A+2B)}}.$$

Now let us consider some potential $V(r)$. At the beginning we consider the spherical case. We suggest the following model:

$$V(r) = \eta\,\text{ctg}^2\left(\frac{r}{R}\right),$$

where η is constant, and for that potential J_r is given by:

$$J_r = \frac{1}{2}\left(\sqrt{(J_\varphi - J_\alpha)^2 + 8m\pi^2\eta R^2} - \sqrt{(J_\varphi + J_\alpha)^2 + 8m\pi^2\eta R^2}\right) - \sqrt{J_\alpha^2 + 8m\pi^2R^2(E+\eta-K)}.$$

In the pseudo-spherical case we take potential in the form:

$$V(r) = \frac{\gamma}{R^2}\,\text{cth}^2\left(\frac{r}{R}\right),$$

where γ is constant. For this potential J_r is given by:

$$J_r = \frac{1}{2}\left(\sqrt{(J_\varphi - J_\alpha)^2 + 8m\gamma\pi^2} - \sqrt{(J_\varphi + J_\alpha)^2 + 8m\gamma\pi^2}\right) - \sqrt{J_\alpha^2 + 8m\pi^2R(RE+\gamma-RK)}.$$

Then for the spherical and pseudo-spherical cases, substituting this to the J_r formula and solving it with respect to the energy parameter E we find in principle the explicit dependence of Hamiltonian on the action variables. As the resulting expression is rather complicated and obscure, we do not quote explicit formulae. The J_r's without potentials look more interesting and have more lucid structure. Description of the dependence of H on J's is more clear:

$$J_r = J_\alpha - 2\pi\sqrt{2mR^2(E-K) + \frac{J_\alpha^2}{4\pi^2}}.$$

3.3 Bohr–Sommerfeld quantization

In 1915 W. Wilson and in 1916 A. Sommerfeld discovered the rules of quantization known today as the old quantum theory. Now we call them the Bohr–Sommerfeld rules of quantization. We can use these rules when our Hamiltonians are cyclic in some variables. Then we have:

$$J_a = \oint p_a da = 2\pi n_a \hbar.$$

Then E_n without potentials for both cases is given by:

$$E_n = \frac{1}{2mR^2} n_r(n_r + 2n_\alpha)\hbar^2 + \frac{1}{4A}(n_\mu - 2n_\alpha)^2\hbar^2 + \sqrt{\frac{\kappa}{4(A+2B)}}\, n_\lambda \hbar.$$

We can do the same for the cases with potentials, but the formulae we obtain are very complicated.

Remark 2. Affinely-rigid body is the simplest generalization of metrically- rigid body with nontrivial deformable degrees of freedom. The model of internal degrees of freedom, which we present in this paper, is thought on as a preliminary step towards relativistic theory and mechanics of continua with microstructure. One can realize some physical applications like the motion of continental plates or oil pollution on oceans. It is very interesting to consider some special cases of incompressible affinely-rigid body, like fat spots on a water surface. It is just a two-dimensional analogue of three-dimensional incompressible objects, like fluid droplet (e.g. a "droplet" of nuclear matter).

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Intertwiners of Pseudo-Hermitian 2×2-Block-Operator Matrices and a No-Go Theorem for Isospectral MHD Dynamo Operators

Uwe GÜNTHER

Research Center Rossendorf, P.O. Box 510119, D-01314 Dresden, Germany
E-mail: *u.guenther@fz-rossendorf.de*

Pseudo-Hermiticity as a generalization of usual Hermiticity is a rather common feature of (differential) operators emerging in various physical setups. Examples are Hamiltonians of PT- and CPT-symmetric quantum mechanical systems [1] as well as the operator of the spherically symmetric α^2-dynamo [2] in magnetohydrodynamics (MHD). In order to solve the inverse spectral problem for these operators, appropriate uniqueness theorems should be obtained and possibly existing isospectral configurations should be found and classified. As a step toward clarifying the isospectrality problem of dynamo operators, we discuss an intertwining technique for η-pseudo-Hermitian 2×2-block-operator matrices with second-order differential operators as matrix elements. The intertwiners are assumed as first-order matrix differential operators with coefficients which are highly constrained by a system of nonlinear matrix differential equations. We analyze the (hidden) symmetries of this equation system, transforming it into a set of constrained and interlinked matrix Riccati equations. Finally, we test the structure of the spherically symmetric MHD α^2-dynamo operator on its compatibility with the considered intertwining ansatz and derive a no-go theorem.

1 Introduction

This article is based on a talk given at the Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" which was held in Kiev, June 23–29, 2003. A more detailed presentation of the material is contained in Ref. [2].

Operator intertwining techniques are one of the basic ingredients of supersymmetric (SUSY) quantum mechanics (QM) [3, 4]. With their help classes of isospectral operators have been constructed for Hermitian Hamiltonians as well as recently for pseudo-Hermitian ones [5, 6]. The success of these techniques in obtaining new and wider classes of isospectral operators raised the natural question whether a suitable extension of them can be developed for analyzing the isospectrality problem [7] of the MHD α^2-dynamo operator [8]

$$\hat{H}_l[\alpha] \equiv \begin{pmatrix} -p^2 - \dfrac{l(l+1)}{r^2} & \alpha(r) \\ p\alpha(r)p + \alpha(r)\dfrac{l(l+1)}{r^2} & -p^2 - \dfrac{l(l+1)}{r^2} \end{pmatrix} \tag{1}$$

which acts on the domain[1]

$$\mathcal{D}(\hat{H}_l[\alpha]) := \left\{ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \ \psi \in \tilde{\mathcal{H}} \equiv \mathcal{H} \oplus \mathcal{H}, \ \mathcal{H} = L_2\big(\Omega, r^2 dr\big), \right.$$
$$\left. \Omega = [0,1], \ \psi(1) = 0, \ r\psi(r)|_{r\to 0} \to 0 \right\}$$

[1]It should be noted that this domain $\mathcal{D}(\hat{H}_l[\alpha])$ corresponds to highly idealized and physically non-realistic boundary conditions (see the corresponding comments in Ref. [2] and the setup of the model in Ref. [8]).

in the Hilbert space $\tilde{\mathcal{H}}$ and where $p = -i(\partial_r + 1/r)$ denotes the radial momentum operator. The scalar function $\alpha(r)$ is the helical turbulence function (α-profile) of the α^2-dynamo [2, 8] and plays a similar role like the potential $V(r)$ in QM models.

Subsequently, it is shown in a sketchy outline that a no-go theorem exists which forbids an extension of the operator intertwining formalism from pseudo-Hermitian QM Hamiltonians to the 2×2-operator matrix (1) of the spherically symmetric α^2-dynamo in its simplest ansatz. The obstruction for such an extension consists in the presence of the centrifugal terms $l(l+1)/r^2$.

2 Operator intertwining ansatz

In SUSY QM the operator intertwining technique heavily relies on the Hermiticity or pseudo-Hermiticity of the corresponding Hamiltonians

$$H = H^\dagger \qquad \text{or} \qquad H = H^\sharp \equiv \eta H^\dagger \eta.$$

The operator η is Hermititian, involutory and unitary

$$\eta = \eta^\dagger, \qquad \eta^2 = I, \qquad \eta^{-1} = \eta^\dagger$$

and, depending on the concrete model, it can be, e.g., the operator of a parity-transformation P [5, 9] or of some other symmetry [10]. For the operator matrix of the α^2-dynamo one obtains the similar relation

$$\hat{H}_l[\alpha] = \hat{H}_l^\sharp[\alpha] \equiv \eta \hat{H}_l^\dagger[\alpha]\eta, \qquad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The structure of the α^2-dynamo operator itself cannot be reduced by appropriate complexification to the simpler setup of one-component pseudo-Hermitian QM Hamiltonians [11].

Due to the matrix structure of the dynamo operator and the α-coupling in its highest (second) order differential expressions we generalize the intertwining technique of QM by an ansatz

$$\hat{H}_{l_0}[\alpha_0] - EI = -\hat{A}\hat{A}^\sharp, \qquad \hat{H}_{l_1}[\alpha_1] - EI = -\hat{A}^\sharp\hat{A} \tag{2}$$

which is based on first-order differential operators

$$\hat{A} := iR(r)p + Q(r), \qquad \hat{A}^\sharp := -ipR^\sharp(r) + Q^\sharp(r). \tag{3}$$

Dynamos with different α-profiles $\alpha_0(r) \neq \alpha_1(r)$ will be isospectral, if appropriate 2×2-matrix functions $R(r)$ and $Q(r)$ can be constructed so that the relations (2), (3) are fulfilled simultaneously. Otherwise, intrinsic contradictions should be found which could be interpreted as a no-go theorem.

For the subsequent analysis it is convenient to introduce the auxiliary matrices $K_{0,1}$, $M_{0,1}$

$$K_{0,1} := I - \alpha_{0,1}\sigma_-,$$
$$M_{0,1} := K_{0,1}\frac{l_{0,1}(l_{0,1}+1)}{r^2} + EI - \alpha_{0,1}\sigma_+$$

with the nilpotent matrices $\sigma_\pm$ defined as $\sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The shifted α^2-dynamo operator matrices in (2) take then the short form

$$\hat{H}_{l_{0,1}}[\alpha_{0,1}] - EI = -pK_{0,1}p - M_{0,1}. \tag{4}$$

As next step, equations (3), (4) can be substituted into the intertwining ansatz (2) and terms of the same type of differential operators p^2, p, $p^0 = I$ can be equated (after appropriate commutations like $[p, R(r)] = -iR'(r)$). As a result, one obtains the following six consistency conditions

$$\hat{H}_{l_0}: \quad p^2: \quad RR^\sharp = K_0, \tag{5}$$

$$p: \quad RQ^\sharp - QR^\sharp - R(R^\sharp)' + R'R^\sharp = 0, \tag{6}$$

$$I: \quad QQ^\sharp - R(R^\sharp)'' + R(Q^\sharp)' - Q(R^\sharp)' = M_0, \tag{7}$$

$$\hat{H}_{l_1}: \quad p^2: \quad R^\sharp R = K_1, \tag{8}$$

$$p: \quad -R^\sharp Q + Q^\sharp R = 0, \tag{9}$$

$$I: \quad Q^\sharp Q - \left(R^\sharp Q\right)' = M_1. \tag{10}$$

For a successful intertwining construction these matrix equations should be fulfilled simultaneously. So, the main task consists in finding explicit solution sets for (5)–(10). Alternatively, intrinsic contradictions within this equation system should be obtained which could be interpreted as a no-go theorem forbidding this construction for α^2-dynamo operator matrices.

3 Consistency analysis

The easiest way for starting the analysis is from equations (5) and (8). From the tautologies $RR^\sharp R = RR^\sharp R$ and $R^\sharp RR^\sharp = R^\sharp RR^\sharp$ follows

$$RK_1 = K_0 R, \qquad K_1 R^\sharp = R^\sharp K_0$$

which with

$$R = \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}, \qquad R^\sharp = \begin{pmatrix} r_{22}^* & r_{12}^* \\ r_{21}^* & r_{11}^* \end{pmatrix}$$

yields

$$r_{12} = 0, \qquad \frac{\alpha_1}{\alpha_0} = \frac{r_{11}}{r_{22}} = \frac{r_{11}^*}{r_{22}^*}. \tag{11}$$

Hence, one can set

$$r_{11} = |r_{11}|e^{i\gamma}, \qquad r_{22} = |r_{22}|e^{i\gamma}, \qquad r_{21} = |r_{21}|e^{i(\gamma+\varepsilon)}.$$

Using this and (11) in

$$RR^\sharp = K_0 = \begin{pmatrix} 1 & 0 \\ -\alpha_0 & 1 \end{pmatrix}, \qquad R^\sharp R = K_1 = \begin{pmatrix} 1 & 0 \\ -\alpha_1 & 1 \end{pmatrix}$$

one finds

$$R = e^{i\gamma} \begin{pmatrix} \sqrt{\dfrac{\alpha_1}{\alpha_0}} & 0 \\ -\dfrac{1}{2}\sqrt{\alpha_0\alpha_1}\,(1 + i\tan\varepsilon) & \sqrt{\dfrac{\alpha_0}{\alpha_1}} \end{pmatrix}, \tag{12}$$

where the phases γ and ε are still undefined.

As next step, equations (6) and (9) will be analyzed. It is easily seen that for the matrices

$$U := R\big[Q^\sharp - (R^\sharp)'\big], \qquad B := R^\sharp Q \tag{13}$$

these equations are equivalent to the J-symmetry relations

$$U = U^\sharp, \qquad B = B^\sharp.$$

Due to the different symmetry content of B and Q it is natural to consider B as primary structural element of the intertwining construction, and Q as a secondary one. So, the subsequent investigation can be performed in terms of B and R. Explicitly, the η-symmetry is realized by the matrix structure

$$B = \begin{pmatrix} b_1 + ib_4 & b_2 \\ b_3 & b_1 - ib_4 \end{pmatrix}, \qquad \Im b_k = 0, \qquad k = 1, \ldots, 4. \tag{14}$$

Furthermore, Q can be excluded from (13) to obtain the interlinking constraint

$$U = RBR^{-1} - R(R^\sharp)'. \tag{15}$$

Introducing the notation $N := R^{-1}R'$ and substituting (15) into the symmetry relation $U = U^\sharp$ yields the additional constraint

$$\big[B, K_1^{-1}\big] = N^\sharp - N. \tag{16}$$

From equation (12) one finds

$$N = i\gamma' I + \begin{pmatrix} -q & 0 \\ f & q \end{pmatrix},$$

$$q = \frac{1}{2}\left(\frac{\alpha_0'}{\alpha_0} - \frac{\alpha_1'}{\alpha_1}\right), \tag{17}$$

$$f = -\frac{\alpha_1}{2}\left[\frac{\alpha_0'}{\alpha_0}\,(1 + i\tan\varepsilon) + i\frac{\varepsilon'}{\cos^2\varepsilon}\right] \tag{18}$$

so that (16) transforms to

$$\alpha_1 \begin{pmatrix} b_2 & 0 \\ -2ib_4 & -b_2 \end{pmatrix} = -2i\gamma' I + \begin{pmatrix} 2q & 0 \\ f^* - f & -2q \end{pmatrix}.$$

Finally, one arrives at the following restrictions on the phase γ and the components b_2 and b_4 of the matrix B:

$$\gamma' = 0, \qquad b_2 = \frac{2q}{\alpha_1}, \qquad b_4 = \frac{\Im f}{\alpha_1} = -\frac{1}{2}\left(\frac{\alpha_0'}{\alpha_0}\tan\varepsilon + \frac{\varepsilon'}{\cos^2\varepsilon}\right). \tag{19}$$

In summary, one finds that the first four consistency conditions are free of intrinsic contradictions. From the initially eight arbitrary complex-valued functions contained in the matrices R and Q only the three real-valued functions (b_1, b_3, ε) are still undefined. Together with the helical turbulence functions (α_0, α_1) and the constants $(\gamma, E, l_0, l_1) \in \mathbb{R}^2 \times \mathbb{Z}_+^2$ they can be expected to be highly fine-tuned by the remaining two consistency conditions (7) and (10).

These conditions can be strongly simplified with the help of the definitions of U and B in (13), their implications

$$Q^\sharp - (R^\sharp)' = R^{-1}U, \tag{20}$$

$$(Q^\sharp)' - (R^\sharp)'' = -R^{-1}R'R^{-1}U + R^{-1}U',$$

$$Q = (R^\sharp)^{-1}B \tag{21}$$

and the relations $RR^\sharp = K_0$, $R^\sharp R = K_1$. According to equations (5), (8), one finds that (7) and (10) transform to the matrix Riccati equations (MREs)

$$U' = M_0 - UK_0^{-1}U, \tag{22}$$
$$B' = -M_1 + BK_1^{-1}B. \tag{23}$$

These MREs can be linearized by an ansatz [12, 13]

$$U = VW^{-1}, \qquad V, W \in \mathbb{C}^{2\times 2}, \qquad \det(W) \neq 0,$$
$$B = XY^{-1}, \qquad X, Y \in \mathbb{C}^{2\times 2}, \qquad \det(Y) \neq 0$$

to give the equation systems

$$\begin{pmatrix} V' \\ W' \end{pmatrix} = \begin{pmatrix} 0 & M_0 \\ K_0^{-1} & 0 \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}, \qquad \begin{pmatrix} X' \\ Y' \end{pmatrix} = -\begin{pmatrix} 0 & M_1 \\ K_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{24}$$

The 4×2 matrices $\begin{pmatrix} V \\ W \end{pmatrix}, \begin{pmatrix} X \\ Y \end{pmatrix} \in \mathbb{C}^{4\times 2}$ are defined up to $GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$-transformations

$$\begin{pmatrix} \tilde{V} \\ \tilde{W} \end{pmatrix} = \begin{pmatrix} VG_0 \\ WG_0 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{X} \\ \tilde{Y} \end{pmatrix} = \begin{pmatrix} XG_1 \\ YG_1 \end{pmatrix}, \qquad G_0, G_1 \in GL(2,\mathbb{C})$$

and can be interpreted as homogeneous coordinates of two points on a complex Grassmann manifold $G_2(\mathbb{C}^4)$ which consists of 2-dimensional complex subspaces in $\mathbb{C}^4$ (see, e.g. [12, 13]). The matrices $U = VW^{-1}$ and $B = XY^{-1}$ are the corresponding affine coordinates of these points.

By differentiating (24) and substituting $V = K_0W'$, $X = -K_1Y'$ it is easily seen that the equation systems (24) are equivalent to the second-order matrix differential equations

$$(\partial_r K_0 \partial_r - M_0)\, W = 0,$$
$$(\partial_r K_1 \partial_r - M_1)\, Y = 0. \tag{25}$$

This implies that the matrices $\tilde{W} = r^{-1}W$, $\tilde{Y} = r^{-1}Y$ should be formal (non-normalized) solutions of the eigenvalue equations for the dynamo operator matrices $\hat{H}_{l_0}[\alpha_0]$, $\hat{H}_{l_1}[\alpha_1]$, respectively

$$\hat{H}_{l_0}[\alpha_0]\tilde{W} = E\tilde{W}, \qquad \hat{H}_{l_1}[\alpha_1]\tilde{Y} = E\tilde{Y}.$$

Similar like in QM models, the intertwining operator matrix $\hat{A}$ can be expressed in terms of W or Y. With the help of (20), (21) and (24) one finds

$$\hat{A} = R\left(ip - Y'Y^{-1}\right) = \left(ip + K_0W'W^{-1}K_0^{-1}\right) R.$$

Additionally, the interlinking constraint (15) induces a product invariant for the matrices W and Y. The latter can be obtained by substitution of

$$U = RR^\sharp W'W^{-1}, \qquad B^\sharp = -(Y^\sharp)^{-1}(Y^\sharp)'R^\sharp R$$

into the slightly modified version of the interlinking constraint (15)

$$U = RB^\sharp R^{-1} - R(R^\sharp)'.$$

As intermediate relation one obtains

$$W'W^{-1} = -(R^\sharp)^{-1}(Y^\sharp)^{-1}(Y^\sharp)'R^\sharp - (R^\sharp)^{-1}(R^\sharp)'$$

which is of Lie algebra conjugation type $g = g_1 n$, $(\partial_r g)g^{-1} = (\partial_r g_1)g_1^{-1} + g_1(\partial_r n)n^{-1}g_1^{-1}$ and which can be integrated to yield the product invariant

$$Y^\sharp R^\sharp W = C, \qquad \det(C) \neq 0,$$

with C a constant non-singular matrix.

So far, the intertwining technique of pseudo-Hermitian QM is generalized to the η-symmetric dynamo operator model. It remains to test whether the MREs of this model are consistent.

4 No-go theorem

In order to test the pair of MREs (22), (23) for consistency, one can use equations (15), (16) as well as the relation

$$N + K_1^{-1}N^\sharp K_1 = K_1^{-1}K_1' = K_1'$$

and transform the MRE for U (equation (22)) into an equivalent MRE for B. As result, one arrives at the following pair of MREs

$$B' = R^{-1}M_0R - K_1^{-1}BB + BK_1' + \left[NN^\sharp + (N^\sharp)'\right]K_1, \tag{26}$$

$$B' = -M_1 + BK_1^{-1}B, \tag{27}$$

which should be satisfied simultaneously. The corresponding consistency test will be performed in two steps:

1. From the limiting behavior at $r \to 0$ a relation between l_0 and l_1 will be derived.
2. From equations (26), (27) a system of non-linear ODEs will be extracted for the helical turbulence functions α_0, α_1 and for the components $b_1, \ldots, b_4$ of the matrix B. By mutual substitutions of these ODEs an inconsistency will be found which can be interpreted as a no-go theorem.

4.1 Limiting behavior at $r \to 0$

From the assumed non-singular behavior of the helical turbulence functions at $r \to 0$ it follows that they can be approximated as

$$\alpha_{0,1}(r \to 0) \approx c_{0,1} + a_{0,1}r + \mathcal{O}\left(r^2\right), \qquad c_{0,1} \neq 0.$$

Substituting this approximation in a slightly rewritten version of the defining equation (25) for the matrix Y

$$\left[I\partial_r^2 - \alpha_1'\sigma_-\partial_r - \frac{l_1(l_1+1)}{r^2}I - \begin{pmatrix} E & -\alpha_1 \\ \alpha_1 & E-\alpha_1^2 \end{pmatrix}\right]Y = 0 \tag{28}$$

one obtains the estimate

$$Y(r \to 0) \approx r^{-l_1}\left(I + \frac{a_1}{2}\sigma_- r + \mathcal{O}\left(r^2\right)\right)\left(r^{2l_1+1}C_+ + C_-\right),$$

where C_+, C_- are arbitrary non-singular constant matrices $\det(C_\pm) \neq 0$. Correspondingly, it holds

$$Z := Y'Y^{-1} \approx -l_1 r^{-1}I + \frac{a_1}{2}\sigma_- + \mathcal{O}(r), \tag{29}$$

$$B = -K_1Y'Y^{-1} \approx l_1 r^{-1}I - \left[c_1 l_1 r^{-1} + (l_1 + 1/2)a_1\right]\sigma_- + \mathcal{O}(r). \tag{30}$$

Comparison of (30) with (14) shows that the components b_2 and b_4 of the matrix B vanish at least as

$$b_2, b_4 \approx \mathcal{O}(r) \qquad \text{for } r \to 0.$$

Furthermore, one finds with the help of equations (17), (18) and (19) that $q \approx \mathcal{O}(r)$ and, hence, $a_0/c_0 = a_1/c_1$, as well as $q', f, f' \approx \mathcal{O}(1)$ what implies $N, N^\sharp, (N^\sharp)' \approx \mathcal{O}(1)$.

Now, a partial consistency test of (26) and (27) can be performed by comparing the singular terms of these equations in the vicinity of the origin $r = 0$. From the MREs (26) and (27) one finds

$$-K_1^{-1}K_1'Z - Z' = \frac{l_0(l_0+1)}{r^2}I - K_1^{-1}ZK_1Z - ZK_1' + \mathcal{O}(1), \tag{31}$$

$$-K_1^{-1}K_1'Z - Z' = -\frac{l_1(l_1+1)}{r^2}I + ZZ + \mathcal{O}(1), \tag{32}$$

respectively. Substituting Z from (29) and equating the coefficients of the r^{-2}-, r^{-1}-terms one obtains from equation (31)

$$l_1 = l_0 + 1, \qquad a_1 = 0$$

and, hence, also $a_0 = 0$. Equation (32) is automatically satisfied, because Y is defined by the corresponding linearized equation (28). The incremental relation $l_1 = l_0 + 1$ is well known from ladder operator constructions for spherically symmetric Hamiltonians in QM [4]. This is not surprising, because this ladder operator construction can be recovered from the intertwining construction (2) for the α^2-dynamo operator matrices by the two-step transition: 1. $\alpha_0 = \alpha_1 = \alpha$, 2. $\alpha \to 0$.

4.2 Systems of coupled non-linear ODEs and their inconsistency

The system of eight coupled non-linear ODEs for the components $b_1, \ldots, b_4$ of the matrix B is easily obtained from the MREs (26), (27), e.g. with the help of the matrix multiplication package of MATHEMATICA©. For the analysis it is sufficient to consider only the simplest four equations of this system, i.e. the σ_+ and I projections of (26) and (27):

$$b_2' = 2b_1b_2 + \alpha_1(1 + b_2^2) \tag{33}$$

$$= -2b_1b_2 - \frac{\alpha_0^2}{\alpha_1}, \tag{34}$$

$$b_1' = b_1^2 + b_2b_3 - b_4^2 - E - \frac{l_1(l_1+1)}{r^2} + \alpha_1b_1b_2, \tag{35}$$

$$= -b_1^2 - b_2b_3 + b_4^2 + E + \frac{l_0(l_0+1)}{r^2} - \alpha_1'b_2 + \frac{\alpha_0^2}{2} + q' - q^2. \tag{36}$$

Equating the right-hand sides of (33), (34) and using $b_2 = 2q/\alpha_1$ from (19) one expresses b_1 as

$$b_1 = -\frac{4q^2 + \alpha_0^2 + \alpha_1^2}{8q}. \tag{37}$$

Taking into account that $q = \partial_r \ln(\alpha_0/\alpha_1)/2$ according to equation (17) and that the helical turbulence functions α_0 and α_1 do not depend on l_0 or l_1 one concludes from equation (37) that b_1 should not depend on l_0 or l_1 too. On the other hand, addition of (35) and (36) together with the relation $l_0 = l_1 - 1$ gives

$$2b_1' = -\frac{2l_1}{r^2} + 2q\left(b_1 - \frac{\alpha_1'}{\alpha_1}\right) + \frac{\alpha_0^2}{2} + q' - q^2$$

what by integration leads to a function b_1 which depends on l_1. I.e. the term depending on l_1 cannot be compensated by a combination of l_1-independent terms. This is an obvious contradiction to (37) and one has to conclude that the consistency conditions (5)–(10) cannot be fulfilled simultaneously. This means that it holds following

No-go theorem. *The structure of the MHD α^2-dynamo operator matrix is incompatible with an operator intertwining technique which is based on first-order differential intertwining operators.*

A similar situation occurs also for three-dimensional spherically symmetric models in QM [4]. There the l-dependent centrifugal term sets so strong restrictions on the form of the allowed potential that an intertwining construction built on first-order differential intertwining operators is only possible for the following three cases: the constant potential $V(r) = \text{const}$, the Coulomb potential $V(r) \propto 1/r$, and the potential of the three-dimensional isotropic harmonic oscillator with $V(r) \propto r^2$. Richer classes of allowed potentials are only found for models in their s states, when $l = 0$. Such states are *a priori* excluded for the α^2-dynamo operator matrix due to its construction [2].

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The Metron Model: a Unified Deterministic Theory of Fields and Particles – a Progress Report

Klaus HASSELMANN and Susanne HASSELMANN

Max-Planck-Institute for Meteorology, Bundestrasse 55, 20146 Hamburg, Germany
E-mail: *klaus.hasselmann@dkrz.de*

Metron computations of the double-slit diffraction of a particle beam are in good agreement with quantum theory, demonstrating that wave-particle duality can be explained quantitatively within the framework of a deterministic particle model. Preliminary forward-modelling computations of the structure the electron are consistent with previous metron predictions from inverse modelling, but work on this complex numerical problem is still in progress.

1 Introduction

In a previous four-part paper [6] (see also [7]) the metron concept of a general unified deterministic theory of fields was developed. The present paper reports on recent numerical computations of some of the basic properties of this model.

The metron model is based on the premise that, in contrast to standard quantum field theory, particles exist as real objects. It is hypothesized that they represent non-linear soliton-type solutions of Einstein's gravitational equations in a higher (at least eight) dimensional matter-free space. The *metron* solutions (*metr*ic solit*ons*) are composed of a non-linear core region, which defines the particle properties (mass, charge, spin, weak and strong coupling constants, etc.), and a linear far-field region, which carries the classical gravitational and electromagnetic fields, as well as a high-frequency periodic field that satisfies the de Broglie dispersion relation. The de Broglie field alone, being oscillatory, exerts no mean force. However, mean forces are generated through quadratic interactions of the de Broglie field with its own scattered field, produced by interactions with other matter, thereby producing wave-like interference phenomena.

Apart from dimensional units defining the background metric (such as the velocity of light), the vacuum Einstein equations contain no physical constants. All particle properties and other physical constants must therefore follow from the structure of the particle solutions themselves.

In the particle rest-frame, the metron solutions are assumed to be periodic with respect to time (x^4) and the extra space coordinates ($x^5, \ldots, x^8$), in accordance with the original Kaluza–Klein method for removing the non-observed dependence of the metric on the extra-space coordinates. The wave-numbers k_4 and $k_5, \ldots, k_8$ define the mass, electroweak and strong coupling constants, respectively.

The bound-state metron solutions are generated through the mutual interaction between oscillatory and mean metric field components. The oscillatory components represent eigenmodes (and forced higher harmonics) which are trapped in the particle core by the mean metric field, which acts as a localized wave-guide. The mean wave-guide field, in turn, is generated by quadratic interactions (radiation stresses or currents) between the trapped eigenmode fields.

In addition to demonstrating that metron solutions exist, the metron model must resolve a number of other fundamental issues. These were addressed in [6] using inverse modelling methods (a more detailed summary is given in [6, Part 1]): Following the demonstration, for a simple scalar analogue of the Einstein equations, that bound-state solutions can indeed be

generated by the wave-mode/wave-guide trapping mechanism [6, Part 1], it was shown that for metron solutions with appropriate polarization structures the metron model is able to

1) recover both the gravitational forces and the electromagnetic forces of charged point-like particles, including spin, in accordance with the classical Einstein–Dirac–Maxwell system of quantum theory [6, Part 2];
2) reproduce qualitatively, through resonant interactions between primary and secondary de Broglie waves, the resonant Bragg scattering of particles at a periodic lattice and the basic structure of discrete atomic spectra [6, Part 3];
3) resolve the EPR paradox, while circumventing the non-existence theorem of Bell for hidden-variable theories (such as the metron model) by invoking the time-reversal symmetry of interactions at the microphysical level, as developed by Wheeler and Feynman [11,12] ([6, Part 3]);
4) explain the symmetries of the Standard Model of Quantum Field Theory in terms of geometrical symmetries of the metron solutions in extra space [6, Part 4].

In the present paper we extend these general results through more detailed numerical computations. First, we show that the metron model is able to reproduce quantitatively the results of the double-slit refraction experiment. We then apply the forward-modelling approach in first steps towards computing the metron solution for the simplest possible particle, the electron (which we have chosen rather than the more enigmatic neutrino, see [6, Part 4]). The paper concludes with an outlook on further work.

2 The double-slit refraction experiment

Few experiments have been more intensively discussed since the conception of quantum theory than the refraction of a particle beam at a double slit. If the beam is sufficiently weak, the position of an individual particle can be located before and after it has passed through the slits. It is readily understandable already from classical physics (through the uncertainties introduced through the measurement process) that the the initial and/or final position of the particle can not be determined with sufficient accuracy to decide through which of the two slits the particle has actually passed. However, the positivistic Copenhagen position, that it is therefore meaningless to state that the particle has passed through one or the other of the slits, represents a radical departure from the classical view that physics can be described by real, existing objects. Physicists have therefore sought alternative explanations of the wave-like interference patterns of the double-slit experiment that are consistent with classical physics.

A well known alternative classical interpretation is the guiding- or pilot-wave model of de Broglie [3–5] and Bohm [2], popularized also by Bell [1]. The basic concept is that the interference patterns are generated through a guiding "pilot wave" that accompanies the particles. However, a basic difficulty of this model is the rather nebulous origin of the pilot wave and its interaction with the particle that it guides. The metron model offers a simple interpretation of the de Broglie–Bohm model: the guiding "pilot wave" is none other than the de Broglie wave component of the metron bound-state solution. Its interaction with the particle that it guides arises through the interaction of the primary de Broglie wave with the scattered or refracted secondary wave that is generated when the primary wave interacts with another object. Following the qualitative application of this concept to Bragg scattering and the structure of discrete atomic spectra in [6, Part 3], we show now that the interaction between the intrinisic primary and refracted secondary de Broglie wave field is able to reproduce quantitatively the detailed refraction pattern of the double-slit experiment.

Consider a homogeneous monochromatic beam of particles travelling in the x-direction that impinges on a screen at $x = 0$ that contains two slits, $a/2 < |y| < a/2 + d$, of width d and separation a centred at $y = 0$. According to quantum theory, the incident beam is described by a state function $\psi_i(x, y) = \exp(ikx)$, where $k = mU/\hbar$, m is the particle mass, U the particle velocity, and we have dropped irrelevant frequency dependent and normalization factors. In the first-order Kirchhoff approximation, the refracted wave function is given, for $y \ll x$, by

$$\psi_r(x, y) = (2d/z) \exp(ikx)[\sin\{(1 + \alpha)z\} - \sin(\alpha z)], \tag{1}$$

where $z = kd\theta$, $\theta = y/x$ and $\alpha = a/2d$.

We compare now quantum theoretical refracted particle flux proportional to $|\psi_r|^2$ with the refracted beam computed using the metron model. An individual particle approaching the screen on a path $x = Ut$, $y = y_0 = \text{const}$ (where y_0 lies in one of the slits) carries with it a de Broglie field $\hat{\psi}_i = f_0(x - Ut, y - y_0)\psi_i$, where ψ_i is the same incident field as in the quantum theoretical case, and $f_0(x - Ut, y - y_0)$ is an additional modulation factor describing the decrease of the (primary) metron de Broglie field with distance from the particle core. For the refracted (secondary) de Broglie field we obtain, in analogy to equation (1), the field $\hat{\psi}_r(x, y) = f_r(x - Ut, y - y_0)\psi_r(x, y)$, where $f_r(x - Ut, y - y_0)$ is a further modulation factor describing the decrease of the refracted field with distance from a (hypothetical) reference particle on the non-refracted particle path. The relations for both $\psi_r(x, y)$ and $\hat{\psi}_r(x, y)$ apply in the far-field region behind the slits.

To compute the path of the refracted particle, we need to compute the force exerted on the particle by the non-linear interaction between the primary and secondary de Broglie fields. This is governed by a potential function Φ, which is determined by the product $\hat{\psi}_r\hat{\psi}_p^*$ of the refracted secondary field $\hat{\psi}_r(x, y)$ with the particle's (complex conjugate) primary field $\hat{\psi}_p^*$, integrated over the core region of the particle (see [6, Parts 2 and 3]). The phase relation between the primary and secondary fields (with respect to the periodicity in $z = kd\theta$) depends on the path of the refracted particle relative to its refracted field.

Assume that in passing through the near-field of the slits, the particle is deflected by the near-field interactions into some initial direction θ_0, therefore propagating first along a path $x = Ut$, $y = y_0 + \theta_0 Ut$ before entering into the far-field region. The evolution of the further particle path $y(x)$ in the far-field region is then determined by the path equations

$$\frac{d\theta}{dx} = \frac{v - \theta}{x}, \qquad \frac{dv}{dx} = U\frac{\partial \Phi}{\partial y}, \tag{2}$$

for $\theta = \theta(x)$, $v = v(x)$, where $y = \theta x$, $v = U dy/dx$, with initial conditions $\theta(0) = \theta_0$, $v(0) = v_0$. The phase of the refracted secondary de Broglie field relative to the intrinsic primary de Broglie field of the particle depends on the point y_0 at which the particle passes the slit, together with the direction θ of the particle path. If one averages over all positions y_0 in the two slits, one obtains the same phase for the intrinsic de Broglie field as for the refraction amplitudes ψ_r and $\hat{\psi}_r(x, y)$. Taking this average phase value as a first approximation, and noting that the gradient of the potential Φ is dominated by the oscillatory dependence on z, we find that the net force acting on the refracted particle is proportional to $\partial\Phi/\partial y \approx \text{const}\, \partial|\psi_r|^2/\partial y$. For const > 0, the force has the effect of focussing the refracted particles towards the maximal values of $|\psi_r|^2$, i.e. into the regions of maximum refracted particle flux as given by quantum theory.

That this is indeed the case is demonstrated in Fig. 1[1]: the metron computation agrees very closely with the quantum-theoretical result. The average-phase approximation of the metron computation was replaced here by a more accurate Monte Carlo computation for an ensemble of 10,000 particles, in which the phase of the primary wave of each particle was computed separately as a function of the point y_0 at which the particle passes through one of the slits and

[1]Figures in colour will be available only in electronic version.

the particle path, given the initial path values θ_0, v_0. The values y_0 were distributed uniformly over the two slits, while the (statistically independent) distributions of the initial values θ_0, v_0 were taken as Gaussian (see Fig. 1). Other adjusted model parameters were the magnitude and rate of decrease of the modulation function f_r with distance behind the slit. A finite spectral broadening of the incident particle flux was assumed with respect to both magnitude and direction of the particle velocities, as in the refraction experiments of Zeilinger *et al* [13] and Tschernitz *et al* [10] using low-energy neutrons. The authors found good agreement between quantum theory and experiment if in addition to the adjustment of these parameters a suitable background neutron flux was fitted to the data. At the present level of accuracy, however, Fig. 1 suggests that quantum theory cannot be distinguished from the metron model.

Since the quantum theoretical and metron computation of the diffracted particle fluxes in neutron diffraction experiments exhibit real differences in detail, an experimental distinction between the two models should become feasible, once all parameters that enter in the computations are accurately determined. Further careful experiments on the diffraction of neutrons under different conditions, together with more detailed computations of the relevant metron parameters (based on a complete representation of the metron solutions, which is presently lacking), should ultimately lead to a clarification of which of the two models provides a more accurate description of particle interference phenomena.

3 Numerical computation of metron solutions: preliminary results

Metron theory hypothesizes that particles represent bound-state solutions of the Einstein vacuum equations

$$E_{LM} = 0, \tag{3}$$

where E_{LM} is the Einstein tensor in a higher- (we assume 8-) dimensional space ($L, M = 1, \ldots, 8$) characterized by a background Minkowski metric $\eta_{LM} = \mathrm{diag}\,(1, 1, 1, -1, 1, 1, 1, 1)$. We seek to construct a solution as a perturbation series

$$g_{LM} = \eta_{LM} + \sum_p g^{(p)}_{LM}, \tag{4}$$

with metric perturbations $g^{(p)}_{LM}$ that are periodic with respect to time and the extra-space coordinates and isotropic with respect to three-dimensional physical space,

$$h^{(p)}_{LM} = f^{(p)}(r) P^{(p)}_{LM} \exp i \left(k^{(p)}_A x^A \right), \tag{5}$$

where

$$h^{(p)}_{LM} := g^{(p)}_{LM} - \frac{1}{2} \eta_{LM} \left(\eta^{NO} g^{(p)}_{NO} \right) \tag{6}$$

is the trace-reversed perturbation, $r = (x^i x_i)^{1/2}$ ($i = 1, 2, 3$), $P^{(p)}_{LM}$ is a constant polarization matrix and the wave-numbers $k^{(p)}_4$ (frequency), and $k^{(p)}_A$, $A = 5, \ldots, 8$ (extra-space wave-numbers) denote the fundamental and higher-harmonic components with respect to a small number (in our application, two, in general, maximally five) base wavenumbers (where "higher harmonic" includes the zero wave-number, $k_A = 0$, associated with the wave-guide).

Substitution of the perturbation expansion into the Einstein equations yields a set of coupled field equations of the general form

$$\left(\partial^N \partial_N + e_n \right) \phi_n = \left\{ \frac{d^2}{dr^2} + 2 \frac{d}{dr} - \hat{\omega}^2 + e_n \right\} \phi_n = q_n, \tag{7}$$

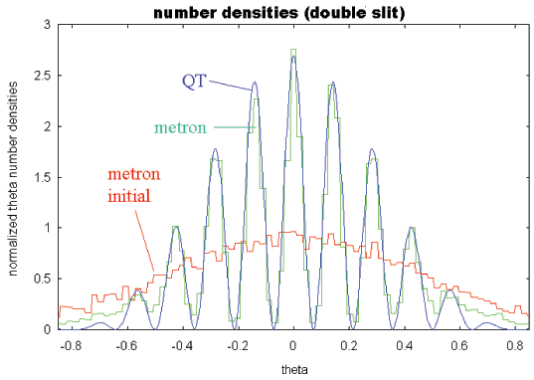

Figure 1. Number densities of refracted particles computed for quantum theory and metron model. Also shown is the assumed near-slit distribution for the metron case.

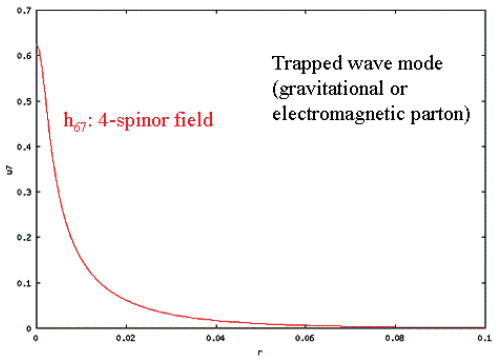

Figure 2. Trapped mode component h_{67} (representing a 4-spinor field) for the gravitational constituent ($k_4 = 999,55$, $k_5 = 1000$, $\hat{\omega} = 30$). The associated wave-guide components are shown in Fig. 3. A similar form is found for the electromagnetic constituent, ($k_4 = 0$, $k_5 = 30$, $\hat{\omega} = 30$), which yields the wave-guide components shown in Fig. 4.

where ϕ_n denotes the radial function $f^{(p)}(r)$ of a given constituent $h^{(p)}_{LM}$, $\hat{\omega}^2 = k_A k^A$, and the eigenvalue operator e_n and forcing term q_n are nonlinear functions of the set of all constituents. The index n distinguishes between first-harmonic ($n = 1$), higher-harmonic ($n > 1$) and wave-guide ($n = 0$) components. In addition, the fields must satisfy appropriate divergence gauge conditions.

For the lowest (cubic) interaction order, the equations for the first-harmonic constituents ($n = 1$) reduce to a quasi-linear eigenvalue problem, with $q_1 = 0$, where the operator e_1 is quadratic in the first-harmonic constituents and linear in the zero- and second-harmonic constituents. The corresponding equations for the wave-guide ($n = 0$) and forced waves ($n = 2$) are characterized by zero eigenvalue operators ($e_n = 0$) and non-zero source functions q_n that depend quadratically on the first-harmonic fields.

The challenge is to find stationary, bound-state solutions of the complete set of equations. To search for such bound-state solutions of the Einstein vacuum equations in eight-dimensional space, a three-layered system of programs was developed. The first program layer translates the Einstein tensor equations into a set of 52 coupled differential equations for the different trapped-mode, wave-guide and forced-wave constituents associated with the different wave-number base vectors. A valuable check on the algebra of the program was the divergence condition $E^M_{L,M} = 0$, valid for any metric field. The second program layer computes the coupling coefficients for the set of differential equations. The third layer, finally, searches for solutions of the non-linear eigenvalue problem, using various iteration methods.

In accordance with the wave-number properties and polarization relations deduced from the inverse modelling analysis of [6, Parts 3 and 4], the electron is characterized by two wave-number vectors lying in the x^4, x^5 plane, while the first-harmonic constituents, corresponding to 4-spinors, were represented by a non-diagonal polarization matrix component P_{67} in the adjacent extra space. The search for bound-state solutions confirmed that this choice was consistent: nonlinear interactions with the zero- and second-order harmonic fields generated no other first-harmonic constituents of different polarization structure.

To obtain periodic far-fields that fall off approximately as r^{-1} rather than exponentially, as required for a de Broglie far field that generates interference effects over large distances, the wave-number components of one of the first-harmonic constituents must be approximately equal: $k_4 \approx k_5$ (so that $\hat{\omega}^2 \approx 0$ in (7)). We refer to this constituent as the gravitational

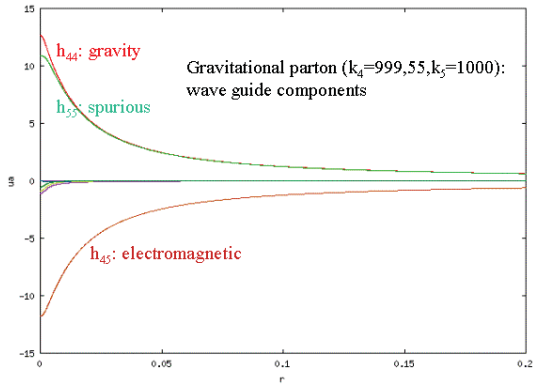

Figure 3. Wave-guide components generated by the gravitational trapped mode shown in Fig. 2. The gravitational (h_{44}), electromagnetic (h_{45}) and spurious extra-space (h_{55}) fields are approximately equal in magnitude.

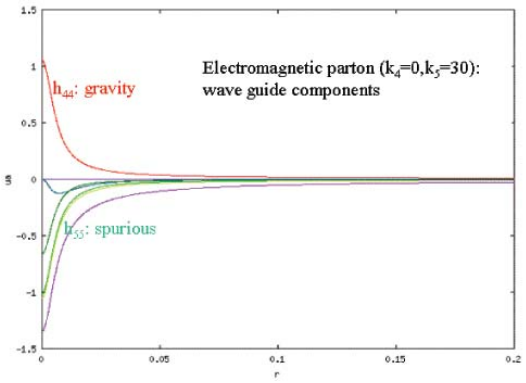

Figure 4. Wave-guide components generated by the electromagnetic trapped mode. There is no electromagnetic field ($h_{45} = 0$), while the gravitational (h_{44}) and spurious extra-space (h_{55}) fields are of comparable magnitude. The remaining extra-space fields do not exert forces on the particles for $k_6, k_7, k_8 = 0$.

fermionic mode. However, since the strength of the forces experienced by a metron particle is proportional to the wave-number components (k_4, corresponding to mass and gravitational coupling, and k_5, corresponding to charge and electromagnetic coupling, cf. [6, Part 2]), the great disparity ($\sim 10^{-39}$) between the strengths of the gravitational and electromagnetic forces requires that for the second first-harmonic constituent, the wave-number components must then satisfy the inequality $k_4 \ll k_5$. We set $k_4 = 0$, and refer to this constituent as the electromagnetic fermionic mode.

As preparation for the computation of the fully coupled system, we computed first the coupling between the first-harmonic fields and their associated zero-harmonic wave-guide and second-harmonic forced-wave fields separately for the gravitational and electromagnetic fermionic modes (Figs. 2–4). Quasi-bound states were computed through an iterative adjustment procedure analogous to that described in [6, Part 1].

Fig. 2 shows the resulting trapped-mode field for the gravitational fermionic mode; the electromagnetic fermionic mode (not shown) has a similar form. The gravitational first-harmonic field (with assumed wave-number vector components $k_4 = 999,55$, $k_5 = 1000$, yielding $\hat{\omega}^2 = k_5^2 - k_4^2 = 30^2$) generates gravitational (h_{44}) and electromagnetic (h_{45}) far fields, as well as a spurious extra-space (h_{55}) far field, all of nearly the same magnitude (Fig. 3). The corresponding zero-harmonic wave-guide components generated by the electromagnetic fermionic mode (with $k_4 = 0$, $k_5 = 30$, chosen to yield the same exponential fall off, $\hat{\omega} = 30$, as the gravitational mode) are shown in Fig. 4. The electromagnetic far-field is zero in this case. We denote nevertheless the first-harmonic fermionic mode that produces the far fields as the electromagnetic fermionic mode, as it carries the charge wave-number component k_5 that generates the electromagnetic forces (see [6, Part 2]). The second-harmonic fields (not shown) have similar structures and magnitudes as the zero-harmonics, but decrease exponentially rather than as r^{-1} for large r. The fields shown represent only quasi-converged bound states: to achieve convergence of our numerical iteration scheme, the r-scales of the first-harmonic fermionic fields needed to be adjusted by about 10–20 % relative to the scales of the forced zero- and second-harmonic fields.

The question of convergence is not critical, however, at this level of analysis, as it needs to be addressed again for the next step: the construction of a realistic model of the electron by combining the gravitational and electromagnetic fermionic modes and the sets of far fields and second-harmonic fields that they generate into a single model, including the cross-coupling between the two systems. For a realistic electron model, the coupled system must reproduce

the 10^{-39} disparity in the magnitudes of the gravitational and electromagnetic forces (requiring very small ratios of the amplitude and/or wave-number components $k_4 \approx k_5$ of the gravitational fermionic mode relative to the amplitude and/or wave-number component k_5 of the electromagnetic fermionic mode) and at the same cancel the spurious h_{55} wave-guide far field, which would otherwise create an additional (non-observed) force by coupling into the charge k_5. (This problem was encountered already by Kaluza [8] in his original five-dimensional generalization of Einstein's equations, considered later also by Klein [9] and revived fifty years later in general higher-dimensional gravity theory.)

The determination of bound-state solutions for the coupled problem of two fermionic modes requires more sophisticated iteration methods, still in development, than applied above. However, preliminary numerical experiments indicate that solutions with the required properties appear feasible; indeed, an amplitude of the gravitational fermionic mode that is considerably smaller than obtained in the analysis of the individual systems (cf. Fig. 3) is a prerequisite for generating a strongly reduced spurious h_{55} wave-guide field by this component, that is then able to cancel the corresponding spurious h_{55} field generated by the electromagnetic fermionic mode, cf. Figs. 3 and 4 (this is also necessary to justify a perturbation approach for the wave-guide fields generated by the gravitational fermionic mode). It is hoped that the validity of the general metron concept explored in this paper, or possibly the need for revisions (for example, by extension to higher dimensions, or the inclusion of higher order terms in the perturbation expansion) can soon be decided.

4 Outlook

The demonstration that the metron model is able to reproduce the properties of the simplest elementary particle, the electron, is clearly an essential first step in the development of the theory. Assuming that this can be achieved, one then faces the challenge of translating the qualitative results listed in the introduction above into quantitative analysis, using forward rather than inverse modelling methods. However, the task is perhaps not so daunting as it appears at first sight. The program codes developed for the analysis of the two-fermion-component electron system are designed to be applicable also for the five-fermion-component system required to reproduce the particle spectrum of the Standard Model, including gravitational, elecro-weak and strong interactions.

Furthermore, all fields that arise in standard quantum field theory, as well as in quantum theory and classical gravity, appear also in the metron model. The only difference is that the metron model contains also a particle-core region, which acts as the source of the fields, and that the fields represent real physical objects rather than abstract mathematical quantities used to predict the probability outcomes of experiments. However, the nonlinear interactions between the different field constituents are basically the same in both theories, so that most of the computations of quantum theory and quantum field theory can be directly translated into equivalent computations for the metron model.

It has been pointed out by Wolfgang Kundt (private communication) that the presentation of the metron model in terms of higher-dimensional gravity can be replaced by an aesthetically more attractive formulation in standard four-dimensional space-time. This is because all fields in the metron model are assumed to be periodic with respect to the extra-space coordinates, so that the only non-trivial spatial dependence is with respect to four-dimensional space-time. The tensor components of the metric fields in higher-dimensional space can therefore be treated as a fiber bundle dependent only on space-time. The transformation relations of the fibre-bundle fields with respect to coordinate transformations in four-dimensional space-time inferred from higher-dimensional gravity then ensure (for a given set of wave-number vectors) that the extended system of Einstein equations, including all components of the fiber bundle, is invariant

with respect to coordinate transformations in four-dimensional space-time. A reformulation of the metron model using this terminology would be an attractive task.

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Dynamical Space-Time Symmetry for Ageing Far from Equilibrium

Malte HENKEL

Laboratoire de Physique des Matériaux CNRS UMR 7556, Université Henri Poincaré Nancy I, B.P. 239, F – 54506 Vandœuvre-lès-Nancy, France
E-mail: *henkel@lpm.u-nancy.fr*

The dynamical scaling of ageing ferromagnetic systems can be generalized to a local scale invariance. This yields a prediction for the causal two-time response function, which has been numerically confirmed in the Glauber–Ising model quenched into the ordered phase. For a dynamical exponent $z = 2$, a new embedding of the Schrödinger group into the conformal group and the resulting conditions for the validity of local scale invariance are discussed.

1 Phenomenology of ageing in simple ferromagnets

Ageing phenomena provide a paradigmatic example of collective behaviour far from equilibrium and have received a lot of attention in recent years [5, 8, 11]. Ageing has been observed first in glassy systems, but for an improved conceptual understanding, it might be more useful to study first ageing phenomena in the simpler ferromagnetic systems, as we shall do here. We consider a ferromagnet with a critical temperature $T_c > 0$ and prepare the system in some initial state (which typically may be disordered). Then the system is suddenly brought into contact (quenched) with a heat bath of temperature $T < T_c$ (or $T \leq T_c$). With T fixed, the system's evolution towards its equilibrium state at temperature T is observed. For definiteness, consider an Ising-like system with a microscopic degree of freedom $\sigma_{\boldsymbol{i}} = \pm 1$ which can only take two possible values, and where $\boldsymbol{i}$ denotes the sites of a (hypercubic) lattice.

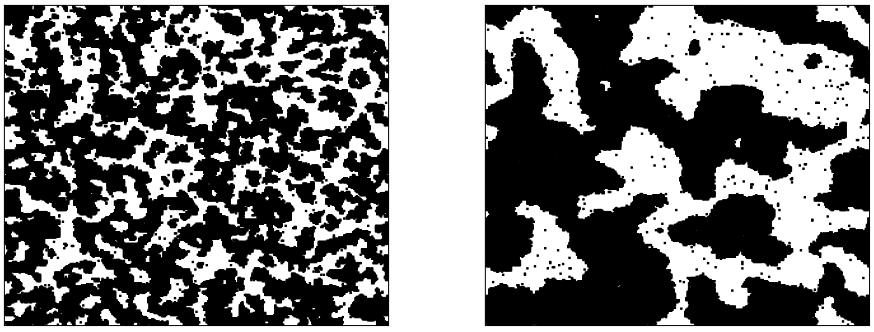

Figure 1. Snapshot of the coarsening of ordered domains in the $2D$ Glauber–Ising model, after a quench to $T = 1.5 < T_c$ from a totally disordered state and at times $t = 25$ (left) and $t = 275$ (right) after the quench. The black and white colors indicate the value of the Ising spins.

A typical example of the microscopic evolution of such a system is shown in Fig. 1. Quite rapidly, ordered domains are formed which slowly move and grow. Empirically, it is found that the typical size of the domains scales with the time t as $L(t) \sim t^{1/z}$, where z is the dynamical exponent. It is known that for dynamical rules chosen such that there is no macroscopic conservation law, $z = 2$ for quenches to $T < T_c$. The slow temporal evolution of macroscopic observables we are interested in results from the slow motion of the domain walls. It is common to consider the coarse-grained order parameter (e.g. mean magnetization for magnetic systems)

$\phi(t,\boldsymbol{r})$ and one tries to capture its time-evolution through a stochastic Langevin equation

$$\frac{\partial\phi}{\partial t} = -\frac{\delta\mathcal{H}[\phi]}{\delta\phi} + \eta, \tag{1}$$

where the Gaussian white noise $\eta = \eta(t,\boldsymbol{r})$ is characterized by its first two moments $\langle\eta(t,\boldsymbol{r})\rangle = 0$ and $\langle\eta(t,\boldsymbol{r})\eta(t',\boldsymbol{r}')\rangle = 2T\delta(t-t')\delta(\boldsymbol{r}-\boldsymbol{r}')$ and $\mathcal{H}$ is the Ginzburg–Landau functional. For systems undergoing a conventional second-order phase-transition, one expects that qualitatively

$$\mathcal{H}[\phi] = \phi\Delta\phi + \mathcal{V}[\phi], \qquad \mathcal{V}[\phi] \sim \begin{cases} \phi^2, & \text{if } T > T_{\mathrm{c}}, \\ \left(\phi^2 - \phi_0^2\right)^2, & \text{if } T < T_{\mathrm{c}}, \end{cases} \tag{2}$$

where $\phi_0 = \phi_0(T)$ are the two equilibrium values of ϕ and Δ is the spatial Laplacian. Physically, systems with $T > T_{\mathrm{c}}$ and $T < T_{\mathrm{c}}$ are very different, since in the first case, there is a single ground-state (where $\mathcal{H}[\phi] = \min!$) while there are two distinct ground states in the second case. Therefore, for $T > T_{\mathrm{c}}$ the system will rapidly relax towards its single ground-state and no ageing occurs. On the other hand, if $T < T_{\mathrm{c}}$ it will depend on the microscopic environment of each spin variable to which of the two possible local ground-states $\pm\phi_0$ the system will evolve locally. The competition between these distinct states then leads to ageing phenomena.

The noisy Langevin equation can be turned into an equivalent field theory through the Martin–Siggia–Rose formalism, see e.g. [8]. Schematically, it may be represented through the effective action

$$S[\phi,\widetilde{\phi}] = \int \mathrm{d}t\,\mathrm{d}\boldsymbol{r}\left[\widetilde{\phi}\left(\frac{\partial\phi}{\partial t} + \frac{\delta\mathcal{H}[\phi]}{\delta\phi}\right) - T\widetilde{\phi}^2\right], \tag{3}$$

where $\widetilde{\phi}$ is the so-called response field conjugate to ϕ. From this, the classical equations of motion take the form

$$\frac{\partial\phi}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\phi} + 2T\widetilde{\phi}, \qquad \frac{\partial\widetilde{\phi}}{\partial t} = \frac{\delta^2\mathcal{H}}{\delta\phi^2}\,\widetilde{\phi} \tag{4}$$

but initial conditions must still be specified. For interacting fields, fluctuation effects are not taken into account by the equations (4), although they are present in the action S and the associated path integral. It turns out, see [11] for a review, that ageing is more fully revealed through the study of two-time correlators $C(t,s)$ and response functions $R(t,s)$ defined by

$$C(t,s) = \langle\phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\rangle, \qquad R(t,s) = \left.\frac{\delta\langle\phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{r})}\right|_{h=0} = \langle\phi(t,\boldsymbol{r})\widetilde{\phi}(s,\boldsymbol{r})\rangle, \tag{5}$$

where $h(s,\boldsymbol{r})$ is the local magnetic field at time s and the position $\boldsymbol{r}$. The last equation comes from Martin–Siggia–Rose theory. Furthermore, $\langle\widetilde{\phi}\widetilde{\phi}\rangle = 0$.

Definition 1. A statistical system described by a Langevin equation (1) or an effective action (3) is said to undergo *ageing*, if $C = C(t,s)$ or $R = R(t,s)$ depend on both the *observation time* t and the *waiting time* s and not merely on the difference $\tau = t - s$.

If the times t, s and $t-s$ become large simultaneously (as compared to some microscopic time scale t_{micro}), one usually finds, for $T < T_{\mathrm{c}}$, the following scaling behaviour, see e.g. [11]

$$\begin{aligned} C(t,s) &\simeq M_{\mathrm{eq}}^2 f_C(t/s), & f_C(x) &\sim x^{-\lambda_C/z}, & x &\to\infty, \\ R(t,s) &\simeq s^{-1-a} f_R(t/s), & f_R(x) &\sim x^{-\lambda_R/z}, & x &\to\infty, \end{aligned} \tag{6}$$

where M_{eq} is the equilibrium magnetization and $\lambda_{C,R}$ are the autocorrelation [9] and autoresponse [23] exponents, respectively. If long-range initial correlations of the form $C_{\mathrm{ini}}(\boldsymbol{r}) \sim$

$|\boldsymbol{r}|^{-d-\alpha}$ are used, where d is the number of space dimensions and α a free parameter, one has for *ferro*magnets the rigorous bound $\lambda_C \geq (d+\alpha)/2$ [27] (it need not hold for disordered systems [26]). Furthermore, if $\alpha < 0$, the relationship $\lambda_C = \lambda_R + \alpha$ has been conjectured [23], while $\lambda_C = \lambda_R$ for a fully disordered initial state is generally accepted. Finally, the value of the exponent a has recently been shown [16] by scaling arguments to depend on the equilibrium spin-spin correlator $C_{\rm eq}$ as follows. Systems of *class S* have short-ranged correlators $C_{\rm eq}(\boldsymbol{r}) \sim e^{-|\boldsymbol{r}|/\xi}$ and systems of *class L* have long-ranged correlators $C_{\rm eq}(\boldsymbol{r}) \sim |\boldsymbol{r}|^{-(d-2+\eta)}$. Then [16]

$$a = \begin{cases} 1/z, & \text{class S,} \\ (d-2+\eta)/z, & \text{class L.} \end{cases} \tag{7}$$

This concludes our review of those main properties of ageing systems which we shall need below.

2 Beyond scale invariance

Our central question about ageing is the following: Is there a general, model-independent way to predict the form of the scaling functions $f_C(x)$ and $f_R(x)$ as defined in (6)?

The possibility of an affirmative answer might be suggested by the following known facts: (i) Ageing phenomena show a dynamical scaling, i.e. they are scale-invariant. (ii) In equilibrium critical phenomena, the invariance of the theory under dilatations $\boldsymbol{r} \mapsto b\boldsymbol{r}$ may be extended to local scale or conformal transformations $\boldsymbol{r} \mapsto b(\boldsymbol{r})\boldsymbol{r}$ (such that angles are kept fixed). It is well-known that conformal invariance is a very powerful principle in two-dimensional equilibrium critical phenomena, see e.g. [13]. More precisely, we inquire whether the global dynamical scale transformations $t \mapsto b^z t$ and $\boldsymbol{r} \mapsto b\boldsymbol{r}$ can be extended, analogously to conformal invariance where $z = 1$, to space-time dependent dilatation factors $b = b(t, \boldsymbol{r})$ in a meaningful way.

Example 1. Let $z = 2$ and consider d space dimensions. The *Schrödinger group* $Sch(d)$ is defined by [20]

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \boldsymbol{r} \mapsto \frac{\mathcal{R}\boldsymbol{r} + \boldsymbol{v}t + \boldsymbol{a}}{\gamma t + \delta}, \qquad \alpha\delta - \beta\gamma = 1, \tag{8}$$

where $\mathcal{R} \in SO(d)$, $\boldsymbol{a}, \boldsymbol{v} \in \mathbb{R}^d$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. It is well-known that $Sch(d)$ is the maximal kinematic group of the free Schrödinger equation $\mathcal{S}\psi = 0$ with $\mathcal{S} = 2mi\partial_t - \partial_{\boldsymbol{r}}^2$ [20] (that is, it maps any solution of $\mathcal{S}\psi = 0$ to another solution). Additional examples of $Sch(d)$ as a kinematic group include certain non-linear Schrödinger equations [21], systems of Schrödinger equations [6] and the Euler equations of fluid dynamics [22]. We denote the Lie algebra of $Sch(d)$ by $\mathfrak{sch}_d$. Specifically, $\mathfrak{sch}_1 = \overline{\{X_{\pm 1,0}, Y_{\pm 1/2}, M_0\}}$ with the non-vanishing commutation relations

$$[X_n, X_{n'}] = (n-n')X_{n+n'}, \qquad [X_n, Y_m] = \left(\frac{n}{2} - m\right) Y_{n+m}, \qquad \left[Y_{1/2}, Y_{-1/2}\right] = M_0, \tag{9}$$

where $n, n' \in \{\pm 1, 0\}$ and $m \in \{\pm 1/2\}$.

Example 2. For a more general dynamical exponent $z \neq 2$, we construct infinitesimal generators of local scale transformations from the following requirements [15] (for simplicity, set $d = 1$): (a) Transformations in time are $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$ with $\alpha\delta - \beta\gamma = 1$. (b) The generator for dilatations is $X_0 = -t\partial_t - z^{-1} r\partial_r - x/z$, where x is the scaling dimension of the fields ϕ, $\widetilde{\phi}$ on which the generators act. (c) Space-translation invariance is required, with generator $-\partial_r$. Starting from these conditions, we can show by explicit construction that there exist generators X_n, $n \in \{\pm 1, 0\}$ and Y_m, $m = -1/z, 1 - 1/z, \ldots$ such that

$$[X_n, X_{n'}] = (n-n')X_{n+n'}, \qquad [X_n, Y_m] = \left(\frac{n}{z} - m\right) Y_{n+m}. \tag{10}$$

For generic values of z, it is sufficient to specify the 'special' generator

$$X_1 = -t^2\partial_t - Ntr\partial_r - Nxt - \widetilde{\alpha}r^2\partial_t^{N-1} - \widetilde{\beta}r^2\partial_r^{2(N-1)/N} - \widetilde{\gamma}\partial_r^{2(N-1)/N}r^2 \tag{11}$$

explicitly, where we wrote $z = 2/N$ and $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\gamma}$ are free constants. Further non-generic solutions exist for $N = 1$ and $N = 2$ [15]. In particular, the generator (11) reproduces for $z = 2$ those of $Sch(1)$. The condition $[X_1, Y_{N/2}] = 0$ is only satisfied if either (I) $\widetilde{\beta} = \widetilde{\gamma} = 0$ which we call *type I* or else (II) $\widetilde{\alpha} = 0$ which we call *type II*.

Definition 2. If a statistical system is invariant under the infinitesimal generators of either type I or type II it is said to be *locally scale-invariant* of type I or type II, respectively.

Only the generators of type II are suitable for applications to ageing phenomena.

Theorem 1. *The generators X_n, Y_m of type II form a kinematic symmetry of the differential equation $\mathcal{S}\psi = 0$, where*

$$\mathcal{S} = -(\widetilde{\beta} + \widetilde{\gamma})\partial_t + \frac{1}{z^2}\partial_r^z. \tag{12}$$

For $z = 2$, we recover the $d = 1$ free Schrödinger equation and its maximal kinematic Lie algebra $\mathfrak{sch}_1$. See [15] for the precise definition of the commuting fractional derivatives ∂_r^z.

In order to be able to apply this kinematic symmetry to ageing phenomena, we must consider the subset of $\{X_n, Y_m\}$ where time translations (generated by $X_{-1} = -\partial_t$) are left out. It can be checked that the initial line $t = 0$ is kept invariant. Ageing systems are not time-translation invariant and local scale invariance for them is meant to exclude the generator X_{-1}.

Theorem 2. *Consider an ageing statistical system which is locally scale-invariant of type II. Then the autoresponse function is*

$$R(t,s) = r_0\left(\frac{t}{s}\right)^{1+a-\lambda_R/z}(t-s)^{-1-a}, \qquad t > s, \tag{13}$$

where r_0 is a normalization constant. Furthermore, consider the following scaling form of the spatio-temporal response

$$R(t,s;\boldsymbol{r}) = \left.\frac{\delta\langle\phi(t,\boldsymbol{r})\rangle}{\delta h(s,\boldsymbol{0})}\right|_{h=0} = R(t,s)\Phi\left(\frac{r}{(t-s)^{1/z}}\right). \tag{14}$$

Then $\Phi(u)$ is a solution of the equation

$$\left(\partial_u + z\left(\widetilde{\beta} + \widetilde{\gamma}\right)u\partial_u^{2-z} + 2z(2-z)\widetilde{\gamma}\partial_u^{1-z}\right)\Phi(u) = 0. \tag{15}$$

In the special case $z = 2$, we have

$$R(t,s;\boldsymbol{r}) = R(t,s)\exp\left(-\frac{\mathcal{M}}{2}\frac{\boldsymbol{r}^2}{t-s}\right), \tag{16}$$

where $\mathcal{M} = \widetilde{\beta} + \widetilde{\gamma}$ is a constant.

Proof. [14,15] Consider first the autoresponse $R = R(t,s) = \langle\phi\widetilde{\phi}\rangle$, where ϕ, $\widetilde{\phi}$ have the scaling dimensions x_ϕ, $x_{\widetilde{\phi}}$, respectively. Local scale invariance means that $X_nR = Y_mR = 0$, with $n \geq 0$. Because of spatial translation invariance and the commutators (10), it is sufficient to check that $X_0R = X_1R = 0$. The explicit form (11) produces two linear differential equations for $R(t,s)$ which are readily solved. Comparison with the scaling forms (6) then establishes (13). Equation (15) is proven similarly, see [15]. Equation (16) had been found earlier [12]. ■

In these explicit forms of $R(t,s;\boldsymbol{r})$ it is also assumed that the system is rotation-invariant. If that is not the case, $\mathcal{M}$ is no longer uniform, but becomes direction-dependent [17].

Equation (13) has been tested and confirmed in several spin systems undergoing ageing, notably the kinetic Ising model with Glauber dynamics in $d = 2, 3$ through intensive simulations [14, 16] and the exactly solvable kinetic spherical model in $d > 2$ dimensions [4, 11, 23], for $T \leq T_\mathrm{c}$, the random walk [7] and finally (up to logarithmic correction factors) the $2D$ XY model with $T < T_\mathrm{c}$ [1] and the $2D$ critical voter model [25]. Furthermore, equation (16) has been numerically confirmed in the Glauber–Ising model, again for $d = 2, 3$ and $T < T_\mathrm{c}$ [17]. However, small corrections with respect to (13) were found at $T = T_\mathrm{c}$ in a two-loop ε-expansion [3] and in a recent self-consistent study [19] of the time-dependent Ginzburg–Landau equation at $T = 0$ and which improves on the approximate Ohta–Jasnow–Kawasaki theory. We refer to the literature for details.

3 On the Schrödinger group

Having reviewed the important phenomenological result of local scale invariance as given in Theorem 2, we now discuss in more detail how a dynamical symmetry such as local scale invariance might come about. We shall do this here for the case $z = 2$ and therefore discuss the Schrödinger group and Schrödinger-invariant systems [18]. For simplicity, we often set $d = 1$.

Under the action of an element $g \in Sch(1)$ of the Schrödinger group, a solution $\phi(t,r)$ of the free equation $(2\mathcal{M}\partial_t - \partial_r^2)\phi = 0$ where $\mathcal{M} = \mathrm{i}m$ is fixed, transforms projectively, viz. $\phi(t,r) \mapsto (T_g\phi)(t,r) = f_g(g^{-1}(t,r))\phi(g^{-1}(t,r))$ with a known [20] companion function f_g.

Following [10], we treat $\mathcal{M}$ as an additional variable and ask about the maximal kinematic group in this case [18]. First introduce a new coordinate ζ and a new wave function ψ through

$$\phi(t,r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}\zeta \, e^{-\mathrm{i}\mathcal{M}\zeta} \psi(\zeta,t,r). \tag{17}$$

We denote time t as the zeroth coordinate and ζ as coordinate number -1. When working out the action of the generators of $\mathfrak{sch}_1$ on the function ψ, it is easily seen that the projective phase factors can be absorbed into certain translations of the variable ζ [18]. Furthermore, the free Schrödinger equation becomes

$$\left(2\mathrm{i}\frac{\partial^2}{\partial\zeta\partial t} + \frac{\partial^2}{\partial r^2} \right) \psi(\zeta,t,r) = 0. \tag{18}$$

In order to find the maximal kinematic symmetry of this equation, we recall that the three-dimensional Klein–Gordon equation $\sum_{\mu=-1}^{1} \partial_\mu \partial^\mu \Psi(\boldsymbol{\xi}) = 0$ has the $3D$ conformal algebra $\mathfrak{conf}_3 \cong so(4,1) \cong B_2$ as maximal kinematic symmetry. By making the following change of variables

$$\zeta = \frac{1}{2}\left(\xi_0 + \mathrm{i}\xi_{-1}\right), \qquad t = \frac{1}{2}\left(-\xi_0 + \mathrm{i}\xi_{-1}\right), \qquad r = \sqrt{\frac{\mathrm{i}}{2}}\,\xi_1 \tag{19}$$

and setting $\psi(\zeta,t,r) = \Psi(\boldsymbol{\xi})$, the $3D$ Klein–Gordon equation reduces to (18). Therefore [18]

Theorem 3. *For variable masses $\mathcal{M}$, the maximal kinematic symmetry algebra of the free Schrödinger equation in d dimensions is isomorphic to the conformal algebra $\mathfrak{conf}_{d+2}$ and one has the inclusion of the complexified Lie algebras $(\mathfrak{sch}_d)_\mathbb{C} \subset (\mathfrak{conf}_{d+2})_\mathbb{C}$.*

For $d = 1$, the Cartan subalgebra is spanned by the generators X_0 and $N := -t\partial_t + \zeta\partial_\zeta$. With respect to the six generators of the Schrödinger algebra $\mathfrak{sch}_1$, there are the four additional ones N, V_+, V_-, W. They are identified in Fig. 2a with the roots in the root space of B_2.

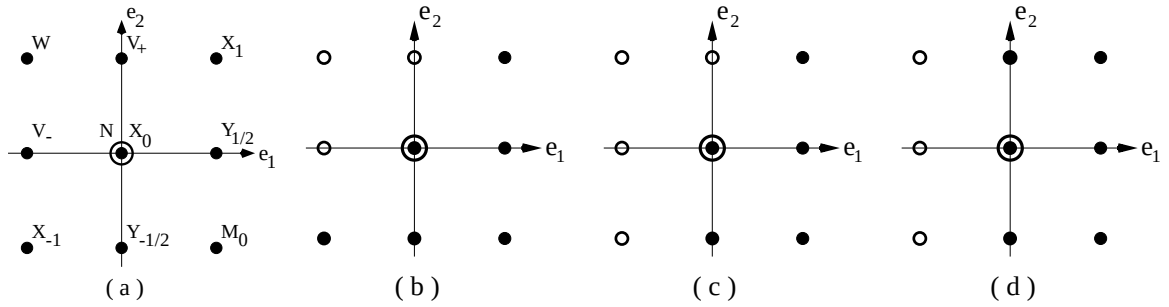

Figure 2. (a) Roots of the complex Lie algebra B_2 and the identification of the generators of the complexified conformal Lie algebra $(\mathfrak{conf}_3)_{\mathbb{C}} \supset (\mathfrak{sch}_1)_{\mathbb{C}}$. The double circle in the center denotes the Cartan subalgebra. The generators belonging to the three non-isomorphic parabolic subalgebras [18] are indicated by the full points, namely (b) $\widetilde{\mathfrak{sch}}_1$, (c) $\widetilde{\mathfrak{age}}_1$ and (d) $\widetilde{\mathfrak{alt}}_1$.

Consider the non-isomorphic parabolic subalgebras of B_2. In Fig. 2, these correspond to convex subsets of roots. There are two maximal parabolic subalgebras, namely (i) $\widetilde{\mathfrak{sch}}_1 := \mathfrak{sch}_1 \oplus \mathbb{C}N$ and (ii) $\widetilde{\mathfrak{alt}}_1 := \mathfrak{alt}_1 \oplus \mathbb{C}N$, where $\mathfrak{alt}_1 := \overline{\{X_0, X_1, Y_{\pm 1/2}, M_0, V_+\}}$ and $V_+ = -2\zeta r\partial_\zeta - 2tr\partial_t - (r^2 + 2\mathrm{i}\zeta t)\partial_r - 2xr$, see also Fig. 2bd. The minimal parabolic subalgebra is $\widetilde{\mathfrak{age}}_1 := \mathfrak{age}_1 \oplus \mathbb{C}N$, where $\mathfrak{age}_1 := \overline{\{X_0, X_1, Y_{\pm 1/2}, M_0\}}$, see Fig. 2c. We see that both $\widetilde{\mathfrak{alt}}_1$ and $\widetilde{\mathfrak{age}}_1$ do not contain the generator X_{-1} of time translations and they may therefore be considered candidates for a dynamical symmetry of ageing systems.

In writing down above the Klein–Gordon equation, we had used units such that the 'speed of light' $c = 1$. It had been claimed in the literature that in a non-relativistic limit $c \to \infty$, there were a group contraction $\mathfrak{conf}_{d+1} \curvearrowright \mathfrak{sch}_d$ provided that an ill-defined "*... transfer of the transformation of mass to the coordinates ...*" is carried out [2]. However, going over first to the variable ζ before letting $c \to \infty$, we do not find a group contraction but rather the map $\mathfrak{conf}_3 \to \widetilde{\mathfrak{alt}}_1 \neq \mathfrak{sch}_1$.

We now discuss some physical consequences of Theorem 3. Consider the effective action $S = S[\psi(\zeta, t, r)]$. For a free field ψ one recovers from the action

$$S = \int \mathrm{d}\zeta\, \mathrm{d}t\, \mathrm{d}r\, \left[2\mathrm{i}\frac{\partial\psi}{\partial\zeta}\frac{\partial\psi}{\partial t} + \left(\frac{\partial\psi}{\partial r}\right)^2 \right] + S_{\mathrm{ini}}, \tag{20}$$

where S_{ini} describes the initial conditions, the Schrödinger equation (18) as equation of motion. For an infinitesimal coordinate transformation parameterized by ε_ν, $\nu = -1, 0, 1$, a theory is said to be *local* if its action S transforms as (the second integral is restricted to the line $t = 0$)

$$\delta S = \int \mathrm{d}\zeta\, \mathrm{d}t\, \mathrm{d}r\, T_\mu{}^\nu \partial^\mu \varepsilon_\nu + \int_{(t=0)} \mathrm{d}\zeta\, \mathrm{d}r\, U^\nu \varepsilon_\nu, \tag{21}$$

where $T_\mu{}^\nu$ is the energy-momentum tensor.

Theorem 4. *If the action S of a local theory is invariant under translations in ζ and r, scale-invariant with $z = 2$ and Galilei-invariant, then $\delta_{X_1} S = 0$ and S in invariant under the action of $\mathfrak{age}_1$.*

Proof. [18] The line $t = 0$ is invariant under the action of $\mathfrak{age}_1$. Furthermore, locality (21) yields several Ward identities. First, translation-invariance in ζ and r implies $U^{-1} = U^1 = 0$. Dilatation invariance gives $T_0{}^0 + \frac{1}{2}T_1{}^1 = 0$ and Galilei-invariance implies $T_0{}^1 - \mathrm{i}T_1{}^{-1} = 0$. Now, for an infinitesimal special Schrödinger transformation,

$$\delta_{X_1} S = -\varepsilon \int \mathrm{d}\zeta\, \mathrm{d}t\, \mathrm{d}r\, \left[\left(2T_0{}^0 + T_1{}^1\right) t + \left(T_0{}^1 - \mathrm{i}T_1{}^{-1}\right) r \right] + \frac{\mathrm{i}\varepsilon}{2} \int_{(t=0)} \mathrm{d}\zeta\, \mathrm{d}r\, r^2 U^{-1} = 0$$

because of the Ward identities derived above. ■

For the free-field action (20), an energy-momentum tensor which satisfies these Ward identities can be explicitly constructed [18]. We stress that Galilei-invariance must be included as a hypothesis and one cannot expect to be able to derive it from weaker assumptions. On the other hand, time-translation invariance is not required.

Finally, we reconsider the derivation of the spatio-temporal response for $z = 2$ [18]. A standard result of conformal invariance (see [13]) gives together with Theorem 3

$$\langle \psi_1(\zeta_1, t_1, r_1)\psi_2(\zeta_2, t_2, r_2)\rangle = \langle \Psi(\boldsymbol{\xi}_1)\Psi(\boldsymbol{\xi}_2)\rangle = \psi_0\, \delta_{x_1,x_2}\, t^{-x_1} \left(\zeta + \frac{\mathrm{i}}{2}\frac{r^2}{t}\right)^{-x_1}, \tag{22}$$

where $x_{1,2}$ are the scaling dimensions of the fields $\psi_{1,2}$, ψ_0 is a normalization constant and $\zeta = \zeta_1 - \zeta_2$ and so on. Using the convention $\mathcal{M}_{1,2} \geq 0$, it turns out that equations (17), (22) together imply that $\langle \phi_1\phi_2\rangle = 0$[1]. However, if we define a *conjugate field* ϕ^* by formal complex conjugation in (17) with $\mathcal{M} \geq 0$, then for $x_1 > 0$ we find

$$\begin{aligned}\langle \phi_1(t_1, r_1)\phi_2^*(t_2, r_2)\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{d}\zeta_1\mathrm{d}\zeta_2\, e^{-\mathrm{i}\mathcal{M}_1\zeta_1 + \mathrm{i}\mathcal{M}_2\zeta_2}\langle \psi_1(\zeta_1, t_1, r_1)\psi_2(\zeta_2, t_2, r_2)\rangle \\ &= \phi_0\, \delta_{x_1,x_2}\, \delta(\mathcal{M}_1 - \mathcal{M}_2)\, \Theta(t) t^{-x_1} \exp\left(-\frac{\mathcal{M}_1}{2}\frac{r^2}{t}\right)\end{aligned} \tag{23}$$

and we stress that the Θ-function expresses the causality condition $t_1 > t_2$ which is required for an interpretation of $\langle \phi_1\phi_2^*\rangle$ as response function. If we recall from Martin–Siggia–Rose theory that $R(t, s; \boldsymbol{r}) = \langle \phi(t, \boldsymbol{r})\widetilde{\phi}(s, \boldsymbol{0})\rangle = \langle \phi(t, \boldsymbol{r})\phi^*(s, \boldsymbol{0})\rangle$, where effectively ϕ has a 'mass' $\mathcal{M} \geq 0$ and ϕ^* has a 'mass' $\mathcal{M}^* \leq 0$, we have the identification

$$\phi^*(s, \boldsymbol{r}) = \widetilde{\phi}(s, \boldsymbol{r}). \tag{24}$$

This result has also been confirmed for the three-point response functions [18].

We now apply this to the parabolic subalgebras. For a system invariant under $\mathfrak{age}_1$, we find

$$\langle \psi_1(\zeta_1, t_1, r_1)\psi_2(\zeta_2, t_2, r_2)\rangle = \psi_0 \left(\frac{t_1}{t_2}\right)^{(x_2-x_1)/2} t^{-(x_1+x_2)/2} \left(\zeta + \frac{\mathrm{i}}{2}\frac{r^2}{t}\right)^{-(x_1+x_2)/2}. \tag{25}$$

If we would consider the extension $\widetilde{\mathfrak{age}}_1 \to \widetilde{\mathfrak{sch}}_1$, then invariance under time translations generated by $X_{-1} = -\partial_t$ fixes $x_1 = x_2$ and we simply recover the result (22) coming form the invariance under $\mathfrak{conf}_3$. On the other hand, the extension $\widetilde{\mathfrak{age}}_1 \to \widetilde{\mathfrak{alt}}_1$ requires that the condition $V_+\langle \psi_1\psi_2\rangle = 0$ holds. It is easy to see from (25) that this is the case. Therefore, for both $\widetilde{\mathfrak{age}}_1$ and $\widetilde{\mathfrak{alt}}_1$ we find, provided $x_1 + x_2 > 0$

$$\langle \phi_1\phi_2^*\rangle = \phi_0\delta(\mathcal{M}_1 - \mathcal{M}_2)\Theta(t) \left(\frac{t_1}{t_2}\right)^{(x_2-x_1)/2} t^{-(x_1+x_2)/2} \exp\left(-\frac{\mathcal{M}_1}{2}\frac{r^2}{t}\right) \tag{26}$$

and we have indeed rederived (16) for ageing systems with $z = 2$ in a model-independent way, including the causality condition. There is not yet a criterion which would allow to decide if $\widetilde{\mathfrak{age}}_1$ or $\widetilde{\mathfrak{alt}}_1$, if any, is the dynamic symmetry of ageing systems with $z = 2$.

In conclusion, the explicit form of the spatio-temporal two-time response function $R(t, s; \boldsymbol{r})$ as given in (16) is a consequence of the assumed Galilei-invariance of the ageing system. The high-precision numerical confirmation of this form in the kinetic Ising model with Glauber dynamics quenched to $T < T_\mathrm{c}$ for $d = 2$ and $d = 3$ dimensions [17] is a strong indication in favour of Galilei-invariance in that model. However, even for $T = 0$ the Langevin equation (1) or the system (4) are for general $\mathcal{H}[\phi]$ not Galilei/Schrödinger-invariant, see [6, 21]. In view of small corrections to (13) suggested by two-loop field-theory calculations [3, 19] and of Theorem 4 on the other hand, it remains to be understood in what precise sense ageing systems might be said to be Galilei-invariant.

[1] This holds true for $T = 0$. For the case $0 < T < T_\mathrm{c}$, see [24].

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Classification of Equilibrium States of Condensed Media with Spontaneously Broken Symmetry

A.P. IVASHIN †, *M.Y. KOVALEVSKY* † *and L.V. LOGVINOVA* ‡

† *National Science Center "Kharkiv Institute of Physics and Technology", 1 Akademicheskaja Str., 61108 Kharkiv, Ukraine*
E-mail: *ivashin@kipt.kharkov.ua, mik@kipt.kharkov.ua*

‡ *Belgorod State University, 85 Pobedy Str., 308007 Belgorod, Russia*

In the present paper the classification of equilibrium states of condensed media with spontaneously broken symmetry is carried out. The used approach is based on concept of the quasiaverages. The admissible properties of equilibrium state symmetry of a quantum fluid and relevant structures of an order parameter are found from requirements of the residual and spatial symmetry at nonzero values of an order parameter. The superfluid in a state with triplet pairing, the superfluid nuclear matter in a state of d-pairing, liquid crystal states of matter are considered in this paper as the examples. Admissible conditions of the spatial symmetry and the general structure of the corresponding generator are found.

1 Introduction

The classification of equilibrium states of condensed media with spontaneously broken symmetry based on the phenomenological Ginzburg–Landau approach requires the exact free energy dependence on the order parameter function and essentially depends on the type of the model we consider. The other group-theory approach is based on the conception of residual symmetry of the degenerate state of equilibrium as the subgroup of normal phase symmetry. The corresponding transformation properties of the order parameter in Hamiltonian symmetry transformations are essential in this approach. This consideration is free from any model assumption about the form of free energy. The classification of homogenous states in terms of both mentioned approaches was carried out in particular for the superfluid ^{3}He [1–3], and for d-pairing in superfluid quantum liquid [4, 5]. In the present paper the general microscopic approach to the classification of equilibrium states of condensed media with spontaneously broken symmetry based on the conception of quasiaverages is suggested. The permissible properties of symmetry of quantum liquid equilibrium state and the corresponding structures of order parameter are found from the conditions of residual symmetry at the nonzero values of order parameter. As the examples of degenerate condensed media with spontaneously broken symmetry we consider the superfluid liquids in state with triple type of pairing (^{3}He), superfluid nuclear matter in d-pairing state and liquid crystals.

2 Symmetry properties of the normal equilibrium state

The theory of many-particle systems that describes the equilibriums of a normal Fermi liquid is based on the statistical Gibbs operator

$$\hat{w} = \exp\left(\Omega - Y_a \hat{\gamma}_a\right), \tag{1}$$

here $\hat{\gamma}_a \equiv \left(\hat{\mathcal{H}}, \hat{\mathcal{P}}_k, \hat{N}, \hat{S}_\alpha\right)$ are additive integrals of motion ($a \equiv 0, k, 4, \alpha$): $\hat{\mathcal{H}}$ is the Hamiltonian, $\hat{\mathcal{P}}$ is the momentum operator, $\hat{N}$ is the particle number operator, $\hat{S}_\alpha$ is the spin operator,

Thermodynamic potential $\Omega = V\omega(Y)$ is defined by the normalization condition $\operatorname{Sp}\hat{w} = 1$. The set of thermodynamic forces Y_a includes the temperature $Y_0^{-1} \equiv T$, the velocity $Y_k^{-1} \equiv v_k$, the chemical potential $-Y_4/Y_0 \equiv \mu_k$, and the effective magnetic field $-Y_\alpha/Y_0 \equiv h_\alpha$.

The additive integrals of motion entering the Gibbs distribution result in a certain symmetry of the equilibrium.The symmetry properties of equilibrium statistical operator (1) have the form

$$\big[\hat{w}, \hat{\mathcal{P}}_k\big] = 0, \qquad \big[\hat{w}, \hat{\mathcal{H}}\big] = 0, \qquad \big[\hat{w}, \hat{N}\big] = 0, \qquad \big[\hat{w}, \hat{\Sigma}_\alpha\big] = 0, \qquad \big[\hat{w}, \hat{L}_k\big] = 0 \tag{2}$$

and reflect the space-time translation invariance and the phase invariance. The requirement for the symmetry under rotations in the spin and configuration spaces implies neglecting the weak dipole and spin-orbit interactions. Here, $\hat{\Sigma}_\alpha$ and $\hat{L}_i$ are the generalized spin and orbital momentum operators

$$\hat{\Sigma}_\alpha \equiv \hat{S}_\alpha + \hat{S}_\alpha^Y, \qquad \hat{S}_\alpha^Y \equiv -i\varepsilon_{\alpha\beta\gamma} Y_\beta \frac{\partial}{\partial Y_\gamma}, \qquad \hat{L}_i \equiv \hat{\mathcal{L}}_i + \hat{\mathcal{L}}_i^Y, \qquad \hat{\mathcal{L}}_i^Y \equiv -i\varepsilon_{ikl} Y_k \frac{\partial}{\partial Y_l}, \tag{3}$$

acting on the Hilbert space and on the thermodynamic parameter space. The action of the differential operators on the vectors $Y_i(Y_\alpha)$ is defined by $i\big[\hat{\mathcal{L}}_i^Y, Y_j\big] = \varepsilon_{ikj} Y_k$, $i\big[\hat{S}_\alpha^Y, Y_\rho\big] = \varepsilon_{\alpha\beta\rho} Y_\beta$. The corresponding means of commutators involving the orbital momentum operators have the form $\operatorname{Sp}\big[\hat{w}, \hat{\mathcal{L}}_i + \mathcal{L}_i^Y\big]\hat{b}(x) = \operatorname{Sp}\hat{w}\big[\hat{\mathcal{L}}_i, \hat{b}(x)\big] + \mathcal{L}_i^Y \operatorname{Sp}\hat{w}\hat{b}(x)$. According to definition (3), the operators $\hat{\Sigma}_\alpha$ and $\hat{L}_i$ satisfy the relations

$$i\big[\hat{L}_i, \hat{L}_k\big] = -\varepsilon_{ikl}\hat{L}_l, \qquad i\big[\hat{\Sigma}_\alpha, \hat{\Sigma}_\beta\big] = -\varepsilon_{\alpha\beta\gamma}\hat{\Sigma}_\gamma.$$

3 Equilibrium degeneracy and quasiaverages

The theoretical foundation of the statistical physics that describes equilibriums of condensed matter with a spontaneously broken symmetry is Bogoliubov's concept of quasiaverages [6]. A constructive point of this concept is inserting an infinitely small source $\nu\hat{F}$ into the equilibrium statistical operator, that reduces the symmetry of the statistical equilibrium in comparison with the symmetry of the Hamiltonian and allows generalizing the Gibbs distribution to condensed matter under spontaneous symmetry-breaking conditions. The quasiaverage value of a quantity $a(x)$ in a statistical equilibrium with a broken symmetry is defined by

$$\langle\hat{a}(x)\rangle \equiv \lim_{\nu\to 0}\lim_{V\to\infty} \operatorname{Sp}\hat{w}_\nu\hat{a}(x), \qquad \hat{w}_\nu \equiv \exp\big(\Omega_\nu - Y_a\hat{\gamma}_a - \nu\hat{F}\big). \tag{4}$$

The operator $\hat{F}$ has the symmetry of the condensed-matter phase under study and removes the degeneracy of the equilibrium. In accordance with the concept of quasiaverages, we choose the source $\hat{F}$ in the form of a linear functional of an order parameter $\hat{\Delta}_a(x)$

$$\hat{F} = \int d^3x\big(f_a(x,t)\hat{\Delta}_a(x) + \text{h.c.}\big),$$

where $f_a(x,t)$ is some function of coordinates and time that is conjugate to the order parameter and defines its equilibrium value $\Delta_a(x,t) = \langle\hat{\Delta}_a(x)\rangle$ in the sense of quasiaverages (4). The structure of the function $f_a(x,t)$ is defined by the symmetry properties of the phase under study. The last circumstance gives a possibility of introducing additional thermodynamic parameters into the Gibbs distribution in the framework of the microscopic theory [7].

A description of condensed matter with a spontaneously broken symmetry essentially rests on the notion of the order parameter. In the language of the secondary quantization, the order-parameter operators $\hat{\Delta}_a(x)$ are built from the field creation and annihilation operators. We form

late the commutation properties of the order-parameter operators. The translation invariance condition is

$$i\big[\hat{\mathcal{P}}_k, \hat{\Delta}_a(x)\big] = -\nabla_k \hat{\Delta}_a(x). \tag{5}$$

The generator of the phase transformation group is the particle number operator $\hat{N}$. The relation

$$\big[\hat{N}, \hat{\Delta}_a(x)\big] = -g_a \hat{\Delta}_a(x) \tag{6}$$

holds for the order-parameter operator $\hat{\Delta}_a(x)$. The constants g_a depend on the tensor dimension of the order-parameter operator.

Under the transformations related to the internal-symmetry group with the generators $\hat{S}_\alpha$ ($\alpha = x, y, z$) the operators $\hat{\Delta}_a(x)$ transform according to the representations of this group, which yields the equality

$$i\big[\hat{S}_\alpha, \hat{\Delta}_a(x)\big] = -g_{\alpha ab} \hat{\Delta}_b(x), \tag{7}$$

or, in compact notation,

$$i\big[\hat{S}_\alpha, \hat{\Delta}(x)\big] = -\hat{g}_\alpha \hat{\Delta}(x),$$

where $(\hat{g}_\alpha)_{ab} \equiv g_{\alpha ab}$– are some constants. Because $i\big[\hat{S}_\alpha, \hat{S}_\beta\big] = -\varepsilon_{\alpha\beta\gamma}\hat{S}_\gamma$, from formula (7) and the Jacobi identity for the operators $\hat{S}_\alpha$ and $\hat{\Delta}\,(x)$, we have

$$\big[\hat{g}_\alpha, \hat{g}_\beta\big] = -\varepsilon_{\alpha\beta\gamma}\hat{g}_\gamma. \tag{8}$$

Under the transformations related to the spatial-rotation group with the generators $\mathcal{L}_i$ ($i = 1, 2, 3$), the order parameter operators $\hat{\Delta}_a(x)$ at the point $x = 0$ transform according to the representations of this group. The equality $i\big[\hat{\mathcal{L}}_i, \hat{\Delta}_a(0)\big] = -g_{iab}\hat{\Delta}_b(0)$ therefore holds, and noting that $\big[\hat{\mathcal{L}}_i, \hat{\mathcal{L}}_j\big] = i\varepsilon_{ijk}\hat{\mathcal{L}}_k$, we obtain a relation similar to (8)

$$\big[\hat{g}_i, \hat{g}_j\big] = -\varepsilon_{ijk}\hat{g}_k.$$

Because $\hat{\Delta}_a(x) = e^{-i\hat{\mathcal{P}}x}\hat{\Delta}_a(0)e^{i\hat{\mathcal{P}}x}$, $e^{-i\hat{\mathcal{P}}x}\hat{\mathcal{L}}_i e^{i\hat{\mathcal{P}}x} = \hat{\mathcal{L}}_i - \varepsilon_{ijk}x_j\hat{\mathcal{P}}_k$, by virtue of (5) we have

$$i\big[\hat{\mathcal{L}}_i, \hat{\Delta}_a(x)\big] = -g_{iab}\hat{\Delta}_b(x) - \varepsilon_{ijk}x_k\nabla_j\hat{\Delta}_a\big(x\big). \tag{9}$$

It is known from the phenomenological theory that, in general, an adequate description of the thermodynamics of non-equilibrium processes in condensed matter with a broken symmetry requires introducing new thermodynamic parameters into the theory; these are not related to conservation laws but are due to the physical nature of the thermodynamic phase. In the case of normal condensed matter, the thermodynamic parameters are defined by only the densities of the additive integrals of motion. We show how the symmetry properties are formulated for the equilibriums of degenerate condensed matter and how the additional thermodynamic parameters are introduced based on this. We consider the translation-invariant subgroups of the residual symmetry H of the total group G. The translation invariance implies that the equilibrium statistical operator satisfies the symmetry relation

$$\big[\hat{w}, \hat{\mathcal{P}}_k\big] = 0. \tag{10}$$

We analyze the translation-invariant subgroups of the residual symmetry for the equilibriums starting from the relation

$$\big[\hat{w}, \hat{T}\big] = 0, \tag{11}$$

where the generator T of the residual symmetry (a generator of the subgroup H) is a linear combination of the integrals of motion (see for comparison (2))

$$\hat{T} \equiv a_i \hat{L}_i + b_\alpha \hat{\Sigma}_\alpha + c\hat{N} \equiv \hat{T}(\xi) \tag{12}$$

with some real parameters $(a_i, b_\alpha, c \equiv \xi)$. The unitary transformations $U(\xi) = \exp[i\hat{T}(\xi)]$ form the continuous subgroup of the residual symmetry $U(\xi)U(\xi') = U(\xi''(\xi, \xi'))$ for the equilibrium state. Using the equalities

$$i\,\mathrm{Sp}\big[\hat{w}, \hat{T}\big]\hat{\Delta}_a(x) = 0, \qquad i\,\mathrm{Sp}\big[\hat{w}, \hat{\mathcal{P}}_k\big]\hat{\Delta}_a(x) = 0,$$

and taking algebraic relations (5)–(7), (9) and definition (12) into account, we obtain the equations

$$a_i \left(g_{iab}\Delta_b + \varepsilon_{ikl} Y_k \frac{\partial \Delta_a}{\partial Y_l} \right) + b_\alpha \left(g_{\alpha ab}\Delta_b + \varepsilon_{\alpha\beta\gamma} Y_\beta \frac{\partial \Delta_a}{\partial Y_\gamma} \right) + i g_a \Delta_a = 0, \qquad \nabla_k \Delta_a = 0.$$

which establish certain relations between the parameters ξ. For simplicity, we consider the case where $Y_\alpha = Y_k = 0$. In this case, we have

$$T_{ab}\Delta_b = 0, \qquad T_{ab} \equiv a_i g_{iab} + b_\alpha g_{\alpha ab} + i g_a \delta_{ab}. \tag{13}$$

The requirement that set of linear equations (13) has a nontrivial solution $\Delta_a \neq 0$ leads to the equality $\det |T_{ab}| = 0$, which imposes restrictions on the admissible values of the parameters ξ of the residual symmetry generator.

4 Translation-invariant equilibrium states of the condensed media with spontaneously broken symmetry

A. *Translation-invariant equilibrium states of the superfluid* ^{3}He. For the order parameter operator $\hat{\Delta}_{\alpha k}(x)$ it is convenient to choose [7]

$$\hat{\Delta}_{\alpha k}(x) \equiv \hat{\psi}(x)\,\sigma_2\sigma_\alpha \nabla_k \hat{\psi}(x) - \nabla_k \hat{\psi}(x)\,\sigma_2\sigma_\alpha \hat{\psi}(x).$$

Here, σ_α are Pauli matrices. The matrices $(\sigma_2\sigma_\alpha)_{\mu\nu} = (\sigma_2\sigma_\alpha)_{\nu\mu}$ are symmetric with respect to indices μ and ν. We see that in accordance with this definition and the canonical commutation relations for Fermi operators, the equalities are valid:

$$\begin{aligned} &i\big[\hat{S}_\alpha, \hat{\Delta}_{\beta i}(x)\big] = -\varepsilon_{\alpha\beta\gamma}\hat{\Delta}_{\gamma i}(x), \qquad \big[\hat{N}, \hat{\Delta}_{\beta i}(x)\big] = -2\hat{\Delta}_{\beta i}(x), \\ &i\big[\hat{\mathcal{P}}_k, \hat{\Delta}_{\alpha i}(x)\big] = -\nabla_k \hat{\Delta}_{\alpha i}(x), \qquad i\big[\hat{\mathcal{L}}_k, \hat{\Delta}_{\alpha i}(x)\big] = -\varepsilon_{kjl} x_j \nabla_l \hat{\Delta}_{\alpha i}(x) - \varepsilon_{kil}\hat{\Delta}_{\alpha l}(x). \end{aligned} \tag{14}$$

By virtue of algebra (14) and relations (13), we obtain the equality defining the equilibrium structure of the order parameter:

$$a_k \varepsilon_{kil}\Delta_{\beta l} + b_\alpha \varepsilon_{\alpha\beta\gamma}\Delta_{\gamma i} + 2ic\Delta_{\beta i} = 0.$$

This system of equations has many anisotropic solutions that are present in Table 1. Let us consider the state corresponding to the isotropic superfluid phase. We introduce the orthogonal rotation matrix that describes the change of the orientation of the spatial coordinate system with respect to the spin $S = 1$ by the equality $b_\alpha = a_i R_{i\alpha}$. Taking (12) into account, we obtain $a_i\big[\hat{w}, \hat{\mathcal{L}}_i + R_{i\alpha}\hat{S}_\alpha\big] = 0$. The isotropy condition implies the validity of the last relation for arbitrary directions of the vector $\vec{a}$. The symmetry property of the states therefore has the form

$$\big[\hat{w}, \hat{\mathcal{L}}_i + R_{i\alpha}\hat{S}_\alpha\big] = 0.$$

This state describes the B phase of the superfluid ^{3}He. For the states with this symmetry, the mean of the order parameter is

$$\Delta_{\alpha k} = \Delta R_{k\alpha}.$$

The resulting classification of the residual symmetry properties and the corresponding order parameter values in the equilibrium of ^{3}He for the translation-invariant case is presented in Table 1. We assume that the projection of the Cooper-pair orbital momentum onto the $\vec{l}$ direction is equal to m_l and the projection of the Cooper-pair spin onto the $\vec{d}$ direction is equal to m_s.

Table 1.

Residual symmetry generator	m_l	m_s	Order parameter	Phase
$\hat{\mathcal{L}}_i + R_{i\alpha}\hat{S}_\alpha$	$-$	$-$	$\Delta R_{\alpha i}$	B
$\vec{l}\hat{\vec{\mathcal{L}}} - \frac{m_l}{2}\hat{N}$	±1	0	$\Delta d_\alpha (m_k \mp i n_k)$	A
	0	±1	$\Delta (e_\alpha \mp i f_\alpha) l_k$	β
$\vec{d}\hat{\vec{S}} - \frac{m_s}{2}\hat{N}$	±1	±1	$\Delta (e_\alpha \mp i f_\alpha)(m_k \mp i n_k)$	A_1
	0	0	$\Delta d_\alpha l_k$	Polar
$\vec{d}\hat{\vec{S}} - 2m_l\lvert m_s\rvert \vec{l}\hat{\vec{\mathcal{L}}} - \frac{1}{2}m_s\hat{N}$	$0, \pm1$	0	$d_\alpha (Am_k + Bn_k + Cl_k)$	
	±1	±1	$A(m_k \mp i n_k)(e_\alpha \mp i f_\alpha)$ $+Bd_\alpha (m_k \mp i n_k)$	$A_1 + A$
	0	±1	$(e_\alpha \mp i f_\alpha)(Am_k + Bn_k + Cl_k)$	A_2
$\vec{l}\hat{\vec{\mathcal{L}}} - 2m_s\lvert m_l\rvert \vec{d}\hat{\vec{S}} - \frac{1}{2}m_l\hat{N}$	0	$0, \pm1$	$(Ae_\alpha + Bf_\alpha + Cd_\alpha) l_k$	
	±1	±1	$A(m_k \mp i n_k)(e_\alpha \mp i f_\alpha) + Bl_k (e_\alpha \mp i f_\alpha)$	$A_1 + \beta$
	$\pm$	0	$(Ae_\alpha + Bf_\alpha + Cd_\alpha)(m_k \mp i n_k)$	B_2
$\vec{l}\hat{\vec{\mathcal{L}}} + \vec{d}\hat{\vec{S}} - \frac{1}{2}(m_l + m_s)\hat{N}$	$0, \pm$	$0, \mp$	$e_\alpha (Am_k + Bn_k) + f_\alpha (-Bm_k + An_k)$ $+Cd_\alpha l_k$	ζ
	0	$\mp$	$Al_k (e_\alpha \mp i f_\alpha) + Bd_\alpha (m_k \mp i n_k)$	ϵ
	$\pm$	0		
	$\pm$	$\pm$	$\Delta (e_\alpha \mp i f_\alpha)(m_k \mp i n_k)$	A_1

B. *The superfluid nuclear matter in a state of d-pairing.* We define the order parameter operator of d-pairing in the terms of the operators of creation and annihilation of Fermi-particle:

$$\hat{\Delta}_{ik}(x) \equiv \nabla_i\hat{\psi}(x)\sigma_2\nabla_k\hat{\psi}(x) + \nabla_k\hat{\psi}(x)\sigma_2\nabla_i\hat{\psi}(x) - \frac{2}{3}\delta_{ik}\nabla_j\hat{\psi}(x)\sigma_2\nabla_j\hat{\psi}(x), \tag{15}$$

where σ_2 is the Pauli matrix.

Using this definitions we obtain the operator relations for the operators of number of particles $\hat{N}$, momentum $\hat{\mathcal{P}}_k$, spin $\hat{S}_\alpha$ and orbital momentum $\hat{\mathcal{L}}_k$

$$\begin{aligned}&\big[\hat{N}, \hat{\Delta}_{ik}(x)\big] = -2\hat{\Delta}_{ik}(x), \qquad \big[\hat{S}_\alpha, \hat{\Delta}_{ik}(x)\big] = 0, \qquad i\big[\hat{\mathcal{P}}_l, \hat{\Delta}_{ik}(x)\big] = -\nabla_l\hat{\Delta}_{ik}(x),\\ &i\big[\hat{\mathcal{L}}_l, \hat{\Delta}_{ik}(x)\big] = -\varepsilon_{lij}\hat{\Delta}_{jk}(x) - \varepsilon_{lkj}\hat{\Delta}_{ji}(x) - \varepsilon_{lkj}x_k\nabla_j\hat{\Delta}_{ik}(x)\end{aligned} \tag{16}$$

The mean values of the order parameter $\Delta_{ik}(x, \hat{\rho}) = \mathrm{Sp}\,\hat{\rho}\hat{\Delta}_{ik}(x)$, ($\hat{\rho}$ is the arbitrary statistical operator) have properties $\Delta_{ik}(x, \hat{\rho}) = \Delta_{ki}(x, \hat{\rho})$, $\Delta_{ii}(x, \hat{\rho}) = 0$.

If we use the general approach developed in Section 3 we get the relation for residual symmetry operator (12)

$$\left[\hat{w}, \frac{a_i}{a}\hat{\mathcal{L}}_i - \frac{M}{2}\hat{N}\right] = 0. \tag{17}$$

Here M is quantum number taking the values 0, ±1, ±2. Let us decompose the vector $\vec{a}$ on the indicated coordinate $\vec{a}/a = \alpha\vec{n} + \beta\vec{m} + \gamma\vec{l}$. The vectors $\vec{m}$, $\vec{n}$, $\vec{l}$ are the axes of an anisotropy, representative unitary and mutually perpendicular vectors and the quantities α, β, γ obey the equality $\alpha^2 + \beta^2 + \gamma^2 = 1$.

From this relation it is easy to obtain the expression for the order parameter, according to formalism developed in Section 3 and taking into account (15), (16). To obtain the final expression for order parameter we define mean value according to the paper [8]

$$\Delta \equiv k_i \Delta_{ij} k_j, \qquad \vec{k} \equiv \vec{m}\sin\theta\sin\varphi + \vec{n}\cos\theta + \vec{l}\sin\theta\cos\varphi.$$

For $M = 0$ in according to (13), (17) we obtain

$$\Delta_1^{(0)} = iE\left\{(\beta k_y + \alpha k_z + \gamma k_x)^2 - \frac{1}{3}\right\}, \qquad \Delta_2^{(0)} = i\widetilde{C}\left\{(\beta k_y + \alpha k_z)^2 - \frac{1}{3}\right\},$$
$$\Delta_3^{(0)} = (A + iC)\left(k_z^2 - 1/3\right), \qquad E^2 = (3/2), \qquad \widetilde{C^2} = (3/2), \qquad A^2 + C^2 = 3/2.$$

These solutions correspond to the "real" state [8].

The solution for $M = \pm1$ in according to (15), (17) gives the equality

$$\Delta^{(1)} = A(k_x + k_y)\big(\pm i\sqrt{2}k_z + k_y - k_x\big), \qquad A^2 = 1/4.$$

At last, in a case ($M = \pm2$) we obtain

$$\Delta^{(2)} = A(k_x \pm ik_y)^2, \qquad A^2 = 1/4.$$

This solution coincides with "axial" state of work [8].

C. *Liquid crystal states of matter.* Let us consider the translation invariant states of liquid crystal equilibrium and define their possible equilibrium structures of order parameter. We choose the order parameter in the form

$$\hat{Q}_{uv}(x) \equiv \nabla_u\hat{\psi}^+(x)\nabla_v\hat{\psi}(x) + \nabla_v\psi^+(x)\nabla_u\hat{\psi}(x) - \frac{2}{3}\delta_{uv}\nabla_j\hat{\psi}^+(x)\nabla_j\hat{\psi}(x). \tag{18}$$

According to definition (1) of operators for number of particles, momentum operator and orbital momentum, we obtain the commutative relations:

$$\big[\hat{N}, \hat{Q}_{uv}(x)\big] = 0, \qquad i\big[\hat{\mathcal{P}}_k, \hat{Q}_{uv}(x)\big] = -\nabla_k\hat{Q}_{uv}(x),$$
$$i\big[\hat{\mathcal{L}}_i, \hat{Q}_{uv}(x)\big] = -\varepsilon_{iuj}\hat{Q}_{jv}(x) - \varepsilon_{ivj}\hat{Q}_{ju}(x) - \varepsilon_{ikl}x_k\nabla_l\hat{Q}_{uv}(x). \tag{19}$$

In case of uniaxial nematic we perform the analysis of the subgroups of unbroken symmetry states in the rest frame reference on the assumption of (11). As the phase invariance in liquid crystal is not broken, the expression for residual generator for homogeneous state has the form

$$\hat{T} \equiv a_i\hat{\mathcal{L}}_i,$$

here a_i are the real parameters. According to (18), (19), (11) we have

$$Q_{ik} = Q\left(n_in_k - \frac{1}{3}\delta_{ik}\right).$$

In the case of biaxial nematic the residual symmetry generator has the form

$$\hat{T}\big(\vec{a}, \vec{m}, \vec{n}\big) \equiv a_i\hat{L}_i\big(\vec{m}, \vec{n}\big). \tag{20}$$

The generalized orbital momentum operator is defined by equality

$$\hat{L}_i(\vec{m},\vec{n}) \equiv \hat{\mathcal{L}}_i + \hat{\mathcal{L}}_i^{\vec{m}} + \hat{\mathcal{L}}_i^{\hat{n}}, \qquad \hat{\mathcal{L}}_i^{\vec{m}} \equiv -i\varepsilon_{ikl} m_k \frac{\partial}{\partial m_l}, \qquad \hat{\mathcal{L}}_i^{\vec{n}} \equiv -i\varepsilon_{ikl} n_k \frac{\partial}{\partial n_l}. \tag{21}$$

The vectors $\vec{n}$ and $\vec{m}$ are unitary and mutually perpendicular vectors. Using the condition of symmetry (11) with the generator in the form (20), (21) we obtain the following order parameter

$$Q_{ik} = Q_1 \left(n_i n_k - \frac{1}{3}\delta_{ik} \right) + Q_2 \left(m_i m_k - \frac{1}{3}\delta_{ik} \right).$$

The description of the equilibrium state of degenerate condensed media with rather complicated order parameter is considered on the basis of microscopic approach. Representation of residual symmetry of the equilibrium state in combination with the conception of quasiaverages allows to suggest an alternative to the Ginzburg–Landau method of classification of possible condensed media phase states. The present approach can be generalized for description of the states without translation invariance. It gives the possibility of consideration of the periodical spatial structures with discrete or continuous residual symmetry.

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A Derivation of Quantum Mechanics from START

Jaime KELLER

Center for Computational Materials Science, Institute of General Physics, Technical University of Vienna, Getreidemarkt 9/134, A-1060 Vienna, Austria;

Department of Physics and Theoretical Chemistry, Facultad de Quomica and Facultad de Estudios Superiores Cuautitlen, Universidad Nacional Autonoma de Mexico, 04510 Mexico D.F., apartado 70-528, Mexico

E-mail: *keller@cms.tuwien.ac.at, keller@servidor.unam.mx*

Quantum Mechanics is a successful model to describe physical phenomena. We show that a simple set of postulates about the description of objects in Nature allows the derivation of the principles of QM, transforming this theory into a deductive approach to Physics. The main postulate is: Objects are described as energy distributions in Space–Time.

1 Quantum Mechanics from START

It is universally accepted that Quantum Mechanics is a successful model to describe physical phenomena, in particular of the electronic structure of atoms, molecules and solids. As such it has been reformulated in several different forms (see for example: *Nine formulations of quantum mechanics*, Daniel F. Styer et al. [15]). The aim of the present paper is to show that a simple set of postulates about the description of objects in Nature allows the derivation of the principles of QM, transforming this theory into a deductive approach to Physics. In our approach the main postulate is: Objects are described as energy distributions in Space–Time. The theory is in fact a systematic mathematical analysis of this postulate and of other general considerations about action, energy–momentum and the role of the theory as a useful description of Nature.

The paper also shows the fundamental and basic status of Density Functional Theory, also contained in our approach.

The initial mathematical structure is a geometric (quadratic form based) union of space, time and action. In physics bundles of trajectories are used to describe the objects of nature: distribution of heights over a surface, a real object is described as a bundle of space-time trajectories, here we use a description as a bundle of action valves in space-time (energy densities in space following light-like lines).

The author is fully aware that the presentation is strongly guided by the knowledge of the usual computational approach to study atoms, molecules and condensed matter electronic structure.

1.1 The quadratic form in physics

Formally our approach, a Space–Time–Action Relativity Theory: START [9, 10], represents the physical world as a 5-D continuum with a quadratic form. Historically the initial use of a quadratic form $l^2 = x^2 + y^2 + z^2$ to describe nature came with the now known as Pythagoras Theorem, which even today is geometrically used in 3-D statics studies. At the beginning of the XXth century the 4th dimension was added in the Minkowski $s^2 = (ct)^2 - l^2$ space-time formalism. For our purpose we consider also a distribution of an action variable $w = \kappa_{(0)}\overline{a}$, $\kappa_{(0)} = \frac{d_{(0)}}{h} = \frac{c}{E_{(0)}}$, $(\overline{a})^2 = \sum\limits_{\mu} a_\mu^2$; $\mu = 0, 1, 2, 3$ over space-time h being Planck's constant and $d_{(0)}$

a system's fundamental length. This is schematically shown in the next table

From Pythagoras to the XXIst Century		
Quadratic Form	Dim/diff. Op	Group
$l^2 = x^2 + y^2 + z^2 \qquad (\triangle t)$	3-D ∇, ∇^2	Statics, Galileo
$s^2 = (ct)^2 - (x^2 + y^2 + z^2)$	4-D $D, \Box^2$	Kinetics, Poincaré
$S^2 = (ct)^2 - \left(x^2 + y^2 + z^2\right) - w^2$	5-D K, $\pentagon^2$	Dynamics, START

In this geometrical setup we use a 5-D Relativity Principle "**For physical systems all trajectories are null for all observers**". Within the geometry, to obtain a comprehensive theory, the development of the theory follows:

The Geometrical Structures of Physics
SPACE + TIME + ACTION
+
Fundamental **Principles** and **Postulates** (Carriers)
$\Downarrow$
START Geometry
$\swarrow \qquad \downarrow \qquad \searrow$
General Relativity **Quantum Mechanics** Standard Model

In this program:

- **Principles** refer to primitive concepts and relations as, for example, the existence of a unified geometry for space, time and action.
- **Postulates**, to complementary concepts as the geometrical union of the manifold of those variables through the use of two constants universal c and κ_0, generating the geometry we have called START, a quadratic space.
- For a given observer the physical objects constituents of a system are the **carriers**, defined as a distribution of energy over space $E_{(c)}(x,t) = \varepsilon_{(c)}\rho_{(c)}(x,t)$, where $\varepsilon_{(c)}$ is the energy per equivalent carrier of type c.
- Two basic additional principles introduce the **unit of action** h in relation to the amount of action that can be given or taken from a system (usually described as energy and angular momentum being exchanged, this is tautologically the definition of a system) and to a **freedom of description** of matter within START.

1.2 Hypothesis and principles in START

Physics is the science which describes the basic phenomena of Nature within the procedures of the Scientific Method. The mathematization of the anthropocentric, primary, concepts of space, time and the existence of the physical objects (action **carriers**), is a suitable point of departure for creating intellectual structures which describe Nature.

We have introduced a set of principles [12, 9, 10]:

Relativity, complex Poincaré group and complex Lorentz transformations.

Existence, physical objects are represented by energy densities.

Least Action, (null trajectories in START).

Quantized Exchange of Action, defines systems or subsystems as those among a quanta of action can be exchanged.

Choice of Descriptions, allows all useful physical models to be employed.

From this we derive, in this and in previous papers, some of, the fundamental structures of Physics.

1.3 Derivation of the structure of Quantum Mechanics

The formal steps to be followed, schematically presented here, are

Steps	Subject	Formal Relation
1a define	Action Density in Space-Time	$a(X) \rightarrow E(x,t)$
1b equivalent to	Energy Density in Space	$E(x,t) = \sum^{(c)} E_{(c)}(x,t)$
2 using a	Definition of (equivalent) Carriers	$E_{(c)}(x,t) = \epsilon_{(c)}\rho_{(c)}(x,t)$
3 and the	Energy-Momentum condition	$p_\mu \doteq \partial_\mu a(X)$
4a imposing an	Analyticity requirement	$\rho_{(c)}(x,t) = \Psi^\dagger(x,t)\Psi(x,t)$
4b also	Least Action requirement: $\pentagon^2\ \Psi = 0$	$\gamma_4 \hbar K\Psi = 0 \rightarrow i\hbar\partial\Psi = mc\Psi$
5 with a	Gauging and Description Choice	$\Psi \rightarrow \Psi \exp(i\phi(X))$

That is matter is considered a (time-dependent) distribution of energy in space as a summation among carriers. Equivalent carriers in a system are attributed by definition the same amount of energy per carrier. If a different amount of energy has to be attributed to two, otherwise considered identical, carriers then they are not equivalent. Energy-momentum is defined as the rate of change of action. The finiteness of the amount of matter and the corresponding energy require that action and then the density obey analytical conditions, to fulfil this condition, as shown below (Keller and Weinberger [11]), the density has to be written as a product of the corresponding pair of square integrable analytical functions. In order to obtain the least action state we follow the Schrödinger procedure (Schrödinger 1926 [14]) to optimize the total action variable functional, which produces the wave equation. Finally because of the analyticity requirement the auxiliary amplitude functions are only defined up to general multivector phase factor s:

$$\phi = \sum^{\text{multivectors } A} \phi_A \gamma^A \quad \text{scalar, vector, bivector, etc.} \quad \text{then} \quad i\hbar\partial\Psi^\circ = mc\Psi^\circ + \overrightarrow{\overrightarrow{\triangle V}}\Psi^\circ.$$

The energy (action) carriers can be given specific names according to the mathematical structure of the corresponding densities. The words simple and composite referring to one term or several terms, the word decomposable to the fact that each of the terms is proportional to one given density (as in the case of equivalent electrons in atomic, molecular, and solid state structure). Because we can use for the description an average description, we can either define the average carrier or an average description. In most cases we are in fact using the definition of the carrier in a system through the introduction of some effective potential acting upon it. Then, as this kind of carrier can not be studied in the absence of that potential, we should really call that carrier a pseudo-carrier. This is summarized

Classification of Carriers	
Carrier Type	**Description Type**
Elementary (Simple)	$\rho_C(\boldsymbol{x},t) = \rho_c(\boldsymbol{x},t)$
Elementary (Composite)	$\rho_C(\boldsymbol{x},t) = \sum\limits_c A^c \rho_c^{\{c\in C\}}(\boldsymbol{x},t)$
Decomposable	$\rho_C(\boldsymbol{x},t) = n^D \rho_D(\boldsymbol{x},t)$
Average Carrier	$\rho_A(\boldsymbol{x},t) = \frac{1}{W}\sum\limits_{c=1}^{N} A^c \rho_c(\boldsymbol{x},t)$
Description Average ($\overline{\rho}$)	$\rho_C(\boldsymbol{x},t) = \frac{1}{\overline{W}}\sum\limits_{c=1}^{N} \frac{1}{\tau}\int_{\lambda=\lambda_0}^{\lambda=\lambda_0+\tau} A^c(\lambda)\rho_c(\boldsymbol{x},t)d\lambda$
Pseudo-Carriers	$\{\rho_c(\boldsymbol{x},t)\} \to \rho_C^{\text{eff}}(\boldsymbol{x},t) \quad [V_{\text{eff}}]$

With corresponding definitions of averaging weights and auxiliary amplitude functions:

$W = \sum\limits_{c=1}^{N} A^c$	$\overline{W} = \sum\limits_{c=1}^{N} \frac{1}{\tau}\int_{\lambda=\lambda_0}^{\lambda=\lambda_0+\tau} A^c(\lambda)d\lambda = 1$
$\boldsymbol{\Psi} = \sum^{l} \boldsymbol{a}_l \sum^{m} \boldsymbol{p}_m \prod^{s} \boldsymbol{\phi}_s^{\{l,m\}}$	$\boldsymbol{\phi}_s^{\{l,m\}} = \sum b_n \prod^{\beta} \phi_\beta^{(n)}$

2 The density

We come back to the conditions to be obeyed by the analytical function $\rho_{\text{c}}(\boldsymbol{x},t)$.

D1. $\rho_{\text{c}}(\boldsymbol{x},t)$ is a real quantity $\rho_{\text{c}}(\boldsymbol{x},t) \subset \mathbb{R}$.

D2. The density $0 \leq \rho_{\text{c}}(\boldsymbol{x},t) < \infty$ in order to represent a finite amount of action.

D3. The derivatives of the density $-\infty < \partial_\mu \rho_{\text{c}}(\boldsymbol{x},t) < +\infty$ in order to represent a finite amount of energy–momentum.

D4. The integral of the density $\int_{\Omega_C} \rho_{\text{c}}(\boldsymbol{x},t)d\Omega = n$ in the n carriers' volume Ω_c.

Theorem 1 (Keller and Weinberger 2002). *If $\Psi(x,t)$ is an analytical quadratic integrable complex or multivector function, conditions D1, D2 and D3 are fulfilled identically if $\rho_{\text{c}}(x,t) = |\Psi_{\text{c}}(\boldsymbol{x},t)|^2$. Here $|f|^2$ means the real quadratic form of any more general function f, even if f itself is not necessarily a real function and we define: if $|f|^2 = f^+ f$ then $\partial_\mu |f|^2 = (\partial_\mu f^+)f + f^+(\partial_\mu f)$.*

Proof. Reality condition D1 is fulfilled by the definition $\rho_{\text{c}}(\boldsymbol{x},t) = |\Psi_{\text{c}}(\boldsymbol{x},t)|^2$, **Carrier Finiteness** D2 by the requirement of quadratic integrability, **Energy Finiteness** D3 by the definition $\partial_\mu |f|^2 = (\partial_\mu f^+)f + f^+(\partial_\mu f)$ and the analytical properties of $\Psi(\boldsymbol{x},t)$. ■

It can be seen that the conditions D1, D2, D3 and **normalization** $\int_V \rho_{\text{c}}(\boldsymbol{x},t)d\boldsymbol{x} = n_{\text{c}}$ correspond to the $\Psi(\boldsymbol{x},t)$ being quadratic integrable **Hilbert functions**. In any Lagrangian type formulation this last definition of ρ can be used as a condition introduced via a Lagrange multiplier.

2.1 The auxiliary amplitude function

There is an additional number of conditions that the auxiliary amplitude function will have to fulfil in order to be adequate to describe different physical situations and systems as discussed below. Besides conditions of purely mathematical origin there will be additional conditions from physical considerations about the system to be described. There are two main ones.

First the description of **alternate possibilities** $\boldsymbol{\phi}_s^{\{l,m\}} = \sum^i a_i \varphi_i^{(S)}$ which are to be included together in the formulation, alternatives which can be either of fundamental physical origin or arising from the choice of describing a system with several alternatives without the selection of only one set of alternatives. This first choice will immediately be related to concepts of either statistics of probabilities in the measurement. That is if an actual measurement will be performed, some statistical distributions would be found. Probabilistic in the sense that if an actual measurement is performed one or one set of states would be found with an attached probability. It is fundamental to distinguish between statistics referred to actual distributions and probabilities referred to possible outcomes of a measurement. Second, the fact that many systems are better described, or should in fact be described, as collections of "carriers" introduces an additional condition which is also called "statistics": the carriers can be equivalent or non-equivalent and in the case of equivalent carriers the descriptions of some carriers of the system can be the same as the description of others (the so-called Bose–Einstein statistics) or alternatively should be all different (the so-called Fermi–Dirac statistics). In the case of the Fermi–Dirac statistics the auxiliary amplitude function must mathematically be built from a combination of at least as many, in the Hilbert function sense, orthogonal functions as there are equivalent carriers to be described.

In the 77 years during which quantum mechanics has been under construction it has been useful to introduce names related to the functions, as for example: basis functions, basis sets, states, configurations, etc. and the every use day of these names can sometimes be felt as corresponding to physical entities and not just to the elements of a mathematical description. This is even more the case because the symmetries relating these mathematical elements and other properties of the solutions of the differential equations obeyed by the functions are in themselves useful to analyze experimental results.

2.2 The geometry of the auxiliary amplitude functions

In the case of a relativistic treatment of the optimization procedure to obtain the least action state we have to include a geometrical Lorentz transformation $R(x)$. This (as discussed in Keller [7]) corresponds to a transformation of the energy-momentum 4-vector p_μ followed by the procedure of obtaining the components from the auxiliary amplitude functions, as follows

Algebraic Equation for the Amplitude Function (Keller 1991)	
$e_\mu p^\mu(\boldsymbol{x}), \quad \boldsymbol{x} = e_\mu x^\mu$	p^μ per unit ρ
$e_\mu p^\mu(\boldsymbol{x}) = p^{(0)} e_0^{(0)}$ (1)	Lorentz Transf.
$e_\mu^{(0)} = R(\boldsymbol{x}) e_\mu R^{-1}(\boldsymbol{x}), \quad R^{-1} = R^\dagger$	Lorentz Transf.
$e_\mu p^\mu(\boldsymbol{x}) = p^{(0)} R(\boldsymbol{x}) e_0 R^{-1}(\boldsymbol{x})$ (2)	Lorentz Transf.
$e_\mu p^\mu(\boldsymbol{x}) R(\boldsymbol{x}) = p^{(0)} R(\boldsymbol{x}) e_0 = m_0 c R(\boldsymbol{x}) e_0,$	$R(\boldsymbol{x})$ on right
$P_{+\uparrow} = e_0 P_{+\uparrow} = P_{+\uparrow} e_0, \quad P_{+\uparrow} = P_{+\uparrow} i e_1 e_2$	Ref. Projectors
$e_\mu p^\mu R(\boldsymbol{x}) P_{+\uparrow} = m_0 c R(\boldsymbol{x}) P_{+\uparrow} i e_0 e_1 e_2,$ (3)	Apply Projectors
$\Psi(\boldsymbol{x}) = A(\boldsymbol{x}) R(\boldsymbol{x}) P_{+\uparrow} \ \in \widehat{C}_{1,3},$ (4)	Square Root of ρ
$-i\hbar \partial^\mu e_\mu \Psi(\boldsymbol{x}) = m_0 c \Psi(\boldsymbol{x}) i e_0 e_1 e_2 = \hbar \partial^4 \Psi(\boldsymbol{x}) e_0 e_1 e_2$	$\Psi(\boldsymbol{x})$ in (3)
$\rho = \rho(\boldsymbol{x}) = A^2(\boldsymbol{x}),$	Define Density
$\rho = j^0 = \Psi(\boldsymbol{x}) e_0 (\Psi(\boldsymbol{x}))^\dagger,$	Carrier Density

The function $\Psi(\boldsymbol{x}) = \Psi^V(\boldsymbol{x})$ above explicitly contains then three main contributions: the **existence of the carriers**' field in $A^V(\boldsymbol{x})$; the observer's description of the **relative motion**

of the field in $R(\boldsymbol{x})$ a (local) Lorentz transformation; and the reference to a **preferred sign of rest mass** m_0 and, writing $R(\boldsymbol{x}) = R^V(\boldsymbol{x})R^{(0)}$ **spin** (through $R^{(0)}$ and $P_{+\uparrow}$). The density $\rho(\boldsymbol{x}) = j^0(\boldsymbol{x})$ must be an acceptable density in the presence of the external potential $V(\boldsymbol{x})$.

As mentioned the formal structure of quantum mechanics allows a statistical analysis of both the structure of the density and the structure of the amplitude functions. This is as far as in probability theory a normalized summation of probabilities corresponds to alternatives in the outcome of the statistical study and a product of probabilities to sequential conditions in the statistical analysis. The amplitude functions can then be written as sums of products:

"Probability" in the Derivation of Quantum Mechanics from START	
$\boldsymbol{\Psi}^S = \sum_{l=1}^{L} \boldsymbol{a}_l \left\{ \sum^m \boldsymbol{p}_m^{(l)} \prod_{s=1}^{n} \boldsymbol{\phi}_s^{\{l,m\}} \right\}$	$\boldsymbol{\phi}_s^{\{l,m\}} = \sum^i a_i \varphi_i^{(S)}$
Sum of L Descriptions of n Carriers $\boldsymbol{\Psi}^S$ = Linear Form of ρ (quadratic scalar) $\lvert\boldsymbol{a}_l\rvert^2 \Rightarrow$ weight of collective description	$\boldsymbol{\phi}_s^{\{l,m\}}$ **describes** 1 carrier in S $\lvert a_i\rvert^2 \Rightarrow$ weight of basis description $\left\{ \sum^m \boldsymbol{p}_m^{(l)} \prod_{s=1}^{n} \boldsymbol{\phi}_s^{\{l,m\}} \right\} \implies$ **n** carriers

3 KKW-DFT

We proceed now to the specific analysis corresponding to density functional theory [13]. The result is that the theorems of Hohenberg and Kohn [2] and of the present author and collaborators appear now as basic natural conditions in the logical structure of the theory [3–6]. Our initial approach is not related with the replacement of the local study of the properties of the density with those of an equivalent density of a gas of carriers (free electron gas for example).

From the considerations above we describe each and any of the n carriers obeying Fermi–Dirac statistics through an auxiliary amplitude function $\psi_i = \boldsymbol{\phi}_s^{\{l,m\}}$ (describing carriers in the system S) where $\langle \psi_i \psi_j \rangle = \delta_{ij}$, then

$$\psi_i = \sum_\lambda a_i^\lambda \varphi_\lambda, \quad \sum_\lambda \left|a_i^\lambda\right|^2 = l \quad \text{for all} \quad i = 1, \ldots, n, \quad \text{where} \quad \sum_i \left|a_i^\lambda\right|^2 = w^\lambda,$$

and the weight of every spin–orbital

$$\sum_\lambda w^\lambda = n \quad \text{number of electrons, such that} \quad \left|a_i^\lambda\right|^2 = \left|a_j^\lambda\right|^2, \quad i, j = 1, \ldots, w,$$

according to the requirement that the density descriptions of all carriers are equivalent. That is

$$\left(a_i^\lambda\right)^* a_j^\lambda + a_i^\lambda \left(a_j^\lambda\right)^* = 2w^\lambda \delta_{ij} \qquad \text{and} \qquad \int \frac{1}{2} \left(\varphi_\lambda^* \varphi_\mu + \varphi_\lambda \varphi_\mu^*\right) dv = \delta_{\lambda\mu},$$

consequently all φ_λ should be solutions of the same differential equation corresponding to different eigenvalues. Also we define

$$\int \rho \, d\text{vol} = n, \qquad \rho = \sum_i |\psi_i|^2 = \sum_i \rho_i = n\rho_i = n\rho_{(1)}, \qquad \rho_i = \rho_j = \rho_{(1)} \quad \text{all } i, j.$$

With these definitions we can compute the carrier–carrier interaction energy as the sum $\varepsilon_{\text{c-c}} = \varepsilon_{\text{coul}} + \varepsilon_{\text{xc}}$, where

$$\varepsilon_{\text{coul}} = \frac{(n-1)}{2} \iint \rho(2)\rho(1) \frac{e^2}{r_{12}} dv_2 dv_1, \qquad \varepsilon_{\text{xc}} = \sum\nolimits_s \frac{n}{2} \iint \rho_{(xc)}^s(2) \rho_{(xc)}^s(1) \frac{e^2}{r_{12}} dv_2 dv_1,$$

$$\rho_{xc}^s(x) = \frac{1}{2} \sum_{\lambda\mu} \frac{1}{2} \left[\left(a_i^\lambda\right)^* a_i^\mu \varphi_\lambda^*(x) \varphi_\mu(x) + a_i^\lambda \left(a_i^\mu\right)^* \varphi_\lambda(x) \ddot{\varphi}_\mu^*(x) \right]^s,$$

where the super-index s stands for spin.

3.1 The Pauli exclusion principle, exchange and correlation

The above formulation of density functional theory includes the crucial concept of carrier statistics in its formulation for a set of carriers, the one corresponding to what, in the standard literature, is called **fermions**. This is, a type of carriers for which, although being indistinguishable, a "exclusion principle" applies stating that "not two identical carriers can be described the same". That is what is usually called the Exclusion Principle after Wolfgang Pauli's formulation in 1923. For these type of carriers even if the density per carrier is the same as far as they are indistinguishable, for each carrier there should be a description, through the auxiliary amplitude function, which must be explicitly different.

- Otherwise stated in the case of fermion carriers the formulation of the exclusion principle can not simply be the textbook type: "two equivalent fermion carriers can not occupy the same state" without specifying that nevertheless they are equivalent. Then the concept "different states" in fact has to refer to different mathematical descriptions of "physically equivalent states". This is in fact present even in the accepted formulation of quantum mechanics where a symmetrization is always performed which renders the carriers equivalent. For example the so-called Hartree–Fock scheme, which after Slater can be synthetically described to that corresponding to a single determinant wave function, is in fact an explicit form of introducing the equivalence of the carriers as a weighted sum of descriptions where every carrier is in turn described as occupying each one of the available "states". The systematic continuation of the Hartree–Fock scheme through the so called Configuration Interaction is introduced as a weighted sum of Slater determinants, assuring thus that each term describes each carrier as equivalent to the other ones. Then the correct formulation of the Pauli exclusion principle states:

- *The description of each carrier of any pair of equivalent carriers should be different.*

In short for fermion carriers the density distribution of each carrier has to be the same but the auxiliary function for each carrier must be different (and is not necessary different for boson carriers).

In our formulation as above this auxiliary condition demands that the auxiliary function for each of the n carriers, a sum over $m \geq n$ descriptions, contains a set of mutually orthogonal coefficients for each description λ where each coefficient squares to the same real number w^λ/n. In our formulation $m > n$ corresponds to what is called Configuration Interaction.

3.1.1 Exchange distribution density

The considerations for the case of fermion carriers define an additional new kind of distribution function: When the real square of the auxiliary function is taken, to obtain the carrier distribution ρ, a sum of cross terms $\rho_{\lambda\lambda'}(x)$ is also obtained. Each individual term $\rho_{\lambda\lambda'}(x)$ integrate to zero, also their sum.

This sum $\rho_{(X)} = \sum_{\lambda\lambda'} \rho_{\lambda\lambda'}(x)$ defines anyhow a new kind of distribution density where two auxiliary description functions are multiplied together, which we will call Exchange Distribution Density. Below these terms contribute to the evaluation of the pair wise interaction energy which corresponds to what is usually denoted as exchange and correlation energy. Then the additional contributions can be summed into two different terms, first, from the usual concept of exchange energy, a sum over n terms with coefficient $1/n$ and the rest of the terms corresponding to the usual concept of correlation contribution.

Then our formulation includes all customary descriptions of the energy contributions in an explicit form.

3.1.2 The START formulation of the exchange and correlation density functionals and local potentials

The existence of the exchange distribution density allows the direct formulation of the density functional for the exchange–correlation energy and of, through derivation with respect to the density, the local exchange–correlation potentials.

The exchange–correlation energy, as stated above, consists of two parts: the pair wise interaction terms arising from the exchange distribution density (here $\sum^{s}$ stands for the two spin values)

$$\varepsilon_{xc} = \sum^{s} \iint \rho_{xc}(2)\rho_{xc}(1)\frac{e^2}{r_{12}}\, dv_2 dv_1 = \iint \rho_{xc}(2)\chi(1)\rho(1)\frac{e^2}{r_{12}}\, dv_2 dv_1,$$

and the self-coulomb part which is, in view or the equivalence of all electrons, the fraction $-1/n$ of the Coulomb interaction energy arising from the density–density pair-wise interaction (this term is already included above as part of the definition of the correct density–density Coulomb interaction energy)

$$\Delta\varepsilon_{\text{self-coul}} = -\frac{1}{2}\iint \rho(2)\rho(1)\frac{e^2}{r_{12}}\, dv_2 dv_1,$$

if a comparison is made with the usual approach the two terms above should be added. From these terms the local potential from the exchange–correlation part is directly obtained through a direct functional derivative:

$$V^{s}_{xc}(1) = \int \rho_{xc}(2)\chi(1)\frac{e^2}{r_{12}}\, dv_2,$$

where the new variable, *exchange ratio* $n\chi(x) = 2\rho^{s}_{xc}(x)/\rho(x)$, the ratio of the (per spin) exchange density to the total (per spin) local electron density, has been introduced. The equations to solve for the auxiliary functions (central charge Ze atomic-like calculation) are

$$-\frac{1}{2m}\nabla^2\psi_i + \left[-\frac{Ze^2}{r_1} - \left(\frac{n}{2}-1\right)\int \rho_{xc}(2)\chi(1)\frac{e^2}{r_{12}}dv_2 + (n-1)\int \rho(2)\frac{e^2}{r_{12}}dv_2\right]\psi_i = \varepsilon_i\psi_i, \quad (1)$$

and the calculation of the total electronic energy

$$E = \sum_i \varepsilon_i - \frac{1}{2}\left[(n-1)\iint \rho(2)\rho(1)\frac{e^2}{r_{12}}dv_2 dv_1 - \left(\frac{n}{2}-1\right)\iint \rho_{xc}(2)\chi(1)\rho(1)\frac{e^2}{r_{12}}dv_2 dv_1\right].$$

As the auxiliary functions corresponding to descriptions of equivalent carriers are a unitary linear transformation (a^{μ}_i) of a basis set $\{\varphi_\mu\}$, the same equation (1) can be used to solve for the basis functions φ_μ. Also the eigenvalues ε_i (the same for the ψ_i) can be given in terms of the eigenvalues ε'_μ of the φ_μ as $\varepsilon_i = \sum_\mu \|a^{\mu}_i\|^2 \varepsilon$.

For systems with a small number of electrons the dominant factor is $((n-1)/n)$ in the electron-electron Coulomb interaction. It is basic for the calculation of the hydrogen atom, it is dominant in the calculation of the Helium atom where each of the two electrons interacts which the other but not with itself, and progressively less important for larger systems where the exchange density and the, to it related, exchange interaction grows.

In the case of the calculation of the Beryllium atom, for example, $((n-1)/n) = 3/4$ and the exchange density (dotted line), shown in the Fig. 1[1], is now as important as the basis functions densities. In the same figure (dashed line) the per electron auxiliary function and its components (1s and 2s thin solid lines) are shown as well as the corresponding electron density (dashed line). The relevant energies for Ne, for example, in this case are:

[1]Figures in colour will be available only in electronic version.

Energy Contribution (Ne atom)	**energy (Hartree units)**
electron-nuclear	−310.0951
electron-electron coulomb	59.20975 (+6.5788)
electron-electron exchange correlation	−13.2254
including self-coulomb	−6.5788
kinetic energy	128.766
total energy (exp. −128.94)	−128.766.

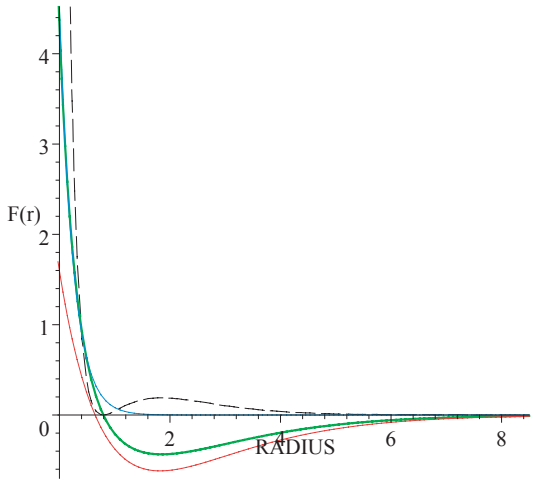

Figure 1. Beryllium atom exchange density.

The calculation is simple. The same procedure and code to obtain the total density can be used to obtain the exchange density from the same auxiliary functions. The same procedure to obtain the electron–electron coulomb potential and energy is used to obtain the (local) exchange potential and exchange energy. The self-consistent algorithm remains the same. The entire procedure is at least as feasible as the currently employed density functional procedures for actual calculations. In the case of a solid the factor $f = \lim_{n\to\infty}((n-1)/n) = 1$ is irrelevant and the summation over pairs of basis functions becomes the starting point for the correct description of exchange and correlation. In condensed matter calculation of the double summation over pairs of auxiliary functions should be replaced by actual double integrals.

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Universal Symmetry of Complexity and Its Manifestations at Different Levels of World Dynamics

Andrei P. KIRILYUK

Institute of Metal Physics, 36 Vernadskogo Ave., 03142 Kyiv-142, Ukraine
E-mail: *kiril@metfiz.freenet.kiev.ua*

The unreduced, universally nonperturbative analysis of arbitrary interaction process, described by a quite general equation, provides the truly complete, "dynamically multivalued" general solution that leads to dynamically derived, universal definitions of randomness, probability, chaoticity, complexity, fractality, self-organisation, and other properties, extending their axiomatic introduction in the conventional, dynamically single-valued theory. Any real system emergence, structure, and behaviour can be expressed now by the universal law of conservation, or symmetry, of complexity that unifies extended versions of any (correct) symmetry, law, or "principle". Particular applications of the universal symmetry of complexity, from fundamental physics to biology and theory of consciousness, provide old mysteries solutions and new research perspectives.

1 Introduction: the unreduced symmetry of nature

The *conventional symmetry* idea used in physics and mathematics (see e.g. [1–3]) assumes the existence of *previously determined* (eventually postulated) structures and properties that appear as dynamical laws relating "variables" $\mathcal{X}$ and "parameters" $\mathcal{P}$: $\mathcal{C}(\mathcal{X};\mathcal{P}) = 0$. A formal symmetry, or "invariance" of the law, is introduced usually through the symmetry transformation "operator", $\hat{\mathcal{S}}$, whose action is compatible with the law in question, $\hat{\mathcal{S}}[\mathcal{C}(\mathcal{X};\mathcal{P})] = 0$. This fact can be used to reduce the law expression to a desired particular form, including the explicit, functional relation between *separated* variables and parameters, or "(exact) solution" of a problem: $\mathcal{X} = f(\mathcal{P})$. However, application of this symmetry concept to the real world dynamics reveals irreducible *ruptures* between various known symmetries and systematic *violations* of almost every exact symmetry, accounting for the real world irregularity, which leads to the "(spontaneously) broken symmetry" concept with quite fundamental consequences.

A much more general interpretation of symmetry is possible [4, 5], within which the *real* world structure *explicitly emerges* as *inevitable realisation* of the *universal dynamic symmetry*, or "conservation law", $C = \text{const}$, where *all* observed entities, properties, and measured quantities are *derived* as forms or manifestations of that universal symmetry, remaining thus *always exact* (unbroken), but producing all the observed *irregularities*. *Any* (correct) dynamical law, $\mathcal{C}(\mathcal{X};\mathcal{P}) = \mathcal{C}_0$, as well as the *unreduced* problem solution, $\mathcal{X} = \mathcal{F}(\mathcal{P})$, is also obtained now, in its essentially extended, *causally complete* form, as a *rigorously derived* (rather than postulated), *totally realistic* manifestation of that single, *intrinsically unified* symmetry. We show that the role of such unified symmetry belongs uniquely to the *universal symmetry (conservation) of dynamic complexity*, where the latter quantity is rigorously derived from the *unreduced* (nonperturbative) analysis of arbitrary (real) *interaction process* [4, 5]. This unreduced complexity, giving rise to the unified dynamical "order of the universe", is essentially different from conventional complexity versions which do not originate from the unreduced problem solution, but reflect a strongly reduced (zero-dimensional) *projection* of real system dynamics.

2 Unreduced dynamic complexity of arbitrary interaction process, its conservation and internal transformation

Mathematical expression of dynamics of a vast variety of interaction processes can be generalised in the form of "existence equation" [4–7] that actually only fixes the fact of interaction within a system of a given configuration:

$$[h_{\rm g}(\xi) + V_{\rm eg}(q,\xi) + h_{\rm e}(q)]\,\Psi(q,\xi) = E\Psi(q,\xi)\,, \tag{1}$$

where $\Psi(q,\xi)$ is the system "state-function", which totally determines its configuration and depends on the degrees of freedom, ξ and q, of the system components, $h_{\rm g}(\xi)$ and $h_{\rm e}(q)$ are "generalised Hamiltonians" of the free (non-interacting) components (i.e. measurable functions eventually expressing dynamic complexity, defined below), $V_{\rm eg}(q,\xi)$ is (arbitrary) interaction potential, E is the generalised Hamiltonian eigenvalue for the whole system, and any number of interacting components can actually be implied behind equation (1), leading to the same results below [5].

It will be convenient to express the problem in terms of the internal system states by performing expansion of the state-function $\Psi(q,\xi)$ over the complete system of eigenfunctions, $\{\phi_n(q)\}$, for all free-state degrees of freedom but one (described here by the ξ variable and usually representing the global system configuration, such as spatial coordinates of its structure):

$$\Psi(q,\xi) = \sum_n \psi_n(\xi)\phi_n(q)\,, \qquad h_{\rm e}(q)\,\phi_n(q) = \varepsilon_n\phi_n(q)\,. \tag{2}$$

Substituting expansion (2) into the existence equation (1), multiplying it by $\phi_n^*(q)$, integrating over q variables (or using other "scalar product" definition), and assuming the orthonormality of eigenfunctions $\{\phi_n(q)\}$, we get a system of equations, which is equivalent to the starting existence equation and includes all its particular cases (e.g. nonlinear or time-dependent forms):

$$\begin{aligned}
&[h_{\rm g}(\xi) + V_{00}(\xi)]\,\psi_0(\xi) + \sum_n V_{0n}(\xi)\,\psi_n(\xi) = \eta\psi_0(\xi)\,,\\
&[h_{\rm g}(\xi) + V_{nn}(\xi)]\,\psi_n(\xi) + \sum_{n'\neq n} V_{nn'}(\xi)\,\psi_{n'}(\xi) = \eta_n\psi_n(\xi) - V_{n0}(\xi)\,\psi_0(\xi)\,,
\end{aligned} \tag{3}$$

where $\eta_n \equiv E - \varepsilon_n$,

$$V_{nn'}(\xi) = \int_{\Omega_q} dq\phi_n^*(q)\,V_{\rm eg}(q,\xi)\,\phi_{n'}(q)\,,$$

and we have separated the equation with $n = 0$ from the system (3), so that other $n \neq 0$ (also below) and $\eta \equiv \eta_0$.

Expressing $\psi_n(\xi)$ from equations (3) through $\psi_0(\xi)$ by the standard Green function technique [8, 9] and inserting the result into the equation for $\psi_0(\xi)$, we restate the problem in terms of *effective existence equation*, formally involving only the selected degrees of freedom ξ:

$$[h_{\rm g}(\xi) + V_{\rm eff}(\xi;\eta)]\,\psi_0(\xi) = \eta\psi_0(\xi)\,, \tag{4}$$

where the *effective (interaction) potential (EP)*, $V_{\rm eff}(\xi;\eta)$, is given by

$$\begin{aligned}
&V_{\rm eff}(\xi;\eta) = V_{00}(\xi) + \hat{V}(\xi;\eta)\,, \qquad \hat{V}(\xi;\eta)\,\psi_0(\xi) = \int_{\Omega_\xi} d\xi' V\left(\xi,\xi';\eta\right)\psi_0\left(\xi'\right)\,,\\
&V\left(\xi,\xi';\eta\right) = \sum_{n,i} \frac{V_{0n}(\xi)\,\psi_{ni}^0(\xi)\,V_{n0}(\xi')\,\psi_{ni}^{0*}(\xi')}{\eta - \eta_{ni}^0 - \varepsilon_{n0}}\,, \qquad \varepsilon_{n0} \equiv \varepsilon_n - \varepsilon_0,
\end{aligned} \tag{5}$$

and $\{\psi^0_{ni}(\xi)\}$, $\{\eta^0_{ni}\}$ is the complete set of eigenfunctions and eigenvalues for an auxiliary, truncated system of equations (where $n, n' \neq 0$):

$$[h_{\rm g}(\xi) + V_{nn}(\xi)]\,\psi_n(\xi) + \sum_{n' \neq n} V_{nn'}(\xi)\psi_{n'}(\xi) = \eta_n \psi_n(\xi)\,. \tag{6}$$

The general solution of the initial existence equation (1) is then obtained as [4–8]:

$$\begin{aligned}
&\Psi(q,\xi) = \sum_i c_i \left[\phi_0(q) + \sum_n \phi_n(q)\,\hat{g}_{ni}(\xi)\right]\psi_{0i}(\xi)\,,\\
&\psi_{ni}(\xi) = \hat{g}_{ni}(\xi)\,\psi_{0i}(\xi) \equiv \int_{\Omega_\xi} d\xi' g_{ni}\left(\xi,\xi'\right)\psi_{0i}\left(\xi'\right),\\
&g_{ni}\left(\xi,\xi'\right) = V_{n0}\left(\xi'\right)\sum_{i'}\frac{\psi^0_{ni'}(\xi)\,\psi^{0*}_{ni'}(\xi')}{\eta_i - \eta^0_{ni'} - \varepsilon_{n0}},
\end{aligned} \tag{7}$$

where $\{\psi_{0i}(\xi)\}$ are the eigenfunctions and $\{\eta_i\}$ the eigenvalues found from equation (4), while the coefficients c_i should be determined by state-function matching on the boundary where the effective interaction vanishes. The observed system density $\rho(q,\xi)$ is given by the squared modulus of the state-function $\rho(q,\xi) = |\Psi(q,\xi)|^2$ (for "quantum" and other "wave-like" levels of complexity), or by the state-function itself $\rho(q,\xi) = \Psi(q,\xi)$ (for "particle-like" levels) [4].

Although the "effective" problem formulation of equations (4)–(7) forms the basis of the well-known optical, or effective, potential method (see e.g. [9]), it is actually used in its reduced, perturbative versions, where the "nonintegrable", nonlinear links in the above EP and state-function expressions are cut in exchange to the closed, "exact" solution. However, this reduction kills the *essential, dynamic nonlinearity* of the real system, together with its intrinsic complexity and chaoticity, and thus replaces the natural symmetry of complexity by an artificial, simplified symmetry of perturbative solutions [4–8]. Indeed, it is not difficult to show that the unreduced "effective" problem has *many* locally complete and therefore *incompatible* solutions, *each* of them being equivalent to the *single*, "complete" solution of the reduced problem, usually attributed also to the initial formulation of equations (1), (3). If N_ξ and N_q are the numbers of terms in sums over i and n in equation (5), then the total number of eigenvalues of equation (4) is $N_{\max} = N_\xi(N_\xi N_q + 1) = (N_\xi)^2 N_q + N_\xi$, which gives the N_ξ-fold redundance of the usual "complete" set of $N_\xi N_q$ eigen-solutions of equations (3) plus an additional, "incomplete" set of N_ξ solutions. Each redundant solution, *intrinsically unstable* with respect to system transitions to other solutions, can be called system *realisation*, since it represents a completely determined system configuration. The total number of "regular", complete system realisations is $N_\Re = N_\xi$, whereas the mentioned additional set of solutions forms a special, "intermediate" realisation that plays the role of transitional state during system jumps between the regular realisations and provides thus the universal, *causally complete* extension of the quantum *wavefunction* and classical *(probability) distribution function* [4, 5].

Thus rigorously derived, qualitatively new property of *dynamic multivaluedness* of the unreduced problem solution is confirmed by its "geometric" analysis and particular applications [4–8]. It provides the intrinsic, omnipresent, and irreducible source of *purely dynamic, or causal, randomness*: the incompatible system realisations, being *equally* real, should *permanently* replace one another, in a *causally random* order, so that the observed density of any real system should be presented as the *dynamically probabilistic* sum of the individual realisation densities, $\{\rho_r(\xi,q)\}$, obtained by solution of the effective existence equation (4):

$$\rho(\xi,Q) = \sum_{r=1}^{N_\Re}{}^{\oplus}\rho_r(\xi,Q)\,, \tag{8}$$

where summation is performed over all (observable) system realisations, numbered by r, and the sign $\oplus$ serves to designate the special, dynamically probabilistic meaning of the sum derived above and consisting in *permanent change* of regular realisations in *dynamically random (chaotic)* order by transition through the intermediate realisation. The dynamically obtained, *a priori probability* of the r-th realisation emergence, α_r, is determined, in general, by the number, N_r, of elementary, experimentally unresolved realisations it contains:

$$\alpha_r\left(N_r\right)=\frac{N_r}{N_{\Re}},\qquad N_r=1,\ldots,N_{\Re},\qquad \sum_r N_r=N_{\Re},\qquad \sum_r \alpha_r=1. \tag{9}$$

According to the "generalised Born's rule", obtained by dynamical matching in the intermediate realisation (wavefunction) phase, the dynamic probability values are determined by the generalised wavefunction obeying the causally derived, *universal Schrödinger equation* [4,5] (see below).

Another important property of the unreduced solution, closely related to the above dynamic multivaluedness, is *dynamic entanglement* between the interacting entities (degrees of freedom) within each realisation, which appears as dynamically weighted products of functions of ξ and q in equations (7) and determines the *tangible new quality* of the emerging interaction results. It leads to the *dynamical* system squeeze, or *reduction*, or *collapse*, to the emerging configuration of each realisation, alternating with the reverse *dynamic disentanglement*, or *extension*, of interacting entities to a quasi-free state in the intermediate realisation (wavefunction), during transitions between realisations [4–7]. The dynamically multivalued entanglement is a totally autonomous *process*, driven only by the system interaction and characterised by the intrinsic *nonseparability* and *irreversible* direction. Nonseparable component entanglement gives rise to the explicitly emerging, physically real *space* (in the form of the squeezed, final realisation configuration, or generalised space "point"), while the irreversible, unceasing and *spatially chaotic* realisation change determines the causal *time* flow [4, 5]. These properties of the dynamically multivalued entanglement between the interacting components are hierarchically reproduced and amplified within the *dynamically fractal* structure of the unreduced problem solution, which can be obtained by application of the same EP method to the truncated system of equations (6) whose solutions are used in the expressions of the first level of solution, equations (5), (7). We obtain thus the causally complete extension of the conventional, dynamically single-valued fractality and the true meaning of (*any* real) system *nonintegrability*, which takes the form of the permanently changing, dynamically probabilistic ("living") fractal hierarchy of the unreduced problem solution, possessing the rigorously obtained properties of *explicit structure emergence (creativity)* and *dynamic adaptability* (self-consistent configuration of the "effective" solution of equations (4)–(7)) [4, 5, 7].

Now that the *dynamically multivalued* structure of the unreduced interaction process has been *explicitly revealed*, we can provide the unrestricted, *universally applicable* definition of *dynamic complexity*, C, as any growing function of realisation number, $C=C(N_{\Re}),\ dC/dN_{\Re}>0$, or the rate of their change, equal to zero for the (unrealistic) case of only one realisation, $C(1)=0$. It is just the latter, unrealistically simplified "model" (zero-dimensional, point-like projection) of reality which is *exclusively* considered in the conventional, dynamically single-valued, or *unitary*, theory, including its concepts of "complexity", "chaoticity", "self-organisation", etc., which explains all its persisting "mysteries" and "difficult" problems, easily finding their dynamically multivalued, causally complete solution within the unreduced complexity concept [4–8] that emerges thus as the direct, qualitative *extension* of the unitary knowledge model to the dynamically multivalued reality. In particular, the properties of dynamic multivaluedness and entanglement show that *chaoticity* is *synonymous to complexity*, in their unreduced, *omnipresent* versions.

In that way the *regular* and *separated* symmetries of the unitary model, *always* (fortunately!) *violated* in the real world (cf. the concept of "spontaneously" broken symmetry), are replaced,

in the unreduced description, by the single, *intrinsically unified, but diverse in manifestations, irregular, but exact (never broken) symmetry of complexity* [4, 5]. Moreover, contrary to the artificially imposed, external origin of the conventional symmetries, mechanistically added to the postulated structures, properties, and "principles", the universal symmetry of complexity emerges as the unique source of existence giving rise, through the explicitly obtained relations, to *all* real entities and (correct) laws, in their causally extended, complex-dynamical (multivalued) version. At a given level of system complexity (described by the above solution (4)–(9)), this irreducibly *dynamic* symmetry appears as equivalence between all (elementary) system realisations meaning their "equal chances" to emerge and permanent actual change, as reflected in the probability expression (9) (also in its relation to the wavefunction values) and dynamically probabilistic sum of the general solution, equation (8). All realisations *differ* in their detailed structure and are taken by the system in a truly random order (equations (4)–(9)), thus reproducing the *real world irregularity*, but the resulting internally *irregular* symmetry between them is *exact* as such (unbroken) and can be expressed simply as fixed realisation number for any given system (interaction process).

However, the universal symmetry of complexity does not stop there: it involves a *qualitative* change of the *form* of complexity that *preserves* its total *quantity*. Namely, the potential, or "hidden" (latent) form of complexity, called *dynamic information* (and generalising "potential energy"), is transformed into the explicit, "unfolded" form of *dynamic entropy* (extending entropy concept to any process), so that their sum, the total system complexity remains unchanged, which gives rise to all emerging entities, their properties and behaviour (reflected in particular "laws" and "principles") [4, 5]. The basic origin of that complexity transformation is revealed by the same, unreduced interaction description, containing the *explicit emergence* of *always internally chaotic* entities and their interactions (given by higher, fine levels of the fractal hierarchy of unreduced interaction development).

The length element, Δx, of a complexity level is obtained from solution of the unreduced "effective" equation (4)–(5) as the distance between the centres of the neighbouring realisation eigenvalues, $\Delta x = \Delta\eta_i^r$, while the time flow rate emerges as intensity (specified as *frequency*, ν) of *realisation change*. Since the emerging space and time represent the two basic, universal forms of complexity, its universal, natural measure should be independently proportional to measures of space and time. It is easy to see that such complexity measure is provided by *action* quantity acquiring thus its extended, *essentially nonlinear*, meaning: $\Delta\mathcal{A} = -E\Delta t + p\Delta x$, where Δx, and $\Delta t = 1/\nu$ are the above dynamic space and time increments, $\Delta\mathcal{A}$ is the corresponding complexity-action increment, while the coefficients, E and p, are identified as energy and momentum. The action value always decreases ($\Delta\mathcal{A} < 0$) and represents the dynamic information, whereas complexity-entropy change is the quantity opposite in sign, $\Delta S = -\Delta\mathcal{A} > 0$, leaving their sum, the total complexity, unchanged, $C = \mathcal{A} + S = \text{const}$. Dividing the differential expression of conservation (symmetry) of complexity by $\Delta t\,|_{x=\text{const}}$, we get the generalised Hamilton–Jacobi equation [4, 5]:

$$\frac{\Delta\mathcal{A}}{\Delta t}\,|_{x=\text{const}} + H\left(x, \frac{\Delta\mathcal{A}}{\Delta x}\,|_{t=\text{const}}, t\right) = 0, \tag{10}$$

where the *Hamiltonian*, $H = H(x, p, t)$, considered as a function of emerging space-structure coordinate x, momentum $p = (\Delta\mathcal{A}/\Delta x)\,|_{t=\text{const}}$, and time t, expresses the implemented, entropy-like form of differential complexity, $H = (\Delta S/\Delta t)\,|_{x=\text{const}}$. Because of a dynamically random order of emerging system realisations, the total time derivative of action, or *Lagrangian*, $L = \Delta\mathcal{A}/\Delta t = pv - H$, should be negative (where $v = \Delta x/\Delta t$ is the global-motion velocity), which provides the rigorously derived, *dynamic* expression of the "arrow of time" orientation to growing entropy:

$$L < 0 \quad \Rightarrow \quad E, H\left(x, \frac{\Delta\mathcal{A}}{\Delta x}\Big|_{t=\text{const}}, t\right) > pv > 0.$$

Realisation change process can be considered also as two adjacent complexity sublevels whose conserved total complexity C equals to the product of complexity-entropy of localised (regular) realisations and "potential" wavefunction complexity, $C = S\Psi = \text{const}$, meaning that $\mathcal{A}\Psi = -S\Psi = \text{const}$, where Ψ is the wavefunction. The total complexity change between two transitional states equals to zero, $\Delta(\mathcal{A}\Psi) = 0$, which expresses the physically evident permanence of the *unique* state of wavefunction, and gives the generalised *causal quantization rule*:

$$\Delta\mathcal{A} = -\mathcal{A}_0 \frac{\Delta\Psi}{\Psi}, \tag{11}$$

where $\mathcal{A}_0$ is a characteristic action value ($\mathcal{A}_0$ may contain also a numerical constant reflecting specific features of a given complexity level). Using equation (11) in equation (10) we obtain the *generalised Schrödinger equation* for Ψ in the form [4, 5]:

$$\mathcal{A}_0 \frac{\partial\Psi}{\partial t} = \hat{H}\left(x, \frac{\partial}{\partial x}, t\right)\Psi, \tag{12}$$

where the Hamiltonian operator, $\hat{H}$, is obtained from the Hamiltonian function $H = H(x, p, t)$ of equation (10) with the help of the causal quantization relation of equation (11).

The generalised Hamilton–Schrödinger formalism, equations (10)–(12), is a universal expression of the symmetry of complexity. Expanding the Hamiltonian in equation (10) in a power series of momentum and action, one obtains a form of the universal Hamilton–Schrödinger formalism that can be reduced to any usual, "model" equation by series truncation [4, 5],

$$\frac{\partial\Psi}{\partial t} + \sum_{\substack{m=0 \\ n=1}}^{\infty} h_{mn}(x,t)\left[\Psi(x,t)\right]^m \frac{\partial^n\Psi}{\partial x^n} + \sum_{m=0}^{\infty} h_{m0}(x,t)\left[\Psi(x,t)\right]^{m+1} = 0,$$

(here the expansion coefficients, $h_{mn}(x,t)$, can be arbitrary functions), which confirms its universality and shows the genuine, unified origin of model equations, semi-empirically guessed and postulated in the unitary theory. All fundamental laws and "principles" of the conventional science, such as relativity (special and general), principle of entropy increase, principle of least action, other "variational" principles, can now be obtained, in their *causally extended, complex-dynamical* versions, from the same unified law of conservation, or symmetry, of complexity [4, 5]. Note, in particular, that the universal complexity conservation, realised by its *unceasing transformation* from decreasing dynamic information (action) to increasing entropy, provides a remarkable unification of the universal, extended versions of least action principle (conventional mechanics) and entropy increase principle ("second law" of thermodynamics), which reveals the true meaning and origin of those "well-known" laws. In a similar way, the "quantum", "classical", and "relativistic" effects and types of behaviour are *causally* explained now as inevitable, and thus universally extendible, manifestations of the unified symmetry of unreduced complexity [4, 5, 11]. The underlying complex (multivalued) dynamics specifies the essential difference of the symmetry of complexity from its conventional imitations: the formal "operators" of the latter are replaced in the former by actual realisation change and complexity unfolding, just forming the real, creative world dynamics (cf. Section 1).

One should emphasize the importance of genuine, *dynamically emerging*, or "essential" *nonlinearity*, defined above and closely related to the dynamic multivaluedness, for the universal symmetry of complexity, as well as its fundamental difference from the conventional, mechanistically defined (non-dynamic) "nonlinearity". The essential nonlinearity *inevitably* emerges as a result of *unreduced* interaction development, even starting from a formally "linear" initial problem formulation (see equations (1)–(7) and the following analysis). On the other hand, formally "nonlinear" equations of the standard approach, being analysed within its reduced,

unitary projection, cannot produce any truly new structure that would not be actually postulated within the starting formulation, and therefore they remain always *basically linear*, as it is confirmed by their invariably perturbative, or *exact*, solutions. The *real* nonlinearity appears as a dynamically fractal network of self-developing interaction feedback loops, explicitly revealed just in the "effective" problem expression of the generalised EP method [4–8]. This *emerging* nonlinear structure forces the system to take, or "collapse" to, *one* of its *multiple* possible realisations, which means that those realisations, *actually* and *unceasingly* replacing one another in a *dynamically random*, or "chaotic", order, are *dynamically symmetric* among them, while they always *differ* in their detailed, partially *irregular* structure. However, the same system in the phase of transition between its normal, "localised" realisations is forced, by the *same* driving interaction, to transiently disentangle its components up to their quasi-free state of "generalised wavefunction" (see above), and that is why the system in this state temporarily behaves as a weekly interacting, quasi-linear one. This remarkable, "intermittent" structure of unreduced interaction process, remaining totally hidden in the dynamically single-valued projection of the conventional theory, explains why and how the real system dynamics naturally unifies the opposed, complementary properties of quasi-linear and highly nonlinear behaviour and symmetry. The realistic, causal explanation of "wave-particle duality" and "complementarity" in quantum systems (and classical "distributed" systems as well) is only one particular consequence of that omnipresent structure of the unreduced symmetry of complexity [4–8].

It would be worthwhile to note finally that the described conceptual transition from the conventional, dynamically single-valued (unitary) to the proposed dynamically multivalued (unreduced) description of system dynamics and the related upgrade of the separated and broken unitary symmetries to the intrinsically unified and exact symmetry of complexity involves important progress in mathematical description of reality, standing as the main, universal tool of science. We have seen that the proposed advance in that description, which practically totally eliminates the existing gap between real phenomena and their unitary "models", is realised simply due to the unreduced, or *really exact* mathematical analysis using quite ordinary particular tools. This is certainly good news for mathematics, which can thus preserve and develop its status as a universal method and basis of *objective* knowledge about reality, the image that has considerably faded in the last period of growing "uncertainty" [10], separations, and untractable technical sophistication. On the other hand, the price that is clearly to be paid for that essential and intrinsically sustainable progress consists in the corresponding considerable, fundamentally rooted upgrade of the scholar framework, which tends traditionally to hide its real difficulties behind the externally "solid" façade of the formally fixed "existence and uniqueness" theorems and other *postulated* constructions. This report presents a brief account of the means and results of elementary realisation of that qualitative transition demonstrating, in our opinion, both its feasibility and inevitability in the future progress of science.

3 Particular manifestations of the unified symmetry of complexity

We can only briefly outline here other manifestations of the universal symmetry of complexity obtained for particular or arbitrary levels of complexity and systems. One of them is universal classification of all possible types of real system behaviour which can vary continuously between the limiting cases of *uniform, or global, chaos* (quasi-homogeneous distribution of probability for sufficiently different realisations) and *multivalued self-organisation, or self-organised criticality* (inhomogeneous realisation probability distribution, close elementary realisations) [4, 5]. It is this latter case that can be more successfully approximated by conventional, dynamically single-valued (intrinsically regular) models, though with irreducible fundamental losses (such as absence

of irreversible time flow). The universal criterion of transition from self-organised (generally ordered) dynamics to the global chaos, in both quantum and classical systems, is obtained in the form of frequency resonance between interacting modes (such as intra- and inter-component dynamics), which extends considerably the concepts of both chaoticity and resonance [4–8]. The observed alternation of globally chaotic and self-organised levels in the hierarchy of complexity is another manifestation of the universal symmetry of complexity.

Application of the unreduced existence equation solution to the simplest system of two attracting, initially homogeneous protofields gives explicitly emerging field-particles, in the form of spatially chaotic quantum beat processes, endowed with the rigorously derived, realistic and unified versions of all "mysterious" quantum features, "relativistic" effects and intrinsic properties (mass, electric charge, spin), obtained as standard, inevitable manifestations of unreduced complexity [4, 5, 11]. The number (four), dynamic origin, properties and intrinsic unification of fundamental interaction forces between particles are obtained within the same picture. The true quantum chaos, passing to classical chaos by the usual semiclassical transition, intrinsically indeterminate quantum measurement, and dynamic emergence of classical, permanently localised behaviour within a closed, bound system (like atom) are obtained as naturally emerging complexity levels, with important practical conclusions for such popular applications as quantum computers, nanotechnology, and quantum many-body systems with irreducibly "strong" interaction [4–6, 11]. The obtained "emergent" and causal world picture includes also natural solution of the problems of cosmology. Finally, symmetry of complexity manifestations for biological and intelligent systems reveal the causal essence of life, intelligence, and consciousness as high enough levels of unreduced complexity, which leads to practically important conclusions [4, 5, 7] and proves once more the universal applicability of the unreduced symmetry of complexity.

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Symmetries of the Jahn–Teller Systems and Their Solvability

Eser KÖRCÜK and Ramazan KOÇ

Dept. of Physics, Faculty of Engineering, University of Gaziantep, 27310 Gaziantep, Turkey
E-mail: *korcuk@gantep.edu.tr, koc@gantep.edu.tr*

We present a method of obtaining the quasi-exact solution of the Jahn–Teller systems in the framework of $osp(2,2)$ superalgebra. The Hamiltonian have been solved in the Bargmann–Fock space by obtaining an expression as linear and bilinear combinations of the generators of $osp(2,2)$. In particular, we have discussed quasi-exact solvability of $E \times \varepsilon$ Jahn–Teller Hamiltonian.

1 Introduction

The Jahn–Teller (JT) distortion problem is an old one, dating back over sixty years [1]. Yet, even today, new contributions to this problem are being made [2]. They appear, however, not to have been fully exploited in the analysis of JT problem. The $E \otimes \varepsilon$ JT problem is a system with doubly degenerate electronic state and doubly degenerate JT active vibrational state. The JT effect describes the interaction of degenerate electronic states through non-totally symmetric, usually non-degenerate, nuclear modes. This effect plays an important role in explaining the structure and dynamics of the solids and molecules in degenerate electronic states.

The studies of the JT effect led Judd to discover a class of exact isolated solutions of the model [2]. The complete description of these solutions have been given by Reik et al [3]. They observed that the isolated solutions could be obtained by using Neumann series of expansions of the eigenvectors in the Bargmann–Fock space described by the boson operators. The same problem has been treated in [4].

On the other hand, the concept of quasi-exactly solvable (QES) systems discovered [5, 7–9] in the 1980's, has received much attention in recent years, both from the viewpoint of physical applications and their inner mathematical beauty. The classification of the 2×2 matrix differential equations in one real variable possessing polynomial solution have been described [6, 10]. The relevant algebraic structure of the $E \otimes \varepsilon$ JT system is the graded algebra $osp(2,2)$ and in this poster, we present a quasi-exact solution of the $E \otimes \varepsilon$ JT Hamiltonian.

2 Symmetry properties of the $E \times \varepsilon$ Jahn–Teller system

In this section a group theoretical treatment of JT distortion, in general case of two-fold degenerate states of various groups is provided. The JT interaction matrices and surface energies have been obtained by using symmetry properties of the system [11]. Let us start by describing the Hamiltonian that generates $D^{\ell} \otimes D^{\ell}$ surface, where D^{ℓ} denotes the irreducible representation. The standard Hamiltonian may be written in the form

$$H = H_0 + H_{\mathrm{JT}}, \tag{1}$$

where H_0 describes free (uncoupled) electron/holes and their vibrational states and H_{JT} is the Jahn–Teller interaction Hamiltonian. It is known that the Hamiltonian of the JT coupling is invariant under the rotational operations of the $SO(3)$ group. The totally symmetric part of direct product of an irreducible representations of a finite group, which describes the properties

of the JT system, can be expressed in the form of

$$\left[D^{\ell} \otimes D^{\ell}\right] = D^{\ell_1} \oplus D^{\ell_2} \oplus \cdots \oplus D^{\ell_n}, \tag{2}$$

where ℓ is the angular momentum quantum number. Decomposition of $\left[D^{\ell} \otimes D^{\ell}\right]$ implies that the JT Hamiltonian can be written in the following way

$$H_{JT} = H^{\ell_1} + H^{\ell_2} + \cdots + H^{\ell_n}, \tag{3}$$

where H^{ℓ_i} is the JT Hamiltonian and it is invariant under the symmetry operations of the corresponding finite group, for the $(2\ell+1)$-dimensional representation. As an example consider an icosahedral symmetric system. The symmetric part of the H_g interaction is given by

$$[H_g \otimes H_g] = A_g \oplus H_g \oplus (H_g \oplus G_g), \tag{4}$$

where A_g, H_g, and G_g are the irreducible representations of the icosahedral group I_h. Since I_h is a subgroup of $O(3)$ decomposition of the coupling of the $\ell = 2$ state can be written as

$$\left[D^2 \otimes D^2\right] = D^0 \oplus D^2 \oplus D^4 \tag{5}$$

and its Hamiltonian is given by

$$H_{JT} = H^0 + H^2 + H^4. \tag{6}$$

The Hamiltonians H^i must be separately invariant under the symmetry group I_h. The symmetric part contains the totally symmetric representation $H^0 = A_g$ can exactly be solved. Before going further we list the decomposition of the symmetric products of the $[E \otimes E]$ JT interaction and corresponding symmetry groups

$$\begin{aligned} O_h: &\quad [E \otimes E] = A_{1g} \oplus E, \qquad T_h: \quad [E \otimes E] = A_g \oplus E, \\ D_{2p}: &\quad [E \otimes E] = A_1 \oplus E, \qquad C_{2p}: \quad [E \otimes E] = A_1 \oplus E, \end{aligned}$$

where O_h, T_h, D_{2p} and C_{2p} denotes octahedral, tetrahedral, dihedral and cyclic groups, respectively. In the following section we discuss the construction of the $E \otimes \varepsilon$ JT Hamiltonian.

3 The $E \otimes \varepsilon$ Jahn–Teller Hamiltonian

The well-known form of the $E \otimes \varepsilon$ JT Hamiltonian describing a two-level fermionic subsystem coupled to two boson modes has been given by Reik [3]

$$H = a_1^+ a_1 + a_2^+ a_2 + 1 + \left(\tfrac{1}{2} + 2\mu\right)\sigma_0 + 2\kappa[(a_1 + a_2^+)\sigma_+ + (a_1^+ + a_2)\sigma_-], \tag{7}$$

where $\frac{1}{2} + 2\mu$ is the level separation, κ is the coupling strength. The Pauli matrices $\sigma_{\pm,0}$ are given by

$$\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{8}$$

The annihilation and creation operators, a_i and a_i^+ satisfy the usual commutation relations

$$[a_i^+, a_j^+] = [a_i, a_j] = 0, \qquad [a_i, a_j^+] = \delta_{ij}. \tag{9}$$

The number operator of the Hamiltonian (7), J_1, represents the angular momentum of the system and is given by

$$J_1 = a_1^+ a_1 - a_2^+ a_2 + \tfrac{1}{2}\sigma_0. \tag{10}$$

Note that J_1 commutes with H and the eigenvalue problem of the angular momentum part can be easily solved and it reads

$$J_1 \left|\psi\right\rangle_{j+\frac{1}{2}} = \left(j+\tfrac{1}{2}\right) \left|\psi\right\rangle_{j+\frac{1}{2}}, \qquad j = 0, 1, 2, \ldots \tag{11}$$

with the eigenfunctions

$$\left|\psi\right\rangle_{j+\frac{1}{2}} = (a_1^+)^j \phi_1(a_1^+ a_2^+) \left|0\right\rangle \left|\uparrow\right\rangle + (a_1^+)^{j+1} \phi_2(a_1^+ a_2^+) \left|0\right\rangle \left|\downarrow\right\rangle, \tag{12}$$

where $|0\rangle$ is the vacuum state for both bosons. Here $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of the σ_0, ϕ_1 and ϕ_2 are arbitrary functions of $a_1^+ a_2^+$. Because the operators H and J_1 commute, the eigenfunctions (12) are also the eigenfunctions of the Hamiltonian (7). Therefore we can write the eigenvalue equation,

$$H \left|\psi\right\rangle_{j+\frac{1}{2}} = E \left|\psi\right\rangle_{j+\frac{1}{2}}, \qquad E = 2\epsilon + j + \tfrac{3}{2}. \tag{13}$$

The Hamiltonian H can be expressed in the Bargmann–Fock space by using the realizations of the bosonic operators

$$a_i^+ = z_i, \qquad a_i = \frac{d}{dz_i}, \qquad i = 1, 2. \tag{14}$$

In this formulation, the Hamiltonian H consists of two independent sets of first-order linear differential equations. Substituting (12) and (14) into (7) and defining $\xi = z_1 z_2$ one can obtain the following two linear differential equations satisfied by the functions ϕ_1 and ϕ_2:

$$\begin{aligned} &\left[\xi \frac{d}{d\xi} - (\epsilon - \mu)\right] \phi_1 + \kappa \left[\xi \frac{d}{d\xi} + (\xi + j + 1)\right] \phi_2 = 0, \\ &\kappa \left[\frac{d}{d\xi} + 1\right] \phi_1 + \left[\xi \frac{d}{d\xi} - (\epsilon + \mu)\right] \phi_2 = 0. \end{aligned} \tag{15}$$

These coupled differential equations represent the Schrödinger equation of the $E \otimes \varepsilon$ JT system in Bargmann's Hilbert space and its isolated exact solution have been obtained by Reik [3]. In this paper we follow a different strategy to solve the Hamiltonian (7) and we show that the Hamiltonian possesses $osp(2,2)$ symmetry.

4 Two-boson one fermion $osp(2,2)$ superalgebra

In order to construct $osp(2,2)$ superalgebra let us start by introducing three generators of the $su(1,1)$ algebra,

$$J_+ = a_1^+ a_2^+, \qquad J_- = a_2 a_1, \qquad J_0 = \tfrac{1}{2}\left(a_1^+ a_1 + a_2^+ a_2 + 1\right). \tag{16}$$

These are the Schwinger representation of $su(1,1)$ algebra and its number operator is given by,

$$N = a_1^+ a_1 - a_2^+ a_2 \tag{17}$$

which commutes with the $su(1,1)$ generators. The superalgebra $osp(2,2)$ might be constructed by extending $su(1,1)$ algebra with the fermionic generators. These are given by

$$V_+ = f^+ a_2^+, \qquad V_- = f^+ a_1, \qquad W_+ = f a_1^+, \qquad W_- = f a_2, \tag{18}$$

where f^+ and f are fermions and they satisfy the anticommutation relation

$$\left\{f, f^+\right\} = 1, \qquad f = \sigma_-, \qquad f^+ = \sigma_+, \qquad f f^+ - f^+ f = \sigma_0. \tag{19}$$

The superalgebra $osp(2,2)$ can be constructed with the generators (16) and (18), as it is discussed in [12]. The total number operator J of the system and it is given by

$$J = \tfrac{1}{2}N + \tfrac{1}{2}\left(f^+ f - f f^+\right). \tag{20}$$

The generators of the $osp(2,2)$ superalgebra satisfy the following commutation and anticommutation relations:

$$\begin{aligned}
&[J_+, J_-] = -2J_0, \qquad [J_0, J_\pm] = \pm J_\pm, \qquad [J, J_\pm] = 0, \qquad [J, J_0] = 0,\\
&[J_0, V_\pm] = \pm\tfrac{1}{2}V_\pm, \qquad [J_0, W_\pm] = \pm\tfrac{1}{2}W_\pm, \qquad [J_\pm, V_\mp] = V_\pm, \quad [J_\pm, W_\mp] = W_\pm,\\
&[J, W_\pm] = -\tfrac{1}{2}W_\pm, \qquad [J, V_\pm] = \tfrac{1}{2}V_\pm, \qquad [J_\pm, V_\pm] = 0, \qquad [J_\pm, W_\pm] = 0,\\
&\{V_\pm, W_\pm\} = J_\pm, \qquad \{V_\pm, W_\mp\} = \pm J_0 - J, \qquad \{V_\pm, V_\pm\} = \{V_\pm, V_\mp\} = 0,\\
&\{W_\pm, W_\pm\} = \{W_\pm, W_\mp\} = 0.
\end{aligned} \tag{21}$$

The Hamiltonian of a physical system, with an underlying $osp(2,2)$ symmetry, has been expressed in terms of the operators of the corresponding algebra.

5 Transformation of the operators

Transformation of the fermion-boson representations of the $osp(2,2)$ algebra and its connection with the QES systems can be done by introducing the following similarity transformation induced by the metrics

$$S = (a_2^+)^{-a_1^+ a_1 - \sigma_+\sigma_-}, \qquad T = (a_2)^{a_1^+ a_1 + \sigma_+\sigma_-}. \tag{22}$$

These transformations lead to the single-variable differential realizations of the $osp(2,2)$ superalgebra. With the operator S, the generators of $osp(2,2)$ takes the form:

$$\begin{aligned}
&J'_+ = SJ_+S^{-1} = a_1^+, \qquad J'_- = SJ_-S^{-1} = a_1(a_2^+ a_2 + a_1^+ a_1 + \sigma_+\sigma_-),\\
&J'_0 = SJ_0S^{-1} = \tfrac{1}{2}\left(2a_1^+ a_1 + a_2^+ a_2 + 1 + \sigma_+\sigma_-\right), \qquad J' = SJS^{-1} = \tfrac{1}{2}(-a_2^+ a_2 - \sigma_-\sigma_+),\\
&V'_+ = SV_+S^{-1} = \sigma_+, \qquad V'_- = SV_-S^{-1} = \sigma_+ a_1,\\
&W'_+ = SW_+S^{-1} = \sigma_- a_1^+, \qquad W'_- = SW_-S^{-1} = \sigma_-(a_2^+ a_2 + a_1^+ a_1 + \sigma_+\sigma_-).
\end{aligned} \tag{23}$$

The representations (23) of $osp(2,2)$ can be characterized by a fixed number $a_2^+ a_2 = -j - 1$. Here j takes integer or half-integer values. Therefore the generators of the $osp(2,2)$ algebra can be expressed as single-variable differential equation in the Bargmann–Fock space and two-component polynomials of degree j and $j+1$ form a basis function for the generators of the $osp(2,2)$ algebra,

$$P_{n+1,n}(x) = \begin{pmatrix} x^0, x^1, \ldots, x^{n+1} \\ x^0, x^1, \ldots, x^n \end{pmatrix}. \tag{24}$$

The general QES operator can be obtained by linear and bilinear combinations of the generators of the $osp(2,2)$ superalgebra. Action of the QES operator on the basis function (24) gives us a recurrence relation, therefore, the wavefunction is itself the generating function of the energy polynomials. Under the transformation T the generators of the $osp(2,2)$ algebra take the form

$$\begin{aligned}
&J'_+ = TJ_+T^{-1} = a_1^+(a_1 a_1 + a_2^+ a_2 + 1 + \sigma_+\sigma_-), \qquad J'_- = TJ_-T^{-1} = a_1,\\
&J'_0 = TJ_0T^{-1} = \tfrac{1}{2}\left(2a_1^+ a_1 + a_2^+ a_2 + 1 + \sigma_+\sigma_-\right) \quad J' = TJT^{-1} = \tfrac{1}{2}\left(-a_2^+ a_2 - \sigma_-\sigma_+\right),\\
&V'_+ = TV_+T^{-1} = \sigma_+(a_2^+ a_2 + a_1^+ a_1 + 1 + \sigma_+\sigma_-), \qquad W'_+ = TW_+T^{-1} = \sigma_- a_1^+,\\
&V'_- = TV_-T^{-1} = \sigma_+ a_1, \qquad W'_- = TW_-T^{-1} = \sigma_-.
\end{aligned} \tag{25}$$

This realization can also be characterized by $a_2^+ a_2 = -j - 1$. The basis function of the realization is given by (24).

6 Solvability of the $E \otimes \varepsilon$ Jahn–Teller Hamiltonian

It will be shown that our approach is relatively very simple when compared to previous approaches. The Hamiltonian (7) can be expressed in terms of the generators of the $osp(2,2)$:

$$H = 2J_0 + \left(\tfrac{1}{2} + 2\mu\right)(2J - N) + 2\kappa\left[V_+ + V_- + W_+ + W_-\right]. \tag{26}$$

The general trend to solve a differential equation quasi-exactly is to express the differential equation in terms of the generators of a given Lie algebra having a finite dimensional invariant subspace and to use the algebraic operations. In the Bargmann–Fock space the Hamiltonian has two different realizations, under the transformations S and T. The first transformation by S leads to the following one-variable differential realization:

$$\begin{aligned} H_1 = &\left(2x\frac{d}{dx} - j + \sigma_+\sigma_-\right) - \left(\tfrac{1}{2} + 2\mu\right)\sigma_-\sigma_+ \\ &+ 2\kappa\left[\sigma_+\left(1 + \frac{d}{dx}\right) + \sigma_-\left(x + x\frac{d}{dx} - j - 1 + \sigma_+\sigma_-\right)\right] \end{aligned} \tag{27}$$

and the second realization can be obtained by transforming the Hamiltonian by T:

$$\begin{aligned} H_2 = &\left(2x\frac{d}{dx} - j + \sigma_+\sigma_-\right) - \left(\tfrac{1}{2} + 2\mu\right)\sigma_-\sigma_+ \\ &+ 2\kappa\left[\sigma_+\left(\frac{d}{dx} + x\frac{d}{dx} - j + \sigma_+\sigma_-\right) + \sigma_-(1 + x)\right]. \end{aligned} \tag{28}$$

The eigenvalue problem can be expressed as

$$H\varphi(x) = E\varphi(x), \qquad \varphi(x) = \begin{bmatrix} v_n(x) \\ \omega_m(x) \end{bmatrix}, \tag{29}$$

where $v_n(x)$ and $\omega_m(x)$ are polynomials of degree n and m respectively. The action of the H_1 on the basis function $\varphi(x)$ gives the following recurrence relation:

$$\begin{aligned} &(2n - j + 1 - E)v_n + 2\kappa(\omega_m + m\omega_{m-1}) = 0, \\ &\left(2m - j - \tfrac{1}{2} - 2\mu - E\right)\omega_m + 2\kappa(v_{n+1} + (n - j)v_n) = 0. \end{aligned} \tag{30}$$

Similarly when the Hamiltonian H_2 acts on the basis function we obtain the recurrence relation:

$$\begin{aligned} &(2n - j + 1 - E)v_n + 2\kappa(m\omega_{m-1} + (m - j)\omega_m) = 0, \\ &\left(2m - j - \tfrac{1}{2} - 2\mu - E\right)\omega_m + 2\kappa(v_n + v_{n+1}) = 0. \end{aligned} \tag{31}$$

It is requiring that the determinant of these sets must be equal to zero giving the compatibility conditions which establish the QES system. According to the (24) one can construct a QES system if $n = m + 1$. Here m takes the values $m = 0, \frac{1}{2}, 1, \ldots, j$. If E_j is a root of the recurrence relations (30) or (31) then the eigenfunction truncates for a certain values of j, and E_j belong to the spectrum of the Hamiltonian. The initial conditions of the recurrence relation is given by

$$v_m = 0 \quad \text{for} \quad j < m < 1 \qquad \text{and} \qquad \omega_m = 0 \quad \text{for} \quad j < m < 0. \tag{32}$$

with these initial conditions solution of (30) gives us the following relation for the energy when $j = 0$:

$$E = \tfrac{1}{4}\left(5 - 4\mu \pm \sqrt{64\kappa^2 + (7 + 4\mu)^2}\right) \tag{33}$$

and for $j = 1/2$:

$$E = \tfrac{1}{4}\left(3 - 4\mu \pm \sqrt{32\kappa^2 + (7 + 4\mu)^2}\right), \qquad E = \tfrac{1}{4}\left(7 - 4\mu \pm \sqrt{64\kappa^2 + (7 + 4\mu)^2}\right). \tag{34}$$

The same energy eigenvalues can be obtained by using the recurrence relation (31). In this case eigenvalues shifted $E \to E - 1$ and j takes negative integer and half-integer values.

7 Conclusion

It is well known that the exact solutions have a direct practical importance. We have presented the quasi-exact solution of the generalized $E \otimes \varepsilon$ JT system. Our paper gives a unified treatment of some earlier works. The method given here can be extended to other JT or multi-dimensional atomic systems. The basic features of our approach is to construct $osp(2,2)$ invariant subspaces. Furthermore, we have presented two different boson-fermion representations and two classes of one variable differential realizations of $osp(2,2)$ algebra. In particular the solution of $E \otimes \varepsilon$ JT system has been constructed.

The suggested approach can be generalized in various directions. Invariant subspaces of the multi-boson and multi-fermion systems can be obtained by extending the method given in this paper. The method given here can be extended to other JT or multi dimensional atomic systems.

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On the Possibility of Faster-Than-Light Motions in Nonlinear Electrodynamics

Gennadii KOTEL'NIKOV

RRC Kurchatov Institute, 1 Kurchatov Sq., 123182 Moscow, Russia
E-mail: *kga@electronics.kiae.ru*

A version of electrodynamics is constructed in which faster-than-light motions of electromagnetic fields and particles with real masses are possible.

1 Introduction

For certainty, by faster-than-light motions we shall understand the motions with velocities $v > 3 \cdot 10^8$ m/sec. The existence of such motions is the question discussed in modern physics.

As already as in 1946–1948 Blokhintsev [1] paid attention to the possibility of formulating the field theory that allows propagation of faster-than-light (superluminal) interactions outside the light cone. Some time later he also noted the possibility of the existence of superluminal solutions in nonlinear equations of electrodynamics [2]. Kirzhnits [3] showed that a particle possessing the tensor of mass $M^i{}_k = \mathrm{diag}\,(m_0, m_1, m_1, m_1)$, $i, k = 0, 1, 2, 3$, $g_{ab} = \mathrm{diag}\,(+, -, -, -)$ can move faster-than-light, if $m_0 > m_1$. Terletsky [4] introduced into theoretical physics the particles with imaging rest masses moving faster-than-light. Feinberg [5] named these particles tachyons and described their main properties.

Research on superluminal tachyon motions opened up additional opportunities which were studied by many authors, for example by Bilaniuk and Sudarshan [6], Recami [7], Mignani (see [7]), Kirzhnits and Sazonov [8], Corben (see [7]), Patty [9], Oleinik [10]. It has led to original scientific direction (several hundred publications). The tachyon movements may formally be described by Special Relativity (SR) expanded to the domain of motions $s^2 < 0$. For comparison, the standard theory describes motions on the light cone $s^2 = 0$ and in the domain $s^2 > 0$ when $v \le 3 \cdot 10^8$ m/sec.

The publications are also known in which the violation of invariance of the speed of light is considered [11–17]. One can note, for example, Pauli monograph [11] where elements of Ritz and Abraham theory are presented; Logunov's lectures [12] in which SR formulation in affine space was given; Glashow's paper [13] discussing experimental consequences of the violation of Lorentz-invariance in astrophysics; publications [14–17] on the violation of invariance of the speed of light in SR.

A version of the theory permitting faster-than-light motions of electromagnetic fields and charged particles with real masses is proposed below as continuation of such investigations.

2 Formal construction of the theory

Let us introduce the space-time $\mathbb{R}^4$, metric properties of which may depend on the velocity of a particle being investigated and take the metric of $\mathbb{R}^4$ in the form:

$$\begin{aligned} ds^2 &= \big(c_0^2 + v^2\big)dt^2 - dx^2 - dy^2 - dz^2 \\ &= \big(c_0'^2 + v'^2\big)(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2 - \text{invariant}. \end{aligned} \tag{1}$$

Here x, y, z are the spatial coordinates, t is the time, c_0, c'_0 are the proper values of the speed of light, v is the velocity of a body under study with respect to the reference frame K. Let us connect the co-moving frame K' with this body. Let the proper speed of light be invariant

$$c_0 = {c'}_0 = 3 \cdot 10^8 \text{ m/sec} - \text{invariant}. \tag{2}$$

As a result, the common time similar to Newton one may be introduced on the trajectory of the frame K' (when $\boldsymbol{v} = d\boldsymbol{x}/dt$) and the velocity of light c, corresponding to the velocity $\boldsymbol{v}$:

$$dt = {dt'}_0 \to t = {t'}_0, \tag{3}$$

$$c = \pm c_0 \sqrt{1 + \frac{v^2}{{c_0}^2}}. \tag{4}$$

We shall name the value $c_0 = 3 \cdot 10^8$ m/sec the speed of light, the value c – the velocity of light[1]. In accordance with the hypothesis of homogeneity and isotropy of space-time, the velocity v of a free particle does not depend on t and $\boldsymbol{x}$. The velocity of light (4) is a constant in this case. Let $dx^0 = cdt$ and

$$x^0 = \int_0^t c d\tau = \pm \int_0^t c_0 \sqrt{1 + \frac{v^2}{{c_0}^2}} d\tau \tag{5}$$

be the "time" x^0 when $v \neq \text{const}$ also. Keeping this in mind, we rewrite the expression (1) in the form

$$ds^2 = \left(dx^0\right)^2 - \left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2, \tag{6}$$

where $x^{1,2,3} = (x, y, z)$. As is known, the metric (6) describes the flat homogeneous Minkowski space-time M^4 with $g_{ik} = \text{diag}\,(+, -, -, -)$, $i, k = 0, 1, 2, 3$. Under the condition (3), infinitesimal space-time transformations, retaining invariance of the form (6), are accompanied by the transformation of the velocity of light [17]:

$$dx'^i = L^i{}_k dx^k, \qquad c' = c(1 - \boldsymbol{\beta} \cdot \boldsymbol{u})/\sqrt{1 - \beta^2}, \qquad i, k = 0, 1, 2, 3. \tag{7}$$

Here $L^i{}_k$ is the matrix of Lorentz group with $\boldsymbol{\beta} = \boldsymbol{V}/c = \textbf{const}$ [11], $\boldsymbol{u} = \boldsymbol{v}/c$. Corresponding homogeneous integral transformations in the case of the one-parametric Lorentz group L_1 are

$$x'^0 = \frac{x^0 - \beta x^1}{\sqrt{1 - \beta^2}}, \qquad x'^1 = \frac{x^1 - \beta x^0}{\sqrt{1 - \beta^2}}, \qquad x'^2 = x^2, \qquad x'^3 = x^3, \qquad c' = c\frac{1 - \beta u^1}{\sqrt{1 - \beta^2}}, \tag{8}$$

with $\boldsymbol{\beta} = (V/c, 0, 0)$, $u^1 = dx^1/dx^0 = v_x/c$. For inertial motions in the space-time (1) we have

$$t' = t, \qquad x' = \frac{x - Vt}{\sqrt{1 - V^2/c^2}}, \qquad y' = y, \qquad z' = z, \qquad c' = c\frac{1 - V v_x/c^2}{\sqrt{1 - V^2/c^2}},$$

where we take into account $v_x = x/t$. The transformations (8) are induced by the operator $X = x_1\partial_0 - x_0\partial_1 - u^1 c\partial_c$ which is the sum of Lorentz group L_1 generator $J_{01} = x_1\partial_0 - x_0\partial_1$ and the generator $D = c\partial_c$ of scale transformations group $\triangle_1$ of the velocity of light $c' = \gamma c$. We can say that these generators act in 5-space $M^4 \times V^1$ where V^1 is a subspace of the velocities of light, and that the transformations (8) belong to the group of direct product $L_1 \times \triangle_1$. The generators J_{01} and D and transformations (8) are respectively the symmetry operators and

[1] In the form of $c' = c(1 - \beta^2)^{1/2}$ expression (4) was obtained by Abraham in the model of Aether [11].

symmetry transformations for the equation of the zero cone $s^2 = 0$ in the 5-space $V^5 = M^4 \times V^1$ where $|c| < \infty$ includes the subset $c_0 < |c| < \infty$:

$$s^2 = \big(x^0\big)^2 - \big(x^1\big)^2 - \big(x^2\big)^2 - \big(x^3\big)^2 = 0, \qquad J_{01}\big(s^2\big) = 0, \quad D\big(s^2\big) = 0, \quad [J_{01}, D] = 0.$$

The relationships between the partial derivatives of the variables (t, x, y, z) and (x^0, x^1, x^2, x^3) are as follows:

$$\frac{\partial}{\partial t} = \frac{\partial x^0}{\partial t}\frac{\partial}{\partial x^0} + \sum_\alpha \frac{\partial x^\alpha}{\partial t}\frac{\partial}{\partial x^\alpha} = c\frac{\partial}{\partial x^0},$$
$$\frac{\partial}{\partial x} = \frac{\partial x^0}{\partial x}\frac{\partial}{\partial x^0} + \sum_\alpha \frac{\partial x^\alpha}{\partial x}\frac{\partial}{\partial x^\alpha} = \left(\int \frac{\partial c}{\partial x} d\tau\right)\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}, \qquad \alpha = 1, 2, 3. \tag{9}$$

The expressions for $\partial/\partial y$ and $\partial/\partial z$ are analogous to the expression $\partial/\partial x$. Further we restrict ourselves by the case of positive values of the velocities of light and by studying a variant of the theory in which the velocity of light in the range of interactions may only depend on the time "t", i.e. $c = c(t) \leftrightarrow c = c(x^0)$. The relationship between x^0 and t may be deduced from the solution of equation (5). Then

$$\nabla c\big(x^0\big) = 0, \quad c = c\big(x^0\big), \quad u^2 = u^2\big(x^0\big) \quad \leftrightarrow \quad \nabla c(t) = 0, \quad c = c(t), \quad v^2 = v^2(t). \tag{10}$$

Let us note some features of motions in this case.

1. As in SR, the parameter $\beta = V/c$ in the present work is in the range $0 \le \beta < 1$.

2. As in SR, the value dx^0 is the exact differential in view of the condition $\nabla c = 0$.

3. As distinct from SR, the "time" $x^0 = ct$ in the present work is a function of the time t only for the case of a free particle. In the range of interaction the velocity of light may depend on time t, and the value x^0 becomes the functional (5) of the function $c(t)$.

4. The parameter $\beta = V/c$ of the transformations (8) may be constant not only at the constant velocity of light, but also with $c = c(t)$. Indeed we may accept that $0 \le \beta = V(t)/c_0(1 + v^2(t)/{c_0}^2)^{1/2} = \text{const} < 1$, which is not in contradiction with $V = V(t)$, $c = c(t)$. This property permits one to use the matrix $L^i{}_k$ from (7) for constructing Lorentz invariants in the range of interaction where $c = c(t)$.

5. The condition (10) is invariant on the trajectory of a particle because $t' = t$ in this case. Replacing space-time variables in the expression $\nabla c' = 0$ ($c' = \gamma c$) we find that the condition $\nabla c = 0$ follows from $\nabla c' = 0$ if $\nabla\gamma = 0 \to (\beta - u^1)\partial\beta/\partial x^\alpha + (\beta^3 - \beta)\partial u^1/\partial x^\alpha = 0$, $\alpha = 1, 2, 3$. The system contains the solution $\beta = \text{const}$, $u^1 = u^1(x^0)$ in agreement with (10) and item 4.

Keeping this in mind, let us construct in the space (6) a theory like SR, reflect it on the space-time (1) by means of the formulas (9), (10) and consider the main properties of the theory. Following [18], we may construct the integral of action in the form:

$$\begin{aligned} S &= S_m + S_{mf} + S_f = -mc_0 \int ds - \frac{e}{c_0}\int A_i dx^i - \frac{1}{16\pi c_0}\int F_{ik}F^{ik} d^4x \\ &= \int \left[-mc_0\sqrt{1 - u^2} + \frac{e}{c_0}(\boldsymbol{A}\cdot\boldsymbol{u} - \phi)\right] dx^0 - \frac{1}{8\pi c_0}\int (E^2 - H^2) d^3x dx^0 \\ &= -mc_0 \int ds - \frac{1}{c_0}\int A_i j^i d^4x - \frac{1}{16\pi c_0}\int F_{ik}F^{ik} d^4x. \end{aligned} \tag{11}$$

Here in accordance with [18] $S_m = -mc_0 \int ds = -mc_0 \int (c_0/c) dx^0 = -mc_0 \int (1 - u^2)^{1/2} dx^0$ is the action for a free particle; $S_f = -(1/16\pi c_0)\int F_{ik}F^{ik} d^4x$ is the action for free electromagnetic field, $S_{mf} = -(e/c_0)\int A_i dx^i = -(1/c_0)\int A_i j^i d^4x$ is the action corresponding to the interaction between the charge e of a particle and electromagnetic field; $A^i = (\phi, \boldsymbol{A})$ is the 4-potential;

$A_i = g_{ik}A^k$; $j^i = (\rho, \rho\boldsymbol{u})$ is the 4-vector of current density; ρ is the charge density; $F_{ik} = \partial A_k/\partial x^i - \partial A_i/\partial x^k$ is the tensor of electromagnetic field; $i,k = 0,1,2,3$; $\boldsymbol{E} = -\partial\boldsymbol{A}/\partial x^0 - \nabla\phi$ is the electrical field; $\boldsymbol{H} = \nabla\times\boldsymbol{A}$ is the magnetic field; $F_{ik}F^{ik} = 2(H^2 - E^2)$; $dx^4 = dx^0dx^1dx^2dx^3$ is the element of the invariant 4-volume. The speed of light c_0, the mass m, the electric charge e are invariant constants of the theory.

In spite of the similarity, the action (11) differs from the action of SR [18]. The current density was taken in the form $j^i = (\rho, \rho\boldsymbol{u}) = (\rho, \rho\boldsymbol{v}/c)$ instead of $j^i = (\rho, \rho\boldsymbol{v})$ [18]. The electromagnetic field was taken in the form $\boldsymbol{E} = -\partial\boldsymbol{A}/\partial x^0 - \nabla\phi = -(1/c)\partial\boldsymbol{A}/\partial t - \nabla\phi$ instead of $\boldsymbol{E} = -(1/c_0)\partial\boldsymbol{A}/\partial t - \nabla\phi$ [18]. The current density in (11) is similar to the one from Pauli monograph [11] with the only difference that the 3-current density in (11) is $\rho\boldsymbol{v}/c$ instead of being $\rho\boldsymbol{v}/c_0$ [11]. Analogously, the velocity of 4-potential propagation in (11) is c from (4) instead of c_0 in [18].

In addition to Lorentz-invariance [18], the action (11) is also invariant with respect to any transformations of the velocity of light and, consequently, with respect to the transformations $c' = \gamma c$, as the value c is not contained in the expression (11). As a result the action (11) is invariant with respect to the transformations (8) from the group of direct product $L_1\times\triangle_1 \subset L_6\times\triangle_1$, containing the Lorentz group L_6 and the scale group $\triangle_1$ ($c' = \gamma c$) as subgroups.

Lagrangian L, generalized 4-momentum $\boldsymbol{P}$ and generalized energy $\mathcal{H}$ of a particle are:

$$L = -mc_0\sqrt{1-u^2} + \frac{e}{c_0}(\boldsymbol{A}\cdot\boldsymbol{u} - \phi), \tag{12}$$

$$\boldsymbol{P} = \frac{\partial L}{\partial\boldsymbol{u}} = \frac{mc_0\boldsymbol{u}}{\sqrt{1-u^2}} + \frac{e}{c_0}\boldsymbol{A} = \boldsymbol{p} + \frac{e}{c_0}\boldsymbol{A}, \tag{13}$$

$$\mathcal{H} = \boldsymbol{P}\cdot\boldsymbol{u} - L = \frac{mc_0}{\sqrt{1-u^2}} + \frac{e\phi}{c_0} = \frac{\mathcal{E}}{c_0} + \frac{e\phi}{c_0}. \tag{14}$$

Here $\boldsymbol{p} = m\boldsymbol{v}$ is the momentum, $\mathcal{E} = mc_0c$ is the energy, $\mathcal{E}_0 = mc_0{}^2$ is the rest energy of a particle. As in SR, the values $\mathcal{E}$ and $\boldsymbol{p}$ may be united into the 4-momentum p^i:

$$p^i = mc_0u^i = \left(\frac{mc_0}{\sqrt{1-u^2}}, \frac{mc_0u^\alpha}{\sqrt{1-u^2}}\right) = \left(\frac{\mathcal{E}}{c_0}, m\boldsymbol{v}\right), \qquad \alpha = 1,2,3. \tag{15}$$

The components of p^i are related by the expressions:

$$p_ip^i = \frac{\mathcal{E}^2}{c_0{}^2} - \boldsymbol{p}^2 = m^2c_0{}^2,$$
$$\boldsymbol{p} = \frac{\mathcal{E}}{c_0c}\boldsymbol{v} \qquad \text{or} \qquad \boldsymbol{p} = \frac{\mathcal{E}}{c_0}\boldsymbol{n}, \quad \boldsymbol{n} = \frac{\boldsymbol{c}}{c}, \quad \text{if} \quad m = 0, \quad \boldsymbol{v} = \boldsymbol{c}. \tag{16}$$

One can see from here that the momentum of a particle with the zero mass $m = 0$ is independent of the particle velocity $v = c$ and only determined by the particle energy. As in SR, for the case of the photon we find: $p^i = (\hbar\omega/c_0, \hbar\omega\boldsymbol{n}/c_0) = (\mathcal{E}/c_0, \hbar\boldsymbol{k})$ where $\mathcal{E} = \hbar\omega$, $\boldsymbol{k} = (\omega/c_0)\boldsymbol{n}$, ω is the frequency of electromagnetic field, $\hbar$ is the Planck constant.

For constructing the equations of motion for a charged particle and electromagnetic field let us start from the mechanical [18] and the field Lagrange equations [19, 20]

$$\frac{d}{dx^0}\frac{\partial L}{\partial\boldsymbol{u}} - \frac{\partial L}{\partial\boldsymbol{x}} = 0, \qquad \frac{\partial}{\partial x^k}\frac{\partial\mathcal{L}}{\partial(\partial A_i/\partial x^k)} - \frac{\partial\mathcal{L}}{\partial A_i} = 0. \tag{17}$$

Here L is Lagrangian (12), $\mathcal{L} = -(1/c_0)A_ij^i - (1/16\pi c_0)F_{ik}F^{ik}$. Taking into account the equality $\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a})$, the permutational ratios of the tensor of electromagnetic field, the expression $\partial(F^{ik}F_{ik})/\partial(\partial A_i/\partial x^k) = -4F^{ik}$ [18] we obtain $d(mc_0\boldsymbol{u}/(1-u^2)^{1/2})/dx^0 = (e/c_0)\boldsymbol{E} + (e/c_0)\boldsymbol{u}\times\boldsymbol{H}$, $\partial F_{ik}/\partial x^l + \partial F_{kl}/\partial x^i + \partial F_{li}/\partial x^k = 0$,

$\partial F^{ik}/\partial x^k + 4\pi j^i = 0$ [18]. With help of relations (9), (10) and expressions $dx^0 = cdt$, $(1 - u^2)^{1/2} = c_0/c$ we may obtain the following equations of motions in the space-time (1) [17]:

$$\begin{aligned}
&\frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}}{dt} = \frac{c}{c_0}e\boldsymbol{E} + \frac{e}{c_0}\boldsymbol{v}\times\boldsymbol{H},\\
&\frac{d\mathcal{E}}{dt} = e\boldsymbol{E}\cdot\boldsymbol{v} \to m\frac{dc}{dt} = \frac{e}{c_0}\boldsymbol{v}\cdot\boldsymbol{E}; \qquad (18)\\
&\nabla\times\boldsymbol{E} + \frac{1}{c}\frac{\partial\boldsymbol{H}}{\partial t} = 0, \qquad \nabla\cdot\boldsymbol{H} = 0,\\
&\nabla\times\boldsymbol{H} - \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t} = 4\pi\rho\frac{\boldsymbol{v}}{c}, \qquad \nabla\cdot\boldsymbol{E} = 4\pi\rho. \qquad (19)
\end{aligned}$$

Equations (18) determine dynamics of motion of a charged particle with a mass m in external field. The set (19) consists of Maxwell equations for the velocity of light $c = c_0(1 + v^2/{c_0}^2)^{1/2}$ with $v = \text{const}$. Considered together, they form the set of nonlinear equations of electrodynamics describing the motion of electrical charge in the field generated by the motion of the this charge.

We may find from here that wave equations for the vector and scalar potentials take the following form under Lorentz gauge $(1/c)\partial\phi/\partial t + \nabla\cdot\boldsymbol{A} = 0$:

$$\Box\boldsymbol{A} - \frac{\dot{c}}{c^3}\frac{\partial\boldsymbol{A}}{\partial t} = 4\pi\rho\frac{\boldsymbol{v}}{c}, \qquad \Box\phi - \frac{\dot{c}}{c^3}\frac{\partial\phi}{\partial t} = 4\pi\rho,$$

where $\dot{c} = dc/dt = \boldsymbol{v}\cdot\dot{\boldsymbol{v}}/c$. For the free field with $\rho = 0$ we have $\dot{c} = 0$, $v^2 = \text{const}$, $\boldsymbol{v}\cdot\dot{\boldsymbol{v}} = 0$, $\Box\boldsymbol{A} = 0$, $\Box\phi = 0$. When $c = c_0$ they coincide with the equations from SR.

The condition $\nabla c = 0$ imposes certain limits on the possible movement of a particle. Acting by the operator ∇ on the equation $\dot{c} = (e/mc_0)\boldsymbol{v}\cdot\boldsymbol{E}$, we find

$$\begin{aligned}
&\rho\left[(\boldsymbol{v}\cdot\nabla)\boldsymbol{E} + (\boldsymbol{E}\cdot\nabla)\boldsymbol{v} - \frac{1}{c}\boldsymbol{v}\times\frac{\partial\boldsymbol{H}}{\partial t}\right]\\
&\quad + \frac{c}{4\pi}\boldsymbol{E}\times\left(\Box\boldsymbol{H} - \frac{\dot{c}}{c^3}\frac{\partial\boldsymbol{H}}{\partial t}\right) - \frac{mc_0}{e}\dot{c}\nabla\rho + \boldsymbol{v}(\boldsymbol{E}\cdot\nabla\rho) = 0.
\end{aligned}$$

For a free particle ($\dot{c} = 0$, $\rho = 0$, e=0) we have $\boldsymbol{E}\times\Box\boldsymbol{H} = 0$, $\dot{\boldsymbol{v}} = 0$.

As follows from the equations $\nabla\cdot\boldsymbol{E} = 4\pi\rho$, $c\nabla\times\boldsymbol{H} - \partial_t\boldsymbol{E} = 4\pi\rho\boldsymbol{v}$ from the set (19), in the present theory the law of the electrical charge conservation is valid:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0.$$

Following [18], we find the well-known relation:

$$\frac{\partial}{\partial t}\left(\frac{E^2 + H^2}{8\pi}\right) = -c\boldsymbol{j}\cdot\boldsymbol{E} - \nabla\cdot\boldsymbol{S}.$$

Here $W = (E^2 + H^2)/8\pi$ is the energy density of electromagnetic field, $\boldsymbol{S} = (c/4\pi)\boldsymbol{E}\times\boldsymbol{H} = (c(0)\mathcal{E}(t)/4\pi\mathcal{E}(0))\boldsymbol{E}\times\boldsymbol{H}$ is the Poynting vector.

At last taking into account the expression for the velocity of light

$$c(t) = c_0\sqrt{1 + \frac{v^2(t)}{{c_0}^2}} = c(0)\left[1 + \frac{e}{mc_0c(0)}\int_0^t \boldsymbol{v}\cdot\boldsymbol{E}d\tau\right] = c(0)\left[1 + \frac{\mathcal{E}(t) - \mathcal{E}(0)}{\mathcal{E}(0)}\right],$$

where $c(0)$ is the velocity of light at the time $t = 0$, $\mathcal{E}(0) = mc_0c(0)$, we may rewrite the sets (18), (19) in the equivalent form:

$$m\frac{d\boldsymbol{v}}{dt} = \frac{c(0)}{c_0}\left[1 + \frac{\mathcal{E}(t) - \mathcal{E}(0)}{\mathcal{E}(0)}\right] e\boldsymbol{E} + \frac{e}{c_0}\boldsymbol{v} \times \boldsymbol{H},$$

$$\frac{d\mathcal{E}}{dt} = e\boldsymbol{v} \cdot \boldsymbol{E} \to \mathcal{E}(t) - \mathcal{E}(0) = e\int_0^t \boldsymbol{v} \cdot \boldsymbol{E} d\tau; \tag{20}$$

$$\left[1 + \frac{\mathcal{E}(t) - \mathcal{E}(0)}{\mathcal{E}(0)}\right] \nabla \times \boldsymbol{E} + \frac{1}{c(0)}\frac{\partial \boldsymbol{H}}{\partial t} = 0, \qquad \nabla \cdot \boldsymbol{E} = 4\pi\rho, \tag{21}$$

$$\left[1 + \frac{\mathcal{E}(t) - \mathcal{E}(0)}{\mathcal{E}(0)}\right] \nabla \times \boldsymbol{H} - \frac{1}{c(0)}\frac{\partial \boldsymbol{E}}{\partial t} = 4\pi\rho\frac{\boldsymbol{v}}{c(0)}, \qquad \nabla \cdot \boldsymbol{H} = 0. \tag{22}$$

The set (20), (22) coincide with the set of equations of Maxwell electrodynamics [18] in the approximation $[\mathcal{E}(t) - \mathcal{E}(0)]/\mathcal{E}(0) \ll 1$ with $c(0) = c_0$.

Further let us pay attention to the expression $v = (\mathcal{E}^2 - m^2{c_0}^4)^{1/2}/mc_0 > c_0$. It follows from here that in the framework of the present work a free particle will move faster-than-light, if the particle energy satisfies the condition [17]

$$\mathcal{E} > \sqrt{2}m{c_0}^2 = \sqrt{2}\mathcal{E}_0. \tag{23}$$

The energy $\mathcal{E} = 2^{1/2}\mathcal{E}_0$ is equal ~ 723 keV for the electron ($\mathcal{E}_0 \sim 511$ keV) and ~ 1327 Mev for the proton and neutron ($\mathcal{E}_0 \sim 938$ MeV). We may conclude from here that in the present work the neutron physics of nuclear reactors may be formulated in the approximation $v < c_0$ as in SR. The electrons with the energy $\mathcal{E} > 723$ keV should be faster-than-light particles (for example, the velocity of the 1 GeV electron should be $\sim 2000\ c_0$). The particle physics on accelerators with the energy of protons more than 1.33 GeV would be physics of faster-than-light motions, if the present theory be realized in the nature.

3 Application to physics

Let us consider how a set of the well-known experiments may be interpreted in the framework of the present theory. As an example chosen may be: the Michelson experiment, Fizeau experiment, aberration of light, appearance of atmospheric μ-mesons near the surface of the Earth, Doppler effect, known tests to check independence of the speed of light from the velocity of light source, decay of unstable particles, creation of new particles in nuclear reactions, possible faster-than light motion of nuclear reactions products, Compton effect, photo-effect. Below we shall consider some of them.

Michelson experiment. Negative result of the Michelson experiment for an observer with a terrestrial source of light (the reference frame K, the speed of light c_0) may be explained by space isotropy. Since the speed of light c_0 is the same in all directions, a shift of the interference pattern is absent with the interferometer's rotation. Analogously, in the case of an extraterrestrial source (the reference frame K' – the star moving inertialy with a velocity v relatively to the Earth), the velocity of light from the star $c = c_0(1 + v^2/{c_0}^2)^{1/2}$ is the same for an observer on the Earth. As a result, the interference pattern does not change with the interferometer's rotation [17].

Aberration of light. By analogy with SR [11, 18] for one-half of the aberration angle (in arc seconds) we have: $\sin\alpha = V/c$, $\alpha \sim (V/c_0)(c_0/c) \sim 20.5(1 + z_\omega)/(1 + z_\lambda)$ [17]. Here $z_\omega = (c - Vn_x)/c_0 - 1, z_\lambda = (c - Vn_x)c/{c_0}^2 - 1$ are the redshift parameters for frequency and wavelength respectively. It follows from here, for quasar Q1158 + 4635 with $z_\lambda = 4.73$ (Carswell and Hewett, 1990) we obtain at $n_x = -1$, that $z_\omega \sim 2.23$, $c \sim 1.77c_0$, $\alpha \sim 11.6$ instead of being $z_\omega = z_\lambda$, $c = c_0$, $\alpha \sim 20.5$ in SR.

Appearance of atmospheric μ-mesons near the surface of the Earth. Since the time dilatation is absent in the present work, the appearance of air μ-mesons near the surface of the Earth could be explained by faster-than-light motion of the mesons with velocity of the order of $6 \cdot 10^6/2.2 \cdot 10^{-6} \sim 3 \cdot 10^{10}$ m/sec, or $100c_0$. This corresponds to the meson energy $\mathcal{E}_\mu = m_\mu c_0 c \sim 10.6$ GeV, where $m_\mu {c_0}^2 \sim 106$ MeV is the rest energy of μ-meson [17].

Tests to check independence of the speed of light from the velocity of light source. It is, for example, Bonch–Bruevich–Molchanov (1956) experiment, in which the velocities of light radiated by the eastern and western equatorial edges of the solar disk were compared; Sadeh experiment (1963) in which the velocities of γ-quanta, arising as a result of the electron-positron annihilation in flight, were compared; Filippas–Fox experiment (1964), where the effect of the velocity of fast π^0-meson on the velocities of γ-quanta from decay $\pi^0 \to \gamma + \gamma$ were investigated. In view of independence of the velocity of light (4) from the direction of emission of light relatively to the vector of velocity $\boldsymbol{v}$ of a light source (the solar disk, center-of-mass of electron and positron, π^0-meson) the result in such type of experiments should be negative [17].

Decay of unstable particles. Because of the equality of the rest energy of particles in SR and in the present work, the condition of spontaneous decay of the particles into fragments is the same for the both theories. In particular, for the case of decay of a particle with the mass M into two fragments with the masses m_1 and m_2, the energy conservation law leads to $M{c_0}^2 = \mathcal{E}_1 + \mathcal{E}_2$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the energies of the particles produced. Since $\mathcal{E}_1 > m_1{c_0}^2$, $\mathcal{E}_2 > m_2{c_0}^2$, the decay is possible if $M > m_1 + m_2$ (as in SR [18]). From the conservation law of energy-momentum $M{c_0}^2 = \mathcal{E}_1 + \mathcal{E}_2$, $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$, and relations (16) it follows that $\mathcal{E}_1 = (M^2 + {m_1}^2 - {m_2}^2)c_0^2/2M$, $\mathcal{E}_2 = (M^2 - {m_1}^2 + {m_2}^2)c_0^2/2M$ as in SR [18]. The difference consists in predicting the fragment velocities. Using formula $\mathcal{E} = mc_0 c$, for these velocities in the present work we obtain $v_1 = [((M^2 + {m_1}^2 - {m_2}^2)^2/4{m_1}^2M^2) - 1]^{1/2}c_0$, $v_2 = [((M^2 - {m_1}^2 + {m_2}^2)^2/4{m_2}^2M^2) - 1]^{1/2}c_0$. One can see that when $(M^2 + {m_1}^2 - {m_2}^2)^2/4{m_1}^2M^2 \geq 2$, $(M^2 - {m_1}^2 + {m_2}^2)^2/4{m_2}^2M^2 \geq 2$, the velocities v_1 and v_2 can exceed or be equal the speed of light c_0 [17].

Creation of new particles. Let us consider the reaction of antiproton creation in the proton-proton collision in a laboratory reference frame K: $p^+(\text{moving}) + p^+(\text{in rest}) = p^+ + p^+ + p^+ + p^-$, where we denote the energy of the moving proton as $\mathcal{E}_1$, the momentum as $\boldsymbol{p}_1$, and the proton mass as m_p. Following [18] and using the relationship between the momentum and energy (16), one can write $(\mathcal{E}_1 + m_p{c_0}^2)^2 - {c_0}^2{p_1}^2 = 16{m_p}^2{c_0}^4$. In view of the relationship $\mathcal{E}_1{}^2 - {c_0}^2{p_1}^2 = {m_p}^2{c_0}^4$, one obtains $2\,\mathcal{E}_1 m_p{c_0}^2 = 14\,{m_p}^2{c_0}^4$. From here we find that the threshold energy of the antiproton creation $\mathcal{E}_1 = 7m_p{c_0}^2 \sim 7$ GeV is the same as in SR [22]. The difference consists in the value of velocities of particles. In particular, according to the formula $p_1 = m_p v_1 = (\mathcal{E}_1{}^2/{c_0}^2 - {m_p}^2{c_0}^2)^{1/2} = (49-1)^{1/2}m_p c_0$, in the present work the velocity of proton possessing the energy 7 GeV is $v_1 = (48)^{1/2}c_0 = 6.9c_0$ [17] is distinct from SR.

Compton effect and photo-effect. Following [21], we find from the energy-momentum the conservation law (15) $\hbar\omega = \hbar\omega' + m_e{c_0}^2[(1 + v^2/{c_0}^2)^{1/2} - 1]$; $\hbar\omega/c_0 = (\hbar\omega'/c_0)\cos\theta + m_e v\cos\alpha$, $0 = (\hbar\omega'/c_0)\sin\theta - m_e v\sin\alpha$. Here $\hbar\omega$, $\hbar\omega'$ are the energies of incident and scattered γ-quanta, α and θ are the angles of scattering the electron and γ-quantum respectively, m_e is the electron mass. As a result the angular distribution of scattered γ-quanta is $\omega' = \omega/[1 + \hbar\omega(1 - \cos\theta)/m_e{c_0}^2]$ as in [21]. But the velocity of forward-scattered electron may exceed the speed of light c_0: $v_e = c_0(\hbar\omega/m_e{c_0}^2)[1 - m_e{c_0}^2/(2\hbar\omega + m_e{c_0}^2)] > c_0$ if $\hbar\omega > 698$ keV, which differs from SR. The velocity of scattered γ-quantum does not depend on the angle θ and is determined by mechanism of scattering (immediately after interaction of γ-quantum with electron, or in the act of absorption-emission by scattered electron).

Analogously, in the case of photo-effect the velocity of photoelectron is equal $v = c_0[((\hbar\omega + m_e{c_0}^2 - U)/m_e{c_0}^2)^2 - 1]^{1/2}$, where U is the energy of ionization. If the energy of photon $\hbar\omega \geq (2^{1/2} - 1)m_e{c_0}^2 + U = 211$ keV $+ U$, the velocity of photoelectron is $v \geq c_0$.

4 Conclusion

The $L_6 \times \triangle_1$ invariant theory has been constructed, where L_6 is the Lorentz group, $\triangle_1$ is the scale transformation group of the velocity of light $c' = \gamma c$. In accordance with Blokhintsev papers [1] we may assume that the proposed theory may prove to be useful in the field of particles physics, when the elementary particle is not a point but possesses some dimensions. Indeed, the elementary particles should be points in the L_6 invariant theory (SR) because of the finiteness of the speed of light c_0. In the $L_6 \times \triangle_1$ theory this requirement is not necessary because of the absence of the limit to the velocity of light c. The postulation $c' = c$ leads to SR.

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Nonlinear Amplitude Maxwell–Dirac Equations. Optical Leptons

Lubomir M. KOVACHEV

Institute of Electronics, Bulgarian Academy of Sciences,
72 Tzarigradcko Shossee, 1784 Sofia, Bulgaria
E-mail: *lkovach@ie.bas.bg*

We apply the method of slowly-varying amplitudes of the electrical and magnetic fields to integro-differential system of nonlinear Maxwell equations. The equations are reduced to system of differential Nonlinear Maxwell amplitude Equations (NME). The electric and magnetic fields are presented as sums of circular and linear components. Thus, NME is written as a set of Nonlinear Dirac Equations (NDE). Exact solutions of NDE with classical orbital momentum $\ell = 1$ and opposite directions of the spin (opposite charge) $j = \pm 1/2$ are obtained. Using the Poynting vector for solutions with spin $j = 1/2$ we find that the energy flow through arbitrary closed surface around our vortex solutions is zero and the localized energy of our solutions circulate in x, y plane. Other important result is that the vortex solutions with spin $j = 1/2$ without external fields are immovable. The initial investigations on stability of these solutions show that vortices with spin $j = 1/2$ are stable while the vortices with opposite spin (charge) $j = -1/2$ are not. The possible generalization of NME to higher number of optical components and higher number of ℓ and j is discussed.

1 Introduction

The interest in the nonlinear generalizations of the quantum field equations [1–3] and in the possibility of obtaining exact stationary solitary solutions of the field equations [4, 5] increases rapidly in the last years. As a rule, different kinds of nonlinearity have been introduced in an *ad-hoc* fashion in the Klein–Gordon equation and also for all four spinor components of the Dirac equations. For the usual case of a cubic nonlinearity, exact $3D + 1$ localized solutions are not found. Our present work, reported in this paper, shows that *the optical analogy* of the Nonlinear Dirac Equations of field (NDE) leads to a nonlinear part *only in the first* coupled equation. This result allows to solve the NDE by separation of variables and to obtain solutions representing optical vortices with classical momenta one and spin one-half.

2 Maxwell's equations with non-stationary linear and nonlinear polarization

Consider the Maxwell's equations in the next case: A source-free medium with non-stationary linear and nonlinear electric polarization and non-stationary magnetic polarization. For this case, the Maxwell's equations can be written:

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \tag{1}$$

$$\nabla \cdot \vec{D} = 0, \qquad \nabla \cdot \vec{B} = \nabla \cdot \vec{H} = 0. \tag{2}$$

The linear and nonlinear electric polarization and the the linear magnetic polarization are presented as:

$$\vec{D} = \vec{P}^{\mathrm{lin}} + 4\pi \vec{P}_{\mathrm{nl}}, \qquad \vec{B} = \vec{H} + 4\pi \vec{M}_{\mathrm{lin}}, \tag{3}$$

where $\vec{E}$ and $\vec{H}$ are the electric and magnetic intensity fields, $\vec{D}$ and $\vec{B}$ are the electric and magnetic induction fields, $\vec{P}^{\rm lin}$, $\vec{P}_{\rm nl}$ are the linear and nonlinear polarizations of the medium respectively and $\vec{M}_{\rm lin}$ is the linear magnetic polarization. The magnetic polarization (magnetization) $\vec{M}_{\rm lin}$ is written as the product of the linear magnetic susceptibility $\eta^{(1)}$ and the magnetic field $\vec{H}$. The nonstationary linear electric polarization can be written as:

$$\begin{aligned}\vec{P}^{\rm lin} &= \int_{-\infty}^{t} \left(\delta(t-\tau) + 4\pi\chi^{(1)}(t-\tau)\right)\vec{E}(\tau,x,y,z)d\tau \\ &= \int_{-\infty}^{t} \varepsilon_0 (t-\tau)\,\vec{E}(\tau,x,y,z)\,d\tau, \end{aligned} \tag{4}$$

where $\chi^{(1)}$ and ε_0 are the linear electric susceptibility and the dielectric constant respectively. Similar expression describes the dependence of $\vec{B}$ on $\vec{H}$ in the case of nonstationary linear magnetic polarization [6]:

$$\begin{aligned}\vec{B} &= \int_{-\infty}^{t} \left(\delta(t-\tau) + 4\pi\eta^{(1)}(t-\tau)\right)\vec{H}(\tau,x,y,z)d\tau \\ &= \int_{-\infty}^{t} \mu_0 (t-\tau)\,\vec{H}(\tau,x,y,z)\,d\tau, \end{aligned} \tag{5}$$

where $\eta^{(1)}$ and μ_0 are the linear magnetic susceptibility and magnetic permeability respectively. In the following, we will study such media with nonstationary cubic nonlinear polarization, where the nonlinear polarization in the case of one carrying frequency can be expressed as:

$$\begin{aligned}\vec{P}^{(3)}_{\rm nl} = \frac{3}{4}\int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} &\chi^{(3)}(t-\tau_1, t-\tau_2, t-\tau_3) \\ &\times \vec{E}(\tau_1,x,y,z)\vec{E}^{*}(\tau_2,x,y,z)\vec{E}(\tau_3,x,y,z)d\tau_1 d\tau_2 d\tau_3. \end{aligned} \tag{6}$$

3 Amplitude equations

We derived in a recent paper [7] the slowly varying amplitude approximation from nonlinear integro-differential Maxwell's equations (1)–(3) in the standard way, as it was done in [10, 11]. The electric and magnetic field amplitudes are determined by the relations:

$$\vec{E}(x,y,z,t) = \vec{A}(x,y,z,t)\exp\left(i\left(\omega_0 t - g(x,y,z)\right)\right), \tag{7}$$

$$\vec{H}(x,y,z,t) = \vec{C}(x,y,z,t)\exp\left(-i\left(\omega_0 t - q(x,y,z)\right)\right), \tag{8}$$

where $\vec{A}$, $\vec{C}$, ω_0, g and q are the amplitudes of the electric and magnetic fields, the optical frequency and the real spatial phase functions respectively. After using Fourier representation of the response functions and of the amplitude functions $\vec{A}$ (and $\vec{C}$), and also the fact that $\nabla\cdot\vec{D} \approx \nabla\cdot\vec{E} \approx \nabla\cdot\vec{A} \approx 0$ [8], we obtain the following system of Nonlinear Maxwell vector amplitude Equations (NME), written in dimensionless variables:

$$\nabla\times\vec{A} = i\alpha_2\vec{C} - \delta\frac{\partial\vec{C}}{\partial t}, \tag{9}$$

$$\nabla\times\vec{C} = i\alpha_1\vec{A} + \frac{\partial\vec{A}}{\partial t} + i\gamma_1\left(\vec{A}\cdot\vec{A}^{*}\right)\vec{A}, \tag{10}$$

$$\nabla\cdot\vec{A} = 0, \qquad \nabla\cdot\vec{C} = 0. \tag{11}$$

where $\alpha_{1,2}$, δ and γ_1 are constants. The gradient of the spatial phase functions g and q satisfied the additional relations:

$$\nabla g \times \vec{A} = 0, \qquad \nabla q \times \vec{C} = 0. \tag{12}$$

The phase functions which satisfied (12) are determined in Section 6.

4 Dirac representation of NME

To solve the NME (9)–(11), we apply the method of separation of variables. The slowly varying amplitude vector of the electric field $\vec{A}$ and the magnetic field $\vec{C}$ are represented as:

$$\vec{A}\,(x,y,z,t) = \vec{F}\,(x,y,z)\exp(i\Delta\alpha t), \qquad \vec{C}\,(x,y,z,t) = \vec{G}\,(x,y,z)\exp(i\Delta\alpha t). \tag{13}$$

Substituting these forms into the NME (9)–(11) we obtain:

$$\nabla \times \vec{F} = -i\nu_2\vec{G}, \tag{14}$$

$$\nabla \times \vec{G} = i\nu_1\vec{F} + i\gamma_1\big(\vec{F}\cdot\vec{F}^{*}\big)\vec{F}, \tag{15}$$

$$\nabla \cdot \vec{F} = 0, \qquad \nabla \cdot \vec{G} = 0, \tag{16}$$

where $\nu_1 = \alpha_1 + \Delta\alpha$, $\nu_2 = \delta\Delta\alpha - \alpha_2 > 0$ are constants. When the electric and magnetic fields are represented as a sum of a linear polarization component and a circular polarized one it is possible to reduce equations (14)–(16) to a system of four nonlinear equations. Substituting [9]:

$$\psi_1 = iF_z, \qquad \psi_2 = iF_x - F_y, \qquad \psi_3 = -G_z, \qquad \psi_4 = -G_x - iG_y \tag{17}$$

into the nonlinear system (14)–(16), we obtain a stationary nonlinear Dirac system of equations (NDE):

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\Psi_4 + \frac{\partial}{\partial z}\Psi_3 = -i\left(\nu_1 + \gamma_1\sum_{i=1}^{2}|\Psi_i|^2\right)\Psi_1, \tag{18}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\Psi_3 - \frac{\partial}{\partial z}\Psi_4 = -i\left(\nu_1 + \gamma_1\sum_{i=1}^{2}|\Psi_i|^2\right)\Psi_2, \tag{19}$$

$$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\Psi_2 + \frac{\partial}{\partial z}\Psi_1 = -i\nu_2\Psi_3, \tag{20}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\Psi_1 - \frac{\partial}{\partial z}\Psi_2 = -i\nu_2\Psi_4. \tag{21}$$

The system (18)–(21) is the optical analog of the nonlinear Dirac equations (NDE). Note that the *optical* NDE are significantly different from the NDE in the field theory. The nonlinear part appears *only* in the first two coupled equations of the system.

5 Hamiltonian representation of the NDE. First integrals for vortex solutions with spin $j = \pm 1/2$

It is not difficult to show that for the NDE system of equations (18)–(21) the Hamiltonian has the form:

$$H = \big(\vec{\sigma}\cdot\vec{P}\,\big) + \sum_{i=1}^{2}|\Psi_i|^2, \tag{22}$$

By virtue of it, equations (18)–(21) can be rewritten:

$$H\Psi = \varepsilon\Psi, \tag{23}$$

where $\varepsilon = (-i\nu_1, -i\nu_1, -i\nu_2, -i\nu_2)$ is the energy operator. Here we investigate the case where the nonlinear part of the equation is represented as a number of spinors with a scalar sum that depends only on the radial component

$$\sum_{i=1}^{2} |\Psi_i(r,\theta,\varphi)|^2 = F(r). \tag{24}$$

We also introduce here the well known orbital momentum operator $\vec{L}$, own orbital (spin) momentum $\vec{S}$, and the full momentum $\vec{J}$. It is straightforward to show that the Hamiltonian (22) of equation (23) commutes with the operators $\vec{J}^2$ and J_z (the z-projections must be x or y). Using these symmetries and the condition that the nonlinearity is of Kerr type, we can solve the NDE equations (23) by a separation of variables technique. We look for solutions in the form:

$$\Psi_1 = a(r)\,\Omega_{jlM}, \tag{25}$$
$$\Psi_2 = a(r)\,\Omega_{jlM}, \tag{26}$$
$$\Psi_3 = ib(r)\,\Omega_{jl'M}, \tag{27}$$
$$\Psi_4 = ib(r)\,\Omega_{jl'M}, \tag{28}$$

where Ω_{jlm} is the spherical spinor, $l + l' = 1$, and $a(r)$ and $b(r)$ are arbitrary radial functions. Using the symmetries of (23) and the fact that the nonlinear parts depend on r we separate variables and obtain the following system of equations for the radial part:

$$\frac{\partial a(r)}{\partial r} + \frac{1+\chi}{r}a(r) = -\nu_2 b(r), \tag{29}$$
$$\frac{\partial b(r)}{\partial r} + \frac{1-\chi}{r}b(r) = \nu_1 a(r) + \gamma|a(r)|^2 a(r), \tag{30}$$

where

$$\chi = l(l+1) - j(j+1) - 1/4. \tag{31}$$

Excluding $b(r)$ from the system (29)–(30), we obtain the equation for $a(r)$:

$$\nu_1\nu_2 a(r) + \frac{\partial^2 a}{\partial r^2} + \frac{2}{r}\frac{\partial a}{\partial r} - \frac{(1+\chi)\chi}{r^2}a + \nu_2\gamma|a|^2 a = 0. \tag{32}$$

The case of optical spinors with spin $j = \pm 1/2$ corresponds to localized solutions with $\chi = 1$ and spherical spinors with $l = 1$ and $j = \pm 1/2$. The spherical spinors are a spinor generalization of usual spherical functions and the spinors (25)–(28) can by written directly in this case:

$$\Psi_1 = a(r)\cos(\theta), \qquad \Psi_2 = a(r)\sin(\theta)\,e^{i\varphi}, \qquad \Psi_3 = -ib(r), \qquad \Psi_4 = 0, \tag{33}$$

for $j = 1/2$ and

$$\Psi_1 = a(r)\sin(\theta)\,e^{-i\varphi}, \qquad \Psi_2 = -a(r)\cos(\theta), \qquad \Psi_3 = 0, \qquad \Psi_4 = -ib(r), \tag{34}$$

for $j = -1/2$.

After substituting solutions (33)–(34) into equations (23) and excluding $b(r)$ from the radial system (29)–(30), we obtained the following equation for $a(r)$:

$$\nu_1\nu_2 a(r) + \frac{\partial^2 a}{\partial r^2} + \frac{2}{r}\frac{\partial a}{\partial r} - \frac{2}{r^2}a + \nu_2\gamma|a|^2 a = 0. \tag{35}$$

This system has exact vortex *de Broglie* soliton solutions [4] in the form:

$$a\left(r\right)=\frac{\sqrt{2}}{i}\frac{\exp\left(i\sqrt{\nu_1\nu_2}r\right)}{r}, \tag{36}$$

if $\nu_2\gamma_1=1$. The complete solutions for these two cases are written:

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\dfrac{\sqrt{2}}{i}\dfrac{\exp(i\sqrt{\nu_1\nu_2}r)}{r}\cos\left(\theta\right)\\ \dfrac{\sqrt{2}}{i}\dfrac{\exp(i\sqrt{\nu_1\nu_2}r)}{r}\sin\left(\theta\right)\exp\left(i\varphi\right)\end{pmatrix}, \tag{37}$$

$$\begin{pmatrix}\Psi_3\\ \Psi_4\end{pmatrix}=\begin{pmatrix}\dfrac{\sqrt{2}}{i\nu_2}\left(\dfrac{i\sqrt{\nu_1\nu_2}\exp\left(i\sqrt{\nu_1\nu_2}r\right)}{r}+\dfrac{\exp i(\sqrt{\nu_1\nu_2}r)}{r^2}\right)\\ 0\end{pmatrix}, \tag{38}$$

for $j=1/2$ and

$$\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}=\begin{pmatrix}\dfrac{\sqrt{2}}{i}\dfrac{\exp(i\sqrt{\nu_1\nu_2}r)}{r}\sin\left(\theta\right)\exp-\left(i\varphi\right)\\ \dfrac{-\sqrt{2}}{i}\dfrac{\exp(i\sqrt{\nu_1\nu_2}r)}{r}\cos\left(\theta\right)\end{pmatrix}, \tag{39}$$

$$\begin{pmatrix}\Psi_3\\ \Psi_4\end{pmatrix}=\begin{pmatrix}0\\ \dfrac{-\sqrt{2}}{i\nu_2}\left(\dfrac{i\sqrt{\nu_1\nu_2}\exp\left(i\sqrt{\nu_1\nu_2}r\right)}{r}+\dfrac{\exp i(\sqrt{\nu_1\nu_2}r)}{r^2}\right)\end{pmatrix}, \tag{40}$$

for $j=-1/2$. The equation (32) admits exact "de Broglie" soliton solutions for arbitrary number of χ, but as we remember our solutions are limited by the conditions (24), the nonlinear part to depend only on the radial components. The condition (24) for a number $\chi\geq 1$ can be fulfilled also for a higher number of fields with different frequencies. This case includes also the parametric processes.

6 Spatial phase functions, Poynting vector and the flow of energy

In a previous paper [7] we investigate the localized energy of our vortex solutions. A spectral region and dispersion parameters of a paramagnetic media were found, where the solutions admit finite energy. The type of the phase functions which satisfied (12) is determined for vortex solutions (33) with spin $j=1/2$. Using again the relations between the spinors of NDE and the amplitude functions (17) we have:

$$F_x=\frac{\psi_2-\psi_2^*}{2i},\qquad F_y=\frac{\psi_2+\psi_2^*}{2},\qquad F_z=\frac{\psi_1-\psi_1^*}{2i}, \tag{41}$$

$$G_x=-\frac{\psi_4+\psi_4^*}{2},\qquad G_y=-\frac{\psi_4-\psi_4^*}{2i},\qquad G_z=-\frac{\psi_3-\psi_3^*}{2i}. \tag{42}$$

Substituting the solutions (33) with spin $j=1/2$ in (41)–(42) for arbitrary real $a(r)$ and $b(r)$ we obtain:

$$F_x=-a(r)\frac{x}{r},\qquad F_y=-a(r)\frac{y}{r},\qquad F_z=-a(r)\frac{z}{r}, \tag{43}$$

$$G_x=0,\qquad G_y=0,\qquad G_z=b(r). \tag{44}$$

We rewrite again the conditions for the spatial phase functions:

$$\nabla g \times \vec{F} = 0, \qquad \nabla q \times \vec{G} = 0. \tag{45}$$

These relations for solutions of kind (43)–(44) are satisfied only when:

$$g = k_0 r \quad \text{or} \quad g = k_0 f(r) \qquad \text{and} \qquad q = k_0 z \quad \text{or} \quad q = k_0 f(z), \tag{46}$$

where k_0 is the carrying wave number. The spatial phase functions of kind $g = k_0 r$ and $q = k_0 z$ correspond to spectral limited pulses which satisfied additional relations $\triangle k \triangle r = \text{const}$. The spatial phase functions of kind $g = k_0 f(r)$ and $q = k_0 f(z)$ correspond to phase modulated pulses and for them the relations $\triangle k \triangle r = \text{const}$ is not satisfied. The Poynting vector can be expressed by the amplitude functions of the electrical and magnetic field:

$$\begin{aligned} \vec{S} &= \vec{E}(x,y,z,t) \times \vec{H}(x,y,z,t) \\ &= \exp\left(i\left(W(t) - q(z) \pm g(r)\right)\right) \vec{F}(x,y,z,t) \times \vec{G}(x,y,z,t), \end{aligned} \tag{47}$$

where q and g are scalar phase functions. Substituting the solutions with spin $j = 1/2$ in above expression we find that:

$$\vec{S} = \exp\left(i\left(W(t) - q(z) \pm g(r)\right)\right) \left(-a(r)b(r)\frac{y}{r};\ a(r)b(r)\frac{x}{r};\ 0\right). \tag{48}$$

We see that the Poynting vector $\vec{S}$ is one circulation vector for solutions with spin $j = 1/2$ and its divergency is zero:

$$\nabla \cdot \vec{S} = 0. \tag{49}$$

The relation (49) determines that the energy flow through arbitrary closed surface around our vortex solutions with spin $j = 1/2$ is zero. The relation (48) shows that flow of energy of our solutions circulate in x, y plane. We generalize the above results for solutions with spin $j = 1/2$: The vortex solutions with spin $j = 1/2$ without external fields are immovable and electromagnetic energy oscillates in x, y plane. The electrical field oscillates spherically, in 'r' direction, while the magnetic field oscillates in z direction. In the same way the Poynting vector was calculated for solutions with spin $j = -1/2$. For them we obtain that $\nabla \cdot \vec{S} \neq 0$ and we expect that they are not stable. Exact investigation of stability requires investigation also of the perturbation of the Poynting vector and will be discussed in a further paper.

7 Conclusion

We have shown that in cases of linear and circularly polarized components of the electric and magnetic fields, the NME reduces to the Nonlinear Dirac system of equations (NDE). The equations are represented in a spinor form. Using the method of separation of variables, exact vortex solutions have been obtained. The optical vortex solutions admit classical orbital momentum $l = 1$ and classical own momentum $j = \pm 1/2$. The energy integral of the vortex solutions is finite only in some special cases of paramagnetic media with suitable conditions on linear electric and magnetic dispersion. Using the Poynting vector for solutions with spin $j = 1/2$ we find that the energy flowing through arbitrary closed surface around our vortex solutions is zero and the localized energy of our solutions circulates in x, y plane. Other important result is, that the vortex solutions with spin $j = 1/2$ without external fields are immovable. The electrical field in the vortices oscillating spherically, while the magnetic field oscillating in z direction. The initial investigations on stability of these solutions show the following: While the vortices with spin $j = 1/2$ are stable, the vortices with opposite spin (charge) $j = -1/2$ are not. All of the above results will be discussed later in relation with nonlinear field theory.

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The Renormalization Group Symmetry for Solution of Integral Equations

Vladimir F. KOVALEV † *and Dmitriy V. SHIRKOV* ‡

† *Institute for Mathematical Modelling, Russian Academy of Science, Moscow, Russia*
E-mail: *kovalev@imamod.ru*

‡ *Joint Institute for Nuclear Research, Dubna, Russia*
Moscow M.V. Lomonosov State University, Russia
E-mail: *shirkovd@thsun1.jinr.ru*

Evolution of the renormgroup algorithm and the related renormgroup symmetry, introduced in mathematical physics for solutions of boundary-value problems based on differential equations, is reviewed. We discuss the essential progress made recently in the application of this algorithm to models with integral equations. Several physical illustrations from nonlinear optics and plasma physics demonstrate the potentialities of the algorithm for models with nonlocal terms in the form of the linear solution functionals.

1 Introduction

The notion of the Renormalization Group Symmetry (RGS) was introduced in mathematical physics in the beginning of the nineties of the last century [1, 2] (see also review papers [3, 4]). It marked the proliferation of the Renormalization Group (RG) ideas [5, 6] from Quantum Field Theory (QFT) to the problems of mathematical physics with the goal that was the same as in QFT: to improve the perturbation theory (PT) solutions and to correct the behavior of these solutions in the vicinity of a singularity. However, the form of the practical implementation of the RG ideas in other fields of theoretical and mathematical physics frequently differs from that used in QFT (see, e.g. the review paper [7]). For boundary value problems (b.v.p.) in mathematical physics that are based on differential equations (DEs) quite new RG algorithm was elaborated [1–4, 8]. In order to clarify this difference we recall that the RG method, as formulated by N.N. Bogoliubov and one of the authors [5] for QFT problems, employs the *exact* group property of a solution with the goal to improve the approximate PT solution. One of the well-known forms of this property is a functional equation (which exhibits the group composition law) for the invariant charge in QFT. Revealing of this symmetry (i.e. the solution group property) in every particular case is a tricky procedure in theoretical physics and is the algorithmic deficiency of the method [9].

In becoming to mathematical physics we usually deal with b.v.p. that are based on DEs. Symmetries of these equations can be obtained in a regular way using the Lie group analysis technique. This fact had a profound impact on creating an algorithm that joined the RG ideology from QFT with a regular procedure of constructing symmetries for b.v.p. It was precisely this algorithm that brought to being the notion of the "renormalization group symmetry" (RGS) for b.v.p. solutions. There were two reasons for calling them RGS: firstly, from mathematical point of view the calculational algorithm for these symmetries is very similar to that used in modern group analysis and, secondly, they are applied to PT solutions with the goal usual to QFT, i.e. to improve these PT solutions.

Early in the development of RGS algorithm in mathematical physics it was mainly applied to mathematical problems based on DEs, though formally this algorithm may be used in any

problem that admit the regular procedure for calculating symmetries for the involved equations. In this sense an application area for RGS algorithm can be substantially expanded to include objects (e.g. integral or integro-differential equations) that traditionally were beyond the scope of the classical group analysis. These integral equations may form the basis of a problem, or they may appear as specific objects for applying RGS algorithm for b.v.p. based on DEs. Frequently we are interested not in a solution itself but in some integral solution characteristic, i.e. the solution functional. This characteristic may appear as a result of integrating with respect to any independent variable or on transition to Fourier variables. In this case RGS algorithm may be applied not for improving a particular solution with the subsequent calculation of some integral characteristic of this solution but for improving the functional of a perturbative solution. In other words this algorithm may restore behavior of an integral characteristic without calculating a solution in an explicit form.

In this report along with the description of the modified RGS algorithm we present also several new illustrations which demonstrate its efficiency.

2 RGS algorithm for nonlocal problems

The scheme for constructing and application of RGS to b.v.p. based on DEs was discussed in details in our recent publications (see, e.g. review papers [4, 8]). Here we briefly touch upon basic stages of this scheme, paying special attention to changes that should be introduced to make the RGS algorithm applicable to nonlocal problems.

We start with a mathematical model, defined by a system of $\nu \geq 1$ *differential* and *integral equations* for functions $u = \{u^\alpha\}$, $\alpha = 1, \ldots, m$ of $x = \{x^i\}$, $i = 1, \ldots, n$,

$$[E]: \quad E_\nu(u(x)) = 0, \tag{1}$$

with appropriate boundary (initial) conditions. Nonlocal terms in these equations depend on integrals of u. We suppose also that we know some approximate solution, U^α, expressed, say, in the form of the truncated PT series in powers of some small parameter or a small distance from the boundary where this solution is given.

Then the general scheme for RGS algorithm, which is depicted on Fig. 1, is realized as a sequence of steps which are:

I constructing the specific RG manifold,

II calculating the symmetry group for this RG manifold,

III restricting this group on a particular (usually approximate) b.v.p. solution that gives the desired RGS,

IV using the RGS obtained to calculate the particular analytical b.v.p. solution.

In the next subsections we consider these steps in more detail.

2.1 Constructing the RG manifold

The key idea of the first step consists in involving in group transformations the parameters and boundary conditions that define the particular solution of the problem. This is achieved by constructing the special RG manifold $\mathcal{RM}$ that we assume to have the form of s differential equations of the k-th order and q nonlocal relations,

$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(k)}) = 0, \qquad \sigma = 1, \ldots, s, \tag{2}$$

$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(r)}, A(u)) = 0, \qquad \sigma = 1 + s, \ldots, q + s. \tag{3}$$

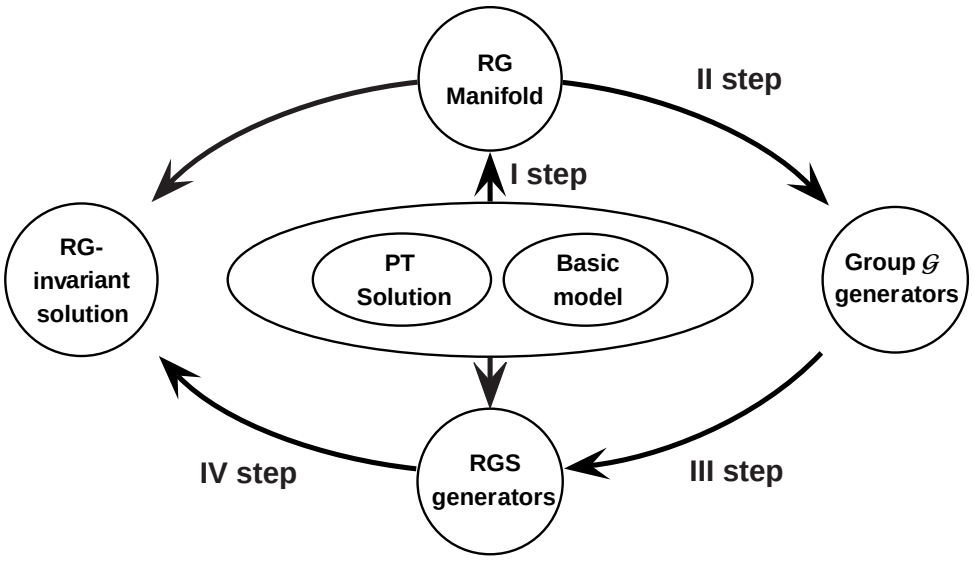

Figure 1. General scheme for construction of RGS.

We use the traditional for the differential algebra notations for independent variables $z = \{z^i\}$, differential variables $u = \{u^\alpha\}$ with the subsequent derivatives $u_{(1)} = \{u^\alpha_i\}$, $u_{(2)} = \{u^\alpha_{ij}\}$, $i, j = 1, \ldots, n + l$. These variables are related by the operator of the total differentiation D_i,

$$u^\alpha_i = D_i\,(u^\alpha)\,, \quad u^\alpha_{ij} = D_j\,(u^\alpha_i)\,D_j D_i\,(u^\alpha)\,, \quad \ldots, \quad D_i = \frac{\partial}{\partial z^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots\,. \quad (4)$$

The parameters $p = \{p^j\}$, $j = 1, \ldots, l$ are included in $z = \{x, p\}$ and nonlocal variables are given by integral relations

$$A(u) = \int \mathcal{F}(u(z))\mathrm{d}z. \quad (5)$$

Nonlocal relations (3) characterize the principal difference of $\mathcal{RM}$ for nonlocal models as compared to b.v.p. for DEs, where we have the differential RG manifold.

The particular form of the realization of the first step depends both on the form of input equations and the form of boundary (initial) conditions and is inspired by PT solution as well. In general case $\mathcal{RM}$ does not coincide with E. We can indicate several possible routines to the problem, that were formulated for local models (see, e.g., [4]) and remain valid also for nonlocal models:

- One can extend the space of variables involved in the group transformations and in fact is realized in equations (2)–(3), where we add parameters p that enter a solution both via equations and/or via boundary (initial) conditions. Adding derivatives with respect to p extends the space of differential variables. The extension of the space of independent and differentiable variables frequently transforms the basic equations (1) into $\mathcal{RM}$, however the non-locality is preserved.
- Another possibility employs reformulating the boundary conditions in terms of *embedding equations* or *differential constraints*. The key idea here is to treat simultaneously the solution of the b.v.p. as an analytic function of the independent variables and the boundary parameters $b = \{x^i_0, u^\alpha_0\}$. Differentiation with respect to these parameters leads to additional DEs (*embedding equations*) that, together with the basic equations (1), form the $\mathcal{RM}$. In general, different boundary conditions for the same input equations lead to

different embedding equations. In the simplest case embedding equations have the form of evolution first-order partial DEs, however, more complicated situations are possible.

A more general approach is based on reformulating the boundary conditions in terms of *differential constraints* that are compatible with the basic DEs. In particular these differential constraints can arise while calculating the higher order (or Lie–Bäcklund) symmetries for the basic or embedding equations.

- In the case when the basic equations (1) contain a small parameter μ, the desired *approximate* manifold $\mathcal{RM}$ can be obtained by simplification of the equations (1) and use of "perturbation methods of group analysis" (see Vol. 3, Chapter 2, p. 31 in [10]). The main idea here is to consider a simplified ($\mu = 0$) model, which admits a wider symmetry group in comparison with the case $\mu \neq 0$. When we take the contributions from small μ into account, this symmetry is *inherited* by the equations (1), which results in some additional terms, corrections in powers of μ, in group generators. As a result the same representation arises for the RGS generators that are calculated at the next steps of the algorithm.

The routines mentioned do not exhaust all possibilities for constructing $\mathcal{RM}$, rather they point to different possibilities for the realization of the first step. In every particular case the choice is defined by the model equations (1) and the form of the PT solution.

2.2 Calculation of the admitted group

The next step is the calculation of the most general symmetry group $\mathcal{G}$, admitted by $\mathcal{RM}$ (2)–(3). Precisely here we need the substantial modification of the algorithm as compared to differential RG manifold. Indeed, in application to $\mathcal{RM}$ defined by DEs, e.g. (2), we deal with a *local group of transformations* in the space of differential functions $\mathcal{A}$ that leaves these equations unaltered. The classical group algorithm for finding these transformations employs the Lie's infinitesimal technique that replaces the procedure of solving original, possibly nonlinear, equations by analyzing linear equations for coordinates of group generators, determining equations. In constructing RG symmetries several generalizations of the classical Lie algorithm that are in use in modern group analysis may be employed (see [10–15] and references therein), i.e. approximate symmetries, non-classical and conditional symmetries, non-local symmetries, discrete symmetries, etc.

This situation drastically changes when moving to RG manifold (3) given by integro-differential equations or to solution functionals, introduced by integral terms. The major obstacle for the application of Lie's infinitesimal techniques to integro-differential equations (or infinite systems of differential equations) is that the *frames* (see, e.g. [13]) of these equations are not locally defined in the space of differential functions. In consequence, the crucial idea of splitting of determining equations into over-determined systems, commonly used in the classical Lie group analysis, fails. The next problem in application of the symmetry group obtained is the problem of prolongation of the symmetry group operator on solution functionals that have the form of nonlocal (integral) variables. Below we indicate several routines to the problem and describe common approaches in application to nonlocal objects.

Loosely speaking, different known approaches to calculating symmetry groups for integro-differential equations can be divided into two large groups: indirect and direct methods.

Algorithms of the first group rest on the possibility to replace in any way input nonlocal (integro-differential) equations by a system of differential equations. Then the resulting system of differential equations is analyzed using standard methods of a classical Lie group analysis. Here we point on two different ways of reducing nonlocal equations to differential ones: a) *method of moments* that was realized to calculate Lie point symmetry group for Vlasov–Maxwell equations in plasma theory [16] and for Benney, Vlasov-type and Boltzmann-type kinetic equations [17,18]

and b) *method of boundary-differential equations* that was developed in [19] on basis of the concept of covering and applied to a coagulation kinetic equation.

With the goal to modify the RGS algorithm we employ methods of the second group, namely, direct methods of finding symmetries that were developed in [20–22] and applied to find symmetries of kinetic Boltzmann equation, the equations of motion of viscoelastic medium and Vlasov–Maxwell equations of plasma theory. In this method the utilization of the Lie–Bäcklund group and the canonical group representation is of principal value, i.e. instead of the traditional form for the group generator

$$X = \xi^i \partial_{z^i} + \eta^\alpha \partial_{u^\alpha} + \zeta_i^\alpha \partial_{u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \partial_{u_{i_1 i_2}^\alpha} + \cdots ,$$
$$\zeta_i^\alpha = D_i(\varkappa^\alpha) + \xi^j u_{ij}^\alpha, \qquad \zeta_{i_1 i_2}^\alpha = D_{i_1} D_{i_2}(\varkappa^\alpha) + \xi^j u_{j i_1 i_2}^\alpha, \qquad \varkappa^\alpha = \eta^\alpha - \xi^i u_i^\alpha, \tag{6}$$

with coordinates $\xi^i([z,u])$, $\eta^\alpha([z,u])$, $\zeta_i^\alpha([z,u])$, ..., we use the equivalent operator Y that is known as the canonical representation for X,

$$X \sim Y = X - \xi^i D_i = \varkappa^\alpha \partial_{u^\alpha}. \tag{7}$$

It is essential that only dependent variables u^α are involved in the group infinitesimal transformation with the generator (7) and the group parameter a whereas independent variables z^i do not vary

$$u'^\alpha = u^\alpha + a\varkappa^\alpha + O\big(a^2\big), \qquad z'^i = z^i. \tag{8}$$

Hence one can define the local group $\mathcal{G}$ of point transformations as a symmetry group of integro-differential equations (3) iff for any a the functions F_σ do not vary [20]. Differentiating the appropriate invariance condition with respect to the group parameter a and assuming $a \to 0$ gives the determining equations. In contrast to the case of input differential equations these determining equations are in general also integro-differential.

The invariance criterion for F_σ with respect to the admitted group can be expressed in an infinitesimal form using the canonical group operator Y,

$$YF_\sigma\Big|_{[F_\sigma]} = 0, \qquad \sigma = 1+s, \ldots, q+s, \qquad \text{where} \qquad Y \equiv \int \mathrm{d}z \varkappa(z) \frac{\delta}{\delta u(z)}, \tag{9}$$

and the symbol $|_{[F_\sigma]}$ means evaluation on the manifold, generated by (3). Here with the goal to generalize the action of a canonical group operator not only on differential functions but on *functionals* as well we use variational differentiation in the definition of Y [22]. One can verify by direct calculation that the action of Y on any differential function and its derivatives, e.g., u, u_z, ... produces the usual result: $Yu = \varkappa$, $Yu_z = D_z(\varkappa)$ and so on. Hence, if $F_\sigma = 0$ describe usual DEs then formulas (9) lead to standard local determining equations, while for $F_\sigma = 0$ having the form of integro-differential equations formulas (9) can be treated as *nonlocal* determining equations as they depend both on local and nonlocal variables.

In order to find solutions of determining equations one can use different approaches, e.g. expanding coordinates of group generator into formal power series and equating coefficients of various powers [20]. However there exists a more traditional way. As we treat local and nonlocal variables in determining equations as independent it is possible to separate these equations into local and nonlocal. The procedure of solving local determining equations is fulfilled in a standard way using Lie algorithm based on splitting the system of over-determined equations with respect to local variables and their derivatives. As a result we get expressions for coordinates of group generator that define the so-called group of *intermediate* symmetry [22]. In the similar manner the solution of nonlocal determining equations is fulfilled using the information borrowed from an intermediate symmetry and by "variational" splitting of nonlocal determining equations using

the procedure of variational differentiation. Therefore, the algorithm of finding symmetries of nonlocal equations appears as an algorithmic procedure that consists of a sequence of several steps:

- defining the set of local group variables,
- constructing determining equations on basis of the infinitesimal criterion of invariance that employs the generalization of the definition of the canonical operator,
- separating determining equations into local and nonlocal,
- solving local determining equations using a standard Lie algorithm,
- solving nonlocal determining equations using the procedure of variational differentiation.

This procedure generalizes the second step of the RGS algorithm in case of the integral or integro-differential manifold (2)–(3). As a result of this step we get infinitesimal operators (group generators) which correspond to the admitted vector field and form a Lie algebra with a general element

$$Y = \sum_j C^j Y_j, \qquad Y_j = \varkappa_j^\alpha \partial_{u^\alpha} \equiv \left(\eta_j^\alpha - \xi_j^i u_i^\alpha\right) \partial_{u^\alpha}, \tag{10}$$

where C^j are arbitrary constants.

To complete we describe the procedure of prolongation of the symmetry group on nonlocal variables, say, in the form of the integral relation (5). To fulfill this procedure one should first rewrite the operator, say, Y, in a canonical form and then formally prolong this operator on the nonlocal variable A

$$Y + \varkappa^A \partial_A \equiv \varkappa \partial_u + \varkappa^A \partial_A. \tag{11}$$

The integral relation between $\varkappa$ and $\varkappa^A$ is obtained by applying the generator (11) to the *definition* of A, i.e. to (5). Substituting the explicit expression for the coordinate $\varkappa$ of the known operator Y and calculating integrals obtained gives the desired coordinate $\varkappa^A$,

$$\varkappa^A = \int \frac{\delta A(u)}{\delta u(z)}\, \varkappa(z)\, \mathrm{d}z \equiv \int \frac{\delta \mathcal{F}(u(z'))}{\delta u(z)}\, \varkappa(z)\, \mathrm{d}z\mathrm{d}z' = \int \mathcal{F}_u \varkappa(z)\, \mathrm{d}z. \tag{12}$$

The group defined by the generators (10) is in general wider than the desired RG, that usually appears as its subgroup. As the RGS is related to a particular b.v.p. solution, it can be revealed by *restricting the admitted group* $\mathcal{G}$ on a manifold defined by this given solution.

2.3 Restriction the group on a solution

Mathematically, this procedure that forms the third step of the RG algorithm appears as checking the vanishing condition for the linear combination of coordinates $\varkappa_j^\alpha$ of the operator (10) on a particular approximate (or exact) b.v.p. solution $U^\alpha(z)$

$$\left\{ \sum_j C^j \varkappa_j^\alpha \equiv \sum_j C^j \left(\eta_j^\alpha - \xi_j^i u_i^\alpha\right) \right\}_{\Big| u^\alpha = U^\alpha(z)} = 0. \tag{13}$$

While the general form of the condition given by (13) is the same for any b.v.p. solution, the specific way of realization of the restriction procedure in every particular case employs a well-defined PT solution for the concrete b.v.p. This is depicted by the additional arrow that links two objects on the general scheme, namely “PT solution” and “RGS generators”.

Evaluating (13) on a particular b.v.p. solution, $U^\alpha(z)$, transforms the system of DEs for the group invariants into algebraic relations. Firstly, it gives relations between the C^j thus "combining" different coordinates of the group generators Y_j admitted by the $\mathcal{RM}$ (2)–(3). Secondly, it eliminates (partially or entirely) the arbitrariness that may appear in the coordinates ξ^i, η^α in the case of an infinite group $\mathcal{G}$.

Generally, the restriction procedure reduces the dimension of $\mathcal{G}$. Hence, the general element (10) of the group $\mathcal{G}$ after the fulfillment of a restriction procedure is expressed as a linear combination of the new generators R_i with the coordinates $\hat{\varkappa}^i$, $\hat{\eta}^\alpha_j$ and $\hat{\xi}^i_j$

$$Y \;\Rightarrow\; R = \sum_j B^j R_j, \qquad R_j = \hat{\varkappa}^\alpha_j \partial_{u^\alpha} \equiv \left(\hat{\eta}^\alpha_j - \hat{\xi}^i_j u^\alpha_i\right) \partial_{u^\alpha}, \tag{14}$$

where the B^j are arbitrary constants. The set of RGS generators R_i, each containing the desired b.v.p. solution in its invariant manifold, define a group of transformations that we also refer to as a *renormgroup* similar to the variant with $\mathcal{RM}$ defined by DEs.

The above prescribed three steps entirely define a regular algorithm for RGS construction but do not touch on how a b.v.p. solution is found. Hence, one more important, fourth step should be added. It consists of using the RGS generators to find analytical expressions for the new, "improved", solution of the b.v.p.

2.4 Construction of the RG invariant solution

Mathematically, the fourth step makes use of the RG *invariance conditions* that are given by a combined system of (2)–(3) and the vanishing condition for the linear combination of coordinates $\hat{\varkappa}^\alpha_j$ of the canonical operator equivalent to (14),

$$\sum_j B^j \hat{\varkappa}^\alpha_j \equiv \sum_j B^j \left(\hat{\eta}^\alpha_j - \hat{\xi}^i_j u^\alpha_i\right) \;= 0. \tag{15}$$

The necessity of using $\mathcal{RM}$ while calculating RG invariant solutions is depicted by the additional arrow that links these two objects on the general scheme.

One can see that the conditions (15) are akin to (13). However, in contrast to the previous step, the differential variables u in (15) should not be replaced by an approximate expression for the b.v.p. solution $U(z)$, but should be treated as usual dependent variables.

For the *one-parameter Lie point renormgroup*, RG invariance conditions (15) lead to the *first-order partial DE* that gives rise to the so-called *group invariants* (such as invariant couplings in QFT) which arise as solutions of associated characteristic equations. The general solution of the b.v.p. is now expressed in terms of these invariants. On the one hand, this is in direct analogy with the structure of RG invariant solutions in QFT. In the general case of arbitrary RGS the group invariance conditions obtained for a b.v.p. are not necessarily characteristic equations for the Lie point group operator. They may appear in a more complicated form, e.g., as a combination of partial DEs and higher order ordinary DEs. Nevertheless, the general idea of finding solutions to the b.v.p. in terms of RG invariant solutions remains valid.

The description of the fourth step completes the presentation of the RGS algorithm. Next we will present several examples that illustrate the application of the algorithm to various physical systems based on nonlocal models.

3 RGS algorithm in application to nonlocal models

3.1 Laser beam refraction in a nonlinear medium

In the section we demonstrate the application of RGS algorithm to solution functionals using a model that describes the evolution of the laser beam intensity $I(z, x)$ and the eikonal deriva-

tive $v(z,x)$ in a nonlinear medium ($z>0$). These functions obey nonlinear optics equations:

$$v_z + vv_x - \alpha I_x = 0, \qquad I_z + vI_x + Iv_x = \nu Iv/x; \qquad v(0,x)=0, \qquad I(0,x)=\mathcal{I}(x). \quad (16)$$

where α is the parameter that defines the nonlinear refraction, z and x are the coordinates along and transverse to the beam axis, respectively; $\nu=1$ for cylindrical beam and $\nu=0$ for the plane beam geometry. Up to now there does not exist a universal algorithm for solving b.v.p. (16) that is suitable for arbitrary boundary conditions and beam geometry. The application of the RGS algorithm to (16) helps to construct new analytical solutions [23,4] and enables to prolong PT solution which is valid in the vicinity of a medium boundary $z\simeq 0$ up to large values of z, and in particular in the vicinity of a solution singularity $z_{\rm sing}$.

The appearance of a singularity in the solution can be obtained using the reduced description via two *solution functionals*, namely the laser beam intensity, $I^0(z)\equiv I(z,0)$, and the second derivative of the eikonal, $W^0(z)\equiv v_x(z,0)$, on the beam axis, that are formally introduces as

$$I^0(z)=\int \mathrm{d}x\delta(x)I(z,x), \quad W^0(z)=\int \mathrm{d}x\delta(x)v_x(z,x), \quad I^0(0)=1, \quad W^0(0)=0. \qquad (17)$$

Here we present two RGS generators that describe the evolution of I^0 and W^0 for the cylindrical laser beam ($\nu=1$) with the parabolic intensity distribution, $\mathcal{I}(x)=1-x^2$, and for the plane laser beam ($\nu=0$) with a "soliton" profile at the boundary, $\mathcal{I}(x)=\cosh^{-2}(x)$,

$$\begin{aligned}
R_{\rm par} &= \left(4\alpha I^0 z - \left(1-2\alpha z^2\right) I^0_z\right)\partial_{I^0} - \left(2\alpha(1-2zW^0) + \left(1-2\alpha z^2\right) W^0_z\right)\partial_{W^0},\\
R_{\rm sol} &= \left(4-5I^0 - zI^0_z + 2(I^0-1)\frac{I^0 I^0_{zz}}{(I^0_z)^2}\right)\partial_{I^0}\\
&\quad + \left(\frac{I^0_z}{I^0} + z\frac{I^0_{zz}}{I^0} - z\left(\frac{I^0_z}{I^0}\right)^2 - 2(I^0-1)\left[\frac{I^0_{zzz}}{(I^0_z)^2} + 2\frac{I^0_z}{(I^0)^2} - 2\frac{(I^0_{zz})^2}{(I^0_z)^3}\right]\right)\partial_{W^0}.
\end{aligned} \qquad (18)$$

These generators result from the corresponding RGS generators [23,24], prolonged on the functionals (17). Utilizing RG invariance conditions (15) for the generator $R_{\rm par}$ gives the behavior of $I^0(z)$ and $W^0(z)$ for the "parabolic" laser beam

$$I^0=\frac{1}{1-2\alpha z^2}, \qquad W^0=-\frac{2\alpha z}{1-2\alpha z^2}, \qquad z_{\rm sing}=1/\sqrt{2\alpha}, \qquad (19)$$

starting from the boundary of the nonlinear medium up to the point of the solution singularity, $z=z_{\rm sing}$, where both the beam intensity and the eikonal derivative go to infinity. Quite similarly RG invariance conditions (15) for the generator $R_{\rm sol}$ in view of the additional constraint, $(I^0_z/\sqrt{I^0-1})_{|z\to 0}=2\sqrt{\alpha}$, yield the behavior of I^0 and W^0 for the "soliton" laser beam

$$z=\frac{\sqrt{I^0-1}}{\sqrt{\alpha}I^0}, \qquad W^0=-\frac{2\alpha z I^0}{1-2\alpha z^2 I^0}, \qquad z_{\rm sing}=1/2\sqrt{\alpha} \qquad (20)$$

from the boundary up to the singularity point $z=z_{\rm sing}$, where $I^0=2$. Though the formulas (19), (20) reproduce the results that were obtained in the context of the RGS algorithm for local models [23,24] (see also [25]) we note that the description of the solution singularity in terms of solution functionals (17) lead to invariance conditions in the form of *ordinary* DEs, in contrast to *partial* DEs, that arise from invariance conditions obtained for b.v.p. solutions.

3.2 RGS for a nonlinear dielectric permittivity of plasma

The next example demonstrates the application of RGS algorithm to calculating the nonlinear dielectric permittivity (NDP) tensors of plasma. In nonlinear electrodynamics the material equation, i.e. the relation between the induced current density $\boldsymbol{j}(t,\boldsymbol{r})$ and the electric field $\boldsymbol{E}(t,\boldsymbol{r})$,

is defined by the dependence of the electric induction $\boldsymbol{D}(t,\boldsymbol{r})$ upon the electric field $\boldsymbol{E}$, that frequently is given by a power series in $\boldsymbol{E}$. Moreover, due to the temporal and spatial dispersion the relation between the induced current and the electric field is integral (nonlocal). Hence, the material equation has the form of an integral power series in $\boldsymbol{E}$ and in Fourier variables (denoted by the sign "tilde") is given by:

$$\tilde{D}_s(k) = \tilde{E}_s(k) + i\frac{4\pi}{\omega}\sum_l \tilde{j}_s^{(l)}(k) = \varepsilon_{sa}(k)\tilde{E}_a(k) + \sum_{n=2}^{\infty}\int \mathrm{d}k_1\cdots\mathrm{d}k_n\delta(k-k_1-\cdots-k_n)$$
$$\times\,\varepsilon_{sj_1\cdots j_n}(k_1;\ldots;k_n)\tilde{E}_{j_1}(k_1)\cdots\tilde{E}_{j_n}(k_n), \quad \tilde{\boldsymbol{j}}^{(l)}(k) \backsim O(\tilde{\boldsymbol{E}}^l), \quad (k)\equiv(\omega,\boldsymbol{k}). \tag{21}$$

Comparison of two parts of this expression gives relations between NDP tensors of plasma and the current density $\tilde{\boldsymbol{j}}^{(l)}(k)$ of the given order $l \geqslant 2$.

In hot plasma NDP tensors are usually obtained by iterating Vlasov–Maxwell equations with the stationary and homogeneous background distribution function $f_0(\boldsymbol{v})$, while in cold plasma we derive NDP tensors from collisionless hydrodynamic equations. It is generally taken that expressions for NDP tensors in hot plasma are of more general type, hence "cold" expressions follow from them in the particular case $f_0(\boldsymbol{v}) = \delta(\boldsymbol{v})$. The use of RGS gives the method of constructing "hot" expression from the "cold" ones [2, 8].

The key idea here is to express the current density $\tilde{\boldsymbol{j}}^{(l)}(k)$ in hot plasma as a convolution of a partial current density $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$ with the equilibrium distribution function $f_0(\boldsymbol{w})$, depending upon the Lagrangian velocity $\boldsymbol{w}$,

$$\tilde{\boldsymbol{j}}^{(l)}(k) = \int \mathrm{d}\boldsymbol{w} f_0(\boldsymbol{w})\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w}). \tag{22}$$

Then the procedure of constructing NDP tensors in hot plasma is realized as a sequence of the following steps: a) calculating $\tilde{\boldsymbol{j}}^{(l)}(k,0)$ in cold plasma using collisionless hydrodynamic equations, b) constructing $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$ from $\tilde{\boldsymbol{j}}^{(l)}(k,0)$ for arbitrary $\boldsymbol{w}\neq 0$, c) integrating $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$ over $\boldsymbol{w}$ with the "weight" function $f_0(\boldsymbol{w})$ to get the desired expression for $\tilde{\boldsymbol{j}}^{(l)}(k)$ in hot plasma, d) making use of $\tilde{\boldsymbol{j}}^{(l)}(k)$ to calculate NDP tensor in hot plasma. It is essential that the step b), i.e. the transition from the "cold" expression for $\tilde{\boldsymbol{j}}^{(l)}(k,0)$ to the "hot" expression $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$, is a group transformation that is defined by the corresponding RGS generator. It is constructed from the Lie point group, admitted by the plasma kinetic equations with Lagrangian velocity. For example, in non-relativistic plasma the RGS generator appears as a linear combination of the generator of Galilean transformations, prolonged on Fourier variables, and translations along the Lagrangian velocity $\boldsymbol{w}$,

$$\boldsymbol{R}_{NDP} = \boldsymbol{k}\partial_\omega + \partial_{\boldsymbol{w}} - \frac{1}{c}\big[\tilde{\boldsymbol{B}},\partial_{\tilde{\boldsymbol{E}}}\big] + \tilde{\varrho}\partial_{\tilde{\boldsymbol{j}}}. \tag{23}$$

The finite transformations defined by this three-dimensional RG have the form[1]

$$\omega = \omega' + \boldsymbol{k}'\boldsymbol{w}, \qquad (\beta_{is}/\omega)\tilde{E}_s = (1/\omega')\tilde{E}'_i, \qquad \tilde{\varrho} = \tilde{\varrho}', \qquad \tilde{j}_i = \beta_{si}\tilde{j}'_s,$$
$$\boldsymbol{k} = \boldsymbol{k}', \qquad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}' = \big(c/\omega'\big)\,[\boldsymbol{k}',\tilde{\boldsymbol{E}}'], \qquad \beta_{is} = \delta_{is} + k_i w_s/(\omega - \boldsymbol{k}\boldsymbol{w}), \tag{24}$$

and relate $\tilde{\boldsymbol{j}}^{(l)}(k,0)$ to $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$ for arbitrary $\boldsymbol{w}\neq 0$. For example, in first order, $l=1$, this procedure gives

$$\tilde{j}_a^{(1)}(k,0) = i\frac{e^2 n_{e0}}{m\omega}\tilde{E}_a(k) \quad \Leftrightarrow \quad \tilde{j}_c{}^{(1)}(k,\boldsymbol{w}) = i\frac{e^2 n_{e0}}{m\omega}\beta_{sc}\beta_{sa}\tilde{E}_a(k), \tag{25}$$

[1]The variables with "primes" are referred to cold plasma limit, i.e. $\boldsymbol{w}=0$.

where e is the electron charge and n_{e0} is the equlibrium density of electrons. By substituting these expressions into (22) and then in (21) we get the one-to-one correspondence between the *scalar* dielectric permittivity in cold plasma and the corresponding *tensor* in hot plasma

$$\varepsilon(k) = 1 - \frac{4\pi e^2 n_{e0}}{m\omega^2} \quad \Leftrightarrow \quad \varepsilon_{ab}(k) = \delta_{ab} - \frac{4\pi e^2 n_{e0}}{m\omega^2} \int \mathrm{d}\boldsymbol{w} f_0(\boldsymbol{w}) \beta_{sa} \beta_{sb}. \tag{26}$$

The generalization of this result for the arbitrary order l is straightforward [8].

3.3 RGS algorithm for the problem of a plasma bunch expansion

The last example deals with application of RGS algorithm to the problem of ion acceleration in an expanding collisionless plasma bunch heated by an ultra short laser pulse. Kinetic treatment of this problem is based on a solution to the Cauchy problem for the coupled Vlasov equations for electrons and ions in a self-consistent electric field. The use of the RGS algorithm here enables to find a general class of kinetic solutions [26–28] for initial conditions of practical interest for laser-plasma interactions which includes a two-temperature Maxwellian electron distribution function (DF), a super-Gaussian electron DF, and a plasma with ion multi-species.

Let us consider an expansion of a plasma slab, which is inhomogeneous in x. The basic system includes kinetic equations for particle DFs f^α,

$$f^\alpha_t + v f^\alpha_x + (e_\alpha/m_\alpha) E(t,x) f^\alpha_v = 0, \qquad f^\alpha\big|_{t=0} = f^\alpha_0(x,v), \tag{27}$$

with the additional quasi-neutrality conditions

$$\int \mathrm{d}v \sum_\alpha e_\alpha f^\alpha = 0, \qquad \int \mathrm{d}v v \sum_\alpha e_\alpha f^\alpha = 0. \tag{28}$$

The solution to the initial-value problem is found perturbatively for $t \to 0$ and is continued in time using the RGS algorithm. The RGS generator is a linear combination of time translations and the projective operator, which arises by virtue of nonlocal relations (28),

$$R_{\text{bunch}} = \big(1 + \Omega^2 t^2\big)\partial_t + \Omega^2 t x \partial_x + \Omega^2 (x - vt)\partial_v. \tag{29}$$

This operator is the only one which selects the spatially symmetric initial DFs with zero mean velocity. The value Ω can be treated as the ratio of the ion acoustic velocity to the gradient length L_0. One can easily find DFs which are invariants of the RG transformation, however for practical applications rough integral characteristics, such as partial ion density, $n_q(t,r)$, might be more useful,

$$n^q(t,x) = \int_{-\infty}^{\infty} \mathrm{d}v f^q(t,x,v). \tag{30}$$

Treating $n^q(t,x)$ as the linear functional of f^q we prolong (29) to get the following RG generator

$$R_{\text{density}} = \big(1 + \Omega^2 t^2\big)\partial_t + \Omega^2 t x \partial_x - \Omega^2 t n^q \partial_{n^q}. \tag{31}$$

Finite group transformations defined by this operator lead to the ion density distribution in terms of a universal function, defined by the initial DFs and in view of the quasi-neutrality conditions (28),

$$n^q = \frac{1}{\sqrt{1+\Omega^2 t^2}} \mathcal{N}_q \left(\frac{x}{\sqrt{1+\Omega^2 t^2}} \right), \qquad \mathcal{N}_q = \int_{-\infty}^{\infty} \mathrm{d}v f^q_0. \tag{32}$$

Therefore, in order to find density distribution we need not know the explicit form of the particle DFs. This example certainly does not exhaust all possible applications of RGS algorithm to plasma dynamics, rather it points to some feasible fields of application (see, e.g. [28]).

4 Conclusion

In this review we have described the expansion of possibilities of the RGS algorithm to various nonlocal models in mathematical physics. In the modern form this algorithm is now applicable both to models based on differential equations and to models that also include nonlocal terms, e.g. integro-differential and integral equations.

The new formulation of the algorithm preserves the general scheme as a sequence of four steps, that were employed for differential models, however the form of their implementation now takes into account the peculiarities introduced by nonlocal terms. Several routines to overcome the difficulties with calculating the symmetry group for nonlocal models are indicated and the procedure of prolongation of the symmetry on to solution functionals is described. The efficiency of the algorithm is illustrated by various physical applications.

We hope that this report serves as an illustration of the universality of the RGS algorithm in application both to models based on DEs and nonlocal (integral) equations. The possibility to prolong the symmetry onto solution functionals enables to investigate the behavior of solution characteristics even in the case when the explicit form of the solution is not known. The results presented gives the promise for the progress in this field and can give rise to many potential applications.

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Renormalization, Ward Identities, Symmetries, and Quantum Anomalies: Logical Aspect

Valentyn I. KUCHERYAVY

Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine
E-mail: *mmtpitp@bitp.kiev.ua*

Logical aspects associated with renormalization methods, Ward identities, symmetries, and quantum anomalies are investigated for ultraviolet divergent VV, AA, AV, and VA mass-anisotropic spinor amplitudes characterizing the most important polarization properties of many-fermion vacuum in any space-time dimension $n = 2r+\delta_n$, $\delta_n = 0, 1$. It is shown clearly that genuine algorithmic cause of perturbation theory anomalies is a logical descrepancy arising in the course of evaluating finite values of UV-divergent Feynman amplitudes (FAs) by means of commonly used renormalization techniques. Some important lessons, which teach us the manner how to construct a logically consistent renormalization technique, i.e. self-consistent renormalization (SCR), are revealed as part of this study.

1 Introduction

This report is a brief exposition of some results by the author on the investigation of a complicated tangle of problems associated on the one hand with renormalization methods and on the other hand with symmetries, their breaking, the Ward identities behavior, the Schwinger terms contributions (STC), and quantum anomalies. The main emphasis of this presentation is made on logical aspects of above problems. It will be shown clearly that genuine algorithmic cause of perturbation theory anomalies is a logical trap arising in the course of evaluating finite values of UV-divergent FAs by means of commonly used renormalization techniques.

The subject is illustrated for some set of quantities appearing in vector and axial vector Ward identities associated with VV, AA, AV, and VA mass-anisotropic UV-divergent spinor FAs characterizing the most important polarization properties of the fermion vacuum for n-dimensional quantum field models in which a mass spectrum of many-fermion sector may be both degenerate and nondegenerate (see reviews [1, 2]). Moreover, these amplitudes have of primary importance for investigating of anomaly problems [3–7][1] as well.

[1]It is appropriate to present Jackiw's point of view (see [6, p. 156]) on this question and on the essence of quantum anomalies at all: "... Thus, in a very precise way, the two-dimensional Abelian anomaly [K. Johnson, PL (1963), see Ref. [3] of our paper] is at the heart of the entire anomaly phenomenon ...

... We have learned much from mathematicians about the topological and cohomological necessity of anomalies [A.S. Schwarz, PL (1977); L. Brown, R. Carlitz and C. Lee, PR (1977); R. Jackiw and C. Rebbi, PR (1977); M. Atiyah and I. Singer, PNAS USA (1984); L. Alvarez-Gaumé and P. Ginsparg, NP (1984)] but perhaps physics can, in its turn, advance mathematical concepts by insisting on the fact that the essence of the anomaly lies beyond present topological/cohomological ideas. The latter involve integrated, global quantities, like the Chern–Pontryagin number, yet the anomaly is local. Moreover, anomalies are present even in the absence of obstructions, like in Abelian [$U(1)$] theories, as in the discussed example which, being two-dimensional, hardly possesses any structure, save the anomaly. The $U(1)$ anomaly, on the other hand, appears to be the heart of the matter, not only for the non-Abelian anomalies,but also for the non-perturbative ones [E. Witten, PL (1982); R. Jackiw, "Relativity, groups, and topology II" (1984), see Ref. [5] of the present paper].

Thus, it seems that we are not yet at the end of the physics nor of the mathematics that can emerge from understanding anomalies ...".

2 VV, AA, AV, and VA mass-anisotropic spinor amplitudes

2.1. Consider VV, AA, AV, and VA mass-anisotropic spinor UV-divergent Feynman amplitudes

$$I^{\alpha\beta}_{JJ'}(m,k)_\epsilon := \int_{-\infty}^{\infty}(d^np)\,\delta(p,k)\,\frac{\mathcal{I}^{\alpha\beta}_{JJ'}(m,p)}{(\mu_{1\epsilon}-p_1^2)(\mu_{2\epsilon}-p_2^2)},\quad J,\,J'\in\{V,\,A\},$$
$$\mathcal{I}^{\alpha\beta}_{JJ'}(m,p) := \mathrm{tr}[\gamma^\alpha\gamma(m_1+\hat p_1)\gamma^\beta\gamma'(m_2+\hat p_2)],\quad \gamma,\gamma'\in\{I_g,\widetilde\gamma^5\},\quad \gamma\gamma'\in\{I_g,\varepsilon_gI_g,\widetilde\gamma^5\},\qquad(1)$$
$$\delta(p,k) := \delta(-k_1+p_2-p_1)\,\delta(-k_2+p_1-p_2),\quad k:=(k_1,k_2),\quad p:=(p_1,p_2),$$
$$(d^np) := d^np_1d^np_2,\quad d^np_l := \prod\nolimits_{\sigma=1}^n dp_l^\sigma,\quad \mu_{l\epsilon}:=m_l^2-i\epsilon_l,\quad \hat p_l := \gamma^\sigma p_{l\sigma},\quad l=1,2,\qquad(2)$$
$$\gamma^\sigma\gamma = (-1)^{\pi_1}\gamma\gamma^\sigma,\quad \gamma^\sigma\gamma' = (-1)^{\pi_2}\gamma'\gamma^\sigma,\quad (-1)^{\pi_i}=\begin{cases}1, & \text{if } \gamma,\gamma'=I_g,\\ (-1)^{n+1}, & \text{if } \gamma,\gamma'=\widetilde\gamma^5.\end{cases}\qquad(3)$$

The nondegenerate metric $g^{\mu\nu}$ with (p,q)-signature, $p+q=n=2r+\delta_n$, $\delta_n=0,1$, are used for each n-dimensional p_l-integration in equations (1)–(3). From now on V, A, S, and P used in subscripts denote vector, axial vector, scalar, and pseudoscalar respectively.

The matrices γ^α, γ^β, $\widetilde\gamma^5$, I_g, act in the N_g-dimensional space of the faithful representation $\pi(g)$ of the lowest dimension for the Clifford algebra $Cl(g)_{\mathbb{K}}$, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, with $\gamma^\sigma\in\Lambda^1(g)$, $\sigma=1,\ldots,n$, being the generating elements of the $Cl(g)_{\mathbb{K}}$-algebra in its matrix representation $\pi(g)$, i.e. $\gamma^\sigma\gamma^\tau+\gamma^\tau\gamma^\sigma=2g^{\sigma\tau}I_g$, and I_g is the unit matrix of the dimension N_g. The n-degree element $\widetilde\gamma^5\in\Lambda^n(g)$, i.e. the dual conjugation matrix, with the obvious but important properties: $\widetilde\gamma^5:=\gamma^1\gamma^2\cdots\gamma^n$, $\gamma^\sigma\widetilde\gamma^5=(-1)^{n+1}\widetilde\gamma^5\gamma^\sigma$, $\sigma=1,\ldots,n$; $(\widetilde\gamma^5)^2=\varepsilon_gI_g$, $\varepsilon_g:=(-1)^q(-1)^{n(n-1)/2}=(-1)^{\varkappa(\varkappa+1)/2}$, $\varkappa:=(q-p)(\mathrm{mod}\,8)$, is the natural n-dimensional analog of the Dirac γ^5-matrix. For more details on properties of the $\widetilde\gamma^5$-matrix and on the self-consistent version of the dimensional regularization with the $\widetilde\gamma^5$-matrix see [8].

Between amplitudes of equations (1)–(3) there exist the following relations

$$I^{\alpha\beta}_{AA}(m,k)_\epsilon = \varepsilon_gI^{\alpha\beta}_{VV}(m,k)_\epsilon+\varepsilon_g\left((-1)^{n+1}-1\right)m_1m_2\mathrm{tr}[\gamma^\alpha\gamma^\beta]I(m,k)_\epsilon,$$
$$I^{\alpha\beta}_{AV}(m,k)_\epsilon = I^{\alpha\beta}_{VA}(m,k)_\epsilon+\left((-1)^{n+1}-1\right)m_1m_2\mathrm{tr}[\widetilde\gamma^5\gamma^\alpha\gamma^\beta]I(m,k)_\epsilon,\quad\forall\, n=2r+\delta_n,\qquad(4)$$

where $I(m,k)_\epsilon$ is given below by equations (9). So, the distinction between either $I^{\alpha\beta}_{AA}(m,k)_\epsilon$ and $I^{\alpha\beta}_{VV}(m,k)_\epsilon$ or $I^{\alpha\beta}_{AV}(m,k)_\epsilon$ and $I^{\alpha\beta}_{VA}(m,k)_\epsilon$ may exist only for even $n=2r$ and if both m_1, $m_2>0$; if so, this distinction is determined by the same quantity, $(-2m_1m_2)I(m,k)_\epsilon$, which distinguishes their functions standing either at tensor $\mathrm{tr}[\gamma^\alpha\gamma^\beta]$ or $\mathrm{tr}[\widetilde\gamma^5\gamma^\alpha\gamma^\beta]$ respectively.

2.2. Owing to the relations $\delta(-k_1+p_2-p_1)(\hat k_1-\hat p_2+\hat p_1)=0$, $\delta(-k_2+p_1-p_2)(\hat k_2-\hat p_1+\hat p_2)=0$, from equations (1)–(3) imply useful decompositions: $\delta(p,k)\hat k_1\gamma=\delta(p,k)(\hat p_2-\hat p_1)\gamma=\delta(p,k)[(-1)^{\pi_1}\gamma(m_1-\hat p_1)-(m_2-\hat p_2)\gamma+(m_2-(-1)^{\pi_1}m_1)\gamma]$, $\delta(p,k)\hat k_2\gamma'=\delta(p,k)(\hat p_1-\hat p_2)\gamma'=\delta(p,k)[(-1)^{\pi_2}\gamma'(m_2-\hat p_2)-(m_1-\hat p_1)\gamma'+(m_1-(-1)^{\pi_2}m_2)\gamma']$, which formally produce the vector and axial vector *canonical Ward identities* (CWIs),

$$k_{1\alpha}I^{\alpha\beta}_{JJ'}(m,k)_\epsilon = D^{\cdot\beta}_{JJ'}(m,k)_\epsilon = P^{\cdot\beta}_{JJ'}(m,k)_\epsilon+M^{\cdot\beta}_{JJ'}(m,k)_\epsilon,$$
$$k_{2\beta}I^{\alpha\beta}_{JJ'}(m,k)_\epsilon = D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon+M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon,\qquad(5)$$

for *main* mass-anisotropic UV-divergent FAs presented by equations (1)–(3); in so doing the vector and axial vector CWIs are given here in the uniform manner. Associated quantities, $D^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $P^{\cdot\beta}_{JJ'}(m,k)_\epsilon:=(-1)^{\pi_1}P^{\cdot\beta}_{JJ';1}(m,k)_\epsilon-P^{\cdot\beta}_{JJ';2}(m,k)_\epsilon$, $P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon:=(-1)^{\pi_2}P^{\alpha\cdot}_{JJ';2}(m,k)_\epsilon-P^{\alpha\cdot}_{JJ';1}(m,k)_\epsilon$, $M^{\cdot\beta}_{JJ'}(m,k)_\epsilon:=(m_2-(-1)^{\pi_1}m_1)I^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon:=(m_1-(-1)^{\pi_2}m_2)\ I^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $\dot J,\dot J'\in\{S,P\}$, entering into the CWIs (5) are similar to the main amplitudes $I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$, and differ from the latest ones only in polynomials of integrands, see Ref. [9], equations (3)–(6).

As is already known from equations (4), the even n are especially important for us. For this reason we shall consider only the case $n = 2r$ in the following. In addition, some simplifications occur since tensors of the type $\mathrm{tr}[\gamma^\alpha\gamma\gamma^\beta\gamma'\gamma^\sigma]$, $\mathrm{tr}[\gamma^\beta\gamma'\gamma]$ are equal zero. So, all amplitudes entering into CWIs (5) can be written in the form ($n = 2r$):

$$
\begin{aligned}
&I^{\alpha\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma_\sigma\gamma^\beta\gamma'\gamma_\tau]I^{\sigma\tau}_{12}(m,k)_\epsilon + \mathrm{tr}[\gamma^\alpha\gamma\gamma^\beta\gamma']\, m_1m_2I(m,k)_\epsilon,\\
&D^{\cdot\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma\gamma_\sigma][I^\sigma_{2;1}(m,k)_\epsilon - I^\sigma_{1;2}(k)_\epsilon + m_1m_2(-1)^{\pi_1}(I^\sigma_2(m,k)_\epsilon - I^\sigma_1(k)_\epsilon)],\\
&D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\gamma_\sigma][I^\sigma_{1;2}(m,k)_\epsilon - I^\sigma_{2;1}(k)_\epsilon + m_1m_2(-1)^{\pi_2}(I^\sigma_1(m,k)_\epsilon - I^\sigma_2(k)_\epsilon)], \qquad (6)\\
&P^{\cdot\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma\gamma_\sigma][P^\sigma_{1;2}(m,k)_\epsilon - P^\sigma_{2;1}(m,k)_\epsilon],\\
&P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\gamma_\sigma][P^\sigma_{2;1}(m,k)_\epsilon - P^\sigma_{1;2}(m,k)_\epsilon], \qquad (7)\\
&M^{\cdot\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma\gamma_\sigma][m_{JJ';1}I^\sigma_2(m,k)_\epsilon - m_{JJ';2}I^\sigma_1(m,k)_\epsilon],\\
&M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\gamma_\sigma][m_{JJ';2}I^\sigma_1(m,k)_\epsilon - m_{JJ';1}I^\sigma_2(m,k)_\epsilon],\\
&\text{where}\quad m_{JJ';l} := (m_1m_2(-1)^{\pi_1} - m_l^2), \quad m_{JJ';l} := (m_1m_2(-1)^{\pi_2} - m_l^2), \quad l = 1,2, \qquad (8)
\end{aligned}
$$

in which we make use of the following quantities:

$$
\begin{aligned}
&I(m,k)_\epsilon := \int_{-\infty}^{\infty}\frac{(d^np)\,\delta(p,k)}{(\mu_{1\epsilon}-p_1^2)(\mu_{2\epsilon}-p_2^2)}, \quad I^\sigma_l(m,k)_\epsilon := \int_{-\infty}^{\infty}\frac{(d^np)\,\delta(p,k)p^\sigma_l}{(\mu_{1\epsilon}-p_1^2)(\mu_{2\epsilon}-p_2^2)}, \quad l = 1,2,\\
&I^{\sigma\tau}_{ll'}(m,k)_\epsilon := \int_{-\infty}^{\infty}\frac{(d^np)\,\delta(p,k)p^\sigma_lp^\tau_{l'}}{(\mu_{1\epsilon}-p_1^2)(\mu_{2\epsilon}-p_2^2)}, \quad l,l' \in \{1,2\},\\
&I^\sigma_{l;l'}(m,k)_\epsilon := \int_{-\infty}^{\infty}\frac{(d^np)\,\delta(p,k)p_l^2p^\sigma_{l'}}{(\mu_{1\epsilon}-p_1^2)(\mu_{2\epsilon}-p_2^2)}, \quad l,l' \in \{1,2\}, \quad l' \neq l, \qquad (9)\\
&P^\sigma_{l;l'}(m,k)_\epsilon := \int_{-\infty}^{\infty}\frac{(d^np)\,\delta(p,k)(m_l^2-p_l^2)p^\sigma_{l'}}{(\mu_{1\epsilon}-p_1^2)(\mu_{2\epsilon}-p_2^2)} = m_l^2I^\sigma_{l'}(m,k)_\epsilon - I^\sigma_{l;l'}(m,k)_\epsilon, \quad l' \neq l,\\
&I^\sigma_1(m,k)_\epsilon - I^\sigma_2(m,k)_\epsilon = k_2^\sigma I(m,k)_\epsilon. \qquad (10)
\end{aligned}
$$

It should be particularly emphasized that $I^\sigma_l(m,k)_\epsilon$ and $I^\sigma_{l;l'}(m,k)_\epsilon$ in its turn are constituents of $P^\sigma_{l;l'}(m,k)_\epsilon$. Next, if $m_1, m_2 > 0$, then it follows from equation (10) that $\lim\limits_{\epsilon_l\to 0}P^\sigma_{l;l'}(m,k)_\epsilon = \int_{-\infty}^{\infty}(d^np)\,\delta(p,k)\,p^\sigma_{l'}/(\mu_{l'\epsilon}-p^2_{l'}) = 0$, as a consequence of the "symmetric integration" (skew-symmetric functions are integrated in symmetric limits).

2.3. The closely similar to $P^\sigma_{l;l'}(m,k)_\epsilon$, but nevertheless different expressions,

$$
\begin{aligned}
P^\sigma_{l\epsilon;l'}(m,k)_\epsilon &:= \int_{-\infty}^{\infty}\frac{(d^np)\,\delta(p,k)}{(\mu_{1\epsilon}-p_1^2)(\mu_{2\epsilon}-p_2^2)}\,(\mu_{l\epsilon}-p_l^2)p^\sigma_{l'} \qquad (11)\\
&= \int_{-\infty}^{\infty}(d^np)\,\delta(p,k)\,p^\sigma_{l'}/(\mu_{l'\epsilon}-p^2_{l'}) = 0, \quad l,l' \in \{1,2\}, \quad l' \neq l; \quad m_1,m_2 \geq 0,
\end{aligned}
$$

are zero due to the "symmetric integration" as well. Equations (11) represent the simplest examples of the *reduction identities*. From equations (9)–(11) it obvious but the very useful relations follows:

$$
P^\sigma_{l;l'}(m,k)_\epsilon = i\epsilon_lI^\sigma_{l'}(m,k)_\epsilon, \quad I^\sigma_{l;l'}(m,k)_\epsilon = \mu_{l\epsilon}I^\sigma_{l'}(m,k)_\epsilon, \quad l,l' \in \{1,2\},\ l' \neq l. \qquad (12)
$$

So, owing to equations (11)–(12) the quantities $P^{\cdot\beta}_{JJ'}(m,k) := \lim\limits_{\epsilon_1,\epsilon_2\to 0}P^{\cdot\beta}_{JJ'}(m,k)_\epsilon$ and $P^{\alpha\cdot}_{JJ'}(m,k) := \lim\limits_{\epsilon_1,\epsilon_2\to 0}P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, closely related to the Schwinger terms contributions (STCs) of current commutators to the CWIs (5), are expressed for all $I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$ in terms of the important quantity

$$
\lim_{\epsilon_1,\epsilon_2\to 0}[P^\sigma_{2;1}(m,k)_\epsilon - P^\sigma_{1;2}(m,k)_\epsilon] = \begin{cases} \lim\limits_{\epsilon_1,\epsilon_2\to 0}[i\epsilon_2I^\sigma_1(m,k)_\epsilon - i\epsilon_1I^\sigma_2(m,k)_\epsilon]\big|_{\substack{m_1=0\\ m_2=0}}, \\ 0, \quad \text{if } m_1,m_2 > 0. \end{cases} \qquad (13)
$$

Consequently, the STCs may be nonzero only for the chiral case, $m_1 = m_2 = 0$.

2.4. Consider now the mass-isotropic case, $m_1 = m_2 = m \geq 0$, $\epsilon_1 = \epsilon_2 = \epsilon \geq 0$, for $n = 2r$. Then, due to equations (6)–(10), (13) and (3) the general CWIs (5) take simpler forms:

$$
\begin{aligned}
&k_{1\alpha} I^{\alpha\beta}_{VJ'}(m,k) = 0, \quad k_{1\alpha} I^{\alpha\beta}_{AJ'}(m,k) = \mathrm{tr}[\gamma^\beta \gamma' \widetilde{\gamma}^5 (-\hat{k}_2)](-2m^2) I(m,k), \quad J' \in \{V,A\},\\
&k_{2\beta} I^{\alpha\beta}_{JV}(m,k) = 0, \quad k_{2\beta} I^{\alpha\beta}_{JA}(m,k) = \mathrm{tr}[\gamma^\alpha \gamma \widetilde{\gamma}^5 \hat{k}_2](-2m^2) I(m,k), \quad J \in \{V,A\}, \quad m > 0,\\
&k_{1\alpha} I^{\alpha\beta}_{JJ'}(m,k)|_{m=0} = \mathrm{tr}[\gamma^\beta \gamma' \gamma (-\hat{k}_2)] \lim_{\epsilon \to 0} [i\epsilon I(m,k)_\epsilon|_{m=0}],\\
&k_{2\beta} I^{\alpha\beta}_{JJ'}(m,k)|_{m=0} = \mathrm{tr}[\gamma^\alpha \gamma \gamma' \hat{k}_2] \lim_{\epsilon \to 0} [i\epsilon I(m,k)_\epsilon|_{m=0}], \quad J, J' \in \{V,A\},
\end{aligned}
\tag{14}
$$

where $I^{\alpha\beta}_{JJ'}(m,k) := \lim\limits_{\epsilon \to 0} I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$, $I(m,k) := \lim\limits_{\epsilon \to 0} I(m,k)_\epsilon$, and we make use of the relation $I^\sigma_1(m,k)_\epsilon - I^\sigma_2(m,k)_\epsilon = k^\sigma_2 I(m,k)_\epsilon$. It should be noted that the forms of mass-isotropic vector and axial vector CWIs (14) are completely determined by equations (8) and (13).

3 Regular values and identities of VV, AA, etc. amplitudes

3.1. The quantities in equations (9) are associated with the s-degree homogeneous p-polynomials in numerators of integrands. The proper divergence indices of them are equal to $\nu(s) := 2\omega + s = n - 4 + s$, $s = 0,1,2,3$, where $\omega := n/2 - 2$. Therefore, the main amplitudes, $I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$, have the maximal divergence index, $\nu := \nu(2) = n - 2$, and the minimal one, $\nu - 2 \equiv \nu(0) = n - 4$, respectively. Associated amplitudes, $D^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $P^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, have corresponding divergence indices $\nu + 1 \equiv \nu(3) = n - 1$ and $\nu - 1 \equiv \nu(1) = n - 3$ respectively, but $M^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$ have merely the proper divergence index $\nu - 1 \equiv \nu(1) = n - 3$.

It is important to stress that $I(m,k)_\epsilon$ is the constituent of $I^{\alpha\beta}_{JJ'}(m,k)_\epsilon$ for all $J, J' \in \{V,A\}$. Similarly, $I^\sigma_1(m,k)_\epsilon$ and $I^\sigma_2(m,k)_\epsilon$ are constituents of $D^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $D^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $P^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $P^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $M^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $M^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$ for all $J, J' \in \{V,A\}$ as well. We have here examples of the general state of affairs, when precisely the same function can be involved in different expressions and relations simultaneously: in our case, in the vector and axial vector CWIs.

3.2. This reasonably raises the following question. Can we construct such regular (finite) values of quantities (1)–(3) which satisfy the relations (4) and the vector and axial vector CWIs (5) simultaneously as well? The answer is yes, we can. Such an universal, high-efficient, and invariant renormalization procedure which is applicable on equal grounds both to renormalizable and nonrenormalizable theories has been constructed by the author [10–13] and was named as the self-consistent renormalization (SCR). The SCR is based on ideas of the Bogoliubov–Parasiuk R-operation [14–16] and is supplemented with recurrence, compatibility, and differential relations fixing a renormalization arbitrariness of the R-operation in some universal way based on mathematical properties of Feynman amplitudes only.

3.3. Regular values $(R^\nu_0 I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 D)^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 D)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 P)^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 P)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 M)^{\cdot\beta}_{JJ'}(m,k)_\epsilon$, $(R^{\nu+1}_0 M)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon$, obtained in the framework of the SCR retain completely the structure of equations (6)–(8) in which the quantities $I(m,k)_\epsilon$, $I^{\sigma\tau}_{ll'}(m,k)_\epsilon$, $I^\sigma_l(m,k)_\epsilon$, $I^\sigma_{l;l'}(m,k)_\epsilon$, $P^\sigma_{l;l'}(m,k)_\epsilon$ are replaced respectively by their regular values that are given by means of the following α-parametric integral representations:

$$
\begin{bmatrix}
(R^\nu_0 I)(m,k)_\epsilon \\
(R^\nu_0 I)^{\sigma\tau}_{ll'}(m,k)_\epsilon \\
(R^{\nu+1}_0 I)^\sigma_l(m,k)_\epsilon \\
(R^{\nu+1}_0 I)^\sigma_{l;l'}(m,k)_\epsilon \\
(R^{\nu+1}_0 P)^\sigma_{l;l'}(m,k)_\epsilon
\end{bmatrix}
= C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}}
\begin{bmatrix}
(R^\nu_0 \mathcal{I})(\alpha,m,k)_\epsilon \\
(R^\nu_0 \mathcal{I})^{\sigma\tau}_{ll'}(\alpha,m,k)_\epsilon \\
(R^{\nu+1}_0 \mathcal{I})^\sigma_l(\alpha,m,k)_\epsilon \\
(R^{\nu+1}_0 \mathcal{I})^\sigma_{l;l'}(\alpha,m,k)_\epsilon \\
(R^{\nu+1}_0 \mathcal{P})^\sigma_{l;l'}(\alpha,m,k)_\epsilon
\end{bmatrix},
\quad C_g := (2\pi)^n \delta(k) b(g),
\tag{15}
$$

where the integrands,

$$(R_0^\nu\mathcal{I})(\alpha,m,k)_\epsilon := (R_0^\nu\mathcal{F})_{00},$$
$$(R_0^\nu\mathcal{I})^{\sigma\tau}_{ll'}(\alpha,m,k)_\epsilon := Y_l^\sigma Y_{l'}^\tau (R_0^\nu\mathcal{F})_{20} + (-2)^{-1} X_{ll'} g^{\sigma\tau} (R_0^\nu\mathcal{F})_{21},$$
$$(R_0^{\nu+1}\mathcal{I})^\sigma_l(\alpha,m,k)_\epsilon := Y_l^\sigma (R_0^{\nu+1}\mathcal{F})_{10},$$
$$(R_0^{\nu+1}\mathcal{I})^\sigma_{l;l'}(\alpha,m,k)_\epsilon := Y_l^2 Y_{l'}^\sigma (R_0^{\nu+1}\mathcal{F})_{30} + (-2)^{-1}\{nX_{ll}Y_{l'}^\sigma + 2X_{ll'}Y_l^\sigma\}(R_0^{\nu+1}\mathcal{F})_{31}, \quad (16)$$
$$(R_0^{\nu+1}\mathcal{P})^\sigma_{l;l'}(\alpha,m,k)_\epsilon := m_l^2 (R_0^{\nu+1}\mathcal{I})^\sigma_{l';l'}(\alpha,m,k)_\epsilon - (R_0^{\nu+1}\mathcal{I})^\sigma_{l;l'}(\alpha,m,k)_\epsilon, \quad (17)$$

are defined in terms of some basic functions, $(R_0^\nu\mathcal{F})_{00}$, $(R_0^\nu\mathcal{F})_{20}$, $(R_0^\nu\mathcal{F})_{21}$, and $(R_0^{\nu+1}\mathcal{F})_{10}$, $(R_0^{\nu+1}\mathcal{F})_{30}$, $(R_0^{\nu+1}\mathcal{F})_{31}$, associated with homogeneous polynomials of the degree $s-2j$, $s = 0,1,2,3$, $j = 0,[s/2]$, in external momenta k_1, k_2. Each monomial of the latter is a product of $s-2j$ linear Kirchhoff forms Y_l, $l=1,2$, and j line-correlator functions $X_{ll'}$, $l,l' \in \{1,2\}$.

The explicit form of the basic functions $(R_0^\nu\mathcal{F})_{sj}$, $(R_0^{\nu+1}\mathcal{F})_{sj}$, and the determining numbers ν_{sj}, λ_{sj}, ν^1_{sj}, λ^1_{sj}, and ω appearing in them are as follows:

$$(R_0^\nu\mathcal{F})_{sj} := M_\epsilon^{\omega+j}\,\Gamma(\lambda_{sj})/\Gamma(2+\nu_{sj}) Z_\epsilon^{1+\nu_{sj}}\,{}_2F_1(1,\lambda_{sj};\,2+\nu_{sj};\,Z_\epsilon),$$
$$\nu_{sj} := [(\nu-s)/2]+j, \quad \lambda_{sj} := -\omega-j+1+\nu_{sj}, \quad \omega := n/2-2, \quad (18)$$
$$(R_0^{\nu+1}\mathcal{F})_{sj} := M_\epsilon^{\omega+j}\,\Gamma(\lambda^1_{sj})/\Gamma(2+\nu^1_{sj}) Z_\epsilon^{1+\nu^1_{sj}}\,{}_2F_1(1,\lambda^1_{sj};\,2+\nu^1_{sj};\,Z_\epsilon),$$
$$\nu^1_{sj} := [(\nu+1-s)/2]+j, \quad \lambda^1_{sj} := -\omega-j+1+\nu^1_{sj}, \quad \omega := n/2-2. \quad (19)$$

The $[(\nu-s)/2]$ and $[(\nu+1-s)/2]$ in equations (18)–(19) are integral parts of the numbers $(\nu-s)/2$ and $(\nu+1-s)/2$ respectively. There exist the following compatibility and recurrence relations:

$$(R_0^\nu\mathcal{F})_{sj} = \mathcal{F}_{sj} := M_\epsilon^{\omega+j}(1-Z_\epsilon)^{\omega+j}\,\Gamma(-\omega-j), \quad \text{if } \nu_{sj} \le -1, \quad (20)$$
$$(R_0^\nu\mathcal{F})_{sj} = (R_0^{\nu+1}\mathcal{F})_{s+1,j}, \quad (21)$$
$$M_\epsilon\,(R_0^\nu\mathcal{F})_{00} - A\,(R_0^\nu\mathcal{F})_{20} + (\omega+1)\,(R_0^\nu\mathcal{F})_{21} = 0,$$
$$M_\epsilon\,(R_0^{\nu+1}\mathcal{F})_{10} - A\,(R_0^{\nu+1}\mathcal{F})_{30} + (\omega+1)\,(R_0^{\nu+1}\mathcal{F})_{31} = 0, \quad (22)$$

between the basic functions $(R_0^\nu\mathcal{F})_{sj}$, $(R_0^{\nu+1}\mathcal{F})_{sj}$. In fact, due to compatibility relations (21) both recurrence relations (22) are different forms of the common one.

The α-parametric functions $Z_\epsilon \equiv Z(\alpha,m,k)_\epsilon$, $M_\epsilon \equiv M(\alpha,m)_\epsilon$, $A \equiv A(\alpha,k)$, $\Delta \equiv \Delta(\alpha)$, $Y_l \equiv Y_l(\alpha,k)$, and $X_{ll'} \equiv X_{ll'}(\alpha)$ entering into equations (16)–(19) have the form:

$$Z_\epsilon := A/M_\epsilon, \quad M_\epsilon := \alpha_1\mu_{1\epsilon} + \alpha_2\mu_{2\epsilon}, \quad A := \Delta\beta_1\beta_2\,k_2^2 = \alpha_1 Y_1^2 + \alpha_2 Y_2^2, \quad \Delta := \alpha_1+\alpha_2,$$
$$Y_1 := \beta_2\,k_2, \quad Y_2 := -\beta_1\,k_2, \quad \beta_l := \alpha_l/\Delta, \quad X_{ll'} := \Delta^{-1}, \; l,l' \in \{1,2\}, \quad Y_2 - Y_1 = -k_2 = k_1,$$
$$Y_1 \cdot Y_2 = -A/\Delta, \quad Y_l^2 = -A/\Delta + \beta_{l'}k_2^2, \quad \alpha_l Y_l^2 = \beta_{l'}A, \quad l' \ne l, \quad \beta_1+\beta_2 = 1. \quad (23)$$

The integration measure $d\mu(\alpha)$, the integration region Σ^1, the metric dependent constant $b(g)$, and the overall δ-function $\delta(k)$ are defined as

$$d\mu(\alpha) := \delta(1-\alpha_1-\alpha_2)d\alpha_1\,d\alpha_2, \quad \Sigma^1 := \{\alpha_l |\, \alpha_l \ge 0,\ \forall\, l,\ \alpha_1+\alpha_2 = 1\},$$
$$b(g) := (\pi^{n/2}i^p)/(2\pi)^n, \quad \delta(k) := \delta(-k_1-k_2), \quad p \text{ – number of positive squares in } g. \quad (24)$$

3.4. From equations (6)–(8), (15)–(17) and (23) more specific formulae follows. So, the regular values of the main FAs (1), reduction identities (11), and their consequences take the form:

$$(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \big\{ \mathrm{tr}[\gamma^\alpha\gamma\gamma^\beta\gamma']\, m_1 m_2 (R_0^\nu\mathcal{F})_{00}$$
$$+ \mathrm{tr}[\gamma^\alpha\gamma\hat{k}_2\gamma^\beta\gamma'\hat{k}_2](-\beta_1\beta_2)(R_0^\nu\mathcal{F})_{20} + \mathrm{tr}[\gamma^\alpha\gamma\gamma^\sigma\gamma^\beta\gamma'\gamma_\sigma](-1/2)\Delta^{-1}(R_0^\nu\mathcal{F})_{21}\big\}, \quad (25)$$

$$(R_0^{\nu+1}P)^{\sigma}_{l\epsilon;l'}(m,k)_\epsilon = (-1)^l k_2^\sigma C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \big\{ \mu_{l\epsilon}\beta_l (R_0^{\nu+1}\mathcal{F})_{10}$$
$$- \beta_l Y_l^2 (R_0^{\nu+1}\mathcal{F})_{30} + [(n/2)\beta_l - \beta_{l'}]\Delta^{-1}(R_0^{\nu+1}\mathcal{F})_{31}\big\} = 0, \quad l,l' \in \{1,2\},\ l' \neq l, \tag{26}$$

$$(R_0^{\nu+1}P_{2\epsilon-1\epsilon})^{\sigma}(m,k)_\epsilon := (R_0^{\nu+1}P)^{\sigma}_{2\epsilon;1}(m,k)_\epsilon - (R_0^{\nu+1}P)^{\sigma}_{1\epsilon;2}(m,k)_\epsilon = 0$$
$$= k_2^\sigma C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \frac{1}{\Delta} \big\{ M_\epsilon (R_0^{\nu+1}\mathcal{F})_{10} - A(R_0^{\nu+1}\mathcal{F})_{30} + (\omega+1)(R_0^{\nu+1}\mathcal{F})_{31}\big\} = 0, \tag{27}$$

$$(R_0^{\nu+1}I)^{\sigma}_1(m,k)_\epsilon - (R_0^{\nu+1}I)^{\sigma}_2(m,k)_\epsilon = k_2^\sigma C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} (R_0^{\nu+1}\mathcal{F})_{10} = k_2^\sigma (R_0^\nu I)(m,k)_\epsilon. \tag{28}$$

3.5. After isolating tensor structures in regular values of all amplitudes involving in CWIs (5), for example, $k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma(-\hat{k}_2)](R_0^\nu I)_{JJ';1}(m,k)_\epsilon$, $k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\hat{k}_2](R_0^\nu I)_{JJ';2}(m,k)_\epsilon$, $(R_0^{\nu+1}D)^{\cdot\beta}_{JJ'}(m,k)_\epsilon = \mathrm{tr}[\gamma^\beta\gamma'\gamma(-\hat{k}_2)](R_0^{\nu+1}D)_{JJ';1}(m,k)_\epsilon$, $(R_0^{\nu+1}D)^{\alpha\cdot}_{JJ'} = \mathrm{tr}[\gamma^\alpha\gamma\gamma'\hat{k}_2](R_0^{\nu+1}D)_{JJ';2}(m,k)_\epsilon$, etc., the scalar factors of them can be represented as follows:

$$\begin{bmatrix} (R_0^\nu I)_{JJ';i}(m,k)_\epsilon \\ (R_0^{\nu+1}D)_{JJ';i}(m,k)_\epsilon \\ (R_0^{\nu+1}P)_{JJ';i}(m,k)_\epsilon \\ (R_0^{\nu+1}M)_{JJ';i}(m,k)_\epsilon \end{bmatrix} = C_g \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^{n/2}} \begin{bmatrix} (R_0^\nu \mathcal{I})_{JJ';i}(\alpha,m,k)_\epsilon \\ (R_0^{\nu+1}\mathcal{D})_{JJ';i}(\alpha,m,k)_\epsilon \\ (R_0^{\nu+1}\mathcal{P})_{JJ';i}(\alpha,m,k)_\epsilon \\ (R_0^{\nu+1}\mathcal{M})_{JJ';i}(\alpha,m,k)_\epsilon \end{bmatrix}, \quad i = 1,2, \tag{29}$$

$$(R_0^\nu \mathcal{I})_{JJ';i}(\alpha,m,k)_\epsilon := m_1 m_2 (-1)^{\pi_i} (R_0^\nu \mathcal{F})_{00} - A/\Delta (R_0^\nu \mathcal{F})_{20} + (\omega+1)/\Delta (R_0^\nu \mathcal{F})_{21}$$
$$\cong \big(m_1 m_2 (-1)^{\pi_i} - M_\epsilon/\Delta\big)(R_0^\nu \mathcal{F})_{00},$$
$$(R_0^{\nu+1}\mathcal{D})_{JJ';i}(\alpha,m,k)_\epsilon := m_1 m_2 (-1)^{\pi_i} (R_0^{\nu+1}\mathcal{F})_{10} - A/\Delta (R_0^{\nu+1}\mathcal{F})_{30} + (\omega+1)/\Delta (R_0^{\nu+1}\mathcal{F})_{31}$$
$$\cong \big(m_1 m_2 (-1)^{\pi_i} - M_\epsilon/\Delta\big)(R_0^{\nu+1}\mathcal{F})_{10},$$
$$(R_0^{\nu+1}\mathcal{P})_{JJ';i}(\alpha,m,k)_\epsilon := \Delta^{-1}\big\{M(R_0^{\nu+1}\mathcal{F})_{10} - A(R_0^{\nu+1}\mathcal{F})_{30} + (\omega+1)(R_0^{\nu+1}\mathcal{F})_{31}\big\}$$
$$\cong iE/\Delta (R_0^{\nu+1}\mathcal{F})_{10}, \quad M := \alpha_1 m_1^2 + \alpha_2 m_2^2, \quad E := \alpha_1\epsilon_1 + \alpha_2\epsilon_2,$$
$$(R_0^{\nu+1}\mathcal{M})_{JJ';i}(\alpha,m,k)_\epsilon := \big(m_1 m_2 (-1)^{\pi_i} - M/\Delta\big)(R_0^{\nu+1}\mathcal{F})_{10}, \quad i = 1,2. \tag{30}$$

The second lines of equations (30) are due to equation (27). The congruence relation $\mathcal{A}(\alpha,m,k) \cong \mathcal{B}(\alpha,m,k)$ denotes the equality $\int_{\Sigma^1} d\mu(\alpha)\Delta^{-n/2}\mathcal{A}(\alpha,m,k) = \int_{\Sigma^1} d\mu(\alpha)\Delta^{-n/2}\mathcal{B}(\alpha,m,k)$.

3.6. Taking into account compatibility relations (21), the identities are verified:

$$(R_0^\nu I)_{JJ';i}(m,k)_\epsilon = (R_0^{\nu+1}D)_{JJ';i}(m,k)_\epsilon = (R_0^{\nu+1}P)_{JJ';i}(m,k)_\epsilon + (R_0^{\nu+1}M)_{JJ';i}(m,k)_\epsilon, \tag{31}$$

$i = 1,2$, which produce the general mass-anisotropic vector and axial vector CWIs,

$$k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = (R_0^{\nu+1}D)^{\cdot\beta}_{JJ'}(m,k)_\epsilon = (R_0^{\nu+1}P)^{\cdot\beta}_{JJ'}(m,k)_\epsilon + (R_0^{\nu+1}M)^{\cdot\beta}_{JJ'}(m,k)_\epsilon,$$
$$k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)_\epsilon = (R_0^{\nu+1}D)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon = (R_0^{\nu+1}P)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon + (R_0^{\nu+1}M)^{\alpha\cdot}_{JJ'}(m,k)_\epsilon, \tag{32}$$

but now for regular values; it is seen that equations (32) retain the form of equations (5).

In the mass-isotropic case, $m_1 = m_2 = m \geq 0$, $\epsilon_1 = \epsilon_2 = \epsilon \geq 0$, from equations (28)–(32) imply the following mass-isotropic CWIs for ϵ-limiting values of amplitudes:

$$k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{VJ'}(m,k) = 0, \quad k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{AJ'}(m,k) = \mathrm{tr}[\gamma^\beta\gamma'\widetilde{\gamma}^5(-\hat{k}_2)](-2m^2)(R_0^\nu I)(m,k),$$
$$k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JV}(m,k) = 0, \quad k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JA}(m,k) = \mathrm{tr}[\gamma^\alpha\gamma\widetilde{\gamma}^5\hat{k}_2](-2m^2)(R_0^\nu I)(m,k),\ m > 0,$$
$$k_{1\alpha}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)|_{m=0} = \mathrm{tr}\,[\gamma^\beta\gamma'\gamma(-\hat{k}_2)] \lim_{\epsilon\to 0}[i\epsilon\,(R_0^\nu I)(m,k)_\epsilon|_{m=0}],$$
$$k_{2\beta}(R_0^\nu I)^{\alpha\beta}_{JJ'}(m,k)|_{m=0} = \mathrm{tr}\,[\gamma^\alpha\gamma\gamma'\hat{k}_2] \lim_{\epsilon\to 0}[i\epsilon\,(R_0^\nu I)(m,k)_\epsilon|_{m=0}], \quad J,J' \in \{V,A\}, \tag{33}$$

which retain the form of equations (14). Recall that the proper divergence index of $I(m,k)_\epsilon$ is $\nu - 2$.

4 Logical trap as a cause of quantum anomalies

4.1. Up till now no explicit forms of the basic functions $(R_0^\nu\mathcal{F})_{sj}$ and $(R_0^{\nu+1}\mathcal{F})_{sj}$ have been used. In fact, we have exploited only the following items: i) $(R_0^\nu\mathcal{F})_{sj}$ and $(R_0^{\nu+1}\mathcal{F})_{sj}$ are regular functions in the vicinity of zero values of external momenta k_2 that is denoted by the subscript 0 on R; ii) *each* $(R_0^\nu\mathcal{F})_{sj}$ *or* $(R_0^{\nu+1}\mathcal{F})_{sj}$, which are defined in some reasonable way according to indices ν or $\nu+1$ that is denoted by superscripts ν or $\nu+1$ on R respectively, take *the same value in all relations*; iii) the compatibility relation (21); iv) the reduction identity (27) which produces the more convenient second form of quantities in equations (30) and predicts the recurrence relation (22) as well. It should be particularly emphasized as to pure mathematical nature of equation (27) associated with some symmetry property of integrals under consideration.

In the mass-isotropic case, $m_1 = m_2 = m > 0$, the mass term functions $(R_0^{\nu+1}M)_{JJ';i}(m,k)$ are zero for vector CWIs and are expressed in terms of the quantity $(-2m^2)(R_0^\nu I)(m,k)$ for axial vector CWIs. In so doing, the Schwinger term contribution (STC) functions $(R_0^{\nu+1}P)_{JJ';i}(m,k)$, closely associated with the reduction identity (27), are zero for both vector and axial vector CWIs. Nevertheless, in the commonly used renormalization techniques (e.g. in the minimal renormalization schemes) the mass term functions for axial vector CWIs are expressed as rule in terms of the quantity $(-2m^2)(R_0^{\nu-2}I)(m,k)$, calculated according to its proper divergence index $\nu-2$, and in addition it is required the zero value for STC functions that provides the conserved vector CWIs. In so doing, we fall into a *logical trap* since *two different regular values*, $(R_0^{\nu-2}I)(m,k)$ and $(R_0^\nu I)(m,k)$, are prescribed to *the same UV-divergent quantity* $I(m,k)$ involving into two different relations that is prohibited. As a result, we come to the well-known quantum anomalies. It remains to show that *just the function* $(R_0^\nu\mathcal{F})_{00}$ *rather than* $(R_0^{\nu-2}\mathcal{F})_{00} = (R_0^\nu\mathcal{F})_{20}$ *to secure realizability of the identity* $M_\epsilon(R_0^\nu\mathcal{F})_{00} - A(R_0^\nu\mathcal{F})_{20} + (\omega+1)(R_0^\nu\mathcal{F})_{21} = 0$.

4.2. If $n = 2r+\delta_n$, $\delta_n = 0,1$, then $\omega = r-2+\delta_n/2$, and from (18) it follows: $\nu_{00} = r-1$, $\lambda_{00} = 2-\delta_n/2$; $\nu_{10} = r-2+\delta_n$, $\lambda_{10} = 1+\delta_n/2$; $\nu_{20} = r-2$, $\lambda_{20} = 1-\delta_n/2$; $\nu_{21} = r-1$, $\lambda_{21} = 1-\delta_n/2$. As a result, the explicit form of basic functions $(R_0^\nu\mathcal{F})_{sj}$ take the form:

$$
\begin{aligned}
(R_0^\nu\mathcal{F})_{00} &= M_\epsilon^\omega\,\Gamma(2-\delta_n/2)/\Gamma(r+1)Z_\epsilon^{\,r}\,{}_2F_1(1,2-\delta_n/2;\,r+1;\,Z_\epsilon),\\
(R_0^\nu\mathcal{F})_{20} &= M_\epsilon^\omega\,\Gamma(1-\delta_n/2)/\Gamma(r)Z_\epsilon^{\,r-1}\,{}_2F_1(1,1-\delta_n/2;\,r;\,Z_\epsilon),\\
(R_0^\nu\mathcal{F})_{21} &= M_\epsilon^{\omega+1}\,\Gamma(1-\delta_n/2)/\Gamma(r+1)Z_\epsilon^{\,r}\,{}_2F_1(1,1-\delta_n/2;\,r+1;\,Z_\epsilon),\\
(R_0^\nu\mathcal{F})_{10} &= \delta_n(R_0^\nu\mathcal{F})_{00} + (1-\delta_n)(R_0^\nu\mathcal{F})_{20}.
\end{aligned}
\tag{34}
$$

Properties of the ${}_2F_1$, see Ref. [17], give rise to the important relations

$$
(R_0^\nu\mathcal{F})_{00} - (R_0^{\nu-2}\mathcal{F})_{00} = (R_0^\nu\mathcal{F})_{00} - (R_0^\nu\mathcal{F})_{20} = -M_\epsilon^\omega\,\Gamma(\lambda_{00}-1)/\Gamma(1+\nu_{00})\,Z_\epsilon^{\nu_{00}}, \tag{35}
$$

$$
(R_0^\nu\mathcal{F})_{00} \overset{M_\epsilon\to 0}{\sim} \frac{(-1)\Gamma(\lambda_{00}-1)\,A^{\,\nu_{00}}}{\Gamma(1+\nu_{00})\,M_\epsilon^{\lambda_{00}-1}}, \quad \text{if } \nu_{00}\geq 0 \ \text{ and } \ \lambda_{00}-1>0. \tag{36}
$$

By using equations (34) and factorizing of $M_\epsilon^{\omega+1}\,Z_\epsilon^{\,r}\Gamma(1-\delta_n/2)/\Gamma(r+1)$ in equation (22) one obtains a particular case of the recurrence relation between contiguous hypergeometric functions ${}_2F_1$:

$$
b\,{}_2F_1(a,b+1;\,c;\,Z_\epsilon) - (c-1)\,{}_2F_1(a,b;\,c-1;\,Z_\epsilon) + (c-b-1)\,{}_2F_1(a,b;\,c;\,Z_\epsilon) = 0, \tag{37}
$$

[see [17], Sec. (2.8)(eq. 42)] for $a=1$, $b=1-\delta_n/2$, $c=r+1$; $c-b-1 = r-1+\delta_n/2 = \omega+1$.

Equations (34)–(37) show clearly that just the function $(R_0^\nu\mathcal{F})_{00}$ rather than $(R_0^{\nu-2}\mathcal{F})_{00} = (R_0^\nu\mathcal{F})_{20}$ satisfy to the recurrence relation (22). In the mass-isotropic case, $m_1 = m_2 = m \geq 0$, from equations (35)–(36) imply the phenomenologically important limiting values:

$$
\begin{aligned}
&\lim_{\epsilon\to 0}[i\epsilon(R_0^\nu I)(m,k)_\epsilon|_{m=0}] = C_g(k_2^2)^{r-1}\Gamma(r)/\Gamma(2r), \quad \lim_{\epsilon\to 0}[i\epsilon(R_0^\nu I)(m,k)_\epsilon|_{m\neq 0}] = 0,\\
&\lim_{m\to 0}[-2m^2(R_0^\nu I)(m,k)|_{m\neq 0}] = C_g(k_2^2)^{r-1}2\Gamma(r)/\Gamma(2r), \quad n = 2r,
\end{aligned}
\tag{38}
$$

of quantities appearing in CWIs (32)–(33) and involving the function $(R_0^\nu I)(m,k)_\epsilon$. In the general mass-anisotropic case, equations (38) must be replaced by

$$(R_0^\nu I)_{JJ';i}(m,k) \overset{m_1,\,m_2\to 0}{\sim} C_g(k_2^2)^{r-1}\Gamma(r)/\Gamma(2r)\big[1-(-1)^{\pi_i}x_2F_1(1,r;\,2r;\,1-x^2)\big], \qquad (39)$$

where $x := m_1/m_2$, see Ref. [9]. In fact, equations (38) are particular cases of (39) since for $a_{2r}(x) := x_2F_1(1,r;\,2r;\,1-x^2)$ there holds $a_{2r}(x) = a_{2r}(1/x)$ and $a_{2r}(0) = a_{2r}(\infty) = 0$, $a_{2r}(1) = 1$.

4.3. So, it is shown that the genuine algorithmic cause of quantum anomalies is of prescribing two different regular values, $(R_0^{\nu-2}I)(m,k)$ and $(R_0^\nu I)(m,k)$, to the same UV-divergent quantity $I(m,k)$ involved into two different Ward identities that is logically prohibited. Quantum anomalies of triangle amplitudes are dictated by the same cause. The SCR [10–13], that makes use of the logically motivated oversubtracted operations, produces such regular values which satisfy the vector and axial vector canonical Ward identities simultaneously.

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D-Branes, Helices, and Proton Decay

Yu.M. MALYUTA and T.V. OBIKHOD

Institute for Nuclear Research of NAS Ukraine, 03022 Kyiv, Ukraine
E-mail: *malyuta@mail.kar.net*

Proton decay is investigated by methods of category theory. The investigation leads to the conclusion that proton decay is forbidden.

1 Introduction

Recently the new description of D-branes was proposed [1–3]. This description is based on methods of category theory [4]. In the present paper we apply these methods to investigation of proton decay.

2 The triangulated category

The triangulated category contains the following data [4]:

1) Distinguished triangles

$$\begin{array}{ccc} & {}^{[1]}C & \\ \swarrow & & \nwarrow \\ A & \longrightarrow & B \end{array} \qquad\qquad C = \mathrm{Cone}(f)$$

(where vertices are complexes of coherent sheaves),

2) Octahedral diagrams

$$\begin{array}{ccccc} F & & \longleftarrow & & E \\ & {}^{[1]}\searrow & \bullet & \nearrow & \\ {\scriptstyle [1]}\downarrow & & B & & \uparrow \\ & \swarrow & \bullet & \nwarrow & \\ C & & \xrightarrow{[1]} & & A \end{array} \qquad \begin{array}{ccccc} F & & \longleftarrow & & E \\ & \nwarrow & & \swarrow & \\ {\scriptstyle [1]}\downarrow & & \bullet\ G\ \bullet & & \uparrow \\ & \nearrow & & {}^{[1]}\searrow & \\ C & & \xrightarrow{[1]} & & A \end{array} \tag{1}$$

(where distinguished triangles are marked by $\bullet$).

These data satisfy Verdier axioms.

3 Helices

Let us consider the special class of distinguished triangles

$$\begin{array}{ccc} & {}^{[1]}\mathrm{Cone}(f) & \\ \swarrow & & \nwarrow \\ V_X^i & \xrightarrow{f} & V_X^j \end{array} \tag{2}$$

where V_X^i and V_X^j are coherent sheaves over the Calabi–Yau manifold X, which are constructed by mutations of helices [5–7].

A collection of coherent sheaves $\{\mathcal{R}_W^i\}$ over the weighted projective space W is called a helix if the following condition is satisfied: The Euler matrix

$$\chi\big(\mathcal{R}_W^i, \mathcal{R}_W^j\big) = \int_W \mathrm{ch}\big(\mathcal{R}_W^{i*} \otimes \mathcal{R}_W^j\big)\mathrm{td}(\mathcal{T}_W)$$

is an upper-triangular matrix with ones on the diagonal.

There exists a mutated helix $\{\mathcal{S}_W^j\}$ over the weighted projective space W if the following orthogonality relation holds

$$\int_W \mathrm{ch}\big(\mathcal{R}_W^i\big)\mathrm{ch}\big(\mathcal{S}_W^j\big)\mathrm{td}(\mathcal{T}_W) = \delta_{ij}.$$

Coherent sheaves V_X^j are obtained by the restriction of $\mathcal{S}_W^j$ to the Calabi–Yau manifold X.

We interpret vertices of distinguished triangles (2) as B-type D-branes if criteria for Π-stability are satisfied [2]. Edges of triangles (2) are interpreted as superstrings.

4 Π-stability

In order to investigate Π-stability of the D-brane Cone(f) against decay into the D-branes V_X^i and V_X^j we need to compute the central charges of V_X^i and V_X^j.

The central charge of V_X^i is determined by [2]

$$Z(V_X^j) = \sum_k Q_k^j \Pi^k = \int_X e^{-B-iJ}\mathrm{ch}\big(V_X^j\big)\sqrt{\mathrm{td}(\mathcal{T}_X)}, \tag{3}$$

where $Q_k^j \in H^3(Y,\mathbb{Z})$ are the RR charges [8] (Y is the mirror of X), Π^k is the Kähler period vector (which describes the Kähler moduli space of X [9]), $B+iJ$ is the complexified Kähler form, $\mathcal{T}_X$ is the tangent sheaf over X.

The grade associated with the central charge (3) is defined by

$$\varphi(V_X^j) = -\frac{1}{\pi}\arg Z\big(V_X^j\big)$$

The D-brane Cone(f) is Π-stable if

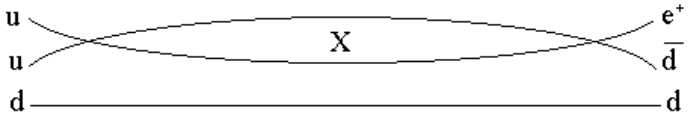
$$\varphi\big(V_X^j\big) - \varphi\big(V_X^i\big) < 0.$$

The application of criteria for Π-stability to distinguished triangles enclosed in the octahedral diagram (1) leads to the following rule of D-brane decays [2]:

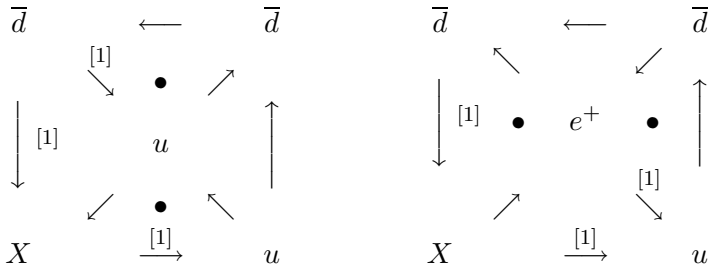
★ If C is stable against decay into A and B, but that B itself is unstable with respect to a decay into E and F, than C will always be unstable with respect to decay into F and some bound state G of A and E.

5 Proton decay

In a grand unified theory [10] proton decay is described by the quark-lepton diagram



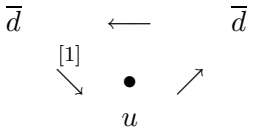
Assuming that quarks, leptons and X-bosons are solitonic excitations in a proton, we can construct the octahedral diagram



(4)

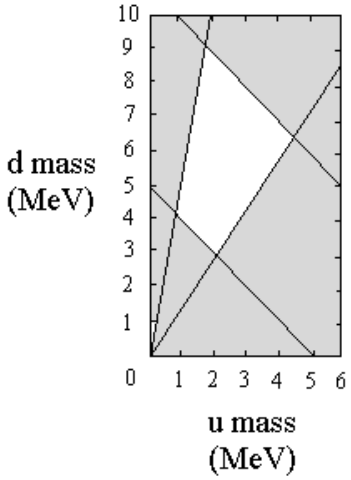
which induces proton decay.

Let us consider the distinguished triangle



(5)

enclosed in the octahedron (4). Taking into account the allowed region (shown in white) for u-quark and d-quark masses [11]



we conclude that in the triangle (5) u is stable with respect to a decay into $\overline{d}$ and $\overline{d}$. This conclusion is incompatible to the rule of decays ★ (where B is unstable with respect to a decay into E and F). Therefore proton decay is forbidden.

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Topological Charges, Prequarks and Presymmetry: a Topological Approach to Quark Fractional Charges

Ernesto A. MATUTE

Departamento de Física, Universidad de Santiago de Chile, Chile
E-mail: *ematute@lauca.usach.cl*

A topological approach to quark fractional charges, based on charge constraints unexplained by the Standard Model of particle physics, is discussed. Charge fractionalization is related to a tunneling process occurring in time between pure gauge field configurations at the far past and future associated with integer-charged bare quarks, named prequarks. This transition conforms to a topologically nontrivial configuration of the weak gauge fields in Euclidean space-time. In this context, an electroweak $\mathcal{Z}_2$ symmetry between bare quarks and leptons, named presymmetry, is revealed. It is shown that an effective topological charge equal to the ratio between baryon number and the number of fermion generations may be associated with baryonic matter. The observed conservation of baryon number is then connected with the conservation of this charge on quarks. Similar results are obtained for leptons in the dual scenario with local quark charges.

1 Introduction

At the level of the Standard Model of elementary particle physics, matter consists of two kinds of particles: quarks and leptons, which are divided into three generations. They interact by means of the strong, weak and electromagnetic forces. These three forces are described by theories of the same general kind: gauge theories with local symmetries. The gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$, which provides with the force particles: gluons, weak bosons and photons. Table 1 gives a classification of the first generation of quarks and leptons according to the representation they furnish of this gauge symmetry group, where the ν_{eR} is introduced because of the experimental signatures for neutrino masses, and the conventional relation

$$Q = T_3 + \frac{1}{2}\, Y \tag{1}$$

between electric charge and hypercharge is chosen.

Table 1. First generation of quarks and leptons.

Fermions	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	3	2	1/3
u_R	3	1	4/3
d_R	3	1	−2/3
$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	1	2	−1
ν_{eR}	1	1	0
e_R	1	1	−2

In the strong interaction sector there are substantial differences between quarks and leptons: quarks come in triplets while leptons do in singlets of the color group $SU(3)_c$. However, in the

electroweak sector their properties are so similar that a deep connection between them seems to exist. Their appearance within each generation is symmetrical: left-handed particles occur in doublets while right-handed ones in singlets of $SU(2)_L$. Their hypercharges are related: their sum within each generation vanishes, i.e.,

$$3\,[Y(u_L)+Y(d_L)+Y(u_R)+Y(d_R)]+Y(\nu_{eL})+Y(e_L)+Y(\nu_{eR})+Y(e_R)=0, \tag{2}$$

where the factor 3 takes into account the number of quark colors. This relation, magically obeyed, is crucial to cancel the triangle gauge anomalies of the Standard Model. Furthermore, it relates the number of colors to the fractional nature of quark hypercharges.

Another striking pattern is the following one-to-one correspondence between the quark and lepton hypercharges:

$$\begin{aligned} Y(u_L)&=Y(\nu_{eL})+\frac{4}{3}, \qquad Y(u_R)=Y(\nu_{eR})+\frac{4}{3},\\ Y(d_L)&=Y(e_L)+\frac{4}{3}, \qquad Y(d_R)=Y(e_R)+\frac{4}{3}, \end{aligned} \tag{3}$$

and similarly for the other two generations. Quark and lepton charges are similarly connected:

$$Q(u)=Q(\nu_e)+\frac{2}{3}, \qquad Q(d)=Q(e)+\frac{2}{3}.$$

Equation (3) is a remarkable and unexpected constraint, more fundamental than (2), which shows that the fractional hypercharge of quarks relies just on a global 4/3 value, independent of flavor and handness. And, if the fractional hypercharge is related to the number of colors as noted above, the factor 1/3 must be there because of quark colors. Besides, it clearly signals for a deeper discrete symmetry between quarks and leptons.

The question is whether there is a way to understand these regularities within the Standard Model. The answer is yes, as I show in this talk, just using the nontrivial topological properties of weak gauge field configurations, as recently proposed [1]. In particular, it is seen that the factor 4 in (3) is a topological Pontryagin index fixed by the requirement of self-consistent gauge anomaly cancellation, so that the number 4/3 is a universal value which in a sense gives structure to the fractional hypercharge of quarks and is conserved by the strong and electroweak interactions.

2 Prequarks and presymmetry

In order to understand the pattern in (3) I pursue a conventional procedure in field theory: start with integer bare local charges for quarks and an exact electroweak symmetry between bare quarks and leptons, and end with fractional charges and a broken or hidden symmetry. I refer to quarks with such bare charges as prequarks and denote them by hats over the symbols that represent the corresponding quarks. The universal hypercharge shift 4/3 may be associated with a bare structure that I denote by X. Thus quarks may conveniently be looked upon as "made" of the following bare mixtures:

$$u=\{\hat{u}X\}, \qquad d=\{\hat{d}X\}, \tag{4}$$

and similarly for the other two generations of quarks. Neither new physical fermions nor new binding forces underlying quarks are introduced. I show that the structure X is associated with an Euclidean configuration of the standard gauge fields having topologically nontrivial properties at infinity; in Minkowski space-time it represents a tunneling process occurring in time by which

prequarks get fractional charges. Leptons have integer charges with no topological contribution from such a gauge field configuration.

What are the quantum numbers of $\hat{q}$ and X? Each prequark has the spin, isospin, color charge, and flavor of the corresponding quark. Its bare weak hypercharge is the same as its lepton partner specified by (3) and its bare electric charge is defined according to (1). In this scenario all bare charges associated with local fields are integral. Finally, the baryon number of prequarks can be fixed through the generalized Gell-Mann–Nishijima formula

$$Q = I_z + \frac{1}{2}(\mathrm{B} + S + C + B^* + T), \tag{5}$$

where I_z, S, C, B^*, T denote (pre)quark flavors. Tables 2 and 3 list the prequark quantum numbers. Quarks and prequarks are identical in all their properties except hypercharge, electric charge and baryon number assignments.

Table 2. First generation of prequarks.

Fermions	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$\begin{pmatrix} \hat{u}_L \\ \hat{d}_L \end{pmatrix}$	3	2	-1
$\hat{u}_R$	3	1	0
$\hat{d}_R$	3	1	-2

Table 3. Prequark additive quantum numbers.

Prequark	$\hat{u}$	$\hat{d}$	$\hat{s}$	$\hat{c}$	$\hat{b}$	$\hat{t}$
B – baryon number	-1	-1	-1	-1	-1	-1
Q – electric charge	0	-1	-1	0	-1	0
I_z – isospin z-component	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	0
S – strangeness	0	0	-1	0	0	0
C – charm	0	0	0	1	0	0
B^* – bottomness	0	0	0	0	-1	0
T – topness	0	0	0	0	0	1

Now regarding the bare quantum numbers associated with the X-configuration, one gets, from (3), (1), and (5),

$$Y(X) = \frac{4}{3}, \qquad Q(X) = \frac{1}{2}Y(X) = \frac{2}{3}, \qquad \mathrm{B}(X) = 2Q(X) = \frac{4}{3}.$$

When these fractional charges are added to the prequark bare ones, the fractional quark charges are obtained.

On the other hand, as readily seen from Tables 1 and 2, a discrete $\mathcal{Z}_2$ symmetry between prequarks and leptons is disclosed in the electroweak sector of the Standard Model. Specifically, this symmetry, which I refer to as presymmetry, means invariance of the classical electroweak Lagrangian of prequarks and leptons under the transformation

$$\hat{u}^i_L \leftrightarrow \nu_{eL}, \qquad \hat{u}^i_R \leftrightarrow \nu_{eR}, \qquad \hat{d}^i_L \leftrightarrow e_L, \qquad \hat{d}^i_R \leftrightarrow e_R,$$

and similarly for the other generations, where i denotes the color degree of freedom. Observe that prequarks and leptons have the same $\mathrm{B}-\mathrm{L} = -1$; i.e., $\mathrm{B}-\mathrm{L}$ is the right fermion number to

be considered under presymmetry. Besides, it is important to realize that presymmetry is not an ad hoc symmetry; as (3) shows, it underlies the electroweak relationships between quarks and leptons. Also, prequarks do not define a new layer of the structure of matter; as shown in the following section, they are just bare quarks whose charges are normalized by the universal contribution of a topologically nontrivial Chern–Simons configuration of the standard gauge fields.

3 Topological currents and gauge anomaly cancellation

Following the standard work [2], it is found that the $U(1)_Y$ gauge current for prequarks and leptons

$$\hat{J}^\mu_Y = \overline{\hat{q}}_L \gamma^\mu \frac{Y}{2} \hat{q}_L + \overline{\hat{q}}_R \gamma^\mu \frac{Y}{2} \hat{q}_R + \overline{\ell}_L \gamma^\mu \frac{Y}{2} \ell_L + \overline{\ell}_R \gamma^\mu \frac{Y}{2} \ell_R,$$

with $\hat{q}_L(\ell_L)$ and $\hat{q}_R(\ell_R)$ uniting the left-handed and right-handed prequarks (leptons) in all generations, respectively, exhibits the $U(1)_Y \times [SU(2)_L]^2$ and $[U(1)_Y]^3$ gauge anomalies generated by the integer hypercharge of prequarks:

$$\partial_\mu \hat{J}^\mu_Y = -\frac{g^2}{32\pi^2} \left[\sum_{\hat{q}_L, \ell_L} \frac{Y}{2} \right] \operatorname{tr} W_{\mu\nu} \tilde{W}^{\mu\nu} - \frac{g'^2}{48\pi^2} \left[\sum_{\hat{q}_L, \ell_L} \left(\frac{Y}{2} \right)^3 - \sum_{\hat{q}_R, \ell_R} \left(\frac{Y}{2} \right)^3 \right] F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (6)$$

where g, g' and $W_{\mu\nu}$, $F_{\mu\nu}$ are the $SU(2)_L$, $U(1)_Y$ couplings and field strengths, respectively. The anomalies are introduced because the sums in (6) do not vanish:

$$\sum_{\hat{q}_L, \ell_L} Y = -8\, N_g, \qquad \sum_{\hat{q}_L, \ell_L} Y^3 - \sum_{\hat{q}_R, \ell_R} Y^3 = 24\, N_g,$$

where N_g is the number of fermion generations.

Now, as often noted in the literature [2], the terms on the right-hand side of (6) are divergences of gauge-dependent currents:

$$\partial_\mu \hat{J}^\mu_Y = -\frac{1}{2} \left(\sum_{\hat{q}_L, \ell_L} \frac{Y}{2} \right) \partial_\mu K^\mu - \frac{1}{2} \left[\sum_{\hat{q}_L, \ell_L} \left(\frac{Y}{2} \right)^3 - \sum_{\hat{q}_L, \ell_L} \left(\frac{Y}{2} \right)^3 \right] \partial_\mu L^\mu, \quad (7)$$

where

$$K^\mu = \frac{g^2}{8\pi^2} \epsilon^{\mu\nu\lambda\rho} \operatorname{tr} \left(W_\nu \partial_\lambda W_\rho - \frac{2}{3} i g W_\nu W_\lambda W \rho \right), \qquad L^\mu = \frac{g'^2}{12\pi^2} \epsilon^{\mu\nu\lambda\rho} A_\nu \partial_\lambda A_\rho, \quad (8)$$

which are the Chern–Simons classes or topological currents related to the $SU(2)_L$ and $U(1)_Y$ gauge groups, respectively. Equation (7) can be rewritten in the form

$$\partial_\mu \hat{J}^\mu_Y = -N_{\hat{q}}\, \partial_\mu J^\mu_X,$$

where $N_{\hat{q}} = 12\, N_g$ is the number of prequarks and

$$J^\mu_X = \frac{1}{4N_{\hat{q}}} K^\mu \sum_{\hat{q}_L, \ell_L} Y + \frac{1}{16 N_{\hat{q}}} L^\mu \left(\sum_{\hat{q}_L, \ell_L} Y^3 - \sum_{\hat{q}_R, \ell_R} Y^3 \right) = -\frac{1}{6}\, K^\mu + \frac{1}{8}\, L^\mu \quad (9)$$

is the current to be associated with the X-configuration of gauge fields introduced in (4). It can be combined with the anomalous fermionic current $\hat{J}^\mu_Y$ to define a new current

$$J^\mu_Y = \hat{J}^\mu_Y + N_{\hat{q}}\, J^\mu_X, \quad (10)$$

which is conserved but it is gauge dependent.

The local counterterms to be added to the Lagrangian of prequarks and leptons, needed to obtain the anomaly-free current of (10), are given by

$$\Delta\mathcal{L} = g' N_{\hat{q}}\, J_X^\mu A_\mu, \tag{11}$$

so that, due to the antisymmetry of $\epsilon^{\mu\nu\lambda\rho}$, only the non-Abelian fields will be topologically relevant, as expected. They produce nontrivial effects as described in the following.

The charge corresponding to the current in (10) is

$$Q_Y(t) = \int d^3x\ \hat{J}_Y^o + N_{\hat{q}} \int d^3x\ J_X^o, \tag{12}$$

which is not conserved after all because of the existence of the topological charge associated with gauge fields. To see this, the change in Q_Y between $t = -\infty$ and $t = +\infty$ is calculated. As usual, it is assumed that the region of space-time where the energy density is nonzero is bounded [3]. Therefore this region can be surrounded by a 3-dimensional surface on which the field configuration becomes pure gauge, i.e.,

$$W_\mu = -\frac{i}{g}(\partial_\mu U)U^{-1}.$$

This field can be obtained from $W_\mu = 0$ by a transformation U that takes values in the corresponding gauge group. In this case, using (12), (9) and (8) for the gauge fields, one ends up with

$$Q_Y(t) = \frac{N_{\hat{q}}}{6}\, n_W(t),$$

where

$$n_W(t) = \frac{1}{24\pi^2} \int d^3x\ \epsilon^{ijk}\ \mathrm{tr}\left(\partial_i U U^{-1} \partial_j U U^{-1} \partial_k U U^{-1}\right) \tag{13}$$

is the winding number of the non-Abelian gauge transformation. This number is integer-valued if we consider a fixed time t and assume that $U(t, \boldsymbol{x})$ equals a direction independent constant at spatial infinity, e.g., $U \to 1$ for $|\boldsymbol{x}| \to \infty$. The usual argument to see this property is based on the observation that this U may be viewed as a map from the 3-dimensional space with all points at infinity regarded as the same onto the 3-dimensional sphere of parameters S^3 of the $SU(2)_L$ group manifold. But 3-space with all points at infinity being in fact one point is topologically equivalent to a sphere S^3 in Minkowski space. Therefore U determines a map $S^3 \to S^3$. These maps are characterized by an integer topological index which labels the homotopy class of the map. This integer is analytically given by (13). For the Abelian case, $n_W = 0$.

Thus, for the field configurations joined to prequarks which at the initial $t = -\infty$ and the final $t = +\infty$ are supposed of the above pure gauge form, the change in charge becomes

$$\Delta Q_Y = Q_Y(t = +\infty) - Q_Y(t = -\infty) = \frac{N_{\hat{q}}}{6}\left[n_W(t = +\infty) - n_W(t = -\infty)\right]. \tag{14}$$

The difference between the integral winding numbers of the pure gauge configurations characterizing the gauge fields at the far past and future can be rewritten as the topological charge or Pontryagin index defined in Minkowski space-time by

$$Q_T = \int d^4x\ \partial_\mu K^\mu = \frac{g^2}{16\pi^2} \int d^4x\, \mathrm{tr}\left(W_{\mu\nu}\tilde{W}^{\mu\nu}\right), \tag{15}$$

where it is assumed that K^i decreases rapidly enough at spatial infinity. This topological index is gauge invariant, conserved and, for arbitrary fields, can take any real value. But, as shown above, for a pure gauge configuration it is integer-valued. Thus

$$Q_T = n = n_W(t = +\infty) - n_W(t = -\infty), \tag{16}$$

so one has from (14) that

$$\Delta Q_Y = N_{\hat{q}}\,\frac{n}{6}. \tag{17}$$

These equations have a special significance when the integral in (15) is analytically continued to Euclidean space-time. The integral topological charge remains the same but it can now be associated with an Euclidean field configuration that at this point we identify with the X in (4). In this case, the meaning of (16) is that the Euclidean X-configuration interpolates in imaginary time between the real time pure-gauge configurations in the far past and future which are topologically inequivalent. The condition is like the one established for instantons. Thus the continuous interpolation is to be considered as a tunneling process, so that a barrier must separate the initial and final gauge field configurations. It should also be noted that if such an analytical continuation to Euclidean space-time is ignored, nonzero topological charge then implies nonvanishing energy density at intermediate real time and so no conservation of energy.

A consequence of (17) is that $N_{\hat{q}}$ prequarks have to change their $Y/2$ in the same amount $n/6$. For each prequark it implies the hypercharge change

$$Y_{\hat{q}} \to Y_{\hat{q}} + \frac{n}{3}. \tag{18}$$

Therefore, the nontrivial topological properties of the X-configuration give an extra contribution to prequark local hypercharge, so that the integer bare values one starts with have to be shifted. Accordingly, the gauge anomalies have to be re-evaluated. It is found that anomalies are cancelled self-consistently for $n = 4$, the number of prequark flavors per generation, since now

$$\sum_{q_L,\ell_L} Y = 0, \qquad \sum_{q_L,\ell_L} Y^3 - \sum_{q_R,\ell_R} Y^3 = 0. \tag{19}$$

The above hypercharge normalization with topological charge $n = 4$ is consistent with (3) and it means restoration of gauge invariance, breaking of the electroweak presymmetry in the Abelian sector, dressing of prequarks into quarks, and the substitution of the bare presymmetric model by the Standard Model. In fact, from (19) and (9) one notes that the gauge-dependent topological current J^μ_X associated with the X-configuration is cancelled. However, the corresponding conserved topological charge is gauge independent and manifests itself as a universal part of the prequark hypercharge according to (3). More precisely, if the topological current introduced in (11) and its induced hypercharge obtained in (17) are considered, an effective prequark current $\hat{J}^\mu_{Y,\text{eff}}$ can be defined by

$$\hat{J}^\mu_{Y,\text{eff}} = \frac{2}{3}\left(\overline{\hat{q}}_L\gamma^\mu\hat{q}_L + \overline{\hat{q}}_R\gamma^\mu\hat{q}_R\right)$$

to absorb the nontrivial effects of such $N_{\hat{q}}J^\mu_X$ current, namely, cancellation of gauge anomalies and inducement of the hypercharge 4/3 on prequarks regardless of flavor and handness. Upon using this, (10) becomes

$$J^\mu_Y = \overline{\hat{q}}_L\gamma^\mu\frac{Y+4/3}{2}\hat{q}_L + \overline{\hat{q}}_R\gamma^\mu\frac{Y+4/3}{2}\hat{q}_R + \overline{\ell}_L\gamma^\mu\frac{Y}{2}\ell_L + \overline{\ell}_R\gamma^\mu\frac{Y}{2}\ell_R.$$

At this point, prequarks with fractional hypercharge, which includes the universal 4/3 part, have to be identified with quarks. The replacement of prequarks by quarks in the strong, weak and Yukawa sectors is straightforward as they have the same color, flavor and weak isospin.

All of this is essentially done at the level of the classical Lagrangian where the specific field configuration X, which only mixes with prequarks, is used. Again, this configuration is a pseudoparticle in Euclidean space-time and a tunneling process in Minkowski space-time by which a prequark hypercharge changes from integer to fractional values. It is also interesting to note that as stated by the model self-consistency is the reason for the "magical" cancellation of gauge anomalies in the Standard Model. Moreover, the factor 1/3 in (18) is due to the number of prequark colors (assuming same number of prequark and lepton families) introduced in (9) through $N_{\hat{q}}$, which predicts, as expected, that quarks carry 1/3-integral charge because they have three colors.

As it is shown above, standard bare quarks instead of prequarks are the fermions to start with in the quantum field theory treatment. The novel news is that bare quarks have fractional charges owing to a universal contribution from a classical gauge field configuration with specific topological properties (i.e., $n = 4$). Underlying this charge structure one has presymmetry as reflected in (3); an electroweak $\mathcal{Z}_2$ symmetry between integer-charged bare quarks and leptons which is broken by the vacuum configuration of gauge fields but it accounts for the electroweak similarities between quarks and leptons.

4 Effective topological charge and baryon number

The underlying topologically nontrivial gauge field configuration in bare quarks suggests to associate an effective topological charge with baryonic matter. If it is considered that an effective fractional topological charge equal to $Q_T = n/N_{\hat{q}} = 1/N_c N_g = 1/9$, where N_c is the number of colors, can be associated with each X-configuration, then, within the bare configuration of (4), it is natural to define an effective topological charge for quarks which is conserved and should be related to its fermion number. As a general rule it is found that

$$Q_T = \frac{\mathrm{B}}{N_g}. \tag{20}$$

Thus $\Delta \mathrm{B} = N_g \Delta Q_T$ for any baryon-number violating process, i.e., only topological effects may violate baryon number.

To see the consistency of the definition for the charge in (20), the baryon plus lepton number (B+L) violating processes induced non-perturbatively by electroweak instanton effects may be considered. According to Ref. [4], for three generations, one electroweak instanton characterized by a topological charge $Q_T = 1$ is associated with quark and lepton number violations in three units: $\Delta \mathrm{B} = \Delta \mathrm{L} = -3$, matching three baryons or nine quarks and three antileptons. A decay such as $p + n + n \to \mu^+ + \bar{\nu}_e + \bar{\nu}_\tau$ is then allowed. Within the bare configuration scheme of (4), this rule implies that one instanton induces a process in which nine X-configurations vanish. The nine X's make precisely the same topological charge of the electroweak instanton. In a sense, the definition in (20) brings back topological charge conservation in quantum flavor dynamics. In Minkowski space-time one electroweak instanton corresponds to a process which has associated the topological charge change $\Delta Q_T = 1$. It induces a process with the effective topological charge change $\Delta Q_T = -1$ associated with the vanishing of nine quarks and baryon number violation $\Delta \mathrm{B} = -3$. It should also be noted that consistency between the topological charge of the instanton and the effective one assigned to quarks corroborates the above value $n = 4$ of the Pontryagin index fixed by gauge anomaly constraints.

Finally, it can be seen from (20) that a baryon number violation $\Delta \mathrm{B} = -1$ in actual experiments means an effective topological-charge change $\Delta Q_T = -1/N_g$. In particular, proton decay would

imply the presence of a background gauge source with topological charge $1/N_g$ [5]. Stability of a free proton is expected anyway because instanton-like events cannot change topological charge by this amount. For three generations, the effective topological charge has a value $Q_T = 1/9$ for quarks and $Q_T = 1/3$ for nucleons.

5 Conclusion

Insights into the classical dynamics of the standard weak gauge fields which give to quarks fractional charge relative to leptons have been given. A self-consistent method to adjust bare local charges with Chern–Simons contributions has been pursued. Presymmetry, a discrete $\mathcal{Z}_2$ symmetry between bare quarks and leptons, has been introduced to understand quark-lepton similarities in the electroweak sector of the Standard Model. I have also presented arguments, based on the topological character of quark fractional charges and conservation of the associated effective topological charge, to explain the observed conservation of baryon number and predict stability of a free proton.

In this talk I have emphasized a picture in which the integral hypercharges of leptons are entirely associated with local fields. An alternative, dual point of view is to consider quark fractional hypercharges as the local ones, so that the integral hypercharge of leptons relies, as seen from (3), just on a universal $-4/3$ value which also has a topological character. In such a case, this number 3 by which the Pontryagin index $n = 4$ is divided cannot correspond to the number of colors because color is not a property of leptons. The number of generations is the only available degree of freedom. This interpretation solves the fermion family problem: presymmetry requires that the number of fermion generations be equal to the number of quark colors. On the other hand, following an analysis similar to the one presented for baryons, it is concluded that leptonic matter has associated an effective topological charge given by $Q_T = -\mathrm{L}/N_g$. Thus, the general result appears to be

$$Q_T = \frac{\mathrm{B} - \mathrm{L}}{N_g}.$$

This charge is absolutely conserved, so that $\Delta(\mathrm{B} - \mathrm{L}) = 0$ in any physical process, as expected.

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Generation of Glauber Coherent State Superpositions via Unitary Transformations

Antonino MESSINA, Benedetto MILITELLO and Anna NAPOLI

INFM, MIUR and Dipartimento di Scienze Fisiche ed Astronomiche dell'Universitá di Palermo, Via Archirafi, 36 - 90123 Palermo, Italy
E-mail: *messina@fisica.unipa.it*

A unitary transformation mapping the vacuum state of a harmonic oscillator into a normalized superposition of two arbitrary Glauber coherent states is reported. Interesting features of the transformation are discussed. Generalizations to the superposition of N coherent states are discussed.

1 Introduction

Since their introduction by Glauber in 1963 [1], coherent states have been deeply studied [2] due to the fact that, in some sense, the statistical properties related to the first momenta of their Fock distribution exhibit features interpretable in the realm of classical probability theory. Of course the quantum nature of a coherent state, and ultimately the granularity of the energy of a quantum harmonic oscillator, becomes more and more evident when higher momenta of the Fock distribution are considered. It is of relevance that coherent states of both a mechanical quantum oscillator or a quantized mode of an electromagnetic resonator may be currently proposed in accordance with well-defined protocols. Even more interesting is the fact that they provide the effective possibility of realizing interesting experiments able to bring to the light peculiarities of quantum mechanics. In particular, the very famous *gedanken* Schrödinger Cat State (SCS) experiment [3] consisting in entangling a microscopic quantum object (an atom) with a macroscopic one (a cat) has been realized. No unfortunate feline has been tortured! Trapped ions have been indeed exploited. In fact, such systems provide the possibility of testing fundamental quantum physics aspects giving at experimentalist's disposal compound quantum systems consisting of a few-level atomic part and a tri-dimensional harmonic oscillator describing the center of mass motion of an ion confined into a Paul trap. Suitably acting upon the system with Raman lasers it is possible to engineer the *Schrödinger Cat-like state* [4, 5]

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\alpha\rangle \, |-\rangle + |-\alpha\rangle \, |+\rangle \right),$$

where $|\pm\rangle$ indicate two internal atomic states and $|\pm\alpha\rangle$ are two macroscopically well distinguishable harmonic oscillator configurations, i.e. *opposite* Glauber coherent states.

Today the definition of SCS is extended to all those situations wherein two macroscopically distinguishable classical states are quantum coherently simultaneously present [6]. Hence the purely vibrational normalized quantum superposition

$$|\phi\rangle = N(\alpha) \left(|\alpha\rangle + |-\alpha\rangle \right) \tag{1}$$

is referred to as a SCS. Here $N(\alpha)$ is the normalizing constant[1]. It is worth to stress that, apart from semantic questions, the superposition in equation (1) is of relevance in the study of fundamental aspects of quantum mechanics.

[1] Due to the non-orthogonality of coherent states, in most of the cases $N(\alpha)$ is not trivially $\frac{1}{\sqrt{2}}$.

In this paper we present, from the mathematical point of view, the construction of a new unitary operator able to generate coherent superpositions of Glauber coherent states when applied to the vibrational vacuum. In particular, dynamical configurations like that in equation (1) may be generated. The operator, initially related to the generation of two coherent states, will be generalized to the case of more than two such states. It is worth noting that, acting with such an operator upon states different from the vacuum but possessing appropriate spatial symmetries, it is possible to generate a wider class of superpositions of translated wave function.

The unitary operator introduced here may be used to solve elegantly and effectively those variational problems involving superpositions of Glauber coherent states as candidate solution.

2 Generation of coherent state superpositions

The physical system we consider is a harmonic oscillator of frequency ω described by the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the annihilation and creation operators respectively. Its eigenstates $|n\rangle$, n being a non-negative integer, describe a spectrum having equidistant levels: $\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$.

It is well known that, applying a displacement operator to the vacuum state, $|0\rangle$, one obtains a so called Glauber coherent state

$$|\alpha\rangle \equiv \hat{D}(\alpha)|0\rangle \equiv e^{\alpha^*\hat{a} - \alpha\hat{a}^\dagger}|0\rangle,$$

where α is a complex number whose real and imaginary parts return the mean values of the position and momentum operators respectively. Observe that, since the vacuum state realizing a non-squeezed minimum position-momentum uncertainty is rigidly translated, the minimal uncertainty is preserved.

Consider now the observable

$$\hat{\Pi} = \cos\left(\pi\hat{a}^\dagger\hat{a}\right)$$

which is a Hermitian and unitary operator commuting with $\hat{H}$ and satisfying the following relations:

$$\hat{\Pi}|n\rangle = (-1)^n|n\rangle \tag{2}$$

and

$$\hat{\Pi}\hat{a}\hat{\Pi} = -\hat{a}, \qquad \hat{\Pi}\hat{a}^\dagger\hat{\Pi} = -\hat{a}^\dagger.$$

Due to equation (2), $\hat{\Pi}$ is called Parity operator. Starting from this point we introduce the operator

$$\hat{D}(\alpha)\,\Pi.$$

It turns out that, like $\hat{\Pi}$, it is both unitary, being the product of two unitary operators, and Hermitian, as easily verifiable:

$$\left[\hat{D}(\alpha)\,\Pi\right]^\dagger = \Pi\hat{D}(\alpha)^\dagger = \Pi\hat{D}(-\alpha) = \Pi\hat{D}(-\alpha)\,\Pi^2 = \hat{D}(\alpha)\,\Pi.$$

Moreover, the square of such an operator is the Identity. Indeed

$$\left[\hat{D}(\alpha)\,\Pi\right]^2 = \hat{D}(\alpha)\,\Pi\hat{D}(\alpha)\,\Pi = \hat{D}(\alpha)\,\hat{D}(-\alpha) = \hat{1}. \tag{3}$$

Taking into account all such properties, we define the following two-parameter unitary operator:

$$\hat{U}(\lambda,\alpha) \equiv e^{i\lambda\hat{D}(\alpha)\Pi}, \tag{4}$$

λ being a real number.

In consideration of equation (3), $\hat{U}(\lambda,\alpha)$ may be cast in the form

$$\begin{aligned}\hat{U}(\lambda,\alpha) &\equiv \sum_{k=0}^{\infty}\frac{\left(i\lambda\hat{D}(\alpha)\Pi\right)^k}{k!}\\ &= \sum_{k=0}^{\infty}\frac{(i\lambda)^{2k}}{(2k)!}\hat{1} + \sum_{k=0}^{\infty}\frac{(i\lambda)^{2k+1}}{(2k+1)!}\hat{D}(\alpha)\Pi\end{aligned} \tag{5}$$

and finally

$$\hat{U}(\lambda,\alpha) = \cos\lambda\,\hat{1} + i\sin\lambda\,\hat{D}(\alpha)\Pi. \tag{6}$$

The action of such an operator on a state we shall refer to as the *state of reference* returns a normalized superposition of the original wave-packet and its rigid translation of $\alpha = x + ip$ in the phase space, x and p being position and momentum expectation value changes. In particular, taking the vibrational vacuum $|0\rangle$ as state of reference one obtains a normalized superposition involving the ground state itself and the coherent state $|\alpha\rangle$:

$$\hat{U}(\lambda,\alpha)|0\rangle = \cos\lambda\,|0\rangle + i\sin\lambda\,|\alpha\rangle\,. \tag{7}$$

It is worth recalling that $|0\rangle$ may be thought of as a coherent state corresponding to the classical situation wherein the harmonically confined particle is *at rest in the origin*. Observe also the peculiarity of the state in equation (7) consisting in the fact that the normalized superposition has two weights expressible as sines and cosine of the same number, i.e. it looks like $|0\rangle$ and $|\alpha\rangle$ are orthogonal[2].

In order to obtain a linear combination of *two arbitrary* Glauber coherent states, it is enough the subsequent action of another displacement operator, $\hat{D}(\beta)$:

$$\hat{D}(\beta)\hat{U}(\lambda,\alpha)|0\rangle = \cos\lambda\,|\beta\rangle + ie^{i\Im(\alpha\beta^*)}\sin\lambda\,|\alpha+\beta\rangle\,. \tag{8}$$

Hence the unitary operator

$$\begin{aligned}\hat{V}(\alpha,\beta,\lambda) &\equiv \hat{D}(\alpha)\hat{U}(\lambda,\beta-\alpha)\\ &= \cos\lambda\hat{D}(\alpha) + ie^{i\Im(\alpha\beta^*)}\sin\lambda\hat{D}(\beta)\hat{\Pi}\end{aligned} \tag{9}$$

applied on the vacuum state generates the normalized superposition

$$\hat{V}(\alpha,\beta,\lambda)|0\rangle = \cos\lambda\,|\alpha\rangle + ie^{i\Im(\alpha\beta^*)}\sin\lambda\,|\beta\rangle\,. \tag{10}$$

Such a result may be easily extended taking as state of reference any eigenfunction of the Parity operator. Indeed, let $|\psi_0\rangle$ be a $\hat{\Pi}$ eigenstate meaning that $\psi_0(x) \equiv \langle x|\,\psi_0\rangle$ is an even or odd function. The action of $\hat{V}(\alpha,\beta,\lambda)$ on such a state generates

$$\hat{V}(\lambda,\alpha,\beta)|\psi_0\rangle = \cos\lambda\hat{D}(\alpha)|\psi_0\rangle + (-1)^\sigma ie^{i\Im(\alpha\beta^*)}\sin\lambda\hat{D}(\beta)|\psi_0\rangle\,, \tag{11}$$

σ being the parity of $\psi_0(x)$.

[2]Generally speaking such a circumstance is realized only in correspondence to a specific phase correlation between the two coefficients involved in the coherent superposition. To this end observe that, given the state $|\psi\rangle = c_1|\alpha_1\rangle + c_2e^{i\phi}|\alpha_2\rangle$ with c_1, c_2, ϕ real numbers, the modulus $\langle\psi|\,\psi\rangle = |c_1|^2 + |c_2|^2 + c_1c_2e^{-\frac{|\alpha-\beta|^2}{2}}\Re\left\{e^{i\phi}e^{i\Im(\alpha^*\beta)}\right\}$ may be expressed as $\left(c_1^2+c_2^2\right)$ only if $\phi = \frac{\pi}{2} - \Im(\alpha^*\beta) + k\pi$.

In the case when α and β are real and $|\psi_0\rangle$ is a localized wave function centered in the origin, the state in equation (10) turns out to be the superposition of two localized packets centered in $x = \alpha$ and $x = \beta$:

$$\langle x|\,\hat{V}\,(\lambda,\alpha,\beta)\,|\psi_0\rangle = \cos\lambda\psi_0\,(x+\alpha) + i\sin\lambda\psi_0\,(x+\beta)\,. \tag{12}$$

3 Superpositions of more than two coherent states

The question of how the number of coherent states in the superposition can be increased immediately arises. To this end the subsequent action of two $\hat{V}$ unitary operators is considered. The relevant transformation is given by

$$\hat{V}^{(2)}\,(\lambda,\mu,\alpha_1,\alpha_2,\alpha_3,\alpha_4) \equiv \hat{V}\,(\lambda,\alpha_1,\alpha_2)\,\hat{V}\,(\mu,\alpha_3,\alpha_4) \tag{13}$$

which may be cast in the form

$$\begin{aligned}\hat{V}^{(2)}\,(\lambda,\mu,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = &\left[\cos\lambda\hat{D}\,(\alpha_1) + ie^{\Im\left(\alpha_1\alpha_2^*\right)}\sin\lambda\hat{D}\,(\alpha_2)\,\hat{\Pi}\right]\\ &\times\left[\cos\mu\hat{D}\,(\alpha_3) + ie^{\Im\left(\alpha_3\alpha_4^*\right)}\sin\mu\hat{D}\,(\alpha_4)\,\hat{\Pi}\right].\end{aligned} \tag{14}$$

Taking into account properties of $\hat{\Pi}$, one obtains

$$\begin{aligned}\hat{V}^{(2)}\,(\lambda,\mu,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = &\cos\lambda\cos\mu\;e^{i\phi_{13}}\hat{D}\,(\alpha_1+\alpha_3) + \cos\lambda\sin\mu\;e^{i\phi_{14}}\hat{D}\,(\alpha_1+\alpha_4)\\ &+\sin\lambda\sin\mu\;e^{i\phi_{24}}\hat{D}\,(\alpha_2-\alpha_4) + \sin\lambda\cos\mu\;e^{i\phi_{23}}\hat{D}\,(\alpha_2-\alpha_3)\,\hat{\Pi},\end{aligned} \tag{15}$$

where ϕ_{jk} are suitable quantum phases.

The four coherent states obtained when $\hat{V}^{(2)}$ is applied upon $|0\rangle$ are

$$|\alpha_1+\alpha_3\rangle\,,\qquad |\alpha_1+\alpha_4\rangle\,,\qquad |\alpha_2-\alpha_3\rangle\,,\qquad |\alpha_2-\alpha_4\rangle\,.$$

Despite appearances, we do not have four independently parametrically changeable states. Indeed, to generate the *arbitrary quadruplet*

$$|c_1\rangle\,,\qquad |c_2\rangle\,,\qquad |c_3\rangle\,,\qquad |c_4\rangle$$

one has to choose the α-parameters satisfying the following equation system:

$$\alpha_1+\alpha_3 = c_1,\qquad \alpha_1+\alpha_4 = c_2,\qquad \alpha_2-\alpha_3 = c_3,\qquad \alpha_2-\alpha_4 = c_4$$

which, unfortunately, has a vanishing determinant:

$$\det\begin{bmatrix}1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\\ 0 & 1 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix} = 0.$$

Nevertheless, the usefulness of the transformation given by equation (15) remains valid, in particular considering the fact that it can be exploited to generate threefold Schrödinger Cat-like states. More in detail

$$\begin{aligned}\hat{V}^{(2)}\,(\lambda,\mu,0,0,0,\alpha)\,|0\rangle = &\cos\mu\left(\cos\lambda\;e^{i\phi_{13}} + \sin\lambda\;e^{i\phi_{23}}\right)|0\rangle\\ &+\cos\lambda\sin\mu\;e^{i\phi_{14}}\,|\alpha\rangle + \sin\lambda\sin\mu\;e^{i\phi_{24}}\,|-\alpha\rangle\,.\end{aligned} \tag{16}$$

In this equation we have three equidistant wave packets corresponding to the same wave function centered in the three positions $x = 0, \pm\Re\{\alpha\}$. A rigid translation of this state is obtained considering $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = 2\alpha$ and $\alpha_4 = \alpha$: $\hat{V}^{(2)}(\lambda, \mu, 0, 2\alpha, 0, \alpha)$. In fact in this case one has a linear combination of $|0\rangle$, $|\alpha\rangle$ and $|2\alpha\rangle$.

The method reported here may be extended to the case of a superposition involving 2^N coherent state just considering

$$\hat{V}^{(N)} \equiv \prod_{k=1}^{N} \hat{V}(\lambda_k, \alpha_k, \beta_k).$$

Of course, like in the case $N = 2$, the number of independently determinable states is smaller than 2^N.

4 Conclusive remarks

In this paper we have presented the construction of a unitary operator, $\hat{V}$ which in some sense behaves like a superposition of phase space translators. Such a circumstance, joint with the well known definition of Glauber coherent states as *displaced vacuum*, has provided an effective and elegant expression for an arbitrary two coherent state superposition. The subsequent application of two such unitary operators produces a four coherent state superposition and so on following the law 2 raised to the number of applications of $\hat{V}$, which is larger than the number of coherent state which may be independently parametrically changed at will.

As a conclusive remark we point out that the possibility of considering a generic linear combination of Glauber states via a unitary transformation of the vibrational vacuum leads to a more elegant and effective way to deal with those variational problems having such wave functions as candidate solutions.

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Space Rotation

Andrey V. NOVIKOV-BORODIN

Institute for Nuclear Research of RAS, 7a 60-th Oct. Anniv. Ave., 117312 Moscow, Russia
E-mail: *novikov@inr.troitsk.ru,* URL: *http://www.inr.troitsk.ru/∼novikov*

An attempt to create a system of the physical reality description applied to the space rotations is undertaken. This approach extends the concept of the space symmetry for the $\mathcal{O}(3)$ space rotation groups on basis of the evolution of the ideas of relativity and on the suggestion of the space rotation equivalence. The basic space-time properties and the physical object description from this point of view are considered. Stable space rotation objects are found, which correlate with the quantum physics models of the elementary particles, quarks and even nuclei. It is shown that the quantification is an essential part of space rotation objects. The introduced ω-invariance hypothesis connects the space rotation methodology with the quantum physics approach. In the framework of the proposed space rotation approach, it is possible to get basic quantum mechanics equations, to interpret basic postulates of quantum physics. It looks like postulates and limitations of quantum physics are coming from the introduced Space rotation theory that can be a foundation of the Unification theory.

1 Physical reality description

If we consider two rotating related to each other frames of reference in "empty", "mathematical" space, they seem to be equivalent and connected only to the observer. Otherwise, we need to declare some *absolute, initial* frame of reference. From the other point of view, it is known that physical laws are not invariant related to space rotations. Today, there are no known proven contradictions to the relativity principle from the experiment. As far as we would like to stay at the relativity principle, we need to extend this principle to space rotations. To overcome these contradictions, we need to review the system of physical reality description (PhRS). We will consider that the PhRS consists of four interconnected components. There are the physical laws, physical objects, reference frames and space-time properties.

Indeed, some attempts to extend the relativity principle to the non-inertial reference frames were made before [1]. Also the physical object in one references frame can be interpreted as another physical object in other frame. For example, the plane electromagnetic wave may be represented in the rotation frame as a set of the spherical waves [2]. So, some physical laws will be valid in both frames, but it will be *different* physical laws for *different* physical objects.

2 Space rotation equivalence and metrics

We will consider three types of space rotations: the axis space rotation (ASR), the multiple space rotation (MSR) and the sum space rotation (SSR).

We will call the space rotation (SR) of the K' reference frame related to K in time t with the frequency ω about some fixed space axis as an *axis* space rotation (ASR). We will also use notations: $\tau = ct$, $\Omega = \omega/c$, where c is a speed of light. Space coordinates in K' will be primed as $(x', y', z') = X'$. The coordinate transformation between the "initial" frame K and rotating

one K' will be written as:

$$X' = X \cdot \left(A_z^{\rm ASR}\right)^{\pm}, \qquad \left(A_z^{\rm ASR}\right)^{\pm} = \begin{pmatrix} \cos(\omega\, t) & \mp \sin(\omega\, t) & 0 \\ \pm \sin(\omega\, t) & \cos(\omega\, t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1}$$

Here $(A_z^{\rm ASR})^{\pm}$ is a transformation matrix of the ASR about z-axis in space (without loosing the generality). At any time the functional determinant of the ASR transformation matrix is equal to unit ($\det A = 1$).

We will call the combination of ASR rotations with transformation matrices $A_1, A_2, \ldots, A_n$ by definition as a *multiple* space rotation (MSR) and a *sum* space rotation (SSR), if the general transformation matrices of these rotations $A^{\rm MSR}$ and $A^{\rm SSR}$ are expressed as:

$$A^{\rm MSR} = \prod_{k=1}^{n} A_k, \qquad A^{\rm SSR} = \sum_{k=1}^{n} A_k. \tag{2}$$

Let the K frame rotate with respect to K' with the transformation matrix A_1 and K'' frame rotates related to K' with the transformation matrix A_2. The transformation between K'' and K will be expressed by the transformation matrix $A^{\rm MSR} = \{a_{ij}\} = A_1 \cdot A_2 = \sum\limits_{k=1}^{n} a_{ik}^1 a_{kj}^2$. The SRs in SSR are considered from the K reference frame.

The ASR is a particular case of the MSR. The MSR transformations is forming the rotation group like $\mathcal{O}(3)$, where time t looks like some parameter. Let us denote this group as an extended rotation group $\widehat{\mathcal{O}}(3)$.

Due to matrix properties, one can conclude that the transformation matrix of *any space rotation* may be represented as a sum of products of the ASR transformation matrices:

$$X' = X \cdot A^{\rm SR} = X \cdot \left\{ \sum_i A_i^{\rm MSR} \right\}. \tag{3}$$

Metrics. Standing on the position of relativity, we need to declare the equivalence of the SR frames. It means that, for example, metrics can be introduced in any SR frame. For some space rotation with the transformation matrix A between K and K' we will analyse the expression for "interval" in the form:

$$ds'^2 = c^2 dt^2 - \|dX'\|^2, \tag{4}$$

where dX' is defined as $dX' = d(XA) = dXA + XdA$ and $\|dX'\|^2 = dX' \cdot dX'^T$. So, for (4) we will get:

$$\left(ds'_{\rm SR}\right)^2 = c^2 dt^2 - XdAdA^T X^T - dXAA^T dX^T - \left\{dXAdA^T X^T + XdAA^T dX^T\right\}. \tag{5}$$

Metrics in rotation point. If we consider the interval (4) from the rotation point $X'_{rp} = X_{rp} = (0,0,0)$ (the same in K and K'), we will get for any time t:

$$\|\Delta X'\|^2 = X' \cdot X'^T = X \cdot A^{\rm MSR}(t)\left[A^{\rm MSR}(t)\right]^T \cdot X^T = X \cdot X^T = \|\Delta X\|^2.$$

It means, according to (4), that K and K' are invariant in rotation point and this invariance has a local (or even point) character. The situation with points differed from the rotation one is another.

The ASR metrics. Let us consider the transformation matrix $\left(A_z^{\rm ASR}\right)^{\pm}$ from (1) without loosing the generality. From (5), we will get the expression:

$$\left(ds'_{\rm ASR}\right)^2 = \left[c^2 - \omega^2\left(x^2 + y^2\right)\right] dt^2 \mp 2\,(ydx - xdy)\,\omega dt - \left(dx^2 + dy^2 + dz^2\right), \tag{6}$$

that in cylindrical coordinates (ρ, ϕ, z) will be:

$$\left(ds'_{\rm ASR}\right)^2 = \left(c^2 - \omega^2\rho^2\right) dt^2 \pm 2\rho^2\omega\, d\phi\, dt - \rho^2\, d\phi^2 - d\rho^2 - dz^2. \tag{7}$$

The metric tensor corresponding to the ASR interval has nondiagonal elements and, so, is immeasurable. Physically, it is clear that on some distance from the rotation axis, the motionless in K' object will have the speed in K equal to speed of light. It means that if we are moving radially in K ($d\phi = 0$), the term in braces at dt^2 in (7) is positive when $\rho^2 < c^2/\omega^2$ and negative when $\rho^2 > c^2/\omega^2$. So the surface, defined by the equation $\rho^2 = c^2/\omega^2$, divides the space into the *internal* and *external* regions correspondingly. This equation with different frequencies ω defines a set of cylindroids about the z-axis.

The physical object in K' from K will be observed as an object, localized in the internal region, stable on its edge and freely movable along with the rotation axis. As far as the rotation frequency with the same value may have two directions of rotation (may have positive or negative sign), it may be two different types of these objects. One can say that this object will have an additional characteristics from the K point of view. This characteristics looks like *spin* that, within this approach, may be directed along or opposite the rotation axis. In elementary particle physics, this model of the ASR physical object has some correspondence with *neutrino*, the possibly massless particle with spin.

The MSR metrics. Let us consider the MSR $A^{\rm MSR} = A_z^{\rm ASR}(\omega_1)\, A_x^{\rm ASR}(\omega_2)\, A_y^{\rm ASR}(\omega_3)$, where ω_1, ω_2 and ω_3 are rotation frequencies. We will analyze the interval, averaged in time, from the point of view of the observer in K and will consider that the period of time of the observation is much longer than the period of any rotation included in the MSR. We will use the following expression for "averaging":

$$\langle f\rangle_t = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{+t} f(t)\, dt.$$

On this way, for the MSR interval one can get:

$$\begin{aligned}\langle\left(ds'_{\rm MSR}\right)^2\rangle_t = \Bigg\{c^2 - \Bigg[\left(x^2+y^2\right)\omega_1^2 + \left(\frac{1}{2}x^2 + \frac{1}{2}\,y^2 + z^2\right)\omega_2^2 \\ + \left(\frac{3}{4}x^2 + \frac{3}{4}y^2 + \frac{1}{2}z^2\right)\omega_3^2\Bigg]\Bigg\} dt^2 - 2\left(ydx - xdy\right)\omega_1 dt - dx^2 - dy^2 - dz^2.\end{aligned} \tag{8}$$

On the analogy of the ASR analysis, we can conclude that there are some regions localized in space, defined by the equation $g_{00} = 0$, where the physical object will be stable in time from the K point of view. In spherical coordinates (r, θ, ϕ), these stable regions will satisfy the equation:

$$r^2 = \left[\left(\frac{\omega_1}{c}\right)^2 \sin^2(\theta) + \left(\frac{\omega_2}{c}\right)^2 \left(1 - \frac{1}{2}\sin^2(\theta)\right) + \frac{1}{2}\left(\frac{\omega_3}{c}\right)^2 \left(1 + \frac{1}{2}\sin^2(\theta)\right)\right]^{-1}. \tag{9}$$

This equation for different frequencies describes ellipsoids in space. The conclusion about the existence of the MSR stable regions is an additional characteristic to the "mathematical" model, the "physical" aspect of the extended rotation group $\widehat{\mathcal{O}}(3)$.

Comparing expression (8) with (6), (7), one can see that although the MSR consists of a few orthogonal ASRs, the MSR object, localized in space, anyway has some axis, picked out in space. Note that it is some pseudo-axis, because it is some "averaged in time" axis. We can suppose, as before, during the ASR object analysis, that it means that the MSR object also needs to have the physical "spin"-characteristic. Here no limitations are seen to the spin direction related to the direction of the particle movement (in contrast to the ASR object). That also corresponds to the experimental data for the massive particles with spin. Such SR objects are similar to *fermions*.

The SSR metrics. We will consider the SSR with $A^{\mathrm{SSR}} = A_z^{\mathrm{ASR}}(\omega_1)+A_x^{\mathrm{ASR}}(\omega_2)+A_y^{\mathrm{ASR}}(\omega_3)$. For the SSR, we need to use the expression (5). For the SSR interval with "averaging", one can get:

$$\langle\left(ds'_{\mathrm{SSR}}\right)^2\rangle_t = \left\{c^2 - \left[\left(x^2+y^2\right)\omega_1^2 + \left(y^2+z^2\right)\omega_2^2 + \left(x^2+z^2\right)\omega_3^2\right]\right\} dt^2 \qquad (10)$$
$$- 2\left[(ydx - xdy)\,\omega_1 - (ydz - zdy)\,\omega_2 + (zdx - xdz)\,\omega_3\right] dt - 3\left(dx^2+dy^2+dz^2\right).$$

The SSR object is localized in space and, also, like the MSR object, has an ellipsoidal structure. It is also possible to separate the space into the internal and external parts. The SSR objects localized in space, apparently, are massive. But, generally, it is impossible to pick out any space axis (even averaged) for the SSR object "as a whole". SSR objects may be "complex", i.e. consisted of a few ASR or MSR objects, but their characteristics "as a whole" differ from the characteristics of the included SR objects. For the SSR objects $\|X'\|^2 \neq \|X\|^2$ (with "averaging" some additional coefficients will appear). It looks like due to the SSR metrics changing the intensity of the "interaction" between the included ASRs is increasing, that, may be, will correspond to electroweak and strong forces. The analogies with objects of chromodynamics (QCD) arise. Within this approach, if we will continue the analogies between SR objects and particles, it is possible to find some correspondence of the SSR objects with models of the particles consisted of *quarks* (the SSR of ASR objects) and even with the *nuclei* (the SSR of MSR objects).

3 Quantum SR objects

If the SR object in K' exists, it needs to be a source of some "influence" in K, otherwise we would not know about it. We consider, that this "influence" is well known in K (we do not mean quantum physics here). Furthermore, this influence cannot be an energy source in K or it has to be localized in space, overwise, the SR object would be a permanent power source or it would be unstable.

We will analyze the stable and localized SR object. Let us assume that it is a source of some influence $u(X,t)$ in K. The source function $\rho(X,t)$, corresponding to SR object, describing some its property, will be equal to zero in external region and non-zero in its internal region $\mathcal{G}$ in K. The wave-like influence $u(X,t)$ needs to satisfy the wave equation:

$$\nabla^2 u(X,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}u(X,t) = -\rho(X,t). \qquad (11)$$

Here v is a speed of the influence wave, δ is the Dirac delta function, θ is the Heaviside staircase function. Considering the SR object as a source of the wave-like influence with the frequency, corresponding to the rotation frequency ω, and supposing $k = \omega/v$ (with $v = c$, $k = \Omega$), $\rho(X,t) = \mathcal{P}(X)\exp(\pm\imath\omega t)$, $u(X,t) = U(X)\exp(\pm\imath\omega t)$ ($\imath$ is an imaginary unit), for $U(X)$ we will have the Helmholtz equation with corresponding fundamental solutions [3]:

$$\nabla^2 U(X) + k^2 U(X) = -\mathcal{P}(X), \qquad (12)$$
$$\mathcal{E}_1(X) = \pm\frac{1}{2\imath k}e^{\pm\imath k|X|}, \qquad \mathcal{E}_2(X) = \mp\frac{\imath}{4}H_0^{(1),(2)}(k|X|), \qquad \mathcal{E}_3(X) = -\frac{1}{4\pi|X|}e^{\pm\imath\, k|X|}.$$

Note that this equation gives the steady-state solutions. The question of the source function expression that we have used, is quite serious, it needs the ω-invariance hypothesis, which is introduced and considered in the next Section 4.

The 1D case. In the one-dimensional case, with $\mathcal{P}(x) = \delta(x-a) \pm \delta(x+a)$, the solution of the equation (12) can be represented as:

$$U(x) = \pm\left(e^{\pm\imath k|x+a|} \pm e^{\pm\imath k|x-a|}\right)/(2\imath k).$$

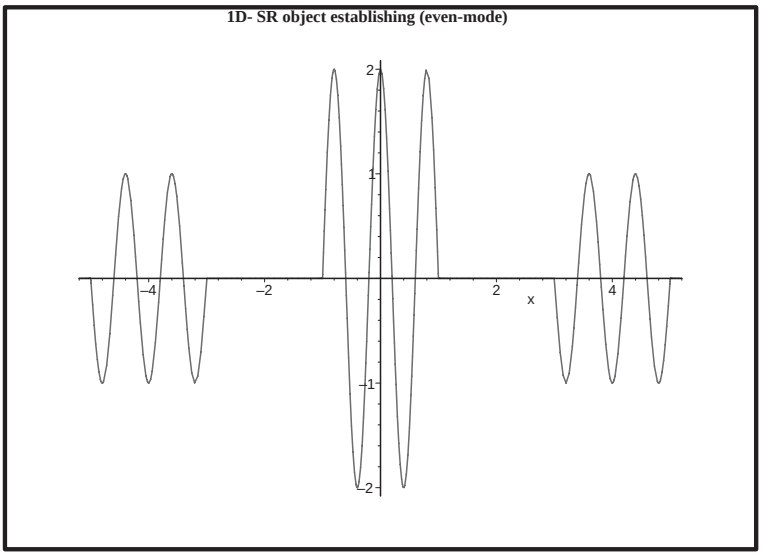

Figure 1. The 1D SR object establishing (even-mode).

In the "internal" $(-a < x < a)$ and the "external" regions $(x < -a)$ and $(x > a)$ these solutions can be represented by even and odd modes:

$$U^{\rm in}_{\rm even}(x) = \pm\frac{1}{\imath k}e^{\pm\imath ka}\cos(kx), \qquad U^{\rm in}_{\rm odd}(x) = \pm\frac{1}{k}e^{\pm\imath ka}\sin(kx),$$
$$U^{\rm ex}_{\rm even}(x) = \pm\frac{1}{\imath k}e^{\mp\imath kx}\cos(ka), \qquad U^{\rm ex}_{\rm odd}(x) = \pm\frac{1}{k}e^{\mp\imath kx}\sin(ka).$$

It follows from here that with $ka = -\pi/2 + \pi n$ for even and with $ka = \pi n$ for odd modes, where n is natural, the SR object influence in "external" regions will be equal to zero at any time, while it will not be zero in the "internal" region. The analysis of the establishing of the found study-states of the SR object by equation (11) shows (see Fig. 1) that there exist objects spreading out the SR object with the speed of influence waves. These additional objects represent the wave trains. The number of trains corresponds to the respective train number of the steady-state object. So one can tell about the "families" or "classes" of the steady-state SR objects and corresponding moving objects. Analogies with the neutrino families are very transparent.

The 3D case. In the three-dimensional case, with $\mathcal{P}(x) = \delta(r - a)$, it is possible to get the steady-state solution of the equation (12) that can be represented as (where $k = \pi n/a$, $n \in \mathcal{N}$):

$$U(r) = \frac{a}{kr}\sin[k(r-a)].$$

So, we have found that the SR object needs to satisfy to some *quantification conditions*. We will call it as a *Quantification principle*. Remarkable, that the quantification takes place without any "external" force field. See also [5].

DeBroglie wave. The SR object influence, taking into account the *quantification condition* $k_n = \omega_n/v$, may be represented as:

$$u(X,t) = \sum_n U(\omega_n, X)\exp(\imath\omega_n t). \tag{13}$$

This expression looks like *Fourier expansion* of the $u(X,t)$. So, the following Fourier expansion properties are valid for influence of the SR object:

$$U(\omega_n, X) = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} u(X,t)e^{\imath\omega_n t}dt, \qquad \left\langle u^2(X,t)\right\rangle_t = \sum_n |U(\omega_n, X)|^2 .$$

We can consider the expression (13) in an inertial to K reference frame K^L, using the Lorentz coordinate transformation (without losing the generality $X^L = (x, y, z^L)$, $\gamma = 1/\sqrt{1-\beta^2}$, $\beta = V/c$), that leads to:

$$u^L\big(X^L, t^L\big) = \sum_n \mathcal{B}(\omega_n, X^L) e^{\imath\omega_n[\gamma(t^L - \beta z^L) - t^L]}, \tag{14}$$

where $\mathcal{B} = U(\omega_n, X^L)e^{\imath\omega_n t^L}$ and (14) looks like the *de Broglie* wave, corresponding to particle. Together with the observed Fourier expansion properties, it may have the probabilistic interpretation. Note, that in (14) influence of the part $e^{\imath\omega_n t^L}$ on the particle behavior is neglected.

4 Description of SR frames

Let us consider the normalized MSR with the periodical rotation matrix $A^{\mathrm{MSR}} = A(t) = A(t + 2\pi n/\omega)$, where n is integer, and fix two events (t_1, X_1) and (t_2, X_2). For corresponding space points X_2' and X_1' in K', one can get by the SR definition:

$$\Delta X'_{|2\pi n/\omega} = X_2' - X_1' = X_2 A(t_2) - X_1 A(t_1) = \Delta X \cdot A_{|2\pi n/\omega}. \tag{15}$$

If the SR matrix is normalized ($\det A = 1$), the distance between two space points in K and K' is found to be equal to each other:

$$\|\Delta X'\|^2_{|2\pi n/\omega} = \|\Delta X\|^2. \tag{16}$$

For any fixed frequency ω, the time points $t_2 = t_1 + 2\pi n/\omega$ create the numerable infinite aggregate $\Lambda : \{t_0 + 2\pi n/\omega\}_{n=0}^{\infty}$ on the t-axis. The rotation frequency ω is the initial parameter of this aggregation. On Λ the equality (16) is true, and, consequently, the interval (4), as it was defined in Minkowski space, is invariant in K and K'. We will call by definition that rotating frames K and K' are *ω-invariant*. It means that SR frames on this infinite aggregate look like Lorentz-invariant and so, we will consider, that they are *measurable* on Λ. The frequency ω defines the *scale* between two reference frames K and K'. On this approach, this parameter seems and needs to be very important in the physical object description in these frames.

Further, let us a physical object (some of its property) is described in K' by the function $\psi'(X', \tau)$ and in K by function $\psi(X, \tau)$. On Λ these functions are measurable, because the interval (4), as it was defined in Minkowski space, is invariant in K and K', and one can write the condition expression, connecting these functions:

$$\psi(X, \tau)_{|2\pi n/\omega} = \sum_l \left[\psi'_l(X', \tau) \prod_{m_l} \exp(\pm\imath m_l \Omega\tau) \right], \tag{17}$$

because for any natural l and integer m_l, $\exp(\pm\imath m_l\Omega\tau)_{|2\pi n/\omega} = 1$, where $\psi'(X', \tau) = \sum\limits_l \psi'_l(X', \tau)$, and in these points $X'_{|2\pi n/\omega} = X$.

Furthermore, we will replace Λ by the real t-axis. It means that we are also replacing the discrete set on X and X', so that $X = X'$. Finally, we have got the expression, connecting functions ψ in K and ψ' in K' on Λ:

$$\psi(X, \tau) = \sum_l \left[\psi'_l(X, \tau) \prod_{m_l} \exp(\pm\imath m_l \Omega\tau) \right]. \tag{18}$$

At any, even very high values of ω, this approximation may be quite accurate, but always not complete.

Following these assumptions, we can conclude that the introduced extended $\widehat{\mathcal{O}}(3)$ group due to ω-invariance hypothesis transforms to the usual $\mathcal{O}(3)$ group and also the Minkowski space metrics is valid for this object. This is already enough to get the basic quantum mechanics equations such as Klein–Gordon–Fock, Schrödinger and Dirac in the fashion standard in quantum theory [4].

The SR theory can explain a lot of difficulties and postulates of the quantum theory. We will illustrate it by obtaining the Klein–Gordon–Fock equation for a scalar particle without spin in the "SR theory fashion style".

KGF equation. For simplicity, we will analyze the case l, $m_l = 1$ of expression (18):

$$\psi(X,\tau) = \psi'(X,\tau)\exp(\pm\imath\Omega\tau). \tag{19}$$

Here we have got the expression (19), we have already used in Section 3 for source function. So, it has a meaning of the SR frames function transformation.

Remind that on Λ $X = X'$ and we can consider $\nabla_X = \nabla_{X'}$. So, the Lorentz invariant second-order differential operator, called the d'Alembertian operator, will be invariant in frames K and K':

$$\Box = \frac{\partial^2}{\partial\tau^2} - \nabla_X^2 = \frac{\partial^2}{\partial\tau^2} - \nabla_{X'}^2 = \Box'.$$

We are able to apply the d'Alembertian operator to both parts of the Equation (19). It will lead to equation:

$$\frac{1}{\psi(X,\tau)}\Box\psi(X,\tau) = \frac{1}{\psi'(X',\tau)}\Box'\psi'(X',\tau) - \Omega^2 + 2\imath\Omega\frac{1}{\psi'(X',\tau)}\frac{\partial\psi'(X',\tau)}{\partial\tau}. \tag{20}$$

The third term in the right-hand side may be neglected in comparison with the second one in case of the stable particle, stable, at least, in comparison with the period of rotation $T = 2\pi/\omega$: $\Omega \gg \left|\frac{1}{\psi'}\frac{\partial\psi'}{\partial\tau}\right|$. If the physical object in K' (it may be, for example, an electromagnetic wave) satisfies the equation $\Box'\psi'(X,\tau) = 0$, then supposing $\Omega = mc/\hbar$, one can get from (20) the *Klein–Gordon–Fock equation*:

$$\frac{\Box\psi}{\psi} + \frac{m^2c^2}{\hbar^2} = 0. \tag{21}$$

The equation (21) is valid for any inertial to K reference frames due to the Lorentz invariance of the d'Alembertian operator, that was directly shown in [5]. Obtaining of the Schrödinger equation was also shown in these papers, which is the non-relativistic approximation to the Klein-Gordon equation. As far as the ω-invariance gives the necessary symmetries, it is possible to obtain the Dirac equation by the usual way [4]. Note that (16) is not satisfied for SSR, so equations, created for SSR, apparently, will correspond to the QCD, QFT equations.

From the SR point of view, the equation (21) contains a new idea – an idea of connecting the physical object properties in different rotating reference frames. It is a consequence of the declared SR equivalence principle.

The "Einstein's formula" for the SR. The invariant $E^2/c^2 - \vec{p}\cdot\vec{p} = m^2c^2$ given by the energy-momentum 4-vector of a particle $p^\mu = (E/c, \vec{p})$ in Quantum mechanics corresponds to the KGF equation. An electromagnetic wave in K' gives for this invariant $E'^2/c^2 - \vec{p}'\cdot\vec{p}' = 0$. For the stable and localized object in K ($\vec{p} = \vec{0}$) this invariant will be $E^2/c^2 = m^2c^2$, so from (20) we can get the well-known Einstein's formula:

$$E^2/c^2 = C(\Omega) = m^2(\Omega)c^2. \tag{22}$$

Note that the "influence" of the SR object (Section 3) may be interpreted as an additional mass to (22), but it has another origin and can be "separated" from the real rest mass.

5 Space rotation theory

It seems that the attempt to create the system of the physical reality description, undertaken in this paper, was quite successful and we may declare the result as a Space rotation theory. The SR theory is based on the SR equivalence, the quantification principle and the ω-invariance hypothesis.

The SR equivalence principle is based on the evolution of the ideas of relativity and on the suggestion of the space rotation equivalence, extends the conception of the space symmetry of the $\mathcal{O}(3)$ space rotation groups. This extended rotation group $\widehat{\mathcal{O}}(3)$ adds some very important physical characteristics to space-time and SR object descriptions. The stable-state objects, correlated with known quantum physics objects, are found.

The quantification principle is based on the usual energy conservation law for found SR objects. It does not need any additional postulates to explain the quantum properties of existing particles and fields.

The ω-invariance hypothesis makes SR frames measurable, but, of course, with some limitations. These limitations are coming from the SR theory; they are not postulated or introduced. They are in a good agreement with the quantum physics postulates and paradoxes.

From the SR theory point of view, the Quantum physics is an effort of the approximate description of the immeasurable SR systems, because in quantum physics the numerable infinite aggregate on t-axis is replaced by the continuous t-axis. At any, even very high values of ω, this approximation may be quite accurate, but always not complete. On this approach, it becomes clear, that the ω-invariance is the reason of the *uncertainties* in quantum physics, its incompleteness and formalism [6]. M. Gell-Mann [7] characterized the quantum physics as a discipline "... full of mysteries and paradoxes, that we do not completely understand, but are able to use. As we know, it perfectly operates in the physics reality description, but as sociologists would say, it is an anti-intuitive discipline. The quantum physics is not a theory, but limits, in which, as we suppose, any correct theory needs to be included". Now, we can clearly see it from SR theory point of view.

All these facts make us declare the "initial", basic origin of the SR theory in comparison with the quantum physics. We may suppose, that "new" interactions, declared in quantum physics, such as strong or electroweak, are reflections of the "usual" forces, existing in rotating frames, to the observer frame and also "influence" of the SR objects in our frame. This is a way to the Unification of these forces.

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The Sigma-Model Representation for the Duality-Symmetric $D = 11$ Supergravity

Alexei J. NURMAGAMBETOV

Institute for Theoretical Physics, NSC "Kharkiv Institute of Physics and Technology", 1 Akademicheskaya Str., 61108 Kharkiv, Ukraine
E-mail: *ajn@kipt.kharkov.ua*

In this paper which is based on the results obtained in collaboration with Igor Bandos and Dmitri Sorokin I present the extention of the "doubled fields" formalism by Cremmer, Julia, Lü and Pope to the supersymmetric case and lifting this construction onto the level of the proper duality-symmetric action for $D = 11$ supergravity. Further extension of the doubled field formulation to include dynamical sources is also discussed.

1 Introduction

This year we can truly commemorate the 25th anniversary from the day when eleven-dimensional supergravity [1] was born. The importance of this event is worth mentioning since supergravity in $D = 11$ space-time dimensions is a kind of special theory possessing a number of non-trivial features. First of all, this theory is the maximally-dimensional supersymmetric theory of gravity interacting with antisymmetric tensor fields for the standard signature of metric with one time-like direction. Second, the theory has relatively simple structure of the action in comparison to other low-dimensional maximal supergravities. The third point consists in the observation that the massless spectrum of a theory derived upon a compactification from $D = 11$ down to $D = 4$ does not contain particles with spin higher than two that is an indication of having a consistent four-dimensional interacting theory. Last but not least the eleven-dimensional supergravity is the low-energy limit of M-theory unifying non-perturbatively five different theories of superstrings. Since not much has been known about M-theory, $D = 11$ supergravity is very useful tool to investigate the M-theory structure.

One of the ways to do so is to consider different routes of dimensional reduction to establish the connection between different low-energy effective actions for superstrings which are nothing but supergravities and to find a relevant scheme that would be in accordance with four-dimensional phenomenology. Following this way another notion, the duality, becomes important and allows us to establish the symmetry structure in reduced theories which is originally hidden. Moreover, the duality gives a fresh look on the symmetry structure in higher-dimensional supergravities that allows one to find new way for describing in a uniform duality invariant manner gauge and internal symmetries of these theories.

It should be emphasized that the notion of duality in the context of searching new symmetries in dimensionally reduced supergravities [2] appeared one year after discovering the eleven-dimensional supergravity. There discovered was a hidden symmetry structure of maximal low-dimensional supergravities that extends the naively expected global symmetry groups arising in the process of dimensional reduction on multidimensional tori. Some important observations were made that were crucial to develop the technique to derive later on the low-dimensional theories of supergravities by reduction on spheres and manifolds with non-trivial holonomy groups.

The aim of this paper is twofold. First, we would like to review the old results reminding ideas and methods that led to the discovery of hidden global symmetry structure in maximal

supergravities and to trace how these ideas and methods naturally get transformed for the description of local symmetries in such theories [4]. And second, we present new results [8] which generalize the recent construction of local symmetries in uniform duality invariant manner. Though we will note only the key points that are necessary for the discussion in what follows, all the details can be read off original papers [2–6, 8].

2 Hidden global symmetries in maximal supergravities

Let me get started with recalling that D-dimensional maximal supergravity is the theory which follows from D=11 supergravity by dimensional reduction on torus $T^{(11-D)}$. Typically, such a theory is expected to be invariant under the product of a non-compact global group and a compact local group $SL(11-D,\mathbb{R})_{\rm Global} \otimes SO(11-D)_{\rm Local}$. The former group acts on tensors and scalars, while the latter one acts on fermions and scalars. However, close inspection reveals the hidden symmetry group structure extending the structure above. This is due to "conspiracy" of scalars and tensors coming in the reduction process from gravitational and tensor gauge fields sectors and of scalars and tensors appearing after dualizing a higher p-rank tensor field with $p \geq D/2$ into a $(D-2-p)$-rank tensor field. Recasting the fields the following symmetry structure can be recovered [6]

D	$G_{\rm Global}$	$H_{\rm Local}$
$D=11$	$E_0 \sim \mathbf{1}$	$\mathbf{1}$
$D=10$	$E_1 \sim SO(1,1)/Z_2$	$\mathbf{1}$
$D=9$	$E_2 = GL(2,\mathbb{R})$	$SO(2)$
$D=8$	$E_{3(+3)} = SL(3,\mathbb{R}) \otimes SL(2,\mathbb{R})$	$U(2) = SO(3) \otimes SO(2)$
$D=7$	$E_{4(+4)} = SL(5,\mathbb{R})$	$Usp(4) = SO(5)$
$D=6$	$E_{5(+5)} = SO(5,5)$	$Usp(4) \otimes Usp(4)$
$D=5$	$E_{6(+6)}$	$Usp(8)$
$D=4$	$E_{7(+7)}$	$SU(8)$
$D=3$	$E_{8(+8)}$	$SO(16)$

Let me now introduce the basic notion of a twisted self-duality condition which will be important in what follows. To this end note that when the dimension D is odd the E_{11-D} symmetry is realized on the gauge potentials and is an invariance of the Lagrangian. However, as soon as the space-time dimension is even, the E_{11-D} symmetry for field strengths of degree $D/2$ is realized on the field strengths rather than on the potentials and is an invariance of the equations of motion and Bianchi identities only. But one can still realize such a symmetry at the level of equations of motion on the potentials if one introduces their additional dual partners. After that the equations of motion take the form of the twisted self-duality conditions [2]

$$\mathcal{V}\mathcal{F}^{(D/2)} = \Omega\mathcal{V} * \mathcal{F}^{(D/2)}, \tag{1}$$

where Ω is the symplectic metric of the Sp group that contains the E_{11-D} as a subgroup, $\mathcal{V}$ is the matrix of scalars in the SL subgroup of the E_{11-D} and $*$ stands for D-dimensional Hodge operator.

The meaning of the twisted self-duality conditions is clear. As soon as we introduced additional dual potentials, the number of degrees of freedom for corresponding tensor fields became twice higher than we need. The twisted self-duality condition reduces this number going back to the standard situation. However, curiously, the corresponding equations of motion for our theory have been encoded in these conditions.

What we have learned from these observations? We should emphasize the role of duality in the establishing the global hidden symmetries in maximal supergravities. Later on dualities

were recognized as a basic tool to establish hidden non-perturbative symmetries of M-theory. The next point is that we have another description for the dynamics of self-dual fields having the field strengths of the degree of $D/2$. Two questions arise: Could we apply similar approach to describe the dynamics of fields dual to each other with respect to the Hodge operation and which are not necessary self-dual? And could we generalize the construction to describe local symmetries of the theory?

The answer came from the observation made by Cremmer, Julia, Lü and Pope [4]. Motivated by studying the global symmetry groups in maximal supergravities in the context of the M-theory conjecture, the authors observed that doubling the gauge fields, as well as dilatonic scalars and axionic scalars by their duals one can present the bosonic equations of motion for this sector of supergravities, starting from $D = 11$ supergravity, in the form which generalizes the twisted self-duality conditions

$$*\mathcal{G} = \mathcal{S}\mathcal{G}. \tag{2}$$

Here $\mathcal{G}$ is a combined single field written in terms of the exponential of linear combinations of generators for each field and its double, with all the potentials as coefficients. $\mathcal{S}$ is a pseudo-involution that exchanges the generators for fields and those for their partners under the doubling. Complete construction is a remnant of G/H coset structure of scalar fields well known in supergravities and can be regarded as its generalization. Since the dynamics of scalars is described by the G/H sigma-model action one can expect the sigma-model structure for "doubled fields action" constructed out of $\mathcal{G}$, if the action does exist. To find such an action, let me be more precise confining myself to the dimension eleven space-time and presenting the details of the construction by Cremmer, Julia, Lü and Pope for $D = 11$ supergravity. Although this case is simple in comparison with even ten-dimensional type IIA supergravity obtained from $D = 11$ supergravity by dimensional reduction, it is generic and possesses all features for the doubled field approach.

3 Doubled field formulation of $D = 11$ supergravity

From the Lagrangian for the bosonic sector of $D = 11$ supergravity [1]

$$\mathcal{L} = -\sqrt{g}\ R - \frac{1}{48}\sqrt{g}\ F_{mnpq}F^{mnpq} - \frac{1}{3}A^{(3)} \wedge F^{(4)} \wedge F^{(4)}, \qquad F^{(4)} = dA^{(3)} \tag{3}$$

we get the second order equation of motion for the $A^{(3)}$ gauge field

$$d(*F^{(4)} - A^{(3)} \wedge F^{(4)}) = 0 \tag{4}$$

that can be presented as the Bianchi identity for the dual field $A^{(6)}$

$$dF^{(7)} = 0, \qquad F^{(7)} = dA^{(6)} + A^{(3)} \wedge F^{(4)}. \tag{5}$$

Now forget for a while the dynamical origin of $F^{(7)} = dA^{(6)} + A^{(3)} \wedge F^{(4)}$ and introduce it as an independent partner of $F^{(4)}$. These field strengths are invariant under the local gauge transformations

$$\delta A^{(3)} = \Lambda^{(3)}, \qquad \delta A^{(6)} = \Lambda^{(6)} - \Lambda^{(3)} \wedge A^{(3)} \tag{6}$$

with closed forms $\Lambda^{(3)}$, $\Lambda^{(6)}$ associated with the so-called large gauge transformations [7]. Because of the presence of "bare" $A^{(3)}$ in $F^{(7)}$ and therefore in $\delta A^{(6)}$ that is traced back to the

presence of the Chern–Simons term in the Lagrangian (3) the large gauge transformations are non-Abelian

$$[\delta_{\Lambda_1^{(3)}}, \delta_{\Lambda_2^{(3)}}] = \delta_{\Lambda^{(6)}}, \qquad [\delta_{\Lambda^{(3)}}, \delta_{\Lambda^{(6)}}] = [\delta_{\Lambda_1^{(6)}}, \delta_{\Lambda_2^{(6)}}] = 0. \tag{7}$$

These relations can be associated with a superalgebra generated by a "Grassmann-odd" generator t_3 and a commuting generator t_6

$$\{t_3, t_3\} = -2t_6, \qquad [t_3, t_6] = [t_6, t_6] = 0 \tag{8}$$

after that one can realize an element of the supergroup

$$\mathcal{A} = e^{t_3 A^{(3)}} e^{t_6 A^{(6)}} \tag{9}$$

and introduce the Cartan form

$$\mathcal{G} = d\mathcal{A}\mathcal{A}^{-1} = F^{(4)} t_3 + F^{(7)} t_6, \tag{10}$$

which by definition satisfies the Maurer–Cartan equation called sometimes the zero-curvature condition

$$d\mathcal{G} + \mathcal{G} \wedge \mathcal{G} = 0. \tag{11}$$

To impose the duality relation between a priori independent field strengths and arriving therefore at the standard number of degrees of freedom one introduces the pseudo-involution operator $\mathcal{S}$ which interchanges the generators t_3 and t_6

$$\mathcal{S}t_3 = t_6, \qquad \mathcal{S}t_6 = t_3, \qquad \mathcal{S}^2 = 1. \tag{12}$$

Using $\mathcal{S}$ and the Hodge operator one can immediately check that the following condition

$$*\mathcal{G} = \mathcal{S}\mathcal{G} \tag{13}$$

reproduces correctly the duality relations between the field strenghts and therefore reduces tensors' degrees of freedom to the correct number. Moreover, when this condition holds the Maurer-Cartan equation amounts to second order equations of motion for $F^{(4)}$ and $F^{(7)}$.

Since the relation (13) enjoys all necessary properties of the condition (1) it also shares the name of the twisted self-duality condition.

So, we have explicitly described the doubled field approach to the bosonic sector of $D = 11$ supergravity. However, we should still add the fermions and construct a proper action for the doubled fields. We are now turning to these points.

4 Generalization of doubled field approach to fermions and lifting it onto the level of action

Adding the fermions is a simple task since what one should do is to extend $\mathcal{G}$ with the superalgebra valued element $\mathcal{C} = -C^{(4)} t_3 + C^{(7)} t_6$ [8]

$$\mathcal{G} \longrightarrow \mathcal{G} + \mathcal{C}, \tag{14}$$

where $C^{(4)}$ and $C^{(7)}$ are defined as

$$C^{(4)} = -\frac{1}{4}\bar{\psi} \wedge \Gamma^{(2)} \wedge \psi, \qquad C^{(7)} = \frac{i}{4}\bar{\psi} \wedge \Gamma^{(5)} \wedge \psi,$$
$$\Gamma^{(n)} = \frac{1}{n!} dx^{m_n} \wedge \cdots \wedge dx^{m_1} \Gamma^{(n)}_{m_1 \cdots m_n}, \tag{15}$$

and ψ^α is the one-form associated with the Majorana spin 3/2 gravitino ψ^α_m.

After this replacement the twisted self-duality condition becomes

$$*(\mathcal{G}+\mathcal{C}) = \mathcal{S}(\mathcal{G}+\mathcal{C}) \longrightarrow (\mathcal{S}-*)(\mathcal{G}+\mathcal{C}) = 0. \tag{16}$$

Therefore, if one finds a way to construct the action from which the twisted self-duality condition (16) will follow as an equation of motion the task will be completed. Fortunately, this way exists and is based on the PST technique [9] developed to construct the Lagrangians in theories with self-dual or duality-symmetric fields.

Applying the PST technique the twisted self-duality condition is reproduced from the following action [8]

$$\begin{aligned} S = S_{EH} + S_{\psi} - \mathrm{Tr}\int_{\mathcal{M}^{11}} \frac{1}{2}\left(\mathcal{G}+\frac{1}{2}\mathcal{C}\right)\wedge(\mathcal{S}-*)\mathcal{C} \\ - \mathrm{Tr}\int_{\mathcal{M}^{11}}\left[\frac{1}{4}*\mathcal{G}\wedge\mathcal{G} - \frac{1}{12}\mathcal{G}\wedge\mathcal{S}\mathcal{G} - \frac{1}{4}*i_v(\mathcal{S}-*)(\mathcal{G}+\mathcal{C})\wedge i_v(\mathcal{S}-*)(\mathcal{G}+\mathcal{C})\right], \end{aligned} \tag{17}$$

where S_{EH} and S_{ψ} stand for the Einstein–Hilbert action and for the fermionic kinetic term [1],

$$v = \frac{da(x)}{\sqrt{-(\partial a)^2}} \tag{18}$$

is the one-form constructed out the PST scalar auxiliary field (cf. [9]) which ensures the covariance of the action and

$$\mathrm{Tr}\,(t_3 t_3) = -\mathrm{Tr}\,(t_6 t_6) = -1, \qquad \mathrm{Tr}\,(t_3 t_6) = 0. \tag{19}$$

As usual we have denoted by i_v the inner product of the vector field v_m with a form.

It is worth mentioning that the action just presented is the generalization of the sigma-model action having typically the form

$$S \propto \int_{\mathcal{M}^D} \mathrm{Tr}\,(*\mathcal{G}\wedge\mathcal{G}) \tag{20}$$

and therefore we have derived the result which we expected from the beginning.

It is an instructive exercise to rewrite the sigma-model action to the standard for $D = 11$ supergravity form. After some manipulations with taking into account the definitions of $\mathcal{G}$ and $\mathcal{S}$ one can arrive at

$$S = S_{CJS} + \int_{\mathcal{M}^{11}} \frac{1}{2} i_v\mathcal{F}^{(4)}\wedge *i_v\mathcal{F}^{(4)} \tag{21}$$

with S_{CJS} being the standard action by Cremmer, Julia and Scherk [1] and $\mathcal{F}^{(4)} = F^{(4)} - *F^{(7)}$. This is the action for the duality-symmetic $D = 11$ supergravity [10] from which one can dynamically derive the duality condition $\mathcal{F}^{(4)} = 0$. Note also that this action depends on both $A^{(3)}$ and $A^{(6)}$ potentials therefore the name duality-symmetric, and on the shell of the duality condition $\mathcal{F}^{(4)} = 0$ it coincides with the Cremmer–Julia–Scherk action.

5 The sigma-model representation of $D = 11$ supergravity coupling to the M-sources

It is well known that a M2 and a M5 branes are dynamical sources for $A^{(3)}$ and $A^{(6)}$ $D = 11$ supergravity gauge fields [10–12] and therefore one may wonder what is the way of extension of

the doubled fields formalism to describe dynamical coupling of $D = 11$ supergravity to the M-branes' sources? To get the answer let me consider the case of coupling of $D = 11$ supergravity to single closed M2 and M5 branes which is naively described by the following set of Bianchi identities and equations of motion (cf. [11, 10, 13, 14])

$$\begin{aligned} &d\hat{F}^{(4)} = *J_m^{(6)}, \\ &d * \hat{F}^{(4)} = \hat{F}^{(4)} \wedge \hat{F}^{(4)} - 2h^{(3)} \wedge *J_m^{(6)} + *J_e^{(3)}, \\ &h^{(3)} = \star h^{(3)} + \text{n. l. t.}, \end{aligned} \tag{22}$$

where

$$\hat{F}^{(4)} = dA^{(3)} + *G_m^{(7)}, \qquad d * G_m^{(7)} = *J_m^{(6)}, \qquad d * G^{(4)} = *J_e^{(3)},$$

and

$$h^{(3)} = db^{(2)} - A^{(3)}$$

is the gauge invariant M5 worldvolume field strength constructing out of the self-dual worldvolume gauge field $b^{(2)}$ and the pullback of the target-space gauge field $A^{(3)}$. The last equation of (22) that generalizes ordinary self-duality condition contains non-linear corrections coming from the Dirac–Born–Infeld-like structure of the M5 kinetic part (see [15–19] for details) and $J_{e,m}$ are the electric and magnetic currents due to the M2 and M5 branes. To distinguish the Hodge operation in target space and on the worldvolume of branes we have denoted the latter by $\star$.

It is easy to verify that the equation of motion of $A^{(3)}$ gauge field leads to the following expression for the field strength dual to $\hat{F}^{(4)}$

$$\hat{F}^{(7)} = dA^{(6)} + A^{(3)} \wedge dA^{(3)} - 2h^{(3)} \wedge *G_m^{(7)} + *G_e^{(4)}, \tag{23}$$

if one admits regularization for ill-defined product of two delta-functions $*G_m^{(7)} \wedge *G_m^{(7)}$ entering the $A^{(3)}$ equation of motion with $*G_m^{(7)} \wedge *G_m^{(7)} = 0$ [20, 21]. Such a regularization has been used in [10] and is good enough as far as the presence of classical gravitational anomaly due to the M5-brane is neglected that we will suppose for the sake of simplicity. However, it should be emphasized that the appearance of product of delta-functions is "a symptom of attempting to treat in classical supergravity what really should be treated in quantum M-theory" [22]. This problem has been approached and solved in a rigorous way using a notion of distributions in [13, 14].

The relevant part of the action which leads to the set of equations (22) has the following generating for the doubled field sigma-model action form

$$\begin{aligned} \mathcal{L}^{(11)} = &\frac{1}{4}\hat{F}^{(4)} * \hat{F}^{(4)} + \frac{1}{4} i_v\hat{\mathcal{F}}^{(4)} * i_v\hat{\mathcal{F}}^{(4)} - \frac{1}{4}\hat{F}^{(7)} * \hat{F}^{(7)} - \frac{1}{4} i_v\hat{\mathcal{F}}^{(7)} * i_v\hat{\mathcal{F}}^{(7)} + \frac{1}{6}F^{(4)}F^{(7)} \\ &+ \left[\mathcal{L}^{(6)}_{\text{M5 kin.}} - \frac{1}{2}(A^{(6)} + db^{(2)}A^{(3)})\right] * J_m^{(6)} + \left[\mathcal{L}^{(3)}_{\text{M2 kin.}} - \frac{1}{2}A^{(3)}\right] * J_e^{(3)} \\ &- \frac{1}{2}h^{(3)}\hat{F}^{(4)} * G_m^{(7)}. \end{aligned} \tag{24}$$

To present the latter as the sigma-model-like action one should extend the Cartan form $\mathcal{G}$ with the superalgebra valued element $\mathbf{g} = t_6\big(* G_e^{(4)} - 2h^{(3)} * G_m^{(7)}\big) + t_3 * G_m^{(7)}$, i.e.

$$\mathcal{G} = d\mathcal{A}\mathcal{A}^{-1} \to \mathcal{G} = d\mathcal{A}\mathcal{A}^{-1} + \mathbf{g}$$

after that the twisted self-duality condition becomes

$$*\mathcal{G} = \mathcal{S}\mathcal{G}, \tag{25}$$

but the zero-curvature condition is realized now as

$$d\tilde{\mathcal{G}} + \tilde{\mathcal{G}} \wedge \tilde{\mathcal{G}} = 0, \qquad \tilde{\mathcal{G}} = \mathcal{G} - \mathbf{g}. \tag{26}$$

When the twisted self-duality condition holds, i.e. when

$$*\hat{F}^{(4)} = \hat{F}^{(7)}, \qquad *\hat{F}^{(7)} = \hat{F}^{(4)}, \tag{27}$$

equations that follow from the zero-curvature condition (26)

$$dF^{(4)} = 0, \qquad dF^{(7)} = F^{(4)} \wedge F^{(4)}$$

are the same second order equations of motion that follow from the action (24).

Therefore, the sigma-model doubled field action describing the coupling of closed M2 and M5 branes to $D = 11$ supergravity has the following form

$$\begin{aligned} S = S_{EH} - \mathrm{Tr} \int_{\mathcal{M}^{11}} \left\{ \frac{1}{4} * \mathcal{G} \wedge \mathcal{G} - \frac{1}{12} \tilde{\mathcal{G}} \wedge \mathcal{S}\tilde{\mathcal{G}} - \frac{1}{4} * i_v(\mathcal{S} - *)\mathcal{G} \wedge i_v(\mathcal{S} - *)\mathcal{G} \right\} \\ + \int_{\mathcal{M}^3} \left[e^a \wedge \star e^b \eta_{ab} - \frac{1}{2} A^{(3)} \right] + \int_{\mathcal{M}^6} \mathcal{L}^{(6)}_{\text{M5 kin.}} - \frac{1}{2} \int_{\mathcal{M}^6} \left[A^{(6)} + db^{(2)} \wedge A^{(3)} \right] \\ - \frac{1}{2} \int_{\mathcal{M}^{11}} h^{(3)} \wedge \hat{F}^{(4)} \wedge * G^{(7)}_m. \end{aligned} \tag{28}$$

6 Discussion

In conclusion let me summarize the obtained results. Following the doubled field approach by Cremmer, Julia, Lü and Pope we have filled the gap in extending this construction to add fermions and have formulated the supersymmeric action in such a way that the twisted self-duality condition which encodes the dynamics of the gauge fields follows from the action as an equation of motion. As the next step towards a fully fledged theory we have extended the construction to include the dynamical sources for antisymmetric gauge fields and have presented the action describing the system of $D = 11$ supergravity interacting with dynamical branes.

The structure of the doubled field action is generic for all maximal supergravities although the relevant superalgebra becomes much more complicated (cf. [4]). The case of $D = 10$ type IIB supergravity, which cannot be recovered by dimensional reduction from $D = 11$ supergravity, also admits the relevant superalgebra structure. This fact motivates understanding the T-duality between type IIA and type IIB theories in framework of the doubled field approach. In our formulation we left the gravity sector out of consideration due to the absence of clear understanding how to present that in a duality-symmetric manner. Having such a recipe it will be interesting to realize the sigma-model construction for the complete sector of fields of $D = 11$ supergravity that possibly could give a new insight on the mysterious M-theory structure.

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Splitting the Kemmer–Duffin–Petiau Equations

Andrzej OKNINSKI

Physics Department, Politechnika Swietokrzyska, Al. 1000-lecia PP7, 25-314 Poland
E-mail: *fizao@tu.kielce.pl*

We study internal structure of the Kemmer–Duffin–Petiau equations for spin-0 and spin-1 mesons. We demonstrate, that the Kemmer–Duffin–Petiau equations can be splitted into constituent equations, describing particles with definite mass and broken Lorentz symmetry. We also show that solutions of the three-component constituent equations fulfill the Dirac equation.

1 Introduction

In recent years there has been a renewed interest in the Kemmer–Duffin–Petiau (KDP) theory describing spin-0 and spin-1 mesons [1] due to discovery of a new conserved four-vector current with positive zeroth component [2], which can be thus interpreted as a probability density. A progress was also made in demonstrating equivalence of the KDP and the Klein–Gordon equations, especially when interactions are taken into account, c.f. [3] and references therein. The KDP equations has been also studied in the context of electromagnetic interactions [4–6], parasupersymmetric quantum mechanics [4], EPR type nonlocality [7], and Riemann–Cartan space-time [8].

It is well known that the KDP equations contain redundant components – only $2\,(2s+1)$ components are needed to describe free spin-s particles with nonzero rest masses [9] while spin-0 and spin-1 KDP equations contain 5 and 10 components, respectively. The presence of redundant components in KDP equations leads for some interactions to nonphysical effects such as superluminal velocities [10, 9]. It is possible, however, to obtain physically acceptable equations for arbitrary spin removing redundant components with use of additional covariant condition [9]. On the other hand, presence of redundant components suggests that the KDP equations possess internal structure. The aim of the present paper is to investigate this inner structure. We shall describe a systematic procedure of splitting (five-component) spin-0 and (ten-component) spin-1 KDP equations by means of the spinor calculus into pairs of constituents equations with smaller numbers of components, such that solutions of the latter equations fulfill the initial KDP equations. Since mesons are spin-0 and spin-1 quark-antiquark bound states it is tempting to recognize the resulting equations as quark equations. Indeed, we shall show that solutions of constituent equations fulfill the Dirac equation.

The paper is organized as follows. In Section 2 the Kemmer–Duffin–Petiau equations for spin 0 and spin 1 are described, and necessary definitions and conventions are given. In Section 3 splitting of the KDP equations into three-component constituent equations is achieved for $s = 0$ (obtaining in a new way our previous result [11]). Main results are described in the last two Sections. In Section 4 we interpret the constituent equations finding direct relation with the Dirac equation. In the last Section we outline the procedure of splitting the KDP equations for $s = 1$, and discuss our results in the light of several current problems of quark theory.

2 Kemmer–Duffin–Petiau equations

In what follows tensor indices are denoted with Greek letters, $\mu = 0, 1, 2, 3$. We shall use the following convention for the metric tensor: $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and we shall always sum over repeated indices. Four-momentum operators are defined in natural units ($c = 1$, $\hbar = 1$) as $p^\mu = i\frac{\partial}{\partial x_\mu}$.

The KDP equations for spin 0 and 1 are written as:

$$\beta_\mu p^\mu \Psi = m\Psi, \tag{1}$$

with 5×5 and 10×10 matrices β^μ, respectively, which fulfill the following commutation relations [1]:

$$\beta^\lambda \beta^\mu \beta^\nu + \beta^\nu \beta^\mu \beta^\lambda = g^{\lambda\mu}\beta^\nu + g^{\nu\mu}\beta^\lambda. \tag{2}$$

In the case of 5×5 (spin-0) representation of β^μ matrices equation (1) is equivalent to the following set of equations:

$$\begin{aligned} &p^\mu \psi = m\psi^\mu, \\ &p_\nu \psi^\nu = m\psi, \end{aligned} \tag{3}$$

$\mu, \nu = 0, 1, 2, 3$, if we define Ψ in (1) as:

$$\Psi = (\psi^\mu, \psi)^T = \left(\psi^0, \psi^1, \psi^2, \psi^3, \psi\right)^T, \tag{4}$$

where $()^T$ denotes transposition of a matrix. Let us note that equation (3) can be obtained by factorizing second-order derivatives in the Klein–Gordon equation $p_\mu p^\mu \, \psi = m^2 \psi$.

In the case of 10×10 (spin-1) representation of matrices β^μ equation (1) reduces to:

$$\begin{aligned} &p^\mu \psi^\nu - p^\nu \psi^\mu = m\psi^{\mu\nu}, \\ &p_\mu \psi^{\mu\nu} = m\psi^\nu, \end{aligned} \tag{5}$$

$\mu, \nu = 0, 1, 2, 3$, with the following definition of Ψ in (1):

$$\Psi = \left(\psi^{\mu\nu}, \psi^\lambda\right)^T = \left(\psi^{01}, \psi^{02}, \psi^{03}, \psi^{23}, \psi^{31}, \psi^{12}, \psi^0, \psi^1, \psi^2, \psi^3\right)^T, \tag{6}$$

where ψ^λ are real and $\psi^{\mu\nu}$ are purely imaginary (in alternative formulation we have $-\partial^\mu \psi^\nu + \partial^\nu \psi^\mu = m\psi^{\mu\nu}$, $\partial_\mu \psi^{\mu\nu} = m\psi^\nu$, where ψ^λ, $\psi^{\mu\nu}$ are real). Because of antisymmetry of $\psi^{\mu\nu}$ we have $p_\nu \psi^\nu = 0$ what implies spin 1 condition. The set of equations (5) was first written by Proca [1].

3 Splitting the spin-0 Kemmer–Duffin–Petiau equations

Equations (3) can be written within spinor formalism as:

$$\begin{aligned} &p^{A\dot{B}} \psi = m\psi^{A\dot{B}}, \\ &p_{A\dot{B}} \psi^{A\dot{B}} = 2m\psi, \end{aligned} \tag{7}$$

$A = 1, 2$, $\dot{B} = \dot{1}, \dot{2}$, where the spinor components are defined as [1]:

$$p^{A\dot{B}} = \left(p^0 \sigma^0 + \sigma \cdot \boldsymbol{p}\right)^{A\dot{B}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}, \tag{8}$$

where σ^0 is the 2×2 unit matrix and σ^i, $i = 1, 2, 3$, are the Pauli matrices (analogously, $\psi^{A\dot{B}} = \left(\psi^0\sigma^0 + \boldsymbol{\psi} \cdot \boldsymbol{p}\right)^{A\dot{B}}$).

Splitting the last of equations (7), $p_{A\dot{B}}\psi^{A\dot{B}} = p_{1\dot{1}}\psi^{1\dot{1}} + p_{2\dot{1}}\psi^{2\dot{1}} + p_{1\dot{2}}\psi^{1\dot{2}} + p_{2\dot{2}}\psi^{2\dot{2}} = 2m\psi$, we obtain two sets of equations involving components $\psi^{1\dot{1}}$, $\psi^{2\dot{1}}$, ψ and $\psi^{1\dot{2}}$, $\psi^{2\dot{2}}$, ψ, respectively:

$$\begin{aligned} &p^{1\dot{1}}\psi = m\psi^{1\dot{1}}, \\ &p^{2\dot{1}}\psi = m\psi^{2\dot{1}}, \\ &p_{1\dot{1}}\psi^{1\dot{1}} + p_{2\dot{1}}\psi^{2\dot{1}} = m\psi; \end{aligned} \tag{9}$$

$$\begin{aligned} &p^{1\dot{2}}\psi = m\psi^{1\dot{2}}, \\ &p^{2\dot{2}}\psi = m\psi^{2\dot{2}}, \\ &p_{1\dot{2}}\psi^{1\dot{2}} + p_{2\dot{2}}\psi^{2\dot{2}} = m\psi, \end{aligned} \tag{10}$$

each of which describes particle with mass m (we check this substituting e.g. $\psi^{1\dot{1}}$, $\psi^{2\dot{1}}$ or $\psi^{1\dot{2}}$, $\psi^{2\dot{2}}$ into the third equations). The splitting preserving $p_\mu p^\mu \psi = m^2\psi$ is possible due to spinor identities

$$p_{1\dot{1}}p^{1\dot{1}} + p_{2\dot{1}}p^{2\dot{1}} = p_{1\dot{2}}p^{1\dot{2}} + p_{2\dot{2}}p^{2\dot{2}} = p_\mu p^\mu, \tag{11}$$

which follow directly from (8).

Thus solutions of equations (9), (10) fulfill the KDP equations (7). We described these equations in [11]. From each of equations(9), (10) an identity follows:

$$p^{2\dot{1}}\psi^{1\dot{1}} = p^{1\dot{1}}\psi^{2\dot{1}}, \tag{12a}$$

$$p^{2\dot{2}}\psi^{1\dot{2}} = p^{1\dot{2}}\psi^{2\dot{2}}. \tag{12b}$$

Equations (9), (10) can be written in matrix form:

$$\rho_\mu p^\mu \Phi = m\Phi, \tag{13}$$

where $\Phi = \left(\psi^{1\dot{1}}, \psi^{2\dot{1}}, \psi\right)^T$,

$$\begin{aligned} &\rho^0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\ &\rho^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & -i & 0 \end{pmatrix}, \qquad \rho^3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{14}$$

and

$$\tilde{\rho}_\mu p^\mu \tilde{\Phi} = m\tilde{\Phi}, \tag{15}$$

where $\tilde{\Phi} = \left(\psi^{1\dot{2}}, \psi^{2\dot{2}}, \psi\right)^T$,

$$\begin{aligned} &\tilde{\rho}^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \tilde{\rho}^1 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ &\tilde{\rho}^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \tilde{\rho}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned} \tag{16}$$

Equations (13), (15) considered together:

$$\begin{pmatrix} \rho_\mu p^\mu & \mathbf{0} \\ \mathbf{0} & \tilde{\rho}_\mu p^\mu \end{pmatrix} \begin{pmatrix} \Phi \\ \tilde{\Phi} \end{pmatrix} = m \begin{pmatrix} \Phi \\ \tilde{\Phi} \end{pmatrix}, \tag{17}$$

are Lorentz covariant since they contain all components of the spinor $\psi^{A\dot{B}}$. Obviously, all solutions of equation (17) satisfy equation (7) but the reverse is not true.

The matrices: ρ^μ, $\tilde{\rho}^\mu$ discussed above, c.f. equations (14), (16), fulfill the Tzou commutation relations [11, 4]

$$\rho^{(\lambda}\rho^{\mu}\rho^{\nu)} = g^{(\lambda\mu}\rho^{\nu)}, \tag{18}$$

more complicated then (2), where $(\lambda\ \mu\ \nu)$ is the symmetrizer. There is, however, no conjugation rule for matrices ρ^μ and $\tilde{\rho}^\mu$, for example there is no such matrix S that $\tilde{\rho}^\mu = S\rho^\mu S^{-1}$. We shall see in the next Section that a conjugation rule (charge conjugation) exists if 3×3 matrices ρ^μ are extended to 4×4 Dirac matrices γ^μ.

4 Subsolutions of the Dirac equation

We shall now interpret the constituent equations (9), (10) together with the identities (12a), (12b). Equation (9) and the identity (12a), as well as equation (10) and the identity (12b) can be written in form of the Dirac equations:

$$\begin{pmatrix} 0 & 0 & p^0+p^3 & p^1-ip^2 \\ 0 & 0 & p^1+ip^2 & p^0-p^3 \\ p^0-p^3 & -p^1+ip^2 & 0 & 0 \\ -p^1-ip^2 & p^0+p^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{1\dot{1}} \\ \psi^{2\dot{1}} \\ \chi \\ 0 \end{pmatrix} = m \begin{pmatrix} \psi^{1\dot{1}} \\ \psi^{2\dot{1}} \\ \chi \\ 0 \end{pmatrix}, \tag{19}$$

$$\begin{pmatrix} 0 & 0 & p^0-p^3 & p^1+ip^2 \\ 0 & 0 & p^1-ip^2 & p^0+p^3 \\ p^0+p^3 & -p^1-ip^2 & 0 & 0 \\ -p^1+ip^2 & p^0-p^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{2\dot{2}} \\ \psi^{1\dot{2}} \\ \chi \\ 0 \end{pmatrix} = m \begin{pmatrix} \psi^{2\dot{2}} \\ \psi^{1\dot{2}} \\ \chi \\ 0 \end{pmatrix}, \tag{20}$$

respectively, with one zero component. Equation (19) can be written as $\gamma^\mu p_\mu \Psi = m\Psi$ with spinor representation of the Dirac matrices, $\gamma^0 = \begin{pmatrix} \mathbf{0} & \sigma^0 \\ \sigma^0 & \mathbf{0} \end{pmatrix}$, $\gamma^j = \begin{pmatrix} \mathbf{0} & -\sigma^j \\ \sigma^j & \mathbf{0} \end{pmatrix}$, $j = 1,2,3$, $\gamma^5 = \begin{pmatrix} \sigma^0 & \mathbf{0} \\ \mathbf{0} & -\sigma^0 \end{pmatrix}$, $\Psi = \left(\psi^{1\dot{1}}, \psi^{2\dot{1}}, \chi, 0\right)^T$. Equation (20) can be analogously written as $\left(\gamma^0 p^0 - \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3\right)\Phi = m\Phi$, $\Phi = \left(\psi^{2\dot{2}}, \psi^{1\dot{2}}, \chi, 0\right)^T$.

We shall demonstrate now that equations (19) and (20) are charge conjugated one to another. Complex conjugation of equation (19) yields:

$$(-1)\begin{pmatrix} 0 & 0 & p^0+p^3 & p^1+ip^2 \\ 0 & 0 & p^1-ip^2 & p^0-p^3 \\ p^0-p^3 & -p^1-ip^2 & 0 & 0 \\ -p^1+ip^2 & p^0+p^3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi^{1\dot{1}} \\ \psi^{2\dot{1}} \\ \chi \\ 0 \end{pmatrix}^* = m \begin{pmatrix} \psi^{1\dot{1}} \\ \psi^{2\dot{1}} \\ \chi \\ 0 \end{pmatrix}^*, \tag{21}$$

i.e. $(-1)\left(\gamma^0 p^0 - \gamma^1 p^1 + \gamma^2 p^2 - \gamma^3 p^3\right)\Psi^* = m\Psi^*$ where * denotes complex conjugation. Acting from the left with matrix γ^3 on equation (21) we obtain equation $\left(\gamma^0 p^0 - \gamma^1 p^1 + \gamma^2 p^2 + \gamma^3 p^3\right) \times \gamma^3\Psi^* = m\gamma^3\Psi^*$, which has the same form as equation (20) (the charge conjugation matrix C is

thus defined as $C\gamma^0 \equiv \gamma^3$ [12]). Hence the initial equations (9), (10)) are charge conjugated one to another in a sense that they are charge conjugated after extension to the Dirac form.

The observations made above can be given representation independent formulation. Let us notice that the three component equations, for instance (9), (10) as well as the identities (12a), (12b), can be obtained by projecting the Dirac equation with projection operator $P_4 = \mathrm{diag}\,(1,1,1,0)$. Incidentally, there are other projection operators which lead to analogous three component equations, $P_1 = \mathrm{diag}\,(0,1,1,1)$, $P_2 = \mathrm{diag}\,(1,0,1,1)$, $P_3 = \mathrm{diag}\,(1,1,0,1)$ but we shall need only the operator P_4.

In general, we can consider subsolutions, of form $P_4\Psi$, of the Dirac equation:

$$\gamma^\mu p_\mu P_4\Psi = mP_4\Psi, \tag{22}$$

which is equivalent to (19) in the case of spinor representation of the Dirac matrices.

Accordingly, acting from the left on (22) with P_4 and $(1-P_4)$ we obtain two equations:

$$P_4\left(\gamma^\mu p_\mu\right)P_4\Psi = mP_4\Psi, \tag{23a}$$
$$(1-P_4)\left(\gamma^\mu p_\mu\right)P_4\Psi = 0. \tag{23b}$$

In the spinor representation of γ^μ matrices equation (23a) is equivalent to (9), while (23b) is equivalent to the identity (12a).

Now the projection operator can be written as $P_4 = \frac{1}{4}\left(3+\gamma^5 - \gamma^0\gamma^3 + i\gamma^1\gamma^2\right)$ (and similar formulae can be given for other projection operators P_1, P_2, P_3), i.e. all equations (22), (23a), (23b) are now given representation independent form. Let us also note that the projection operator P_4 commutes with two generators of Lorentz transformations $\gamma^0\gamma^3$ and $\gamma^1\gamma^2$ (and does not commute with other generators), i.e. is invariant under boosts in x^0x^3 plane and rotations in x^1x^2 plane. Accordingly, the three component equations are covariant with respect to such Lorentz transformations only. Let us note finally that all three component equations describe particles with definite mass and partly undefined spin.

5 Discussion

We shall now approach the problem of splitting the KDP equations for $s = 1$. Equations (5) can be written in spinor form as [1]:

$$\begin{aligned} &p_A^{\ \dot B}\zeta_{C\dot B} + p_C^{\ \dot B}\zeta_{A\dot B} = 2m\eta_{AC},\\ &p^A_{\ \dot B}\zeta_{A\dot D} + p^A_{\ \dot D}\zeta_{A\dot B} = 2m\chi_{\dot B\dot D},\\ &p_A^{\ \dot C}\chi_{\dot B\dot C} + p^C_{\ \dot B}\eta_{AC} = -2m\zeta_{A\dot B}. \end{aligned} \tag{24}$$

It is possible to split the spinor form of the KDP equations (24) to get two equations for spinors $\chi_{\dot B\dot D}$, $\zeta_{A\dot B}$ and η_{AC}, $\zeta_{A\dot B}$:

$$p_A^{\ \dot B}\zeta_{C\dot B} = m\eta_{AC}, \qquad \eta_{AC} = \eta_{CA}, \qquad p^C_{\ \dot B}\eta_{AC} = -m\zeta_{A\dot B}, \tag{25}$$

$$p^A_{\ \dot B}\zeta_{A\dot D} = m\chi_{\dot B\dot D}, \qquad \chi_{\dot B\dot D} = \chi_{\dot D\dot B}, \qquad p_A^{\ \dot D}\chi_{\dot B\dot D} = -m\zeta_{A\dot B}, \tag{26}$$

respectively. The splitting is possible due to spinor identities:

$$p^C_{\ \dot B}p_A^{\ \dot B} = -\delta^C_A p_\mu p^\mu, \qquad p_A^{\ \dot D}p^A_{\ \dot B} = -\delta^{\dot D}_{\dot B}p_\mu p^\mu. \tag{27}$$

Thus solutions of equations (25), (26) fulfill the KDP equations (24). The spinor equations (25), (26) describe spin-1 bosons [13] where spinors η_{CA}, $\chi_{\dot D\dot B}$ correspond to selfdual or antiselfdual

antisymmetric tensors $\psi^{\mu\nu}$, respectively. Each of the above equations is covariant except from space reflection, but both equations taken together are fully covariant. These equations written in tensor form, $\beta^\mu p_\mu \Psi = m\Psi$, $\Psi = [\psi_{01}, \psi_{02}, \psi_{03}, \psi_0, \psi_1, \psi_2, \psi_3]^T$ where $\psi^{\mu\nu}$ are selfdual or antiselfdual antisymmetric tensors, with 7×7 matrices β^μ fulfilling equation (18), are the Hagen–Hurley equations [14, 4].

The spinor form of either of the Hagen–Hurley equations (25), (26) can be splitted again to obtain two 3×3 equations analogous to equations (9) and (10) and a Klein–Gordon equation [15].

Let us conclude with several general remarks. We have shown that spinor formalism discloses internal structure of KDP equations which manifests itself by presence of redundant components – there are special three-component solutions of these equations. Accordingly, the meson spin-0 and spin-1 KDP equations split into pairs of three-component constituent equations, each equation describing a particle with definite mass and partly undefined spin (all three-component constituent equations discussed above are similar in a sense that their matrices ρ^μ fulfill the same commutation relations (18) [11, 4]) and in the case of spin-1 KDP equations an additional wave-function fulfilling the Klein-Gordon equation is present. Moreover, solutions of the constituent equations are subsolutions of appropriate Dirac equations and pairs of such Dirac equations, corresponding to pairs of constituent equations, are charge conjugated one to another. This last finding entitles us to conclude that Kemmer–Duffin–Petiau equations describe mesons as composed from quark-antiquark pairs. These results are consistent with quark theory of mesons [16].

Let us stress that the separation of a meson into constituents is imperfect, since although each of the constituent equations describes a massive particle, its Lorentz symmetry is broken. This offers explanation of quark confinement different than in quantum chromodynamics where linearly increasing potential energy between a quark and other quarks in a hadron is responsible for confinement [17]). We hope that our results can also cast light on the problem of spin crisis [18] since meson constituents in our theory have undefined spins. Let us assume that proton constituents (quarks) have the same nature as the meson constituents of our theory. It follows that the proton spin cannot be obtained as a sum of spins of its constituents since the constituents have (partly) undefined spins.

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Diagonal Representation of Density Matrix Using q-Coherent States

R. PARTHASARATHY and R. SRIDHAR

The Institute of Mathematical Sciences, Chennai 600 113, India
E-mail: *sarathy@imsc.res.in, sridhar@imsc.res.in*

A q-analogue of the diagonal representation of the density matrix is derived using q-boson coherent states. The q-generalization of the self-reproducing property of $\rho(z', z)$ and a self consistent self-reproducing kernel $K(z', z)$ are obtained. Some applications to non-linear issues in optics are indicated

1 Introduction

A diagonal representation of the quantum mechanical density matrix by means of standard bosonic oscillator coherent states was obtained by Sudarshan [1] and Glauber [2]. A remarkable feature of this representation is that the average expectation value of normal ordered operators becomes the same as that of a classical function for a probability distribution over complex plane, thereby bringing one-to-one correspondence between classical complex representation and quantum mechanical density matrices. The study of Quantum Groups has led to a non-linear realization of Lie algebras and this resulted in the construction of $SU_q(2)$ algebra using q-bosonic oscillators by Macfarlane [3] and Biedenharn [4]. A corresponding q-fermion oscillator and $SU_q(2)$ algebra had been proposed by Parthasarathy and Viswanathan [5]. Such q-oscillators effectively deal with non-ideal systems which have interactions and provide a method to study non-linear excitations of EM fields. The use of q-bosonic oscillator to describe density matrix has been made by Nelson and Fields [6]. In this contribution, we reconsider this issue in more detail and present some new results regarding the self-reproducing property and relation between the expansion coefficients of the density matrix in the Fock space description and q-coherent state description.

2 q-boson coherent states

The q-boson oscillator algebra for q-annihilation, q-creation and q-number operators $\{a, a^\dagger, N\}$ is [3, 4]

$$aa^\dagger - qaa^\dagger = 1, \qquad [N, a] = -a, \qquad [N, a^\dagger] = a^\dagger, \tag{1}$$

and the deformation parameter q is real and $0 < q \leq 1$. Further $N \neq a^\dagger a$. The Fock space $\mathcal{F}$ is built upon a vacuum state $|0\rangle$ with $a|0\rangle = 0$. The n q-boson normalized Fock space state is given by

$$|n\rangle = \frac{1}{\sqrt{[n]!}}(a^\dagger)^n|0\rangle, \tag{2}$$

where $\{|n\rangle \mid n \in \mathbb{N} \cup \{0\}\}$ and $[n]$ is the q-number

$$[n] = \frac{1 - q^n}{1 - q}. \tag{3}$$

Further it follows,

$$a|n\rangle = \sqrt{[n]}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{[n+1]}|n+1\rangle,$$
$$N|n\rangle = n|n\rangle, \qquad a^\dagger a|n\rangle = [n]|n\rangle. \tag{4}$$

And $[n]! = [n][n-1][n-2]\cdots[2][1]$. A q-boson coherent state is defined as

$$|z\rangle = \left\{e_q^{|z|^2}\right\}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}}|n\rangle, \tag{5}$$

where z is a complex variable and e_q^x is the q-exponential function. This is a meromorphic function [7] with no zeroes and simple poles at $x_k = q^{-k}/(1-q)$. The first pole occurs at $x = 1/(1-q)$. Further it can be seen that $|z\rangle$ is an eigenstate of a with eigenvalue z. The most important property of $|z\rangle$ is that if $|z'\rangle$ is another such q-coherent state, then

$$\langle z|z'\rangle = \left\{e_q^{|z|^2} e_q^{|z'|^2}\right\}^{-\frac{1}{2}} e_q^{\bar{z}z'}, \tag{6}$$

which means that such states are not orthogonal. But $\langle z|z\rangle = 1$. Further, by writing $z = re^{i\theta}$ and treating the integration over the bounded variable θ as ordinary integration while that over r^2 as q-integration, it follows from (5)

$$\frac{1}{\pi}\int d^2z|z\rangle\langle z| = \frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^{R_q^2} \frac{1}{2} dr^2 |z\rangle\langle z| = 1, \tag{7}$$

where $R_q^2 = 1/(1-q)$.

3 Diagonal representation of density matrix

The density matrix has the expansion

$$\rho = \sum_{n,m=0}^{\infty} \rho(n,m)|n\rangle\langle m|, \tag{8}$$

in terms of Fock space states. From the definition of the q-coherent state (5), it can be shown [8] that

$$|n\rangle\langle m| = \frac{\sqrt{[n]![m]!}}{[n+m]!} \left\{ \left(\frac{d}{dr}\right)_q^{(n+m)} \int_0^{2\pi} \frac{d\theta}{2\pi} e_q^{r^2} e^{i(m-n)\theta} |re^{i\theta}\rangle\langle re^{i\theta}| \right\}_{r=0}, \tag{9}$$

where $(\frac{d}{dr})_q$ stands for q-differentiation. Substituting this in (8), we obtain the diagonal representation of the density matrix in terms of q-boson coherent states as

$$\rho = \sum_{n,m=0}^{\infty} \rho(n,m) \frac{\sqrt{[n]![m]!}}{[n+m]!} \left\{ \left(\frac{d}{dr}\right)_q^{(n+m)} \int_0^{2\pi} \frac{d\theta}{2\pi} e_q^{r^2} e^{i(m-n)\theta} |re^{i\theta}\rangle\langle re^{i\theta}| \right\}_{r=0}, \tag{10}$$

a q-analogue of the result of [1, 2]. Expanding ρ as

$$\rho = \int d^2z \phi(z)|z\rangle\langle z|,$$

with $\int d^2z\phi(z) = 1$ (to ensure that $\mathrm{Tr}\,\rho = 1$), the expansion coefficients $\rho(n,m)$ in $\mathcal{F}$ and $\phi(z)$ in q-coherent states, are related by

$$\begin{aligned}\rho(n,m) &= \int d^2z\phi(z)\frac{z^n\bar{z}^m}{\sqrt{[n]![m]!}}\left\{e_q^{|z|^2}\right\}^{-1} \\ &= \pi\int_0^{R_q^2} d\left(r^2\right)\phi(r^2)\frac{r^{2n}}{[n]!}\left\{e_q^{r^2}\right\}^{-1}\delta_{n,m},\end{aligned} \tag{11}$$

where in the last step the θ integration is performed. It follows

$$\sum_n^{\infty}\rho(n,n) = 1.$$

4 Some applications

As a first application, we consider the expansion of normal ordered operators, $F = \sum\limits_{p,s} C_{p,s}(a^\dagger)^p a^s$. Then using $a|z\rangle = z|z\rangle$, and taking matrix elements of F in between q-coherent states, we find

$$C_{p,s} = \sum_{k=0}^{\min(p,s)}\frac{(-1)^k q^{k(k-1)/2}}{[k]!\sqrt{[s-k]![p-k]!}}\langle p-k|F|s-k\rangle, \tag{12}$$

relating $C_{p,s}$ to the Fock space matrix elements of F.

As a second application, we consider the average expectation value of F.

Using $\rho = \int d^2z\phi(z)|z\rangle\langle z|$, we find,

$$\mathrm{Tr}\,(\rho F) = \sum_{p,s} C_{p,s}\int d^2z\phi(z)\bar{z}^p z^s. \tag{13}$$

The following results are to be noted. First, $\mathrm{Tr}\,(\rho F)$ is now the expectation value of complex classical function $\bar{z}^p z^s$ for a distribution $\phi(z)$. Second for $\phi(z) = \phi\left(r^2\right)$, this becomes,

$$\pi\sum_{p,s} C_{p,s}\int_0^{R_q^2} d\left(r^2\right)\phi\left(r^2\right)r^{2p}.$$

For $\phi(r^2) = \frac{1}{\pi}e_q^{-r^2}$, q-Gaussian, $\mathrm{tr}\,(\rho F) = \sum\limits_p C_{p,p}[p]!$.

As a third application, we consider an operator $O = \sum\limits_n B_n(a^\dagger a)^n$. First of all, it is possible to express

$$(a^\dagger a)^n = \sum_s C_s^n(a^\dagger)^s a^s. \tag{14}$$

The coefficients C_s^n satisfy a recursion relation

$$C_s^{n+1} = C_{s-1}^n q^{s-1} + [s]C_s^n.$$

Then we find,

$$\mathrm{Tr}\,(\rho O) = \pi\sum_n B_n\sum_{s=0}^n C_s^n\int_0^{R_q^2} d\left(r^2\right)\phi\left(r^2\right)r^{2s}. \tag{15}$$

For $\phi(r^2)$ a q-Gaussian, (15) simplifies to

$$\mathrm{Tr}\,(\rho O) = \sum_n B_n\sum_{s=0}^n C_s^n[s]!.$$

5 Self reproducing property

Now we take up the important feature of ρ, namely, the self-reproducing property. The matrix elements of ρ in between two q-coherent states is calculated to be

$$\langle z'|\rho|z\rangle = \sum_{n,m} \rho(n,m) \left\{ e_q^{|z|^2} e_q^{|z'|^2} \right\}^{-\frac{1}{2}} \frac{\bar{z}'^m z^n}{\sqrt{[m]![n]!}}. \tag{16}$$

Introduce

$$\rho(z',z) = \sum_{n,m} \rho(n,m) \frac{\bar{z}'^m z^n}{\sqrt{[m]![n]!}} \left\{ e_q^{|z'|^2} e_q^{|z|^2} \right\}. \tag{17}$$

Then, $\rho(z',z) = \langle z'|\rho|z\rangle$. Now we use the completeness relation (7) for the q-coherent states, to write

$$\begin{aligned} \rho(z',z) &= \langle z'|\rho|z\rangle \\ &= \int d^2\xi \langle z'|\rho|\xi\rangle\langle\xi|z\rangle \frac{1}{\pi} \\ &= \int d^2\xi K(\xi,z)\rho(z',\xi), \end{aligned} \tag{18}$$

where the reproducing kernel $K(\xi,z)$ is defined as

$$K(\xi,z) = \frac{1}{\pi}\langle\xi|z\rangle. \tag{19}$$

(18) gives the self-reproducing property of $\rho(z',z)$ with $K(\xi,z)$ as the self-reproducing kernel. The theory of self-reproducing kernel has been studied by Aronszajn [9]. The above kernel (19) satisfies the required properties of a self-reproducing kernel. Namely, first, it satisfies the matrix multiplication property,

$$\int d^2\xi \; K(z,\xi)K(\xi,z') = K(z,z'). \tag{20}$$

Second, since ρ is Hermitian, the kernel satisfies the Hermiticity property,

$$K(z,\xi)^* = K(\xi,z). \tag{21}$$

The explicit expression for the kernel is,

$$K(\xi,z) = \frac{1}{\pi} \left\{ e_q^{|\xi|^2} e_q^{|z|^2} \right\}^{-\frac{1}{2}} \; e_q^{\bar{\xi}z}, \tag{22}$$

from which it follows

$$K(z,z) > 0. \tag{23}$$

6 *q*-coherent states in physical situations

We now give an attempt to use q-coherent states to some physical situations.

The fluctuations of photon number in the standard Fock space description is zero. However, optic field described by the standard coherent state has large uncertainty in photon number.

This is seen by considering $\langle z|a^\dagger a|z\rangle = |z|^2 = \langle N\rangle$ for an ordinary coherent state. Also, $\langle N^2\rangle = \langle z|a^\dagger aa^\dagger a|z\rangle = |z|^2 + |z|^4$. Then the uncertainty in the photon number is

$$\triangle n = \sqrt{\langle N^2\rangle - \langle N\rangle^2} = |z|. \tag{24}$$

For large $|z|$, the variance of the photon number is large. If we choose to describe the optic field by a q-coherent state, the photon number is taken as photon q-number. Then using q-coherent states, we have $\langle z|a^\dagger a|z\rangle = \langle N\rangle = |z|^2$. On the other hand, $\langle z|a^\dagger aa^\dagger a|z\rangle = \langle N^2\rangle = |z|^2 + q|z|^4$. Then,

$$\triangle n = |z|\sqrt{1 + (q-1)|z|^2}. \tag{25}$$

By comparing this with (24), we see that the uncertainity in photon q-number depends non-linearly on $|z|$. For $q < 1$, the photon q-number has less uncertainity. Further, the probability of finding the optic field in number state is $|\langle n|z\rangle|^2$, which for ordinary coherent state is

$$P_n = e^{-|z|^2}\frac{|z|^{2n}}{n!},$$

which is just a Poisson distribution. If we use q-coherent states, then

$$P_n = \{e_q^{|z|^2}\}^{-1}\frac{|z|^{2n}}{[n]!},$$

which is q-Poisson distribution.

We now consider the propagation of light in a non-linear medium like Kerr medium. This is usually described by the Hamiltonian [10] as

$$H = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\chi(a^\dagger)^2a^2, \tag{26}$$

where a, $a^\dagger$ are the usual annihilation and creation operators with $aa^\dagger - a^\dagger a = 1$ and χ represents the interaction strength of the light with the non-linear medium. Using photon number states,

$$\langle n|H|n\rangle = \hbar\omega\left\{n + \frac{\chi}{2\omega}n(n-1)\right\}. \tag{27}$$

Instead of describing the propagation by the Hamiltonian (26), we desire to use a Hamiltonian

$$\mathcal{H} = \hbar\omega a^\dagger a, \tag{28}$$

where now a, $a^\dagger$ are q-annihilation and creation operators (1). Then it follows

$$\langle n|\mathcal{H}|n\rangle = \hbar\omega[n]. \tag{29}$$

The important observation is that (29) will describe the content of (27). By setting $q = 1 - \epsilon$, we have from (3),

$$\langle n|\mathcal{H}|n\rangle = \hbar\omega\left\{n + \frac{\epsilon}{2}n(n-1)\right\}, \tag{30}$$

keeping terms up to ϵ. Thus (30) has the same content of (27), for $\epsilon = \frac{\chi}{\omega}$. However $\mathcal{H}$ is simple, like a free Hamiltonian and the interaction is shifted to q-number. As we have kept terms linear in ϵ, such description will be valid for high frequency propagation.

7 Summary

We have obtained an explicit diagonal representation of the quantum mechanical density matrix using q-coherent states. Certain classes of operators are studied for their statistical averages. The expansion coefficients of ρ in Fock space and q-coherent state representation are related. The C_s^n coefficients introduced in (14) with their recursion relation, are q-Stirling numbers of the second kind and they enter in q-coherent state description. The self-reproducing property of $\rho(z', z)$ is established. In this, the completeness relation of q-coherent states (7) plays a crucial role. The self-reproducing kernel is found and it satisfies the required Hermiticity property and matrix multiplication rule. Applications representing the use of q-coherent states in the description of light propagating in non-linear medium are illustrated.

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Multi-Soliton Solutions with Discrete Symmetries in the Chiral Quark Soliton Model

Nobuyuki SAWADO, Noriko SHIIKI and Shinsho ORYU

Department of Physics, Tokyo University of Science, Noda, Chiba 278-8510, Japan
E-mail: *sawado@ph.noda.tus.ac.jp, norikoshiiki@mail.goo.ne.jp, oryu@ph.noda.tus.ac.jp*

We discuss multi-soliton solutions with discrete symmetries in the chiral quark soliton model using the rational map ansatz. The solutions exhibit degenerate bound spectra of the quark orbits depending on the background pion field configurations. It is shown that resultant baryon densities inherit the same discrete symmetries as the chiral fields. Evaluating the radial component of the baryon density, shell-like structure of the valence quark spectra is also observed.

1 Introduction

The chiral quark soliton model (CQSM) was developed in 1980's as a low-energy effective theory of QCD. Since it includes the Dirac Sea quark contribution and explicit valence quark degrees of freedom, the model interpolates between the constituent quark model and the Skyrme model [1–4]. The CQSM is derived from the instanton liquid model of QCD vacuum and incorporates the non-perturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. It has been shown that the $B = 1$ solution provides correct observables such as a nucleon including mass, electromagnetic value, spin carried by quarks, parton distributions and octet SU(3) baryon spectra. For $B = 2$, the stable axially symmetric soliton solution was found in [5]. The solution exhibits doubly degenerate bound spectrum of the quark orbits in the background of axially symmetric chiral fields with winding number two. Upon quantization, various dibaryon spectra were obtained, showing that the quantum number of the ground state coincide with that of physical deuteron [6, 7]. For $B > 2$, the Skyrme model predicts that minimum energy solutions have only discrete, crystal-like symmetries [8–10]. According to this prediction, we studied the CQSM with $B = 3$ tetrahedrally symmetric chiral fields and obtained triply degenerate spectrum of the quark orbits [11]. Its large degeneracy indicates that the tetrahedrally symmetric solution may be the lowest-lying configuration. Thus, for $B > 3$, one can also expect that the lowest-lying solutions in the CQSM inherits the discrete symmetries predicted in the Skyrme model.

Following the $B = 3$ case, we shall study soliton solutions with $B \geq 3$ in the CQSM using the rational map ansatz with discrete symmetries obtained in the Skyrme model. We will show obtained classical self-consistent soliton solutions with $B = 1$–$9, 17$. These solutions exhibit various degenerate spectra of the quark orbits depending on the symmetry of the background chiral fields. Such degeneracy generates large shell gaps, which suggests that the solutions are stable local minima. It is shown that resultant baryon number densities inherit the same symmetries as the chiral fields. Evaluating the radial component of the baryon density, shell-like structure of the valence quarks can be observed.

2 The model

The CQSM is derived from the instanton liquid model of the QCD vacuum and incorporates the nonperturbative feature of the low-energy QCD, spontaneous chiral symmetry breaking. The

vacuum functional is defined by [1]

$$\mathcal{Z} = \int \mathcal{D}\pi\mathcal{D}\psi\mathcal{D}\psi^\dagger \exp\left[i\int d^4x\,\bar{\psi}\left(i\partial\!\!\!/ - MU^{\gamma_5}\right)\psi\right], \tag{1}$$

where the SU(2) matrix

$$U^{\gamma_5} = \frac{1+\gamma_5}{2}U + \frac{1-\gamma_5}{2}U^\dagger \qquad \text{with} \qquad U = \exp\left(i\boldsymbol{\tau}\cdot\boldsymbol{\pi}/f_\pi\right)$$

describes chiral fields, ψ is quark fields and M is the constituent quark mass. f_π is the pion decay constant and experimentally $f_\pi \sim 93\,\mathrm{MeV}$.

In the CQSM, the number of valence quark is associated with the baryon number such that a soliton with baryon number B consists of $N_c \times B$ valence quarks. If the correlation between quarks is sufficiently strong, their binding energy becomes large and the valence quarks cannot be observed as positive energy particles [12, 13]. Thus, one gets the picture of the topological soliton model in the sense that the baryon number coincides with the winding number of the background chiral field when the valence quarks occupy all the levels diving into negative energy region.

The vacuum functional in (1) can be integrated over the quark fields to obtain the effective action

$$S_{\mathrm{eff}}[U] = -iN_c\,\mathrm{lndet}\,\left(i\partial\!\!\!/ - MU^{\gamma_5}\right) \tag{2}$$

$$= -\frac{i}{2}N_c\,\mathrm{Spln}\,D^\dagger D, \tag{3}$$

where $D = i\partial\!\!\!/ - MU^{\gamma_5}$. We introduce the eigenequation

$$H(U^{\gamma_5})\phi_\mu(\boldsymbol{x}) = E_\mu\phi_\mu(\boldsymbol{x}), \qquad H(U^{\gamma_5}) = -i\alpha\cdot\nabla + \beta MU^{\gamma_5}. \tag{4}$$

The effective action $S_{\mathrm{eff}}(U)$ is ultraviolet divergent and hence must be regularized. Using the proper-time regularization scheme [14], we can write

$$S_{\mathrm{eff}}^{\mathrm{reg}}[U] = \frac{i}{2}N_c\int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\,\mathrm{Sp}\left(\mathrm{e}^{-D^\dagger D\tau} - \mathrm{e}^{-D_0^\dagger D_0\tau}\right)$$

$$= \frac{i}{2}N_cT\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\,\mathrm{Sp}\left[\mathrm{e}^{-\tau(H^2+\omega^2)} - \mathrm{e}^{-\tau(H_0^2+\omega^2)}\right], \tag{5}$$

where T is the Euclidean time separation, D_0 and H_0 are operators with $U = 1$.

At $T \to \infty$, we have $\mathrm{e}^{iS_{\mathrm{eff}}} \sim \mathrm{e}^{-iE_{\mathrm{field}}T}$. The total energy then is given by

$$E_{\mathrm{static}}[U] = E_{\mathrm{val}}[U] + E_{\mathrm{field}}[U] - E_{\mathrm{field}}[U=1], \tag{6}$$

where

$$E_{\mathrm{val}} = N_c\sum_i E_{\mathrm{val}}^{(i)} \tag{7}$$

is the valence quark contribution with the valence energy of the i-th valence quark $E_{\mathrm{val}}^{(i)}$, and the vacuum part is

$$E_{\mathrm{field}} = N_c\sum_\mu\left\{\mathcal{N}(E_\mu)|E_\mu| + \frac{\Lambda}{\sqrt{4\pi}}\exp\left[-\left(\frac{E_\mu}{\Lambda}\right)^2\right]\right\}$$

with

$$\mathcal{N}(E_\mu) = -\frac{1}{\sqrt{4\pi}}\Gamma\left(\frac{1}{2}, \left(\frac{E_\mu}{\Lambda}\right)^2\right).$$

Λ is a cutoff parameter evaluated by the condition that the derivative expansion of (5) reproduces the pion kinetic term with the correct coefficient, *i.e.*,

$$f_\pi^2 = \frac{N_c M^2}{4\pi^2}\int_{1/\Lambda^2}^{\infty}\frac{d\tau}{\tau}\,\mathrm{e}^{-\tau M^2}. \tag{8}$$

The constituent quark mass M is only one free parameter and we adopt the value $M = 400\,\mathrm{MeV}$, which reproduces the observables of the nucleon and the delta. From (8), one obtains $\Lambda \sim 637\,\mathrm{MeV}$. The contribution for the total energy in the absence of the chiral fields ($U = 1$) can be estimated using the eigenstates of eigenequation,

$$H_0\phi_\mu^{(0)}(\boldsymbol{x}) = E_\mu^{(0)}\phi_\mu^{(0)}(\boldsymbol{x}), \qquad H_0 = -i\alpha\cdot\nabla + \beta M. \tag{9}$$

In the Skyrme model, it is known that solitons with $B \geq 3$ have particular discrete symmetries [8]. Therefore, we expect that soliton solutions of the CQSM inherit the same discrete symmetry as skyrmions. Houghton, Manton and Sutcliffe proposed remarkable ansatz for the chiral fields, the rational map ansatz [10]. According to this ansatz, the chiral fields are expressed in a rational map as

$$U(r,z) = \exp(iF(r)\hat{\boldsymbol{n}}_R\cdot\boldsymbol{\tau}), \qquad \hat{\boldsymbol{n}}_R = \frac{1}{1+|R(z)|^2}(2\mathrm{Re}\,[R(z)], 2\mathrm{Im}\,[R(z)], 1-|R(z)|^2) \tag{10}$$

and $R(z)$ is the rational map. The complex coordinate z is given by $z = \tan(\theta/2)e^{i\varphi}$ via stereographic projection. Rational maps are maps from $CP(1)$ to $CP(1)$ (equivalently, from S^2 to S^2) classified by winding number. In [10], Manton *et al.* showed that $B = N$ skyrmions can be well-approximated by rational maps with winding number N. The rational map with winding number N possesses $(2N+1)$ complex parameters whose values can be determined by imposing the symmetry of the skyrmion. We shall use this ansatz for the background chiral fields in the CQSM. We employ the explicit forms of $R(z)$ for various B proposed in [10] to our analysis. Since the chiral fields in (10) is parameterized by polar coordinates, one can apply the numerical technique developed for $B = 1$ to find solutions with higher B.

Field equations for the chiral fields can be obtained by demanding that the total energy in equation (6) be stationary with respect to variation of the profile function $F(r)$,

$$\frac{\delta}{\delta F(r)}E_{\mathrm{static}} = 0,$$

which gives

$$S(r)\sin F(r) = P(r)\cos F(r), \tag{11}$$

where

$$S(r) = N_c\sum_\mu\big(n_\mu\theta(E_\mu) + \mathrm{sign}(E_\mu)\mathcal{N}(E_\mu)\big)\langle\mu|\gamma^0\delta(|x|-r)|\mu\rangle, \tag{12}$$

$$P(r) = N_c\sum_\mu\big(n_\mu\theta(E_\mu) + \mathrm{sign}(E_\mu)\mathcal{N}(E_\mu)\big)\langle\mu|i\gamma^0\gamma^5\hat{\boldsymbol{n}}_R\cdot\boldsymbol{\tau}\delta(|x|-r)|\mu\rangle. \tag{13}$$

The procedure to obtain self-consistent solutions of equation (11) is that 1) solve the eigenequation in (4) under an assumed initial profile function $F_0(r)$, 2) use the resultant eigenfunctions and

eigenvalues to calculate $S(r)$ and $P(r)$, 3) solve equation (11) to obtain a new profile function, 4) repeat 1)–3) until the self-consistency is attained.

The baryon density $b(\boldsymbol{x})$ is defined by the zeroth component of the baryon current [2];

$$b(\boldsymbol{x}) = \frac{1}{N_c}\langle\bar{\psi}\gamma_0\psi\rangle = b_{\rm val}(\boldsymbol{x}) + b_{\rm field}(\boldsymbol{x}), \tag{14}$$

where

$$b_{\rm val}(\boldsymbol{x}) = \sum_i b^{(i)}_{\rm val}(\boldsymbol{x}) = \frac{1}{N_c}\sum_i \phi_i(\boldsymbol{x})^\dagger\phi_i(\boldsymbol{x}),$$
$$b_{\rm field}(\boldsymbol{x}) = \frac{1}{N_c}\sum_\mu \Big[{\rm sign}(E_\mu)\mathcal{N}(E_\mu)\phi_\mu(\boldsymbol{x})^\dagger\phi_\mu(\boldsymbol{x}) - {\rm sign}(E^{(0)}_\mu)\mathcal{N}(E^{(0)}_\mu)\phi^{(0)}_\mu(\boldsymbol{x})^\dagger\phi^{(0)}_\mu(\boldsymbol{x})\Big]. \tag{15}$$

To examine the shell structure of the quarks, we evaluated the radial density for the ith valence quark $\rho^{(i)}(r)$ in which the angular degrees of freedom are integrated via,

$$\rho^{(i)}(r) = \int d\varphi \int \sin\theta\, d\theta\, b^{(i)}_{\rm val}(r,\theta,\varphi) \tag{16}$$

with the baryon number

$$B = \sum_i \int dr\, r^2 \rho^{(i)}(r). \tag{17}$$

3 Results and discussions

Let us first show some results of the spectral flow analysis. For convenience, we shall take

$$F(r) = \begin{cases} -\pi + \pi r/X & \text{for } \ r < X, \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

as a trial function for the profile function. In Figs. 1, 2, we show the spectral flow with $B = 3, 7$. As can be seen, the number of B positive energy levels are diving into negative energy region and thus we obtain the baryon number B soliton solutions. In Table 1 there are the results for the valence quark levels as well as the vacuum sea contributions. The valence quark spectra show various degenerate patterns depending on the background configuration. The total energy indicates that all the solitons are deep bound states. From (15), we estimated the baryon number density (see Fig. 3). The density inherits the same symmetry as the corresponding skyrmion.

In Fig. 4, we display the valence quark spectra together with the results of $B = 1, 2$ [6]. It is interesting that the results strongly suggest the existence of shell structure for the valence quarks. The spectra show (i) four-fold degeneracy of the ground state labeled by $\mathcal{G}$ and various degenerate pattern for excited levels labeled by $\mathcal{A}_1, \mathcal{A}_2, \ldots$, (ii) a large energy gap between the ground state $\mathcal{G}$ and the first excited level $\mathcal{A}_1$. We suspect that these large degeneracy should contribute to the minimization of the total energy. Note that the small dispersions of the spectra observed here are caused by the finite size effect of the basis. Increasing the size $r_{\rm max}$ together with increase of the number of the basis, more accurate degeneracy will be attained.

The bunch of the valence spectra caused by symmetry have also been observed within the study of heavier nuclear system. As discussed in Ref. [10], the group theory should predict the level structure for pion fluctuation. However, our problem is more complicated due to the presence of quarks.

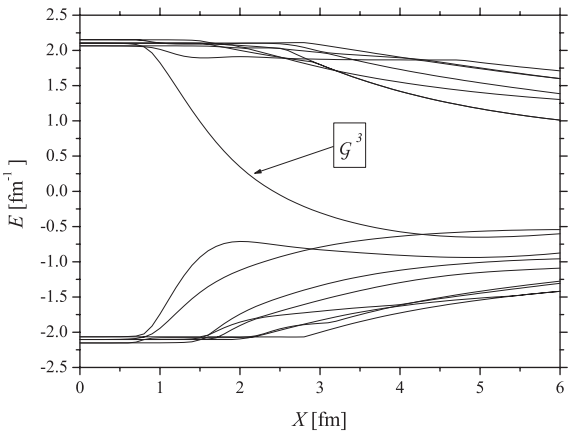

Figure 1. Spectral flow of $B = 3$ with the occupation number.

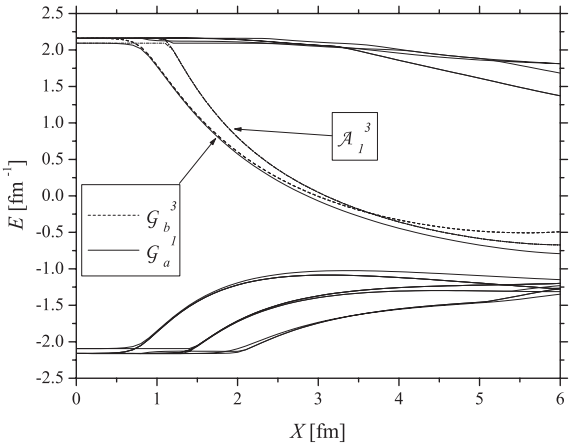

Figure 2. Spectral flow of $B = 7$ with the occupation number.

Table 1. Mass spectra for $B = 1$–$9, 17$ (in MeV). The data for $B = 1, 2$ are taken from Ref. [6]. The ratio of the mass $E_{\rm static}$ to $B \times E_{\rm static}^{(B=1)}$ are compared to that of the Skyrme model [10].

B	$E_{\rm val}^{(i)}$									$E_{\rm field}$	$E_{\rm static}$	$E_{\rm static}/BE_{\rm static}^{(B=1)}$	
												Ours	Skyrme
1	173									674	1192	1.00	1.00
2	173	173								1166	2204	0.92	0.95
3	210	210	210							1633	3522	0.98	0.96
4	144	146	146	146						2628	4378	0.92	0.92
5	123	131	131	139	210					3265	5467	0.92	0.93
6	120	124	150	150	206	206				3740	6603	0.92	0.92
7	115	120	120	120	166	166	166			4554	7478	0.90	0.90
8	97	97	115	120	139	139	203	203		5229	8565	0.90	0.91
9	99	105	105	121	142	142	210	210	210	5700	9742	0.91	0.89
17	83	95	95	95	153	156	157	173	175				
	177	178	179	192	194	194	196	196		10586	18650	0.93	0.88
5*	157	157	157	232	232					2874	5680	0.95	1.00

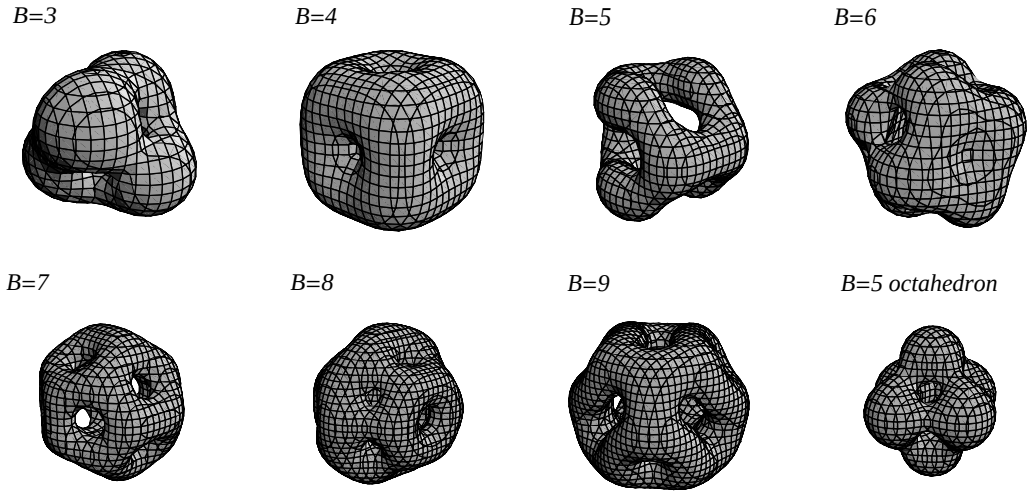

Figure 3. Baryon number densities.

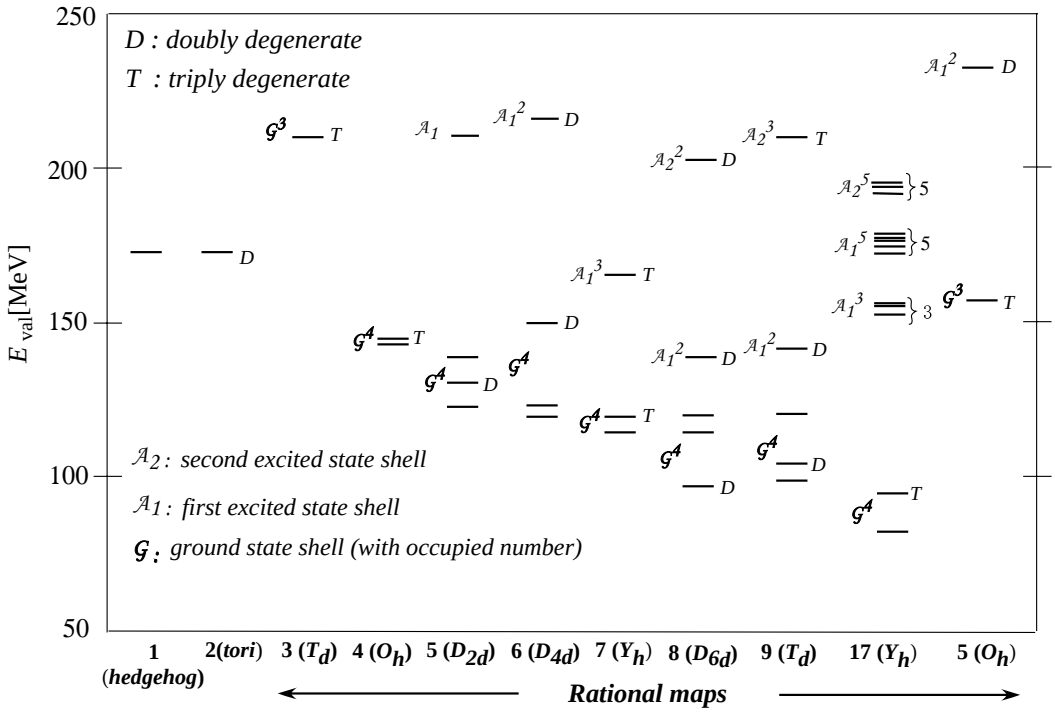

Figure 4. Valence quark spectra for $B = 1$–$9, 17$.

Let us show how the shell deformation is related to the degeneracy of the spectrum. In general, if eigenequation

$$H\psi_\mu = E_\mu \psi_\mu \tag{19}$$

is invariant under a symmetric operation $R \in G$, the equation transforms as

$$RH\psi_\mu = H(R\psi_\mu) = E_\mu \psi_\mu. \tag{20}$$

Thus the states $\{\psi_\mu, R\psi_\mu\}$ have the same energy E_μ. The set of d_μ eigenfunctions $\{\psi_i^{(\mu)}\}$ $(i = 1, \ldots, d_\mu)$ belonging to a given eigenvalue E_μ will provide the basis for an irreducible representation of the group G of the Hamiltonian [15]:

$$R\psi_j^{(\mu)} = \sum_i \psi_i^{(\mu)} D_{ij}^{(\mu)}(R). \tag{21}$$

The operator R are constructed as follows. If chiral fields have some particular point group symmetry i.e., $U(\boldsymbol{x}') = G(\boldsymbol{a})U(\boldsymbol{x})G(\boldsymbol{a})^\dagger$ $(G(\boldsymbol{a}) \in SU(2)_I)$, the Dirac equation is invariant under the rotation

$$x' = ax \tag{22}$$

with

$$x' = (t, \boldsymbol{x}'), \qquad a = (1, \boldsymbol{a}), \qquad x = (t, \boldsymbol{x}), \tag{23}$$

accompanying the iso-rotation

$$(i\gamma^\mu \partial_\mu - MU(\boldsymbol{x}))\psi(x) = 0 \quad \Rightarrow \quad (i\gamma^\nu \partial'_\nu - MU(\boldsymbol{x}'))\psi'(x') = 0 \tag{24}$$

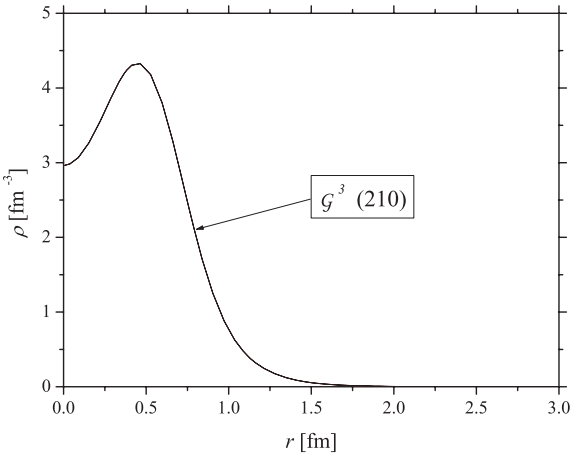

Figure 5. Baryon number density of ith valence quarks $\rho^{(i)}(r)$ of $B = 3$, with the occupation numbers and their eigenvalues (in MeV).

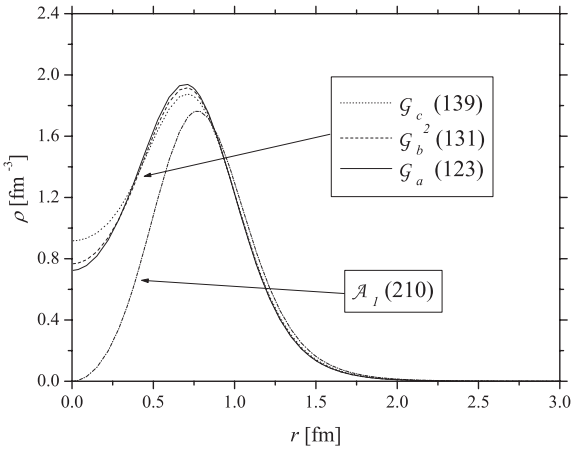

Figure 6. Baryon number density of ith valence quarks $\rho^{(i)}(r)$ of $B = 5$, with the occupation numbers and their eigenvalues (in MeV).

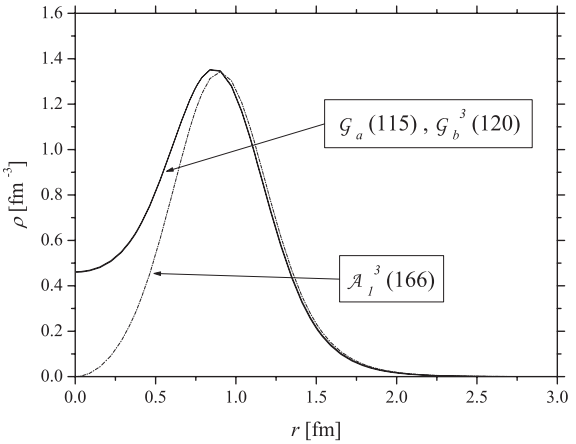

Figure 7. Baryon number density of ith valence quarks $\rho^{(i)}(r)$ of $B = 7$, with the occupation numbers and their eigenvalues (in MeV).

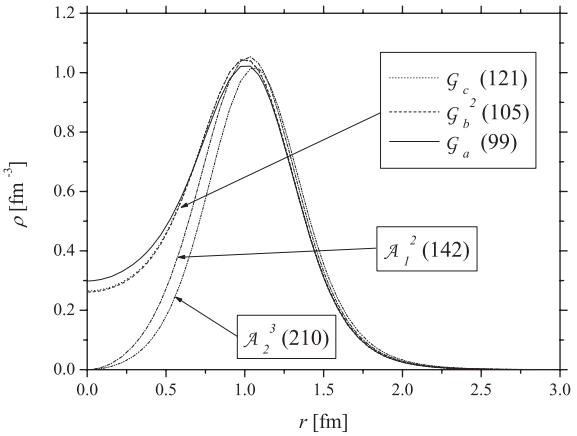

Figure 8. Baryon number density of ith valence quarks $\rho^{(i)}(r)$ of $B = 9$, with the occupation numbers and their eigenvalues (in MeV).

and

$$\psi'(x') = S(\boldsymbol{a}) \times G(\boldsymbol{a})\psi(x), \tag{25}$$

where $S(\boldsymbol{a})$ satisfies $a_\mu^\nu \gamma^\mu = S^{-1}\gamma^\nu S$. The operator R corresponding to this rotation is defined by

$$\psi'(x) \stackrel{\text{def}}{=} R\psi(x) = S(\boldsymbol{a}) \times G(\boldsymbol{a})\psi(a^{-1}x). \tag{26}$$

One can easily check that R commutes with the Hamiltonian in (4). Constructing R for each symmetries, one should be able to deduce the degeneracy structure of the spectra occurring in the valence levels. The four-fold degeneracy of the lowest states may be ascribed to the chiral symmetry $SU(2)_L \times SU(2)_R$ of the Hamiltonian. The degenerate structure will be understood if symmetric operators of the Hamiltonian which consist of the angular momentum, spin, isospin and winding number, are explicitly constructed.

In Figs. 5–8, we present the results of $\rho^{(i)}(r)$ for $B = 3, 5, 7$ and 9. The behavior of the density near the origin confirms at least three shells $(\mathcal{G}, \mathcal{A}_1, \mathcal{A}_2)$. $\mathcal{G}$ behaves like "S-wave", and others

like "P-, D-wave" in a hydrogen-like atom. Although the shell structure is emerged, most of the densities are nearly on the same surface and very small (not zero) near the origin. The plateau in the density observed at the center of the nucleus [16] can not be attained in our solutions. Therefore, one may need to employ the multi-shell ansatz [17] even in the case of light nuclei.

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Symmetries in Semiclassical Mechanics

Oleg Yu. SHVEDOV

Sub-Department of Quantum Statistics and Field Theory, Department of Physics, Moscow State University, Vorobievy Gory, 119992 Moscow, Russia
E-mail: *shvedov@qs.phys.msu.su*

Symmetry properties in quantum, classical and semiclassical mechanics are considered. Geometric structures in the semiclassical (Maslov complex-WKB) theory are investigated. Infinitesimal properties of semiclassical symmetry transformations are discussed.

1 Introduction

Semiclassical approximation is widely used in quantum mechanics and field theory. Many different approaches to constructing semiclassical theories (functional integral approach, operator approach etc.) have been developed. However, the most appropriate for mathematical justification approach is based on direct substitution of the wave function to the quantum evolution equation.

Historically, the WKB wave function was the first substitution that obeyed the Schrödinger equation in the semiclassical approximation. Now, many different semiclassical substitutions (arised in the Maslov theory of wave packets, in the Maslov theory of Lagrangian manifolds with complex germs etc) to the Schrödinger equation are known. The simplest one is the wave packet function which is specified by classical variables (numbers: S – phase, P_i – momenta, Q_i – coordinates) and quantum function f specifying a shape of the wave packet.

Generally, a semiclassical theory may be viewed as follows. A semiclassical state is a point on the space of a bundle ("semiclassical bundle"), which can be denoted as (X, f), where X is a classical state (for ordinary quantum mechanics, $X = (S, P, Q)$ and f is a quantum state in the given classical background). Set of all $\{X\}$ may be considered as a base of the bundle, while f belongs to a Hilbert space $\mathcal{F}_X$, which may be generally X-dependent.

If the quantum theory model is symmetric under a Lie group, the corresponding semiclassical theory should be also symmetric. This means that the symmetry Lie group acts on the semiclassical bundle; an automorphism of the bundle corresponds to each element of the Lie group and the group property is satisfied.

When one is interested whether the quantum theory is symmetric under a Lie group, provided that the corresponding classical theory is symmetric, a first step to solve the problem is to investigate the corresponding semiclassical theory. Quantum anomalies may be investigated even in the semiclassical level.

The purpose of the talk is to investigate the properties of the semiclassical mechanics symmetric under Lie groups. The following problems are to be discussed: infinitesimal properties; correspondence between Lie groups and algebras in the semiclassical mechanics; properties of semiclassical gauge theories; applications to quantum field theory.

2 Symmetries in classical and quantum mechanics

There are many classical and quantum systems with symmetries. The simplest example of the symmetry transformation is evolution. Group of Poincaré transformations is also a typical example of symmetry group. More complicated symmetries (such as gauge) are also known.

Although the mathematical formalism of quantum mechanics differs from the classical one, and the spaces of states are quite different, the symmetry notions are similar. For example, the evolution is viewed as a transformation mapping the initial state to the state at time t. In quantum mechanics (see, for example, [3]) it is an unitary operator $\hat{U}_t = e^{-i\hat{H}t}$ acting in the Hilbert space $\mathcal{H}$ (quantum state space); classically, it is a symplectic transformation of the classical state space (phase space) generated by a Hamiltonian vector field (see, for example, [1]).

The comparison of classical and quantum symmetries is presented in Table 1.

Table 1. Symmetries in classical and quantum mechanics.

	Classical mechanics	Quantum mechanics
State space	Symplectic manifold $\mathcal{M}$ $(X = (p, q))$	Hilbert space $\mathcal{H}$
Symmetry transformation under Lie group G	$g \in G \mapsto u_g : \mathcal{M} \to \mathcal{M}$ is smooth symplectic transformation; $u_{g_1 g_2} = u_{g_1} u_{g_2}$; $u_e = 1$	$g \in G \mapsto \hat{U}_g : \mathcal{H} \to \mathcal{H}$ is unitary operator; $\hat{U}_{g_1 g_2} = \hat{U}_{g_1} \hat{U}_{g_2}$; $\hat{U}_e = 1$
Constructing transformations for $g(t) = e^{t\delta g}$, δg belongs to the Lie algebra	$u_{g(t)}$ takes the initial condition for equation $\frac{dX(t)}{dt} = \nabla[\delta g](X(t)), \quad X(t) \in \mathcal{M}$ to the solution of the Cauchy problem: $u_{g(t)} : X(0) \mapsto X(t)$; $\nabla[\delta g](X)$ is Hamiltonian vector field on $\mathcal{M}$	$\hat{U}_{g(t)} = \exp[-i\hat{H}[\delta g]t]$ takes the initial condition for equation $i\frac{d\psi}{dt} = \hat{H}[\delta g]\psi, \quad \psi(t) \in \mathcal{H}$ to the solution of the Cauchy problem $\hat{U}_{g(t)} : \psi(0) \mapsto \psi(t)$; $\hat{H}[\delta g]$ is self-adjoint operator in $\mathcal{H}$
Properties of infinitesimal generators	For operator $\nabla[\delta g] = (\nabla[\delta g])^i \frac{\partial}{\partial X^i}$, $[\nabla[\delta g_1]; \nabla[\delta g_2]] = -\nabla[\delta g_1; \delta g_2]$	$[\hat{H}[\delta g_1]; \hat{H}[\delta g_2]] = -i\hat{H}[\delta g_1; \delta g_2]$

3 What is semiclassical approximation?

It follows from Table 1 that main notions of classical and quantum mechanics are different. However, both theories describe nature. They should not contradict each other then; quantum systems should be interpreted from the classical viewpoint under certain conditions.

Such a correspondence is possible, provided that quantum theory depends on the parameter h, and $h \to 0$. This is a "small parameter" of the semiclassical expansion. It is proportional to the Planck constant. The dependence of self-adjoint generator $\hat{H}$ of the symmetry transformation $e^{-i\hat{H}t}$ on h should be as follows:

$$\hat{H}_h = \frac{1}{h} H\left(x, -ih\frac{\partial}{\partial x}\right), \qquad x \in \mathbb{R}^n.$$

The definition of function of non-commuting operators x and $-ih\frac{\partial}{\partial x}$ can be found in [4].

Therefore, the equation to be investigated as $h \to 0$ reads

$$ih\frac{\partial \psi^t(x)}{\partial t} = H\left(x, -ih\frac{\partial}{\partial x}\right)\psi^t(x). \tag{1}$$

Historically, the first approximate solution of equation (1) was (see, for example, [3]) the WKB solution of the form

$$\psi^t(x) = \varphi^t(x) e^{\frac{i}{h}S^t(x)}. \tag{2}$$

Substituting expression (2) to equation (1), one obtains the Hamilton–Jacobi equation for $S^t(x)$ and the transport equation for $\varphi^t(x)$.

Now, a lot of approximate solutions for equation (1) are known. For example, there is the Maslov substitution [4, 5] of the form:

$$\psi^t(x) = \text{const}\, e^{\frac{i}{\hbar}S^t} e^{\frac{i}{\hbar}P^t(x-Q^t)} f^t\left(\frac{x-Q^t}{\sqrt{h}}\right) \equiv (K^h_{S^t,P^t,Q^t} f^t)(x), \tag{3}$$

with $S^t \in \mathbb{R}$, $P^t \in \mathbb{R}^n$, $Q^t \in \mathbb{R}^n$, $f^t \in S(\mathbb{R}^n)$. The normalizing factor should be chosen to be const $\sim h^{-n/4}$ in order to make the norm $\|\psi^t\|$ to be of the order $O(1)$.

One finds by the direct calculation that the wave packet ψ^t approximately satisfies equation (1) under conditions

$$\frac{dS^t}{dt} = P^t \frac{dQ^t}{dt} - H(Q^t, P^t), \tag{4}$$

$$\frac{dQ^t_j}{dt} = \frac{\partial H}{\partial P_j}(Q^t, P^t), \qquad \frac{dP^t_j}{dt} = -\frac{\partial H}{\partial Q_j}(Q^t, P^t), \tag{5}$$

$$i\frac{\partial f^t(\xi)}{\partial t} = \sum_{js}\left[\frac{1}{2}\frac{1}{i}\frac{\partial}{\partial \xi_j}\frac{\partial^2 H}{\partial P_j \partial P_s}\frac{1}{i}\frac{\partial}{\partial \xi_s} + \xi_j \frac{\partial^2 H}{\partial Q_j \partial P_s}\frac{1}{i}\frac{\partial}{\partial \xi_s} + \frac{1}{2}\xi_j \frac{\partial^2 H}{\partial P_j \partial P_s}\xi_s\right] f^t(\xi). \tag{6}$$

Let us compare the substitutions (2) and (3) at fixed time moment t. One can notice that the width of the WKB wave function is of the order $O(1)$; the same conclusion is valid for its Fourier transformation as well. On the other hand, the Maslov wave packet (3), as well as its Fourier transformation, is of the width $O(\sqrt{h})$ being small as $h \to 0$. This means that the uncertainties of coordinates and momenta are of the order $O(\sqrt{h})$. The quantum state (3) can be interpreted as a classical particle with coordinates $Q^t \in \mathbb{R}^n$ and momenta $P^t \in \mathbb{R}^n$ then.

It is remarkable that all known semiclassical solutions (see [4]) of equation (1) (including WKB) can be constructed [6] as superpositions of the Maslov wave packets (3):

$$\psi(x) = \text{const} \int d\alpha e^{\frac{i}{\hbar}S(\alpha)} e^{\frac{i}{\hbar}P(\alpha)(x-Q(\alpha))} f\left(\alpha, \frac{x-Q(\alpha)}{\sqrt{h}}\right), \qquad \alpha \in \mathbb{R}^k. \tag{7}$$

It happens that the state (7) is nontrivial, only if the Maslov isotropic condition

$$\frac{\partial S}{\partial \alpha_a} = \sum_j P_j \frac{\partial Q_j}{\partial \alpha_a} \tag{8}$$

is satisfied. Otherwise,

$$\|\psi\| = O(h^\infty).$$

Under condition (8), one should choose const $\sim h^{-\frac{n+k}{4}}$, and

$$(\psi, \psi) \simeq \int d\alpha d\xi f^*(\alpha, \xi) \prod_a 2\pi\delta\left(\sum_j \left\{\frac{\partial P_j}{\partial \alpha_a}\xi_j - \frac{\partial Q_j}{\partial \alpha_a}\frac{1}{i}\frac{\partial}{\partial \xi_j}\right\}\right) f(\alpha, \xi). \tag{9}$$

Notice also that the wave function (7) is not small only in the vicinity (of the width $\sqrt{h}$) around the k-dimensional surface $\{Q(\alpha)\}$.

4 Geometry of semiclassical mechanics

The wave-packet semiclassical solutions (3) can be interpreted geometrically as follows [7]. First of all, notice that the state $K^h_{S,P,Q} f$ can be viewed as a point on a bundle ("semiclassical

bundle"). Its base $\mathcal{X}$ is a classical phase space extended by adding a new variable S; it is $\mathcal{X} = \mathbb{R}^{2n+1} = \{X = (S, P, Q)\}$. All fibres $\mathcal{F}_X = L^2(\mathbb{R}^n) = \{f\}$ being Hilbert spaces of quantum states in a given classical background $X \in \mathcal{X}$ are identical.

The evolution transformation takes the initial data $(X^0 = (S^0, P^0, Q^0), f^0)$ to the solution of the Cauchy problem for system of equations (4)–(6) $(X^t = (S^t, P^t, Q^t), f^t)$:

$$X^t = u_t X^0, \qquad f^t = U_t(X^t \leftarrow X^0) f^0.$$

It is an automorphism of the semiclassical bundle, since the evolution of X does not depend on f.

Classical geometric structures appear in the semiclassical theory as follows. The wave functions

$$K^h_{X^t} f^t \equiv K^h_{S^t, P^t, Q^t} f^t, \tag{10}$$

$$K^h_{X^t + h\delta X^t}(f^t + h\delta f^t) \equiv K^h_{S^t + h\delta S^t, P^t + h\delta P^t, Q^t + h\delta Q^t}(f^t + h\delta f^t) \tag{11}$$

should both be approximate solutions of equation (1) as $h \to 0$, provided that $(X^t + h\delta X^t, f^t + h\delta f^t)$ satisfies the system (4)–(6). However, as $h \to 0$,

$$K^h_{X^t + h\delta X^t}(f^t + h\delta f^t) \simeq e^{-i\omega_{X^t}[\delta X^t]} K^h_{X^t} f^t, \tag{12}$$

with

$$\omega_X[\delta X] = \sum_j P_j \delta Q_j - \delta S, \tag{13}$$

so that both functions (10) and (12) may be approximate solutions only if

$$\omega_{X^t}[\delta X^t] = \sum_j P^t_j \delta Q^t_j - \delta S^t = \text{const}. \tag{14}$$

Analogously, one finds

$$K^h_{X^t + \sqrt{h}\delta X^t}(f^t + \sqrt{h}\delta f^t) = \text{const}\, K^h_{X^t} e^{i\Omega_{X^t}[\delta X^t]} f^t, \tag{15}$$

with

$$(\Omega_X[\delta X] f)(\xi) = \sum_j \left(\delta P_j \xi_j - \delta Q_j \frac{1}{i} \frac{\partial}{\partial \xi_j} \right) f(\xi), \tag{16}$$

so that the operator (16) should take solutions of equation (6) to solutions, i.e.:

$$\Omega_{X^t}[\delta X^t] U_t(X^t \leftarrow X^0) = U_t(X^t \leftarrow X^0) \Omega_{X^0}[\delta X^0]. \tag{17}$$

Thus, the introduced 1-forms ω (number-valued) and Ω (operator-valued) enter to important relations (14) and (17), i.e. they are conserved under time evolution. Here δX^t is any solution of variation system for equations (4), (5), i.e.

$$u_t(X^0 + \delta X^0) \simeq X^t + \delta X^t.$$

Another important property is

$$[\Omega_X[\delta X_1]; \Omega_X[\delta X_2]] = i d\omega_X(\delta X_1, \delta X_2).$$

The superposition state (7) of the form

$$\psi(x) = \text{const} \int d\alpha K^h_{X(\alpha)} f(\alpha) \tag{18}$$

may be viewed as a k-dimensional surface on the semiclassical bundle $\{(X(\alpha), f(\alpha))\}$; the Maslov isotropic condition (8) and inner product formula (9) can be rewritten as

$$\omega_X \left[\frac{\partial X}{\partial \alpha_a}\right] = 0, \tag{19}$$

$$(\psi, \psi) \simeq \int d\alpha \left(f(\alpha), \prod_a 2\pi\delta \left(\Omega_X \left[\frac{\partial X}{\partial \alpha_a}\right]\right) f(\alpha)\right). \tag{20}$$

The comparison of geometric properties of classical, quantum and semiclassical mechanics is presented in Table 2.

Table 2. Classical, quantum and semiclassical mechanics.

	Classical mechanics	Quantum mechanics	Semiclassical mechanics
State space	Symplectic manifold	Hilbert space	Semiclassical bundle: base – symplectic manifold (classical states); fibres – Hilbert spaces (quantum states in classical background)
Geometric structures	Symplectic 2-form	Inner product	1-form on base ("action"); operator-valued 1-form on base (values are operators in fibre); inner products in fibres
Symmetry transformations	Lie group acts on manifold	Lie group is represented by unitary operators	Lie group is represented by automorphisms of the bundle

5 Semiclassical symmetries and their infinitesimal properties

Symmetry group G should act on the semiclassical bundle as a group of automorphisms. This means that for each $g \in G$ classical symmetry transformation $u_g : \mathcal{X} \to \mathcal{X}$ is specified; the properties

$$u_e = 1, \qquad u_{g_1 g_2} = u_{g_1} u_{g_2} \tag{21}$$

should be specified. Unitary operators $U_g(u_g X \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{u_g X}$ such that

$$U_{g_1 g_2}(u_{g_1 g_2} X \leftarrow X) = U_{g_1}(u_{g_1 g_2} X \leftarrow u_{g_2} X) U_{g_2}(u_{g_2} X \leftarrow X) \tag{22}$$

are also specified. The geometric structures (1-forms ω and Ω) should conserve under symmetry transformations. Therefore, if

$$u_g(X + \delta X_0) = u_g X + \delta X_g + \cdots,$$

one should have

$$\omega_{u_g X}[\delta X_g] = \omega_X[\delta X_0], \tag{23}$$

$$\Omega_{u_g X}[\delta X_g] U_g(u_g X \leftarrow X) = U_g(u_g X \leftarrow X) \Omega_X[\delta X_0]. \tag{24}$$

Properties (23) and (24) provide conservation of Maslov isotropic condition (19) and inner product (20) for superposition states (18).

Consider now the infinitesimal properties of the semiclassical group transformations [8]. Let the semiclassical bundle be trivial (i.e. $\mathcal{F}_X = \mathcal{F}$. For $g(t) = \exp[t\delta g]$, transformations $u_{g(t)}$ are constructed similarly to the classical theory (Table 1): $u_{g(t)}$ takes the initial condition for the equation

$$\frac{dX(t)}{dt} = \nabla[\delta g](X(t)), \qquad X(t) \in \mathcal{X} \tag{25}$$

to the solution at time t. Here $\nabla[\delta g](X)$ is a Hamiltonian vector field. The unitary operator $U_g(u_g X \leftarrow X)$ takes the initial condition for the equation

$$i\frac{df(t)}{dt} = H[\delta g|X(t)]f(t), \qquad f(t) \in \mathcal{F} \tag{26}$$

to the solution at time t:

$$U_{g(t)}(u_{g(t)}X \leftarrow X) : f(0) \mapsto f(t).$$

Here $H[\delta g|X]$ is a self-adjoint X-dependent operator in $\mathcal{F}$.

The infinitesimal analog of the group property (22) is

$$[i\nabla[\delta g_1] - H[\delta g_1|X]; i\nabla[\delta g_2] - H[\delta g_2|X]] = i(i\nabla[\delta g_1; \delta g_2] - H[[\delta g_1, \delta g_2]|X). \tag{27}$$

Invariance of 1-forms implies that

$$\nabla[\delta g]\omega = 0, \tag{28}$$

$$(\nabla[\delta g]\Omega)_X[\delta X] = i[\Omega_X[\delta X]; H[\delta g|X]]. \tag{29}$$

Algebraic property (27) can be interpreted in terms of operators acting in the space of sections of the semiclassical bundle. Namely, let Ψ be a section

$$\Psi = \{\Psi_Y \in \mathcal{F}_Y, Y \in \mathcal{X}\}.$$

Set $\check{U}_g : \Psi \mapsto \check{U}_g\Psi$, with

$$(\check{U}_g\Psi)_Y = U_g(Y \leftarrow u_{g^{-1}}Y)\Psi_{u_{g^{-1}}Y}. \tag{30}$$

Then the group property (22) is simplified:

$$\check{U}_{g_1g_2} = \check{U}_{g_1}\check{U}_{g_2}, \tag{31}$$

so that for $g(t) = \exp[t\delta g]$ one has $\check{U}_g = \exp[-it\check{H}[\delta g]$, with

$$\check{H}[\delta g] = H(\delta g|X) - i\nabla[\delta g].$$

The property (27) reads

$$[\check{H}[\delta g_1]; \check{H}[\delta g_2]]\Psi = -i\check{H}[\delta g_1; \delta g_2]\Psi. \tag{32}$$

The mathematical formulations of the results can be found in [9, 10].

6 Further developments and applications

The obtained results can be applied to quantum field theories. The proof of Poincaré invariance of semiclassical scalar field theory is presented in [11].

One can investigate the semiclassical theory [12, 13] of gauge theories [14, 2]. For such constrained systems, the classical extended phase space is not flat ($\mathcal{X} \neq \mathbb{R}^{2n+1}$). It is an m-dimensional surface

$$\Lambda_a(X) = 0, \qquad a = \overline{1,m}. \tag{33}$$

Here the constraints Λ_a obey the relations

$$\{\Lambda_a; \Lambda_b\} = 0$$

on the constraint surface (33).

Moreover, there is an equivalence relation on the classical phase space. The functions Λ_a are generators of classical gauge transformations. The m-dimensional gauge group $\mathcal{L}$ acts on the phase space

$$\alpha \in \mathcal{L} \mapsto \lambda_\alpha : \mathcal{X} \to \mathcal{X},$$
$$\lambda_e = 1, \qquad \lambda_{\alpha_1\alpha_2} = \lambda_{\alpha_1}\lambda_{\alpha_2},$$

and classical states $X \sim \lambda_\alpha X$ are set to be equivalent.

In the semiclassical theory, the semiclassical bundle is nontrivial: the inner products in $\mathcal{F}_X$ are different for different X. The gauge group acts on the semiclassical bundle. The unitary operators

$$V_\alpha(\lambda_\alpha X \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{\lambda_\alpha X}$$

such that

$$V_{\alpha_1\alpha_2}(\lambda_{\alpha_1\alpha_2}X \leftarrow X) = V_{\alpha_1}(\lambda_{\alpha_1\alpha_2}X \leftarrow \lambda_{\alpha_2}X)V_{\alpha_2}(\lambda_{\alpha_2}X \leftarrow X)$$

are given. The semiclassical states

$$(X, f) \sim (\lambda_\alpha X, V_\alpha(\lambda_\alpha X \leftarrow X)f) \tag{34}$$

are set to be equivalent.

Introduce a notion of a symmetry group for gauge theories. For $g \in G$, the transformations

$$u_g : \mathcal{X} \to \mathcal{X}; \qquad U_g(u_g X \leftarrow X) : \mathcal{F}_X \to \mathcal{F}_{u_g X}$$

should be given. Gauge equivalent states (34) should be taken to gauge equivalent, i.e.

$$(X_1, f_1) \sim (X_2, f_2)$$

implies

$$(u_g X_1, U_g(u_g X \leftarrow X)f_1) \sim (u_g X_2, U_g(u_g X \leftarrow X)f_2). \tag{35}$$

Requirements (21) and (22) are too strong for gauge theories. One should impose a weaker condition

$$(u_{g_1g_2}X, U_{g_1g_2}(u_{g_1g_2}X \leftarrow X)f) \sim (u_{g_1}u_{g_2}X, U_{g_1}(u_{g_1}u_{g_2}X \leftarrow u_{g_2}X)U_{g_2}(u_{g_2}X \leftarrow X)f). \tag{36}$$

To investigate infinitesimal properties, consider the gauge invariant sections of the semiclassical bundle, which satisfies the condition

$$V_\alpha(\lambda_\alpha Y \leftarrow Y)\Psi_Y = \Psi_{\lambda_\alpha Y}.$$

Introduce the operator $\check{U}_g$ with the help of equation (30). The first requirement (35) means that gauge invariant sections are taken to gauge invariant. The property (36) means that relation (31) is satisfied. Then the infinitesimal property (32) is obeyed for gauge-invariant sections Ψ.

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Alternative Gauge Procedure with Fields of Various Ranks

Douglas SINGLETON †, *Atsushi YOSHIDA* ‡ *and Akira KATO* †

† *Physics Dept., CSU Fresno, Fresno, CA 93740-8031, USA*
E-mail: *dougs@csufresno.edu, ak086@csufresno.edu*

‡ *Science Dept., Saint Anne's Belfield School, Charlottesville, VA 22903, USA*
E-mail: *ayoshida@stab.org*

Making a global phase symmetry local usually involves the introduction of a rank 1, vector field in the definition of the covariant derivative. In this paper we show how a symmetry can be gauged using fields of various ranks in the definition of the covariant derivative. Some of these higher rank gauge theories share similarities with general relativity in that the covariant derivative has terms which are derivatives of some more fundamental quantity. Most of the Lagrangians that we find under these higher rank gauge theories lead to nonrenormalizable quantum theories which is also similar to general relativity.

1 Standard gauge procedure with a rank 1 field

Turning global symmetries into local ones (i.e. gauging a symmetry) is an important feature of most modern field theories. For example, Maxwell's theory can be derived by gauging an Abelian $U(1)$ phase symmetry. In this section the standard gauge procedure is summarized using a complex scalar matter field φ. Throughout the paper the complex scalar field φ will be used to illustrate the higher rank gauge procedures. The same procedure applies equally to other matter field (e.g. a spinor or vector fields). The Lagrange density for φ is

$$\mathcal{L}_{\text{scalar}} = (\partial_\mu \varphi)^* (\partial^\mu \varphi) + \cdots . \tag{1}$$

The ellipses leave off non-derivative terms (e.g. mass $m^2 \varphi^* \varphi$ or self-interaction terms $\lambda(\varphi^* \varphi)^2$). This Lagrange density satisfies the global phase symmetry

$$\varphi(x) \to e^{-ig\Lambda} \varphi(x), \qquad \varphi^*(x) \to e^{ig\Lambda} \varphi^*(x), \tag{2}$$

where g is the coupling and Λ is a constant phase. This phase symmetry can be made local ($g\Lambda \to g\Lambda(x)$) by replacing the ordinary derivative with the covariant derivative

$$\partial_\mu \to D^{[1]}_\mu \equiv \partial_\mu - ig\sigma_{\mu\nu} A^\nu. \tag{3}$$

Throughout the paper the bracketed superscript will indicate the rank of the gauge field. We have introduced a rank-2 operator $\sigma_{\mu\nu}$ since this will allow us to give the general gauging procedure for gauge fields of other ranks. The newly introduced four-vector gauge field A_μ is required to transform as

$$A_\mu \to A_\mu - \Gamma_\mu. \tag{4}$$

In order for the local symmetry version of (2) to be valid Γ_μ, $\sigma_{\mu\nu}$ and $\Lambda(x)$ must satisfy

$$\sigma_{\mu\nu} \Gamma^\nu - \partial_\mu \Lambda = 0. \tag{5}$$

The Maxwell theory case corresponds to choosing

$$\sigma_{\mu\nu} = \eta_{\mu\nu}, \qquad \Gamma_\mu = \partial_\mu \Lambda. \tag{6}$$

However, it is also possible to make the following choice

$$\sigma_{\mu\nu} = \partial_\mu \partial_\nu, \qquad \Lambda = \partial_\mu \Gamma^\mu + f(x), \tag{7}$$

where $f(x)$ is a divergenceless function, $\partial_\mu f(x) = 0$. An interesting difference between the two options is that in (6) Γ_μ is defined in terms of the local phase factor $\Lambda(x)$, while for (7) $\Lambda(x)$ is defined in terms of Γ_μ.

Next one constructs a "kinetic" energy term for A_μ by introducing the field strength tensor

$$F^{[1]}_{\mu\nu} = A_1 \partial_\mu A_\nu + B_1 \partial_\nu A_\mu. \tag{8}$$

This is invariant under (4), (6) if $A_1 + B_1 = 0$, that is to say, if $F^{[1]}_{\mu\nu}$ is anti-symmetric between μ and ν. The reason for writing $F^{[1]}_{\mu\nu}$ as in (8) is that it makes the connection with the field strength tensors of the higher rank cases more transparent. $F^{[1]}_{\mu\nu}$ is invariant in the case of (6), however for the case of (7) it is not since Γ_μ is arbitrary. Because of this arbitrariness one can always make $A_\mu \to 0$ and $F^{[1]}_{\mu\nu} = 0$ by taking $\Gamma_\mu = A_\mu$. If we take case (6) in the construction of $F^{[1]}_{\mu\nu}$ the following Lagrange density

$$\mathcal{L}'_{\rm scalar} = (D^{[1]}_\mu \varphi)^* (D^{[1]\mu} \varphi) - \tfrac{1}{4} F^{[1]}_{\mu\nu} F^{[1]\mu\nu} + \cdots, \tag{9}$$

is invariant under the combined transformations of equations (4), (6) and the local version of equation (2).

The gravitational interaction follows a similar, although not exactly identical path. One can take the global space-time symmetries of special relativity and make them local [1] to arrive at a theory of the gravitational interaction. Here one again replaces ordinary derivatives with covariant derivatives. For example, the covariant derivative of a vector field V_ν is

$$\partial_\mu V_\nu \to \partial_\mu V_\nu + \Gamma^\alpha_{\mu\nu} V_\alpha \qquad \text{with} \qquad \Gamma^\alpha_{\mu\nu} = \tfrac{1}{2} g^{\alpha\sigma} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) \tag{10}$$

Unlike the gauge field in equation (3), $\Gamma^\alpha_{\mu\nu}$ is not fundamental, but is defined in terms of the first derivatives of the metric tensor $g_{\mu\nu}$.

2 Gauge procedure with a rank 0 field

Here we present the generalized gauge procedure for a rank 0 field. We begin with the Lagrange density of a complex scalar matter field of equation (1) and a local phase transformation (i.e. (2) with $\Lambda \to \Lambda(x)$.) The space-time dependence of $\Lambda(x)$ means that the derivative of φ and φ^* are no longer invariant under (2), but become $\partial_\mu \varphi \to \partial_\mu \varphi - ig(\partial_\mu \Lambda)\varphi$ and $\partial_\mu \varphi^* \to \partial_\mu \varphi^* + ig(\partial_\mu \Lambda)\varphi^*$. As in the case of the gauging procedure with a vector field we want to find a generalization of the derivative operator ∂_μ, which is invariant under the local version of (2). We define this generalized rank 0 derivative operator as

$$D^{[0]}_\mu \equiv \partial_\mu - ig\partial_\mu \Phi. \tag{11}$$

Φ is real, scalar gauge field which is required to undergo the transformation

$$\Phi(x) \to \Phi(x) - \Lambda(x). \tag{12}$$

These transformations of the scalar field Φ are similar to the toy model considered in [4]. With replacement of the ordinary derivative in equation (1) with $D^{[0]}_\mu$ the Lagrangian of equation (1) becomes invariant under the local versions of (2), (12). As in the non-standard rank 1 case

of (7) there is no kinetic term, since from the transformation of (12) it is always possible to take $\Phi = 0$ by choosing $\Lambda = \Phi$. Thus the Lagrangian

$$\mathcal{L}_{\text{scalar}} = (D_\mu^{[0]}\varphi)^*(D^{\mu\,[0]}\varphi) = (\partial_\mu\varphi^* + ig\varphi^*\partial_\mu\Phi)(\partial^\mu\varphi - ig\varphi\partial^\mu\Phi) \tag{13}$$

is invariant under (2), (12). This Lagrangian has no kinetic term for Φ. In contrast to the standard covariant derivative $\partial_\mu - ieA_\mu$, the covariant derivative of (11) involves the derivative of the fundamental gauge field Φ. This can be compared to the covariant derivative of general relativity which involves derivatives of a more fundamental quantity (the metric tensor $g_{\mu\nu}$).

One can apply this rank 0 procedure starting with matter fields other than a complex scalar field φ. In [5] this was done starting with a vector field, and this could be interpreted as a gauging of the electromagnetic dual symmetry [6]. In [7] a related localizing of the Schwarz–Sen [8] dual symmetry was given. The idea of having a scalar gauge field has also been considered by other authors. In [9] a unified version of the Standard Model was constructed via the introduction of a generalized covariant derivative which involved *both* vector and scalar gauge fields.

3 Classification of Gauge procedures

At this stage we have encountered three different categories of gauging procedures:

• **Trivial Case:** This is illustrated by the rank 0 case of (12). The gauge field transformation of (12) allows one to transform away the gauge field by taking $\Phi = \Lambda$. This case can be seen as a special case of the standard gauge procedure with the association $A_\mu \propto \partial_\mu\Phi$. This is a pure gauge case since for such an A_μ one finds $F_{\mu\nu}^{[1]} = 0$. For this trivial case the phase factor and gauge transformation function are related without the need of a derivative operator.

• **Semi-Trivial Case:** This is illustrated by the rank 1 case of (7). Here the transformation function of the gauge field Γ_μ is arbitrarily given. By choosing $A_\mu = \Gamma_\mu$ it is always possible to transform away the gauge field, making it non-dynamical. Both this case and the trivial case are marked by having covariant derivatives of the form

$$\partial_\mu - ig\partial_\mu(\text{Scalar}), \tag{14}$$

where Scalar is some scalar quantity. In contrast to the trivial case in the semi-trivial case the phase factor and gauge transformation function are related using derivative operator(s).

• **Non-Trivial Case:** This is illustrated by the rank 1 Maxwell Theory case of (6). In contrast to the previous semi-trivial case the phase factor Λ is arbitrary while the gauge transformation function Γ_μ takes a restricted form in terms of the phase function. The gauge field is dynamical and the covariant derivative does not take the form of (14).

In the following sections we show that for the abstract phase symmetries it is possible to formulate a gauge procedure with fields of various ranks as gauge fields. The procedure will employ generalizations of (3), (4) of the form

$$D_\mu = \partial_\mu - ig\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}A^{\mu_1\mu_2\cdots\mu_n}, \qquad A^{\mu_1\mu_2\cdots\mu_n} \to A^{\mu_1\mu_2\cdots\mu_n} - \Gamma^{\mu_1\mu_2\cdots\mu_n}. \tag{15}$$

As with general relativity the covariant derivative of this generalized gauge procedure will have connections, $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}A^{\mu_1\mu_2\cdots\mu_n}$, which are defined in terms of the derivative of a more fundamental object. Many of the theories obtained from this generalized gauging will have dimensionful coupling constants and lead to nonrenormalizable theories as is also the case with general relativity. Here we do not attempt to give a phenomenological application of the field theories that result from the generalized gauging procedure. We simply demonstrate an alternative, general way in which a global phase symmetry can be made local. This may give some insight into a connection between the way in which one usually gauges abstract phase symmetries via a fundamental vector field and the way in which space-time symmetries are made local. There has been previous work on higher rank (i.e. higher spin) gauge fields [2, 3]. The differences between the present work and [2,3] are discussed in the concluding section.

4 Gauge procedure with a rank 2 field

In this section we will gauge the local version of the symmetry of equation (2) for the Lagrange density in equation (1) using a rank 2 gauge field. We define a covariant derivative as

$$D_\mu^{[2]} \equiv \partial_\mu - ig\sigma_{\mu\nu\rho}A^{\nu\rho}, \tag{16}$$

where we have introduced a rank 3 operator $\sigma_{\mu\nu\rho}$, and a rank 2 gauge field $A_{\nu\rho}$. Here we consider the case when the gauge field indices are symmetric. The antisymmetric case will be considered in a longer work [10].

For symmetric $A_{\nu\rho}$ ($A_{\nu\rho} = A_{\rho\nu}$) the operator $\sigma_{\mu\nu\rho}$ has the partial symmetry $\sigma_{\mu\nu\rho} = \sigma_{\mu\rho\nu}$. Constructing $\sigma_{\mu\nu\rho}$ from $\eta_{\mu\nu}$ and ∂_μ, we can write a general form as

$$\sigma_{\mu\nu\rho} = \tfrac{1}{2}a_2(\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu) + b_2\eta_{\nu\rho}\partial_\mu + c_2\partial_\mu\partial_\nu\partial_\rho, \tag{17}$$

where a_2, b_2, c_2 are constants with the subscript indicating the rank of $A_{\mu\nu}$. The last term in (17) has a greater symmetry (it is symmetric in all three indices) than required. In conjunction with (2) we require that $A_{\mu\nu}$ transforms as

$$A_{\mu\nu} \to A_{\mu\nu} - \Gamma_{\mu\nu}. \tag{18}$$

If the rank 2 function $\Gamma_{\mu\nu}$ and Λ satisfy

$$\sigma_{\mu\nu\rho}\Gamma^{\nu\rho} - \partial_\mu\Lambda = 0, \tag{19}$$

then the Lagrange density $\mathcal{L}_{\text{scalar}} = (D_\mu^{[2]}\varphi)^*(D^{[2]\ \mu}\varphi) + \cdots$ will be invariant under the combined (2), (18). We will consider three special cases.

1. $[\boldsymbol{a_2 = 1,\ b_2 = c_2 = 0}]$. The covariant derivative and the condition in equation (19) become

$$D_\mu^{[2]} = \partial_\mu - ig\partial_\nu A_\mu^\nu, \qquad \partial_\nu\Gamma_\mu^\nu - \partial_\mu\Lambda = 0. \tag{20}$$

The symmetry of $A_{\mu\nu}$ was used in finding $D_\mu^{[2]}$. A general solution to the last equation in (20) is

$$\Gamma_{\mu\nu} = \eta_{\mu\nu}\Lambda + h_{\mu\nu}, \tag{21}$$

where it is required that $\partial^\nu h_{\mu\nu} = 0$. An example of a solution is

$$\Gamma_{\mu\nu} = \partial_\mu\partial_\nu f(x) + \eta_{\mu\nu}g(x), \qquad \Lambda = \partial_\nu\partial^\nu f(x) + g(x) + h(x), \tag{22}$$

where $f(x)$, $g(x)$ and $h(x)$ are arbitrary scalar functions. The function $h(x)$ must satisfy $\partial_\mu h(x) = 0$, which corresponds to the choice $h_{\mu\nu} = \partial_\mu\partial_\nu f(x) - \eta_{\mu\nu}(\partial_\rho\partial^\rho f(x) + h(x))$ in (21). The explicit gauge field transformation is

$$A_{\mu\nu} \to A_{\mu\nu} - \partial_\mu\partial_\nu f(x) - \eta_{\mu\nu}g(x). \tag{23}$$

2. $[\boldsymbol{b_2 = 1,\ a_2 = c_2 = 0}]$. The covariant derivative and the condition in equation (19) become

$$D_\mu^{[2]} = \partial_\mu - ig\partial_\mu A_\nu^\nu, \qquad \partial_\mu\Gamma_\nu^\nu - \partial_\mu\Lambda = 0. \tag{24}$$

The last equation in (24) can be satisfied by taking

$$\Lambda = \Gamma_\nu^\nu + f(x), \tag{25}$$

with $\partial_\mu f(x) = 0$. The gauge transformation function $\Gamma_{\mu\nu}$, and phase factor Λ are arbitrary, and are related without a derivative operator so this is a trivial case. Since $\Gamma_{\mu\nu}$ is arbitrary it should always be possible to transform the gauge field away via $A_{\mu\nu} \to A_{\mu\nu} - \Gamma_{\mu\nu}$ with $A_{\mu\nu} = \Gamma_{\mu\nu}$.

3. $[\boldsymbol{c_2 = 1,\ a_2 = b_2 = 0}]$. The covariant derivative and the condition in equation (19) become

$$D_\mu^{[2]} = \partial_\mu - ig\partial_\mu\partial_\nu\partial_\rho A^{\nu\rho}, \qquad \partial_\mu\partial_\nu\partial_\rho\Gamma^{\nu\rho} - \partial_\mu\Lambda = 0. \tag{26}$$

The last equation above can be satisfied by taking

$$\Lambda = \partial_\nu\partial_\rho\Gamma^{\nu\rho} + f(x), \tag{27}$$

with $\partial_\mu f(x) = 0$. This is a semi-trivial case since the gauge function is arbitrary, but the relationship between it and the phase factor involves the derivative operator.

For cases 2 and 3 the rank 2 gauge field is arbitrary, and any form for $\Gamma_{\mu\nu}$ works. For case 1 the specific form given in (22) is necessary. This difference can be traced to the different relationships between $\Gamma_{\mu\nu}$ and Λ given in the second equations in (20), (24), (26). Equations (24) and (26) involve the same index on the derivative, while for equation (20) the indices on the derivative operator are different. This is connected with the fact that cases 2 and 3 are trivial and semi-trivial as previously discussed, and have covariant derivatives of the form of equation (14).

Next we want to add a kinetic term involving $A_{\mu\nu}$ alone. Cases 2 and 3 are trivial and semi-trivial, so that $\Gamma_{\mu\nu}$ has a completely arbitrary form. In these cases $A_{\mu\nu}$ is not dynamical since it is possible to transform it away by taking $A_{\mu\nu} = \Gamma_{\mu\nu}$. Thus we only consider case 1 in constructing of an invariant field strength tensor. Also we will work with the special example given in (22) with $g(x) = 0$. Under these conditions the following rank 3 object

$$F_{\mu\nu\rho}^{[2]} = A_2\partial_\mu A_{\nu\rho} + B_2\partial_\nu A_{\mu\rho} + C_2\partial_\rho A_{\mu\nu}, \tag{28}$$

is invariant under the gauge field transformation if the constants obey $A_2 + B_2 + C_2 = 0$. $F_{\mu\nu\rho}^{[2]}$ is neither symmetric nor antisymmetric. Its defining feature is the permutation of the indices which generalizes the form of the rank 2 field strength tensor given in (8). The following Lagrangian is invariant under the local phase and gauge field transformations

$$\mathcal{L}_{\text{scalar}} = (D_\mu^{[2]}\varphi)^*(D^{\mu[2]}\varphi) + KF_{\mu\nu\rho}^{[2]}F^{[2]\mu\nu\rho} + \cdots, \tag{29}$$

where K is a constant. The kinetic energy term involving only the gauge field $A_{\mu\nu}$ is more complex than the rank 1 kinetic term in equation (9).

For all three cases (20), (24) and (26) the coupling g is dimensionful implying that the Lagrangian in equation (29) is nonrenormalizable. The mass dimension of g can be determined from equation (16). The derivative operator has mass dimension +1 and in the usual way the gauge field $A_{\mu\nu}$ is taken to have mass dimension +1. Thus for case 1 and 2 from (20) and (24) the coupling g has mass dimension -1, while for case 3 from (26) g has mass dimension -3.

5 Gauge procedure with a rank 3 field

In this section we will gauge the local version of the symmetry of equation (2) for the Lagrange density in equation (1) using a rank 3 gauge field. As n becomes large the number of possible terms in the definition of $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$ and in the construction of a kinetic term for the gauge field becomes larger and more complex. We define a covariant derivative as

$$D_\mu^{[3]} \equiv \partial_\mu - ig\sigma_{\mu\nu\rho\tau}A^{\nu\rho\tau}, \tag{30}$$

where we have introduced a rank 4 operator $\sigma_{\mu\nu\rho\tau}$, and a rank 3 gauge field $A_{\nu\rho\tau}$. In this paper we consider only the rank 3 gauge field with complete symmetry in its indices. Antisymmetric and mixed symmetric cases will be considered in a longer work in preparation [10]. For the totally symmetric gauge field the operator $\sigma_{\mu\nu\rho\tau}$ is symmetric in the last three indices.

Using the operators $\eta_{\mu\nu}$ and ∂_μ we can write a general form for $\sigma_{\mu\nu\rho\tau}$

$$\sigma_{\mu\nu\rho\tau} = \tfrac{1}{3}a_3(\eta_{\mu\nu}\eta_{\rho\tau} + \eta_{\mu\rho}\eta_{\nu\tau} + \eta_{\mu\tau}\eta_{\rho\nu}) + \tfrac{1}{3}b_3(\eta_{\mu\nu}\partial_\rho\partial_\tau + \eta_{\mu\rho}\partial_\nu\partial_\tau + \eta_{\mu\tau}\partial_\rho\partial_\nu) + \tfrac{1}{3}c_3(\eta_{\rho\nu}\partial_\mu\partial_\tau + \eta_{\tau\rho}\partial_\mu\partial_\nu + \eta_{\nu\tau}\partial_\mu\partial_\rho) + d_3\partial_\mu\partial_\nu\partial_\rho\partial_\tau, \tag{31}$$

where a_3, b_3, c_3, d_3 are constants with the subscript indicating the rank of $A_{\mu\nu\tau}$. The last term in (5) has a greater index symmetry than required: it is symmetric in all four indices. In conjunction with the transformation of equation (2) we require that $A_{\mu\nu\tau}$ transforms as

$$A_{\mu\nu\tau} \to A_{\mu\nu\tau} - \Gamma_{\mu\nu\tau}. \tag{32}$$

If the rank 3 gauge function $\Gamma_{\mu\nu\tau}$ and Λ satisfy

$$\sigma_{\mu\nu\rho\tau}\Gamma^{\nu\rho\tau} - \partial_\mu\Lambda = 0, \tag{33}$$

then the Lagrange density $\mathcal{L}_{\rm scalar} = (D^{[3]}_\mu\varphi)^*(D^{[3]\ \mu}\varphi) + \cdots$ will be invariant under the combined transformation (2), (32). As in the previous section we will consider four special cases.

1. $[\boldsymbol{a_3 = 1,\ b_3 = c_3 = d_3 = 0}]$. The covariant derivative and the condition in equation (33) become

$$D^{[3]}_\mu = \partial_\mu - igA_{\mu\rho}{}^{\rho}, \qquad \Gamma_{\mu\rho}{}^{\rho} - \partial_\mu\Lambda = 0, \tag{34}$$

where the symmetry of $A_{\mu\nu\rho}$ was used in determining $D^{[3]}_\mu$. An example of a solution to the last equation of (34) is

$$\Gamma_{\mu\nu\rho} = \partial_\mu\partial_\nu\partial_\rho f(x) + \tfrac{1}{6}(\eta_{\mu\nu}\partial_\rho + \eta_{\nu\rho}\partial_\mu + \eta_{\rho\mu}\partial_\nu)g(x), \qquad \Lambda = \partial_\rho\partial^\rho f(x) + g(x) + h(x), \tag{35}$$

where $f(x)$, $g(x)$, $h(x)$ are arbitrary functions with $\partial_\mu h(x) = 0$. Equation (35) appears as a generalization of the rank 2 example solution of (22). The explicit gauge field transformation is

$$A_{\mu\nu\rho} \to A_{\mu\nu\rho} - \partial_\mu\partial_\nu\partial_\rho f(x) - \tfrac{1}{6}(\eta_{\mu\nu}\partial_\rho + \eta_{\nu\rho}\partial_\mu + \eta_{\rho\mu}\partial_\nu)g(x). \tag{36}$$

This gauge field transformation appears as a generalized version of the rank 2 gauge field case given in (23). This case is closer to the standard vector gauge procedure of (6) than any of the previous rank 2 cases, since the covariant derivative in (34) only involves the gauge field and not derivatives of the gauge field.

2. $[\boldsymbol{b_3 = 1,\ a_3 = c_3 = d_3 = 0}]$. The covariant derivative and the condition in equation (33) become

$$D^{[3]}_\mu = \partial_\mu - ig\partial^\rho\partial^\tau A_{\mu\rho\tau}, \qquad \partial^\nu\partial^\rho\Gamma_{\mu\nu\rho} - \partial_\mu\Lambda = 0. \tag{37}$$

The last equation in (37) can be satisfied by taking the same $\Gamma_{\mu\nu\rho}$ as in (35), and a phase factor of the form

$$\Lambda = \partial_\rho\partial^\rho\partial_\tau\partial^\tau f(x) + \tfrac{1}{2}\partial_\tau\partial^\tau g(x) + h(x), \tag{38}$$

where $f(x)$, $g(x)$, $h(x)$ are arbitrary functions with $\partial_\mu h(x) = 0$. The gauge transformation of $A_{\mu\nu\rho}$ is identical to (36).

3. $[\boldsymbol{c_3 = 1,\ a_3 = b_3 = d_3 = 0}]$. The covariant derivative and the condition in equation (33) become

$$D^{[3]}_{\mu} = \partial_\mu - ig\partial_\mu\partial^\tau A^{\rho}_{\rho\tau}, \qquad \partial_\mu\partial^\tau\Gamma^{\rho}_{\rho\tau} - \partial_\mu\Lambda = 0. \tag{39}$$

The last equation above can be satisfied by taking

$$\Lambda = \partial^\tau\Gamma^{\rho}_{\rho\tau} + f, \tag{40}$$

where f is gradient-less. Since the covariant derivative is of the form (14) and the gauge function and phase factor are related using derivatives this case is characterized as semi-trivial. $\mathcal{L}_{\text{scalar}} = (D^{[3]}_{\mu}\varphi)^*(D^{[3]\ \mu}\varphi) + \cdots$ is invariant under the phase transformation of $\varphi(x)$ and an arbitrary gauge transformation $A_{\mu\nu\rho} \to A_{\mu\nu\rho} - \Gamma_{\mu\nu\rho}$.

4. $[\boldsymbol{d_3 = 1,\ a_3 = b_3 = c_3 = 0}]$. The covariant derivative and the condition in equation (33) become

$$D^{[3]}_{\mu} = \partial_\mu - ig\partial_\mu\partial_\nu\partial_\rho\partial_\tau A^{\nu\rho\tau}, \qquad \partial_\mu\partial_\nu\partial_\rho\partial_\tau\Gamma^{\nu\rho\tau} - \partial_\mu\Lambda = 0. \tag{41}$$

This last equation above can be satisfied by taking

$$\Lambda = \partial_\nu\partial_\rho\partial_\tau\Gamma^{\nu\rho\tau} + f, \tag{42}$$

where f is gradient-less. As for the preceding case this is also semi-trivial.

As before, we want to add a kinetic term involving $A_{\mu\nu\rho}$ alone. Cases 1 and 2 are both non-trivial and have the same transformation of $A_{\mu\nu\rho}$ given by (36). Thus both cases will have the same pure gauge invariant field strength tensors. Cases 3 and 4 are both semi-trivial and have arbitrary forms for the gauge transformation. Thus we only consider cases 1 and 2 in constructing the field strength tensor. Furthermore, as in the rank 2 fields of the previous section, we will work with the special case $g(x) = 0$ in (36) and (38). Thus, the rank 4 object

$$F^{[3]}_{\mu\nu\rho\tau} = A_3\partial_\mu A_{\nu\rho\tau} + B_3\partial_\nu A_{\rho\tau\mu} + C_3\partial_\rho A_{\tau\mu\nu} + D_3\partial_\tau A_{\mu\nu\rho}, \tag{43}$$

is invariant under (36) and (38) (given $g(x) = 0$) if the constants obey $A_3 + B_3 + C_3 + D_3 = 0$. The common feature between this invariant field strength tensor and the rank 1 and 2 cases of (8) and (28) is the permutation of indices. Thus the following Lagrangian is invariant under the local phase transformation and the gauge field transformation equation (36).

$$\mathcal{L}_{\text{scalar}} = (D^{[3]}_{\mu}\varphi)^*(D^{\mu[3]}\varphi) + KF^{[3]}_{\mu\nu\rho\tau}F^{[3]\mu\nu\rho\tau} + \cdots, \qquad K = \text{const.} \tag{44}$$

For the symmetric rank 3 gauge field the dimension of the coupling, and therefore whether the theory is renormalizable or not is different for each of the four cases above. For case 1, there are no derivative terms which appear in the second term in the covariant derivative in (34). Taking $A_{\mu\nu\rho}$ to have the usual mass dimension of $+1$, then g is dimensionless. This case may lead to a renormalizable theory. For cases 2 and 3 two derivatives appear in the second term of the covariant derivative in (37), (39), which implies that g should have mass dimension -2. Finally, for case 4 four derivatives appear in the second term of the covariant derivative in (41), which gives g a mass dimension of -4.

6 Summary and conclusion

Here we summarize the overall structure of the various gauging procedures presented here.

1. Starting from some initial matter field a local version of the phase symmetry in equation (2) is imposed. A covariant derivative of the form of the first equation in (15) and a rank n gauge field which transforms like the second equation in (15) are introduced.

2. The definition of the covariant derivative in (15) involves the introduction of a rank $n+1$ operator $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$, which is constructed from ∂_μ and $\eta_{\mu\nu}$. The freedom in the construction of $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$ can be seen from (6), (7), (17), and (5).

3. There were three different categories of gauge procedures: trivial, semi-trivial, or non-trivial. The trivial and semi-trivial cases involve introduction of an arbitrary gauge function $\Gamma_{\mu_1\mu_2\cdots}$ in terms of which the phase factor Λ was defined either without (trivial case) or with (semi-trivial) use of the derivative operator. The non-trivial case involves the introduction of an arbitrary phase factor Λ in terms of which the gauge function was defined. For the trivial and semi-trivial cases the gauge field could always be transformed away. For the non-trivial case one can construct an invariant field strength tensor and a kinetic energy term.

4. For the rank n symmetric gauge field one can define a rank $n+1$ field strength tensor (as in equations (8), (28), (43)) which was invariant under just the transformation of the gauge field, equations (23) or (36). This allows the construction of kinetic terms for the gauge field in the Lagrangian. The rank $n+1$ field strength tensor has a permutation symmetry among its indices.

5. The coupling constant g in general will have a non-zero mass dimension which can be determined by the covariant derivative, the structure of the $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$ and taking the gauge field $A_{\mu_1\mu_2\cdots\mu_m}$ to have the standard mass dimension $+1$. The mass dimension of g is then the inverse of $\sigma_{\mu_1\mu_2\cdots\mu_{n+1}}$. For example, (20) and (24) imply g has a mass dimension of -1; equations (7) and (37) imply g has a mass dimension of -2. These theories having a dimensionful coupling are nonrenormalizable. There are cases (e.g. the standard rank 1 case of (6) or the rank 3 case of (34)), where g has mass dimension 0 and should therefore be renormalizable.

There have been other studies of higher rank (i.e. higher spin) gauge fields. In particular, the work of Fronsdal sought to extend a gauge procedure to higher rank fields of both integer [2] and half-integer spin [3]. In these works the gauge transformation of the gauge fields is different from that in the present work. In [2] the transformation of the rank n gauge field involves one derivative operator acting on rank $n-1$ gauge parameters. Here the transformation of the rank n gauge field involves up to n derivative operators acting on a scalar function (for example see equations (32) and (38)). In both the present work and in [2] the rank $n \geq 2$ gauge fields are totally symmetric under exchange of indices. However, in [2] the gauge fields must satisfy a traceless condition that is not necessary in the present work. A current review of these higher spin gauge theories, and connections to supersymmetry and string theory can be found in [11].

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Charge Fractionalization in a Planar System with a Point-Like Vortex Defect at Finite Temperature

Yurii A. SITENKO † *and Volodymyr M. GORKAVENKO* ‡

† *Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine*
E-mail: *yusitenko@bitp.kiev.ua*

‡ *Department of Physics, Taras Shevchenko National University of Kyiv, 6 Academician Glushkov Ave., 03680 Kyiv, Ukraine*
E-mail: *gorka@univ.kiev.ua*

States with fractional charge can appear in quantum systems with topological defects. We consider an ideal gas of two-dimensional Dirac fermions in the background of a point-like magnetic vortex with arbitrary flux and find that this system acquires fractional electric charge at finite temperature. The functional dependence of the thermal average and quadratic fluctuation of the charge on the temperature, the vortex flux, and the continuous parameter of the boundary condition at the location of the vortex defect is determined.

1 Introduction

Spontaneous breakdown of continuous symmetries can give rise to topological defects (texture solitons) with rather interesting properties. A topological defect in three-dimensional space, which is characterized by the nontrivial second homotopy group, is known as a magnetic monopole [1, 2], see also genuine Ref. [3]. Vacuum fluctuations of quantized Dirac fields result in the monopole becoming a CP symmetry violating dyon, i.e. acquiring nonzero (and fractional) electric charge [4–6]. More recently the effect of thermal fluctuations of quantized Dirac fields in the presence of the monopole has been considered, yielding the temperature dependence of the induced charge [7, 8].

A topological defect in two-dimensional space, which is characterized by the nontrivial first homotopy group, is a cross-section of the Abrikosov–Nielsen–Olesen magnetic vortex [9, 10]. The vortex defect is described in terms of a spin-0 field which condenses and a spin-1 field corresponding to the spontaneously broken gauge group; the former is coupled to the latter in the minimal way with constant $e_{\rm cond}$. Single-valuedness of the condensate field and finiteness of the vortex energy imply that the vortex flux is related to $e_{\rm cond}$:

$$\Phi = \frac{1}{2\pi} \oint d\boldsymbol{x}\, \boldsymbol{V}(\boldsymbol{x}) = \frac{1}{e_{\rm cond}}, \tag{1}$$

where $\boldsymbol{V}(\boldsymbol{x})$ is the vector potential of the spin-1 field, and the integral is over a path enclosing once the vortex tube. The quantized fermion field is coupled minimally to the spin-1 field with constant e – the elementary charge; thus, quantum effects depend on the value of $e\Phi$. The case of $e_{\rm cond} = 2e$ ($e\Phi = 1/2$) is realized in ordinary Bardeen–Cooper–Schrieffer superconductors where the Cooper pair field condenses and, in addition, there are normal electron (pair-breaking) excitations. It remains still to be elucidated, whether other values of $e\Phi$ are realized in nature, although there are claims that vortices with fractional $e\Phi \neq 1/2$ exist in chiral superfluids and chiral and two-gap superconductors [11].

The aim of the present paper is to consider the effect of thermal fluctuations of quantized Dirac fields[1] in the presence of the vortex defect with arbitrary value of $e\Phi$, which results in the vortex acquiring fractional electric charge; the zero-temperature effect was considered earlier [13–15]. Since continuous symmetry is not spontaneously broken at the core of the defect, it seems reasonable to exclude the region of the defect and to impose a boundary condition for quantized fields at the edge of this region. Thus, quantum effects depend both on $e\Phi$ and real continuous quantity Θ which parameterizes the most general varieties of boundary conditions (for more details see next Section). This setup should not be confused with the setup when fermions are quantized in the presence of an extensive magnetic field with finite flux and the region of the nonvanishing field strength is not excluded. The induced charge in the latter case was considered in Refs. [16, 17] (zero temperature) and Ref. [18] (nonzero temperature), and we shall compare the results of both setups in Section 3.

The operator of the second-quantized fermion field in a static background can be presented in the form

$$\Psi(\boldsymbol{x},t)=\sum\!\!\!\!\!\!\int_{E>0} e^{-iEt}\langle \boldsymbol{x}|E,\lambda\rangle a_{E\lambda}+\sum\!\!\!\!\!\!\int_{E<0} e^{-iEt}\langle \boldsymbol{x}|E,\lambda\rangle b^{+}_{E\lambda}, \tag{2}$$

where $a^{+}_{E\lambda}$ and $a_{E\lambda}$ ($b^{+}_{E\lambda}$ and $b_{E\lambda}$) are the fermion (antifermion) creation and destruction operators satisfying anticommutation relations,

$$\left[a_{E\lambda},a^{+}_{E'\lambda'}\right]_{+}=\left[b_{E\lambda},b^{+}_{E'\lambda'}\right]_{+}=\langle E,\lambda|E',\lambda'\rangle, \tag{3}$$

and $\langle \boldsymbol{x}|E,\lambda\rangle$ is the solution to the stationary Dirac equation,

$$H\langle \boldsymbol{x}|E,\lambda\rangle=E\langle \boldsymbol{x}|E,\lambda\rangle, \tag{4}$$

H is the Dirac Hamiltonian, E is the energy and λ is the set of other parameters (quantum numbers) specifying a state; symbol $\sum\!\!\!\!\!\!\int$ means the summation over discrete and the integration (with a certain measure) over continuous values of all quantum numbers. Conventionally, the operators of dynamical invariants are defined as bilinears of the fermion field operators, and, thus, comprizing: the energy operator (temporal component of the energy-momentum vector),

$$\hat{P}^0=\frac{i}{4}\int d^dx\left(\left[\Psi^{+},\partial_t\Psi\right]_{-}-\left[\partial_t\Psi^{+},\Psi\right]_{-}\right), \tag{5}$$

and the fermion number operator,

$$\hat{N}=\frac{1}{2}\int d^dx\left[\Psi^{+},\Psi\right]_{-}, \tag{6}$$

where d is the space dimension. Operators (5) and (6) commute and are thus diagonal in the fermion and antifermion creation and destruction operators.

The thermal average of the fermion number operator over the canonical ensemble is defined as (see, e.g., Ref. [19])

$$\langle\hat{N}\rangle=\frac{\mathrm{Sp}\,\hat{N}\exp(-\beta\hat{P}^0)}{\mathrm{Sp}\exp(-\beta\hat{P}^0)},\qquad \beta=(k_BT)^{-1}, \tag{7}$$

[1]This may be relevant for various particle physics models with applications ranging from early Universe cosmology to hot nuclear matter phenomenology, and even for condensed matter models, because effectively quasirelativistic fermions arise, in particular, in d-wave type II superconductors (see, e.g., Ref. [12]).

where T is the equilibrium temperature, k_B is the Boltzmann constant, and Sp is the trace or the sum over the expectation values in the Fock state basis created by operators in equation (3). Appropriately, the electric charge of the quantum fermionic system in thermal equilibrium is given by expression

$$Q(T) \equiv e\langle \hat{N} \rangle = -\frac{e}{2} \int_{-\infty}^{\infty} dE\, \tau(E) \tanh\left(\frac{1}{2}\beta E\right), \tag{8}$$

where the last equality is obtained by transforming the right-hand side of equation (7) into an integral over the spectrum of the Dirac Hamiltonian (see, e.g., Ref. [18]), and the spectral density of the Dirac Hamiltonian (or density of states) is

$$\tau(E) = \frac{1}{\pi} \operatorname{Im} \operatorname{Tr} \frac{1}{H - E - i0}, \tag{9}$$

where Tr is the trace of an integro-differential operator in functional space:

$$\operatorname{Tr} U = \int d^d x \operatorname{tr} \langle \boldsymbol{x}|U|\boldsymbol{x}\rangle;$$

tr denotes the trace over spinor indices only; note that the functional trace should be regularized and renormalized by subtraction, if necessary.

Similarly, one gets expression for the quadratic fluctuation of the electric charge:

$$\Delta^2_{Q(T)} \equiv e^2 \left[\langle \hat{N}^2 \rangle - \left(\langle \hat{N} \rangle\right)^2\right] = \frac{e^2}{4} \int_{-\infty}^{\infty} dE\, \frac{\tau(E)}{\cosh^2\left(\frac{1}{2}\beta E\right)}. \tag{10}$$

Evidently, if the quadratic fluctuation becomes nonvanishing, then the corresponding dynamical invariant ceases to be a sharp quantum observable.

In the present paper we shall find electric charge (8) and its fluctuation (10) in the $d = 2$ quantum fermionic system in the background of a single static topological defect which is a two-dimensional cross section of the magnetic vortex.

2 Thermal average and fluctuation of the charge

Taking into account equation (9), one can get the following contour integral representation for induced charge (8) and its quadratic fluctuation (10):

$$Q(T) = -\frac{e}{2} \int_C \frac{d\omega}{2\pi i} \tanh\left(\frac{1}{2}\beta\omega\right) \operatorname{Tr}(H - \omega)^{-1}, \tag{11}$$

and

$$\Delta^2_{Q(T)} = \frac{e^2}{4} \int_C \frac{d\omega}{2\pi i} \operatorname{sech}^2\left(\frac{1}{2}\beta\omega\right) \operatorname{Tr}(H - \omega)^{-1}, \tag{12}$$

where C is the contour $(-\infty + i0, +\infty + i0)$ and $(+\infty - i0, -\infty - i0)$ in the complex ω-plane.

Thus we obtain:

$$\begin{aligned} Q(T) = -\frac{e}{2} \operatorname{sgn}(m) \Bigg\{ & \frac{1}{2} \left[\operatorname{sgn}\left(1 + A^{-1}\right) - \operatorname{sgn}(1 + A)\right] \tanh\left(\frac{1}{2}\beta |E_{BS}|\right) \\ & + \frac{2\sin(F\pi)}{\pi} \int_0^{\infty} \frac{du}{u\sqrt{u+1}} \tanh\left(\frac{1}{2}\beta |m| \sqrt{u+1}\right) \\ & \times \frac{FAu^F - (1-F)A^{-1}u^{1-F} - u\cos(F\pi) + \left(F - \frac{1}{2}\right) u \left(Au^F + A^{-1}u^{1-F}\right)}{\left[Au^F - A^{-1}u^{1-F} + 2\cos(F\pi)\right]^2 + 4(u+1)\sin^2(F\pi)} \Bigg\}, \end{aligned} \tag{13}$$

and

$$\Delta^2_{Q(T)} = \frac{e^2}{4}\left\{\frac{1}{2}[1-\mathrm{sgn}\,(A)]\,\mathrm{sech}^2\left(\frac{1}{2}\beta|E_{BS}|\right) - F(1-F)\,\mathrm{sech}^2\left(\frac{1}{2}\beta|m|\right)\right.$$
$$+\frac{2\sin(F\pi)}{\pi}\int_0^\infty \frac{du}{u}\,\mathrm{sech}^2\left(\frac{1}{2}\beta|m|\sqrt{u+1}\right)$$
$$\left.\times\frac{FAu^F+(1-F)A^{-1}u^{1-F}-(2F-1)u\cos(F\pi)}{\left[Au^F-A^{-1}u^{1-F}+2\cos(F\pi)\right]^2+4(u+1)\sin^2(F\pi)}\right\}, \tag{14}$$

where

$$F = e\Phi - [\![e\Phi]\!], \qquad 0\le F<1, \tag{15}$$

is the fractional part of $e\Phi$,

$$A = 2^{1-2F}\frac{\Gamma(1-F)}{\Gamma(F)}\tan\left(s\frac{\Theta}{2}+\frac{\pi}{4}\right), \tag{16}$$

and Θ is the continuous real parameter of the boundary condition at the location of the vortex defect; recall that bound state energy E_{BS} is determined implicitly by [15]

$$\frac{\left(1+m^{-1}E_{BS}\right)^{1-F}}{\left(1-m^{-1}E_{BS}\right)^{F}} = -A. \tag{17}$$

In the cases of $A=0$ and $A^{-1}=0$ expressions for the charge and its fluctuation simplify:

$$Q(T) = -\frac{e}{2}\left(F-\frac{1}{2}\pm\frac{1}{2}\right)\tanh\left(\frac{1}{2}\beta m\right), \qquad \Theta = \pm s\frac{\pi}{2}\,(\mathrm{mod}\,2\pi), \tag{18}$$

and

$$\Delta^2_{Q(T)} = \frac{e^2}{4}\left(F-\frac{1}{2}\pm\frac{1}{2}\right)^2\mathrm{sech}^2\left(\frac{1}{2}\beta|m|\right), \qquad \Theta = \pm s\frac{\pi}{2}\,(\mathrm{mod}\,2\pi); \tag{19}$$

note that equation (18) at $\Theta = s\frac{\pi}{2}\,(\mathrm{mod}\,2\pi)$ was obtained in Ref. [7].

In the limit $T\to 0$ ($\beta\to\infty$) the charge tends to finite value (see Ref. [15]):

$$Q(0) = \left\{\begin{array}{ll} \frac{e}{2}\,\mathrm{sgn}\,(m)(1-F), & -1<A<\infty \\ -\frac{e}{2}\,\mathrm{sgn}\,(m)F, & A^{-1}=-1,\ A^{-1}=0 \\ -\frac{e}{2}\,\mathrm{sgn}\,(m)(1+F), & -\infty<A<-1 \end{array}\right\}, \quad 0<F<\frac{1}{2},$$
$$Q(0) = \left\{\begin{array}{ll} -\frac{e}{2}\,\mathrm{sgn}\,(m)F, & -1<A^{-1}<\infty \\ \frac{e}{2}\,\mathrm{sgn}\,(m)(1-F), & A=-1,\ A=0 \\ \frac{e}{2}\,\mathrm{sgn}\,(m)(2-F), & -\infty<A^{-1}<-1 \end{array}\right\}, \quad \frac{1}{2}<F<1, \tag{20}$$

$$Q(0) = \left\{\begin{array}{ll} -\frac{e}{\pi}\,s\,\mathrm{sgn}\,(m)\arctan\left(\tan\frac{\Theta}{2}\right), & \Theta\neq\pi\,(\mathrm{mod}\,2\pi) \\ 0, & \Theta=\pi\,(\mathrm{mod}\,2\pi) \end{array}\right\}, \quad F=\frac{1}{2}, \tag{21}$$

whereas the fluctuation tends exponentially to zero for almost all values of Θ with the exception of one corresponding to the zero bound state energy, $E_{BS} = 0$ ($A = -1$):

$$\Delta^2_{Q(0)} = \begin{cases} 0, & A \neq -1, \\ \dfrac{e^2}{4}, & A = -1. \end{cases} \tag{22}$$

In the high-temperature limit the charge tends to zero:

$$Q(T \to \infty) = \begin{cases} \dfrac{e}{2}\operatorname{sgn}(m)\dfrac{\sin(F\pi)}{\pi}\dfrac{\Gamma(1-F)}{\Gamma(1+F)}\tan\left(s\dfrac{\Theta}{2}+\dfrac{\pi}{4}\right)\left(\dfrac{|m|}{k_BT}\right)^{1-2F}, & 0 < F < \dfrac{1}{2}, \\ -\dfrac{e}{8}\dfrac{sm}{k_BT}\sin\Theta, & F = \dfrac{1}{2}, \\ -\dfrac{e}{2}\operatorname{sgn}(m)\dfrac{\sin(F\pi)}{\pi}\dfrac{\Gamma(F)}{\Gamma(2-F)}\cot\left(s\dfrac{\Theta}{2}+\dfrac{\pi}{4}\right)\left(\dfrac{|m|}{k_BT}\right)^{2F-1}, & \dfrac{1}{2} < F < 1. \end{cases} \tag{23}$$

whereas the fluctuation tends to finite value:

$$\lim_{T\to\infty}\Delta^2_{Q(T)} = \begin{cases} \left.\begin{array}{ll} \dfrac{e^2}{4}(1-F)^2, & \Theta \neq s\dfrac{\pi}{2}\,(\mathrm{mod}\,2\pi) \\ \dfrac{e^2}{4}F^2, & \Theta = s\dfrac{\pi}{2}\,(\mathrm{mod}\,2\pi) \end{array}\right\}, & 0 < F \leq \dfrac{1}{2}, \\ \left.\begin{array}{ll} \dfrac{e^2}{4}F^2, & \Theta \neq -s\dfrac{\pi}{2}\,(\mathrm{mod}\,2\pi) \\ \dfrac{e^2}{4}(1-F)^2, & \Theta = -s\dfrac{\pi}{2}\,(\mathrm{mod}\,2\pi) \end{array}\right\}, & \dfrac{1}{2} \leq F < 1. \end{cases} \tag{24}$$

At half-integer values of $e\Phi$ one has

$$A|_{F=\frac{1}{2}} = \tan\left(s\frac{\Theta}{2}+\frac{\pi}{4}\right), \tag{25}$$

and the charge and its fluctuation take the form

$$Q(T)|_{F=\frac{1}{2}} = -\frac{e}{4}s\Bigg\{[1-\operatorname{sgn}(\cos\Theta)]\tanh\left(\frac{1}{2}\beta m\sin\Theta\right) + \frac{\sin 2\Theta}{2\pi}\int_1^\infty \frac{dv}{\sqrt{v(v-1)}}\frac{\tanh\left(\frac{1}{2}\beta m\sqrt{v}\right)}{v-\sin^2\Theta}\Bigg\}, \tag{26}$$

and

$$\Delta^2_{Q(T)}\Big|_{F=\frac{1}{2}} = \frac{e^2}{8}\Bigg\{[1-\operatorname{sgn}(\cos\Theta)]\operatorname{sech}^2\left(\frac{1}{2}\beta|m\sin\Theta|\right) - \frac{1}{2}\operatorname{sech}^2\left(\frac{1}{2}\beta|m|\right) + \frac{\cos\Theta}{\pi}\int_1^\infty \frac{dv}{\sqrt{v-1}}\frac{\operatorname{sech}^2\left(\frac{1}{2}\beta|m|\sqrt{v}\right)}{v-\sin^2\Theta}\Bigg\}. \tag{27}$$

An alternative representation for the charge and its fluctuation is obtained by deforming contour C to encircle poles of the $\tanh\left(\frac{1}{2}\beta\omega\right)$ and $\operatorname{sech}^2\left(\frac{1}{2}\beta\omega\right)$ functions, which occur along the imaginary axis at the Matsubara modes $\left(\omega_n = (2n+1)\frac{i\pi}{\beta}\right)$:

$$Q(T) = -e\operatorname{sgn}(m)\Bigg\{\frac{1}{2}\left(F-\frac{1}{2}\right)\tanh\left(\frac{\pi}{2\xi}\right) \tag{28}$$

$$+\frac{\xi}{\pi}\sum_{\substack{n\in\mathbb{Z}\\ n\geq 0}}\frac{2(2F-1)(2n+1)^2\xi^2+A[1+(2n+1)^2\xi^2]^F-A^{-1}[1+(2n+1)^2\xi^2]^{1-F}}{[1+(2n+1)^2\xi^2]\{A[1+(2n+1)^2\xi^2]^F+2+A^{-1}[1+(2n+1)^2\xi^2]^{1-F}\}}\Bigg\},$$

and

$$\Delta^2_{Q(T)}=\frac{e^2}{8}[1-2F(1-F)]\,\mathrm{sech}^2\left(\frac{\pi}{2\xi}\right)$$
$$+e^2\frac{\xi^2}{\pi^2}\sum_{\substack{n\in\mathbb{Z}\\ n\geq 0}}\frac{1}{[1+(2n+1)^2\xi^2]^2\{A[1+(2n+1)^2\xi^2]^F+2+A^{-1}[1+(2n+1)^2\xi^2]^{1-F}\}}$$
$$\times\Bigg\{(2F-1)[(2n+1)^2\xi^2-1]\{A[1+(2n+1)^2\xi^2]^F-A^{-1}[1+(2n+1)^2\xi^2]^{1-F}\} \tag{29}$$
$$+2\{1-[3-4F(1-F)](2n+1)^2\xi^2\}-4(2n+1)^2\xi^2$$
$$\times\frac{(2F-1)\{A[1+(2n+1)^2\xi^2]^F-A^{-1}[1+(2n+1)^2\xi^2]^{1-F}\}-1+(2F-1)^2(2n+1)^2\xi^2}{A[1+(2n+1)^2\xi^2]^F+2+A^{-1}[1+(2n+1)^2\xi^2]^{1-F}}\Bigg\},$$

where $\xi=\pi/(\beta|m|)$.

3 Discussion

In the present paper we consider an ideal gas of two-dimensional relativistic massive electrons in the background of a static point-like magnetic vortex. This system at thermal equilibrium is found to acquire electric charge: its average $Q(T)$ is given by equation (13), and its quadratic fluctuation $\Delta^2_{Q(T)}$ is given by equation (14). The most general boundary conditions (parametrized by the self-adjoint extension parameter Θ) at the location of the vortex are employed, and arbitrary values of the vortex flux Φ are permitted; our results are periodic in Θ with period 2π at fixed Φ and periodic in Φ with period e^{-1} at fixed Θ (e is the electron charge). Note that equations (13) and (14) can be regarded as the Sommerfeld–Watson transforms of the infinite sum representation, equations (28) and (29). Note also that the charge is odd and its fluctuation is even under transition to the inequivalent representation of the Clifford algebra ($m\to -m$).

Equation (13) can rewritten in the form

$$Q(T)=Q(0)+\tilde{Q}(T), \tag{30}$$

where $Q(0)$ is given by equations (20)–(21) [15], and

$$\tilde{Q}(T)=\frac{e}{2}\,\mathrm{sgn}\,(m)\Bigg\{\frac{\mathrm{sgn}\,\left(1+A^{-1}\right)-\mathrm{sgn}\,(1+A)}{\exp\left(\beta|E_{BS}|\right)+1}+\frac{2F-1}{\exp(\beta|m|)+1} \tag{31}$$
$$+\frac{\beta|m|}{2\pi}\int_1^\infty dw\,\mathrm{sech}^2\left(\frac{1}{2}\beta|m|w\right)\arctan\left[\frac{A(w^2-1)^F-A^{-1}(w^2-1)^{1-F}+2\cos(F\pi)}{2w\sin(F\pi)}\right]\Bigg\}.$$

Our result should be compared with the result of Ref. [18]

$$Q(T)=-\frac{e^2}{2}s\Phi\tanh\left(\frac{1}{2}\beta m\right), \tag{32}$$

where Φ is the flux of a magnetic field with an extensive support, and it is implied that the region of the support is not excluded. Thus, result (32) describes the direct effect of the field

strength, whereas our result describes the indirect, through the vector potential, effect of the field strength from the excluded region. In contrast to equation (32), our expressions for $Q(T)$ and $\Delta^2_{Q(T)}$ are periodic in the value of the flux, vanishing at integer values of $e\Phi$, and this can be regarded as a manifestation of the Bohm–Aharonov effect [20] in quantum field theory at nonzero temperature.

The nonvanishing of the charge quadratic fluctuation signifies that the charge of the system is not a sharp quantum observable and has to be understood as a thermal expectation value only. In the high-temperature limit the average charge tends to zero (23) and the fluctuation tends to finite value (24). In the zero-temperature limit quantities $\Delta^2_{Q(T)}$ and $\tilde{Q}(T)$ tend exponentially to zero and the charge becomes a sharp quantum observable with finite value $Q(0)$ (20)–(21). However, the last statement is true for almost all values of Θ with the exception of one corresponding to the zero bound state energy, $E_{BS} = 0$ ($A = -1$), since in this case the zero-temperature fluctuation is nonzero, see equation (22).

At half-integer values of $e\Phi$ the average charge takes form of equation (26) which coincides (after substituting s for $2eg$, where g is the magnetic monopole charge, $2eg = n$ is the Dirac quantization condition) with the expression for the thermally induced charge in the monopole background in three-dimensional space [7, 8]. It should be emphasized that at non-half-integer values of $e\Phi$ the behavior of the charge as a function of Θ differs drastically from the one at half-integer $e\Phi$.

In the $F \neq \frac{1}{2}$ case the charge at zero temperature is given by a step function with two jumps. As temperature increases, the jump corresponding to the zero bound state energy ($A = -1$) is smoothed out, while another jump is persisting. The charge at $A = -1$ is not a sharp quantum observable even at zero temperature, which is explicated by the nonvanishing of the fluctuation in this case. As temperature departs from zero, the fluctuation develops a maximum at $A = -1$ and a minimum close to the position of the persisting jump of the charge, but out of the region where bound state exists. With the increase of temperature the maximum is widening and disappearing, while the minimum is narrowing with its position approaching the position of the charge jump and its width tending to zero in the high-temperature limit.

In the $F = \frac{1}{2}$ case the charge at zero temperature is linear in Θ with one jump at $A = -1$ ($\Theta = s\pi \,(\mathrm{mod}\, 2\pi)$) where the charge is not a sharp quantum observable. As temperature increases, this jump is smoothed out. Appropriately, the fluctuation is symmetric with respect to the position of this jump, and a maximum of the fluctuation is smoothed out with the increase of temperature.

In conclusion we note that the system considered can acquire, in addition to the charge, also other quantum numbers. In the case of zero temperature this issue is comprehensively elucidated in Refs. [15, 21, 22], and an appropriate generalization to the case of nonzero temperature will be studied elsewhere.

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Double Symmetry and Infinite-Component Field Theory

L.M. SLAD

D.V. Skobeltsyn Institute of Nuclear Physics, Moscow State University, 119899 Moscow, Russia
E-mail: *slad@theory.sinp.msu.ru*

Qualitative characteristics and the rigorous definition of the concept of the double symmetry is given. We use double symmetry for constructing a theory of fields not investigated before which transform as the proper Lorentz group representations decomposable into infinite direct sums of finite-dimensional irreducible representations. All variants of the double-symmetric free Lagrangian of such fields are described in brief. The solution of the problem of changing Lagrangian mass terms due to a spontaneous breaking of the secondary symmetry is stated. The general properties of the mass spectra of fermions in the considered theory are given. A region of free parameters is pointed out where the theoretical mass spectra qualitatively correspond to a picture typical for the parton model of hadrons.

1 Qualitative characteristics and the rigorous definition of the double symmetry

The main objective of our work presented here is a study of the possibility of an alternative description of hadrons and their interactions.

It is supposed that the composite particles can have an effective description with the help of monolocal fields which transform as representations of the proper Lorentz group $L_+^\uparrow$ decomposable into infinite direct sums of finite-dimensional irreducible representations. We call such infinite-component fields the ISFIR-class fields. Until recently, there was no study of them. It is necessary to note that the physical quantities, including the amplitudes of processes with participation of the particles described by ISFIR-class fields, are expressed through infinite functional series whose terms are proportional to the matrix elements of finite transformations of the proper Lorentz group. Due to this circumstance we can expect any behavior of a given physical quantity as the function of its variables, including such behavior which holds in hadron physics.

The elimination of an infinite number of arbitrary parameters in the relativistically invariant Lagrangians of the ISFIR-class fields is achieved due to an additional invariance of the theory. Initial (primary) and additional (secondary) symmetries of the theory form together the double symmetry. This concept can be considered as generalization for the long-known symmetry of the Gell-Mann–Levy σ-model [1] and supersymmetry.

In the σ-model, particles with different values of the spatial parity, namely, π- and σ-mesons, being a pseudoscalar and a scalar, accordingly, with respect to orthochronous Lorentz group $L^\uparrow$ are united in one multiplet. In order to ensure the invariance of the theory under space reflection P, Gell-Mann and Levy have evidently declared that the parameters of transformations, connecting between themselves the π- and σ-mesons, were the space pseudoscalars. The σ-model symmetry group $SU(2)_L \otimes SU(2)_R$ has found broad applications in the field theory and particle physics. However, except for Gell-Mann and Levy nobody, spoke in explicit form about the transformation properties of these group parameters under the P-reflection.

If one reformulates the left-right symmetric model of electroweak interactions [2–4] in the manner of Gell-Mann–Levy, identifying some parameters of the local gauge group with space scalars and the others with space pseudoscalars, it is possible to provide the initial P-invariance

of such model and to reproduce all experimentally observed consequences of the Weinberg–Salam model. In this case the nature of the P-symmetry violation appears logically simple and clear. If the Higgs field has both scalar and pseudoscalar components, and they both have nonzero vacuum expectation values whose relative phase is not equal to $\pm\pi/2$, then either left or right weak charged current dominates depending on the relative signs of the quantities which are present in gauge transformations. It means that, as a result of spontaneous breaking of the local symmetry, the physical vacuum does not possess a certain P-parity. The fields of all intermediate bosons constitute a superposition of polar and axial 4-vectors, these vectors having equal weights in the fields of W-bosons. Transformation properties of the gauge bosons with respect to the orthochronous Lorentz group have the same powerful significance as the form of weak currents. However, they yield no experimental detection in contrast to the interaction vertices.

In Ref. [5], the grain of new group theoretical approach sown by Gell-Mann and Levy has received its generalization in the concept of double symmetry consisting of the primary and secondary symmetries.

The secondary symmetry has three essential features. First, the parameters of the transformations generating a global or local secondary-symmetry group H_T belong to the given T-representation space of the global group of the primary symmetry G. Second, the H_T-group transformations operate on the vectors of some S-representation space of the group G and do not take them out from this space, i.e. the secondary symmetry does not violate the primary one. Third, the primary- and secondary-symmetry groups G and H_T have no common elements except for the unity, and even in that case when they are isomorphic.

We restrict ourselves with the consideration of the global symmetries. The above-listed features find their realization in the following rigorous definition.

Definition 1. Let the group $\mathcal{G}_T$ contain the subgroup G and the invariant subgroup H_T, with $G \neq \mathcal{G}_T$, $H_T \neq \mathcal{G}_T$, and be their semidirect production $\mathcal{G}_T = H_T \circ G$. If any element $h_T \in H_T$ can be written in the form

$$h_T = h(\theta_1)h(\theta_2)\cdots h(\theta_n), \qquad n \in \{1, 2, \ldots\}, \tag{1}$$

where θ_i $(i = 1, 2, \ldots, n)$ is some vector of the representation space of T of the group G, and if for any element $g \in G$

$$gh(\theta)g^{-1} = h(T(g)\theta), \tag{2}$$

then the group G will be called the primary symmetry group, and the groups H_T and $\mathcal{G}_T$ will be respectively called the secondary and double symmetry groups generated by the representation T of the group G.

Constructing the double-symmetric field theories is based on the following consequence of the secondary-symmetry definition.

Corollary 1. *Let $\Psi(x)$ be any field vector in the S-representation space, and $\theta = \{\theta_a\}$ be a vector in the T-representation space of the group G. Then the transformation*

$$\Psi'(x) = \exp(-iD^a\theta_a)\Psi(x) \tag{3}$$

will be a secondary-symmetry transformation iff the condition

$$D^a = S^{-1}(g)D^bS(g)[T(g)]_b{}^a \tag{4}$$

is fulfilled, i.e. the operators D^a are the T-operators (transform as the representation T) of the group G.

In our construction [6,7] of the theory of infinite-component fields of the ISFIR-class the secondary symmetry is generated by transformations (3), with the index μ instead of the index a, in which the parameters θ_μ are the components of the polar or axial 4-vectors of the orthochronous Lorentz group, and the operators D^μ have the matrix realization. We postulate no closing of the algebra of the operators D^μ. It arises automatically in each variant of the considered theory due to those restrictions which are imposed by the requirement that the corresponding Lagrangians possess the double invariance.

2 Double-symmetric Lagrangians of the free ISFIR-class fields

The general structure of relativistically invariant free Lagrangians of the form

$$\mathcal{L}_0 = \frac{i}{2}[(\Psi, \Gamma^\mu \partial_\mu \Psi) - (\partial_\mu \Psi, \Gamma^\mu \Psi)] - (\Psi, R\Psi) \tag{5}$$

was found and described by Gelfand and Yaglom [8,9]. Let us remind that, according to Gelfand and Yaglom, the finite-dimensional irreducible representation τ of the proper Lorentz group is defined by such a pair of numbers (l_0, l_1) that $2l_0$ and $2l_1$ are integers of the same parity and $|l_1| > |l_0|$. The canonical basis vectors of this representation space are related to the subgroup $SO(3)$ and are denoted by $\xi_{(l_0,l_1)lm}$ where l is a spin, m is its projection onto the third axis, $m = -l, -l+1, \ldots, l$, and $l = |l_0|, |l_0|+1, \ldots, |l_1|-1$. The pairs (l_0, l_1) and $(-l_0, -l_1)$ define the same representations, $(l_0, l_1) \sim (-l_0, -l_1)$.

The requirement that the Lagrangian (5) is invariant under the group $L_+^\uparrow$ gives for the operator Γ^μ a condition which is equivalent to the relation (4) (at $g \in L_+^\uparrow$) for the operator D^μ from the secondary-symmetry transformation. It has the consequences expressed through the following equalities

$$\Gamma^i = \left[I^{i0}, \Gamma^0\right] \qquad (i = 1, 2, 3), \tag{6}$$

$$\begin{aligned}\Gamma^0 \xi_{(l_0,l_1)lm} = {} & c(l_0+1, l_1; l_0, l_1)\sqrt{(l+l_0+1)(l-l_0)}\,\xi_{(l_0+1,l_1)} \\ & + c(l_0-1, l_1; l_0, l_1)\sqrt{(l+l_0)(l-l_0+1)}\,\xi_{(l_0-1,l_1)lm} \\ & + c(l_0, l_1+1; l_0, l_1)\sqrt{(l+l_1+1)(l-l_1)}\,\xi_{(l_0,l_1+1)lm} \\ & + c(l_0, l_1-1; l_0, l_1)\sqrt{(l+l_1)(l-l_1+1)}\,\xi_{(l_0,l_1-1)lm},\end{aligned} \tag{7}$$

where I^{i0} are the infinitesimal operators of the proper Lorentz group, and $c(l_0', l_1'; l_0, l_1) \equiv c_{\tau'\tau}$ are arbitrary parameters. The same equality are valid for the operator D^μ with arbitrary quantities $d_{\tau'\tau}$.

Infinite number of arbitrary parameters $c_{\tau'\tau}$ in the relativistically invariant Lagrangians of the free ISFIR-class fields was the serious reason for that until recently there were no research of such field theory.

Involved into our consideration are the $L_+^\uparrow$-group representations S, in whose decomposition the multiplicity of each finite-dimensional irreducible representation does not exceed unity.

The Lagrangian (5) (at $R \neq 0$) will be invariant under the secondary-symmetry transformations (3) generated by the polar or axial 4-vector of the group $L^\uparrow$, if the condition

$$[\Gamma^\mu, D^\nu] = 0 \tag{8}$$

is satisfied. This condition is reduced to an algebraic system of the equations with respect to quantities $c_{\tau'\tau}$ and $d_{\tau'\tau}$.

As the analysis shows this system of the equations has nontrivial solutions only for the quite certain countable set of the $L_+^\uparrow$-group representations S. The fermionic field can be described

either by any of the representations

$$S^{k_1}=\sum_{n_1=0}^{+\infty}\sum_{n_0=-k_1+1/2}^{k_1-3/2}\oplus\left(\frac{1}{2}+n_0,k_1+n_1\right), \tag{9}$$

where $k_1\geq 3/2$, or by the representation S^F which contains all finite-dimensional irreducible half-integer spin representations of the group $L_+^\uparrow$. The bosonic field can correspond either to any of the representations

$$S^{k_1}=\sum_{n_1=0}^{+\infty}\sum_{n_0=-k_1+1}^{k_1-1}\oplus(n_0,k_1+n_1), \tag{10}$$

where $k_1\geq 1$, or to the representation S^B which contains all finite-dimensional irreducible integer spin representations of the group $L_+^\uparrow$.

Three variants of the theory are assigned to each of representations (9). In one of them the secondary symmetry is generated by a polar 4-vector of the group $L^\uparrow$, the operators D^μ commute with each other

$$[D^\mu,D^\nu]=0, \tag{11}$$

i.e. the group H_T is the Abelian one, and the quantities $c_{\tau'\tau}$ are given by equalities

$$\begin{aligned}c(l_0+1,l_1;l_0,l_1)&=c(l_0,l_1;l_0+1,l_1)\\&=c_0D(l_1)\sqrt{\frac{(k_1-l_0-1)(k_1+l_0)}{(l_1-l_0)(l_1-l_0-1)(l_1+l_0)(l_1+l_0+1)}},\end{aligned} \tag{12}$$

$$\begin{aligned}c(l_0,l_1+1;l_0,l_1)&=c(l_0,l_1;l_0,l_1+1)\\&=c_0D(l_0)\sqrt{\frac{(k_1-l_1-1)(k_1+l_1)}{(l_1-l_0)(l_1-l_0+1)(l_1+l_0)(l_1+l_0+1)}},\end{aligned} \tag{13}$$

where $D(j)=1$, and c_0 is a real constant. In the second and third variants of the theory the secondary symmetry is generated by an axial 4-vector of the group $L^\uparrow$. In the second variant, the formulas (9)–(11), where $D(j)=(-1)^{j-1/2}j$, are valid. In the third variant the following inequality holds

$$[G^{\mu\nu},G^{\rho\sigma}]\neq 0, \tag{14}$$

in which the antisymmetric operator $G^{\mu\nu}$ is involved. This operator is determined as follows

$$G^{\mu\nu}=[D^\mu,D^\nu]. \tag{15}$$

Four variants of the theory at $k_1\geq 2$ and two variants at $k_1=1$ are assigned to each of representations (10). All variants divides into pairs in which the secondary symmetry is generated either by a polar, or by an axial 4-vector of the group $L^\uparrow$. In one pair of the variants $k_1\geq 1$ and the equality (11) is valid, and in other pair $k_1\geq 2$ and the inequality (14) holds.

The representation S^F is assigned by one variant of the theory, and the representation S^B is by two variants. In these cases the operators $G^{\mu\nu}$ are nonzero and commute with each other, and the secondary-symmetry transformations of fermionic (bosonic) fields is generated by the axial (polar or axial) 4-vector of the group $L^\uparrow$.

In all variants of the double-symmetric theory the operator R, specifying the mass term of Lagrangian (5), is a multiple of the identity operator.

3 A spontaneous secondary-symmetry breaking

In complete accordance with the known Coleman–Mandula theorem [10] relating to consequences of the Lorentz-group extension, the mass spectrum corresponding to any of the variants of the infinite-component field theory with the double symmetry is infinitely degenerate with respect to spin. This degeneracy forces us to suppose that the secondary symmetry is spontaneously broken, namely, that the scalar (with respect to the group $L^{\uparrow}$) components of one or several bosonic infinite-component ISFIR-class fields have nonzero vacuum expectation values λ^i. A spontaneous secondary-symmetry breaking introduces only one correction into the free-field theory with the initial double symmetry which consists in changing the operator R in the Lagrangian (5). So, in the fermionic-field Lagrangian, this operator takes the form

$$R = \kappa E + \lambda^i Q_i^{(0,1)00}, \tag{16}$$

where κ is a constant. The P-even operators $Q_i^{(0,1)00}$ come to relation (16) from the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = \left(\psi(x), Q_{i'}^{\tau lm}\varphi_{\tau lm}^{i'}(x)\psi(x)\right), \tag{17}$$

where $\psi(x)$ is the fermionic field, $\varphi_{\tau lm}^{i'}(x)$ is the bosonic-field component, and $Q_i^{\tau lm}$ are the matrix operators.

The Lagrangian (17) containing nonzero operators $Q_i^{(0,1)00}$ possesses the considered double symmetry only in two cases denoted by $\mathcal{A}$ and $\mathcal{B}$. In each of them the secondary-symmetry group is the four-parametrical Abelian group, the fermionic field transforms as any of the representations S^{k_1} (9), and bosonic fields transform as the representation S^1 (10). In the case $\mathcal{A}$, the secondary symmetry is generated by the polar 4-vector of group $L^{\uparrow}$, and $D(j) = 1$. In the case $\mathcal{B}$, the secondary symmetry is generated by the axial 4-vector of group $L^{\uparrow}$ and $D(j) = (-1)^{j-1/2}j$. In both cases the relations

$$R\xi_{(l_0,l_1)lm} = r(l_0,l_1)\xi_{(l_0,l_1)lm} = [\kappa + \lambda^i q_i(l_0,l_1)]\xi_{(l_0,l_1)lm}, \tag{18}$$

$$\begin{aligned}&(k_1 - l_0 - 1)(k_1 + l_0)q_i(l_0+1,l_1) + (k_1 - l_0)(k_1 + l_0 - 1)q_i(l_0-1,l_1)\\ &\quad - (k_1 - l_1 - 1)(k_1 + l_1)q_i(l_0,l_1+1) - (k_1 - l_1)(k_1 + l_1 - 1)q_i(l_0,l_1-1)\\ &\quad = z_i(l_1 - l_0)(l_1 + l_0)q_i(l_0,l_1)\end{aligned} \tag{19}$$

are valid for all irreducible representations $(l_0, l_1) \in S^{k_1}$. The parameter z_i is expressed through the ratio of normalization constants of the operators D^μ in the secondary-symmetry transformations (3) of the bosonic and fermionic fields. It can have any values. If $z_i = 2$, then the bosonic-field transformation (3) is trivial ($D^\mu = 0$), i.e. it is not involved in the secondary symmetry.

In what follows we will deal only with fermionic fields which transform according to the "lowest" of the representations S^{k_1} (9) of the proper Lorentz group, namely, according to the representation $S^{3/2}$. In this case the equation (19) has a unique solution, which can be written in the form

$$q_i\left(-\frac{1}{2}, l_1\right) = q_i\left(\frac{1}{2}, l_1\right) = 2q_{i0}\frac{u_i^N(u_iN + N + 1) - w_i^N(w_iN + N + 1)}{N(N+1)(u_i - w_i)(2 + u_i + w_i)}, \tag{20}$$

where $N = l_1 - 1/2$, $u_i = \left(z_i + \sqrt{z_i^2 - 4}\right)/2$ and $w_i = \left(z_i - \sqrt{z_i^2 - 4}\right)/2$.

4 Properties of the mass spectra in the ISFIR-class field theory

We have all what is necessary for solving of the first physical problem which concerns the general properties of the mass spectra in the double-symmetric theory of the ISFIR-class fermionic fields.

In the rest system of a particle with the mass M the Gelfand–Yaglom equation corresponding to the Lagrangian (5) has the form

$$(M\Gamma^0 - R)\psi_{M0} = 0. \tag{21}$$

Its solutions must satisfy the selection condition at which the amplitudes of various processes are finite. We formulate such condition of finiteness of the amplitudes as the relation

$$|(\psi_{M0}, R\psi_{Mp})| < +\infty \tag{22}$$

or

$$(\psi_{M'0}, R\psi_{Mp}) = a(p)\delta(M' - M), \tag{23}$$

where ψ_{Mp} is the particle state vector with the momentum p directed along the third axis, and $a(p)$ is a nonzero number.

In the formula (22) or (23) the bilinear form is expressed through the infinite series whose terms are linear with respect to the components of vector ψ_{M0}, to the components of other vector ψ_{M0} or $\psi_{M'0}$, and to the matrix elements of finite transformations of the proper Lorentz group. These matrix elements for irreducible representation (l_0, l_1) behave as $T_0 \exp(\alpha l_1)/l_1$ at $l_1 \to +\infty$ where $\tanh\alpha = p/\sqrt{M^2c^2 + p^2}$, and the quantity T_0 does not depend on the number l_1 [11].

In the space of the $L_+^{\uparrow}$-group representation $S^{3/2}$ the equation (21) for the field vector components $(\psi_{M0})_{(\pm 1/2, l_1)lm} \equiv \chi_{lm}(l_1)$ with the spatial parity $(-1)^{l-1/2}$ takes the form

$$D\left(\frac{1}{2}\right)\frac{\sqrt{(l_1 - l)(l_1 + l + 1)}}{2l_1 + 1}\chi_{lm}(l_1 + 1) + D\left(\frac{1}{2}\right)\frac{\sqrt{(l_1 - l - 1)(l_1 + l)}}{2l_1 - 1}\chi_{lm}(l_1 - 1)$$
$$- \left[\frac{D(l_1)(2l + 1)}{4l_1^2 - 1} - \frac{1}{2Mc_0}r(l_1)\right]\chi_{lm}(l_1) = 0, \tag{24}$$

where $l_1 \geq l$, $r(l_1) \equiv r(\pm 1/2, l_1)$. At the replacement of the value M by $-M$ the P-parity of solution of this equation changes to the opposite one.

In a situation when the secondary symmetry of the theory is broken spontaneously we failed either to find solutions of the equation (24) in the form of elementary or special functions, or to find analytical formulas for mass spectra of the theory. However, we are able to make a number of conclusions concerning the mass spectra on the basis of asymptotic behavior of some quantities. On the basis of on these conclusions it is possible to find any number of the lower values of the mass M with the help of numerical methods.

We have analyzed the mass spectrum characteristics in all details in both cases $\mathcal{A}$ and $\mathcal{B}$ of the considered theory provided that the spontaneous breaking of the secondary symmetry is caused by one ISFIR-class bosonic field, and $\kappa = 0$ in formulas (16) and (18).

In the region of the parameter values $z_1 \in (-2, 2)$ the solutions of equation (24) cannot satisfy the condition of finiteness of the amplitudes (22) or (23) at any values of the quantity M, i.e. the mass spectrum is empty.

In the region of the parameter values $z_1 \in (-\infty, -2] \cup (2, +\infty)$ the set of all masses of the theory is restricted below by some positive number. At $l_1 \to +\infty$ the quantities $r(l_1)$ and $\chi_{lm}(l_1)$

have the following asymptotic behavior

$$r(l_1) = r_0 \frac{v^{l_1+\frac{1}{2}}}{l_1+\frac{1}{2}} \left(1 + \mathcal{O}\left(l_1^{-1}\right)\right), \tag{25}$$

$$\chi_{lm}(l_1) = A_0 G(l_1) \left(1 + \mathcal{O}\left(l_1^{-1}\right)\right) + B_0 G^{-1}(l_1) \left(1 + \mathcal{O}\left(l_1^{-1}\right)\right), \tag{26}$$

where

$$G(l_1) = \frac{x^{l_1-\frac{1}{2}} v^{\frac{4l_1^2-1}{8}}}{\left(l_1-\frac{1}{2}\right)!}, \qquad x = -\frac{r_0}{Mc_0 D\left(\frac{1}{2}\right)}, \qquad v = \begin{cases} w_1, & \text{if} \quad z_1 \in (-\infty, -2], \\ u_1, & \text{if} \quad z_1 \in (2, +\infty). \end{cases} \tag{27}$$

If the quantity A_0 is not equal to zero at some value of M, then neither condition (22) nor condition (23) cannot be fulfilled. If $A_0 = 0$ at some value of M, then the condition (22) is fulfilled. Hence, for all z-parameter values from the region $z \in (-\infty, -2)$, the mass spectrum is discrete, if it is not empty. As the calculations show in the whole parameter region $z > 2$, each value of spin and parity corresponds to an accountable set of the masses rising to infinity. The lowest values of mass levels, corresponding to a definite spin, grow with a spin. Thus, in the qualitative sense the theoretical mass spectra correspond to a picture typical for the quark-gluon model of hadrons.

We compare the theoretical mass spectrum with nucleon resonance levels provided that the spontaneous breaking of the secondary symmetry is caused by two ISFIR-class bosonic fields, and $\kappa = 0$ in formulas (16) and (18). In this situation we pay attention to the following property of the particle state vectors in the considered theory.

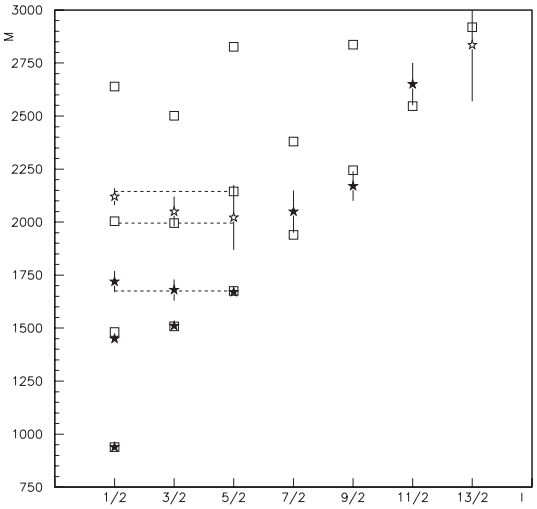

Figure 1.

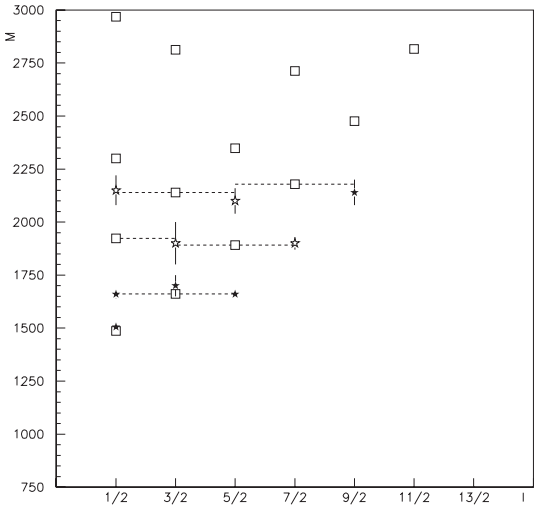

Figure 2.

In the reference system where the ground fermion (boson) of the theory possesses a nonzero velocity, its state vector has nonzero components with all half-integer (integer) spins, irrespective to what spin corresponds to this particle in its rest system. This results in that the amplitude of the resonance decay into ground particles will contain any partial wave in its decomposition. Certainly, only those of them may be observable in experiments which have an appreciable weight. Thus, one resonance can be shown in experiments as a group of several resonances with the same mass and differing only in their spin.

The results given in Figs. 1 and 2 correspond to the case $\mathcal{A}$ of the considered theory at the following choice of parameters $\lambda^1 q_{10}/c_0 = -939/2.4686$ MeV, $z_1 = 2.036$, $z_2 = 0.14$, $\lambda^2 q_{20}/\lambda^1 q_{10} = -0.6724$. The levels with the partial parities $(-1)^{l-1/2}$ and $(-1)^{l+1/2}$ are represented in Figs. 1 and 2 correspondingly. The theoretical masses are shown by the square markers,

and the experimental masses (the pole positions) are done by the dark- and light-star markers for, accordingly, well and badly established nucleon resonances [12]. The Breit–Wigner mass is taken for the resonance $N(1440)$ because two poles are found around 1440 MeV, $1370-114i$ and $1360-120i$, whose parameters are much different from the conventional ones $M = 1470$ MeV and $\Gamma = 545$ MeV [13]. The first approximation of the theory to experiments can be estimated as satisfactory.

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Invariant Geodetic Problems on the Projective Group $\mathbf{Pr}\,(n, \mathbb{R})$

Jan J. SŁAWIANOWSKI and Vasyl KOVALCHUK

Institute of Fundamental Technological Research, Polish Academy of Sciences,
21 Świętokrzyska Str., 00-049 Warsaw, Poland
E-mail: *jslawian@ippt.gov.pl, vkoval@ippt.gov.pl*

The concept of n-dimensional projectively-rigid body is introduced and its connection to the concept of $(n+1)$-dimensional incompressible affinely-rigid body is analysed. The equations of geodetic motion for such a projectively-rigid body are obtained. As an instructive example, the special case of $n = 1$ is investigated.

1 Introduction

The concept of the metrically-rigid body have played a very important role in the theoretical and applied mechanics (see e.g. [1]), mainly because our macroscopic environment is dominated by objects which are approximately rigid. But what would happen if we went forward and got rid of the metrical properties keeping the concept of rigidness? Such problems in the application to the theory of continuous media were investigated in [2–10], where the concept of an affinely-rigid body as a medium the deformative behaviour of which is restricted to performing homogeneous deformations only was developed. Other applications are also possible, e.g. in the theory of large oscillations of molecules, small mono-crystals, atomic nuclei and even in the theory of elementary particles. In fact, an affinely-rigid body in an amorphous affine space is an obvious counterpart of the usual metrically-rigid body in a Euclidean space. But we need not to get rid of the metric once and for all, we may introduce it in our consideration at any step, and that is what makes this approach attractive. For example, to be able to introduce the notion of the kinetic energy, we should have some fixed Euclidean metric.

Let us explain this more precisely and consider the classical system of points (discrete or continuous), which is placed in the physical space M. We assume that the material points are distinguishable and label them by means of points of an auxiliary space N, which is called the material space (e.g. we may choose these labels as initial positions of all points at the moment t_0). Let V and U be the linear spaces of translations (free vectors) in M and N, respectively, then $(M, V, \rightarrow)$ and $(N, U, \rightarrow)$ are affine spaces. The position of the a-th material point at the time instant t is denoted by $x(t, a)$ ($x \in M$, $a \in N$). If the system is continuous, the label a and position x become the Lagrangian and Eulerian radius-vectors (material and physical variables), respectively. If the dimensions of M and N are equal, then we can impose such constraints of affine rigidness that the connection between material and physical variables is as follows:

$$x^i(t, a) = \varphi^i_B(t) a^B + r^i(t), \qquad i, B = \overline{1, n}, \qquad n = \dim(M) = \dim(N). \tag{1}$$

Now we have two opposite possibilities to introduce the metric: a) in the physical space (then $g \in V^* \otimes V^*$ is the metric tensor and $(M, V, \rightarrow, g)$ is the corresponding Euclidean space) or b) in the material space (then $\eta \in U^* \otimes U^*$ is the metric tensor and $(N, U, \rightarrow, \eta)$ is the corresponding Euclidean space). In the case a) distances in N are measured by means of the configuration-dependent Green deformation tensor $G = \varphi^* g$ and in the case b) in M they are measured by means of the configuration dependent Cauchy deformation tensor $C = \varphi_* \eta$. Although the first

possibility is more physical, the second one also has attractive features (e.g. then we have the situation very similar to the one in the general relativity, i.e. in the physical space there is no fixed metric geometry at all and the components of the metric tensor are included in the physical degrees of freedom and dynamically coupled with matter distribution).

In the present paper we would like to consider even more "amorphous" case, which we obtain from (1) by generalizing our affine constraints to the projective ones:

$$x^i(t,a) = \frac{A^i_B(t)a^B + b^i(t)}{c_D(t)a^D + d(t)}, \qquad Ad - bc \neq 0, \qquad i, B, D = \overline{1,n}. \tag{2}$$

Then at any fixed time $t \in \mathbb{R}$ the configuration space Q of our problem is identical with the projective group $\mathrm{Pr}\,(n,\mathbb{R}) \supset \mathrm{GAf}\,(n,\mathbb{R})$ (for $N = M = \mathbb{R}^n$) and such a system of material points is called the projectively-rigid body. Due to the isomorphism $\mathrm{Pr}\,(n,\mathbb{R}) \simeq \mathrm{SL}\,(n+1,\mathbb{R})$ we may rewrite those constraints (2) in n dimensions as the constraints defining an incompressible affinely-rigid body in $n+1$ dimensions, i.e.

$$x^\mu(t,a) = \varphi^\mu_\nu(t)a^\nu, \qquad \varphi = \begin{pmatrix} A & b \\ c & d \end{pmatrix}, \qquad \det\varphi = 1, \qquad \mu,\nu = \overline{0,n}.$$

The additional variable is called 0-th only for convenience reasons. The new configuration space $\widetilde{Q}$ is identical with the unimodular linear group $\mathrm{SL}\,(n+1,\mathbb{R})$. We may introduce also non-holonomic velocities instead of the very $\dot\varphi$, i.e. $\Omega = \dot\varphi\varphi^{-1}$ and $\hat\Omega = \varphi^{-1}\dot\varphi$. They are related to each other by the configuration-dependent similarity transformation: $\Omega = \varphi\hat\Omega\varphi^{-1}$.

2 Left and right invariant geodetic problems

Let us consider the left-invariant geodetic problem on the projective group $\mathrm{Pr}\,(n,\mathbb{R})$ (or equivalently on the unimodular linear group $\mathrm{SL}\,(n+1,\mathbb{R})$), which is also right-invariant under the orthogonal subgroup $\mathrm{SO}\,(n+1,\mathbb{R})$:

$$T_{\mathrm{left}} = \frac{J}{2}\,\mathrm{Tr}\,\big(\hat\Omega^T\hat\Omega\big) + \frac{\alpha}{2}\,\mathrm{Tr}\,\big(\hat\Omega^2\big) + \frac{\beta}{2}\big(\mathrm{Tr}\,\hat\Omega\big)^2, \tag{3}$$

and the right-invariant geodetic problem on the projective group, which is left-invariant under the orthogonal subgroup:

$$T_{\mathrm{right}} = \frac{J}{2}\,\mathrm{Tr}\,\big(\Omega^T\Omega\big) + \frac{\alpha}{2}\,\mathrm{Tr}\,\big(\Omega^2\big) + \frac{\beta}{2}\,(\mathrm{Tr}\,\Omega)^2, \tag{4}$$

where J, α, β are generalized inertial constants, the second and third terms for both kinetic energies are identical and are the Casimir invariants.

The most adequate description of internal degrees of freedom is that based on the two-polar decomposition, i.e. we split the system of degrees of freedom into three subsystems: $\varphi = LDR^T$, where $L, R \in \mathrm{SO}\,(n+1,\mathbb{R})$ are special orthogonal matrices ($L^TL = R^TR = \hat I$, $\det L = \det R = 1$) and D is diagonal, positive and $\det D = 1$. If we take $D^{\mu\mu} = \exp(q^\mu)$, then $\sum_\mu q^\mu = 0$. In this way our system is formally represented as a composition of two $(n+1)$-dimensional fictitious rigid bodies (systems of principal axes of the Cauchy and Green deformation tensors) and n independent material points oscillating along the straight line $\mathbb{R}$. Orthogonal transformations L and R diagonalize the Cauchy and Green deformation tensors: $C = LD^{-2}L^T$ and $G = RD^2R^T$. If there is no coincidence of diagonal elements of D, i.e. if the spectra of C and G are simple, then the two-polar decomposition is finitely non-unique: $\varphi = LDR^T = \widetilde L\widetilde D\widetilde R^T$, where $\widetilde L = LO_\pi$, $\widetilde D = O_\pi^{-1}DO_\pi$, $\widetilde R^T = RO_\pi$ and O_π is an orthogonal representation of the permutation group with the restriction $O_\pi\ \mathrm{Diag}\,\big(q^1,\ldots,q^{n+1}\big)\,O_\pi^{-1} = \mathrm{Diag}\,\big(q^{\pi(1)},\ldots,q^{\pi(n+1)}\big)$, i.e. $S_{n+1} \ni \pi \mapsto$

$O_\pi \in \mathrm{SO}\,(n+1,\mathbb{R})$. The components of matrices O_π are only 0 or ± 1, i.e. in any row and column there is only one non-vanishing element. If the spectrum of C and G is non-simple, then the non-uniqueness of decomposition becomes continuous. An extreme situation occurs when D is completely degenerate (proportional to the unit matrix), then no meaning may be assigned separately to L and R, only LR^T is well-defined.

It is convenient to introduce non-holonomic variables adapted to the two-polar decomposition, i.e. velocity $\dot q$ and non-holonomic angular velocities of the L and R rigid bodies in the co-moving representation: $\lambda = L^T\dot L = -\lambda^T$ and $\rho = R^T\dot R = -\rho^T$. Then the kinetic energies (3) and (4) may be rewritten in the combined form (the upper expression is related to the left-invariant and the lower one to the right-invariant problems) as follows:

$$T_{\text{left/right}} = \frac{J+\alpha}{2}\,\mathrm{Tr}\left(\dot D^2 D^{-2}\right) + (J-\alpha)\,\mathrm{Tr}\left(\lambda D\rho D^{-1}\right) + \frac{\alpha}{2}\,\mathrm{Tr}\left(\lambda^2+\rho^2\right) - \frac{J}{2}\,\mathrm{Tr}\begin{pmatrix}\lambda D^2\lambda D^{-2}+\rho^2\\ \lambda^2+\rho D^2\rho D^{-2}\end{pmatrix}.$$

We see that the constant β is absent in $T_{\text{left/right}}$ because of the condition $\det D = 1 \sim \sum_\mu q^\mu = 0$. If we explicitly substitute this condition into the expression for the kinetic energy, i.e. $q^0 = -\sum\limits_{i=1}^{n} q^i$, then we can rewrite the previous formula in the following form:

$$T_{\text{left/right}} = \frac{J+\alpha}{2}\left[\sum_{i=1}^{n}\left(\dot q^i\right)^2 + \langle\dot q\rangle^2\right] + \sum_{i=1}^{n} V_{\text{left/right}}(|q^i+\langle q\rangle|,\lambda_{0i},\rho_{0i}) + \frac{1}{2}\sum_{i,j=1}^{n} U_{\text{left/right}}\big(|q^i-q^j|,\lambda_{ij},\rho_{ij}\big), \tag{5}$$

where $\langle\bullet\rangle = \sum\limits_{j=1}^{n}\bullet^j$, one-point and binary effective interaction potentials are as follows:

$$V_{\text{left/right}} = (J-\alpha)\left[(\lambda_{0i})^2+(\rho_{0i})^2-2\lambda_{0i}\rho_{0i}\,\mathrm{ch}\left(q^i+\langle q\rangle\right)\right] + 2J\begin{Bmatrix}\lambda_{0i}\\ \rho_{0i}\end{Bmatrix}^2 \mathrm{sh}^2\left(q^i+\langle q\rangle\right),$$

$$U_{\text{left/right}} = (J-\alpha)\left[(\lambda_{ij})^2+(\rho_{ij})^2-2\lambda_{ij}\rho_{ij}\,\mathrm{ch}\left(q^i-q^j\right)\right] + 2J\begin{Bmatrix}\lambda_{ij}\\ \rho_{ij}\end{Bmatrix}^2 \mathrm{sh}^2\left(q^i-q^j\right).$$

Let us define the canonical affine momenta (p,j,k), which are conjugate to $(\dot q,\lambda,\rho)$, respectively. Here j and k are skew-symmetric matrices expressing the co-moving representation of the angular momenta of L and R rigid bodies. The Legendre transformation reads:

$$p_i = \frac{\partial T_{\text{left/right}}}{\partial\dot q^i} = (J+\alpha)\left[\dot q^i+\langle\dot q\rangle\right] \Longleftrightarrow \dot q^i = \frac{1}{J+\alpha}\left[p_i - \frac{1}{n+1}\langle p\rangle\right],$$

$$j_{0i} = \frac{\partial T_{\text{left/right}}}{\partial\lambda_{0i}} = 2(J-\alpha)\left[\lambda_{0i}-\rho_{0i}\,\mathrm{ch}\left(q^i+\langle q\rangle\right)\right] + 4J\begin{Bmatrix}\lambda_{0i}\\ 0\end{Bmatrix}\mathrm{sh}^2\left(q^i+\langle q\rangle\right),$$

$$k_{0i} = \frac{\partial T_{\text{left/right}}}{\partial\rho_{0i}} = 2(J-\alpha)\left[\rho_{0i}-\lambda_{0i}\,\mathrm{ch}\left(q^i+\langle q\rangle\right)\right] + 4J\begin{Bmatrix}0\\ \rho_{0i}\end{Bmatrix}\mathrm{sh}^2\left(q^i+\langle q\rangle\right),$$

$$j_{ij} = \frac{\partial T_{\text{left/right}}}{\partial\lambda_{ij}} = 2(J-\alpha)\left[\lambda_{ij}-\rho_{ij}\,\mathrm{ch}\left(q^i-q^j\right)\right] + 4J\begin{Bmatrix}\lambda_{ij}\\ 0\end{Bmatrix}\mathrm{sh}^2\left(q^i-q^j\right), \qquad i<j,$$

$$k_{ij} = \frac{\partial T_{\text{left/right}}}{\partial\rho_{ij}} = 2(J-\alpha)\left[\rho_{ij}-\lambda_{ij}\,\mathrm{ch}\left(q^i-q^j\right)\right] + 4J\begin{Bmatrix}0\\ \rho_{ij}\end{Bmatrix}\mathrm{sh}^2\left(q^i-q^j\right), \qquad i<j.$$

The non-vanishing basic Poisson brackets have the following form: $\{q^i, p_j\} = \delta^i_j$, $\{j_{\mu\nu}, j_{\kappa\sigma}\} = j_{\mu\sigma}\delta_{\kappa\nu} - j_{\kappa\nu}\delta_{\mu\sigma} + j_{\sigma\nu}\delta_{\mu\kappa} - j_{\mu\kappa}\delta_{\sigma\nu}$ and $\{k_{\mu\nu}, k_{\kappa\sigma}\} = k_{\mu\sigma}\delta_{\kappa\nu} - k_{\kappa\nu}\delta_{\mu\sigma} + k_{\sigma\nu}\delta_{\mu\kappa} - k_{\mu\kappa}\delta_{\sigma\nu}$. It is more convenient later on to use the auxiliary variables $R := -j - k = -R^T$ and $A := j - k = -A^T$ instead of the very j and k:

$$R_{0i} = 4\left[(J-\alpha)(\lambda_{0i}+\rho_{0i})\,\mathrm{sh}^2\left(\frac{q^i+\langle q\rangle}{2}\right) - J\left\{\begin{array}{c}\lambda_{0i}\\ \rho_{0i}\end{array}\right\}\mathrm{sh}^2\left(q^i+\langle q\rangle\right)\right],$$

$$A_{0i} = 4\left[(J-\alpha)(\lambda_{0i}-\rho_{0i})\,\mathrm{ch}^2\left(\frac{q^i+\langle q\rangle}{2}\right) + J\left\{\begin{array}{c}\lambda_{0i}\\ -\rho_{0i}\end{array}\right\}\mathrm{sh}^2\left(q^i+\langle q\rangle\right)\right],$$

$$R_{ij} = 4\left[(J-\alpha)(\lambda_{ij}+\rho_{ij})\,\mathrm{sh}^2\left(\frac{q^i-q^j}{2}\right) - J\left\{\begin{array}{c}\lambda_{ij}\\ \rho_{ij}\end{array}\right\}\mathrm{sh}^2\left(q^i-q^j\right)\right], \qquad i<j,$$

$$A_{ij} = 4\left[(J-\alpha)(\lambda_{ij}-\rho_{ij})\,\mathrm{ch}^2\left(\frac{q^i-q^j}{2}\right) + J\left\{\begin{array}{c}\lambda_{ij}\\ -\rho_{ij}\end{array}\right\}\mathrm{sh}^2\left(q^i-q^j\right)\right], \qquad i<j.$$

They satisfy the following Poisson bracket rules: $\{R_{\mu\nu}, R_{\kappa\sigma}\} = \{A_{\mu\nu}, A_{\kappa\sigma}\} = -R_{\mu\sigma}\delta_{\kappa\nu} + R_{\kappa\nu}\delta_{\mu\sigma} - R_{\sigma\nu}\delta_{\mu\kappa} + R_{\mu\kappa}\delta_{\sigma\nu}$ and $\{R_{\mu\nu}, A_{\kappa\sigma}\} = -A_{\mu\sigma}\delta_{\kappa\nu} + A_{\kappa\nu}\delta_{\mu\sigma} - A_{\sigma\nu}\delta_{\mu\kappa} + A_{\mu\kappa}\delta_{\sigma\nu}$.

Geodetic Hamiltonians corresponding to our kinetic energies (5) may be written as follows:

$$\Im_{\text{left/right}} = \frac{1}{2(J+\alpha)}\left[\sum_{i=1}^{n}(p_i)^2 - \frac{1}{n+1}\langle p\rangle^2\right] + \frac{1}{32(J-\alpha)}\sum_{\mu,\nu=0}^{n}\left[R^2_{\mu\nu} + A^2_{\mu\nu}\right] \tag{6}$$

$$\pm\frac{J}{8(J^2-\alpha^2)}\sum_{\mu,\nu=0}^{n}R_{\mu\nu}A_{\mu\nu} + \sum_{i=1}^{n}V_{\text{eff}}(|q^i+\langle q\rangle|, R_{0i}, A_{0i}) + \frac{1}{2}\sum_{i,j=1}^{n}U_{\text{eff}}(|q^i-q^j|, R_{ij}, A_{ij}),$$

where even in the purely geodetic problems we have the "internal" effective interaction potentials:

$$V_{\text{eff}} = \frac{1}{16(J+\alpha)}\left\{R^2_{0i}\,\mathrm{cth}^2\left(\frac{q^i+\langle q\rangle}{2}\right) + A^2_{0i}\,\mathrm{th}^2\left(\frac{q^i+\langle q\rangle}{2}\right)\right\},$$

$$U_{\text{eff}} = \frac{1}{16(J+\alpha)}\left\{R^2_{ij}\,\mathrm{cth}^2\left(\frac{q^i-q^j}{2}\right) + A^2_{ij}\,\mathrm{th}^2\left(\frac{q^i-q^j}{2}\right)\right\}.$$

The Hamilton equations of motion may be expressed in terms of Poisson brackets as follows:

$$\frac{dp_i}{dt} = \{p_i, H\} = \frac{1}{4(J+\alpha)\,\mathrm{sh}\,(q^i+\langle q\rangle)}\left\{R^2_{0i}\,\mathrm{cth}^2\left(\frac{q^i+\langle q\rangle}{2}\right) - A^2_{0i}\,\mathrm{th}^2\left(\frac{q^i+\langle q\rangle}{2}\right)\right\}$$
$$+\sum_{j=1}^{n}\frac{1}{8(J+\alpha)\,\mathrm{sh}\,(q^i-q^j)}\left\{R^2_{ij}\,\mathrm{cth}^2\left(\frac{q^i-q^j}{2}\right) - A^2_{ij}\,\mathrm{th}^2\left(\frac{q^i-q^j}{2}\right)\right\},$$

$$\frac{dR_{\mu\nu}}{dt} = \{R_{\mu\nu}, R_{\kappa\sigma}\}\frac{\partial\Im_{\text{left/right}}}{\partial R_{\kappa\sigma}} + \{R_{\mu\nu}, A_{\kappa\sigma}\}\frac{\partial\Im_{\text{left/right}}}{A_{\kappa\sigma}},$$

$$\frac{dA_{\mu\nu}}{dt} = \{A_{\mu\nu}, R_{\kappa\sigma}\}\frac{\partial\Im_{\text{left/right}}}{\partial R_{\kappa\sigma}} + \{A_{\mu\nu}, A_{\kappa\sigma}\}\frac{\partial\Im_{\text{left/right}}}{A_{\kappa\sigma}},$$

where the corresponding partial derivatives of the kinetic energies (6) have the following form:

$$\frac{\partial\Im_{\text{left/right}}}{\partial R_{0i}} = \frac{\left[J\,\mathrm{ch}\left(q^i+\langle q\rangle\right)-\alpha\right]R_{0i} \pm 2JA_{0i}}{8(J^2-\alpha^2)\,\mathrm{sh}^2\left[\left(q^i+\langle q\rangle\right)/2\right]},$$

$$\frac{\partial\Im_{\text{left/right}}}{\partial A_{0i}} = \frac{\left[J\,\mathrm{ch}\left(q^i+\langle q\rangle\right)-\alpha\right]A_{0i} \pm 2JR_{0i}}{8(J^2-\alpha^2)\,\mathrm{sh}^2\left[\left(q^i+\langle q\rangle\right)/2\right]},$$

$$\frac{\partial\Im_{\text{left/right}}}{\partial R_{ij}} = \frac{\left[J\,\mathrm{ch}\left(q^i-q^j\right)-\alpha\right]R_{ij} \pm 2JA_{ij}}{8(J^2-\alpha^2)\,\mathrm{sh}^2\left[\left(q^i-q^j\right)/2\right]},$$

$$\frac{\partial\Im_{\text{left/right}}}{\partial A_{ij}} = \frac{\left[J\,\mathrm{ch}\left(q^i-q^j\right)-\alpha\right]A_{ij} \pm 2JR_{ij}}{8(J^2-\alpha^2)\,\mathrm{sh}^2\left[\left(q^i-q^j\right)/2\right]}.$$

3 Geodetic problems on the projective line

In n dimensions a projectively-rigid body is such a body all projective relations between constituents of which during any admissible motion are invariant, i.e. material straight lines remain straight lines and all cross-ratios of any four points placed on the same straight lines are constant. It is interesting that the cross-ratio of four points on the line, i.e.

$$(x_1,x_2,x_3,x_4)=\frac{x_4-x_1}{x_4-x_2}:\frac{x_3-x_1}{x_3-x_2},$$

plays here the same role as the usual mutual ratio of segments for the affine and the distance for the metrical geometries (see e.g. [11]). After choosing the appropriate homogeneous coordinates and adding to the consideration a set of non-proper points in the infinity, the cross-ratio is constant under the action of the whole projective group $\mathrm{Pr}\,(n,\mathbb{R})$.

Let us consider now the very simple and in some sense trivial but nevertheless very illustrative example of the one-dimensional left and right invariant geodetic problems on the projective group $\mathrm{Pr}\,(1,\mathbb{R})\simeq \mathrm{SL}\,(2,\mathbb{R})$. Hence:

$$D=\begin{bmatrix} e^{-q} & 0\\ 0 & e^{q}\end{bmatrix},\qquad L=\begin{bmatrix}\cos\gamma & -\sin\gamma\\ \sin\gamma & \cos\gamma\end{bmatrix},\qquad R=\begin{bmatrix}\cos\delta & -\sin\delta\\ \sin\delta & \cos\delta\end{bmatrix},$$
$$\lambda=L^{-1}\dot L=\begin{bmatrix}0 & -\lambda\\ \lambda & 0\end{bmatrix},\qquad \rho=R^{-1}\dot R=\begin{bmatrix}0 & -\rho\\ \rho & 0\end{bmatrix},\qquad \lambda=\dot\gamma,\qquad \rho=\dot\delta.$$

The kinetic energy (5) now has the following form:

$$T^{n=1}_{\mathrm{left/right}}=(J+\alpha)\dot q^2+(J-\alpha)\begin{bmatrix}\rho-\mathrm{ch}\,(2q)\lambda\\ \lambda-\mathrm{ch}\,(2q)\rho\end{bmatrix}^2+(J+\alpha)\,\mathrm{sh}^2(2q)\begin{bmatrix}\lambda^2\\ \rho^2\end{bmatrix}. \tag{7}$$

The canonical momenta (p,j,k) (equivalently, the auxiliary valuables R, A) and corresponding velocities $(\dot q,\lambda,\rho)$ are connected by the Legendre transformation in the following way:

$$p=2(J+\alpha)\dot q,\qquad j_{\mathrm{left/right}}=2(J-\alpha)\,[\lambda-\rho\,\mathrm{ch}(2q)]+4J\begin{Bmatrix}\lambda\\ 0\end{Bmatrix}\mathrm{sh}^2(2q),$$
$$k_{\mathrm{left/right}}=2(J-\alpha)\,[\rho-\lambda\,\mathrm{ch}\,(2q)]+4J\begin{Bmatrix}0\\ \rho\end{Bmatrix}\mathrm{sh}^2(2q),$$
$$R_{\mathrm{left/right}}=2(J-\alpha)(\lambda+\rho)\,\mathrm{sh}^2q-4J\begin{Bmatrix}\lambda\\ \rho\end{Bmatrix}\mathrm{sh}^2(2q),$$
$$A_{\mathrm{left/right}}=2(J-\alpha)(\lambda-\rho)\,\mathrm{ch}^2q+4J\begin{Bmatrix}\lambda\\ -\rho\end{Bmatrix}\mathrm{sh}^2(2q).$$

Geodetic Hamiltonians corresponding to our kinetic energies (7) are as follows:

$$\begin{aligned}\Im^{n=1}_{\mathrm{left/right}}&=\frac{p^2}{4(J+\alpha)}+\frac{1}{4(J-\alpha)}\begin{Bmatrix}k^2\\ j^2\end{Bmatrix}+\frac{1}{4(J+\alpha)\,\mathrm{sh}^2(2q)}\begin{bmatrix}j+\mathrm{ch}\,(2q)k\\ k+\mathrm{ch}\,(2q)j\end{bmatrix}^2\\ &=\frac{p^2}{4(J+\alpha)}+\frac{R^2+A^2}{16(J-\alpha)}\pm\frac{JRA}{4(J^2-\alpha^2)}+V^{n=1}_{\mathrm{eff}}(q,R,A),\end{aligned}$$

where the one-dimensional effective potential and corresponding effective force have the following form:

$$V^{n=1}_{\mathrm{eff}}(q,R,A)=\frac{R^2\,\mathrm{cth}^2q+A^2\,\mathrm{th}^2q}{16(J+\alpha)},\qquad F^{n=1}_{\mathrm{eff}}:=-\frac{\partial V^{n=1}_{\mathrm{eff}}}{\partial q}=\frac{R^2\,\mathrm{cth}^2q-A^2\,\mathrm{th}^2q}{4(J+\alpha)\,\mathrm{sh}(2q)}.$$

The canonical momenta j and k (equivalently R and A) in the one-dimensional case are the constants of motion, so the Newton equation of motion of the fictitious particle on the line $\mathbb{R}$ in the "internal" effective potential $V_{\text{eff}}^{n=1}$ is as follows:

$$\ddot{q} = \frac{1}{2(J+\alpha)} F_{\text{eff}}^{n=1} = \frac{R^2 \operatorname{cth}^2 q - A^2 \operatorname{th}^2 q}{8(J+\alpha)^2 \operatorname{sh}(2q)}, \tag{8}$$

Assigning some special values to such constants of motion as energy E and canonical momenta R, A we can write the first-order differential equation on the q variable as follows:

$$\dot{q}^2 = \frac{1}{J+\alpha} E_{\text{eff}}^{\pm} - \frac{R^2 \operatorname{cth}^2 q + A^2 \operatorname{th}^2 q}{16(J+\alpha)^2}, \qquad E_{\text{eff}}^{\pm} = E - \frac{R^2 + A^2}{16(J-\alpha)} \mp \frac{JRA}{4(J^2-\alpha^2)},$$

and finally we have the following solution of the (8):

$$\begin{aligned} t(q) &= \int dq \frac{4(J+\alpha)}{\sqrt{16(J+\alpha)E_{\text{eff}}^{\pm} - R^2 \operatorname{cth}^2 q - A^2 \operatorname{th}^2 q}} \\ &= \frac{2(J+\alpha)}{\sqrt{\Theta}} \ln\left[R^2 - A^2 - \Theta \operatorname{ch}^2(2q) - 2\sqrt{\Theta} \operatorname{sh} q \sqrt{16(J+\alpha)E_{\text{eff}}^{\pm} - R^2 \operatorname{cth}^2 q - A^2 \operatorname{th}^2 q}\right], \end{aligned}$$

where $\Theta = 16(J+\alpha)E - 2J(R \pm A)^2/(J-\alpha)$.

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Invariance of Maxwell's Equations under Nonlinear Representations of Poincaré Algebra

Stanislav SPICHAK

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *spichak@imath.kiev.ua*

It is shown that the Maxwell equations are conditionally invariant with respect to two nonlinear representations of Poincaré algebra. The corresponding groups of finite transformations are constructed. The nonlinear generalization of Maxwell equations invariant under the above-mentioned nonlinear representations of Poincaré algebra is found.

1 Conditional invariance

Let us consider the Maxwell equations in vacuum written in the complex form:

$$\vec{\nabla}\vec{\Sigma} = 0, \qquad \partial_0\vec{\Sigma} + i\vec{\nabla}\times\vec{\Sigma} = 0, \tag{1}$$

where $\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$, $\partial_0 = \frac{\partial}{\partial t}$, $\vec{\Sigma} = \vec{E} + i\vec{H}$ is a complex-valued vector of electromagnetic field. As it is known [1, 2], the system (1) is invariant under the linear representation of the Poincaré algebra $AP(1,3)$ with basis operators

$$\begin{aligned} &AP^{\rm lin}(1,3) = \langle P_\mu, J^{\rm lin}_{ab}, J^{\rm lin}_{0a}\rangle, \\ &P_\mu = \partial_\mu, \qquad J^{\rm lin}_{ab} = x_a\partial_b - x_b\partial_a + \Sigma_a\partial_{\Sigma_b} - \Sigma_b\partial_{\Sigma_a}, \qquad J^{\rm lin}_{0a} = x_0\partial_a + x_a\partial_0 + N^{\rm lin}_a, \\ &\vec{N}^{\rm lin} = (N^{\rm lin}_1, N^{\rm lin}_2, N^{\rm lin}_3) = i\vec{\Sigma}\times\vec{\nabla}_\Sigma, \qquad \mu = \overline{0,3}, \end{aligned} \tag{2}$$

where $\vec{\nabla}_\Sigma = \left(\frac{\partial}{\partial\Sigma_1}, \frac{\partial}{\partial\Sigma_2}, \frac{\partial}{\partial\Sigma_3}\right)$.

A nonlinear representation for the algebra $AP(1,3)$ and for the electromagnetic field $\vec{E}$, $\vec{H}$ that has the following form for complex-valued vectors was found in [3]:

$$\begin{aligned} &AP^{\rm nli}(1,3) = \langle P_\mu, J^{\rm nli}_{ab}, J^{\rm nli}_{0a}\rangle, \\ &P_\mu = \partial_\mu, \qquad J^{\rm nli}_{ab} = J^{\rm lin}_{ab}, \qquad J^{\rm nli}_{0a} = x_0\partial_a + x_a\partial_0 + N^{\rm nli}_a, \\ &\vec{N}^{\rm nli} = \left(N^{\rm nli}_1, N^{\rm nli}_2, N^{\rm nli}_3\right) = \vec{\nabla}_\Sigma - \vec{\Sigma}(\Sigma_k\partial_{\Sigma_k}), \qquad k = 1,2,3 \end{aligned} \tag{3}$$

(summation by k is implied). Representations (2) and (3) are not equivalent as there is only one independent invariant for the operators (3) and two functionally independent invariants for the operators (2). Let us consider invariance properties of the system (1) with respect to the operators (3).

Theorem 1. *The Maxwell equations* (1) *are invariant with respect to the representation* (3) *of* $AP(1,3)$ *iff the following conditions are satisfied:*

$$\begin{aligned} &\left[\partial_0 + \vec{\Sigma}\vec{\nabla}\right]\Sigma^k = 0, \\ &\left[\vec{\nabla} + \vec{\Sigma}\partial_0 - i\vec{\Sigma}\times\vec{\nabla}\right]\Sigma^k = 0, \qquad k = 1,2,3. \end{aligned} \tag{4}$$

Besides, the system (1), (4) *is invariant with respect to the linear representation* $AP^{\rm lin}(1,3)$ (2).

Corollary. *There exists another representation of the Poincaré algebra that is an invariance algebra of the system under consideration.*

Let us consider the set of operators

$$J^3_{0a} = x_0\partial_a + x_a\partial_0 + \alpha N^{\rm lin}_a + \beta N^{\rm nli}_a, \qquad a = 1,2,3 \tag{5}$$

and its completion by means of commutation. As a result a Lie algebra is generated (generally speaking it is infinitely dimensional). It is not difficult to prove the following

Statement. *The completion of the set* (5) *will be the Poincaré algebra iff either* $(\alpha,\beta) = \left(\frac{1}{2},\frac{1}{2}\right)$, *or* $(\alpha,\beta) = \left(\frac{1}{2},-\frac{1}{2}\right)$. *These representations are equivalent with respect to the substitution* $\vec{\Sigma} \to -\vec{\Sigma}$.

The explicit form of the operators for $\alpha = \beta = \frac{1}{2}$ is as follows:

$$AP^3(1,3) = \langle P_\mu, J^3_{ab}, J^3_{0a}\rangle, \tag{6}$$
$$J^3_{ab} = x_a\partial_b - x_b\partial_a - \frac{i}{2}\varepsilon_{abc}\left(N^{\rm lin}_c + N^{\rm nli}_c\right), \qquad J^3_{0a} = x_0\partial_a + x_a\partial_0 + N^3_a = \frac{1}{2}\left(J^{\rm lin}_{0a} + J^{\rm nli}_{0a}\right).$$

As $J^{\rm lin}_{0a}$ and $J^{\rm nli}_{0a}$ are symmetry operators of the system (1), (4), this system is invariant with respect to the operators J^3_{0a}, and whence with respect to the representation (6) of $AP^3(1,3)$. Let us point out that operators of space rotations J^3_{ab}, as distinct from representations (2), (3), are nonlinear with respect to $\vec{\Sigma}$.

It is not difficult to show that the following relation is true on the solutions of the system (4):

$$\vec{\Sigma}^2 = \Sigma^a\Sigma_a = 1, \tag{7}$$

or, in the notations of electromagnetic field $\vec{E}$, $\vec{H}$,

$$\vec{E}\cdot\vec{H} = 0, \qquad \vec{E}^2 - \vec{H}^2 = 1.$$

It is interesting to note that if Σ_a satisfy the conditions (7), then the representations (2), (3) and (6) are connected by the following expressions:

$$\vec{N}^{\rm nli} = \left[i\vec{\Sigma}\times\vec{N}^{\rm lin}\right], \qquad \vec{N}^{\rm lin} = \left[i\vec{\Sigma}\times\vec{N}^{\rm nli}\right], \qquad \vec{N}^3_a = \left[i\vec{\Sigma}\times\vec{N}^3_a\right].$$

The system of equations (4) is much overdetermined: there are 12 equations with respect to 3 unknown functions. However, we can go on to 4 equations for 2 unknown functions. Actually it is not difficult to check that the system (4) is equivalent to its first equation (for Σ_k), first "component" of the second equation

$$[\partial_1 + \Sigma_1\partial_0 - i\Sigma_2\partial_3 + i\Sigma_3\partial_2]\Sigma_k = 0,$$

and the condition (7). Whence making the substitution $\Sigma_3 = \pm\sqrt{1-\Sigma_1^2-\Sigma_2^2}$, we go on to the system of 4 equations with respect to Σ_1, Σ_2.

The system (1), (4) is compatible, and it has, for example, the following solution

$$\vec{\Sigma} = (1, i\varphi(t,x), \varphi(t,x)),$$
$$\partial_2\varphi + i\partial_3\varphi = 0, \qquad \partial_0\varphi + \partial_1\varphi = 0, \qquad \varphi\in\mathbb{C}. \tag{8}$$

Among solutions of (8) there are functions of the following form

$$\vec{\Sigma} = (1, if(t-x_1), f(t-x_1)), \tag{9}$$

where f is an arbitrary differentiable function.

Thus, at the certain subset of solutions of the Maxwell equations that include flat wave solutions (9), the electromagnetic field $\vec{E}$, $\vec{H}$ can be transformed by transformations in the 4-dimensional space, not only as an electromagnetic tensor, but also in a nonlinear way.

2 Finite transformations

Having solved the respective system of the Lie equations, we can find the finite group transformations corresponding to operators of Lorentz boosts $J^3_{\mu\nu}$ (6) (for the representation $AP^{\rm nli}(1,3)$ these transformations were found in [4]):

$$\begin{aligned}
J^3_{ab}:\ & x_a \to \tilde{x}_a = x_a\cos(2\theta) - x_b\sin(2\theta),\\
& x_b \to \tilde{x}_b = x_b\cos(2\theta) + x_a\sin(2\theta),\\
& x_c \to \tilde{x}_c = x_c, \quad c \neq a, b,\\
& \Sigma_k \to \tilde{\Sigma} = \frac{\Sigma_k\cos\theta + (\delta_{kb}\Sigma_a - \delta_{ka}\Sigma_b - i\varepsilon_{abk})\sin\theta}{\cos\theta - i\varepsilon_{abc}\Sigma_c\sin\theta};
\end{aligned} \tag{10}$$

$$\begin{aligned}
J^3_{0a}:\ & x_0 \to \tilde{x}_0 = x_0\,\mathrm{ch}\,(2\theta) + x_a\,\mathrm{sh}\,(2\theta),\\
& x_a \to \tilde{x}_a = x_a\,\mathrm{ch}\,(2\theta) + x_0\,\mathrm{sh}\,(2\theta),\\
& x_k \to \tilde{x}_k = x_k, \quad k \neq a,\\
& \Sigma_k \to \tilde{\Sigma} = \frac{\Sigma_k\,\mathrm{ch}\,\theta + (\delta_{ka} - i\varepsilon_{akl}\Sigma_l)\,\mathrm{sh}\,\theta}{\mathrm{ch}\,\theta + \Sigma_a\,\mathrm{sh}\,\theta}.
\end{aligned} \tag{11}$$

The transformations (10), (11) can be used for group generations of solutions for the Maxwell equations by means of e.g. initial solution (8). The newly obtained solutions can be further multiplied by the symmetry groups of the Maxwell equations.

3 Generalizations of the Maxwell equations

In the paper [4] the following nonlinear generalizations of the Maxwell equations is considered:

$$\begin{aligned}
&\vec{\nabla}\vec{\Sigma} = A(\omega)\partial_0\omega,\\
&\partial_0\vec{\Sigma} + i\vec{\nabla}\times\vec{\Sigma} = A(\omega)\vec{\nabla}(\omega),
\end{aligned} \tag{12}$$

where ω are invariants of the representation (2) of the algebra $AP^{\rm lin}(1,3)$. Let us consider the following system of equations

$$\begin{aligned}
&\vec{\nabla}\vec{\Sigma} = A(\Omega)\partial_0\Omega + B(\Omega)\vec{\Sigma}\vec{\nabla}\Omega,\\
&\partial_0\vec{\Sigma} + i\vec{\nabla}\times\vec{\Sigma} = A(\Omega)\vec{\nabla}(\Omega) + B(\Omega)\big(\vec{\Sigma}\partial_0\Omega - i\vec{\Sigma}\times\vec{\nabla}\Omega\big),
\end{aligned} \tag{13}$$

where $\Omega = f\big(\Sigma_a^2\big)$. Without losing generality we can assume that $\Omega = \ln\sqrt{\Sigma_a^2 - 1}$. If $B = 0$ then the system (13) has the form (12). The following theorem is true.

Theorem 2. *The system of equations* (13) *is invariant with respect to the linear representation of the Poincaré algebra* (2) *with any* $A(\Omega)$, $B(\Omega)$, *and with respect to the nonlinear representation* (3) *if and only if* $A = B = 1$. *In this case the right-hand parts of the equations* (13) *coincides with left-hand parts of the equations* (4), *if the substitution* $\Sigma^k \to \Omega$ *is made.*

It follows from the Theorem that the system (13) with $A = B = 1$ is invariant also with respect to the representation of the Poincaré algebra $AP^3(1,3)$ (6).

Let us write the equations (13) in the covariant form. Let us introduce the following notations for this purpose. $G^{\mu\nu}$ is an antisymmetric tensor:

$$G^{0a} = \Sigma_a, \qquad G^{ab} = -i\varepsilon_{abc}\Sigma_c.$$

In terms of the tensor of the electromagnetic field $F^{\mu\nu}$, $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\delta}F^{\rho\delta}$ we can write $G^{\mu\nu} = F^{\mu\nu} + i\tilde{F}^{\mu\nu}$. Then (13) can be written in the same notations as:

$$\partial_\mu G^{\mu\nu} = G^{\mu\nu}\partial_\mu\varphi - \partial^\nu\varphi, \qquad \varphi = \ln\sqrt{-\frac{G^{\mu\nu}G_{\mu\nu}}{4} - 1}.$$

For the Maxwell equation (1) $\varphi \equiv 0$.

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Massive Gauge Field Theory without Higgs Mechanism

Junchen SU

Center for Theoretical Physics, Department of Physics, Jilin University, Changchun 130023, People's Republic of China
E-mail: *junchens@public.cc.jl.cn*

It is argued that the massive gauge field theory without the Higgs mechanism can well be set up on the gauge-invariance principle based on the viewpoint that a massive gauge field must be viewed as a constrained system and the Lorentz condition, as a constraint, must be introduced from the beginning and imposed on the Yang–Mills Lagrangian. The quantum theory for the massive gauge fields may perfectly be established by the quantization performed in the Hamiltonian or the Lagrangian path-integral formalism by means of the Lagrange undetermined multiplier method and shows good renormalizability and unitarity.

It is the prevailing viewpoint that the massive gauge field theory can not be set up without introducing the Higgs mechanism. The first obstacle is the gauge-non-invariance of the mass term in the massive Yang–Mills Lagrangian for a massive gauge field. On the contrary, we present an argument to show that the conventional viewpoint is not true [1]. In fact, a certain massive gauge field theory can be well established on the basis of gauge-invariance principle without recourse to the Higgs mechanism. The basic ideas are stated in the following.

(1) A massive gauge field must be viewed as a constrained system. In the previous attempt of building up the massive non-Abelian gauge field theory, the massive Yang–Mills Lagrangian density written below was chosen to be the starting point [2, 3]

$$\mathcal{L} = -\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2}m^2A^{a\mu}A^a_\mu, \tag{1}$$

where A^a_μ are the vector potentials for the non-Abelian massive gauge fields,

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{2}$$

are the field strengths and m is the mass of gauge bosons. The first term in the Lagrangian is the ordinary Yang–Mills Lagrangian which is gauge-invariant under a whole Lie group and used to determine the form of interactions among the gauge fields themselves. The second term in the Lagrangian is the mass term which is not gauge-invariant and only affects the kinematic property of the fields. The above Lagrangian itself was ever considered to give a complete description of the massive gauge field dynamics. This consideration is not correct because the Lagrangian is not only not gauge-invariant, but also contains redundant unphysical degrees of freedom. As one knows, a full vector potential $A^\mu(x)$ can be split into two Lorentz-covariant parts: the transverse vector potential $A^\mu_T(x)$ and the longitudinal vector potential $A^\mu_L(x)$, $A^\mu(x) = A^\mu_T(x) + A^\mu_L(x)$. The transverse vector potential $A^{a\mu}_T(x)$ contains three independent spatial components which is sufficient to represent the polarization states of a massive vector boson. Whereas, the longitudinal vector potential $A^{a\mu}_L$ appears to be a redundant unphysical variable which must be constrained by introducing the Lorentz condition

$$\varphi^a \equiv \partial^\mu A^a_\mu = 0, \tag{3}$$

whose solution is $A^{a\mu}_L = 0$. With this solution, the massive Yang–Mills Lagrangian may be expressed in terms of the independent dynamical variables $A^{a\mu}_T(x)$,

$$\mathcal{L} = -\tfrac{1}{4}F^{a\mu\nu}_T F^a_{T\mu\nu} + \tfrac{1}{2}m^2A^{a\mu}_T A^a_{T\mu}, \tag{4}$$

which gives a complete description of the massive gauge field dynamics. If we want to represent the dynamics in the whole space of the full vector potential as described by the massive Yang–Mills Lagrangian in equation (1), the massive gauge field must be treated as a constrained system. In this case, the Lorentz condition in equation (3), as a constraint, is necessary to be introduced from the onset and imposed on the Lagrangian in equation (1) so as to guarantee the redundant degrees of freedom to be eliminated from the Lagrangian.

(2) The gauge-invariance should generally be required for the action written in the physical subspace. Usually, the gauge-invariance is required to the Lagrangian. From the dynamical viewpoint, as we know, the action is of more essential significance than the Lagrangian. This is why in Mechanics and Field Theory, to investigate the dynamical and symmetric properties of a system, one always starts from the action of the system. Similarly, when we examine the gauge-symmetric property of a field system, in more general, we should also see whether the action of the system is gauge-invariant or not. In particular, for a constrained system such as the massive gauge field, we should see whether or not the action represented by the independent dynamical variables is gauge-invariant. This point of view is familiar to us in the mechanics for constrained systems. Certainly, in some special cases, the Lagrangian itself is gauge-invariant in the physical subspace so that the gauge-invariance of the action is ensured. This situation happens for the massless gauge fields and the massive Abelian gauge field. For the non-Abelian gauge fields, the infinitesimal gauge transformation usually is given by [3]

$$\delta A^a_\mu = D^{ab}_\mu \theta^b, \tag{5}$$

where

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu. \tag{6}$$

This gauge transformation is different from the Abelian one in that in the physical subspace defined by the Lorentz condition, i.e., spanned by the transverse vector potential $A^{a\mu}_T$, the fields still undergo nontrivial gauge transformations

$$\delta A^a_{T\mu} = D^{ab}_{T\mu} \theta^b, \tag{7}$$

where

$$D^{ab}_{T\mu} = \delta^{ab}\partial_\mu - g f^{abc} A^c_{T\mu}. \tag{8}$$

Therefore, the mass term in the massive Yang–Mills Lagrangian written in equation (4) is not gauge-invariant. But, the action given by this Lagrangian is invariant with respect to the gauge transformation shown in equations (7) and (8). In fact, noticing the identity: $f^{abc} A^{a\mu}_T A^b_{T\mu} = 0$ and the transversity condition (an identity): $\partial^\mu A^a_{T\mu} = 0$, it is easy to see

$$\delta S = -m^2 \int d^4x \theta^a \partial^\mu A^a_{T\mu} = 0.$$

This shows that the dynamics of massive non-Abelian gauge fields is gauge-invariant. Alternatively, the gauge-invariance may also be seen from the action given by the Lagrangian in equation (1) which is constrained by the Lorentz condition in equation (3). Under the gauge transformation written in equation (5) and (6), noticing the identity $f^{abc} A^{a\mu} A^b_\mu = 0$ and the Lorentz condition, it can be found that

$$\delta S = -m^2 \int d^4x \theta^a \partial^\mu A^a_\mu = 0.$$

This suggests that the massive non-Abelian gauge field theory may also be set up on the basis of gauge-invariance principle.

(3) Only infinitesimal gauge transformations need to be considered in the physical subspace. In examining the gauge invariance of the action for the massive non-Abelian gauge fields, we confine ourself to consider the infinitesimal gauge transformation only. The reason for this arises from the fact that the Lorentz condition limits the gauge transformation only to take place in the vicinity of the unity of the gauge group. In other words, the residual gauge degrees of freedom existing in the physical subspace are characterized by the infinitesimal gauge transformations. This fact was pointed out in the pioneering article by Faddeev and Popov for the quantization of massless non-Abelian gauge fields [4]. Usually, the dynamics of massless gauge fields is described by the Yang–Mills Lagrangian. It is well-known that the Yang–Mills Lagrangian itself is not quantizable, namely, it cannot be used to construct a convergent generating functional of Green's functions even though it is gauge-invariant with respect to the whole gauge group. This is because the Yang–Mills Lagrangian contains redundant unphysical degrees of freedom and hence is not complete for describing the massless gauge field dynamics unless a suitable constraint such as the Lorentz condition is introduced to eliminate the unphysical degrees of freedom. In the article by Faddeev and Popov, the Lorentz condition is introduced through the following identity [4]

$$\Delta[A]\int D(g)\delta[\partial^\mu A^g_\mu]=1,$$

which is inserted into the functional integral representing the vacuum-to-vacuum transition amplitude. After doing this, the authors said that "We must know $\Delta[A]$ is only for the transverse fields and in this case all contributions to the last integral are given in the neighborhood of the unity element of the group". The delta-functional in the above identity implies $\partial^\mu(A^g)^a_\mu=\partial^\mu A^a_\mu=0$ which represents the gauge-invariance of the Lorentz condition because the Lorentz condition is required to hold for all the field variables including the ones before and after gauge transformations. In the physical subspace where only the transverse fields are allowed to appear, only infinitesimal gauge transformations around the unity element are permitted and needed to be considered in the course of Faddeev–Popov's quantization even though the Yang–Mills Lagrangian used is invariant under the whole gauge group. Obviously, there are no reasons of considering the gauge-transformation property of the fields in the region beyond the physical subspace because the fields do not exist in that region. By this point, it can be understood why in the ordinary quantum gauge field theories such as the standard model, the BRST-transformations are all taken to be infinitesimal.

According to the general procedure, the Lorentz condition (3) may be incorporated into the Lagrangian (1) by the Lagrange undetermined multiplier method to give a generalized Lagrangian. In the first order formalism, this Lagrangian is written as [5]

$$\mathcal{L}=\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}-\tfrac{1}{2}F^{a\mu\nu}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu)+\tfrac{1}{2}m^2A^{a\mu}A^a_\mu+\lambda^a\partial^\mu A^a_\mu, \tag{9}$$

where A^a_μ and $F^a_{\mu\nu}$ are now treated as the mutually independent variables and λ^a are chosen to represent the Lagrange multipliers. Using the canonically conjugate variables defined by

$$\Pi^a_\mu(x)=\frac{\partial\mathcal{L}}{\partial\dot{A}^{a\mu}}=F^a_{\mu0}+\lambda^a\delta_{\mu0}=\begin{cases}F^a_{k0}=E^a_k, & \text{if } \mu=k=1,2,3,\\ \lambda^a=-E^a_0, & \text{if } \mu=0,\end{cases}$$

the Lagrangian in equation (9) may be rewritten in the canonical form

$$\mathcal{L}=E^{a\mu}\dot{A}^a_\mu+A^a_0C^a-E^a_0\varphi^a-\mathcal{H}, \tag{10}$$

where

$$C^a=\partial^\mu E^a_\mu+gf^{abc}A^b_kE^{ck}+m^2A^a_0, \tag{11}$$

$$\mathcal{H}=\tfrac{1}{2}(E^a_k)^2+\tfrac{1}{4}(F^a_{ij})^2+\tfrac{1}{2}m^2\left[(A^a_0)^2+(A^a_k)^2\right], \tag{12}$$

here $E^a_\mu = (E^a_0, E^a_k)$ is a Lorentz vector, $\mathcal{H}$ is the Hamiltonian density in which F^a_{ij} are defined in equation (2). In the above, the four-dimensional and the spatial indices are respectively denoted by the Greek and Latin letters. From equation (10), it is clearly seen that the second and third terms are given respectively by incorporating the constraint condition

$$C^a = 0, \tag{13}$$

where C^a was represented in equation (11) and the Lorentz condition in equation (3) into the Lagrangian.

Now, let us first perform the quantization of the massive non-Abelian gauge fields in the Hamiltonian path-integral formalism [5]. In accordance with the general procedure of the quantization, we should first write the generating functional of Green's functions via the independent canonical variables which are now chosen to be the transverse parts of the vectors A^a_μ and E^a_μ

$$Z[J] = \frac{1}{N}\int D(A^{a\mu}_T, E^{a\mu}_T)\exp\left\{i\int d^4x\left[E^{a\mu}_T\dot{A}^a_{T\mu} - \mathcal{H}^*(A^{a\mu}_T, E^{a\mu}_T) + J^{a\mu}_T A^a_{T\mu}\right]\right\}, \tag{14}$$

where $\mathcal{H}^*(A^{a\mu}_T, E^{a\mu}_T)$ is the Hamiltonian which is obtained from the Hamiltonian in equation (12) by replacing the constrained variables $A^{a\mu}_L$ and $E^{a\mu}_L$ with the solutions of equations (3) and (13)

$$\mathcal{H}^*(A^{a\mu}_T, E^{a\mu}_T) = \mathcal{H}(A^{a\mu}, E^{a\mu})\mid_{\varphi^a=0, C^a=0}.$$

As mentioned before, equation (3) leads to $A^{a\mu}_L = 0$. Noticing this solution and the decomposition $E^{a\mu}(x) = E^{a\mu}_T + E^{a\mu}_L(x)$, when setting

$$E^{a\mu}_L(x) = \partial^\mu_x Q^a(x), \tag{15}$$

where $Q^a(x)$ is a scalar function, one may get from equation (13) an equation obeyed by the scalar function $Q^a(x)$

$$K^{ab}(x)Q^b(x) = W^a(x), \tag{16}$$

where

$$K^{ab}(x) = \delta^{ab}\Box_x - gf^{abc}A^{c\mu}_T(x)\partial^x_\mu \quad \text{and} \quad W^a(x) = gf^{abc}E^{b\mu}_T(x)A^c_{T\mu}(x) - m^2A^{a0}_T(x).$$

With the aid of the Green's function $G^{ab}(x-y)$ (the ghost particle propagator) which satisfies the following equation

$$K^{ac}(x)G^{cb}(x-y) = \delta^{ab}\delta^4(x-y)$$

one may find the solution to the equation in equation (16) as follows

$$Q^a(x) = \int d^4y G^{ab}(x-y)W^b(y). \tag{17}$$

From the expressions given in equations (15) and (17), one can see that the $E^{a\mu}_L(x)$ is a complicated functional of the variables $A^{a\mu}_T$ and $E^{a\mu}_T$ so that the Hamiltonian $\mathcal{H}^*(A^{a\mu}_T, E^{a\mu}_T)$ is of much more complicated functional structure which is not convenient for constructing the diagrammatic technique in the perturbation theory. Therefore, it is better to express the generating functional in equation (14) in terms of the variables A^a_μ and E^a_μ. For this purpose, it is necessary to insert the following delta-functional into equation (14) [5]

$$\delta[A^{a\mu}_L]\delta[E^{a\mu}_L - E^{a\mu}_L(A^{a\mu}_T, E^{a\mu}_T)] = \det M\delta[C^a]\delta[\varphi^a], \tag{18}$$

where M is the matrix whose elements are defined by the Poisson bracket

$$M^{ab}(x,y) = \left\{C^a(x), \varphi^b(y)\right\} = \int d^4x \left\{ \frac{\delta C^a}{\delta A^a_\mu(x)} \frac{\delta \varphi^b}{\delta E^{a\mu}(x)} - \frac{\delta C^a}{\delta E^a_\mu(x)} \frac{\delta \varphi^b}{\delta A^{a\mu}(x)} \right\}$$
$$= D^{ab}_\mu(x)\partial^\mu_x \delta^4(x-y).$$

The relation in equation (18) is easily derived from equations (3) and (13) by applying the property of delta-functional. Upon inserting equation (18) into equation (14) and utilizing the Fourier representation of the delta-functional

$$\delta[C^a] = \int D(\eta^a/2\pi) e^{i\int d^4x \eta^a C^a},$$

we have

$$Z[J] = \frac{1}{N}\int D(A^a_\mu, E^a_\mu, \eta^a)$$
$$\times \det M \delta[\partial^\mu A^a_\mu] \exp\left\{ i\int d^4x [E^{a\mu}\dot{A}^a_\mu + \eta^a C^a - \mathcal{H}(A^{a\mu}, E^{a\mu}) + J^{a\mu}A^a_\mu] \right\}.$$

In the above exponential, there is a E^a_0-related term $E^a_0(\partial_0 A^a_0 - \partial_0 \eta^a)$ which permits us to perform the integration over E^a_0, giving a delta-functional

$$\delta[\partial_0 A^a_0 - \partial_0 \eta^a] = \det|\partial_0|^{-1} \delta[A^a_0 - \eta^a].$$

The determinant $\det|\partial_0|^{-1}$, as a constant, may be put in the normalization constant N and the delta-functional $\delta[A^a_0 - \eta^a]$ will disappear when the integration over η^a is carried out. The integral over E^a_k is of Gaussian-type and hence easily calculated. After these manipulations, we arrive at

$$Z[J] = \frac{1}{N}\int D(A^a_\mu) \det M \delta[\partial^\mu A^a_\mu]$$
$$\times \exp\left\{ i\int d^4x \left[-\tfrac{1}{4} F^{a\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2} m^2 A^{a\mu}A^a_\mu + J^{a\mu}A^a_\mu \right] \right\}. \tag{19}$$

When employing the familiar expression [4]

$$\det M = \int D(\bar{C}^a, C^a) \exp\left\{ i\int d^4x d^4y \bar{C}^a(x) M^{ab}(x,y) C^b(y) \right\},$$

where $\bar{C}^a(x)$ and $C^a(x)$ are the mutually conjugate ghost field variables and the following limit for the Fresnel functional

$$\delta[\partial^\mu A^a_\mu] = \lim_{\alpha\to 0} C[\alpha] e^{-\frac{i}{2\alpha}\int d^4x (\partial^\mu A^a_\mu)^2},$$

where $C[\alpha] \sim \prod_x (\frac{i}{2\pi\alpha})^{1/2}$ and supplementing the external source terms for the ghost fields, the generating functional in equation (19) is finally given in the form

$$Z[J, \overline{\xi}, \xi] = \frac{1}{N}\int D(A^a_\mu, \bar{C}^a, C^a) \exp\left\{ i\int d^4x [\mathcal{L}_{\text{eff}} + J^{a\mu}A^a_\mu + \overline{\xi}^a C^a + \bar{C}^a \xi^a] \right\}, \tag{20}$$

where

$$\mathcal{L}_{\text{eff}} = -\tfrac{1}{4} F^{a\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2} m^2 A^{a\mu}A^a_\mu - \tfrac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 - \partial^\mu \bar{C}^a D^{ab}_\mu C^b, \tag{21}$$

which is the effective Lagrangian for the quantized massive non-Abelian gauge field in which the third and fourth terms are the so-called gauge-fixing term and the ghost term respectively. In equation (20), the limit $\alpha \to 0$ is implied. Certainly, the theory may be given in arbitrary gauges ($\alpha \neq 0$). In this case, as will be seen soon later, the ghost particle will acquire a spurious mass $\mu = \sqrt{\alpha} m$.

To confirm the result of the quantization given above, let us turn to quantize the massive non-Abelian gauge fields in the Lagrangian path-integral formalism. For later convenience, the massive Yang–Mills Lagrangian in equation (1) and the Lorentz constraint condition in equation (3) are respectively generalized to the following forms

$$\mathcal{L}_\lambda = -\tfrac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \tfrac{1}{2} m^2 A^{a\mu} A^a_\mu - \tfrac{1}{2}\alpha(\lambda^a)^2 \tag{22}$$

and

$$\partial^\mu A^a_\mu + \alpha \lambda^a = 0, \tag{23}$$

where $\lambda^a(x)$ are the extra functions which will be identified with the Lagrange multipliers and α is an arbitrary constant playing the role of gauge parameter. According to the general procedure for constrained systems, the constraint condition in equation (23) may be incorporated into the Lagrangian in equation (22) by the Lagrange undetermined multiplier method, giving a generalized Lagrangian

$$\mathcal{L}_\lambda = -\tfrac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + \tfrac{1}{2} m^2 A^{a\mu} A^a_\mu + \lambda^a \partial^\mu A^a_\mu + \tfrac{1}{2}\alpha(\lambda^a)^2. \tag{24}$$

This Lagrangian is obviously not gauge-invariant. However, for building up a correct gauge field theory, it is necessary to require the action given by the Lagrangian in equation (24) to be invariant with respect to the gauge transformations shown in equations (5) and (6) so as to guarantee the dynamics of the gauge field to be gauge-invariant. By this requirement, noticing the identity $f^{abc} A^{a\mu} A^b_\mu = 0$ and applying the constraint condition in equation (23), we have

$$\delta S_\lambda = -\frac{1}{\alpha} \int d^4x \partial^\nu A^a_\nu(x) \partial^\mu (\mathcal{D}^{ab}_\mu(x) \theta^b(x)) = 0,$$

where

$$\mathcal{D}^{ab}_\mu(x) = \delta^{ab} \frac{\mu^2}{\Box_x} \partial^x_\mu + D^{ab}_\mu(x),$$

in which $\mu^2 = \alpha m^2$ and $D^{ab}_\mu(x)$ was defined in equation (6). From equation (23), we see $\frac{1}{\alpha}\partial^\nu A^a_\nu = -\lambda^a \neq 0$. Therefore, to ensure the action to be gauge-invariant, the following constraint condition on the gauge group is necessary to be required

$$\partial^\mu_x (\mathcal{D}^{ab}_\mu(x) \theta^b(x)) = 0. \tag{25}$$

These are the coupled equations satisfied by the parametric functions $\theta^a(x)$ of the gauge group. Since the Jacobian of the equations is not singular, $\det M \neq 0$ where M is the matrix whose elements are

$$M^{ab}(x,y) = \frac{\delta(\partial^\mu_x \mathcal{D}^{ac}_\mu(x)\theta^c(x))}{\delta\theta^b(y)}\Big|_{\theta=0} = \delta^{ab}(\Box_x + \mu^2)\delta^4(x-y) - g f^{abc} \partial^\mu_x (A^c_\mu(x)\delta^4(x-y)),$$

the above equations are solvable and would give a set of solutions which express the functions $\theta^a(x)$ as functionals of the vector potentials $A^a_\mu(x)$. The constraint conditions in equation (25) may also be incorporated into the Lagrangian in equation (24) by the Lagrange undetermined

multiplier method. In doing this, it is convenient, as usually done, to introduce ghost field variables $C^a(x)$ in such a fashion [3–5]: $\theta^a(x) = \varsigma C^a(x)$, where ς is an infinitesimal Grassmann's number. In accordance with this relation, the constraint condition in equation (25) can be rewritten as

$$\partial^\mu(\mathcal{D}^{ab}_\mu C^b) = 0, \tag{26}$$

which usually is called ghost equation. When this constraint condition is incorporated into the Lagrangian in equation (24) by the Lagrange multiplier method, we obtain a more generalized Lagrangian as follows

$$\mathcal{L}_\lambda = -\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2}m^2A^{a\mu}A^a_\mu + \lambda^a\partial^\mu A^a_\mu + \tfrac{1}{2}\alpha(\lambda^a)^2 + \bar{C}^a\partial^\mu(\mathcal{D}^{ab}_\mu C^b), \tag{27}$$

where $\bar{C}^a(x)$, acting as Lagrange multipliers, are the new scalar variables conjugate to the ghost variables $C^a(x)$.

At present, we are ready to formulate the quantization of the massive gauge field in the Lagrangian path-integral formalism. As we learn from the Lagrange multiplier method, the dynamical and constrained variables as well as the Lagrange multipliers in equation (27) can all be treated as free ones, varying arbitrarily. Therefore, we are allowed to use this kind of Lagrangian to construct the generating functional of Green's functions

$$\begin{aligned} Z[J^{a\mu},\overline{\xi}^a,\xi^a] = \frac{1}{N}\int D(A^a_\mu,\bar{C}^a,C^a,\lambda^a) \\ \times\exp\left\{i\int d^4x[\mathcal{L}_\lambda(x) + J^{a\mu}(x)A^a_\mu(x) + \overline{\xi}^a(x)C^a(x) + \overline{C}^a(x)\xi^a(x)]\right\}, \end{aligned} \tag{28}$$

where $D(A^a_\mu,\ldots,\lambda^a)$ denotes the functional integration measure, J^a_μ, $\overline{\xi}^a$ and ξ^a are the external sources coupled to the gauge and ghost fields and N is the normalization constant. Looking at the expression of the Lagrangian in equation (27), it is seen that the integral over $\lambda^a(x)$ is of Gaussian-type. Upon completing the calculation of this integral, we finally arrive at

$$\begin{aligned} Z[J^{a\mu},\overline{\xi}^a,\xi^a] = \frac{1}{N}\int D(A^a_\mu,\bar{C}^a,C^a,) \\ \times\exp\left\{i\int d^4x[\mathcal{L}_{\text{eff}}(x) + J^{a\mu}(x)A^a_\mu(x) + \overline{\xi}^a(x)C^a(x) + \overline{C}^a(x)\xi^a(x)]\right\}, \end{aligned} \tag{29}$$

where

$$\mathcal{L}_{\text{eff}} = -\tfrac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \tfrac{1}{2}m^2A^{a\mu}A^a_\mu - \tfrac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 - \partial^\mu\bar{C}^a\mathcal{D}^{ab}_\mu C^b. \tag{30}$$

is the effective Lagrangian given in the general gauges. In the Landau gauge ($\alpha \to 0$), The Lagrangian in equation (30) just gives the result in equation (21). It has been proved [1] that the above quantization carried out by means of the Lagrange multiplier method is equivalent to the Faddeev–Popov approach of quantization [4].

There are three points we would like to emphasize: (1) In the quantization by the Lagrange multiplier method, the gauge-invariance is always to be required even in the arbitrary gauge. Moreover, it has been found that the action given by the Lagrangian in equation (30) is invariant under a kind of BRST-transformations [6]. Thus, the quantum non-Abelian gauge field theory is set up from beginning to end on the firm basis of gauge-invariance. (2) In the Lagrangian path-integral formalism, as shown before, the quantized result is derived by utilizing the infinitesimal gauge transformations. This result is, as mentioned before, identical to that obtained by the quantization in the Hamiltonian path-integral formalism. In the latter quantization, we only need to calculate the classical Poisson brackets without concerning any gauge transformation.

This fact reveals that to get a correct quantum theory in the Lagrangian path-integral formalism, the infinitesimal gauge transformations are only necessary to be taken into account and thereby confirms the fact that in the physical subspace restricted by the Lorentz condition, only the infinitesimal gauge transformations are possible to exist. (3) From the generating functional shown in equation (29) and (30), one may derive the gauge boson propagator as follows

$$iD^{ab}_{\mu\nu}(k) = -i\delta^{ab}\left\{\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{k^2 - m^2 + i\varepsilon} + \frac{\alpha k_\mu k_\nu/k^2}{k^2 - \mu^2 + i\varepsilon}\right\},$$

which is of good renormalizable behavior. In the zero-mass limit, this propagator with the massive Yang–Mills Lagrangian and the generating functional together all go over to the results given in the massless gauge field theory, different from the quantum theory established previously from the massive Yang–Mills Lagrangian alone without any constraint [7–10]. For the previous theory, there occurs a severe contradiction in the zero-mass limit that the massive Yang–Mills Lagrangian in equation (1) is converted to the massless one, but, the propagator is not and of a singular behavior. In particular, the previous theory was shown to be nonrenormalizable [3, 7–9] because the unphysical longitudinal fields and residual gauge degrees of freedom are not excluded from the theory.

Up to the present, we limit ourself to discuss the gauge fields themselves without concerning fermion fields. For the gauge fields, in order to guarantee the mass term in the action to be gauge-invariant, the masses of all gauge bosons are taken to be the same. If fermions are included, obviously, the QCD with massive gluons fulfils this requirement because all the gluons can be considered to have the same mass. Such a QCD, as has been proved, is not only renormalizable, but also unitary [6]. The renormalizability and unitarity are warranted by the fact that the unphysical degrees of freedom in the theory have been removed by the constraint conditions in equations (23) and (26). The gauge-fixing term and the ghost term in equation (30) just play the role of counteracting the unphysical degrees of freedom contained in the massive Yang–Mills Lagrangian as verified by the perturbative calculations [6].

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CPT Symmetry and Properties of the Exact and Approximate Effective Hamiltonians for the Neutral Kaon Complex[1]

Krzysztof URBANOWSKI

University of Zielona Góra, Institute of Physics, ul. Podgórna 50, 65-246 Zielona Góra, Poland
E-mail: *K.Urbanowski@proton.if.uz.zgora.pl, K.Urbanowski@if.uz.zgora.pl*

We show that the diagonal matrix elements of the exact effective Hamiltonian governing the time evolution in the subspace of states of an unstable particle and its antiparticle need not be equal at for $t > t_0$ (t_0 is the instant of creation of the pair) when the total system under consideration is CPT invariant but CP noninvariant. The unusual consequence of this result is that, contrary to the properties of stable particles, the masses of the unstable particle "1" and its antiparticle "2" need not be equal for $t \gg t_0$ in the case of preserved CPT and violated CP symmetries.

1 Introduction

The problem of testing CPT-invariance experimentally has attracted the attention of physicist, practically since the discovery of antiparticles. CPT symmetry is a fundamental theorem of axiomatic quantum field theory which follows from locality, Lorentz invariance, and unitarity [2]. Many tests of CPT-invariance consist in searching for decay process of neutral kaons. All known CP- and hypothetically possible CPT-violation effects in neutral kaon complex are described by solving the Schrödinger-like evolution equation [3–7] (we use $\hbar = c = 1$ units)

$$i\frac{\partial}{\partial t}|\psi;t\rangle_{\|} = H_{\|}|\psi;t\rangle_{\|} \tag{1}$$

for $|\psi;t\rangle_{\|}$ belonging to the subspace $\mathcal{H}_{\|} \subset \mathcal{H}$ (where $\mathcal{H}$ is the state space of the physical system under investigation), e.g., spanned by orthonormal neutral kaons states $|K_0\rangle$, $|\overline{K}_0\rangle$, and so on, (then states corresponding with the decay products belong to $\mathcal{H} \ominus \mathcal{H}_{\|} \stackrel{\text{def}}{=} \mathcal{H}_{\perp}$), and non-Hermitian effective Hamiltonian $H_{\|}$ obtained usually by means of Lee–Oehme–Yang (LOY) approach (within the Weisskopf–Wigner approximation (WW)) [3–5, 7]:

$$H_{\|} \equiv M - \frac{i}{2}\Gamma, \tag{2}$$

where $M = M^+$, $\Gamma = \Gamma^+$, are (2×2) matrices.

Solutions of equation (1) can be written in matrix form and such a matrix defines the evolution operator (which is usually nonunitary) $U_{\|}(t)$ acting in $\mathcal{H}_{\|}$:

$$|\psi;t\rangle_{\|} = U_{\|}(t)|\psi;t_0 = 0\rangle_{\|} \stackrel{\text{def}}{=} U_{\|}(t)|\psi\rangle_{\|}, \tag{3}$$

where,

$$|\psi\rangle_{\|} \equiv q_1|\mathbf{1}\rangle + q_2|\mathbf{2}\rangle, \tag{4}$$

[1] This paper is a shortened version of [1].

and $|\mathbf{1}\rangle$ stands for the vectors of the $|K_0\rangle$, $|B_0\rangle$ type and $|\mathbf{2}\rangle$ denotes antiparticles of the particle "1": $|\overline{K}_0\rangle$, $|\overline{B}_0\rangle$, $\langle \boldsymbol{j} \,|\, \boldsymbol{k}\rangle = \delta_{jk}$, $j,k = 1,2$.

In many papers it is assumed that the real parts, $\Re(\cdot)$, of the diagonal matrix elements of $H_{\|}$, $\Re(h_{jj}) \equiv M_{jj}$, $(j = 1,2)$, where

$$h_{jk} = \langle \boldsymbol{j} \,|\, H_{\|} | \boldsymbol{k}\rangle, \qquad (j,k = 1,2), \tag{5}$$

correspond to the masses of particle "1" and its antiparticle "2" respectively [3–7], (and such an interpretation of $\Re(h_{11})$ and $\Re(h_{22})$ will be used in this paper), whereas the imaginary parts, $\Im(\cdot)$, $\Im(h_{jj}) \equiv -\frac{1}{2}\Gamma_{jj}$, $(j = 1,2)$ interpreted as the decay widths of these particles [3–7]. Such an interpretation seems to be consistent with the recent and the early experimental data for neutral kaon and similar complexes [8].

Relations between matrix elements of $H_{\|}$ implied by CPT-transformation properties of the Hamiltonian H of the total system, containing neutral kaon complex as a subsystem, are crucial for designing CPT-invariance and CP-violation tests and for proper interpretation of their results. The aim of this talk is to examine the properties of the approximate and exact $H_{\|}$ generated by the CPT-symmetry of the total system under consideration and independent of the approximation used.

2 H_{LOY} and CPT-symmetry

Now, let us consider briefly some properties of the LOY model. Let H be total (self-adjoint) Hamiltonian, acting in $\mathcal{H}$ – then the total unitary evolution operator $U(t)$ fulfills the Schrödinger equation

$$i\frac{\partial}{\partial t}U(t)|\phi\rangle = HU(t)|\phi\rangle, \qquad U(0) = I, \tag{6}$$

where I is the unit operator in $\mathcal{H}$, $|\phi\rangle \equiv |\phi; t_0 = 0\rangle \in \mathcal{H}$ is the initial state of the system:

$$|\phi\rangle \equiv |\psi\rangle_{\|}. \tag{7}$$

In our case $U(t)|\phi\rangle \equiv |\phi; t\rangle$.

Let P denote the projection operator onto the subspace $\mathcal{H}_{\|}$: $P\mathcal{H} = \mathcal{H}_{\|}$, $P = P^2 = P^+$, then the subspace of decay products $\mathcal{H}_{\perp}$ equals $\mathcal{H}_{\perp} = (I - P)\mathcal{H} \stackrel{\mathrm{def}}{=} Q\mathcal{H}$, and $Q \equiv I - P$. For the case of neutral kaons or neutral B-mesons, etc., the projector P can be chosen as follows:

$$P \equiv |\mathbf{1}\rangle\langle\mathbf{1}| + |\mathbf{2}\rangle\langle\mathbf{2}|. \tag{8}$$

We assume that time-independent basis vectors $|K_0\rangle$ and $|\overline{K}_0\rangle$ are defined analogously to corresponding vectors used in LOY theory of time evolution in neutral kaon complex [3]. In the LOY approach it is assumed that vectors $|\mathbf{1}\rangle$, $|\mathbf{2}\rangle$ considered above are eigenstates of $H^{(0)}$ for 2-fold degenerate eigenvalue m_0:

$$H^{(0)}|\boldsymbol{j}\rangle = m_0|\boldsymbol{j}\rangle, \qquad j = 1,2, \tag{9}$$

where $H^{(0)}$ is a so called free Hamiltonian, $H^{(0)} \equiv H_{\mathrm{strong}} = H - H_W$, and H_W denotes weak and other interactions which are responsible for transitions between eigenvectors of $H^{(0)}$, i.e., for the decay process.

The condition guaranteeing the occurrence of transitions between subspaces $\mathcal{H}_{\|}$ and $\mathcal{H}_{\perp}$, i.e., a decay process of states in $\mathcal{H}_{\|}$, can be written as follows $[P, H_W] \neq 0$, that is

$$[P, H] \neq 0. \tag{10}$$

Usually, in LOY and related approaches, it is assumed that $\Theta H^{(0)}\Theta^{-1} = {H^{(0)}}^{+} \equiv H^{(0)}$, where Θ is the antiunitary operator, $\Theta \stackrel{\text{def}}{=} \mathcal{CPT}$. The subspace of neutral kaons $\mathcal{H}_{\|}$ is assumed to be invariant under Θ:

$$\Theta P \Theta^{-1} = P^{+} \equiv P. \tag{11}$$

In the kaon rest frame, the time evolution is governed by the Schrödinger equation (6), where the initial state of the system has the form (7), (4). Within assumptions (9), (10) the WW approach, which is the source of the LOY method, leads to the following formula for $\mathrm{H}_{\mathrm{LOY}}$ (e.g., see [3–5, 7]):

$$H_{\mathrm{LOY}} = m_0 P - \Sigma(m_0) \equiv PHP - \Sigma(m_0) = M_{\mathrm{LOY}} - \frac{i}{2}\Gamma_{\mathrm{LOY}}, \tag{12}$$

where it has been assumed that $\langle \mathbf{1} \,|\, H_W \,|\, \mathbf{2}\rangle = \langle \mathbf{1} \,|\, H_W \,|\, \mathbf{2}\rangle^{*} = 0$ (see [3–7]). Here

$$\Sigma(\epsilon) = PHQ \frac{1}{QHQ - \epsilon - i0}\, QHP. \tag{13}$$

The matrix elements h_{jk}^{LOY} of H_{LOY} are

$$h_{jk}^{\mathrm{LOY}} = H_{jk} - \Sigma_{jk}(m_0) = M_{jk}^{\mathrm{LOY}} - \frac{i}{2}\Gamma_{jk}^{\mathrm{LOY}} \qquad (j,k = 1,2), \tag{14}$$

where, in this case,

$$H_{jk} = \langle \boldsymbol{j} \,|\, H \,|\, \boldsymbol{k}\rangle \equiv \langle \boldsymbol{j} \,|\, (H^{(0)} + H_W) \,|\, \boldsymbol{k}\rangle \equiv m_0 \delta_{jk} + \langle \boldsymbol{j} \,|\, H_W \,|\, \boldsymbol{k}\rangle, \tag{15}$$

and $\Sigma_{jk}(\epsilon) = \langle \boldsymbol{j} \,|\, \Sigma(\epsilon) \,|\, \boldsymbol{k}\rangle$.

Now, if $\Theta H_W \Theta^{-1} = H_W^{+} \equiv H_W$, that is if

$$[\Theta, H] = 0, \tag{16}$$

then using, e.g., the following phase convention [4–7]

$$\Theta|\mathbf{1}\rangle \stackrel{\text{def}}{=} -|\mathbf{2}\rangle, \qquad \Theta|\mathbf{2}\rangle \stackrel{\text{def}}{=} -|\mathbf{1}\rangle, \tag{17}$$

and taking into account that $\langle \psi \,|\, \varphi\rangle = \langle \Theta\varphi \,|\, \Theta\psi\rangle$, one easily finds from (12)–(15) that

$$h_{11}^{\mathrm{LOY}^{\Theta}} - h_{22}^{\mathrm{LOY}^{\Theta}} = 0, \tag{18}$$

and thus

$$M_{11}^{\mathrm{LOY}} = M_{22}^{\mathrm{LOY}}, \tag{19}$$

(where $h_{jk}^{\mathrm{LOY}^{\Theta}}$ denotes the matrix elements of $H_{\mathrm{LOY}}^{\Theta}$ – of the LOY effective Hamiltonian when the relation (16) holds), in the CPT-invariant system. This is the standard result of the LOY approach and this is the picture which one meets in the literature [3–6, 8].

3 Beyond the LOY approximation

The more accurate approximate formulae for $H_{\|}(t)$ have been derived in [9] using the Krolikowski–Rzewuski equation for the projection of a state vector [10], which results from the Schrödinger

equation (6) for the total system under consideration, and, in the case of initial conditions of the type (7), takes the following form

$$\left(i\frac{\partial}{\partial t} - PHP\right) U_{\|}(t) = -i\int_0^\infty K(t-\tau)U_{\|}(\tau)d\tau, \tag{20}$$

where $U_{\|}(0) = P$, $K(t) = \Theta(t)PHQ\exp(-itQHQ)QHP$, and $\Theta(t) = \{1 \text{ for } t \geq 0, \;\; 0 \text{ for } t < 0\}$.

The integro-differential equation (20) can be replaced by the following differential one (see [9–13])

$$\left(i\frac{\partial}{\partial t} - PHP - V_{||}(t)\right) U_{\|}(t) = 0, \tag{21}$$

where $PHP + V_{||}(t) \stackrel{\text{def}}{=} H_{||}(t)$. Taking into account (20) and (21) or (1) one finds

$$V_{\|}(t)U_{\|}(t) = -i\int_0^\infty K(t-\tau)U_{\|}(\tau)d\tau. \tag{22}$$

Using this relation, one finds to the lowest nontrivial order that [9]

$$V_{\|}(t) \cong V_{\|}^{(1)}(t) \stackrel{\text{def}}{=} -i\int_0^\infty K(t-\tau)\exp\left[i(t-\tau)PHP\right]d\tau. \tag{23}$$

From (22) one can conclude that [9, 13]

$$H_{\|}(0) \equiv PHP, \qquad V_{\|}(0) = 0, \qquad V_{\|}(t\to 0) \simeq -itPHQHP. \tag{24}$$

In the case of conserved CPT and of H such that $H_{12} = H_{21} = 0$, from (23) one finds that $H_{||} = m_0 P - \Sigma(m_0) \equiv H_{\text{LOY}}$.

On the other hand, in the case

$$H_{12} = H_{21}^* \neq 0, \tag{25}$$

the form of $H_{||}$ is much more complicated. For example in the case of conserved CPT, formula (23) leads to matrix elements of $H_{||}$ [7, 14], which differ from (14). Indeed, in the case of preserved CPT-symmetry (16), one finds $H_{11} = H_{22}$, which implies that $H_{11} \equiv H_{22} \stackrel{\text{def}}{=} H_0$, and [9] $\Sigma_{11}(\varepsilon = \varepsilon^*) \equiv \Sigma_{22}(\varepsilon = \varepsilon^*) \stackrel{\text{def}}{=} \Sigma_0(\varepsilon = \varepsilon^*)$. Therefore matrix elements $v_{jk}^{\Theta} \equiv \langle \boldsymbol{j} \,|\, V_{||}^{\Theta} \,|\, \boldsymbol{k}\rangle$, $(j,k = 1,2)$ of operator $V_{\|}^{\Theta}$ (here $V_{\|}^{\Theta}$ denotes $V_{\|}$ when (16) occurs and $V_{||} \stackrel{\text{def}}{=} \lim\limits_{t\to\infty} V_{||}^{(1)}(t)$) take the following form

$$\begin{aligned}
v_{j1}^{\Theta} &= -\frac{1}{2}\Big\{\Sigma_{j1}(H_0 + |H_{12}|) + \Sigma_{j1}(H_0 - |H_{12}|) \\
&\quad + \frac{H_{21}}{|H_{12}|}\Sigma_{j2}(H_0 + |H_{12}|) - \frac{H_{21}}{|H_{12}|}\Sigma_{j2}(H_0 - |H_{12}|)\Big\}, \\
v_{j2}^{\Theta} &= -\frac{1}{2}\Big\{\Sigma_{j2}(H_0 + |H_{12}|) + \Sigma_{j2}(H_0 - |H_{12}|) \\
&\quad + \frac{H_{12}}{|H_{12}|}\Sigma_{j1}(H_0 + |H_{12}|) - \frac{H_{12}}{|H_{12}|}\Sigma_{j1}(H_0 - |H_{12}|)\Big\}.
\end{aligned} \tag{26}$$

Finally, assuming $|H_{12}| \ll |H_0|$, and using relation (5) and the expression (14), we obtain for the CPT–invariant system [15, 16]

$$h_{j1}^{\Theta} \simeq h_{j1}^{\text{LOY}} - H_{21}\frac{\partial\Sigma_{j2}(x)}{\partial x}\bigg|_{x=H_0}, \qquad h_{j2}^{\Theta} \simeq h_{j2}^{\text{LOY}} - H_{12}\frac{\partial\Sigma_{j1}(x)}{\partial x}\bigg|_{x=H_0}, \tag{27}$$

where $h_{jk} = H_{jk} + v_{jk}$ and $j = 1, 2$. From these formulae we conclude that, e.g., the difference between diagonal matrix elements of $H_{||}^{\Theta}$, which plays an important role in designing CPT-invariance tests for the neutral kaons system, equals

$$\Delta h \stackrel{\text{def}}{=} h_{11} - h_{22} \simeq H_{12} \frac{\partial \Sigma_{21}(x)}{\partial x}\bigg|_{x=H_0} - H_{21} \frac{\partial \Sigma_{12}(x)}{\partial x}\bigg|_{x=H_0} \neq 0, \tag{28}$$

which differs from the LOY results (18), (19).

4 CPT and the exact effective Hamiltonian

The aim of this Section is to show, that contrary to the LOY conclusion (18), diagonal matrix elements of the exact effective Hamiltonian $H_{||}$ cannot be equal when the total system under consideration is CPT invariant but CP noninvariant. This will be done by means of the method used in [17].

Universal properties of the (unstable) particle–antiparticle subsystem of the system described by the Hamiltonian H, for which the relation (16) holds, can be extracted from the matrix elements of the exact $U_{||}(t)$ appearing in (3). Such $U_{||}(t)$ has the following form

$$U_{||}(t) = PU(t)P, \tag{29}$$

where P is defined by the relation (8), and $U(t)$ is the total unitary evolution operator $U(t)$, which solves the Schrödinger equation (6). Operator $U_{||}(t)$ acts in the subspace of unstable states $\mathcal{H}_{||} \equiv P\mathcal{H}$. Of course, $U_{||}(t)$ has nontrivial form only if (10) holds, and only then transitions of states from $\mathcal{H}_{||}$ into $\mathcal{H}_{\perp}$ and vice versa, i.e., decay and regeneration processes, are allowed.

Using the matrix representation one finds

$$U_{||}(t) \equiv \begin{pmatrix} \boldsymbol{A}(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad \boldsymbol{A}(t) = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix}, \tag{30}$$

where $\mathbf{0}$ denotes the suitable zero submatrices and a submatrix $\boldsymbol{A}(t)$ is the (2×2) matrix acting in $\mathcal{H}_{||}$ and $A_{jk}(t) = \langle \boldsymbol{j} \,|\, U_{||}(t) \,|\, \boldsymbol{k} \rangle \equiv \langle \boldsymbol{j} \,|\, U(t) \,|\, \boldsymbol{k} \rangle$, $(j, k = 1, 2)$.

Now assuming (16) and using the phase convention (17), [3–5], one easily finds that [6,18,19]

$$A_{11}(t) = A_{22}(t). \tag{31}$$

Note that assumptions (16) and (17) give no relations between $A_{12}(t)$ and $A_{21}(t)$.

The important relation between amplitudes $A_{12}(t)$ and $A_{21}(t)$ follows from the famous Khalfin's Theorem [6, 18, 19]. This Theorem states that in the case of unstable states, if amplitudes $A_{12}(t)$ and $A_{21}(t)$ have the same time dependence

$$r(t) \stackrel{\text{def}}{=} \frac{A_{12}(t)}{A_{21}(t)} = \text{const} \equiv r, \tag{32}$$

then it must be $|r| = 1$.

For unstable particles the relation (31) means that decay laws $p_j(t) \stackrel{\text{def}}{=} |A_{jj}(t)|^2$, (where $j = 1, 2$), of the particle $|\mathbf{1}\rangle$ and its antiparticle $|\mathbf{2}\rangle$ are equal, $p_1(t) \equiv p_2(t)$. The consequence of this latter property is that the decay rates of the particle $|\mathbf{1}\rangle$ and its antiparticle $|\mathbf{2}\rangle$ must be equal too. From (31) it does not follow that the masses of the particle "1" and the antiparticle "2" should be equal.

More conclusions about the properties of the matrix elements of $H_{||}$ one can infer analyzing the following identity [20, 11–13]

$$H_{||} \equiv H_{||}(t) = i\frac{\partial U_{||}(t)}{\partial t}[U_{||}(t)]^{-1} \equiv i\frac{\partial \boldsymbol{A}(t)}{\partial t}[\boldsymbol{A}(t)]^{-1}. \tag{33}$$

Relation (33) must be fulfilled by the exact as well as by every approximate effective Hamiltonian governing the time evolution in every two-dimensional subspace $\mathcal{H}_{||}$ of states $\mathcal{H}$ [20, 11–13].

It is easy to find from (33) the general formulae for the diagonal matrix elements, $h_{jj}(t)$, of $H_{||}(t)$, in which we are interested. We have

$$h_{11}(t) = \frac{i}{\det \boldsymbol{A}(t)}\left(\frac{\partial A_{11}(t)}{\partial t}A_{22}(t) - \frac{\partial A_{12}(t)}{\partial t}A_{21}(t)\right), \tag{34}$$

$$h_{22}(t) = \frac{i}{\det \boldsymbol{A}(t)}\left(-\frac{\partial A_{21}(t)}{\partial t}A_{12}(t) + \frac{\partial A_{22}(t)}{\partial t}A_{11}(t)\right). \tag{35}$$

Now, assuming (16) and using the consequence (31) of this assumption, one finds

$$h_{11}(t) - h_{22}(t) = -i\frac{A_{12}(t)A_{21}(t)}{\det \boldsymbol{A}(t)}\frac{\partial}{\partial t}\ln\left(\frac{A_{12}(t)}{A_{21}(t)}\right). \tag{36}$$

This result means that in the considered case for $t > 0$ the following Theorem holds:

$$h_{11}(t) - h_{22}(t) = 0 \;\Leftrightarrow\; \frac{A_{12}(t)}{A_{21}(t)} = \text{const}, \qquad (t > 0). \tag{37}$$

Thus for $t > 0$ the problem under studies is reduced to the Khalfin's Theorem (see the relation (32)).

From (34) and (35) it is easy to see that at $t = 0$

$$h_{jj}(0) = \langle \boldsymbol{j} \,|\, H \,|\, \boldsymbol{j}\rangle, \qquad (j = 1, 2), \tag{38}$$

which means that in a CPT invariant system (16) in the case of pairs of unstable particles, for which transformations of type (17) hold, the unstable particles "1" and "2" are created at $t = t_0 \equiv 0$ as particles with equal masses: $M_{11}(0) = M_{22}(0) \equiv \langle \mathbf{1} \,|\, H \,|\, \mathbf{1}\rangle$.

Now let us go on to analyze conclusions following from the Khalfin's Theorem. CP noninvariance requires that $|r| \neq 1$ [6, 19] (see also [3–5, 8]). This means that in such a case it must be $r = r(t) \neq \text{const}$. So, if in the system considered the property (16) holds but $[\mathcal{CP}, H] \neq 0$, and the unstable states "1" and "2" are connected by a relation of type (17), then at $t > 0$ it must be $(h_{11}(t) - h_{22}(t)) \neq 0$ in this system.

Let us focus our attention on $\Re\,(h_{11}(t) - h_{22}(t))$. Following the method used in [12] and using assumption (11) and the identity (33), after some algebra, one finds [21]

$$[\Theta, H_{\|}(t)] = \mathcal{A}(t) + \mathcal{B}(t), \tag{39}$$

where:

$$\mathcal{A}(t) = P[\Theta, H]U(t)P(U_{\|}(t))^{-1}, \qquad \mathcal{B}(t) = \big\{PH - H_{\|}(t)P\big\}[\Theta, U(t)]P(U_{\|}(t))^{-1}. \tag{40}$$

From (39) we find

$$\Theta H_{\|}(t)\Theta^{-1} - H_{\|}(t) \equiv (\mathcal{A}(t) + \mathcal{B}(t))\Theta^{-1}. \tag{41}$$

Using this latter relation one finds that

$$h_{11}(t)^* - h_{22}(t) = \langle \mathbf{2} \,|\, (\mathcal{A}(t) + \mathcal{B}(t))\Theta^{-1} \,|\, \mathbf{2}\rangle. \tag{42}$$

Adding expression (42) to its complex conjugate one gets

$$\Re(h_{11}(t) - h_{22}(t)) = \Re\langle \mathbf{2} \,|\, (\mathcal{A}(t) + \mathcal{B}(t))\Theta^{-1} \,|\, \mathbf{2}\rangle. \tag{43}$$

We are considering the case of unstable states, i.e., states $|\mathbf{1}\rangle$, $|\mathbf{2}\rangle$, which lead to such projection operator P (8) that condition (10) holds. In this case we have $\Theta U(t) = U^{+}(t)\Theta$, which gives $\Theta U_{\|}(t) = U_{\|}^{+}(t)\Theta$, $\Theta U_{\|}^{-1}(t) = (U_{\|}^{+}(t))^{-1}\Theta$, and $[\Theta, U(t)] = -2i(\Im U(t))\Theta$. This relation leads to the following result in the case of the conserved CPT-symmetry

$$\mathcal{B}(t) = -2iP\big\{H - H_{\|}(t)P\big\}(\operatorname{Im} U(t))P(U_{\|}^{+}(t))^{-1}\Theta. \tag{44}$$

Formula (44) allows us to conclude that $\langle \mathbf{2} \,|\, \mathcal{B}(0)\Theta^{-1} \,|\, \mathbf{2}\rangle = 0$ and $\Re(\langle \mathbf{2} \,|\, \mathcal{B}(t>0)\Theta^{-1} \,|\, \mathbf{2}\rangle) \neq 0$, if condition (16) holds. This means that in this case it must be $\Re(h_{11}(t)) \neq \Re(h_{22}(t))$ for $t > 0$. So, there is no possibility for $\Re(h_{11})$ to equal $\Re(h_{22})$ for $t > 0$ in the considered case of P fulfilling the condition (10) (i.e., for unstable states) when CPT-symmetry is conserved: It must be $\Re(h_{11}) \neq \Re(h_{22})$.

Assuming the LOY interpretation of $\Re(h_{jj}(t))$, $(j = 1, 2)$, one can conclude from the Khalfin's Theorem and from the properties (37), (43), (44) that if $A_{12}(t)$, $A_{21}(t) \neq 0$ for $t > 0$ and if the total system considered is CPT-invariant, but CP-noninvariant, then $M_{11}(t) \neq M_{22}(t)$ for $t > 0$, that is, that contrary to the case of stable particles (the bound states), the masses of the simultaneously created unstable particle "1" and its antiparticle "2", which are connected by the relation (17), need not be equal for $t > t_0 = 0$. Of course, such a conclusion contradicts the standard LOY result (18), (19). However, one should remember that the LOY description of neutral K mesons and similar complexes is only an approximate one, and that the LOY approximation is not perfect. On the other hand the relation (37) and the Khalfin's Theorem follow from the basic principles of the quantum theory and are rigorous. Consequently, their implications should also be considered as rigorous.

5 Final remarks

Note that properties of the more accurate approximation described in Section 3 are consistent with the general properties and conclusions obtained in Section 4 for the exact effective Hamiltonian – compare (24) and (38) and relations (37) with (28). From the result (28) it follows that in the case of the approximate $H_{||}$, $\Delta h = 0$ can be achieved only if $H_{12} = H_{21} = 0$. This means that if the first order $|\Delta S| = 2$ interactions are forbidden in the $K_0, \overline{K}_0$ complex then predictions following from the use of the mentioned more accurate approximation and from the LOY theory should lead to the the same masses for K_0 and for $\overline{K}_0$. This does not contradict the results of Section 3 derived for the exact $H_{||}$: the mass difference is very, very small and should arise at higher orders of the more accurate approximation. On the other hand from (28) it follows that $\Delta h \neq 0$ if and only if $H_{12} \neq 0$. This means that if measurable deviations from the LOY predictions concerning the masses of, e.g. K_0, $\overline{K}_0$ mesons are ever detected, then the most plausible interpretation of this result will be the existence of first order $|\Delta S| = 2$ interactions in the system considered.

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The Pseudoinstanton – a Lorentzian Analogue of the Instanton

Dmitri VASSILIEV

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
E-mail: *D.Vassiliev@bath.ac.uk*

In abstract Yang–Mills theory the standard instanton construction relies on the Hodge star having real eigenvalues which makes it inapplicable in the Lorentzian case. We show that for the affine connection (i.e. connection on vectors) an instanton-type construction can be carried out in the Lorentzian setting. Namely, the Lorentzian analogue of an instanton is a metric compatible connection whose curvature is irreducible and simple ("pseudoinstanton"). We prove that a pseudoinstanton is a solution of the Yang–Mills equation for the affine connection. In fact, we prove a much stronger result: a pseudoinstanton is a stationary point of any Lorentz-invariant quadratic action with respect to the independent variation of the metric and the connection. We present examples of pseudoinstantons and discuss them within the context of non-Riemannian theories of gravity.

1 Mathematical model

We consider space-time to be a connected real 4-manifold M equipped with a Lorentzian metric g and an affine connection Γ. The 10 independent components of the (symmetric) metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^{\lambda}{}_{\mu\nu}$ are the unknowns of our theory. This approach is known as metric-affine gravity. Its origins lie in the works of authors such as É. Cartan, A.S. Eddington, A. Einstein, T. Levi-Civita, E. Schrödinger and H. Weyl. A review of the more recent work in this area can be found in [1].

We define our action as

$$S := \int q(R) \tag{1}$$

where q is an $\mathrm{O}(1,3)$-invariant quadratic form on curvature R. Independent variation of the metric g and the connection Γ produces Euler–Lagrange equations which we will write symbolically as

$$\partial S/\partial g = 0, \tag{2}$$

$$\partial S/\partial \Gamma = 0. \tag{3}$$

Our objective is the study of the combined system of field equations (2), (3). This is a system of $10+64$ real nonlinear partial differential equations with $10+64$ real unknowns.

Our motivation comes from Yang–Mills theory. The Yang–Mills action for the affine connection is a special case of (1) with

$$q(R) = q_{\mathrm{YM}}(R) := R^{\kappa}{}_{\lambda\mu\nu}\, R^{\lambda}{}_{\kappa}{}^{\mu\nu}. \tag{4}$$

With this choice of q equation (3) is the Yang–Mills equation for the affine connection. There is a substantial bibliography devoted to the study of the system (2), (3) in the special case (4); see, for example, references in [2].

The idea of modelling gravity by means of a quadratic action goes back to H. Weyl, see end of his paper [3]. Weyl also pointed out that such an action should contain all possible invariant

quadratic combinations of curvature, say, the square of Ricci curvature, the square of scalar curvature, etc. It turns out (see Appendix B.4 in [1]) that in the metric-affine setting curvature has 11 irreducible pieces. There are 16 ways of squaring these irreducible pieces to a scalar. The reason why the number of different invariant quadratic combinations is greater than the number of pieces is that some of the irreducible subspaces of curvature are isomorphic. Namely, all three 6-dimensional subspaces are isomorphic, and there are also two pairs of isomorphic 9-dimensional subspaces. The explicit formula for a general $\mathrm{O}(1,3)$-invariant quadratic form $q(R)$ with 16 coupling constants is given in Appendix B of [4].

The paper has the following structure. In Section 3 we state and prove our main result, Theorem 1. In Sections 4 and 5 we use this theorem to construct solutions to our field equations (2), (3). Finally, in Section 6 we discuss the possible physical interpretation of our solutions.

2 Notation

Our notation follows [2, 5]. In particular, we denote local coordinates by x^μ, $\mu = 0, 1, 2, 3$, and write $\partial_\mu := \partial/\partial x^\mu$. We define the covariant derivative of a vector function as $\nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda{}_{\mu\nu} v^\nu$, torsion as $T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$, curvature as $R^\kappa{}_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\kappa{}_{\mu\eta}\Gamma^\eta{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\eta}\Gamma^\eta{}_{\mu\lambda}$, and Ricci curvature as $Ric_{\lambda\nu} := R^\kappa{}_{\lambda\kappa\nu}$. Given a scalar function $f : M \to \mathbb{R}$ we write for brevity $\int f := \int_M f \sqrt{|\det g|}\, dx^0 dx^1 dx^2 dx^3$ where $\det g := \det(g_{\mu\nu})$. The Christoffel symbol is $\left\{ {\lambda \atop \mu\nu} \right\} := \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$. We define the action of the Hodge star on a rank q antisymmetric tensor as $(*Q)_{\mu_{q+1}\dots\mu_4} := (q!)^{-1} \sqrt{|\det g|}\, Q^{\mu_1\dots\mu_q} \varepsilon_{\mu_1\dots\mu_4}$ where ε is the totally antisymmetric quantity.

3 Main result

The following definition plays a crucial role in our construction.

Definition 1. We call a space-time $\{M, g, \Gamma\}$ a *pseudoinstanton* if the connection is metric compatible and curvature is irreducible and simple.

Here irreducibility of curvature means that all irreducible pieces but one are identically zero. Simplicity means that the given irreducible subspace is not isomorphic to any other irreducible subspace. Metric compatibility means, as usual, that $\nabla g \equiv 0$.

It is easy to see that there are only three possible types of pseudoinstantons:

- *scalar* pseudoinstanton (all pieces of curvature apart from the scalar piece are identically zero),
- *pseudoscalar* pseudoinstanton (all pieces of curvature apart from the pseudoscalar piece are identically zero), and
- *Weyl* pseudoinstanton (all pieces of curvature apart from the Weyl piece are identically zero).

Here scalar, pseudoscalar, and Weyl curvatures are defined by the conditions

$$R_{\kappa\lambda\mu\nu} \sim g_{\kappa\mu} g_{\lambda\nu} - g_{\lambda\mu} g_{\kappa\nu}, \tag{5}$$

$$R_{\kappa\lambda\mu\nu} \sim \varepsilon_{\kappa\lambda\mu\nu}, \tag{6}$$

$$R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda}, \qquad Ric = 0, \qquad \varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0 \tag{7}$$

respectively, with $\sim$ standing for proportionality.

Our main result is

Theorem 1. *A pseudoinstanton is a solution of the field equations* (2), (3).

Proof. Let R_{pseudo} be the irreducible piece of curvature corresponding to the type of our pseudoinstanton; that is, R_{pseudo} denotes scalar, pseudoscalar, or Weyl curvature depending on whether we are proving Theorem 1 for a scalar, pseudoscalar, or Weyl pseudoinstanton. Then for *any* curvature R we have $q(R) = q(R_{\text{pseudo}}) + q(R - R_{\text{pseudo}})$. Here we used the fact that the piece R_{pseudo} is simple: if not, then we would have cross-over terms of the type $R_{\text{pseudo}} \times (R - R_{\text{pseudo}})$.

When we start our variation from a space-time with $R - R_{\text{pseudo}} \equiv 0$ the resulting variation of $\int q(R - R_{\text{pseudo}})$ is zero. Thus, the proof of Theorem 1 reduces to proving that our pseudoinstanton is a stationary point of the action $\int q(R_{\text{pseudo}})$. But examination of the explicit formula for $q(R)$ given in Appendix B of [4] shows that $q(R_{\text{pseudo}}) = \text{const}(R_{\text{pseudo}}, R_{\text{pseudo}})_{\text{YM}}$ where $(R, Q)_{\text{YM}} := R^{\kappa}{}_{\lambda\mu\nu} Q^{\lambda}{}_{\kappa}{}^{\mu\nu}$ is the Yang–Mills inner product on curvatures. Thus, the action $\int q(R_{\text{pseudo}})$ is of the type studied in [2] and the result follows from Theorem 2.1 of that paper. ■

4 Riemannian pseudoinstantons

Definition 2. We call a space-time $\{M, g, \Gamma\}$ *Riemannian* if the connection is Levi-Civita (i.e. $\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\}$), and *non-Riemannian* otherwise.

Note that the word "Riemannian" has a different meaning in mathematics and theoretical physics. In mathematical literature the connection is usually Levi-Civita by default and "Riemannian" indicates that the metric is definite, whereas in theoretical physics literature the metric is usually Lorentzian by default (as it is in our paper) and "Riemannian" indicates that the connection is Levi-Civita. In Definition 2 we adopt the theoretical physics terminology.

In the Riemannian case the first and third conditions (7) are automatically fulfilled, so a Riemannian space-time is a Weyl pseudoinstanton if and only if

$$Ric = 0. \tag{8}$$

Therefore, according to Theorem 1, Riemannian space-times satisfying the vacuum Einstein equation (8) are solutions to our field equations (2), (3).

The above argument demonstrates both the power and the limitations of the pseudoinstanton technique. This technique allowed us to obtain an important class of solutions without having to write down explicitly the field equations. On the other hand, it did not give us all the Riemannian solutions: it is known [4] that Einstein spaces with arbitrary cosmological constant are solutions as are pp-spaces with parallel Ricci curvature, but such space-times are not necessarily pseudoinstantons.

5 Non-Riemannian pseudoinstanton

We know only one non-Riemannian solution to our field equations (2), (3). It is constructed as follows.

Let us define Minkowski space $\mathbb{M}^4$ as a real 4-manifold equipped with global coordinates (x^0, x^1, x^2, x^3) and metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. Working in $\mathbb{M}^4$, let us consider a complex-valued vector field

$$A(x) = a\, \mathrm{e}^{-\mathrm{i} l \cdot x} \tag{9}$$

which is a plane wave solution of the polarized Maxwell equation

$$*dA = \pm \mathrm{i} dA. \tag{10}$$

Define torsion

$$T = \frac{1}{2}\operatorname{Re}(A \otimes dA) = -\frac{1}{2}\operatorname{Re}\left(\mathrm{i}\, a \otimes (l \wedge a)\, \mathrm{e}^{-2\mathrm{i}l\cdot x}\right) \tag{11}$$

and let Γ be the corresponding metric compatible connection. Straightforward calculations give the following explicit formula for curvature:

$$R = \operatorname{Re}(dA \otimes dA) = -\operatorname{Re}\left((l \wedge a) \otimes (l \wedge a)\, \mathrm{e}^{-2\mathrm{i}l\cdot x}\right). \tag{12}$$

It is easy to see that formulae (9), (10), (12) imply (7). Therefore, the space-time $\{\mathbb{M}^4, \Gamma\}$ is a Weyl pseudoinstanton, and hence, by Theorem 1, a solution of our field equations (2), (3).

For the Yang–Mills case (4) the "torsion wave" solution described above was first obtained by Singh and Griffiths: see last paragraph of Section 5 in [6] and put $k = 0$, $N = \text{const}\ \mathrm{e}^{-2\mathrm{i}l\cdot x}$. Our contribution is the observation that this torsion wave remains a solution for a general quadratic action (1) and that this fact can be established without having to write down explicitly the field equations.

6 Discussion

Riemannian pseudoinstantons (see (8)) clearly model vacuum. So below we examine the non-Riemannian pseudoinstanton from Section 5 with the aim of assigning a physical meaning.

We first list some remarkable properties of our non-Riemannian pseudoinstanton.

The curvature of our non-Riemannian pseudoinstanton (see (12)) satisfies the conditions

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\nu\mu} = R_{\mu\nu\kappa\lambda}, \qquad \varepsilon^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu} = 0.$$

These are the standard symmetries of curvature in Riemannian geometry. Thus, an observer of our non-Riemannian pseudoinstanton might think that they live in a Riemannian world.

The torsion of our non-Riemannian pseudoinstanton (see (11)) satisfies the conditions

$$T^{\lambda}{}_{\lambda\nu} = 0, \qquad T^{\kappa\lambda\mu}\varepsilon_{\kappa\lambda\mu\nu} = 0.$$

This means that the torsion is irreducible (purely tensor), see Appendix B in [2].

The Ricci curvature of our non-Riemannian pseudoinstanton is identically zero, which may be interpreted classically as zero mass. Also, the wave vector $2l$ is light-like, which may be interpreted quantum mechanically as zero mass.

Further on we assume that $l \neq 0$ and $a \notin \operatorname{span} l$, which are the necessary and sufficient conditions for non-flatness.

It is easy to check (see Section 10 in [4] for details) that our non-Riemannian pseudoinstanton has holonomy

$$B^2 := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \quad b \in \mathbb{C} \right\}. \tag{13}$$

Here we use the standard identification of the proper orthochroneous Lorentz group with $\mathrm{SL}(2,\mathbb{C})$, and our notation for subgroups follows that of Section 10.122 of [7]. Note that the group (13) is, up to conjugation, the unique nontrivial Abelian Lie subgroup of $\mathrm{SL}(2,\mathbb{C})$; in this statement "nontrivial" is understood as "not 1-dimensional and not a product of 1-dimensional subgroups", with dimension understood as real dimension. Note also that the (restricted) holonomy group of a (metric compatible) space-time is a subgroup of the group (13) if and only if the space-time admits a nonvanishing parallel rank 1 spinor field.

Our non-Riemannian pseudoinstanton is determined by a pair of orthogonal isotropic vectors, namely, a real vector l and a complex vector a. (We call a vector v isotropic if $v_\mu v^\mu = 0$.) By choosing different l and a we obtain a set of non-Riemannian pseudoinstantons. Any two pseudoinstantons from this set can be obtained from one another by a rescaling and a Lorentz transformation. This, however, ceases being true if we restrict ourselves to proper orthochroneous Lorentz transformations: with respect to such transformations our set of non-Riemannian pseudoinstantons has two disjoint connected components. In other words, the construction from Section 5 gives us two essentially different solutions. It is tempting to view these two essentially different solutions as a particle and antiparticle.

Assuming that our non-Riemannian pseudoinstanton is indeed a very basic model for some elementary particle, we nominate three particles as possible candidates: the photon, the graviton, and the neutrino. Our (highly speculative) arguments in favour of each of these three interpretations go as follows.

The photon interpretation is based on the observation that our non-Riemannian pseudoinstanton is constructed out of a vector field A which is a solution of Maxwell's equation $\delta dA = 0$ (this is a consequence of (10)). However, an argument against the photon interpretation is that at a quantum mechanical level the electromagnetic field should be associated with a U(1)-connection, and the presence of this structure group is not evident in our construction.

The arguments in favour of the graviton interpretation are as follows. Torsion is not an accepted physical observable but curvature is, so we base our interpretation on the analysis of the curvature of our non-Riemannian pseudoinstanton. Examination of the explicit formula (12) indicates that it is more convenient to deal with the complexified curvature $dA \otimes dA$; note also that complexification is in line with the traditions of quantum mechanics. Our complex curvature is polarized, ${}^*(dA \otimes dA) = (dA \otimes dA)^* = \pm \mathrm{i}(dA \otimes dA)$, and purely Weyl, hence it is equivalent to a (symmetric) rank 4 spinor ζ; see subsection 1.2.3 in [8] or Appendix C in [2] for details. A rank 4 spinor corresponds to a spin 2 particle, and one naturally thinks of the graviton.

Finally, the neutrino interpretation is based on the observation that our rank 4 spinor (see previous paragraph) has additional algebraic structure: it is the 4th tensor power of a rank 1 spinor, $\zeta = \xi \otimes \xi \otimes \xi \otimes \xi$. Direct calculations [5,9] show that the rank 1 spinor field ξ satisfies Weyl's equation, which is the accepted mathematical model for the neutrino. An attractive feature of the neutrino interpretation of our non-Riemannian pseudoinstanton is that it explains [5,9] why neutrinos are always left-handed whereas antineutrinos are always right-handed.

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Calculation of $\mathcal{C}$ Operator in $\mathcal{PT}$-Symmetric Quantum Mechanics

Qinghai WANG

Department of Physics, Washington University, St. Louis, MO 63130, USA
E-mail: *qwang@hbar.wustl.edu*

If a Hamiltonian H has an unbroken $\mathcal{PT}$ symmetry, then it also possesses a hidden symmetry represented by the linear operator $\mathcal{C}$. The operator $\mathcal{C}$ commutes with both H and $\mathcal{PT}$. The inner product with respect to $\mathcal{CPT}$ is associated with a positive norm and the quantum theory built on the associated Hilbert space is unitary. In this paper it is shown how to construct the operator $\mathcal{C}$ for the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + i\epsilon x^3$ using perturbative techniques. It is also shown how to construct the operator $\mathcal{C}$ for $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 - \epsilon x^4$ using nonperturbative methods [1].

1 Introduction and Background

It was observed in 1998 [2] that with properly defined boundary conditions the Sturm–Liouville differential equation eigenvalue problem associated with the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian

$$H = p^2 + x^2(ix)^\nu \qquad (\nu > 0) \tag{1}$$

exhibits a spectrum that is *real and positive*. By $\mathcal{PT}$ symmetry we mean the following: The linear parity operator $\mathcal{P}$ performs spatial reflection and thus reverses the sign of the momentum and position operators: $\mathcal{P}p\mathcal{P}^{-1} = -p$ and $\mathcal{P}x\mathcal{P}^{-1} = -x$. The antilinear time-reversal operator $\mathcal{T}$ reverses the sign of the momentum operator and performs complex conjugation: $\mathcal{T}p\mathcal{T}^{-1} = -p$, $\mathcal{T}x\mathcal{T}^{-1} = x$, and $\mathcal{T}i\mathcal{T}^{-1} = -i$. The non-Hermitian Hamiltonian H in (1) is not symmetric under $\mathcal{P}$ or $\mathcal{T}$ separately, but it *is* invariant under their combined operation; such Hamiltonians are said to possess *space-time reflection symmetry* ($\mathcal{PT}$ symmetry). We say that the $\mathcal{PT}$ symmetry of a Hamiltonian H is *not spontaneously broken* if the eigenfunctions of H are simultaneously eigenfunctions of the $\mathcal{PT}$ operator.

In a recent letter it was shown that any Hamiltonian that possesses an unbroken $\mathcal{PT}$ symmetry also has a hidden symmetry [3]. This new symmetry is represented by the linear operator $\mathcal{C}$, which commutes with both the Hamiltonian H and the $\mathcal{PT}$ operator. In terms of $\mathcal{C}$ one can construct an inner product whose associated norm is positive definite. Observables exhibit $\mathcal{CPT}$ symmetry and the dynamics is governed by unitary time evolution. Thus, $\mathcal{PT}$-symmetric Hamiltonians give rise to new classes of fully consistent complex quantum theories.

The purpose of the present paper is to present an explicit calculation of $\mathcal{C}$ for two nontrivial Hamiltonians. First, we consider the case of the $\mathcal{PT}$-symmetric Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 + i\epsilon x^3, \tag{2}$$

for which we give a perturbative calculation of the operator $\mathcal{C}$ correct to *third* order in powers of ϵ. Second, we calculate $\mathcal{C}$ for the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}x^2 - \epsilon x^4, \tag{3}$$

for which ordinary perturbative methods are ineffective and nonperturbative methods must be used. The organization of this paper is straightforward. In Section 2 we review the formal construction, first presented in Ref. [3], of the $\mathcal{C}$ operator. In Section 3 we calculate $\mathcal{C}$ for the Hamiltonian in (2) and in Section 4 we calculate $\mathcal{C}$ for the Hamiltonian in (3).

2 Formal derivation of the $\mathcal{C}$ operator

In this section we present a formal discussion of $\mathcal{PT}$-symmetric Hamiltonians and we show how to construct the $\mathcal{C}$ operator. In general, for any $\mathcal{PT}$-symmetric Hamiltonian H we must begin by solving the Sturm–Liouville differential equation eigenvalue problem associated with H:

$$H\phi_n(x) = E_n\phi_n(x) \qquad (n = 0,\, 1,\, 2,\, 3,\, \ldots). \tag{4}$$

For Hamiltonians like those in (1)–(3) the differential equation (4) must be imposed on an infinite contour in the complex-x plane. For large $|x|$ the contour lies in wedges that are placed symmetrically with respect to the imaginary-x axis. These wedges are described in Ref. [2]. The boundary conditions on the eigenfunctions are that $\phi(x) \to 0$ exponentially rapidly as $|x| \to \infty$ on the contour. For H in (2) the contour may be taken to be the real-x axis, but for H in (3) the contour lies in the two wedges $-\pi/3 < \arg x < 0$ and $-\pi < \arg x < -2\pi/3$.

For all n, the eigenfunctions $\phi_n(x)$ are simultaneously eigenstates of the $\mathcal{PT}$ operator. We can choose the phase of $\phi_n(x)$ such that the eigenvalue of $(\mathcal{PT})$ is unity:

$$\mathcal{PT}\phi_n(x) = \phi_n(x). \tag{5}$$

Next, we observe that there is an inner product, called the $\mathcal{PT}$ inner product, with respect to which the eigenfunctions $\phi_n(x)$ for two different values of n are orthogonal. For the two functions $f(x)$ and $g(x)$ the $\mathcal{PT}$ inner product (f, g) is defined by

$$(f, g) \equiv \int_{\mathrm{C}} dx\ [\mathcal{PT}f(x)]\, g(x), \tag{6}$$

where $\mathcal{PT}f(x) = [f(-x^*)]^*$ and the contour C lies in the wedges described above. For this inner product the associated norm (f, f) is independent of the overall phase of $f(x)$ and is conserved in time.

We then normalize the eigenfunctions so that $|(\phi_n, \phi_n)| = 1$ and we discover the apparent problem with using a non-Hermitian Hamiltonian. While the eigenfunctions are orthogonal, the $\mathcal{PT}$ norm is not positive definite:

$$(\phi_m, \phi_n) = (-1)^n\delta_{m,n} \qquad (m,\, n = 0,\, 1,\, 2,\, 3,\, \ldots). \tag{7}$$

Despite the fact that this norm is not positive definite, the eigenfunctions are complete. For real x and y the statement of completeness in coordinate space is[1]

$$\sum_n (-1)^n\phi_n(x)\phi_n(y) = \delta(x - y). \tag{8}$$

This is a nontrivial result that has been verified numerically to extremely high accuracy [4].

[1]It is important to remark here that the argument of the Dirac delta function in (8) must be *real* because the delta function is only defined for real argument. This may seem to be in conflict with the earlier remark in this section that the Schrödinger equation (4) must be solved along a contour that lies in wedges in the complex-x plane. To resolve this apparent conflict we specify the contour as follows. We demand that the contour lie on the real axis until it passes the points x and y. Only then may it veer off into the complex-x plane and enter the wedges. This choice of contour is allowed because the wedge conditions are *asymptotic* conditions. The positions of the wedges are determined by the boundary conditions.

We construct the linear operator $\mathcal{C}$ that expresses the hidden symmetry of the Hamiltonian H. The position-space representation of $\mathcal{C}$ is

$$\mathcal{C}(x,y)=\sum_n \phi_n(x)\phi_n(y). \tag{9}$$

The properties of the operator $\mathcal{C}$ are easy to verify using (7). First, the square of $\mathcal{C}$ is unity:

$$\int dy\,\mathcal{C}(x,y)\mathcal{C}(y,z)=\delta(x-z). \tag{10}$$

Second, the eigenfunctions $\phi_n(x)$ of the Hamiltonian H are also eigenfunctions of $\mathcal{C}$ and the corresponding eigenvalues are $(-1)^n$:

$$\int dy\,\mathcal{C}(x,y)\phi_n(y)=(-1)^n\phi_n(x). \tag{11}$$

Third, the operator $\mathcal{C}$ commutes with both the Hamiltonian H and the operator $\mathcal{PT}$. Note that while the operators $\mathcal{P}$ and $\mathcal{C}$ are unequal, both $\mathcal{P}$ and $\mathcal{C}$ are square roots of the unity operator $\delta(x-y)$. Last, the operators $\mathcal{P}$ and $\mathcal{C}$ do not commute. Indeed, $\mathcal{CP}=(\mathcal{PC})^*$.

The operator $\mathcal{C}$ does not exist as a distinct entity in conventional Hermitian quantum mechanics. Indeed, we will see that as the parameter ϵ in (2) and (3) tends to zero the operator $\mathcal{C}$ becomes identical to $\mathcal{P}$.

We can now define an inner product $\langle f|g\rangle$ whose associated norm is positive:

$$\langle f|g\rangle\equiv\int dx\,[\mathcal{CPT}f(x)]g(x). \tag{12}$$

The $\mathcal{CPT}$ norm associated with this inner product is positive because $\mathcal{C}$ contributes -1 when it acts on states with negative $\mathcal{PT}$ norm.

3 Perturbative calculation of $\mathcal{C}$ in a cubic theory

In this section we use perturbative methods to calculate the operator $\mathcal{C}(x,y)$ for the Hamiltonian $H=\frac{1}{2}p^2+\frac{1}{2}x^2+i\epsilon x^3$. We perform the calculations to third order in perturbation theory. We begin by solving the Schrödinger equation

$$-\frac{1}{2}\phi_n''(x)+\frac{1}{2}x^2\phi_n(x)+i\epsilon x^3\phi_n(x)=E_n\phi_n(x) \tag{13}$$

as a series in powers of ϵ.

The perturbative solution to this equation has the form

$$\phi_n(x)=\frac{i^n a_n}{\pi^{1/4}2^{n/2}\sqrt{n!}}e^{-\frac{1}{2}x^2}\left[H_n(x)-iP_n(x)\epsilon-Q_n(x)\epsilon^2+iR_n(x)\epsilon^3\right], \tag{14}$$

where $H_n(x)$ is the nth Hermite polynomial and $P_n(x)$, $Q_n(x)$, and $R_n(x)$ are polynomials in x of degree $n+3$, $n+6$, and $n+9$, respectively. These polynomials can be expressed as linear combinations of Hermite polynomials [1].

The energy E_n to order ϵ^3 is

$$E_n=n+\frac{1}{2}+\frac{1}{8}\left(30n^2+30n+11\right)\epsilon^2+\mathrm{O}\left(\epsilon^4\right). \tag{15}$$

The expression for $\phi_n(x)$ must be $\mathcal{PT}$-normalized according to (7) so that its square integral is $(-1)^n$:

$$\int_{-\infty}^{\infty}dx\,\left[\phi_n(x)\right]^2=(-1)^n+\mathrm{O}\left(\epsilon^4\right). \tag{16}$$

This determines the value of a_n in (14):

$$a_n = 1 + \frac{1}{144}(2n+1)\left(82n^2+82n+87\right)\epsilon^2 + \mathrm{O}\left(\epsilon^4\right). \tag{17}$$

We calculate the operator $\mathcal{C}(x,y)$ by substituting the wave functions $\phi_n(x)$ in (14) into (9). We then use the completeness relation for Hermite polynomials to evaluate the sum.

To third order in ϵ the result is

$$\begin{aligned}\mathcal{C}(x,y) = \Bigg\{1 - i\epsilon\left(\frac{4}{3}\frac{\partial^3}{\partial x^3} + 2xy\frac{\partial}{\partial x}\right) - \epsilon^2\left[\frac{8}{9}\frac{\partial^6}{\partial x^6} + \frac{8}{3}xy\frac{\partial^4}{\partial x^4} + \left(2x^2y^2 - 12\right)\frac{\partial^2}{\partial x^2}\right] \\ + i\epsilon^3\left[\frac{32}{81}\frac{\partial^9}{\partial x^9} + \frac{16}{9}xy\frac{\partial^7}{\partial x^7} + \left(\frac{8}{3}x^2y^2 - \frac{176}{5}\right)\frac{\partial^5}{\partial x^5}\right. \\ \left. + \left(\frac{4}{3}x^3y^3 - 48xy\right)\frac{\partial^3}{\partial x^3} + \left(-8x^2y^2 + 64\right)\frac{\partial}{\partial x}\right] + \mathrm{O}\left(\epsilon^4\right)\Bigg\}\delta(x+y).\end{aligned} \tag{18}$$

Hence, the coordinate-space representation of the operator $\mathcal{C}(x,y)$ is expressed as a derivative of a Dirac delta function. From this expression for $\mathcal{C}(x,y)$ we can verify the following properties: First, to order ϵ^3 the operator $\mathcal{C}(x,y)$ satisfies (10). That is,

$$\int_{-\infty}^{\infty} dy\, \mathcal{C}(x,y)\mathcal{C}(y,z) = \delta(x-z) + \mathrm{O}\left(\epsilon^4\right). \tag{19}$$

Second, to order ϵ^3 the operator $\mathcal{C}(x,y)$ satisfies (11); the wave functions $\phi_n(x)$ are eigenstates of $\mathcal{C}(x,y)$ with eigenvalue $(-1)^n$. That is,

$$\int_{-\infty}^{\infty} dy\, \mathcal{C}(x,y)\phi_n(y) = (-1)^n\phi_n(x) + \mathrm{O}\left(\epsilon^4\right). \tag{20}$$

Third, in the limit as $\epsilon \to 0$, the operator $\mathcal{C}(x,y)$ becomes the coordinate-space representation of the parity operator $\mathcal{P}(x,y) = \delta(x+y)$.

There is a somewhat simpler way to express the operator $\mathcal{C}(x,y)$. The derivative operator in (18) that is acting on $\delta(x+y)$ can be exponentiated so that to order ϵ^4 (and not just ϵ^3) we have

$$\mathcal{C}(x,y) = e^{-i\epsilon A - i\epsilon^3 B}\delta(x+y) + \mathrm{O}\left(\epsilon^5\right), \tag{21}$$

where the derivative operators A and B are given by

$$\begin{aligned}A &= \frac{4}{3}\frac{\partial^3}{\partial x^3} - 2x\frac{\partial}{\partial x}x, \\ B &= \frac{128}{15}\frac{\partial^5}{\partial x^5} - \frac{40}{3}x\frac{\partial^3}{\partial x^3}x + 8x^2\frac{\partial}{\partial x}x^2 - 32\frac{\partial}{\partial x}.\end{aligned} \tag{22}$$

4 Nonperturbative calculation of $\mathcal{C}$ in a quartic theory

In this section we explain briefly the nonperturbative methods that must be used to calculate the operator $\mathcal{C}(x,y)$ for the Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}x^2 - \epsilon x^4$. We follow the approach taken in Ref. [5], in which nonperturbative methods were used to calculate the one-point Green's function for this Hamiltonian.

4.1 Failure of perturbation theory

We begin by explaining why perturbation theory fails to produce the operator $\mathcal{C}(x,y)$. Following the approach taken in Section 3, we expand the solution to the Schrödinger equation

$$-\frac{1}{2}\phi_n''(x)+\frac{1}{2}x^2\phi_n(x)-\epsilon x^4\phi_n(x)=E_n\phi_n(x) \tag{23}$$

as a series in powers of ϵ:

$$\phi_n(x)=\frac{i^n a_n}{\pi^{1/4}2^{n/2}\sqrt{n!}}e^{-\frac{1}{2}x^2}\left[H_n(x)+P_n(x)\epsilon\right]+\mathrm{O}\left(\epsilon^2\right), \tag{24}$$

where $H_n(x)$ is the nth Hermite polynomial and $P_n(x)$ is a polynomial in x of degree $n+4$. The polynomial $P_n(x)$ is a linear combination of Hermite polynomials [1].

The energy E_n to order ϵ is

$$E_n=n+\frac{1}{2}-\frac{3}{4}\left(2n^2+2n+1\right)\epsilon+\mathrm{O}\left(\epsilon^2\right). \tag{25}$$

We must also $\mathcal{PT}$ normalize the expression for $\phi_n(x)$ according to (7) so that its square integral is $(-1)^n$:

$$\int_{-\infty}^{\infty}dx\ [\phi_n(x)]^2=(-1)^n+\mathrm{O}\left(\epsilon^2\right). \tag{26}$$

This determines the value of a_n in (24). The result is very simple; to order ϵ we have

$$a_n=1+\mathrm{O}\left(\epsilon^2\right). \tag{27}$$

Finally, we substitute $\phi_n(x)$ in (24) into (9). However, we obtain the trivial result that only the leading term (zeroth-order in powers of ϵ) survives. More generally, we can show by a parity argument that the coefficients of *all* higher powers of ϵ vanish. Thus, we get the (wrong) result that

$$\mathcal{C}(x,y)=\delta(x+y)\qquad\text{(WRONG!)}. \tag{28}$$

We know that this result is wrong because the operator $\mathcal{C}(x,y)$ is complex and the result in (28) is real. An alternative way to see this is to note (28) implies that $\mathcal{C}(x,y)$ and $\mathcal{P}(x,y)$ coincide; but in this $\mathcal{PT}$-symmetric theory, $\mathcal{C}(x,y)$ and $\mathcal{P}(x,y)$ are distinct operators. We will see that the difference between $\mathcal{C}(x,y)$ and $\mathcal{P}(x,y)$ is a nonperturbative term of order $e^{-1/(3\epsilon)}$, which is smaller than any integer power of ϵ.

4.2 Nonperturbative analysis

We will now show how to perform a nonperturbative analysis of the Schrödinger equation (23). We decompose the eigenfunction $\phi_n(x)$ into its perturbative part on the right side of (24) and a nonperturbative part:

$$\phi_n(x)=\phi_n^{\mathrm{pert}}(x)+\phi_n^{\mathrm{nonpert}}(x). \tag{29}$$

The nonperturbative part of $\phi_n(x)$ is exponentially small compared with the perturbative part, but these two contributions can be easily distinguished because for real argument x, one is real while the other is imaginary.

Following the WKB analysis in Ref. [5], we break the real-x axis into three regions: In region I, where $|x| \ll \epsilon^{-1/4}$, we have

$$\phi_n^{\rm pert}(x) \sim \frac{i^n}{\pi^{1/4}\sqrt{n!}} D_n\big(x\sqrt{2}\,\big),$$
$$\phi_n^{\rm nonpert}(x) \sim i b_n C_n\big(x\sqrt{2}\,\big), \tag{30}$$

where the coefficient of D_n is taken from (24) and the coefficient ib_n of C_n will be determined by asymptotic matching. Note that for nonnegative integer index the parabolic cylinder function D_n is expressed in terms of a Hermite polynomial H_n as

$$D_n(x\sqrt{2}) = 2^{-n/2} e^{-\frac{1}{2}x^2} H_n(x). \tag{31}$$

Also, for nonnegative integer index the functions D_n and C_n are a pair of linearly independent solutions to the parabolic cylinder equation. They can be expressed in terms of parabolic cylinder functions as follows:

$$D_n(z) \equiv \frac{n!}{\sqrt{2\pi}} \left[i^n D_{-n-1}(iz) + (-i)^n D_{-n-1}(-iz) \right],$$
$$C_n(z) \equiv \frac{i}{\sqrt{2\pi}} \left[i^n D_{-n-1}(iz) - (-i)^n D_{-n-1}(-iz) \right]. \tag{32}$$

In region II, where $1 \ll |x| \ll \epsilon^{-1/2}$, we can obtain the eigenfunction using WKB theory. We write the Schrödinger equation (23) in the form $\phi_n''(x) = \omega_n(x)\phi_n(x)$ where, to leading order in ϵ, we have $\omega_n(x) = -2\epsilon x^4 + x^2 - 2n - 1$. Then, for positive x the physical-optics WKB approximation reads

$$\phi_n^{\rm pert}(x) \sim f_n [\omega_n(x)]^{-1/4} \exp\left[- \int_{x_1}^{x} ds\, \sqrt{\omega_n(s)} \right],$$
$$\phi_n^{\rm nonpert}(x) \sim g_n [\omega_n(x)]^{-1/4} \exp\left[+ \int_{x_1}^{x} ds\, \sqrt{\omega_n(s)} \right], \tag{33}$$

where the constants f_n and g_n will be determined by asymptotic matching. The lower endpoint of integration, $x_1 = \sqrt{2n+1}$, is the approximate location of the inner turning point.

In region III x is near the outer turning points at $\pm 1/\sqrt{2\epsilon}$. For positive x we define the variable r by $x = x_2\left(1 - 2^{1/3}\epsilon^{2/3} r\right)$, where $x_2 = 1/\sqrt{2\epsilon}$. The condition that x is near x_2 is that $r \ll \epsilon^{-2/3}$. In this region the Schrödinger equation becomes an Airy equation in the variable r: $\phi_n''(r) = r\phi_n(r)$. The solution in this region reads

$$\phi_n^{\rm pert}(r) \sim h_n {\rm Bi}(r),$$
$$\phi_n^{\rm nonpert}(r) \sim -i h_n {\rm Ai}(r), \tag{34}$$

where ${\rm Ai}(r)$ and ${\rm Bi}(r)$ are the exponentially decaying and growing Airy functions for large positive r. The fact that the same coefficient h_n multiplies both Bi and Ai is a nontrivial result that is established in Ref. [5].

By asymptotically matching the solutions in regions I and II and the solutions in regions II and III we obtain the formula for the coefficient of the nonperturbative part of the solution in (30):

$$b_n = -\frac{i^n \pi^{1/4}}{\sqrt{2\,n!}} \left(\frac{4}{\epsilon}\right)^{n+\frac{1}{2}} e^{-\frac{1}{3\epsilon}}. \tag{35}$$

Finally, using the wave function in region I we can construct the operator $\mathcal{C}(x,y)$ according to (9):

$$\mathcal{C}(x,y) = \sum_{n=0}^{\infty} \left[\phi_n^{\rm pert}(x)\phi_n^{\rm pert}(y) + \phi_n^{\rm pert}(x)\phi_n^{\rm nonpert}(y) + \phi_n^{\rm nonpert}(x)\phi_n^{\rm pert}(y) + \phi_n^{\rm nonpert}(x)\phi_n^{\rm nonpert}(y)\right]. \tag{36}$$

The first sum in this equation gives $\delta(x+y)$ to all orders in powers of ϵ as explained above in Subsection 4.1. The last sum is negligible compared with the second and third sums. We thus obtain

$$\mathcal{C}(x,y) = \delta(x+y) - ie^{-\frac{1}{3\epsilon}}\sqrt{\frac{2}{\epsilon}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{4}{\epsilon}\right)^n\left[D_n(x\sqrt{2})C_n(y\sqrt{2}) + C_n(x\sqrt{2})D_n(y\sqrt{2})\right], \tag{37}$$

where C_n and D_n are defined in (32). Observe that the correction to the delta function (that is, the difference between the $\mathcal{P}$ operator and the $\mathcal{C}$ operator) is nonperturbative; it is exponentially small and imaginary.

The summation in (37) can be converted to a double integral:

$$\mathcal{C}(x,y) = \delta(x+y) + i\sqrt{\frac{2}{\pi^3\epsilon}}\,e^{-\frac{1}{3\epsilon}}e^{\frac{1}{2}(x^2+y^2)}\left\{\frac{\partial}{\partial x}\int_0^{\pi}d\theta\int_0^1\frac{ds}{\sqrt{1+s^2}} \times \exp\left[\frac{\left(2\sqrt{2s/\epsilon}\,\cos\theta - ix - isy\right)^2}{1+s^2}\right] + (x\leftrightarrow y)\right\}. \tag{38}$$

This is the leading-order nonperturbative approximation to the coordinate-space representation of the operator $\mathcal{C}$.

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Generally Covariant Quantum Mechanics

Jarosław WAWRZYCKI

Institute of Nuclear Physics, ul. Radzikowskiego 152, 31-342 Kraków, Poland
E-mail: *waw@alf.ifj.edu.pl, jwaw@th.if.uj.edu.pl*

We present a complete theory, which is a generalization of Bargmann's theory of factors for ray representations. We apply the theory to the generally covariant formulation of the Quantum Mechanics.

1 Problem

In the standard Quantum Mechanics (QM) and the Quantum Field Theory (QFT) the space-time coordinates are pretty classical variables. Therefore the question about the general covariance of QM and QFT emerges naturally just like in the classical theory:

what is the effect of a change in space-time coordinates in QM and QFT when the change does not form any symmetry transformation?

It is a commonly accepted belief that there are no substantial difficulties if we refer the question to the wave equation. We simply treat the wave equation, and do not say why, in such a manner as if it was a classical equation. The only problem arising is to find the transformation rule $\psi \to T_r\psi$ for the wave function ψ. This procedure, which on the other hand can be seriously objected, does not solve the above-stated problem. The heart of the problem as well as that of QM and QFT, lies in the Hilbert space of states, and specifically in finding the representation T_r of the covariance group in question. The trouble gets its source in the fact that the covariance transformation changes the form of the wave equation such that ψ and $T_r\psi$ do not belong to the same Hilbert space, which means that T_r does not act in the ordinary Hilbert space. This is not compatible with the paradigm worked out in dealing with symmetry groups.

2 Resume of the reformulation of the Quantum Mechanics

We have shown that covariance group acts in a Hilbert bundle $\mathcal{R}\triangle\mathcal{H}$ over time in the non-relativistic theory and in a Hilbert bundle $\mathcal{M}\triangle\mathcal{H}$ over space-time $\mathcal{M}$ in the relativistic case. The wave functions are the appropriate cross sections of the bundle in question. The exponent $\xi(r,s,p)$ in the formula

$$T_rT_s = e^{i\xi(r,s,p)}T_{rs},$$

depends on the point p of the base of the bundle in question: that is, ξ depends on the time t in the nonrelativistic theory and on space-time point p in the relativistic theory if there exists a nontrivial gauge freedom.

Moreover, we argue that the bundle $\mathcal{M}\triangle\mathcal{H}$ is more appropriate for treating the covariance as well as the symmetry groups then the Hilbert space itself. Namely, we show that from the more general assumption that the representation T_r of the Galilean group acts in $\mathcal{R}\triangle\mathcal{H}$ and has an exponent $\xi(r,s,t)$ depending on the time t we reconstruct the nonrelativistic Quantum Mechanics. Even more, in the less trivial case of the theory with nontrivial time-dependent

gauge describing a spin less quantum particle in the Newtonian gravity we are able to infer the wave equation and prove the equality of the inertial and gravitational masses.

In doing it we apply extensively the classification theory for exponents $\xi(r,s,t)$ of T_r acting in $\mathcal{R}\triangle\mathcal{H}$ and depending on the time.

In the presented theory which is slightly more general then the standard one the gauge freedom emerges from the very nature of the fundamental laws of Quantum Mechanics. By this it opens a new perspective in solving the troubles in QFT caused by the gauge freedom.

Interpretation. The physical interpretation ascribed to the cross-section ψ is as follows. Each experiment is, out of its very nature, a spatiotemporal event. To each act of measurement carried out at the space-time point p_0 we ascribe a self-adjoint operator Q_{p_0} acting in the Hilbert space $\mathcal{H}_{p_0}$ and ascribe to the spectral theorem for Q_{p_0} the standard interpretation. Hence, assuming for simplicity that Q_{p_0} is bounded, if $\phi_0 \in \mathcal{H}_{p_0}$ and $\lambda_0 = \lambda_o(p_0)$ is a characteristic vector and its corresponding characteristic value of Q_{p_0} respectively then we have the following statement. *If the experiment corresponding to Q_p was performed at the spatiotemporal event p_0 on a system in the state described by the cross section ψ, then the probability of the measurement value to be $\lambda_0(p_0)$ and the system to be found in the state described by ϕ such that $\phi(p_o) = \phi_0$ after the experiment is given by the square of the absolute value of the Borel function $|f_\psi(p_0,\phi_0)|^2 = |(\phi_0,\psi_{p_0})|^2$ induced by the cross section ψ.* In the nonrelativistic case the above statement is a mere rephrasing of the well established knowledge.

See [1] for detail treatment, where the theory was proposed.

3 Resume of the generalization of the Bargmann's theory of factors

Our main task was to construct the general classification theory of space-time dependent exponents $\xi(r,s,p)$ of representations acting in $\mathcal{M}\triangle\mathcal{H}$, see [1]. On the other hand the presented theory can be viewed as a generalization of the Bargmann's [2] classification theory of exponents $\xi(r,s)$ of representations acting in ordinary Hilbert spaces, which are independent of $p \in \mathcal{M}$.

Definition 1. By an *isomorphism* of the Hilbert bundle $\mathcal{M}\triangle\mathcal{H}$ with the Hilbert bundle $\mathcal{M}'\triangle\mathcal{H}'$ we shall mean a Borel isomorphism T of $\mathcal{M}\triangle\mathcal{H}$ on $\mathcal{M}'\triangle\mathcal{H}'$ such that for each $p \in \mathcal{M}$ the restriction of T to $p \times \mathcal{H}_p$ has some $q \times \mathcal{H}'_q$ for its range and is unitary when regarded as a map of $\mathcal{H}_p$ on $\mathcal{H}'_q$. The induced map carrying p into q is clearly a Borel isomorphism of $\mathcal{M}$ with $\mathcal{M}'$ and we denote it by T^π. The above defined T is said to be an *automorphism* if $\mathcal{M}\triangle\mathcal{H} = \mathcal{M}'\triangle\mathcal{H}'$. Note that for any automorphism T we have $(T\psi,T\phi)_{T^\pi p} = (\psi,\phi)_p$, but in general $(T\psi,T\phi)_p \neq (\psi,\phi)_p$. By this any automorphism T is what is frequently called a *bundle isometry*.

(We use the Hilbert bundle with the ordinary Borel structure in the total space and with the ordinary manifold structure in the base $\mathcal{M}$, see e.g. [3].)

Definition 2. The function $r \to T_r$ from a group G into the set of automorphisms (bundle isometries) of $\mathcal{M}\triangle\mathcal{H}$ is said to be a *general factor representation* of G associated to the action $G\times\mathcal{M} \ni r,p \to r^{-1}p \in \mathcal{M}$ of G on $\mathcal{M}$ if $T^\pi_r(p) \equiv r^{-1}p$ for all $r \in G$, and T_r satisfy the condition

$$T_rT_s = e^{i\xi(r,s,p)}T_{rs}.$$

Here we give only the summing up

Theorem 1. *(1) On a Lie group G, every local exponent $\xi(r,s,p)$ is equivalent to a canonical local exponent $\xi'(r,s,p)$ which, on some canonical neighborhood $\mathfrak{N}_0$, is analytic in canonical*

coordinates of r and s and and vanishes if r and s belong to the same one-parameter subgroup. Two canonical local exponents ξ, ξ' are equivalent if and only if $\xi'(r,s,p) = \xi(r,s,p) + \Lambda(r,p) + \Lambda\big(s, r^{-1}p\big) - \Lambda(rs,p)$ on some canonical neighborhood, where $\Lambda(r,p)$ is a linear form in the canonical coordinates of r such that $\Lambda(r,sp)$ does not depend on s if s belongs to the same one-parameter subgroup as r. (2) To every canonical local exponent of G corresponds uniquely an infinitesimal exponent $\Xi(a,b,p)$ on the Lie algebra $\mathfrak{G}$ of G, i.e. a bilinear antisymmetric form which satisfies the identity[1] $\Xi([a,a'],a'',p) + \Xi([a',a''],a,p) + \Xi([a'',a],a',p) = \boldsymbol{a}\Xi(a',a'',p) + \boldsymbol{a'}\Xi(a'',a,p) + \boldsymbol{a''}\Xi(a,a',p)$. *The correspondence is linear. (3) Two canonical local exponents ξ, ξ' are equivalent if and only if the corresponding Ξ, Ξ' are equivalent, i.e.*[2] $\Xi'(a,b,p) = \Xi(a,b,p) + \boldsymbol{a}\Lambda(b,p) - \boldsymbol{b}\Lambda(a,p) - \Lambda([a,b],p)$ *where $\Lambda(a,p)$ is a linear form in a on $\mathfrak{G}$ such that $\tau \to \Lambda(a,(\tau b)p)$ is constant if $a = b$. (4) There exist a one-to-one correspondence between the equivalence classes of local exponents ξ (global in $p \in \mathcal{M}$) of G and the equivalence classes of infinitesimal exponents Ξ of $\mathfrak{G}$.*

4 The physical motivation

It will be instructive to investigate the problem for a free particle in the flat Galilean space-time. The set of solutions ψ of the Schrödinger equation which are admissible in Quantum Mechanics is precisely given by

$$\psi(\vec{x},t) = (2\pi)^{-3/2} \int \varphi(\vec{k}) e^{-i\frac{t}{2m}\vec{k}\cdot\vec{k} + i\vec{k}\cdot\vec{x}}\, d^3k,$$

where $p = \hbar k$ is the linear momentum and $\varphi(\vec{k})$ is any square integrable function. The functions φ (wave functions in the "Heisenberg picture") form a Hilbert space $\mathcal{H}$ with the inner product

$$(\varphi_1, \varphi_2) = \int \varphi_1^*(\vec{k})\varphi_2(\vec{k})\, d^3k.$$

The correspondence between ψ and φ is one-to-one.

But in general the construction fails if the Schrödinger equation possesses a nontrivial gauge freedom. We explain it.

We need not to use the Fourier transform. **What is the role of the Schrödinger equation in the above construction of $\mathcal{H}$?** In the above construction the Hilbert space $\mathcal{H}$ is isomorphic to the space of square integrable functions $\varphi(\vec{x}) \equiv \psi(\vec{x},0)$ – the set of square integrable space of initial data for the Schrödinger equation. The connection between ψ and φ is given by the time evolution $U(0,t)$ operator (by the Schrödinger equation):

$$U(0,t)\varphi = \psi.$$

The correspondence between φ and ψ has all formal properties such as in the above Fourier construction. Denote the space of the initial square integrable data φ on the simultaneity hyperplane $t(X) = t$ by $\mathcal{H}_t$. The space of wave functions $\psi(\vec{x},t) = U(0,t)\varphi(\vec{x})$ isomorphic to the Hilbert space $\mathcal{H}_0$ of φ's is called in the common "jargon" the "Schrödinger picture".

However, the connection between $\varphi(\vec{x})$ and $\psi(\vec{x},t)$ is not unique in general, if the wave equation possesses a gauge freedom. Namely, consider the two states φ_1 and φ_2 and ask the question: when the two states are equivalent and by this indistinguishable? The answer is as

[1] $\boldsymbol{a}\Xi(b,c,p)$ stands for the differential operator $d/d\tau\Xi(b,c,(\tau a)p)|_{\tau=0}$.

[2] $\tau \to \tau a$ is a one-parameter group generated by $a \in \mathfrak{G}$.

follows: they are equivalent if

$$|(\varphi_1,\varphi)| \equiv \Big|\int \psi_1^*(\vec{x},t)\psi(\vec{x},t)\,d^3x\Big| = |(\varphi_2,\varphi)| \equiv \Big|\int \psi_2^*(\vec{x},t)\psi(\vec{x},t)\,d^3x\Big|, \tag{1}$$

for any state φ from $\mathcal{H}$, or for all $\psi = U\varphi$ (ψ_i are defined to be $= U(0,t)\varphi_i$). Substituting φ_1 and then φ_2 for φ and making use of the Schwarz's inequality one gets: $\varphi_2 = e^{i\alpha}\varphi_1$, where α is any constant[3]. The situation for ψ_1 and ψ_2 is however different. In general the condition (1) is fulfilled if

$$\boldsymbol{\psi_2 = e^{i\Lambda(t)}\psi_1}$$

and the phase factor can depend on time. Of course it has to be consistent with the wave equation, that is, together with a solution ψ to the wave equation the wave function $e^{i\Lambda(t)}\psi$ also is a solution to the appropriately gauged wave equation. *A priori* one cannot exclude the existence of such a consistent time evolution. This is not a new observation, it was noticed by J. von Neumann[4], but it seems that it has never been deeply investigated (probably because the ordinary nonrelativistic Schrödinger equation has a gauge symmetry with constant Λ). The space of waves ψ describing the system cannot be reduced in the above way to any fixed Hilbert space $\mathcal{H}_t$ with a fixed t. So, the existence of the nontrivial gauge freedom leads to the

Hypothesis. *The two waves ψ and $e^{i\Lambda(t)}\psi$ are quantum-mechanically indistinguishable.*

Moreover, we are obliged to use the whole Hilbert bundle $\mathcal{R}\triangle\mathcal{H} : t \to \mathcal{H}_t$ over the time instead of a fixed Hilbert space $\mathcal{H}_t$, with the appropriate cross sections as the waves ψ.

Consider now an action T_r of a group G in the space of waves ψ. From our analysis it follows that it is natural to replace the ordinary postulate:

Classical-like postulate. *The group G is a symmetry group if and only if the wave equation is invariant under the transformation $x' = rx$, $r \in G$ of independent variables and the transformation $\psi' = T_r\psi$ of the wave function.*

by the more appropriate alternative:

Quantum postulate. *The group G is a symmetry group if and only if the transformation $x' = rx$, $r \in G$ of independent variables and the transformation $\psi' = T_r\psi$ of the wave function transform the wave equation into a gauge-equivalent one.*

Acceptation of the **Quantum postulate** gives a new perspective for solving the two very difficult problems [1]:

(a) generally covariant formulation of Quantum Mechanics,

(b) the troubles in the Quantum Field Theory caused by the gauge freedom.

Moreover, with the help of the **Hypothesis** we can see that both **(a)** and **(b)** are deeply connected [1].

[3]This gives the conception of the ray, introduced to Quantum Mechanics by H. Weyl [4]: a physical state does not correspond uniquely to a normed state $\varphi \in \mathcal{H}$, but it is uniquely described by a *ray*, two states belong to the same ray if they differ by a constant phase factor.

[4]J. von Neumann, Mathematical Principles of Quantum Mechanics, Princeton, University Press, 1955. He did not mention about the gauge freedom on that occasion. But the gauge freedom is necessary for the equivalence of ψ_1 and $\psi_2 = e^{i\Lambda(t)}\psi_1$.

5 Connection to the troubles with the gauge freedom

Now, we return to the problem **(b)**. It should be mentioned at this place that the troubles in QFT generated by the gauge freedom are of general character, and are well known. For example, there do not exist vector particles with helicity = 1, which is a consequence of the theory of unitary representations of the Poincaré group, as was shown by J. Łopuszański [5]. This is apparently in contradiction with the existence of vector particles with helicity = 1 in nature – the photon, which is connected with the electromagnetic four-vector potential. The connection of the problem with the gauge freedom is well known [5]. We omit however the difficulty if we allow the inner product in the "Hilbert space" to be not positively defined, see [6], or [7]. Due to [5], the vector potential (promoted to be an operator valued distribution in QED) cannot be a vector field, if one wants to have the inner product positively defined then together with the coordinate transformation the gauge transformation has to be applied, which breaks the vector character of the potential. Practically it means that any gauge condition which brings the theory into the canonical form such that the quantization procedure can be consequently applied (with the positively defined inner product in the Hilbert space) breaks the four-vector character of the electromagnetic potential, the Coulomb gauge condition is an example. To achieve the Poincaré symmetry of Maxwell equations with such a gauge condition (the Coulomb gauge condition for example), it is impossible to preserve the vector character of the potential – together with the coordinate transformation a well defined (by the coordinate transformation) gauge transformation f has to be applied:

$$A_\mu \to A'_{\mu'} = \frac{\partial x^\nu}{\partial x^{\mu'}}(A_\nu + \partial_\nu f).$$

This means that the electromagnetic potential can form a generalized ray representation T_r of the Poincaré group at most, with the space-time-dependent factor $e^{i\xi}$ if the scalar product is positively defined. One may ask: how possible is it if the Poincaré group is not only a covariance group but at the same time a symmetry group? The solution of this paradox on the grounds of the existing theory is rather obscure. We propose the following solution. The factor $e^{i\xi}$ is space-time independent for the symmetry group but under the assumption that the fundamental space describing the states of a quantum system is the ordinary Hilbert space and the **Classical-like postulate** is true. But we have presented serious objections to this assumption. Moreover, the nonrelativistic quantum theory can be reconstructed from the more general assumption about the space of quantum mechanical states saying that it compose the space of appropriate cross sections of the Hilbert bundle $\mathcal{R}\triangle\mathcal{H}$ over time $t \in \mathcal{R}$. The Schrödinger equation can be uniquely reconstructed from the generalized ray representations of the Galilean group. We also watch for a more fundamental justification of this assumption in the presumption that time is a purely classical variable in the nonrelativistic quantum mechanics or so to speak a parameter. The most general unitary representation of the locally compact commutative group of the time real line acts in a Hilbert bundle $\mathcal{R}\triangle\mathcal{H}$ over time, see Mackey [3]. So, the assumption about the "classicity" of the time t fixes the structure of space of quantum states to be a subset of cross sections of a Hilbert bundle over the time. This is the peculiar property of the Galilean group structure that the whole construction degenerates as if we were started from the ordinary ray representation over the ordinary Hilbert space and the theorem that the phase $e^{i\xi}$ is space-time independent is true in this case, but only accidentally. The generalization to the relativistic case is natural. First we postulate the space-time coordinates to be classical commutative variables, which leads to the Hilbert bundle $\mathcal{M}\triangle\mathcal{H}$ over space-time manifold $\mathcal{M}$. The factor of the representation of the Poincaré group acting in the bundle $\mathcal{M}\triangle\mathcal{H}$ has not to be a constant with respect to space-time coordinates even when it is a symmetry group. This solves the paradox.

On the other hand H. Weyl (pages 272–276 of the Dover ed. of [4]) showed that the Quantum Kinematics represented by the Heisenberg commutation relations is nothing else but a unitary

factor representation of an Abelian group. This Weyl's construction can be viewed as emerging from the gauge freedom with the constant phase (constant Λ), i.e. a special case of the Bargmann' theory, see [2]. This is just the case of the Heisenberg' analysis of commutation relations for the atom which is determined (in the first approximation) by the Coulomb field of its nuclei charge. That is, the field which is naturally connected with the constant phase gauge freedom. But the general electromagnetic interactions possess the space-time dependent gauge freedom. So, by the construction our generalization seems to be natural for treating the quantum electromagnetic interactions.

6 Connection to the troubles with the position operator

It was shown in Section 4 that our generalization with states as cross sections in a Hilbert bundle $\mathcal{M}\triangle\mathcal{H}$ over the whole space-time $\mathcal{M}$ looses physical sense in the nonrelativistic theory. In this case we have the bundle $\mathcal{R}\triangle\mathcal{H}$ over time. It should be noted that we can recover this fact from the more general point of view. Namely, any exponent $\xi(r,s,p)$ of a generalized factor representation of the Galilean group acting in $\mathcal{M}\triangle\mathcal{H}$ is equivalent to a time dependent one: $\xi(r,s,p) = \xi(r,s,t)$, such that we can restrict the whole analysis to the bundle $\mathcal{R}\triangle\mathcal{H}$ over the time. This is a peculiar property of the Galilean group. This is not the case for the Poincaré group. The time, being a classical variable, does not posses any quantum mechanical measurement operator. But, contrary to the time, the space coordinates do possess operators in the nonrelativistic theory. In the relativistic case all the space-time coordinates play the role of the time and do not possess quantum mechanical operators. This is in agreement to the well established knowledge and reflects the standard problem in construction of the position operator in the relativistic theory, compare e.g. [3] and [8].

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Extended Relativistic Dynamics with n-degree Characteristic Polynomial

Robert M. YAMALEEV

FES-C UNAM, Edo. de Mexico, C.P.54740, México;
Joint Institute for Nuclear Research, 141980 Dubna, Russia
E-mail: *iamaleev@servidor.unam.mx*

It is shown that the relativistic equations of motion can be decomposed into two Newtonian equations. The mapping between momenta of the system of Newtonian and relativistic equations is given by Vieta's formulae of a quadratic polynomial – the *characteristic polynomial* of the relativistic dynamics. In the present contribution we extend this scheme to obtain an extension of the relativistic mechanics. We suggest extended dynamic equations, which, with respect to the definite set of evolution parameters, are decomposed into three (n) Newtonian equations. The corresponding mapping is built on the basis of Vieta's formulae of n-degree polynomial. This polynomial we define as the *characteristic polynomial* of the extended mechanics. In the polar representation the solutions of the dynamic equations are given by Jacobi and Weierstrass (hyper) elliptic functions.

1 Introduction

The *principle of relativity* does not necessarily imply the quadratic metric in the space-time plane. The Lorentz-invariance may be violated in the same way as it has been violated the Galilei-invariance. A violation, rather, will be stipulated with the increasing of the energy of motion. That is to say, the relativistic mechanics is not consummate step of the extension of the Newtonian mechanics. The former can be extended in the same manner, namely, by involving the energy conservation law into the system of dynamic equations.

In this paper we present one faithful way of extension of the relativistic equations of motion. First of all (Section 2) we show that the relativistic equations of motion can be decomposed into two Newtonian equations with different evolution parameters. For this decomposition we use the Vieta's mapping of a quadratic polynomial. The last is defined as a characteristic polynomial of the relativistic dynamics. Further (Section 3), we extend the relativistic dynamics in such a way that the extended equations of motion will decomposed into three Newtonian equations. The dynamic equations of the extended mechanics are constructed by using Vieta's mapping of the characteristic cubic polynomial. The solutions in the polar representation are given by Jacobi and Weierstrass elliptic functions. In Section 4, the scheme elaborated in Section 3 generalized in order to obtain an extended mechanics with n-degree characteristic polynomial.

2 Decomposition of the relativistic equation into two Newtonian equations

Consider a motion of the relativistic particle under potential field $V(r)$. With respect to the proper-time these equations are written as

$$\frac{d\vec{P}}{d\tau}=\frac{1}{mc}\vec{E}\,P_0,\qquad \frac{dP_0}{d\tau}=\frac{1}{mc}(\vec{E}\cdot\vec{P}),\qquad \vec{E}=-\vec{\nabla}V(r), \tag{1}$$

$$\frac{d\vec{r}}{d\tau} = \frac{\vec{P}}{m}, \qquad \frac{dt}{d\tau} = \frac{P_0}{mc}, \tag{2}$$

These equations imply the first integral of motion:

$$P_0^2 - P^2 = M^2c^2. \tag{3}$$

Let us represent the relationship (3) in the following factorized form

$$c^2P^2 = c^2P_0^2 - m^2c^4 = \left(cP_0 - mc^2\right)\left(cP_0 + mc^2\right). \tag{4}$$

In the non-relativistic limit

$$cP_0 - mc^2 \to \frac{p^2}{2m}.$$

Introduce two kinetic energies by definition

$$\frac{p^2}{2m} := \left(cP_0 - Mc^2\right), \qquad \frac{q^2}{2\mu} := \left(cP_0 + Mc^2\right), \tag{5}$$

Here we use the following units: p has a dimension of a momentum, q and μ have a dimension of an energy.

In these terms the square of momentum P in (4) is represented as a product of two kinetic energies

$$c^2P^2 = \frac{p^2}{2m}\frac{q^2}{2\mu}. \tag{6}$$

The linear combinations of the formulae (5) give the following expressions for P_0 and Mc^2:

$$cP_0 = \frac{1}{2}\left(\frac{q^2}{2\mu} + \frac{p^2}{2m}\right), \qquad Mc^2 = \frac{1}{2}\left(\frac{q^2}{2\mu} - \frac{p^2}{2m}\right). \tag{7}$$

The expressions for P and P_0 from (6) and (7) substitute into projection of (1), (2), on the direction of momentum

$$\vec{n} = \frac{\vec{P}}{P}.$$

On making equal the expressions at p and q we come to the following evolution equations for p and q [1]:

$$\text{(a)}\ \ \frac{dp}{d\tau} = -(\vec{n}\cdot\vec{\nabla})V(r)\frac{q}{\mu}, \qquad \text{(b)}\ \ \frac{dq}{d\tau} = -(\vec{n}\cdot\vec{\nabla})V(r)\frac{p}{m}, \qquad \text{(c)}\ \ \frac{d\vec{r}}{d\tau} = \vec{n}\frac{p}{m}\frac{q}{\mu}, \tag{8}$$

with the first integral

$$\frac{q^2}{2\mu} - \frac{p^2}{2m} = 2Mc^2.$$

Now, let us exchange the parameter of evolution τ in (8) by a new time-like parameter of evolution $dt_p = d\tau\frac{q}{\mu}$. With respect to the new time-parameter (8) are reduced into Newtonian equations

$$\frac{dp}{dt_p} = -(\vec{n}\cdot\vec{\nabla})V(r), \qquad \frac{dr}{dt_p} = \frac{p}{m}. \tag{9}$$

In a same manner we obtain another Newtonian equation:

$$\frac{d}{d(ct_q)}q = -(\vec{n}\cdot\vec{\nabla})V(r), \qquad \frac{dr}{d(ct_q)} = \frac{q}{\mu}, \qquad d\tau\frac{p}{m} = d(ct_q). \tag{10}$$

Thus, (8) are decomposed into two Newtonian equations of motion: (9) and (10) with different evolution parameters (times).

Equations (1), (2) can be written in the *polar representation*:

$$P = Mc\sinh(\phi), \qquad P_0 = Mc\cosh(\phi),$$

where ϕ obeys the equation

$$\frac{d\phi}{d\tau} = \frac{e}{mc}(\vec{E}\cdot\vec{n}).$$

In this representation for p and q we obtain

$$\frac{p}{\sqrt{2m}} = \sqrt{2Mc}\sinh\left(\frac{\phi}{2}\right), \qquad \frac{q}{\sqrt{2\mu}} = \sqrt{2Mc}\cosh\left(\frac{\phi}{2}\right).$$

Renormalize the values p, q by

$$p = p\frac{1}{\sqrt{2m}}, \qquad q = q\frac{1}{\sqrt{2\mu}}.$$

Within these designations the formulae (6) and (7) are written as

$$c^2P^2 = p^2q^2, \qquad 2cP_0 = \left(q^2+p^2\right),$$

which are nothing else than Vieta's formulae for the quadratic equation

$$X^2 - 2cP_0X + c^2P^2 = 0.$$

This polynomial is the *characteristic polynomial* of the relativistic mechanics [2].

3 Extended relativistic mechanics with cubic characteristic polynomial

In the previous section we have shown that the relativistic equations of motion can be decomposed into two Newtonian equations. In this section we seek such an extension of the relativistic equations of motion which in a similar manner can be decomposed into three Newtonian equations.

Let us start from the set of three Newtonian equations written with respect to different evolution parameters ($c = 1$):

$$\frac{dp}{dt_p} = -(\vec{n}\cdot\vec{E}), \qquad \frac{dq}{dt_q} = -(\vec{n}\cdot\vec{E}), \qquad \frac{dh}{dt_h} = -(\vec{n}\cdot\vec{E}). \tag{11}$$

In order to obtain this set of Newtonian equations from unique system of equations we should extend (8) in the following way

$$\frac{dp}{ds} = (\vec{n}\cdot\vec{E})\frac{q}{\mu}\frac{h}{\mu_h}, \qquad \frac{dq}{ds} = (\vec{n}\cdot\vec{E})\frac{p}{m}\frac{h}{\mu_h}, \qquad \frac{dh}{ds} = (\vec{n}\cdot\vec{E})\frac{p}{m}\frac{q}{\mu}, \tag{12}$$

$$\frac{dr}{ds} = \frac{p}{m}\frac{q}{\mu}\frac{h}{\mu_h}. \tag{13}$$

It is easily seen that these equations are reduced into (8) by exchanging the evolution parameter ds by $d\tau = \frac{h}{\mu_h} ds$. From (12) we obtain three Newtonian equations (12) by using the following formulae for the evolution parameters

$$d\tau \frac{qh}{\mu\mu_h} = dt_p, \qquad d\tau \frac{hp}{m\mu_h} = dt_q, \qquad d\tau \frac{pq}{m\mu} = dt_h.$$

Now, let us construct analogous mapping as we have built in the previous section, but for the triplet of variables $\left\{p^2, q^2, h^2\right\}$. The expression for the momentum we obtain from (13)

$$P = p\frac{h}{\mu_h}\frac{q}{\mu}. \tag{14}$$

In the polar representation momenta p, q, h are represented by Jacobi elliptic functions of imaginary argument:

$$p = \sqrt{m\mu}\,\mathrm{sc}\,(\phi, k), \qquad q = \mu\,\mathrm{nc}\,(\phi, k), \qquad h = \mu_h\,\mathrm{dc}\,(\phi, k),$$

where $k = \mu/\mu_h$, and the "angle" ϕ obeys the equation

$$\frac{d\phi}{ds} = (\vec{n}\cdot\vec{E})\frac{1}{\sqrt{m\mu}}.$$

In the sequel we shall use the set of dimensionless variables. For that purpose define

$$p_3 := \frac{1}{\sqrt{k}}\frac{h}{\mu_h}, \qquad p_2 := \frac{q}{\mu}, \qquad p_1 := \frac{p}{\sqrt{m\mu}}.$$

Then (12) in the polar representation are written in the following symmetric form

$$\frac{1}{\sqrt{k}}\frac{dp_1}{d\phi} = p_2p_3, \qquad \frac{1}{\sqrt{k}}\frac{dp_2}{d\phi} = p_3p_1, \qquad \frac{1}{\sqrt{k}}\frac{dp_3}{d\phi} = p_1p_2. \tag{15}$$

Correspondingly, re-define (14) by

$$P := \frac{1}{\sqrt{k}}\frac{1}{\sqrt{m\mu}}\,P = p_1p_2p_3. \tag{16}$$

Evolution equations for the squares p_1^2, p_2^2, p_3^2 are derived from (15):

$$\frac{1}{\sqrt{k}}\frac{dp_k^2}{d\phi} = 2p_1p_2p_3 = 2P, \qquad k = 1, 2, 3. \tag{17}$$

By using these equations calculate derivative of P with respect to ϕ. In the process of evaluation we shall introduce additional variables P_1, P_2 as algebraic functions of p_1^2, p_2^2, p_3^2. The first equation is:

$$\frac{1}{\sqrt{k}}\frac{dP}{d\phi} = 2P_2, \tag{18}$$

where

$$P_2 = \frac{1}{2}\left(p_1^2p_2^2 + p_2^2p_3^2 + p_3^2p_1^2\right). \tag{19}$$

Further, by evaluating the derivative of P_2, we get

$$\frac{1}{\sqrt{k}}\frac{dP_2}{d\phi} = 6PP_1, \tag{20}$$

where

$$P_1 = \frac{1}{3}\left(p_1^2 + p_2^2 + p_3^2\right). \tag{21}$$

The next differentiation of P_1 brings up the system of equations for the set of variables $\{P, P_1, P_2\}$:

$$\frac{1}{\sqrt{k}}\frac{dP_1}{d\phi} = 2P. \tag{22}$$

Notice that the mapping given by formulae (16), (19), (21) is nothing else than Vieta's formulae for the cubic polynomial

$$X^3 - 3P_1X^2 + 2P_2X - P^2 = 0.$$

Notice, (18), (20), (22) admit two first integrals of motion which do not depend of the potential function $V(r)$:

$$R_1 = -2P_2 + 3P_1^2, \qquad R_0 = P_1^3 - R_1P_1 - P^2.$$

From equations (18), (20), (22) we obtain

$$\frac{1}{\sqrt{k}}\frac{dP_1}{d\phi} = \sqrt{4P_1^3 - 4R_1P_1 - 4R_0}.$$

The integral of this equation is given by the Weierstrass elliptic integral

$$\sqrt{k}\phi = \int \frac{dy}{\sqrt{4y^3 - 4R_1y - 4R_0}}.$$

Consequently, the solution $y = P_1(\sqrt{k}\phi; 4R_1, 4R_0)$ is expressed via Weierstrass elliptic function [4].

4 Extended relativistic mechanics with n-degree characteristic polynomial

The theory we built in the previous sections can be extended to the case of mechanics with n-degree characteristic polynomial.

Consider the set of n Newtonian equations written with respect to different evolution parameters ($c = 1$):

$$\frac{dp}{dt_k} = -(\vec{n}\cdot\vec{E}).$$

By working in a similar manner as it has been in the previous section we come to the following equations of motion

$$\frac{dp}{ds} = (\vec{n}\cdot\vec{E})\frac{q}{\mu}\prod_{l=1}^{n-2}\frac{p_l}{\mu_l}, \qquad \frac{dq}{ds} = (\vec{n}\cdot\vec{E})\frac{p}{m}\prod_{l=1}^{n-2}\frac{p_l}{\mu_l},$$

$$\frac{dp_k}{ds} = (\vec{n}\cdot\vec{E})\frac{q}{\mu}\frac{p}{m}\prod_{(l\neq k)}^{n-2}\frac{p_l}{\mu_l}, \qquad k = 1, 2, \ldots, n-2, \tag{23}$$

$$\frac{dx}{ds} = \frac{p}{m}\frac{q}{\mu}\prod_{l=1}^{n-2}\frac{p_l}{\mu_l}. \tag{24}$$

Let P be momentum of the particle. Then from (24) it follows

$$P = p\frac{q}{\mu}\prod_{l=1}^{n-2}\frac{p_l}{\mu_l}. \tag{25}$$

In the sequel we shall use renormalized values

$$p := \frac{p}{\sqrt{m\mu}}, \qquad q := \frac{q}{\mu}, \qquad p_l = \frac{p_l}{\sqrt{k_l}\mu_l}, \qquad l = 1, \ldots, n-2.$$

Then, in the polar representation (23) will be written as

$$\begin{aligned}
&\frac{dp}{d\phi} = q\prod_{l=1}^{n-2}p_l, \qquad \frac{dq}{d\phi} = p\prod_{l=1}^{n-2}p_l,\\
&\frac{dp_k}{d\phi} = pq\prod_{(l\neq k)}^{n-2}p_l, \qquad k = 1, 2, \ldots, n-2.\\
&\frac{d\phi}{ds} = -k(\vec{n}\cdot\vec{E})\frac{1}{\sqrt{m\mu}},
\end{aligned} \tag{26}$$

where $k = \prod_{l=1}^{n-2}\sqrt{\mu_l}$.

It is convenient to use the following set of designations $p_1 := p$, $p_2 := q$, $p_k := p_{k-2}$, $k = 3, \ldots, n$. The formula (25) is written now as

$$P^2 = \prod_{k=1}^{n}p_k^2. \tag{27}$$

The other dynamic variable is defined by:

$$P_1 = \frac{1}{n}\sum_{k=1}^{n}p_k^2. \tag{28}$$

The values P^2, P_1 are the first members of a mapping which we are seeking. These formulae are nothing else than the Vieta's formulae for the polynomial:

$$X^n + \sum_{k=1}^{n}(-1)^k(n-k+1)P_kX^{n-k} = 0, \qquad \text{with} \quad P_n = P^2. \tag{29}$$

The roots of this polynomial are given by the set of momenta p_k^2 $(k = 1, \ldots, n)$, whereas the coefficients of the polynomial will be used in the capacity of dynamic variables of the extended dynamics.

The evolution equations for the squares of the momenta are given by

$$\frac{dp_k^2}{ds} = (\vec{E}\cdot\vec{P})\frac{1}{m}, \qquad k = 1, \ldots, n.$$

By using the mapping given by formulae (27), (28) and formulae Vieta of the equation (29) from equations (26) we deduce the following evolution equations for the values $P_1, P_2, \ldots, P_n$:

$$\frac{dP_{n-k}}{ds} = (\vec{E}\cdot\vec{P})P_{n-k-1}\frac{k+2}{m}, \qquad k = 0, \ldots, n-1, \qquad \text{with} \quad P_0 = \frac{1}{n+1}. \tag{30}$$

These equations admit $(n-1)$ invariants no containing the potential $V(r)$. These invariants are given by algebraic expressions of $P_1, P_2, \ldots, P_n$. It is not easy to construct these expressions, therefore let us outline the method of construction of these invariants. The method is given by the following theorem [3]

Theorem 1. *The coefficients of the polynomial equation of n-degree*

$$Y^n + R_1 Y^{n-2} + \cdots + R_k Y^{n-k+1} + \cdots + R_{n-2}Y + R_0 = 0,$$

which is obtained from the polynomial equation (29) *by replacing X with $X = Y + P_1$, are invariants of equations of motion* (30).

In this contribution, we did not touch the problem of Extended Lorentz-kinematics. Here, let us only notice that due to the addition theorem for the elliptic functions still is possible to formulate *a group of transformation*. However, for higher order elliptic functions an addition theorem is not more valid. In this aspect the Extended Relativistic Mechanics with cubic characteristic polynomial is the last step of possible extensions via elliptic functions.

In the conclusion let us mention about some connection of this contribution with the results of [5], where hyperelliptic Lax representation for the generalized model of Euler's top has been done. In [2] we have shown that the relativistic oscillator model formally coincides with the model of Euler's top, whereas the oscillator models of the extended relativistic dynamics (see, for example, [3]), give generalizations of the Euler's top model. It is expected that the mathematical tool developed by Skrypnyk will be helpful within the mechanics elements of which we have presented in this contribution.

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Symmetry in Relativistic Measurements

C.Y. YANG

G.R.A.A., Greenbelt, Maryland, USA
E-mail: *chaoyyang@netscape.net*

The covarying scale effect in the observational relativistic measurement found a few years ago is that the Lorentz factor is a scale conversion ratio between the virtual covarying scale of measurement in relative motion and the real invariant scale at rest with respect to the observer. We find this scale conversion ratio being the bridge between two relative frames, and a logical symmetry for the measurement of the spatial distance and the measurement of time interval by using the same scale conversion. Hence the two relative frames are not only reciprocal to each other, but also symmetric with respect to the scale conversion to each other.

1 Introduction

A few years ago, we studied the space-time invariance in the Special Theory of Relativity [1]. To conserve the invariance, a scale conversion has been found necessary in between two inertial frames in relative uniform motion. The inseparable relationship between the relativistic measurement and the virtual covarying scale was discovered.

The scale is the size of the measuring unit. The scale used in any measurement is part of it. When extended to uniform relative motion, this concept calls for a measurement scale for the relativistic interval measurement. Such a scale can be called a "virtual covarying scale" which is part of the relativistic measurement. In other words, this is a principle of inseparability between relativistic measurement and virtual covarying scale.

This inseparability provides a sound logical foundation to reconfirm the Theory of Special Relativity, since any paradox simply can not happen when scale conversion exists.

The inseparability between the relativistic measurent and the virtual covarying scale serves as a bridge between two relative frames. Not only the relativistic measurements are reciprocal to each other between any two relative frames, but also symmetric to each other with respect to the scale conversion between the virtual covarying scale and the real invariant scale.

In the following sections we will show that the virtual covarying scale effect can explain the signal delay for the time interval measurement. With the same virtual covarying scale, one can explain why a linear accelerator needs not be built like a trombone physically for various energy levels, but functions like a virtual trombone.

2 The time interval measurement

To facilitate the time interval measurement in relative motion, let us use the reference frames as in the following Fig. 1.

Let us consider the two identical light clocks illustrated in Figs. 2 and 3 with photon emitters and detectors separated at a distance d from a mirror; c is the speed of light. One of the light clocks is stationary with respect to an observer and the other is moving at a uniform speed v relative to to the observer.

These gedanken Light Clocks have been used by Leighton [2] and others for certain demonstrations. We apply here to introduce a new concept of virtual covarying scale and the scale conversion as follows:

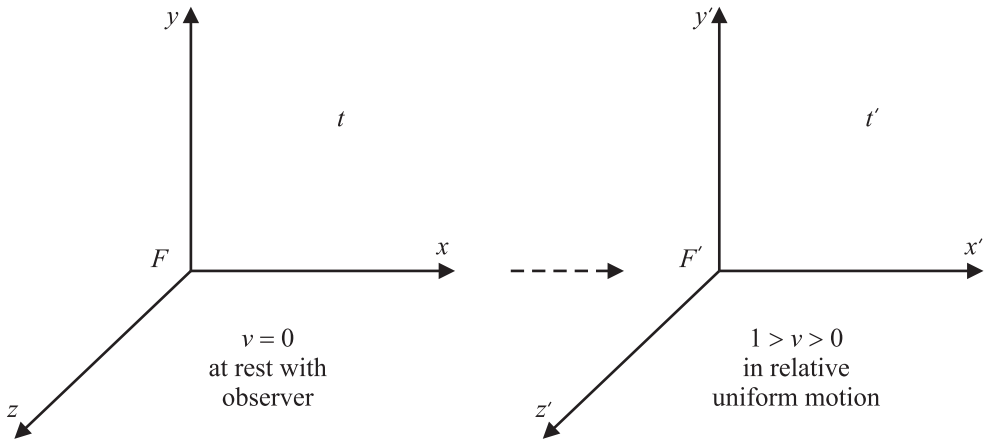

Figure 1. The inertial reference frames in relative uniform motion.

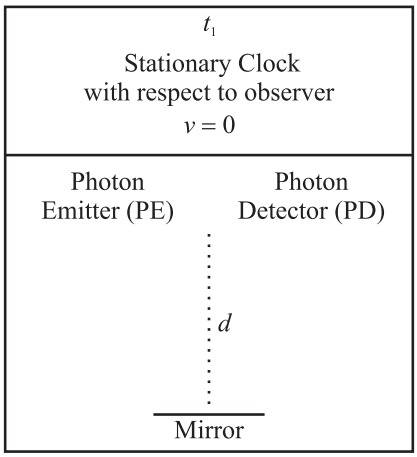

Figure 2. The cycle duration (t_1) of one period for a stationary clock.

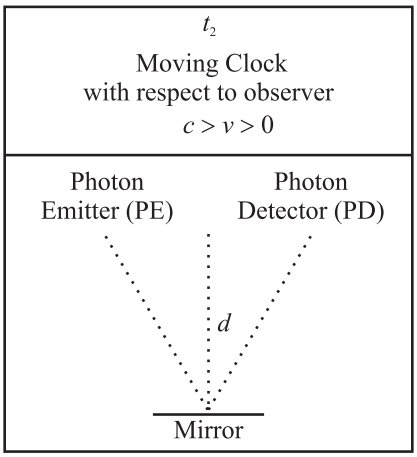

Figure 3. The cycle duration (t_2) of one period for a moving clock.

a) For the clock at rest with respect to the observer: For one period of proper time interval of resting clock cycle (the real invariant scale of the resting clock cycle),

$$t_1 = 2d/c. \tag{1}$$

b) For the uniformly moving clock with respect to the observer: For one period of improper time interval of moving clock cycle (the virtual covarying scale of the moving clock cycle),

$$t_2 = 2d/\left(c^2 - v^2\right)^{1/2}. \tag{2}$$

Thus the ratio between the virtual covarying and the real invariant scales is:

$$t_2/t_1 = \left[1 - (v/c)^2\right]^{-1/2}. \tag{3}$$

This ratio is exactly the Lorentz factor r, derived here in simple terms from comparison of scales of moving and stationary clock period. The Lorentz factor is thus the converter between the covarying and the invariant scales, the scale conversion ratio. The size of invariant real scale is amplified by Lorentz factor to become the virtual covarying scale which covaries with the relative speed automatically. Hence the relativistic time interval measurement is always inseparable from this virtual covarying scale [3], and is reciprocally symmetric to each other

in between relative frames. Therefore, the well known "twin paradox" simply cannot happen because of the necessarily scale conversion.

It is worthwhile to notice that the real time duration is indeed independent of path as Sachs proved mathematically [4]. Lorentz factor is the scale converter, and is the equalizer for real time duration in all reference frames.

3 The spatial interval measurement

The relativistic spatial interval measurement is also inseparable from the virtual covarying scale [3] whose size is amplified by the Lorentz factor automatically. Any fixed linear accelerator for particles can accommodate those particle streams at various energy levels without being built like a trombone physically can demonstrate this automatic scale effect.

A Real Example can be found in the Stanford Linear Accelerator conducted experiments which began in 1966 with the completion of the 3-km-long linear accelerator (Linac), a machine capable of producing an electron beam with an energy up to 20 GeV initially. Experiments directed these electrons onto stationary targets to study the structure of matter. The maximum energy of the Linac was increased over the years to 50 GeV as part of an extensive upgrade required for its use in the Stanford Linear Collider (SLC). The 3-km-long accelerator continues to generate high intensity beams of electrons at the same fixed length.

These Linac experiments demonstrate clearly how a fixed length accelerator can be used to accelerate particle beams to different energy levels, and to accommodate different length contractions virtually without being built like a trombone. The logic behind these experiments was overlooked due to the traditional paradigm of misinterpretation and mismatched reference frame in thinking of material contraction [5, 6]. We now see that there is no material contraction but measurement change due to covarying scale change, and that inseparable relationship between relativistic measurement and covariant scale works naturally.

We now also understand that readings of the length measurement becomes smaller due to amplified covariant scale which covaries automatically with relative speed. The Lorentz factor is automatic conversion ratio between the covarying and the invariant scales.

Consider the kinetic energy K of the accelerated particle:

$$K = mc^2(r-1), \tag{4}$$

where m is the rest mass of particle; c is the speed of light; r is the Lorentz factor.

We compute using equation (4), and list covariant length measurements at various kinetic energy (K.E.) level of particles along with the covarying scale size computed by using the Lorentz factor as scale conversion ratio in the 3 km long Linac:

Table 1. Measurements & scales

(A)	(B)	(C)	(D)
Kinetic energy (in GeV)	Covarying scale size (in real cm)	Covarying length (in virtual cm) (vir. cm) = (real cm) $\cdot\, r$	Real length $(B) \cdot (C)$ (in real km)
50	97848.36	3.07	3
40	78278.89	3.83	3
30	58709.41	5.11	3
20	39139.94	7.66	3
10	19570.47	15.33	3

Why do we perform such numerous simple computations? Because we would like to illustrate the Principle of Inseparability between the relativistic interval measurement and the virtual covarying scale.

It also shows that the product of covarying scale size and the covarying length measurement always equals exactly to the same invariant length (3km) of Linac for each and every kinetic energy level of particle streams inside. If the traditional paradigm of interpretation with material contraction were correct, and with correctly matched reference frame and the necessary condition, then fixed length Linac would not have worked, but a trombone-like design for the tunnel would have been required to accommodate particle beams inside the accelerator at different relative speeds.

The fact that a fixed length on the solid ground works for a linear accelerator helps us see clearly that the virtual trombone works correctly without any material contraction. The accelerator accommodates relativistic lengths for accelerated particle beams at different speeds automatically. The covarying scale is amplified by the Lorentz factor, i.e.,

$$\text{Covarying scale} = \text{Invariant scale} \cdot \text{Lorentz factor}.$$

We are convinced that this logical interpretation is correct. Therefore, it warrants textbook correction of the traditional paradigm of interpretation in which the reference frame and the necessary condition of simultaneity were mismatched.

How lucky the mismatch might have been to provide the working design for Linac by lucky coincidence which is rare by doubling (i.e., the mismatched frame and the misinterpretation) to achieve a good working design and the construction of the accelerator. Thereby, unfortunately, the true logic became hidden. The conundrum of logical interpretation has been lingering.

We hope the understanding of the inseparability principle and correction of the traditional paradigm of interpretation will help resolve the conundrum of relativistic length measurement. The logical interpretation is now possible by the inseparable relationship between the relativistic measurement and the virtual covarying scale without any material contraction.

4 Conclusion

The symmetry with scale conversion for the virtual covarying scale and the real invariant scales between inertial frames in relative uniform motion provides the beauty of logical simplicity and a satisfactory interpretation of the relativistic measurement free from any paradox or any logical conundrum, and thereby readily reconfirms the theory.

More details and discussion on the subject of this paper are available at the following webpage `http://www.geocities.com/astronomer.geo/QNA.HTM`.

Acknowledgements

I am grateful for discussions with many friends, especially with Professor R.W. Nicholls of York University, Ontario, Canada.

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New Type of Exact Solvability and of a Hidden Nonlinear Dynamical Symmetry in Anharmonic Oscillators

Miloslav ZNOJIL † *and Denis YANOVICH* ‡

† *Ústav jaderné fyziky AV ČR, 250 68 Řež, Czech Republic*
E-mail: *znojil@ujf.cas.cz*

‡ *Lab. Inf. Tech., Joint Inst. Nucl. Research (JINR) 141980 Dubna, Moscow Region, Russia*
E-mail: *yan@jinr.ru*

Schrödinger bound-state problem in D dimensions is considered for a set of central polynomial potentials containing $2q$ arbitrary coupling constants. Its polynomial (harmonic-oscillator-like, quasi-exact, terminating) bound-state solutions of degree N are sought at an $(q+1)$-plet of exceptional couplings/energies, the values of which comply with (the same number of) termination conditions. We revealed certain hidden regularities in these coupled polynomial equations and in their roots. A particularly impressive simplification of their pattern occurred at the very large spatial dimensions $D \gg 1$ where all the "multi-spectra" of exceptional couplings/energies proved equidistant. In this way, one generalizes one of the key features of the elementary harmonic oscillators to (presumably, all) non-vanishing integers $q > 0$.

1 Introduction: quasi-exact terminating solutions

The never-ending story of the search for exact bound-state solutions started with the very emergence of quantum mechanics. Its part which pays attention to the polynomial central potentials $V(r)$ in the ordinary differential "radial" Schrödinger equation

$$-\psi''(r) + \frac{\ell(\ell+1)}{r^2}\,\psi(r) + V(r)\,\psi(r) = E\,\psi(r), \qquad \psi \in L_2(0,\infty) \tag{1}$$

is not much younger. Indeed, the elementary nineteen-century mathematics proves sufficient for the construction of $\psi(r)$ from (1) in analytic form with, say, a power-series ansatz for components $A(r)$ and $B(r)$ in

$$\psi(r) = r^{\ell+1}\,A(r)\,e^{B(r)}. \tag{2}$$

The most common harmonic-oscillator model $V^{(\mathrm{HO})}(r) = \omega^2 r^2$ provides a particularly appealing illustration of such an approach because the semiclassical exponent $B^{(\mathrm{HO})}(r) = -\frac{1}{2}\omega\,r^2$ in equation (2) describes the correct asymptotic decrease of $\psi^{(\mathrm{HO})}(r)$ while the Taylor series for $A^{(\mathrm{HO})}(r)$ degenerates to a polynomial as well.

A broad family of polynomial potentials admits a similar specification of their "asymptotically optimal" polynomial exponents $B(r)$. *Vice versa*, for all the "canonical" polynomial WKB-like exponents

$$B^{(\mathrm{WKB})}(r) = \frac{1}{2}\alpha_0 r^2 + \frac{1}{4}\alpha_1 r^4 + \cdots + \frac{1}{2q+2}\alpha_q r^{2q+2} \tag{3}$$

and for all the "canonical" power-series choices of the ansatz (2),

$$\psi(r) = \sum_{n=0}^{\infty} h_n\,r^{2n+\ell+1}\exp\left[-B_{\mathrm{WKB}}(r)\right] \tag{4}$$

potentials may be polynomials with $2q+1$ *arbitrary* couplings,

$$V(r) = V^{[q]}(r) = g_0\, r^2 + g_1\, r^4 + \cdots + g_{2q}\, r^{4q+2} = \left[\Omega^{(q)}(r)\right]^2 r^2 + S^{(q)}(r). \tag{5}$$

The first, asymptotically dominating auxiliary factor

$$\Omega^{(q)}(r) = \alpha_0 + \alpha_1 r^2 + \cdots + \alpha_q\, r^{2q} \tag{6}$$

is determined precisely by the $(q+1)$-plet of the WKB-related free parameters while

$$S^{(q)}(r) = G_0 r^2 + G_1 r^4 + \cdots + G_{q-1} r^{2q}$$

carries just the asymptotically less relevant information about the full force $V(r)$ at any $q \geq 1$. All the relevant details may be found in our older review of the related, so called Hill-determinant bound-state method [1].

Due to the one-to-one correspondence $g_{2q} = {\alpha_q}^2$, $g_{2q-1} = g_{2q-1}(\alpha_q, \alpha_{q-1}) = 2\,\alpha_{q-1}\,\alpha_q$, ... etc (or, in opposite direction, $\alpha_q = \sqrt{g_{2q}} > 0$, $\alpha_{q-1} = g_{2q-1}/(2\alpha_q)$ etc), we may work with both the old and new couplings. Moreover, using the trivial changes of variables in our differential equation (1) ($r^2 = x$ etc, with all details described again thoroughly in the above-mentioned review [1]), the canonical potential (5) generates the whole series of its mathematical equivalents,

$$U^{[q]}(x) = f_0\, x^{-1} + f_1\, x + f_2\, x^2 + \cdots + f_{2q}\, x^{2q}, \qquad f_{2q} > 0, \tag{7}$$

$$W^{[q]}(z) = h_0\, z^{-3/2} + h_1\, z^{-1} + \cdots + h_{2q-1}\, z^{q-3/2} + h_{2q}\, z^{q-1}, \qquad h_{2q} > 0, \tag{8}$$

etc. Thus, the well known one-to-one mapping between harmonic oscillator and Coulombic spectra of bound states exemplifies the transition from (5) to (7) at $q = 0$. Similarly, we shall not distinguish, at any $q \geq 0$, between the wave functions pertaining to the symmetric well (5) and to its descendants (7) or (8).

Returning to the simplest $q = 0$ models, let us emphasize that they are extremely exceptional, possessing

- *all* their wave functions in terminating Taylor-series form (note that their factors $A^{(\mathrm{HO})}(r)$ are Laguerre *polynomials*);
- *all* their energies in closed form (note that the HO set forms an *equidistant* family).

As a consequence, one should not be surprised by the existence of numerous symmetries (and even supersymmetries [2]) in the underlying Hamiltonians at $q = 0$.

At $q \neq 1$, many (though not all) of these symmetries become hidden or lost (see the monograph [3] for a wealth of details). Still, one reveals that the exceptional polynomial solutions exist in the form

$$\psi(r) = \sum_{n=0}^{N-1} h_n^{(N)}\, r^{2n+\ell+1} \exp\left[-B_{\mathrm{WKB}}(r)\right] \tag{9}$$

at all the finite integers $N \geq 1$ and $q \geq 1$ (see the review of this point in our recent paper [4]).

2 Schrödinger equation at large ℓ

For our canonical potential (5), the use of the quasi-exact solution ansatz (9) converts the differential equation (1) in algebraic recurrences

$$\begin{pmatrix}
B_0 & C_0 & & & & & \\
A_1^{(1)} & B_1 & C_1 & & & & \\
\vdots & & \ddots & \ddots & & & \\
A_q^{(q)} & \dots & A_q^{(1)} & B_q & C_q & & \\
 & A_{q+1}^{(q)} & \dots & A_{q+1}^{(1)} & B_{q+1} & C_{q+1} & \\
 & & \ddots & & & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
h_0^{(N)} \\ h_1^{(N)} \\ \vdots \\ h_{N-1}^{(N)} \\ 0 \\ \vdots
\end{pmatrix} = 0 \tag{10}$$

with coefficients

$$
\begin{aligned}
&C_n = (2n+2)\,(2n+2\ell+3), \qquad B_n = E - \alpha_0\,(4n+2\ell+3),\\
&A_n^{(1)} = -\alpha_1\,(4n+2\ell+1) + \alpha_0^2 - g_0, \qquad A_n^{(2)} = -\alpha_2\,(4n+2\ell-1) + 2\alpha_0\alpha_1 - g_1,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots ,\\
&A_n^{(q)} = -\alpha_q\,(4n+2\ell+3-2q) + (\alpha_0\alpha_{q-1} + \alpha_1\alpha_{q-2} + \cdots + \alpha_{q-1}\alpha_0) - g_{q-1},\\
&n = 0, 1, \dots .
\end{aligned}
\tag{11}
$$

They form a finite set of algebraic equations which can hardly be solved non-numerically at the generic q and N. In most cases, people only pay attention to their very first "square-matrix" special case at $q = 1$ [3].

In what follows, let us admit an arbitrary pair of integers q and N and, for simplification, accept merely the assumption that the spatial dimension D is very large. In the other words, on the basis of the well known formula

$$
\ell = \frac{D-3}{2}, \frac{D-1}{2}, \frac{D+1}{2}, \frac{D+3}{2}, \dots
\tag{12}
$$

we postulate that these numbers are *all* very large, $\ell \gg 1$. This is a key assumption of our forthcoming considerations, inspired by the well known fact that for *any* potential $V(r)$, the practical solution of radial Schrödinger equations is easier in the domain of the large angular momenta (a deeper explanation may be found, say, in the randomly selected paper [5] or in many other relevant papers with citations listed therein).

3 Terminating solutions at large ℓ

In our present very specific context of the incomplete exact solvability, we should not be misled by the observation that virtually all the contemporary $\ell \gg 1$ calculations are based on the perturbation expansions using the "most natural" artificial expansion parameter $1/\ell$. Rather, we shall follow our older paper [6] (on the $q = 2$ partial solvability) as our most relevant guidance in what follows, having in mind the use of a generalized expansion parameter $1/\ell^{\text{const}}$.

In its spirit, our first step will consist in a re-scaling of our over-complete linear set (10), $Q(E)h = 0$, in accord with the simple rule

$$
h_n^{(N)} = p_n/\mu^n, \qquad \mu = \mu(D) = \left(\frac{D}{2\alpha_q}\right)^{1/(q+1)} .
\tag{13}
$$

In this way, all the elements of our non-square band-matrix "Hamiltonian" $Q(E)$ become tremendously simplified in the leading order in $D \gg 1$. In effect [4], we then have to solve the much easier algebraic problem with N columns and $N + q - 1$ rows,

$$
\left(
\begin{array}{cccccc}
s_1 & 1 & & & & \\
s_2 & s_1 & 2 & & & \\
\vdots & & \ddots & \ddots & & \\
s_q & \vdots & & s_1 & N-2 & \\
N-1 & s_q & & & s_1 & N-1 \\
 & N-2 & s_q & & \vdots & s_1 \\
 & & \ddots & \ddots & & \vdots \\
 & & & 2 & s_q & s_{q-1} \\
 & & & & 1 & s_q
\end{array}
\right)
\left(
\begin{array}{c}
p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1}
\end{array}
\right) = 0,
\tag{14}
$$

where we merely re-scaled the energy E ($= -g_{-1}$) and couplings $\{g_0, \ldots, g_{q-2}\}$ in linear manner,

$$g_{k-2} = -\alpha_{k-1} D - \frac{\tau}{\mu^{k-1}}\, s_k, \qquad k = 1, 2, \ldots, q, \qquad \tau = \left(2^{q+2}\, D^q\, \alpha_q\right)^{1/(q+1)}. \tag{15}$$

At this stage of development, one does not see any perceivable progress yet. The required "multi-spectrum" of q different "multi-eigenvalues" $s_1, \ldots, s_q$ seems obtainable only by purely numerical means at all the larger q or N.

4 A brief summary of the known non-numerical results

Besides the well known non-numerical $q = 0$ solutions of the harmonic oscillator, also the first nontrivial $q = 1$ case does not need to be discussed too thoroughly. One just solves the *linear algebraic* eigenvalue problem with spectrum which proves equidistant in the limit $D \to \infty$ (see [7]). For inspiration, let us briefly return to this $q = 1$ model in more detail: In equation (14), the unknown quantities $s = s_1$ represent either the energies of the sextic oscillator of equation (5) or, *mutatis mutandis*, the charges of the spiked and shifted harmonic oscillator [3], the values of which form a finite set,

$$s_1 = N-1, N-3, N-5, \ldots, -N+3, -N+1, \qquad q = 1. \tag{16}$$

We may also very quickly recollect the next $q = 2$ case where the solution of our problem has been found and discussed thoroughly and with direct reference to the quartic oscillator potential (7) in 1999 [6]. The first nontrivial form of our equation (14) has been solved there in closed form for so many values of N that the results could be extrapolated to all $N = 1, 2, \ldots$. In particular, the resulting energies were shown there to form the multiplets

$$\begin{aligned} &s_1 = s_2 = N-1, N-4, N-7, \ldots, -K+2, \qquad N = 2K, \\ &s_1 = s_2 = N-1, N-4, N-7, \ldots, -K, \qquad N = 2K+1, \qquad q = 2 \end{aligned} \tag{17}$$

i.e., $s = N+2-3j$, $j = 1, 2, \ldots, [(N+1)/2]$ at any wave-function degree $N = 1, 2, \ldots$. In the next step of development, an "optimal" calculation method has been discovered in our subsequent study in 2003 [4]. There, we succeeded in the re-interpretation and re-calculation of all the above $q = 2$ energies as special *real* roots selected out of "hidden-symmetric" complex triplets $E_m = s^{1/3} e^{2\pi m/3}$, $m = 1, 2, 3$. The intermediate, auxiliary variables s were produced again, numerically, as roots of a set of polynomials $s^6 - 7s^3 - 8 = 0$ ($N = 3$), $s^{10} - 27s^7 + 27s^4 - 729s = 0$ ($N - 4$) etc.

In the same paper, the complete solution of the next problem with $q = 3$ has been offered. The very similar sequence of the secular polynomials $F[s^4]$ has been obtained there, with $F[s^4] = s^9 - 12s^5 - 64s = 0$ at $N = 3$, with $F[s^4] = s^{16} - 68s^{12} + \cdots + 50625 = 0$ at $N = 4$ etc. This gives the result

$$s_2 = N-1, N-5, N-9, \ldots, N+3-4\left[\frac{N+1}{2}\right], \qquad q = 3, \tag{18}$$

the presentation of which proves hindered by the occurrence of the other two independent eigenvalues s_1 and s_3. Incidentally, the latter quantities coincide and may be specified by a closed formula. For our present purposes it is sufficient to elucidate the N-dependence of the

resulting multi-spectrum via its first few examples,

$$\left|\begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array}\right| = \left|\begin{array}{rrr} -2 & 0 & 2 \\ 2 & 2 & -2 \\ -2 & 0 & 2 \end{array}\right|, \qquad N=3, \tag{19}$$

$$\left|\begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array}\right| = \left|\begin{array}{rrrrr} -3 & -1 & 1 & 3 & -1 & 1 \\ 3 & 3 & 3 & 3 & -1 & -1 \\ -3 & -1 & 1 & 3 & -1 & 1 \end{array}\right|, \qquad N=4, \tag{20}$$

$$\left|\begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array}\right| = \left|\begin{array}{rrrrrrrrr} -4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 0 \\ 4 & 4 & 4 & 4 & 4 & 0 & 0 & 0 & -4 \\ -4 & -2 & 0 & 2 & 4 & -2 & 0 & 2 & 0 \end{array}\right|, \qquad N=5 \tag{21}$$

etc. As long as we solved equation (14) for a sufficiently long series of "dimensions" N, we succeeded in determining the general, extrapolated pattern for s. The paper itself should be consulted for more details since the latter spectrum proves to have an impressively compact representation in integer arithmetic, with $s_2 = N+3-4j$, $j = 1, 2, \ldots, [(N+1)/2]$ etc.

At the time of its derivation, this feature looked ephemeral as definitely failing to hold at the next degree $q = 4$ of the potential. At the same time, although the computer-assisted solution of equation (14) ceased to be feasible, the $q = 4$ problem looked extremely interesting as long as it involves not only a less appealing polynomial of the symmetric well (5) of the eighteen degree, but also much more interesting octic-polynomial anharmonic oscillator (7) and, first of all, the phenomenologically most important case of the asymptotically cubic force (8).

5 Brand new result: The case of $q = 4$

Before a thorough description of our present continuation of the systematic and efficiently computerized symbolic-manipulation study we should re-emphasize that at any $q \geq 2$, our algebraic set of $N+q-1$ equations (14) is nonlinear. It is formed by the sums of the one- and two-term products of the $N+q$ unknown quantities. In the latter role we selected the $N-1$ arbitrarily normalized Taylor coefficients p_j and the q multi-eigenvalues $s_1, s_2, \ldots, s_q$.

In order to convey the feeling of what happens when one chooses the different strategies of the usual elimination, we may start experimenting at $N = 1$ an find that the only real solution is trivial, $s_1 = s_2 = s_3 = s_4 = 0$. The solution at $N = 2$ is also unambiguous. Once we abbreviate $s_1 = t$, $s_2 = r$, $s_3 = \tilde{r}$, $s_4 = \tilde{t}$, we may fix the norm by setting $p_1 = 1$ and proceed, recurrently, in an upwards direction in (14). This gives $p_0 = -\tilde{t}$ while $\tilde{t}^5 = 1$, providing finally the unique real root $\tilde{t} = 1$ and, subsequently, full solution with $\tilde{r} = r = t = 1$.

We have seen in ref. [8] that the similar construction is also feasible at $N = 3$. Proceeding in an upward-downward-symmetric recurrent manner we now normalize $p_1 = 1$ and infer that $p_0 = -1/t$ while $p_2 = -1/\tilde{t}$. Next we abbreviate $t\,\tilde{t} = \xi$ and re-write the remaining four lines of equation (14) in the following form,

$$r\tilde{t}/t = \xi - 2, \qquad \tilde{r}\tilde{t}/t^2 = \xi - 3, \qquad rt/\tilde{t}^2 = \xi - 3, \qquad \tilde{r}t/\tilde{t} = \xi - 2. \tag{22}$$

The ratio of the two odd or two even lines eliminates r or $\tilde{r}$, respectively, and we get the same quantity $(\xi-2)/(\xi-3)$. Its next-step elimination gives the desired simplification $\tilde{t}^5 = t^5$, with the only real solution $\tilde{t} = t$. Then the first and last line of equation (22) define easily $r = r(\xi)$ and $\tilde{r} = \tilde{r}(\xi)$ while, finally, the appropriate insertions in one of the middle lines results in the "secular" equation

$$\mathcal{P}(t) = t^3 - t^2 - 3\,t + 2 = 0 \tag{23}$$

with the following three real roots,

$$t_1 = 2, \qquad t_{2,3} = \frac{1}{2}\left(-1 \pm \sqrt{5}\right) \tag{24}$$

(cf. also Table 1 below).

Table 1. Columns of energy-roots $s_4 = s_4(N) = \frac{1}{2}\left(P_N \pm \sqrt{5}\cdot Q\right)$ at $q = 4$.

	P_N													Q
$N =$	1	2	3	4	5	6	7	8	9	10	11	12	...	*all N*
	0	2	4	6	8 −2	10 0	12 2	14 4	16 6 −4	18 8 −2	20 10 0	22 12 2	... ...	0 0 0 ⋮
			−1	1	3	5	7 −3	9 −1	11 1	13 3	15 5 −5	17 7 −3	... ...	1 1 1
					−2	0	2	4	6 −4	8 −2	10 0	12 2	... ...	2 2 ⋮
							−3	−1	1	3	5 −5	7 −3	... ...	3 3
									−4	−2	0	2	... ...	4 ⋮
											−5	−3	...	5
													...	⋮
total #	1	1	3	3	6	6	10	10	15	15	21	21	...	⋮

The elimination of the unknowns becomes almost prohibitively tedious from $N = 5$ on. The comparatively high complexity of the (necessarily, computerized) reduction of our multi-polynomial problem (14) to the single polynomial "secular" equation $\mathcal{P}(s) = 0$ is accompanied by an extremely quick growth of the degree of our secular polynomials with N. At the same time, there exists an empirically observed fact [4] that, paradoxically, the Gröbner-based solution of the next $q = 5$ problem is in fact more easy than its $q = 4$ predecessor. This underlines the key importance of the revealed "missing pattern" in the $q = 4$ roots as presented here in Table 1.

For compensation, the impression produced by the high degree of our polynomials $\mathcal{P}(s)$ is again strongly weakened when we notice that these functions depend in effect just on the powers of the new auxiliary variable $z = s^{q+1}$. This has several consequences. Firstly, we see that even if all the auxiliary roots z themselves were real, the final number of the complex roots s would still be much higher than that of their real and, hence, "physically acceptable" partners. Secondly,

the formidable task of the search for the real roots in the closed form did not prove to be as prohibitively difficult as it might have appeared at first sight.

In Table 1 summarizing the results of our $q = 4$ construction, a climax of our present effort is perceived in an absolute regularity of all its items. The pattern of extrapolation of these results beyond their boundaries set by the computer is already fully obvious,

$$
\begin{aligned}
&s = s_4(N) = \frac{1}{2}\left(P_N \pm \sqrt{5}\cdot Q\right),\\
&P(N) = P(N)_{(j,k)} = 2N + 13 - 5j - 10k, \qquad Q = Q_{(j,k)} = j - 1,\\
&j, k = 1, 2, \ldots, \qquad 2j + 4k \le N + 5
\end{aligned}
\tag{25}
$$

and does not seem to create any doubts and/or unanswered questions. With respect to the non-doubling of the $j = 1$ (i.e., $Q = 0$) roots, the elementary formula

$$
\text{total}\,\# = \left(\begin{array}{c} K+1 \\ 2 \end{array}\right), \qquad K = \left[\frac{N+1}{2}\right], \tag{26}
$$

also expresses the total number of the separate items in each column of Table 1, i.e., of the real energy roots at each fixed N.

6 Summary and outlook

We reported the progress achieved in the field where the quasi-exact solutions are sought for the radial equations where the potentials are "next-to-most-common". Our main result is that we were able to construct the energies for the class of the $q = 4$ models which involves the important and very popular cubic and octic anharmonic oscillators.

Our main task lied in the necessity of making the form of our exact and polynomial wave functions $\psi(r)$ closed and explicit for *all* their integer degrees $N = 1, 2, \ldots$. The main difficulty in this direction emerges from the fact that the dimensions of the matrices we need (or degrees $\mathcal{N}(N)$ of the "effective" secular polynomials) seem to grow extremely quickly with N. Unfortunately, we did not find any regularity in the series $\mathcal{N}(5) = 70$, $\mathcal{N}(6) = 126$, $\mathcal{N}(7) = 210$, $\mathcal{N}(8) = 330$, $\mathcal{N}(9) = 495$, $\mathcal{N}(10) = 715$ etc.

Our task was quite challenging formally, and we must admit that we did not even expect that its solution could appear very soon. Our biggest surprise occurred in the form of the explicit factorizability of all the polynomials $\mathcal{P}(s)$ over the (sometimes called "surdic") field of the quasi-complex numbers $a + b\sqrt{5}$ with rational coefficients.

One cannot resist to re-emphasize here that after a certain suitable re-numbering and re-grouping of levels, the spectrum of our "solvable" $q = 4$ couplings/energies remains expressible directly in terms of integers. Such a type of a generalized equidistance re-emerges also in the next, $q = 5$ case (i.e., for the class of potentials involving the square-root-power-series form of the quartic oscillator, etc). The analysis of $q = 5$ already lies beyond the scope of our present study. Even in the purely formal setting, it lies on the very boundary of the capacity of the computers and software which are at our disposal at present. We were still able to factorize the corresponding effective secular polynomials at a few N in ref. [4], and we obtained the regular recipe $s_5 = N - 1, N - 2, N - 3, \ldots, -N + 1$ there. One feels how this achievement was formidable since at $N = 7$, the extreme coefficient c in the secular polynomial $F(s) = s^{127} - 60071s^{121} + \cdots + cs$ possesses as many as 72 decimal digits and, hence, looks like a candidate for being placed in the Guiness' book of records in the factorization context.

In conclusion, let us point out that our results sample a nice mathematics in interplay with a useful physics. Thus, in physics, the equidistance and representation of the energies in integer arithmetics in the $D \to \infty$ limit will enable us to work, in any "realistic" dimension $D < \infty$,

with perturbation theory *without rounding errors*. In mathematics, the ease of the factorization of polynomials almost certainly reflects a hidden symmetry of the Schrödinger equation, but in the light of the nonlinearity of its present "algebraization", we still do not dare to predict any form of its possible "explicit manifestation" in the future.

Besides that "new horizon", let us also stress once more that our present study has been motivated by the disturbing paradox (revealed in [8]) that "phenomenologically the simplest" cubic oscillator (such that $V(x) \approx x^3$ for $x \gg 1$) belongs, in terms of mathematics, among "the most difficult" examples when its incomplete but exact $D \gg 1$ solvability is concerned. In this sense, we described here a resolution of this paradox, showing that the existence of the elementary and exact wave functions $\psi(x)$ in the large-D regime *and for any degree* N is admitted not only by the standard anharmonic Schrödinger equation with $q = 2$ (involving the quartic potentials) but also by its cubic analogue with $q = 4$.

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Dynamical Symmetries and Well-Localized Hydrogenic Wave Packets

Vladimir ZVEREV † *and Boris RUBINSTEIN* ‡

† *Ural State Technical University, 620002 Ekaterinburg, Russia*
E-mail: *zverev@dpt.ustu.ru*

‡ *Department of Mathematics, University of California, Davis, CA 95616, USA*
E-mail: *boris@math.ucdavis.edu*

A new method for constructing of composite coherent states of the hydrogen atom, based on the dynamical group approach and various schemes of reduction to subgroups, is presented. A wide class of well-localized (Gaussian) hydrogenic wave packets for circular and elliptic orbits is found using the saddle-point method.

1 Introduction

In recent years, new experimental techniques opened way to creation and study of high energy (Rydberg) states in atoms. These states are described by approximate hydrogenic wave functions with very large principal quantum numbers. Some new effects, as the dynamical localization and the dynamical chaos, have attracted considerable interest. Explanation of these phenomena uses classical equations of motion [1]. It is reasonable to look for an alternative quantum description on the basis of semi-classical approximations, which is naturally provided by a coherent states (CS) formalism.

In Section 2, starting from the $\mathrm{O}(4,2)$ dynamical group approach [2] and using three schemes of reduction to subgroups [3]: $\mathrm{O}(4,2) \supset \mathrm{O}(4) \sim \mathrm{O}(3) \otimes \mathrm{O}(3)$, $\mathrm{O}(4,2) \supset \mathrm{O}(2,2) \sim \mathrm{O}(2,1) \otimes \mathrm{O}(2,1)$, $\mathrm{O}(4,2) \supset \mathrm{O}(3) \otimes \mathrm{O}(2,1)$, we construct composite CS in physical and auxiliary ("tilted") representations [4]. We use two types of generating operators of CS with different procedures of transition to a classical limit. In particular, the generating operators for Perelomov $\mathrm{SO}(3)$ and $\mathrm{SO}(2,1)$ CS [5], Barut–Girardello $\mathrm{SO}(2,1)$ CS [2,6], generalized hypergeometric CS [7], Brif $\mathrm{SO}(3)$ and $\mathrm{SO}(2,1)$ algebra eigenstates [8], may be used for this purpose. The CS are separated into *two classes* with different semi-classical behavior.

The hydrogenic CS wave functions have a complicated form, so it is reasonable to use simplified asymptotic expressions. In Section 3 we describe a method for asymptotic estimate and obtain well-localized hydrogenic wave packets for circular and elliptic orbits. A similar asymptotic estimate method is used in the theory of CS path integrals. We believe that the approach discussed in this paper can be applied to computation of the CS path integrals for the hydrogen atom and other systems with known dynamical symmetry.

2 The hydrogen atom: O(4, 2) dynamical group and reductions to subgroups

Denote generators of the dynamical group $\mathrm{O}(4,2)$ of the hydrogen atom [2, 4] (the group of rotations in six-dimensional pseudo-Euclidean space with a metric $g = \mathrm{diag}(1,1,1,1,-1,-1)$) as $\mathcal{L}_{\alpha\beta}$ (in the notation used in [3]):

$$\mathcal{L}_{\alpha 0} = (\boldsymbol{r} \times \boldsymbol{p})_\alpha, \qquad \mathcal{L}_{\alpha 1} = \left(r_\alpha p^2 - 2p_\alpha \boldsymbol{r}\boldsymbol{p} + r_\alpha\right)/2,$$
$$\mathcal{L}_{\alpha 2} = -r p_\alpha, \qquad \mathcal{L}_{\alpha 3} = \left(r_\alpha p^2 - 2p_\alpha \boldsymbol{r}\boldsymbol{p} - r_\alpha\right)/2, \tag{1}$$

$$\mathcal{L}_{01} = \left(rp^2 - r\right)/2, \qquad \mathcal{L}_{02} = \boldsymbol{rp} - i, \qquad \mathcal{L}_{03} = \left(rp^2 + r\right)/2.$$

The Schrödinger equation for the hydrogen atom reads $(\mathcal{H} - E)|\Psi\rangle = 0$ with $\mathcal{H} = p^2/2 - a/r$. After multiplication by r it can be rewritten in terms of the generators (1) [4, 9, 10]:

$$r(\mathcal{H} - E)|\Psi\rangle = \{(\mathcal{L}_{03} + \mathcal{L}_{01})/2 - E(\mathcal{L}_{03} - \mathcal{L}_{01}) - a\}|\Psi\rangle = 0. \tag{2}$$

The unitary transformation

$$|\overline{\Psi}\rangle = \exp(i\theta\mathcal{L}_{02})|\Psi\rangle, \qquad \tanh\theta = (1 + 2E)/(1 - 2E),$$

reduces (2) to

$$\left(\mathcal{L}_{03} - a/\sqrt{-2E}\,\right)|\overline{\Psi}\rangle = 0.$$

The eigenstates of $\mathcal{L}_{03}$ may be chosen in this *auxiliary representation* (AR) as $|\overline{n, l, m}\rangle_{\rm sph}$ or $|\overline{n, n_1, n_2, m}\rangle_{\rm par}$ with $n = a/\sqrt{-2E}$, in the spherical or parabolic coordinates. An inverse transformation

$$\left\{ \begin{array}{l} |n, l, m\rangle_{\rm sph} \\ |n, n_1, n_2, m\rangle_{\rm par} \end{array} \right\} = \exp(-i\theta\mathcal{L}_{02}) \left\{ \begin{array}{l} |\overline{n, l, m}\rangle_{\rm sph} \\ |\overline{n, n_1, n_2, m}\rangle_{\rm par} \end{array} \right\},$$

produces the *physical representation* (PR) of the eigenvectors.

Consider three types of reduction to subgroups and corresponding Lie algebras:

$$\begin{array}{ll} (a) & \mathrm{O}(4,2) \supset \mathrm{O}(4) \sim \mathrm{O}(3) \otimes \mathrm{O}(3), \quad \mathrm{o}(3) \oplus \mathrm{o}(3) = \{\mathcal{P}_\alpha^{(+)}\} \oplus \{\mathcal{P}_\alpha^{(-)}\}, \\ (b) & \mathrm{O}(4,2) \supset \mathrm{O}(2,2) \sim \mathrm{O}(2,1) \otimes \mathrm{O}(2,1), \quad \mathrm{o}(2,1) \oplus \mathrm{o}(2,1) = \{\mathcal{Q}_\alpha^{(+)}\} \oplus \{\mathcal{Q}_\alpha^{(-)}\}, \\ (c) & \mathrm{O}(4,2) \supset \mathrm{O}(3) \otimes \mathrm{O}(2,1), \quad \mathrm{o}(3) \oplus \mathrm{o}(2,1) = \{\mathcal{L}_{\alpha 0}\} \oplus \{\mathcal{L}_{0\alpha}\}, \end{array} \tag{3}$$

where

$$\mathcal{P}_\alpha^{(\pm)} = (\mathcal{L}_{\alpha 0} \pm \mathcal{L}_{\alpha 3})/2, \qquad \mathcal{Q}_\alpha^{(\pm)} = (\mathcal{L}_{0\alpha} \pm \mathcal{L}_{3\alpha})/2.$$

For these cases the space R of bound states of the hydrogen atom is decomposed as follows:

$$\begin{array}{ll} (a) & R = \oplus_{n\geq 1} \left\{ R(^{\mathbf{O(3)}}n - 1, (n_1)) \otimes R(^{\mathbf{O(3)}}n - 1, (n_2)) \right\}, \\ (b) & R = \oplus_{m\in Z} \left\{ R(^{\mathbf{O(2,1)}}1 + |m|, (n_1 + \tfrac{1}{2}(m - |m|))) \right. \\ & \qquad \left. \otimes R(^{\mathbf{O(2,1)}}1 + |m|, (n_2 + \tfrac{1}{2}(m - |m|))) \right\}, \\ (c) & R = \oplus_{l\geq 0} \left\{ R(^{\mathbf{O(3)}}2l, (l - m)) \otimes R(^{\mathbf{O(2,1)}}2l + 2, (n - 1 - l)) \right\}, \end{array}$$

where $R(^{\mathbf{G}}S, (K)) = \left\{|^{\mathbf{G}}S, (K)\rangle\right\}$ denotes the irreducible representation space of a group $G = \mathrm{O}(3)$ or $\mathrm{O}(2,1)$ with generators $\{E_\alpha\}$, for which we have

$$\begin{array}{l} E_3|^{\mathbf{G}}S, (K)\rangle = (S/2 - \epsilon_{\mathbf{G}}K)|^{\mathbf{G}}S, (K)\rangle, \\ (E_1^2 + E_2^2 + \epsilon_{\mathbf{G}}E_3^2)|^{\mathbf{G}}S, (K)\rangle = (S/2 - \epsilon_{\mathbf{G}}S^2/4)|^{\mathbf{G}}S, (K)\rangle, \end{array}$$

where $\epsilon_{\mathbf{O(3)}} = 1$, $\epsilon_{\mathrm{O}(2,1)} = -1$, and the representation space $R(^{\mathbf{G}}S, (K))$ is spanned by the set of orthonormal eigenvectors $|^{\mathbf{G}}S, (K)\rangle$. The vector subspaces

$$\begin{array}{l} R_0^{[+]} = \{|n, n-1, n-1\rangle_{\rm sph}\} = \{|n, 0, 0, n-1\rangle_{\rm par}\}, \\ R_0^{[-]} = \{|n, n-1, -n+1\rangle_{\rm sph}\} = \{|n, n-1, n-1, -n+1\rangle_{\rm par}\}, \end{array}$$

with maximal $m = n - 1$ or minimal $m = -n + 1$ values of the magnetic quantum number correspond to the circular orbits [11]. The vectors from $R_0^{[\pm]}$ play the role of reference vectors in decompositions of R. On the other hand, $R_0^{[-]}$ and $R_0^{[+]}$ are irreducible spaces of $\mathrm{O}(2,1)$ ($S = 1$). The corresponding $\mathrm{O}(2,1)$ Lie algebras have the form

$$\mathcal{H}_1^{[\pm]} = (\pm\mathcal{L}_{11} - \mathcal{L}_{22})/2, \qquad \mathcal{H}_2^{[\pm]} = (\mathcal{L}_{21} \pm \mathcal{L}_{12})/2, \qquad \mathcal{H}_3^{[\pm]} = (\mathcal{L}_{03} \pm \mathcal{L}_{30})/2,$$

where $\pm$ corresponds to the subspace $R_0^{[\pm]}$ with $m = \pm(n-1)$. Following [12], we construct coherent states in two stages. At first, using the operators $\{\mathcal{H}_\alpha^{[\pm]}\}$, we construct the circular orbits CS in $R_0^{[\pm]}$. Next, we construct CS in the global space R using one of the generator sets from (3). Each reduction scheme corresponds to a specific construction method of the hydrogen atom CS:

$$\begin{aligned}
&(a)\ \ |{}^{\mathbf{O(4)}}\omega^{[\pm]}, \tau, \eta\rangle = \mathcal{D}_{\mathbf{I}}({}^{\mathbf{O(3)}}\tau, (\mathcal{P}_\alpha^{(+)}))\mathcal{D}_{\mathbf{I}}({}^{\mathbf{O(3)}}\eta, (\mathcal{P}_\alpha^{(-)}))|\omega^{[\pm]}\rangle,\\
&(b)\ \ |{}^{\mathbf{O(2,2)}}\omega^{[\pm]}, \tau, \eta\rangle = \mathcal{D}_{\mathbf{I}}({}^{\mathbf{O(2,1)}}\tau, (\mathcal{Q}_\alpha^{(+)}))\mathcal{D}_{\mathbf{I}}({}^{\mathbf{O(2,1)}}\eta, (\mathcal{Q}_\alpha^{(-)}))|\omega^{[\pm]}\rangle,\\
&(c)\ \ |{}^{\mathbf{O(3)}\otimes\mathbf{O(2,1)}}\omega^{[\pm]}, \tau, \eta\rangle = \mathcal{D}_{\mathbf{I}}({}^{\mathbf{O(3)}}\tau, (\mathcal{L}_{\alpha 0}))\mathcal{D}_{\mathbf{I}}({}^{\mathbf{O(2,1)}}\eta, (\mathcal{L}_{0\alpha}))|\omega^{[\pm]}\rangle,
\end{aligned} \tag{4}$$

where

$$|\omega^{[\pm]}\rangle = \mathcal{D}_{\mathbf{II}}({}^{\mathbf{O(2,1)}}\omega, (\mathcal{H}_\alpha^{[\pm]}))|1,0,0,0\rangle_{\mathrm{par}} \tag{5}$$

is the circular orbit CS. We use uniform notation for the generating operator of CS:

$$|{}^{\mathbf{G}}S, (\omega)\rangle = \mathcal{D}_\Omega({}^{\mathbf{G}}\omega, (E_\alpha))|{}^{\mathbf{G}}S, (0)\rangle,$$

where $G = \mathrm{O}(3)$ or $\mathrm{O}(2,1)$, and $\Omega = \mathbf{I}$ or $\mathbf{II}$. An important point is that one should use two types of generating operators of CS, with different procedures for transition to the classical limit: $S \to \infty$ ($\Omega = \mathbf{I}$); $|\omega| \to \infty$ ($\Omega = \mathbf{II}$).

There exist many algorithms to construct coherent states of dynamical groups and quantum physical systems (see review [13]). In most cases these CS provide a transition to the "classical limit" in some sense. However, a selection of the CS for a specific physical system must be done with great care. It is essential to take into account not only the symmetry of a model, but also the type of semiclassical behavior. We consider some special types of CS and clarify conditions of quantum-classical correspondence.

Usually CS $|\xi\rangle$ is called semiclassical if an overlapping distribution function $|\langle\xi\,|\xi + \delta\xi\rangle|^2$ has a sharp peak for small $|\delta\xi|$, and becomes singular delta-shaped function in the classical limit, which can be written in the form

$$\langle\xi\,|\eta\rangle \overset{\text{cl. lim.}}{\longrightarrow} 0, \quad \xi \neq \eta, \qquad \langle\xi\,|\xi\rangle = 1. \tag{6}$$

Satisfaction of the condition

$$\langle\xi|\,E_\alpha^2\,|\xi\rangle \,/\, (\langle\xi|\,E_\alpha\,|\xi\rangle)^2 \overset{\text{cl. lim.}}{\longrightarrow} 1, \tag{7}$$

where E_α are elements of the Lie algebra L, gives another evidence of semiclassical properties of the CS $|\xi\rangle$. The conditions (6), (7) are satisfied in the cases: the Barut–Girardello $\mathrm{O}(2,1)$ CS, $|\xi| \to \infty$, $L = \mathrm{o}(2,1)$; the Perelomov $\mathrm{O}(2,1)$ CS, $S \to \infty$, $L = \mathrm{o}(2,1)$; the Perelomov $\mathrm{O}(3)$ CS, $S \to \infty$, $L = \mathrm{o}(3)$.

An extended class of semiclassical CS follows from a definition of the *generalized hypergeometric* CS [7]:

$$|\xi\rangle = N_\xi^{-1} \sum_{n=0}^{\infty} \sqrt{\frac{\prod_{i=1}^{p} (\alpha_i)_n}{\prod_{j=1}^{q} (\rho_j)_n}}\, \frac{\xi^n}{\sqrt{n!}}\, |n\rangle, \tag{8}$$

where $(\alpha)_n$ is the Pochhammer symbol and ${}_pF_q$ denotes the generalized hypergeometric function; ξ is a complex argument; integers α_i are negative for $i = 1, \ldots, l$ and positive for $i = l+1, \ldots, p$; $0 \le l \le p$; ρ_j are positive real numbers. The overlapping function for this states has the form:

$$\langle \xi | \zeta \rangle = N_\xi^{-1} N_\zeta^{-1} {}_pF_q((\alpha_p), (\rho_q), (-1)^l \xi^* \zeta), \qquad \langle \xi | \xi \rangle = 1.$$

Starting from the definition (8), let us consider a number of special realizations:

(i) $p = q + 1 \ge 1$, $\rho_j = 1$, $\alpha_j = S$, $S = 1, 2, \ldots$; in the special case $p = 1$ the state (8) is the Perelomov O(2, 1) CS [5];

(ii) $p = q + 1 \ge 1$, $\rho_j = 1$, $\alpha_j = -S$, $S = 1, 2, \ldots$; in the special case $p = 1$ the state (8) is the Perelomov O(3) CS [5];

(iii) $q \ge p$, $\rho_j = 1$, $\alpha_j = S$, $S = 1, 2, \ldots$; in the case $p = q = 0$ the state (8) is the CS of a harmonic oscillator [14]; in the case $p = -1$, $q = 0$ this is the Barut–Girardello CS [6].

It can be shown directly that for the cases (i) and (ii) the requirement (6) is satisfied for the limiting condition $S \to \infty$ ($\Omega = \mathbf{I}$), while for the case (iii) the proper condition is $|\xi| \to \infty$ ($\Omega = \mathbf{II}$).

For comparison, we consider briefly an alternative extension of a family of CS for O(3) and O(2, 1) groups – the Brif's *algebra eigenstates* [8]:

$$(\beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3) \, |\zeta, \beta_1, \beta_2, \beta_3\rangle = \zeta \, |\zeta, \beta_1, \beta_2, \beta_3\rangle \, ,$$

where $|\zeta, \beta_1, \beta_2, \beta_3\rangle$ is a linear superposition of $\left|{}^{\mathbf{G}}S, (K)\right\rangle$ with different K. In the *general* case $b = (\beta_1^2 + \beta_2^2 + \epsilon_{\mathbf{G}} \beta_3^2)^{1/2} \ne 0$ and for $\zeta = (l - \epsilon_{\mathbf{G}} S/2)b$, $l = 0, 1, 2, \ldots$, the condition (6) is satisfied at $S \to \infty$ ($\Omega = \mathbf{I}$). At the same time, in the *degenerate* case $b = 0$ the O(2, 1) algebra eigenstates are semiclassical at the both $S \to \infty$ and $|\zeta| \to \infty$ limiting conditions ($\Omega = \mathbf{I}$ and $\mathbf{II}$).

These examples do not cover all diversity of CS with semiclassical behavior.

3 Semiclassical asymptotics of coherent states wave functions

We use the standard saddle-point method of asymptotic estimate for integrals

$$F(\lambda, \boldsymbol{x}) = \int_c \exp\{\lambda f(\boldsymbol{x}, \boldsymbol{t})\} \phi(\boldsymbol{x}, \boldsymbol{t}) d\boldsymbol{t},$$

where $\lambda \gg 1$, $\boldsymbol{x} = \{x_1, x_2, \ldots, x_n\}$ and $\boldsymbol{t} = \{t_1, t_2, \ldots, t_m\}$; $f(\boldsymbol{x}, \boldsymbol{t})$ and $\phi(\boldsymbol{x}, \boldsymbol{t})$ are holomorphic functions; c is a n-dimensional smooth manifold deformed to reach a minimax (a saddle point is nonsingular). The asymptotic formula reads [15]

$$F(\lambda, \boldsymbol{x}) \sim (2\pi/\lambda)^{m/2} \exp\{\lambda f(\boldsymbol{x}, \boldsymbol{t}_0(\boldsymbol{x}))\} \phi(\boldsymbol{x}, \boldsymbol{t}_0(\boldsymbol{x})) / \sqrt{\det\left[-\mu\left(\boldsymbol{x}, \boldsymbol{t}_0\left(\boldsymbol{x}\right)\right)\right]}, \tag{9}$$

where $\boldsymbol{t} = \boldsymbol{t}_0(\boldsymbol{x})$ is a solution of the system of equations $\partial f / \partial t_i = 0$, $i = 1, \ldots, m$, for a stationary point, and $\mu_{ij} = \partial^2 f / \partial t_i \partial t_j$. We also assume that $|F(\lambda, \boldsymbol{x})|$ has the delta-shaped peak near the point $\boldsymbol{x} = \boldsymbol{x}_{00}$. Since $|F(\lambda, \boldsymbol{x})| \sim |\exp\{\lambda f(\boldsymbol{x}, \boldsymbol{t}_0(\boldsymbol{x}))\}| \sim \exp\{\lambda \mathrm{Re} f(\boldsymbol{x}, \boldsymbol{t}_0(\boldsymbol{x}))\}$, we can find $\boldsymbol{x}_{00}$ as an extremal point of the real part of the exponent. The supplementary equations for the stationary point have the form

$$\frac{\partial f(\boldsymbol{x}, \boldsymbol{t}_0(\boldsymbol{x}))}{\partial x_i} + \text{c.c.} = \frac{\partial f(\boldsymbol{x}, \boldsymbol{t})}{\partial x_i}\Big|_{\boldsymbol{t} = \boldsymbol{t}_0(\boldsymbol{x})} + \text{c.c.} = 0.$$

Substituting the second-order expansion of $f(\boldsymbol{x}, \boldsymbol{t}_0(\boldsymbol{x}))$ at $\boldsymbol{x} = \boldsymbol{x}_{00}$ into (9), we find the desired expression for the asymptotic estimate

$$F(\lambda, \boldsymbol{x}) \sim (2\pi/\lambda)^{m/2} \sqrt{\det\left[-\mu_{00}\right]}^{-1} \phi_{00}$$

$$\times \exp\left\{\lambda f_{00} + i\lambda \boldsymbol{h}^{\mathrm{T}}\delta\boldsymbol{x} + (\lambda/2)\,\delta\boldsymbol{x}^{\mathrm{T}}\left(\sigma_{00} - \gamma_{00}^{\mathrm{T}}\,\mu_{00}^{-1}\,\gamma_{00}\right)\delta\boldsymbol{x}\right\},$$

with

$$\sigma_{00} = \left\{\frac{\partial^2 f_{00}}{\partial x_k \partial x_l}\right\}, \qquad \gamma_{00} = \left\{\frac{\partial^2 f_{00}}{\partial x_l \partial t_r}\right\}, \qquad \boldsymbol{h} = \left\{\mathrm{Im}\left[\frac{\partial f_{00}}{\partial x_k}\right]\right\},$$

where we use the notation $(\cdot)_{00} \equiv (\cdot)|_{x=x_{00},\,t=t_{00}}$ for different functions given in the point $(\boldsymbol{x}_{00}, \boldsymbol{t}_{00})$, $\boldsymbol{t}_{00} \equiv \boldsymbol{t}_0(\boldsymbol{x}_{00})$.

Consider in detail a special case of hydrogenic CS wave function. Using the formulae (4c) and (5) with generating operators for the Perelomov CS (P) and the Barut–Girardello CS (BG), we obtain

$$|\omega, 0, \eta\rangle = \mathcal{D}_{\boldsymbol{I}}(\overset{\mathbf{O(2,1)}}{\mathrm{P}}\eta, (\mathcal{L}_{0\alpha}))\mathcal{D}_{\mathbf{II}}(\overset{\mathbf{O(2,1)}}{\mathrm{BG}}\omega, (\mathcal{H}_{\alpha}^{[\pm]}))|1, 0, 0\rangle_{\mathrm{sph}}.$$

Noting that variation of τ in (4c) implies only a rotation of the CS wave function without change of its shape, we can set $\tau = 0$. Using the coordinate hydrogenic wave function in the *auxiliary representation*

$$\langle \boldsymbol{r}|\overline{n, l, m}\rangle = \frac{2}{(2l+1)!}\sqrt{\frac{(n+l)!}{(n-l-1)!}}(2r)^l e^{-r}\,{}_1F_1(l+1-n, 2l+1, 2r)Y_{lm}(\theta, \phi),$$

and in the physical representation

$$\langle \boldsymbol{r}|n, l, m\rangle = \frac{1}{n^2}\langle \frac{\boldsymbol{r}}{n}|\overline{n, l, m}\rangle,$$

we obtain the explicit form for the CS:

$$\langle \boldsymbol{r}|\overline{\omega, 0, \eta}\rangle = \frac{1}{\sqrt{I_0(2|\omega|)}}\sum_{l=0}^{\infty}\sum_{n=l+1}^{\infty}\frac{\omega^l}{l!}\frac{(1-|\eta|^2)^{l+1}}{\sqrt{(2l+1)!}}\sqrt{\frac{(n+l)!}{(n-l-1)!}}\eta^{n-l-1}\langle \boldsymbol{r}|\overline{n, l, m}\rangle.$$

Performing the summation we arrive at

$$\langle \boldsymbol{r}|\overline{\omega, 0, \eta}\rangle = \frac{1}{\sqrt{\pi I_0(2|\omega|)}}\frac{1-|\eta|^2}{(1-\eta)^2}\exp\left(-r\frac{1+\eta}{1-\eta}\right)I_0\left(\frac{2\sqrt{-\omega r_+(1-|\eta|^2)}}{1-\eta}\right),$$

where $r_+ = r\exp(i\phi)$, $r = |\boldsymbol{r}|$ and I_0 denotes the modified Bessel function. The asymptotic estimate (obtained directly or by the saddle-point method) for the absolute value of the CS wave function has the form:

$$|\langle \boldsymbol{r}|\overline{\omega, 0, \eta}\rangle|^2 \approx \frac{1}{2\pi^{3/2}|\omega|^{1/2}}\frac{(1-|\eta|^2)^2}{|1-\eta|^4}\exp\left\{-\frac{1}{2|\omega|}\frac{(1-|\eta|^2)^2}{|1-\eta|^4}\left(\Delta x^2 + \Delta y^2 + 2\Delta z^2\right)\right\},$$

where $\Delta x = x - x_0$, $\Delta y = y - y_0$; the point with coordinates

$$x_0 = \langle n\rangle_\infty(e - \cos\theta), \qquad y_0 = \langle n\rangle_\infty\sqrt{1-e^2}\sin\theta$$

lies on the *elliptic* orbit with an eccentricity e:

$$\langle n\rangle_\infty = \frac{|\omega|(1+|\eta|^2)}{1-|\eta|^2}, \qquad e = \frac{2|\eta|}{1+|\eta|^2}.$$

In a similar way one can find the asymptotic estimate for the physical CS wave function, but it is omitted here due to its complexity. In this case only the saddle-point method with interchange

of summation and integration can be used. It is remarkable that in this case the system of equations for the saddle point admits analytical solution. In a special case of *circular* orbits the asymptotic estimate for auxiliary and physical representations simplifies to

$$\langle \boldsymbol{r}_0 + \Delta\boldsymbol{r}|\overline{\omega,0,0}\rangle \approx \frac{1}{\sqrt{2}\pi^{3/4}|\omega|^{1/4}} \exp\left\{ i|\omega|\Delta\phi - \frac{z^2}{2|\omega|} - \frac{|\omega|\Delta\phi^2}{4} - \frac{\Delta\rho^2}{4|\omega|} + \frac{i\Delta\rho\Delta\phi}{2} \right\}, \quad (10)$$

where $\boldsymbol{r} = (\rho\cos\phi, \rho\sin\phi, z), \Delta\phi = \phi - \phi_0, \Delta\rho = \rho - |\omega|$, and

$$\langle \boldsymbol{r}_0 + \Delta\boldsymbol{r}|\omega,0,0\rangle \approx \frac{1}{\sqrt{5}\pi^{3/4}|\omega|^{9/4}} \exp\left\{ i|\omega|\Delta\phi - \frac{z^2}{2|\omega|^3} - \frac{|\omega|\Delta\phi^2}{10} - \frac{\Delta\rho^2}{10|\omega|^3} + \frac{2i\Delta\rho\Delta\phi}{5|\omega|} \right\}, \quad (11)$$

where $\boldsymbol{r} = (\rho\cos\phi, \rho\sin\phi, z)$, $\Delta\phi = \phi - \phi_0$, $\Delta\rho = \rho - |\omega|^2$. The functions (10) and (11) are related by the following equation:

$$\langle \boldsymbol{r}_1|\omega,0,0\rangle = \int \frac{d\boldsymbol{r}_2}{r_2} K(\boldsymbol{r}_1,\boldsymbol{r}_2)\langle \boldsymbol{r}_2|\overline{\omega,0,0}\rangle,$$

where

$$K(\boldsymbol{r}_1,\boldsymbol{r}_2) \approx \frac{1}{\sqrt{10\pi^3\rho_2^5}} \exp\left\{ i\rho_2\Delta\phi - \frac{z_2^2}{2\rho_2} - \frac{z_1^2}{2\rho_2^3} - \frac{\rho_2\Delta\phi^2}{10} - \frac{\Delta\rho^2}{10\rho_2^3} + \frac{2i\Delta\rho\Delta\phi}{5\rho_2} \right\},$$

and $\Delta\phi = \phi_1 - \phi_2$, $\Delta\rho = \rho_1 - \rho_2^2$.

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Дане видання є третью частиною і включає праці, присвячені задачам алгебри, теорії груп та теорії зображень, а також іншим проблемам пов'язанним з симетрією в математичній фізиці.

Розраховано на наукових працівників, аспірантів, які цікавляться новими тенденціями симетрійного аналізу та його застосувань.

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This issue is the third part which contains papers devoted to problems of algebra, group and representation theory, and to other problems related to symmetry in mathematical physics.

The book may be useful for researchers and post graduate students who are interested in modern trends in symmetry analysis and its applications.

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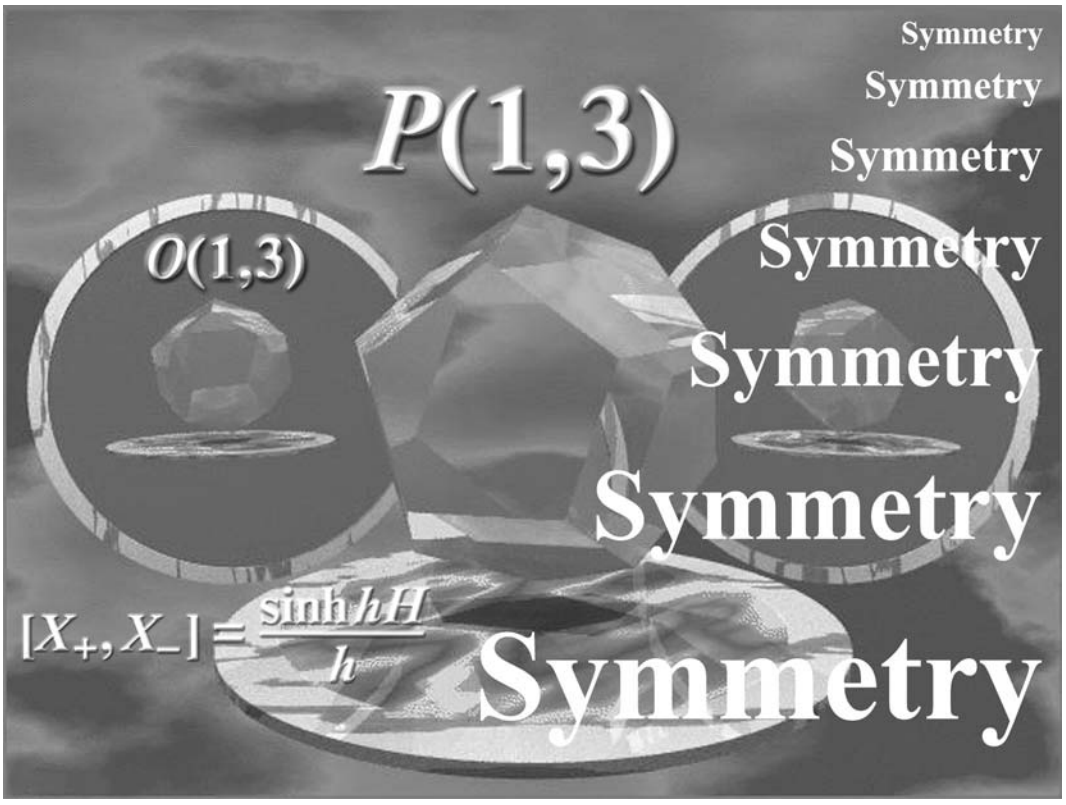
Algebras, Groups and Representation Theory



Symmetry in Nonlinear Mathematical Physics

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Hamiltonian Type Operators in Representations of the Quantum Algebra $\mathrm{su}_q(1,1)$

Natig ATAKISHIYEV † *and Anatoliy KLIMYK* †‡

† *Instituto de Matemáticas, UNAM, CP 62210 Cuernavaca, Morelos, México*
E-mail: *natig@matcuer.unam.mx*

‡ *Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine*
E-mail: *anatoliy@matcuer.unam.mx, aklimyk@bitp.kiev.ua*

We study some classes of symmetric operators for the discrete series representations of the quantum algebra $\mathrm{su}_q(1,1)$, which may serve as Hamiltonians of various physical systems. These operators are expressed in the canonical basis by a Jacobi matrix. The problem of diagonalization of these operators (eigenfunctions, spectra, overlap coefficients, etc.) is solved by using the connection of such operators with the theory of orthogonal polynomials.

1 Introduction

The theory of representations of the Lie group $SU(1,1) \simeq SL(2,\mathbb{R})$ and its Lie algebra has been extensively employed in various branches of physics and mathematics. Representations of the Lie algebra $\mathrm{su}(1,1)$ have been particularly useful in studying the isotropic harmonic oscillator, non-relativistic Coulomb problem, relativistic Schrödinger equation, Dirac equation with the Coulomb interaction, and so on. The Hamiltonian H in the interacting boson model is represented as a linear combination of the operators, corresponding to generating elements J_+^{cl}, J_-^{cl}, J_0^{cl} of the Lie algebra $\mathrm{su}(1,1)$. For this reason, the diagonalization of representation operators, corresponding to such linear combinations, is an important problem.

After the appearance of quantum groups and quantum algebras, most problems of the representation theory for Lie groups and Lie algebras have been transferred to the representation theory of quantized groups and algebras. This development is also very important from the point of view of possible applications both in mathematics and in physics. In particular, the diagonalization of representation operators for simplest quantum algebras (especially, such as $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$) is of great significance.

Representation operators for the quantum algebras $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$ find wide applications in physics. For example, some models in quantum optics, such as Raman and Brillouin scattering, parametric conversion and the interaction of two-level atoms with a single-mode radiation field (Dicke model), can be described by interaction Hamiltonians, which are representation operators for $\mathrm{su}_q(2)$ or $\mathrm{su}_q(1,1)$ (see, for example, [1] and references therein).

A great interest to spectral analysis of the operators for the positive discrete series of $\mathrm{su}_q(1,1)$ appears in the analysis on noncommutative (quantum) spaces. For example, the Laplace operator and the squared radius (together with the third operator, which serves as the operator J_0) generate the algebra $\mathrm{su}_q(1,1)$, acting on the space of polynomials on the n-dimensional Manin space or on the quantum complex vector space (see [2–4]). These operators realize irreducible representations of the algebra $\mathrm{su}_q(1,1)$, which belong to the positive discrete series and form a q-analogue of the oscillator representations of the Lie algebra $\mathrm{su}(1,1)$. To construct Hamiltonians of physical systems, existing in the Manin space or in the quantum complex vector space (for example, Hamiltonians for harmonic oscillators in these spaces), one thus needs to deal with operators of the positive discrete series representations of $\mathrm{su}_q(1,1)$. Consequently, the

diagonalization of these Hamiltonians reduces to the problem of diagonalization of representation operators for the quantum algebra $\mathrm{su}_q(1,1)$.

In the present paper we discuss the problem of the diagonalization (eigenfunctions, spectra, transition coefficients, etc.) of some classes of operators for the discrete series representations of the quantum algebra $\mathrm{su}_q(1,1)$, related to q-orthogonal polynomials. We restrict ourselves by the discrete series representations of $\mathrm{su}_q(1,1)$, because these very representation operators are often used as Hamiltonians of physical models and these representations are related to the q-oscillator algebra.

Throughout the sequel we always assume that q is a fixed positive number such that $q < 1$. We extensively use the theory of q-special functions and notations of the standard q-analysis (see, for example, [5] and [6]). In particular, we use q-numbers defined as

$$[a]_q := \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}},$$

where a is any complex number.

2 The algebra $\mathrm{su}_q(1,1)$ and its discrete series representations

The classical Lie algebra $\mathrm{su}(1,1)$ is generated by the elements J_0^{cl}, J_1^{cl}, J_2^{cl}, satisfying the relations

$$[J_0^{\mathrm{cl}}, J_1^{\mathrm{cl}}] = \mathrm{i}J_2^{\mathrm{cl}}, \qquad [J_1^{\mathrm{cl}}, J_2^{\mathrm{cl}}] = -\mathrm{i}J_0^{\mathrm{cl}}, \qquad [J_2^{\mathrm{cl}}, J_0^{\mathrm{cl}}] = \mathrm{i}J_1^{\mathrm{cl}}.$$

In terms of the raising and lowering operators $J_\pm^{\mathrm{cl}} = J_1^{\mathrm{cl}} \pm \mathrm{i}J_2^{\mathrm{cl}}$ these commutation relations can be written as

$$[J_0^{\mathrm{cl}}, J_\pm^{\mathrm{cl}}] = \pm J_\pm^{\mathrm{cl}}, \qquad [J_-^{\mathrm{cl}}, J_+^{\mathrm{cl}}] = 2J_0^{\mathrm{cl}}.$$

The discrete series representations T_l^+ of $\mathrm{su}(1,1)$ with lowest weights are given by a positive number l and they are realized on the spaces $\mathcal{L}_l$ of polynomials in x. The basis in $\mathcal{L}_l$ consists of the monomials

$$g_n^l(x) = \{(2l)_n/n!\}^{1/2} x^n, \qquad n = 0, 1, 2, 3, \ldots .$$

Assuming that this basis consists of orthonormal elements, one defines a scalar product in $\mathcal{L}_l$. The closure of $\mathcal{L}_l$ leads to a Hilbert space, on which the representation T_l^+ acts.

We consider an explicit realization of representation operators J_i^{cl}, $i = 0, 1, 2$, in terms of the first-order differential operators:

$$J_0^{\mathrm{cl}} = x\frac{d}{dx} + l, \qquad J_1^{\mathrm{cl}} = \frac{1}{2}(1+x^2)\frac{d}{dx} + lx, \qquad J_2^{\mathrm{cl}} = \frac{i}{2}(1-x^2)\frac{d}{dx} - ilx.$$

Then

$$J_0^{\mathrm{cl}} g_n^l = (l+n)\, g_n^l, \qquad J_+^{\mathrm{cl}} g_n^l = \sqrt{(2l+n)(n+1)}\, g_{n+1}^l, \qquad J_-^{\mathrm{cl}} g_n^l = \sqrt{(2l+n-1)n}\, g_{n-1}^l.$$

The quantum algebra $\mathrm{su}_q(1,1)$ and its irreducible representations are obtained by deformation of the corresponding relations for the Lie algebra $\mathrm{su}(1,1)$ and its irreducible representations. The algebra $\mathrm{su}_q(1,1)$ is defined as the associative algebra, generated by the elements J_+, J_-, and J_0, which satisfy the commutation relations

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_-, J_+] = \frac{q^{J_0} - q^{-J_0}}{q^{1/2} - q^{-1/2}} \equiv [2J_0]_q, \tag{1}$$

and the conjugation relations

$$J_0^* = J_0, \qquad J_+^* = J_-. \tag{2}$$

(Observe that here we have replaced J_- by $-J_-$ in the usual definition of the algebra $\mathrm{sl}_q(2)$.)

We are interested in the discrete series representations of $\mathrm{su}_q(1,1)$ with lowest weights. These irreducible representations will be denoted by T_l^+, where l is a lowest weight, which can be any positive number (see, for example, [7]). These representations are obtained by deforming the corresponding representations of the Lie algebra $\mathrm{su}(1,1)$.

As in the classical case, the representation T_l^+ can be realized on the space $\mathcal{L}_l$ of all polynomials in x. We choose a basis for this space, consisting of the monomials

$$f_n^l \equiv f_n^l(x) := c_n^l\, x^n, \qquad n = 0, 1, 2, \ldots, \tag{3}$$

where

$$c_0^l = 1, \qquad c_n^l = \prod_{k=1}^{n} \frac{[2l+k-1]_q^{1/2}}{[k]_q^{1/2}} = q^{(1-2l)n/4} \frac{(q^{2l};q)_n^{1/2}}{(q;q)_n^{1/2}}, \qquad n = 1, 2, 3, \ldots, \tag{4}$$

and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$. The representation T_l^+ is then realized by the operators

$$J_0 = x\frac{d}{dx} + l, \qquad J_\pm = x^{\pm 1}[J_0 \pm l]_q\,. \tag{5}$$

As a result of this realization, we have

$$J_0\, f_n^l = (l+n)\, f_n^l, \qquad J_+\, f_n^l = \sqrt{[2l+n]_q\,[n+1]_q}\, f_{n+1}^l, \tag{6}$$

$$J_-\, f_n^l = \sqrt{[2l+n-1]_q\,[n]_q}\, f_{n-1}^l. \tag{7}$$

Obviously, these operators satisfy the commutation relations (1).

We know that the discrete series representations T_l^+ can be realized on a Hilbert space, on which the conjugation relations (2) are satisfied. In order to obtain such a Hilbert space, we assume that the monomials $f_n^l(x)$, $n = 0, 1, 2, \ldots$, constitute an orthonormal basis for this Hilbert space. This introduces a scalar product $\langle\cdot,\cdot\rangle$ into the space $\mathcal{L}_l$. Then we close this space with respect to this scalar product and obtain a Hilbert space, which will be denoted by $\mathcal{H}_l$. The Hilbert space $\mathcal{H}_l$ consists of functions (series)

$$f(x) = \sum_{n=0}^{\infty} b_n f_n^l(x) = \sum_{n=0}^{\infty} b_n c_n^l x^n = \sum_{n=0}^{\infty} a_n x^n,$$

where $a_n = b_n c_n^l$. Since $\langle f_m^l, f_n^l\rangle = \delta_{mn}$ by definition, for $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and $\tilde f(x) = \sum\limits_{n=0}^{\infty} \tilde a_n\, x^n$ we have $\langle f, \tilde f\rangle = \sum\limits_{n=0}^{\infty} a_n\, \tilde a_n/|c_n^l|^2$, that is, the Hilbert space $\mathcal{H}_l$ consists of analytical functions $f(x) = \sum\limits_{n=0}^{\infty} a_n\, x^n$, such that

$$\|f\|^2 \equiv \sum_{n=0}^{\infty} |a_n/c_n^l|^2 < \infty.$$

It is directly checked that for a function $f(x) \in \mathcal{H}_l$ we have $q^{cx\frac{d}{dx}} f(x) = f(q^c x)$. Therefore, taking into account formulas (5), we conclude that

$$q^{J_0/2} f(x) = q^{\frac{1}{2}(x\frac{d}{dx}+l)} f(x) = q^{l/2} f(q^{1/2}x), \tag{8}$$

$$J_+ f(x) = \frac{x}{q^{1/2}-q^{-1/2}} \left[q^l f(q^{1/2}x) - q^{-l} f(q^{-1/2}x)\right], \tag{9}$$

$$J_- f(x) = \frac{1}{(q^{1/2}-q^{-1/2})x} \left[f(q^{1/2}x) - f(q^{-1/2}x)\right]. \tag{10}$$

3 Hamiltonian type operators

We are interested in spectra, eigenfunctions and overlap functions for operators in the representations T_l^+, which correspond to elements of the quantum algebra $\mathrm{su}_q(1,1)$ of the form

$$H := q^{p\,J_0}(J_+ + J_-)\, q^{p\,J_0} + f(q^{J_0}), \qquad p \in \mathbb{R}, \tag{11}$$

where f is some polynomial (or function). These operators have the following properties:

(i) They are representable in the basis (3) by a Jacobi matrix.

(ii) They are symmetric operators.

(iii) They are not necessarily self-adjoint operators.

Recall that a Jacobi matrix is a matrix, with all entries vanishing except for those which occur on the main diagonal and on two neighbouring (upper and lower) diagonals.

The most important example of the operators of the form (11) is

$$H^{(p)} := q^{pJ_0/4}(J_+ + J_-)q^{pJ_0/4}, \qquad p \in \mathbb{R}.$$

For these operators the following theorem is true [8].

Theorem 1. *If $p > 1$, then the operator $H^{(p)}$ is bounded and has a discrete simple spectrum. Zero is a unique point of accumulation of the spectrum. If $p < 1$, then the closure $\overline{H^{(p)}}$ of the symmetric operator $H^{(p)}$ is not a self-adjoint operator and it has deficiency indices $(1,1)$, that is, $\overline{H^{(p)}}$ has a one-parameter family of self-adjoint extensions. These extensions have discrete simple spectra. If $p = 1$, then $H^{(p)}$ has a continuous simple spectrum, which covers the interval $(-b, b)$, $b = 2/(q^{-1/2} - q^{1/2})$.*

4 Applications of Hamiltonian type operators

There are many applications of the operators (11). Below we exhibit only some of these applications.

4.1. Quantum mechanics in noncommutative world. A big interest in spectral analysis of the operators for the positive discrete series of $\mathrm{su}_q(1,1)$ appears in the analysis on noncommutative (quantum) spaces. The n-dimensional Manin space can serve as a simple example of such a space. This space is defined by elements (quantum coordinates) $x_1, x_2, \ldots, x_n$, satisfying the relations $x_i x_j = q x_j x_i$ for $i < j$. A certain q-deformation $U_q'(\mathrm{so}_n)$ of the universal enveloping algebra $U(\mathrm{so}_n)$ of the Lie algebra so_n (not coinciding with the Drinfeld–Jimbo quantum algebra $U_q(\mathrm{so}_n)$) acts on the Manin space (it replaces the action of the rotation group $SO(n)$ on the n-dimensional Euclidean space; for details see [2]).

The squared radius on the Manin space is given by the formula

$$Q = x_1 + q^{-1}x_2^2 + \cdots + q^{-n+1}x_n^2.$$

A Laplace operator on it is defined as

$$\Delta_q = q^{n-1}\partial_1^2 + q^{n-2}\partial_2^2 + \cdots + \partial_n^2$$

(the definition of the derivatives ∂_i are given in [2] and [3]). The operators Δ_q and $\hat{Q}$ (the operator of the multiplication by Q) are invariant with respect to the algebra $U_q'(\mathrm{so}_n)$.

It is known [2] that Δ_q and $\hat{Q}$ generate the quantum algebra $\mathrm{sl}_{q^2}(2)$. Moreover, $J_+ = cQ$ and $J_- = c'\Delta_q$ (where c and c' are constants). In the usual Euclidean space, Hamiltonians of many physical systems are constructed by means of a linear combination of the Laplace operators and the radius. Similarly, Hamiltonians of physical systems in the Manin space are constructed as combinations of the Laplace operator Δ_q and the operator $\hat{Q}$. Then we led to the fact that these Hamiltonians are combinations of the operators J_+ and J_- (see [9]). However, we wish that a Hamiltonian H_q in the Manin space tend to a corresponding Hamiltonian H in the Euclidean space. In order to achieve this, we have to multiply the operator $J_+ + J_-$ from the left and from the right by the operator q^{pJ_0} with an appropriate p. (This multiplication is explained by the fact that q-deforming the classical algebra $U(g)$, where g is a Lie algebra, into the quantum algebra $U_q(g)$, one performs twisting of the elements, corresponding to positive and negative root elements.) Thus, we come to the consideration of Hamiltonians, which are operators of the form (11).

4.2. Quantum optics. Hamiltonians of the type (11) appear in the quantum optics. Examples of such Hamiltonians and further details can be found in [1].

4.3. q-Oscillators, based on $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$. This q-oscillator, which is different from the q-oscillators of Biedenharn and Macfarlane, is introduced in [10]. It can be described as follows. Let us take in a finite dimensional representation of the algebra $\mathrm{su}_q(2)$ or in a discrete series representation of the algebra $\mathrm{su}_q(1,1)$ the operators

$$B_1 := \frac{1}{2}\, q^{J_0/4}[J_+ + J_-]q^{J_0/4}, \qquad B_2 := \frac{1}{2\mathrm{i}}\, q^{J_0/4}[J_+ - J_-]q^{J_0/4}.$$

Both operators B_1 and B_2 are symmetric and bounded. These operators satisfy the relations

$$[J_0, B_1] = \mathrm{i}B_2, \qquad [J_0, B_2] = \mathrm{i}B_1, \qquad [B_1, B_2] = f(J_0, C), \tag{12}$$

where f is some (explicitly known) function and C is the Casimir operator. Clearly, the operator $f(J_0, C)$ is diagonal in the canonical basis.

Now we introduce a Hamiltonian, a position operator and a momentum operators by the formulas

$$P := -B_2, \qquad Q := B_1, \qquad H = J_0 - l + \frac{1}{2},$$

where l is an index of the representation. Then if f_n^l are basis elements of the representation space, then for the Hamiltonian H we have

$$Hf_n^l = \left(n + \frac{1}{2}\right) f_n^l.$$

It follows from (12) that the position and momentum operators satisfy the relations

$$[H, Q] = -\mathrm{i}P, \qquad [H, P] = \mathrm{i}Q,$$

where $[\cdot, \cdot]$ is the usual commutator.

Thus, using the algebras $\mathrm{su}_q(2)$ and $\mathrm{su}_q(1,1)$, we have obtained Hamiltonian systems of quantum (not q-deformed) mechanics.

4.4. q-Orthogonal polynomials. As will be explained in the next section, operators of the form (11) are closely related to q-orthogonal polynomials. With the aid of these operators we established a notion of duality of polynomials, orthogonal on some countable sets. By means of this duality we derived new orthogonality relations for some instances of polynomials (see [11] and [12]).

5 Relation to moment problem and q-orthogonal polynomials

There exists close relationship between the following directions, which we use for studying Hamiltonian type operators (see [13] and [14]):

(i) the theory of symmetric operators L, representable by a Jacobi matrix;

(ii) the theory of orthogonal polynomials;

(iii) the theory of classical moment problem.

Let us describe this relationship. Let L be a closed symmetric operator on a Hilbert space $\mathcal{H}$. Let $e_1, e_2, \dots$ be a basis in $\mathcal{H}$ such that

$$Le_n = a_n e_{n+1} + b_n e_n + a_{n-1} e_{n-1}.$$

Let $|x\rangle = \sum\limits_{n=0}^{\infty} p_n(x) e_n$ be an eigenvector of L with the eigenvalue x, that is, $L|x\rangle = x|x\rangle$. Then

$$L|x\rangle = \sum_{n=0}^{\infty} (p_n(x) a_n e_{n+1} + p_n(x) b_n e_n + p_n(x) a_{n-1} e_{n-1}) = x \sum_{n=0}^{\infty} p_n(x) e_n.$$

Equating coefficients of the vector e_n, one comes to a recurrence relation for the coefficients $p_n(x)$:

$$a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) = x p_n(x).$$

Since $p_{-1}(x) = 0$, then setting $p_0(x) \equiv 1$, we see that this relation completely determines the coefficients $p_n(x)$. Moreover, $p_n(x)$ are polynomials in x of degree n. If coefficients a_n and b_n are real, then all coefficients of the polynomials $p_n(x)$ are real and they are orthogonal with respect to some positive measure $\mu(x)$. If the operator L is self-adjoint, then this measure is uniquely determined and

$$\int p_m(x) p_n(x) d\mu(x) = \delta_{mn},$$

where the integration is taken over some subset (possibly discrete) of $\mathbb{R}$. Moreover, the spectrum of the operator L is simple and coincide with the set, on which the polynomials are orthogonal. The measure $\mu(x)$ determines the spectral measure for the operator L (for details see [14], Chapter VII).

If a closed symmetric operator L is not self-adjoint, then the measure $\mu(x)$ is not determined uniquely. Moreover, in this case there exist infinitely many measures, with respect to which the polynomials are orthogonal. Among these measures there are so-called extremal measures (that is, such that a set of polynomials $\{p_n(x)\}$ is complete in the Hilbert space L^2 with respect to the corresponding measure). These measures determine self-adjoint extensions of the symmetric operator L.

On the other side, with the polynomials $p_n(x)$, $n = 0, 1, 2, \dots$, the classical moment problem is associated [13]. Namely, with these polynomials (that is, with the coefficients a_n and b_n of the corresponding recurrence relation) positive numbers c_n, $n = 0, 1, 2, \dots$, are related, which

enter into the classical moment problem. The definition of classical moment problem consists in the following. There is a set of positive numbers c_n, $n = 0, 1, 2, \ldots$. We are looking for a measure $\mu(x)$, such that

$$\int x^n d\mu(x) = c_n, \qquad n = 0, 1, 2, \ldots, \tag{13}$$

where the integration is taken over some fixed subset of $\mathbb{R}$. There are two principal questions:

(i) Does exist a measure $\mu(x)$, such that relations (13) are satisfied?

(ii) If such a measure exists, is it determined uniquely?

The answer to the first question is positive, if the numbers c_n, $n = 0, 1, 2, \ldots$, are those, which correspond to a particular family of orthogonal polynomials. Moreover, a measure $\mu(x)$ then coincides with the measure, with respect to which these polynomials are orthogonal.

If a measure in (13) is determined uniquely, then we say that we deal with determined moment problem. It is the case when the region of integration is bounded. If there are many measures, with respect to which relations (13) hold, then we say that we deal with indetermined moment problem. In this case there exist infinitely many measures $\mu(x)$ for which (13) take place. In the second case the corresponding polynomials are orthogonal with respect to all these measures and the corresponding symmetric operator L is not self-adjoint.

Thus, we see that one can study the operator L by investigating the corresponding sets of orthogonal polynomials and their moment problems. We do that here for Hamiltonian type operators.

6 Spectrum and eigenfunctions of Hamiltonians $H^{(\varphi)}$

This section deals with eigenfunctions $\xi^l_\lambda(x;\varphi)$ and eigenvalues of a one-parameter family of the self-adjoint operators

$$H^{(\varphi)} := \frac{1}{2}(q^{1/4}J_+ + q^{-1/4}J_-)\, q^{J_0/2} + \frac{\cos\varphi}{q^{-1/2} - q^{1/2}}\, q^{J_0} \tag{14}$$

of the representation T^+_l of the algebra $\mathrm{su}_q(1,1)$: $H^{(\varphi)}\, \xi^l_\lambda(x;\varphi) = \lambda\, \xi^l_\lambda(x;\varphi)$. Using the relations (8)–(10) we find that

$$H^{(\varphi)} f(x) = c(x^{-1} - 2q^{(2l+1)/4}\cos\varphi + q^{l+1/2}x)f(qx) - c(x^{-1} + q^{1/2-l}x)f(x),$$

where $c = (q^{(2l-1)/4})/2(q^{1/2} - q^{-1/2})$. By using this expression we find (details are given in [15]) that the eigenfunctions of $H^{(\varphi)}$ are

$$\xi^l_\lambda(x;\varphi) = \frac{(axe^{i\varphi};q)_\infty (axe^{-i\varphi};q)_\infty}{(bxe^{i(\theta-\varphi)};q)_\infty (bxe^{i(\varphi-\theta)};q)_\infty}\,, \qquad \lambda = \frac{\cos(\theta-\varphi)}{q^{-1/2} - q^{1/2}}\,,$$

where $a = q^{(2l+1)/4}$ and $b = q^{(1-2l)/4}$. A relation between the eigenfunctions $\xi^l_\lambda(x;\varphi)$ and the basis functions $f^l_n(x)$ is now an easy consequence of the generating function

$$\frac{(ae^{i\varphi}t;q)_\infty (ae^{-i\varphi}t;q)_\infty}{(e^{i(\theta-\varphi)}t;q)_\infty (e^{i(\varphi-\theta)}t;q)_\infty} = \sum_0^\infty P_n\left(\cos(\theta-\varphi);a|q\right)\, t^n$$

for the q-Meixner–Pollaczek polynomials $P_n(y;a|q)$, defined (see [16], section 3.9) as

$$P_n(\cos(\theta+\varphi);a|q) = a^{-n}\, e^{-in\varphi}\frac{(a^2;q)_n}{(q;q)_n}\, {}_3\phi_2\left(q^{-n}, ae^{i(\theta+2\varphi)}, ae^{-i\theta}\; a^2, 0\,; q, q\right).$$

Thus

$$\xi_\lambda^l(x;\varphi)=\sum_{n=0}^{\infty}\frac{q^{(1-2l)n/4}}{c_n^\nu}P_n(\cos(\theta-\varphi);q^l|q)f_n^l(x).$$

To find a spectrum of the operator $H^{(\varphi)}$, we note that the q-Meixner–Pollaczek polynomials $P_n(\cos(\theta-\varphi))\equiv P_n\left(\cos(\theta-\varphi);q^l|q\right)$ are orthogonal and the orthogonality relation has the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_m(\cos(\theta-\varphi))P_n(\cos(\theta-\varphi))w_\varphi(\cos(\theta-\varphi))d\theta=\frac{(q^{2l};q)_n}{(q;q)_n}\delta_{mn},$$

where

$$w_\varphi(\cos(\theta-\varphi))=(q;q)_\infty(q^{2l};q)_\infty\left|\frac{(e^{2\mathrm{i}(\theta-\varphi)});q)_\infty}{q^l(e^{\mathrm{i}(\theta-\varphi)});q)_\infty(q^le^{\mathrm{i}\theta};q)_\infty}\right|^2$$

(see formula (3.9.2) in [16]). This orthogonality relation can be written as

$$\int_a^b\frac{(q;q)_n}{(q^{2l};q)_n}P_m(\lambda(q^{-1/2}-q^{1/2}))P_n(\lambda(q^{-1/2}-q^{1/2})))\hat{w}(\lambda)d\lambda=\delta_{mn},$$

where

$$\hat{w}(\lambda)=\frac{w_\varphi(\lambda(q^{-1/2}-q^{1/2}))(q^{-1/2}-q^{1/2})}{\sin(\varphi-\theta)},\quad a=\frac{-\cos(\pi+\varphi)}{q^{-1/2}-q^{1/2}},\quad b=\frac{\cos(\pi-\varphi)}{q^{-1/2}-q^{1/2}}.\quad(15)$$

Therefore, we may formulate the following theorem:

Theorem 2. *The operator $H^{(\varphi)}$ has continuous and simple spectrum, which completely covers the interval (a,b), where a and b are given by* (15).

Continuity of the spectrum means that the eigenfunctions $\xi_\lambda^l(x;\varphi)$ do not belong to the Hilbert space $\mathcal{H}_l$. They belong to the space of functionals on the linear space $\mathcal{L}_l$, which can be considered as a space of generalized functions on $\mathcal{L}_l$.

7 A limit to the classical case

The classical limit (that is, the limit $q\to 1$) has sense only for the operator

$$H^{(\pi/2)}=\frac{1}{2}(q^{1/4}J_++q^{-1/4}J_-)q^{J_3/2}.$$

When $q\to 1$ the operator $H^{(\pi/2)}$ tends to the operator $J_1^{(\mathrm{cl})}$: $\lim\limits_{q\to 1}H^{(\pi/2)}=J_1^{(\mathrm{cl})}$. In this limit the basis elements (3) turn into the basis elements $g_n^l(x)$ of the representation space for the Lie algebra su(1,1) and the eigenfunctions $\xi_\lambda^l(x;\pi/2)$ of $H^{(\pi/2)}$ into the eigenfunctions

$$\xi_\lambda^l(x):=(1+ix)^{-l-i\lambda}(1-ix)^{-l+i\lambda}$$

of the operator $J_1^{(\mathrm{cl})}$. They are related to the eigenfunctions of the operator $J_0^{(\mathrm{cl})}$ as

$$\xi_\lambda^l(x)=\sum_{n=0}^{\infty}\left(\frac{n!}{(2l)_n}\right)^{1/2}P_n^{(l)}(\lambda;\pi/2)\,f_n^l(x)\equiv\sum_{n=0}^{\infty}P_n^{(l)}(\lambda;\pi/2)\,x^n\,,$$

where $P_n^{(l)}(\lambda;\pi/2)$ are the classical Meixner–Pollazcek polynomials, defined by the formula

$$P_n^{(l)}(x;\varphi):=\frac{(2l)_n}{n!}\,e^{in\varphi}\,{}_2F_1(-n,\,\nu+ix;\,2l;\,1-e^{-2i\varphi}),\qquad l>0,\quad 0<\varphi<\pi.$$

The Meixner–Pollazcek polynomials in the expression for $\xi_\lambda^l(x)$ are a limit case of the corresponding q-Meixner–Pollazcek polynomials (see [16], section 5.9).

8 A limit to the q-oscillator algebra

Let us consider the operator $I := 2H^{(0)}$ from Section 6. It acts upon the canonical basis f_k^l, $k = 0, 1, 2, \ldots$, as

$$If_k^l = a_k f_{k+1}^l + a_{k-1} f_{k-1}^l + \frac{q^{k+l}}{q^{-1/2} - q^{1/2}} f_k^l, \qquad a_k = \frac{(1 - q^{k+1})^{1/2}(1 - q^{k+2l+2})^{1/2}}{q^{-1/2} - q^{1/2}}.$$

Clearly, the operator I depends on the index l of the representation T_l^+. Taking the limit $l \to +\infty$, we obtain the operator $Q = \lim\limits_{l \to +\infty} (q^{-1} - 1)^{1/2} I$ such that

$$Qf_k = \left(\frac{1 - q^{k+1}}{1 - q}\right)^{1/2} f_{k+1} + \left(\frac{1 - q^k}{1 - q}\right)^{1/2} f_{k-1}.$$

Here f_k is a vector, which must be a limit of the vectors f_k^l as $l \to +\infty$. However, such limit does not exist. In order to have a well-defined limit of basis vectors we change a basis. To this end, we take in $\mathcal{H}_l$ another scalar product. Namely, we take in $\mathcal{L}_l$ the basis consisting of the monomials

$$\tilde{f}_n^l(x) = b_n^l x^n, \quad n = 0, 1, 2, \ldots, \qquad b_0^l = 1, \qquad b_n^l = \frac{(q^{2l}; q)_k^{1/2}}{(q; q)_k^{1/2}}.$$

One can take in $\mathcal{L}_l$ the scalar product such that these vectors constitute an orthonormal basis and close $\mathcal{L}_l$ with respect to this scalar product. The closure of this space will be denoted by $\hat{\mathcal{H}}_l$. Then the operator I is realized by the formula

$$I = \frac{1 - q^{J_0 + l}}{q^{-1/2} - q^{1/2}} x + \frac{1 - q^{J_0 - l}}{q^{-1/2} - q^{1/2}} x^{-1} + \frac{q^{J_0}}{q^{-1/2} - q^{1/2}}$$

and we have

$$I\tilde{f}_k^l(x) = a_k \tilde{f}_{k+1}^l(x) + a_{k-1} \tilde{f}_{k-1}^l(x) + \frac{q^{l+k}}{q^{-1/2} - q^{1/2}} \tilde{f}_k^l,$$

where a_k are the same as above.

Repeating the reasoning of Section 6, we obtain that the eigenfunctions of the operator I in the space $\hat{\mathcal{H}}_l$ are

$$\tilde{\xi}_\lambda^l(x) = \frac{(q^l x; q)_\infty (q^l x; q)_\infty}{(x e^{\mathrm{i}\theta}; q)_\infty (x e^{-\mathrm{i}\theta}; q)_\infty}, \qquad \lambda = \frac{2\cos\theta}{q^{-1/2} - q^{1/2}},$$

and these eigenfunctions are decomposed in terms of the basis $\{\tilde{f}_n^l(x)\}$ as

$$\tilde{\xi}_\lambda^l(x) = \sum_{n=0}^{\infty} \frac{1}{b_n^l} P_n(\lambda(q^{-1/2} - q^{1/2}); q^l | q) \tilde{f}_k^l(x).$$

Due to the orthogonality relation for q-Meixner–Pollazcek polynomials, spectrum of the operator I is simple and coincides with the interval $(-a, a)$, $a = 2q^{1/4}(1 - q)^{-1/2}$.

Now we take the limit $l \to +\infty$. Then the basis vectors $\tilde{f}_n^l(x)$ turn into the basis vectors

$$e_n(x) = (q; q)_n^{-1/2} x^n, \qquad n = 0, 1, 2, \ldots.$$

The operator $Q = \lim\limits_{l\to+\infty}(q^{-1}-1)^{1/2}I$ has the form

$$Qe_n(x) = \left(\frac{1-q^{n+1}}{1-q}\right)^{1/2} e_{n+1}(x) + \left(\frac{1-q^n}{1-q}\right)^{1/2} e_{n-1}(x) \equiv (a^+ + a_-)e_n(x),$$

where

$$a^+e_n(x) = \left(\frac{1-q^{n+1}}{1-q}\right)^{1/2} e_{n+1}(x), \qquad a^-e_n(x) = \left(\frac{1-q^n}{1-q}\right)^{1/2} e_{n-1}(x).$$

The operator $q^{J_0+l}\tilde{f}^l_k(x) = q^k\tilde{f}^l_k(x)$ of the representation T^+_l of $\mathrm{su}_q(1,1)$ turns in the limit as $l\to+\infty$ into the operator

$$q^Ne_n(x) = q^ne_n(x).$$

It is easy to see that the operators a^+, a^- and q^N satisfy the relations

$$a^-a^+ - qa^+a^- = 1, \qquad q^Na^+ = qa^+q^N, \qquad q^Na^- = q^{-1}a^-q^N,$$

that is, they generate the well-known q-oscillator algebra introduced by Biedenharn and Macfarlane. Clearly, $Q = a^+ + a^-$ is the position operator for this q-oscillator.

The limit $l\to+\infty$ turns the eigenfunctions $\tilde{\xi}^l_\lambda(x)$ into eigenfunctions of the operator Q having the form

$$\tilde{\xi}_\lambda(x) = \frac{1}{(xe^{\mathrm{i}\theta};q)_\infty(xe^{-\mathrm{i}\theta};q)_\infty}, \qquad \lambda = \frac{2\cos\theta}{(1-q)^{1/2}}.$$

We have $Q\tilde{\xi}_\lambda(x) = \lambda\tilde{\xi}_\lambda(x)$ and a spectrum of the operator Q is simple and coincides with the interval $(-b,b)$, $b = 2/(1-q)^{1/2}$.

9 Hamiltonian with bounded discrete spectrum

In this section we consider the operator

$$H_1 = q^{3J_0/4}(J_+ + J_-)q^{3J_0/4} - \left([J_0-l]_q\,q^{l/2} + [J_0+l]_q\,q^{-l/2}\right)q^{3J_0/2},$$

(note that this operator depends on the index l of the representation T^+_l). It acts on the basis elements (3) by the formula

$$H_1\,f^l_k = a_{k+1}f^l_{k+1} + a_kf^l_{k-1} - q^{3(l+k)/2}d_kf^l_k,$$

where

$$a_k = q^{3(l+k)/2-3/4}\sqrt{[k]_q\,[2l+k-1]_q}, \qquad d_k = [k]_q\,q^{(l-1)/2} + [2l+k]_q\,q^{-(l-1)/2}.$$

By using this action it is easy to check that the H_1 is a bounded self-adjoint operator. Eigenfunctions of the operator H_1, $H_1\chi^l_\lambda(x) = \lambda\,\chi^l_\lambda(x)$, can be represented in the form

$$\chi^l_\lambda(x) = \sum_{k=0}^{\infty} P_k(\lambda)f^l_k(x).$$

It is not hard to prove that

$$P_k(\lambda) = \left(\frac{(q^{2l};q)_k}{(q;q)_k}\right)^{1/2} q^{-lk}p_k(q^y;q^{2l-1}|q), \qquad q^y = (1-q^{-1})\lambda,$$

where $p_k(q^y;q^{2l-1}|q) = {}_2\phi_1(q^{-k},0;\ q^{2l};\ q;q^{y+1})$ are the so-called little q-Laguerre (Wall) polynomials (see, for example, [16], section 3.20).

Due to the orthogonality relation

$$(q^{2l};q)_\infty \sum_{k=0}^{\infty} \frac{q^{2lk}}{(q;q)_k}\, p_m(q^k;q^{2l-1}|q)\, p_n(q^k;q^{2l-1}|q) = \frac{q^{2ln}(q;q)_n}{(q^{2l};q)_n}\,\delta_{mn}$$

for little q-Laguerre polynomials (see formula (3.20.2) in [16]), a spectrum of the operator H_1 coincides with the set of points $q^n/(1-q^{-1})$, $n=0,1,2,\ldots$. This means that the eigenfunctions

$$\chi^l_{\lambda_n}(x;q) \equiv \Xi^l_n(x), \qquad n=0,1,2,\ldots, \qquad \lambda_n = \frac{q^n}{1-q^{-1}},$$

constitute a basis in the representation space. We thus proved the following theorem.

Theorem 3. *The operator H_1 has a simple discrete spectrum, which consists of the points $q^n/(1-q^{-1})$, $n=0,1,2,\ldots$. The corresponding eigenfunctions $\Xi^l_n(x)$ constitute an orthogonal basis in the space $\mathcal{H}_l$.*

10 Hamiltonian related to big q-Laguerre polynomials

In this section we are interested in the operator

$$H_2 := \alpha q^{J_0/4}(\sqrt{1-bq^{J_0-l}}J_+ q^{(J_0-l)/2} + q^{(J_0-l)/2}J_-\sqrt{1-bq^{J_0-l}})q^{J_0/4} - \beta_1 q^{2J_0} + \beta_2 q^{J_0-l}$$

of the representation T^+_l, where $b<0$ and

$$\alpha = (-b)^{1/2}q^l(1-q), \qquad \beta_1 = b(1+q), \qquad \beta_2 = bq + q^{2l}(b+1).$$

Since the bounded operator q^{J_0} is diagonal in the canonical basis (3) without zero diagonal elements, the operator H_2 is well defined.

We have the following expression for the symmetric operator H_2 in the canonical basis (3):

$$H_2\, f^l_n = (-ab)^{1/2}q^{(n+2)/2}\Big[\sqrt{(1-q^{n+1})(1-aq^{n+1})(1-bq^{n+1})}f^l_{n+1} + q^{-1/2}\sqrt{(1-q^n)(1-aq^n)(1-bq^n)}f^l_{n-1}\Big] - [abq^{2n+1}(1+q) - q^{n+1}(a+ab+b)]f^l_n,$$

where $a=q^{2l-1}$. Since $q<1$ the operator H_2 is bounded. Therefore, one can close this operator, so we assume that H_2 is a closed (and, consequently, defined on the whole space $\mathcal{H}_l$) operator. Since H_2 is symmetric, its closure is a self-adjoint operator.

For eigenfunctions $\xi_\lambda(x)$ of the operator H_2, $H_2\xi_\lambda(x) = \lambda\xi_\lambda(x)$, we have the expression

$$\xi_\lambda(x) = \sum_{n=0}^{\infty}(-ab)^{-n/2}q^{-n(n+3)/4}\left(\frac{(aq,bq;q)_n}{(q;q)_n}\right)^{1/2} P_n(\lambda;a,b;q)f^l_n(x),$$

where $P_n(\lambda;a,b;q)$ are big q-Laguerre polynomials, defined by the formula

$$P_n(\lambda;a,b;q) := {}_3\phi_2(q^{-n},0,\lambda;\ aq,bq;\ q,q) = (q^{-n}/b;q)^{-1}_n\ {}_2\phi_1(q^{-n},\ aq/\lambda;\ aq;\ q,\lambda/b).$$

The orthogonality relation for these polynomials is known to be of the form

$$\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(aq^{n+1}/b;q)_\infty}{(aq^{n+1};q)_\infty}q^n P_m(aq^{n+1};a,b;q)P_{m'}(aq^{n+1};a,b;q) - \frac{b}{a}\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_\infty(bq^{n+1}/a;q)_\infty}{(bq^{n+1};q)_\infty}q^n P_m(bq^{n+1};a,b;q)P_{m'}(bq^{n+1};a,b;q) = \frac{(q,b/a,aq/b;q)_\infty}{(aq,bq;q)_\infty}\frac{(q;q)_m}{(aq,bq;q)_m}(-ab)^m q^{m(m+3)/2}\delta_{mm'}.$$

Therefore, we arrive to the following assertion:

Theorem 4. *The spectrum of the operator H_2 coincides with the set of points aq^{n+1}, bq^{n+1}, $n = 0, 1, 2, \ldots$. The spectrum is simple and has only one accumulation point at 0.*

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On the Gauge Groups of Particle Physics and Their Quantum Deformations[1]

Jules BECKERS

Institute of Physics, University of Liege, Liege, Belgium
E-mail: *Jules.Beckers@ulg.ac.be*

All the forces of Nature are actually understood as mediated by an exchange of gauge fields of the symmetry group $SU(3) \otimes SU(2) \otimes U(1)$. Such a gauge principle then describes the very important field called "particle (or high energy) physics" and its famous "*Standard Model*" relating strong, electromagnetic and weak (or strong and electroweak) interactions. Knowing that the *quantum* deformations of such unitary local symmetry groups as well as their irreducible representations have been studied in a relatively recent past, we propose to visit a corresponding "*q-Standard Model*". Such a step is seen as the first part of our general programme and will be useful in a second one coming back on "old current algebras and associated sum rules" based on *chiral* symmetry groups such as $SU(2) \otimes SU(2)$ and $SU(3) \otimes SU(3)$. Here we plan to introduce the main useful ingredients for these purposes but by taking the q-deformed version of $SU(2)$ as the typical example.

The series of unitary $U(n, \mathbb{C})$- or more often unimodular and unitary $SU(n, \mathbb{C})$-Lie groups have already played very important roles in particle physics by characterizing weak, electromagnetic and (or) strong interactions at different occasions. If we limit ourselves to recalling the famous gauge principle [1, 2] saying that all the forces of Nature are actually understood as mediated by an exchange of gauge fields, we immediately think of the following direct product of Lie groups $SU(3) \otimes SU(2) \otimes U(1)$, where $SU(3)$ refers to the colour group of QCD (Quantum Chromodynamics) for strong interactions while $SU(2) \otimes U(1)$ is necessary for taking account of electroweak ones. More recently since the pioneering works of Jimbo [3] and Drinfeld [4], quantum groups and their representations have been extensively studied so that their impact inside particle physics developments [5] is a natural inquiry in order to exploit new possibilities according to recent experimental results.

In particular, the quantum deformation of $SU(2)$ has to be visited in connection with the electroweak interactions as already suggested by Finkelstein [6] for example but also inside chiral algebras (and their associated sum rules) such as $SU(2) \times SU(2)$ developed elsewhere [7].

This lecture is mainly devoted to the introduction of some necessary and useful ingredients dealing with the standard model on the one side and with the representations of the quantum deformation of $SU(2)$ on the other one in order to enter our general programme inside the chiral algebras more particularly.

We insist on Q.E.D. (Quantun Electrodynamics) seen as an Abelian gauge theory based on the gauge group $U(1)$. Then we go to the non-Abelian context through the example of Yang–Mills developments based on the gauge group $SU(2)$. One of the main points necessary for the understanding of the standard model is the introduction and the construction of current densities inside the Lagrangian of the model describing electroweak interactions: we thus come back on the Puppi triangle for simplicity but also on the Gell-Mann tetrahedron in weak interaction. Finally some information on quantum deformations (their generalities and representations) may be given by taking the simplest context of q-$SU(2)$ as the example of particular interest.

Such ingredients being essentially proposed for information to non-specialists in the already mentioned fields, we have decided to give here some specific references where the useful tools can be found and understood while the contents of this report are limited.

[1] It is short review of plenary talk given by Prof. J. Beckers at the Conference.

We want to insist on the very compact and precise overview of the standard model reported by Kazakov [2]. Its necessary elements can be found in standard textbooks in Q.E.D., in Q.C.D. and in weak interactions like those described by Weinberg [1] and by Marshak et al. [8], for example, that I strongly recommend. There the introduction of covariant derivatives and the way to understand how the gauge principle is a dynamical one are very simple, as well as the extension of Abelian to non-Abelian developments according to those given by Yang–Mills [9], for example. In the same context the very well-known beta-decay can help us to understand the construction of current densities in the hadronic and leptonic worlds by including not only Dirac matrices but also isospin ones inside these densities entering into ad-hoc Lagrangians for the V-A theory of weak interactions.

Trough the simplest non-Abelian Lie algebra corresponding to the group $SU(2,\mathbb{C})$ as an example, we have also shown how to get very easily its quantum group by replacing the Jacobi identity by the famous Yang–Baxter equation in order to get the typical entries of the corresponding quantum deformation of $SU(2,\mathbb{C})$ and, more importantly its q-Wigner functions entering in the corresponding irreducible representations.

Then let us take the 2-dimensional Borel factorization in order to show how to obtain, following Finkelstein [6], the corresponding dual algebras of special interest here. After Finkelstein we get the q-dependent commutation relations varying with the $(2j+1)$-dimensional representation. We see that the fundamental 2-dimensional representation is not altered but that the 3-dimensional adjoint one is: this is the important property by going back to the Standard Model where we have to recall that the (spin 1/2) fermion sector corresponding to the matter fields (leptons and quarks) deals with the fundamental representation(s) of the gauge group(s) while the (spin 1) gauge sector corresponding to the gauge bosons (spin 1-vector particles) deals with their adjoint representation(s). Such a last point has an important impact in the corresponding gauge theory and could be included by q-modifying the corresponding "structure constants" typical of such representations in accordance with the existing q-modified commutation relations in that "adjoint" context in particular!

This motivates the q-electroweak theory developed by Finkelstein [6]: the corresponding Lagrangian density consequently contains quadrilinear terms (Section 5) showing such a q-dependence which should be related to "experimental results" appearing, maybe, in particular in an isotopic context.

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Description of Anti-Fock Representation Set of $*$-Algebras Generated by the Relation $XX^* = f(X^*X)$

Andrew BONDARENKO

Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, Kyiv, Ukraine
E-mail: *bondarenkoa@ukr.net*

In present paper the method of description of anti-Fock representations of the relation $XX^* = f(X^*X)$ for unimodal mapping f was devised. Examples of description of anti-Fock representation sets for piecewise linear-fractional mappings under condition of existence of attracting cycle were given.

1 Introduction

There were many investigations on the representations of $*$-algebras

$$\mathcal{A}_f = \mathbb{C}\langle X, X^* \,|\, XX^* = f(X^*X)\rangle \tag{1}$$

and their enveloping C^*-algebras. Some of them are purely mathematical, some are motivated by physics [4,5]. For example, a general framework was constructed in the paper [6]. Recently there have appeared papers where attention was focused on $*$-algebras connected with a non-bijective simple dynamical system (f, I). In particular, quantum inharmonic oscillator algebra [7] and "unimodal deformation" of a quantum disk algebra [2] were considered. In the present paper we eliminate the condition of simplicity of a dynamical system, and try to classify the anti-Fock representations of $\mathcal{A}_f$.

Description of representations of $\mathcal{A}_f$ is closely connected with description of positive orbits of dynamical system $(f, \mathbb{R})$ [6]. By positive orbits of dynamical system $(f, \mathbb{R})$ we imply sequences

1) $\omega = (x_k)_{k\in\mathbb{Z}}:\ f(x_k) = x_{k+1},\ x_k > 0$ for all k;
2) $\omega = (x_k)_{k\in\mathbb{N}}:\ x_1 = 0,\ f(x_k) = x_{k+1},\ x_k \geq 0$ for $k > 1$ (Fock orbit);
3) $\omega = (x_{-k})_{k\in\mathbb{N}}:\ x_{-1} = 0,\ f(x_{-k}) = x_{-k+1},\ x_{-k} \geq 0$ for $k > 1$ (anti-Fock orbit).

For every positive orbit one may put in correspondence an irreducible representation of $\mathcal{A}_f$ [6], in particular each anti-Fock representation corresponds to each anti-Fock orbit.

The idea of anti-Fock representation description is to construct one-to-one correspondence among anti-Fock orbits and the paths on a certain graph. To realize this idea we introduce in the Section 2 the definitions of P-partition of the dynamical system (X, σ) and corresponding transition graph Γ_P. After that we prove Theorem 1 that establishes one-to-one correspondence between the set of orbits passing through point $x \in X$ and a set of certain paths on a graph Γ_P.

In the Section 3 of this paper we introduce the definition of negative elements of P-partition and prove Theorem 3. It establishes one-to-one correspondence between the set of anti-Fock representations of $\mathcal{A}_f$ and the set of infinite paths on the graph Γ_P that start from the vertex corresponding to point 0 and do not pass through this vertex and vertices of the set N.

In the Section 4 we give description of anti-Fock representation set of $\mathcal{A}_f$, where $f = f_{l,s,t}$ is piecewise linear-fractional mapping depending on parameters.

2 Partitions

Definition 1. Let (X, f) be a dynamical system. We say that a partition $X = \bigcup\limits_{j\in T} S_j$, $S_i \cap S_j = \varnothing$, $i \neq j$, where index set T generally is not countable, is a P-partition, if the following conditions are held:

1) $\forall\ i \in T$ either $f^{-1}(S_i) = \varnothing$ or there exists a partition $f^{-1}(S_i) = \bigcup\limits_{j\in Q_i} U_j$, $U_i \cap U_j = \varnothing$, $i \neq j$ such that $\forall\ j \in Q_i$ $f(U_j) = S_i$ and f is one to one on U_j;

2) for any two sets U_{j_1}, U_{j_2}, $j_1, j_2 \in Q_i$ there are sets S_{i_1} and S_{i_2}, such that $U_{j_1} \subseteq S_{i_1}$, $U_{j_2} \subseteq S_{i_2}$, $i_1, i_2 \in T$.

We will also assume in this paper that

2′) if $j_1 \neq j_2$ then $i_1 \neq i_2$.

The following graph can be associated with P-partition: $\Gamma_P = \{\Gamma_{P0}, \Gamma_{P1}\}$, where $\Gamma_{P0} = \{v_j : j \in T\}$ is the set of vertices, and $\Gamma_{P1} = \{s_{ij} : i, j \in T,\ S_i \subseteq \sigma(S_j)\}$ is the set of ages.

Theorem 1. *Let (X, σ) be a dynamical system that possesses P-partition. And let Γ_P be a corresponding graph. Then there exists one-to-one correspondence between the set of orbits passing through the point $x \in X$ and the set of infinite paths on the graph Γ_P starting from the vertex that corresponds to the element of the partition containing point x.*

Proof. Let $\Delta_x = \{\{x_k\}_{k\in\mathbb{Z}} : x_{-1} = x\}$ be a set of orbits passing through the point x. Denote $\{v_{j_i}\}_{i\in\mathbb{N}}$, $v_{j_i} \in \Gamma_{P0}$ is a path that starts at vertex v_{j_1}, K_v is the element of P-partition that corresponds to vertex v and $W_x = \{\{v_{j_i}\}_{i\in\mathbb{N}}, x \in K_{v_{j_1}}\}$, is a set of paths that start at vertex v_{j_1} such that corresponding element of P-partition contains point x. Define mapping $\phi : \Delta_x \to W_x$ in the following way: $\phi(\{x_k\}_{k\in\mathbb{Z}}) = \{v_j\}_{j\in\mathbb{N}}$, $x_{-j} \in K_{v_j}$. From conditions 1, 2 in the definition of P-partition it follows that mapping ϕ is well-defined. Let now $\delta_1, \delta_2 \in \Delta_x$, $\delta_1 \neq \delta_2$ $\delta_1 = \{x_k\}_{k\in\mathbb{Z}}$, $\delta_2 = \{y_k\}_{k\in\mathbb{Z}}$. Let k_0 be the least integer such that $x_k = y_k$ for all $k \geqslant k_0$. Then from conditions 2, 2′ in the definition of P-partition it follows that x_{k_0-1}, y_{k_0-1} belong to different elements of P-partition, therefore $\phi(\delta_1) \neq \phi(\delta_2)$. So ϕ is injection.

Let now $p = \{v_{j_i}\}_{i\in\mathbb{N}} \in W_x$ be a path. Then from construction of the graph and from condition 2 in definition of P-partition it follows that there exists a sequence $\{x_{-k}\}_{k\in\mathbb{N}}$, $\sigma(x_{-k-1}) = x_{-k}$, $x_{-1} = x$ such that $x_{-k} \in v_{j_k}$. Therefore $\phi(\dots, x_{-3}, x_{-2}, x_{-1}, \sigma(x_{-1}), \sigma^{(2)}(x_{-1}), \dots) = p$. So mapping ϕ is bijection. ■

Leaving the description of P-partition construction in general case for further investigations we will concentrate on the partitions of interval.

Consider a unimodal dynamical system $(f, [c, d])$ such that

$$f(x) = \begin{cases} f_1(x), & x \in [c, b], \\ f_2(x), & x \in (b, d], \end{cases} \tag{2}$$

$c \leq f(c) < d$, $f_1(b) = f_2(b) = d$, $f(d) = c$.

To simplify our work we rewrite the definition of P-partition in more convenient and simplified form just by choosing the partition of preimage.

Definition 2. Let (f, I) be a dynamical system with mapping f defined in (2). We assume that $f_i^{-1}(x)$ is undefined for some $x \in I$, when there does not exist $x' \in I$ such that $f(x') \stackrel{\text{def}}{=} f_i(x') = x$, $i = 1, 2$.

We say that dynamical system (f, I) possesses a P-partition whenever $I = \bigcup\limits_j S_j$, $j \in T$, $S_i \cap S_j = \varnothing$, $i \neq j$, where index set T generally is not countable, and the following conditions are being held.

1. $\forall\ j \in T$ $f_1^{-1}(x)$ and $f_2^{-1}(x)$ are either undefined for all $x \in S_j$ or defined for all $x \in S_j$.

2. a) $\forall\ j \in T$ whenever mappings are defined $f_1^{-1}(S_j) \subseteq S_s$ and $f_2^{-1}(S_j) \subseteq S_t$ for some $s, t \in T$, b) $s \neq t$.

Denote $\delta_{(x)}$ unilateral orbit $(x, f(x), f^{(2)}(x), \dots)$ of the point x.

Definition 3. We say that the partition of the interval on the disjoint elements *is defined by the closed set* S, whenever

1) any point at S is the element of the partition,

2) other elements of the partition are opened intervals.

Proposition 1. *Let (f, I) be a dynamical system with mapping f defined in* (2)*. Then any closed set S, such that $f(S) \subseteq S$ and $\delta_{(b)} \subseteq S$ defines a P-partition.*

Proof. First condition in definition of P-partition is fulfilled because $f^{(n)}(d) \in S$, $n \geqslant 0$.

Let's check the second condition. For one-point elements of partition it is clear. Let $I_{ij} = (x_i, x_j)$, $x_i, x_j \in S$, $i \neq j$ be a certain interval of partition. Let suppose that $f_1^{-1}(I_{ij}) \not\subseteq I_{kl}$ for some k and l, hence there exists a point $x \in f_1^{-1}(I_{ij})$, $x \in S$. Since $f(S) \subset S$ then there exists a point $f(x) \in I_{ij}$, $f(x) \in S$. Thus we came to contradiction. For f_2^{-1} is similar. f_1^{-1}, f_2^{-1} can't map different intervals into one because $b \in S$. ■

3 Anti-Fock representation set description

The following theorem (see [6]) connects representations of $*$-algebra A_f with certain orbits of dynamical system $(f, \mathbb{R}_+)$.

Theorem 2 (V. Ostrovskyi V. and Yu. Samoilenko [6]). *There exists the following correspondence between positive orbits of dynamical system $(f, \mathbb{R}_+)$ and irreducible representations of $*$-algebra $\mathcal{A}_f = \mathbb{C}\langle X,\ X^* \mid XX^* = f(X^*X)\rangle$.*

1. There is an irreducible representation π_ω in Hilbert space $l_2(Z)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k \in Z$ and $X = UC$ is a polar decomposition that corresponds to every positive non-cyclic orbit $\omega(x_k)_{k\in Z}$.

2. There is an irreducible representation π_ω in Hilbert space $l_2(N)$ given by the formulae: $Ue_0 = 0$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$ that corresponds to positive non-cyclic Fock-orbit $\omega = (x_k)_{k\in N}$.

3. There is an irreducible representation π_ω in Hilbert space $l_2(N)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$ that corresponds to positive non-cyclic anti-Fock-orbit $\omega = (x_{-k})_{k\in N}$.

4. There is a family of m-dimensional irreducible representations $\pi_{\omega,\phi}$ in Hilbert space $l_2(\{1, \dots, m\})$ given by the formulae: $Ue_0 = e^{i\phi}e_{m-1}$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k = 1, \dots, m$; $0 \leq \phi \leq 2\pi$ and $X = UC$ that corresponds to cyclic positive orbit $\omega = (x_k)_{k\in N}$ of length m.

Note that in the case when mapping f is not polynomial the $*$-algebra $\mathcal{A}_f$ is undefined. Instead of it we may consider $C^*(\mathcal{A}_f)$, a C^*-algebra obtained from free $*$-algebra $\mathcal{F}(X, X^*)$ generated by X with sub-norm $\|b\| = \sup_\pi \|\pi(b)\|$, where supremum is taken over all $\pi \in \mathrm{Rep}(\mathcal{F}(X, X^*))$ such that $\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure.

Definition 4. We call the element S_i of P-partition positive or respectively negative if $S_i \subset [0, +\infty)$ or respectively $S_i \subset (-\infty, 0)$. Let N be a set of vertices of the graph Γ_P that correspond to negative elements of the partition.

Theorem 3. *Let (f, I) be a dynamical system with mapping f defined in (2), and let Γ_P be a graph associated with P-partition defined by the set $S = \{\delta_{(b)} \cup \delta_{(0)} \cup \omega(\delta_{(b)}) \cup \omega(\delta_{(0)})\}$. Then there exists one-to-one correspondence between the set of anti-Fock representations of $\mathcal{A}_f$ and the set of infinite paths on the graph Γ_P that start from the vertex corresponding to point 0 and do not pass through this vertex and vertices of the set N.*

Proof. By Proposition 1 the set S defines a P-partition. By Theorem 1 there exists one-to-one correspondence between the set of anti-Fock orbits and the set of infinite paths on the graph Γ_P starting from the vertex that corresponds to the element of the partition containing point 0. Since point 0 belongs to S, all elements of the partition are either positive or negative. Therefore anti-Fock orbits correspond to paths that do not pass through vertex corresponding to 0 and vertices of the set N. ■

Remark 1. We can reduce the number of elements in the partition mentioned in Theorem 3 just by splitting elements $f^n(0)$ with appropriate intervals. For example, 0 with interval $(0, x)$.

Corollary 1. *Let (f, I) be a dynamical system with mapping f defined in (2) and let Γ_P be a graph associated with P-partition defined by the set $S = \{\delta_{(b)} \cup \omega(\delta_{(b)})\}$. Let Q be an element of P-partition such that $0 \in Q$. And let any preimage of zero do not belong to Q or equivalently no closed path starting from corresponding vertex. Then there exists one-to-one correspondence between the set of anti-Fock representations of $\mathcal{A}_f$ and the set of infinite paths on the graph Γ_P that start from the vertex corresponding to element Q and do not pass through vertices of the set N.*

4 Linear-fractional mappings

Many important examples of $*$-algebras, C^*-algebras and their representations arising in physical models are connected with dynamical systems. In particular, the two-parameter unit quantum disk algebra [4]

$$\mathbb{C}\langle z, z^* \mid qzz^* - z^*z = q - 1 + \mu(1 - zz^*)(1 - z^*z)\rangle,$$
$$0 \le \mu \le 1, \qquad 0 \le q \le 1, \qquad (\mu, q) \ne (0, 1),$$

that can be rewritten [6] in the form $\mathcal{A}_f = \mathbb{C}\langle X,\, X^* \mid XX^* = f(X^*X)\rangle$, where

$$F(\lambda) = \frac{(q+\mu)\lambda + 1 - q - \mu}{\mu\lambda + 1 - \mu}.$$

In present paper we investigate "unimodal deformation" of the relation written above and describe anti-Fock representations of a corresponding $*$-algebra.

Consider a continuous unimodal piecewise linear-fractional map $f : \mathbb{R} \to \mathbb{R}$ that consists of two hyperbolae:

$$f(x) = \begin{cases} f_1(x) = \dfrac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}, & x \leqslant b, \\ f_2(x) = \dfrac{\alpha_2 x + \beta_2}{\gamma_2 x + \delta_2}, & x > b. \end{cases} \tag{3}$$

According to paper [3] we restrict ourselves to the following conditions

$$f_1(x) \text{ is increasing}, \quad f_2(x) \text{ is decreasing}, \quad b > 0, \quad f(b) > b, \quad f(f(b)) < b,$$
$$0 \geqslant f^{(2)}(b), \quad f^{(3)}(b) \geqslant f^{(2)}(b). \tag{4}$$

Under these conditions anti-Fock orbit sets of dynamical systems $(f, \mathbb{R})$ and $(f, [f(f(b)), f(b)])$ coincide. Hence we will consider dynamical systems as defined in (2).

Considering mapping (3) up to the topological conjugacy [8, 1] under condition $(\alpha_1 - \delta_1)^2 + 4\gamma_1\beta_1 \geqslant 0$ we may assume $\gamma_1 = 0$, $\alpha_2 = 1$.

Thus we restrict ourselves to the following set of mappings (see Fig. 1): $f = f_{l,s,t} : [0, 1] \to [0, 1]$, $l > 0$,

$$f_{l,s,t}(x) = \begin{cases} f_1(x) = lx + a, & x \in [0, b], \\ f_2(x) = \dfrac{x - 1}{sx + t}, & x \in (b, 1], \end{cases} \tag{5}$$

Conditions $f_1(b) = f_2(b) = 1$, $f_2(1) = 0$ give us formulae for a and b via l, s, t:

$$a = 1 - l\frac{t + 1}{1 - s}, \quad b = \frac{t + 1}{1 - s}.$$

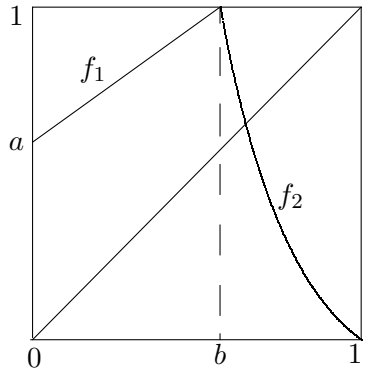


Figure 1

In paper [1] the conditions of existence of stable cycle $\gamma_n = \{x_1, \ldots, x_n\}$ such that

$$x_i < x_{i+1}, \qquad f(x_i) = x_{i+1}, \qquad i = 1, \ldots, n - 1, \qquad f(x_n) = x_1$$

were obtained.

Let us display some necessary results from paper [1].

Conditions $b \in (0, 1)$, $f_2'(x) < 0$ and $a \in [0, 1)$ give us following restrictions:

$$s < 1, \qquad -1 < t < -s, \qquad t \leqslant \frac{1 - s}{l} - 1.$$

So we will assume that parameters l, s, t lie in the area

$$\Pi = \left\{ (l, s, t) : \ l > 0, \ s < 1, \ t \in \left(-1, \min\left\{ -s, \frac{1 - s}{l} - 1 \right\} \right) \right\}.$$

Denote

$$L_n = 1 + l + l^2 + \cdots + l^n = \frac{1 - l^{n+1}}{1 - l}.$$

Theorem 4. *Mapping f has the cycle γ_n iff*

$$t \geqslant -\frac{l^{n-2} + sL_{n-3}}{L_{n-2}} \quad \textit{and it is attracting iff} \quad \begin{cases} s < \dfrac{1}{L_{n-1}}, \\ t < \dfrac{l^{n-1} + s(1 - 2l^{n-1}) - s^2 L_{n-2}}{sL_{n-1} - 1}. \end{cases}$$

We denote the corresponding areas of parameters by Π_n

$$\Pi_n = \left\{(l,s,t):\ \frac{l^{n-1}+s(1-2l^{n-1})-s^2L_{n-2}}{sL_{n-1}-1} > t \geqslant \frac{-l^{n-2}-sL_{n-3}}{L_{n-2}}, \frac{1}{L_{n-1}} > s\right\}.$$

The lower border of each area Π_n is defined by the surface of existence, we denote it by E_n. The upper border is defined by surface of stability, we denote it by S_n.

On the Fig. 2 we give section of these areas for $s = -2$.

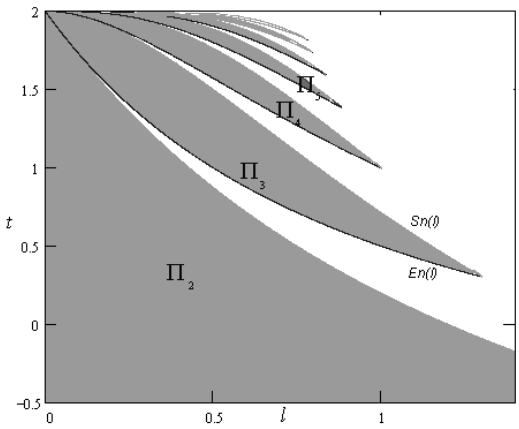


Figure 2.

It is well-known [2], [7] that the structure of anti-Fock representation set as well as the structure of $C^*(\mathcal{A}_f)$ heavily depends on the dynamics of (f, I). We will show that anti-Fock representation set has a good description whenever the corresponding dynamical system has an attracting cycle.

The following theorem from [8] helps us to describe possible attracting cycles of dynamical system generated by piecewise linear mappings ($s = 0$). In Theorem 6 we will give such a description.

Theorem 5. *Let $f : I \to I$ be a unimodal mapping, $f \in C^3(I\backslash\{c\})$ and $Sf(x) = 0$, $x \in I\backslash\{c\}$ ($Sf(x)$ is Schwarzian derivative of $f(x)$, $Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$) then the set of all not repellent cycles is either empty or consists of one attracting or semiattracting cycle or represents the cycle of closed intervals $B = \{J_0, J_1, \ldots, J_{n-1}\}$ of some period $n \geqslant 1$ such that $f^{(n)}(x) = x$, $x \in J_0$ and the point c is one of the ends of interval J_0.*

Theorem 6. *Let $f : I \to I$, $I = [c, d]$ be a continuous piecewise linear unimodal mapping such that ($f(b) = \max\limits_{x\in I} f(x) = d$, $f(d) = c$, $f(c) \geq c$). Whenever corresponding dynamical system has a stable cycle then this cycle is of type γ_n.*

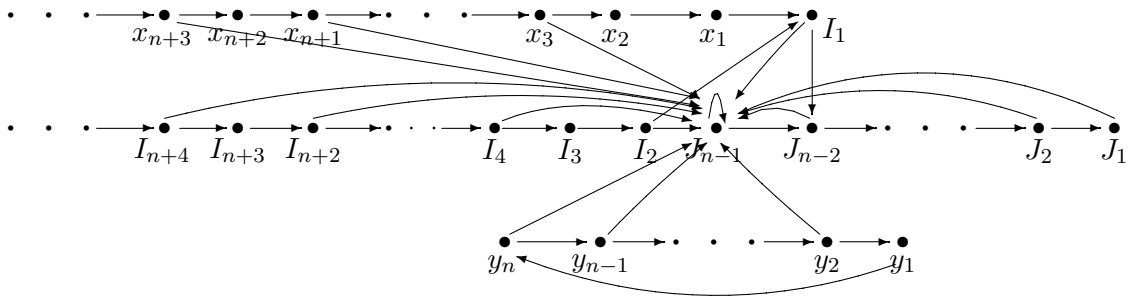
Proof. First of all we recall that cycle γ_n exists iff $s-2$ iterations of point cv are smaller than b. Let the stable cycle $\beta = \{x_1, x_2, \ldots x_k\}$, $k > 2$ exist. It is easy to check that there exists a sequence of points $x_j, x_{j+1}, \ldots, x_{j+s}$ such that $x_i < b$, $i = j, \ldots, j+s-1$, $x_{j+s} > b$, $s \geq 1$, $1 \leq j \leq k-s$ and product of derivatives at these points equals to u, $|u| < 1$. Also $s-2$ iterations of point c are smaller than b. Therefore the cycle γ_s exists. And product of derivatives at points of this cycle also equals u. Hence cycle γ_s is stable. According to Theorem 5 there exists only one stable cycle. Therefore $\beta = \gamma_s$. ■

The following proposition gives description of anti-Fock representation set of $C^*(\mathcal{A}_{f_{l,s,t}})$ under condition of existing of stable cycle.

Proposition 2. *Let point (l, s, t) be an inner point of area P_n. Then there exists countable P-partition of the following form:*

Let $I_1 = (f^{(2n-1)}(1), b]$, $I_2 = (f^{(2n)}(1), 1)$ and $I_j = f^{(j-2)}(I_2)$, $j \geqslant 3$. Then $I = \underset{j}{\cup} I_j \cup (f^{(n+1)}(1), f^{(2)}(1)) \cup (f^{(n+2)}(1), f^{(3)}(1)) \cup \cdots \cup (f^{(2n-2)}(1), f^{(n-1)}(1)) \cup (b, f^{(n)}(1)) \cup S = \underset{j}{\cup} I_j \cup J_1 \cup J_2 \cup \cdots \cup J_{n-1} \cup S$, where set S consists of the points of unilateral orbit $\delta_{(1)} = (f^{(k)}(1), k \geqslant 0) = (x_i)_{i \in \mathbb{N}}$ and points of the cycle $\gamma_n = \{y_1, y_2, \ldots, y_n\}$.

Corresponding transition graph has the following form:



There exists one-to-one correspondence between the paths on the graph that start at point x_2, and do not pass this point, and irreducible anti-Fock representations of $C^*(\mathcal{A}_f)$.

Proof. Structure of P-partition and corresponding transition graph follows from $\omega(\delta_{(1)}) = \gamma_n$ and can be checked by the direct calculations.

From $f_1^{-1}(x) = \varnothing$, $x \in [0, f(0))$ it follows that: $f_1^{-1}(I_{nk+3}) = \varnothing$, $f_1^{-1}(x_{nk+2}) = \varnothing$, $k \geqslant 0$, and $f_1^{-1}(J_1) = \varnothing$, $f_1^{-1}(y_1) = \varnothing$.

We also have:

$$
\begin{aligned}
&f_1^{-1}(I_t) = I_{t-1}, \quad t \neq nk+3, \quad t \geqslant 2, \quad k \geqslant 0,\\
&f_1^{-1}(x_t) = x_{t-1}, \quad t \geqslant 2, \quad t \neq nk+2, \quad k \geqslant 0,\\
&f_1^{-1}(J_k) = J_{k-1}, \quad 2 \leqslant k \leqslant n-1,\\
&f_1^{-1}(y_k) = y_{k-1}, \quad k \neq 1, f_2^{-1}(I_{nk+3}) = I_{nk+2}, \quad f_2^{-1}(x_{nk+2}) = x_{nk+1}, \quad k \geqslant 0,\\
&f_2^{-1}(I_1) \in J_{n-1}, \quad f_1^{-1}(I_1) \in J_{n-2},\\
&f_2^{-1}(x_1) = \varnothing, \quad f_1^{-1}(x_1) = b \in I_1,\\
&f_2^{-1}(y_1) = y_n, \quad f_2^{-1}(1) = \varnothing.
\end{aligned}
$$

From $f_2^{-1}(x) \in (b, f^{(n)}(1))$ when $x \in (f^{(n+1)}(1), 1)$ it follows that:

$$
\begin{aligned}
&f_2^{-1}(I_t) \in J_{n-1}, \quad t \neq nk+3, \quad k \geqslant 1,\\
&f_2^{-1}(x_t) \in J_{n-1}, \quad t \neq 1, \quad t \neq nk+2, \quad k \geqslant 1,\\
&f_2^{-1}(J_t) \in J_{n-1}, \quad 1 \leqslant t \leqslant n-1,\\
&f_2^{-1}(y_k) \in J_{n-1}, \quad k \neq 1.
\end{aligned}
$$

One-to-one correspondence between paths and representations follows from Theorem 3. ■

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On Finiteness of Critical Tits Forms of Posets

Vitalij M. BONDARENKO and Andrij M. POLISHCHUK

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *vit-bond@imath.kiev.ua*

In the paper we consider the Tits form of (finite and infinite) posets. We call such a form critical (with respect to positivity) if it is not positive but each of its proper subform is positive, and prove that a critical Tits form is finite (i.e. is of the Tits form of a finite poset).

The quadratic Tits form, introduced by P. Gabriel [1] for quivers, Yu.A. Drozd [2] for posets, and M.M. Kleiner, A.V. Roiter [3] and Yu.A. Drozd [4] for a wide class of classification problems, plays an important role in representation theory. In particular, there are many results on connections between representation types of various objects and properties of the Tits forms. The reader interested in this topic is refereed to the papers of [5,6], the monographs [7,8] and, e.g., [9–15] (with the bibliographies therein). Above all one must mention the well known result that a quiver is of finite type if and only if its Tits form is positive [1]; in the case of posets the Tits form must be weakly positive [2] (recall that representations of posets were introduced in [16]). It follows from the results of [2] that the Tits form of a poset S is weakly positive if and only if S contains no subposet isomorphic to a critical, with respect to finiteness of type, poset (critical posets are indicated in [17]; their number is 5).

Our paper is devoted to study critical, with respect to positivity, Tits forms of posets.

Formulate first the main result.

Let S be a (finite or infinite) poset and $\mathbb{Z}$ the integer numbers. Denote by $\mathbb{Z}_0^{S\cup 0}$ the subset of the cartesian product $\mathbb{Z}^{S\cup 0}$ consisting of all vectors $z=(z_i)$, $i\in S\cup 0$ with only finitely many non-zero coordinates. The quadratic Tits form of S is by definition the form $q_S:\mathbb{Z}_0^{S\cup 0}\to\mathbb{Z}$ defined by the equality

$$q_S(z)=z_0^2+\sum_{i\in S}z_i^2+\sum_{i<j,i,j\in S}z_iz_j-z_0\sum_{i\in S}z_i.$$

This form (as an arbitrary one) is called positive if it takes a positive value on every nonzero vector $z\in\mathbb{Z}_0^{S\cup 0}$, and non-positive if otherwise.

A quadratic form is called finite if the number of the variables is finite. In the case of the Tits form $q_S(z)$ it means that S is finite.

Denote by $\mathcal{P}$ the set of all (finite and infinite) posets and set $\mathcal{F}=\{q_S(z)|\,S\in\mathcal{P}\}$. A form $=q_S(z)$ from $\mathcal{F}$ is said to be critical with respect positivity (or simply critical) if f is not positive, but $f'=q_{S'}(z)$ is positive for any proper subposet S' of S.

Theorem 1. *Any critical Tits form $q_S(z)$ is finite.*

Before we prove Theorem 1, we describe infinite posets with positive Tits form.

Let S be a poset. For nonempty subsets X,Y of S, we write $X\lhd Y$ if $x<y$ for some $x\in X$, $y\in Y$, and $X\not\lhd Y$ if otherwise. We say that S is a sum of subsets A and B, and write $S=A+B$, if $A\cap B=\varnothing$ and $S=A\cup B$. If any two elements $a\in A$ and $b\in B$ are incomparable, we call this sum direct; we denote such sum by $S=A\coprod B$. The sum $S=A+B$ is called one-sided if $B\not\lhd A$ or $A\not\lhd B$ (see [18]). Finally, the sum $S=A+B$ is called minimax if $x<y$ with x and y belonging to different summands implies that x is minimal and y maximal in S (see also [18]).

For a sum S of posets A and B let $R^<(A,B)$ denotes the set of pairs $(x,y)\in A\times B$ with $x<y$. Such a pair (a,b) is called short if there is no other such a pair (a',b') satisfying $a\leqslant a'$,

$b' \leqslant b$. By $R_0^<(A,B)$ we denote the subset of all short pairs from $R^<(A,B)$. We call the order $r_0 = r_0(A,B)$ of $R_0(A,B) = R_0^<(A,B) \cup R_0^<(B,A)$ the rank of the sum S.

Note that direct sums are particular cases of sums and, in particular, one-sided sums, minimax sums, etc. (more precisely, they are sums of zero rank). We could exclude them from others, considering sums of ranks 0 and greater than 0 separately (as in [18]), but here we do not do it.

Recall that a linear ordered set is also called a chain. A poset with the only pair of incomparable elements is called an almost chain.

We have the following theorem.

Theorem 2. *Let S be an infinite poset. Then the Tits form of S is positive if and only if one of the following condition holds*:

1) *S is a direct sum of two chains;*

2) *S is a direct sum of a chain and an almost chain;*

3) *S is a one-sided minimax sum (of rank 1) of two chains.*

This theorem describes completely infinite posets with positive Tits form. It was proved by the authors in [18].

We now present the scheme of the proof of the necessity of Theorem 2, calling particular attention to critical posets.

Step 1. It is proved that the Tits form is not positive for the following posets:

$T_1 = \{1,2,3,4\}$ (without comparable $i \neq j$);

$T_2 = \{1,2,3,4 \,|\, 1 \prec 4, 2 \prec 4, 3 \prec 4\}$;

$T_3 = \{1,2,3,4,5,6,7,8 \,|\, 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;

$T_4 = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 7, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7\}$;

$T_5 = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 5 \prec 6 \prec 7 \prec 8, 2 \prec 4, 2 \prec 5, 3 \prec 4\}$;

$T_6 = \{1,2,3,4 \,|\, 1 \prec 3, 1 \prec 4, 2 \prec 3, 2 \prec 4\}$;

$T_7 = \{1,2,3,4,5,6,7 \,|\, 1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 5\}$;

$T_8 = \{1,2,3,4,5,6,7 \,|\, 1 \prec 5, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7\}$;

$T_9 = \{1,2,3,4,5,6,7 \,|\, 1 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 2 \prec 6\}$;

$T_{10} = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;

$T_{11} = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\}$;

$T_{12} = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 7, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;

$T_{13} = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 4, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$;

$T_{14} = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 7\}$;

$T_{15} = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\}$;

$T_{16} = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 2 \prec 5, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\}$;

$T_{17} = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 3 \prec 4 \prec 5 \prec 6 \prec 8, 1 \prec 2 \prec 8\}$.

It is also proved that all these posets are critical.

By $T_i^*, i = 1, \ldots, 17$, we denote the poset dual to T_i. Obviously, the posets T_i^* are also critical.

Step 2. Let S be a poset. Recall that the width of S is defined to be the maximum number $w(S)$ of pairwise incomparable elements of S. An element of S is called nodal if it is comparable to any other element. The set of all nodal elements of S is denoted by S_0.

Let S be an infinite poset and $w(S) = 2$. It is proved that, if S has no subposet isomorphic to T_i or T_i^* with $i > 5$, then the following assertions hold.

Proposition 1. *If S_0 is infinite, then S is an almost chain.*

Proposition 2. *If S_0 is empty, then S is a one-sided minimax sum of two chains.*

Proposition 3. *If S_0 is finite, but not empty, then S is a one-sided minimax sum of a chain and an almost chain.*

From these proposition it follows the necessity of Theorem 2 in the case $w(S) = 2$.

Step 3. Let S be an infinite poset and $w(S) \neq 2$. The case $w(S) = 1$ is trivial and the case $w(S) > 3$ is imposable (because the Tits form of T_1 is not positive). So it remains to consider the case $w(S) = 3$. It is proved that if S has no subposet isomorphic to T_i or T_i^* with $i > 2$, then the following assertions hold.

Proposition 4. *If R is a direct summand of S of order larger than 1, then $S \setminus R$ is an almost chain.*

Proposition 5. *S is not a one-sided minimax sum of a chain and an almost chain.*

The necessity of Theorem 2 in the case $w(S) = 3$ follows from that in the case $w(S) = 2$ and Propositions 4, 5.

It is easy to see that from the proof of the necessity of Theorem 2 it follows Theorem 1. Indeed, from that the next proposition follows.

Proposition 6. *Let S be an infinite poset with non-positive Tits form. Then S contains a subposet isomorphic to T_i or T_i^* for some $l = 1, 2, \ldots, 17$.*

And it follows from this proposition that a critical poset may not be infinite.

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On 2 + 2 Locally Compact Quantum Groups

Y.A. CHAPOVSKY †, *A.A. KALYUZHNYI* † *and G.B. PODKOLZIN* ‡

† *Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *yc@imath.kiev.ua, kalyuz@imath.kiev.ua*

‡ *Institute of Applied System Analysis, National Technical University of Ukraine ("KPI"), 37 Pobedy Ave., 03056 Kyiv, Ukraine*
E-mail: *gpod@mmsa.ntuu-kpi.kiev.ua*

We construct new examples of cocycle bicrossed products of Lie groups by using a relation between 3-cocycles on a Lie algebra and the cocycle-matched pair of the Lie algebras.

1 Introduction

A method for constructing examples of non-commutative and non-cocommutative examples of finite quantum groups, that is Kac algebras, was proposed by G.I. Kac in [1]. This method consists in extending a finite group G_1 with a dual to a finite group G_2. Each such an extension can be given by a group G that contains G_1 and G_2 as subgroups subject to the conditions $G_1 \cap G_2 = \{e\}$ and $G_1 G_2 = G$, called a matched pair of groups, and a pair of 2-cocycles. This method, now known as a cocycle bicrossed product construction, was generalized to bialgebras in [2], on the one hand, and to locally compact quantum groups [3] in [4], on the other hand. The cohomology theory for cocycle bicrossed products was developed in [5].

In the case where G_1 and G_2 are Lie groups, the corresponding notion is that of a cocycle matched pair of Lie algebras [2]. A cohomology theory for cocycle matched pairs of Lie algebras was developed in [6].

In terms of cocycle matched pairs of Lie algebras, an extensive study of low-dimensional locally compact quantum groups has been carried out in [7] for the cocycle bicrossed products. In particular, a complete classification of all locally compact quantum groups that arise from the bicrossed product construction starting with 2- and 1-dimensional Lie algebras is given there. It turned out that one of the cocycles in the constructed cocycle matched pair of groups was equivalent to a trivial one.

The purpose of this article is to construct nontrivial examples of cocycle bicrossed products, in particular, the ones where both cocycles are not equivalent to the trivial ones. We do it by considering two 2-dimensional Lie groups, find the cocycles on the level of the Lie algebras, and then lift them to the matched pairs of the Lie groups. In Section 2 we recall necessary definitions and known facts. We also give results needed for easy calculation of cocycles for matched pairs of Lie algebras. In Section 3 we construct three examples of cocycle bicrossed products. In the first example, we consider $G = \mathbb{R}^2$ and $H = \mathbb{R}^2$ with trivial actions, the second example deals with $G = \mathbb{R} \times \mathbb{R}_+$ and $H = \mathbb{R}_+ \times \mathbb{R}$ with a nontrivial action. In the third example, we take G to be the "$ax + b$" group and $H = \mathbb{R}^2$ with a nontrivial action of G on H.

2 General definitions and properties

Definition 1 ([8]). Let K be a locally compact group, G, H subgroups of K satisfying the conditions

$$G \cdot H = K, \qquad \text{and} \qquad G \cap H = \{e\}. \tag{1}$$

Then (G, H) is called a matched pair of locally compact groups.

Remark 1. For a more general definition of a matched pair of locally compact groups, see [4].

Everywhere in the sequel, elements of G and H are denoted by g, g_1, etc., h, h_1, etc., correspondingly.

Let (G, H) be a matched pair of locally compact groups. It is known [4] that there are right and left actions, $\triangleleft : H \times G \to H$ and $\triangleright : H \times G \to G$, given by

$$h \cdot g = (h \triangleright g) \cdot (h \triangleleft g) \tag{2}$$

and satisfying

$$\begin{aligned}(h_1h_2) \triangleleft g &= (h_1 \triangleleft (h_2 \triangleright g))(h_2 \triangleleft g),\\ h \triangleright (g_1g_2) &= (h \triangleright g_1)((h \triangleleft g_1) \triangleright g_2).\end{aligned} \tag{3}$$

Definition 2 ([4]). Let (G, H) be a matched pair of locally compact groups, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. A pair of measurable maps (u, v), $u : H \times G \times G \to \mathbb{T}$, $v : H \times H \times G \to \mathbb{T}$ is called a pair of cocycles for the pair (G, H) if

$$u(h \triangleleft g_1; g_2, g_3)u(h; g_1, g_2g_3) = u(h; g_1, g_2)u(h; g_1g_2, g_3), \tag{4}$$

$$v(h_1, h_2; h_3 \triangleright g)v(h_1h_2, h_3; g) = v(h_1, h_2h_3; g)v(h_2, h_3; g), \tag{5}$$

$$\begin{aligned}v(h_1, h_2; g_1g_2)u(h_1h_2; g_1, g_2) = v(h_1, h_2; g_1)u(h_2; g_1, g_2) \cdot v(h_1 \triangleleft (h_2 \triangleright g_1), h_2 \triangleleft g_1; g_2)\\ \cdot u(h_1; h_2 \triangleright g_1, (h_2 \triangleleft g_1) \triangleright g_2).\end{aligned} \tag{6}$$

Definition 3 ([4]). Two pairs of cocycles (u_1, v_1) and (u_2, v_2) for (G, H) are called equivalent if there exists a measurable function $r : H \times G \to \mathbb{T}$ such that

$$\begin{aligned}u_1(h; g_1, g_2)u_2(h; g_1, g_2)^{-1} &= r(h; g_1)r(h \triangleleft g_1; g_2)r(h; g_1g_2)^{-1},\\ v_1(h_1, h_2; g)v_2(h_1, h_2; g)^{-1} &= r(h_1h_2; g)r(h_1; h_2 \triangleright g)^{-1}r(h_2; g)^{-1}.\end{aligned} \tag{7}$$

Let now G and H be Lie groups and $\mathfrak{g}$, $\mathfrak{h}$ be the corresponding Lie algebras. For Lie algebras there is a corresponding notion of a matched pair.

Definition 4 ([2]). A pair $(\mathfrak{g}, \mathfrak{h})$ is called a matched pair of Lie algebras if there are left and right actions $\triangleright : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{g}$ and $\triangleleft : \mathfrak{h} \otimes \mathfrak{g} \to \mathfrak{h}$ satisfying

$$\begin{aligned}X \triangleright [A_1, A_2] &= [X \triangleright A_1, A_2] + [A_1, X \triangleright A_2] + (X \triangleleft A_1) \triangleright A_2 - (X \triangleleft A_2) \triangleright A_1,\\ [X_1, X_2] \triangleleft A &= [X_1 \triangleleft A, X_2] + [X_1, X_2 \triangleleft A] + X_1 \triangleleft (X_2 \triangleright A) - X_2 \triangleleft (X_1 \triangleright A),\end{aligned} \tag{8}$$

where $A, A_1, A_2 \in \mathfrak{g}$ and $X, X_1, X_2 \in \mathfrak{h}$.

Definition 5 ([2]). For a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, a pair of linear mappings (U, V), $U : \mathfrak{h} \otimes (\mathfrak{g} \wedge \mathfrak{g}) \to \mathbb{R}$ and $V : (\mathfrak{h} \wedge \mathfrak{h}) \otimes \mathfrak{g} \to \mathbb{R}$ is called a pair of cocycles for $(\mathfrak{g}, \mathfrak{h})$ if

$$\begin{aligned}&U(X \triangleleft A_1; A_2, A_3) + U(X \triangleleft A_2; A_3, A_1) + U(X \triangleleft A_3; A_1, A_2)\\ &\quad = -U(X; A_1, [A_2, A_3]) - U(X; A_2, [A_3, A_1]) - U(X; A_3, [A_1, A_2]),\end{aligned} \tag{9}$$

$$\begin{aligned}&V(X_1, X_2; X_3 \triangleright A) + V(X_2, X_3; X_1 \triangleright A) + V(X_3, X_1; X_2 \triangleright A)\\ &\quad = V(X_1, [X_2, X_3]; A) + V(X_2, [X_3, X_1]; A) + V(X_3, [X_1, X_2]; A),\end{aligned} \tag{10}$$

$$\begin{aligned}&U([X_1, X_2]; A_1, A_2) + V(X_1, X_2; [A_1, A_2])\\ &\quad = U(X_1; X_2 \triangleright A_1, A_2) + U(X_1; A_1, X_2 \triangleright A_2) - X_1 \leftrightarrow X_2\\ &\qquad + V(X_1, X_2 \triangleleft A_1; A_2) + V(X_1 \triangleleft A_1, X_2; A_2) - A_1 \leftrightarrow A_2,\end{aligned} \tag{11}$$

where $X_1 \leftrightarrow X_2$ and $A_1 \leftrightarrow A_2$ mean that the corresponding variables are interchanged in the preceding expression.

Definition 6 ([2]). Two pairs of cocycles (U_1, V_1) and (U_2, V_2) for a matched pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ are called equivalent if there is a linear map $R : \mathfrak{h} \otimes \mathfrak{g} \to \mathbb{R}$ such that, for all $A, A_1, A_2 \in \mathfrak{g}$ and $X, X_1, X_2 \in \mathfrak{h}$,

$$\begin{aligned} U_1(X; A_1, A_2) - U_2(X; A_1, A_2) &= R(X \triangleleft A_1; A_2) - R(X \triangleleft A_2; A_1) - R(X; [A_1, A_2]), \\ V_2(X_1, X_2; A) - V_1(X_1, X_2; A) &= R(X_1; X_2 \triangleright A) - R(X_2; X_1 \triangleright A) - R([X_1, X_2]; A). \end{aligned} \tag{12}$$

Let $(\mathfrak{g}, \mathfrak{h})$ be a matched pair of Lie algebras. Then it is known [2] that the vector space $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g}$ can be endowed with a Lie algebra structure defined by

$$[X, A] = X \triangleright A + X \triangleleft A \tag{13}$$

for $A \in \mathfrak{g}$, $X \in \mathfrak{h}$, and extending it to $\mathfrak{k}$ by linearity.

Let now $(\mathfrak{g}, \mathfrak{h})$ be a matched pair of Lie algebras and $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g}$ be the Lie algebra with the bracket defined by (13). We consider $\mathbb{R}$ as a trivial $\mathfrak{k}$-module.

Proofs of the following two propositions are straightforward, so we omit them.

Proposition 1. *Let $F : \mathfrak{k} \wedge \mathfrak{k} \wedge \mathfrak{k} \to \mathbb{R}$ be a 3-cocycle on $\mathfrak{k}$ with coefficients in $\mathbb{R}$ such that*

$$F(X_1, X_2, X_3) = 0 \qquad \text{and} \qquad F(A_1, A_2, A_3) = 0 \tag{14}$$

for all $X_1, X_2, X_3 \in \mathfrak{h}$ and $A_1, A_2, A_3 \in \mathfrak{g}$. Then $U : \mathfrak{h} \otimes (\mathfrak{g} \wedge \mathfrak{g}) \to \mathbb{R}$ and $V : (\mathfrak{h} \wedge \mathfrak{h}) \otimes \mathfrak{g} \to \mathbb{R}$ defined by

$$U(X; A_1, A_2) = F(X, A_1, A_2), \qquad V(X_1, X_2; A) = F(X_1, X_2, A) \tag{15}$$

make a pair of cocycles for the matched pair $(\mathfrak{g}, \mathfrak{h})$.

Conversely, let (U, V) be a pair of cocycles for the matched pair $(\mathfrak{g}, \mathfrak{h})$. Define $F : \mathfrak{k} \wedge \mathfrak{k} \wedge \mathfrak{k} \to \mathbb{R}$ by

$$\begin{aligned} F(A_1 + X_1, A_2 + X_2, A_3 + X_3) &= U(X_3; A_1, A_2) - U(X_2; A_1, A_3) + U(X_1; A_2, A_3) \\ &\quad + V(X_2, X_3; A_1) - V(X_1, X_3; A_2) + V(X_1, X_2; A_3), \end{aligned} \tag{16}$$

where $A_1, A_2, A_3 \in \mathfrak{g}$ and $X_1, X_2, X_3 \in \mathfrak{h}$. Then F is a 3-cocycle on $\mathfrak{k}$ satisfying (14).

Proposition 2. *Let F_1 and F_2 be two 3-cocycles on $\mathfrak{k}$ satisfying (14). Assume that F_1 and F_2 are cohomologous such that $F_1 - F_2 = dS$, where S is a 2-cochain on $\mathfrak{k}$ satisfying*

$$S(A_1, A_2) = 0 \qquad \text{and} \qquad S(X_1, X_2) = 0 \tag{17}$$

for all $A_1, A_2 \in \mathfrak{g}$ and $X_1, X_2 \in \mathfrak{h}$. Then the corresponding pairs of cocycles given by (15) for the matched pair $(\mathfrak{g}, \mathfrak{h})$ are equivalent with

$$R(X; A) = S(X, A). \tag{18}$$

Conversely, let pairs (U_1, V_1) and (U_2, V_2) be equivalent. Then the 3-cocycles F_1 and F_2 on $\mathfrak{k}$ defined by (16) are cohomologous, $F_1 - F_2 = dS$, where S satisfies (17) and is given by

$$S(A_1 + X_1, A_2 + X_2) = R(X_1; A_2) - R(X_2; A_1). \tag{19}$$

3 Examples of cocycle bicrossed products of Lie groups

As was mentioned in Introduction, the bicrossed product construction [4] can be carried out by starting with a matched pair of Lie groups (G, H) and then finding a pair of cocycles satisfying equations of (4), (5), and (6). These cocycles will be sought for in the form

$$u(h; g_1, g_2) = e^{i\tilde{u}(h;g_1,g_2)}, \qquad v(h_1, h_2; g) = e^{i\tilde{v}(h_1,h_2;g)}, \tag{20}$$

where $\tilde{u} : H \times G \times G \to \mathbb{R}$, $\tilde{v} : H \times H \times G \to \mathbb{R}$, so that (4), (5), and (6) become

$$\tilde{u}(h \triangleleft g_1; g_2, g_3) + \tilde{u}(h; g_1, g_2 g_3) = \tilde{u}(h; g_1, g_2) + \tilde{u}(h; g_1 g_2, g_3) \mod 2\pi \tag{21}$$

$$\tilde{v}(h_1, h_2; h_3 \triangleright g) + \tilde{v}(h_1 h_2, h_3; g) = \tilde{v}(h_1, h_2 h_3; g) + \tilde{v}(h_2, h_3; g) \mod 2\pi, \tag{22}$$

$$\begin{aligned}\tilde{v}(h_1, h_2; g_1 g_2) + \tilde{u}(h_1 h_2; g_1, g_2) &= \tilde{v}(h_1, h_2; g_1) + \tilde{u}(h_2; g_1, g_2) \\ &+ \tilde{v}(h_1 \triangleleft (h_2 \triangleright g_1), h_2 \triangleleft g_1; g_2) + \tilde{u}(h_1; h_2 \triangleright g_1, (h_2 \triangleleft g_1) \triangleright g_2) \mod 2\pi.\end{aligned} \tag{23}$$

Then we construct a pair of cocycles (U, V) for the corresponding matched pair of the Lie algebras $(\mathfrak{g}, \mathfrak{h})$ by using Proposition 1, find nonequivalent cocycles in terms of Proposition 2, and consider the corresponding left-invariant forms ω_u and ω_v on $K = GH$. Following the procedure of finding a 3-cocycle on a Lie group from a 3-cocycle on the Lie algebra [9], we consider the 3-simplices σ_u and σ_v in K given by

$$\sigma_v(h_1, h_2, g) = (h_1 h_2^{s_2})^{s_1} g^{s_1 s_2 s_3}, \qquad \sigma_u(h, g_1, g_2) = h^{s_1} (g_1 g_2^{s_3})^{s_1 s_2}, \tag{24}$$

where $s_i : \Delta_3 \to \mathbb{R}$ are some differentiable functions on the standard 3-simplex $\Delta_3 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 : t_1, t_2, t_3 \geq 0,\ t_1 + t_2 + t_3 \leq 1\}$. Then the functions $\tilde{u}$, $\tilde{v}$ are found in the form

$$\tilde{u}(h; g_1, g_2) = \int_{\sigma_u(h,g_1,g_2)} \omega_u, \qquad \tilde{v}(h_1, h_2; g) = \int_{\sigma_v(h_1,h_2,g)} \omega_v. \tag{25}$$

3.1 $G = \mathbb{R}^2$, $H = \mathbb{R}^2$ with trivial actions

Let $G = \mathbb{R}^2$, $H = \mathbb{R}^2$ be a matched pair of Lie groups with addition as the group operation and trivial mutual actions. Then $K = GH = \mathbb{R}^4$ is an Abelian group. We will denote by $g(a, b)$ and $h(x, y)$, $a, b, x, y \in \mathbb{R}$, elements of the groups G and H, respectively.

Let $(\mathfrak{g}, \mathfrak{h})$ be the corresponding matched pair of Abelian two-dimensional Lie algebras with trivial mutual actions. Then $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$ is an Abelian Lie algebra of dimension 4. Let A_1, A_2 form a basis in $\mathfrak{g}$ and X_1, X_2 a basis in $\mathfrak{h}$. Propositions 1 and 2 give that non-equivalent pairs of cocycles are of the following form:

$$U(X_i; A_1, A_2) = \lambda_i, \qquad V(X_1, X_2; A_j) = \mu_j, \tag{26}$$

where $\lambda_i, \mu_j \in \mathbb{R}$, $i, j = 1, 2$. The corresponding left-invariant forms on K are given by

$$\omega_u = \lambda_1 da \wedge db \wedge dx + \lambda_2 da \wedge db \wedge dy, \qquad \omega_v = \mu_1 da \wedge dx \wedge dy + \mu_2 db \wedge dx \wedge dy.$$

The following proposition describes cocycles for the matched pair (G, H).

Proposition 3. *The functions*

$$\tilde{u}(h; g_1, g_2) = (\lambda_1 x + \lambda_2 y) \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \qquad \tilde{v}(h_1, h_2; g) = (\mu_1 a + \mu_2 b) \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}, \tag{27}$$

where $\lambda_i, \mu_i \in \mathbb{R}$, $i = 1, 2$, and $h = h(x, y)$, $h_i = h(x_i, y_i)$, $g = g(a, b)$, $g_i = g(a_i, b_i)$, $i = 1, 2$, are solutions of (21), (22), *and* (23) *thus giving pairs of cocycles for the matched pair of the Lie groups (G, H).*

Proof. One can easily check that if

$$s_1 = 1 - t_1, \qquad s_2 = \frac{1 - t_1 - t_2}{1 - t_1}, \qquad s_3 = \frac{1 - t_1 - t_2 - t_3}{1 - t_1 - t_2},$$

then the functions $\tilde{u}$ and $\tilde{v}$ defined by the formulas (25) and (24) are found to be as in (27) and satisfy the equalities (21), (22), and (23). ■

3.2 $G = \mathbb{R} \times \mathbb{R}_+$, $H = \mathbb{R}_+ \times \mathbb{R}$ with nontrivial actions

Denote

$$g(a,b) = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad h(x,y) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The groups $G = \{g(a,b) : a \in \mathbb{R}, b > 0\}$ and $H = \{h(x,y) : x > 0, y \in \mathbb{R}\}$ form a matched pair of Abelian Lie groups with the mutual actions

$$h(x,y) \triangleright g(a,b) = g(ax,b), \qquad h(x,y) \triangleleft g(a,b) = h\left(x, \frac{y}{b}\right).$$

Thus, the group $K = GH$ is a direct product of two copies of the "$ax + b$" groups, that is, K is the group of matrices of the form

$$\begin{pmatrix} x & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider the corresponding matched pair of the Abelian two-dimensional Lie algebras $(\mathfrak{g}, \mathfrak{h})$. Let A_1, A_2 form a basis in $\mathfrak{g}$ and X_1, X_2 a basis in $\mathfrak{h}$ such that the non-trivial actions are

$$X_1 \triangleright A_1 = A_1, \qquad X_2 \triangleleft A_2 = X_2.$$

Thus, the Lie algebra $\mathfrak{k} = \mathfrak{g} + \mathfrak{h}$ has the following non-zero commutation relations:

$$[X_1, A_1] = A_1, \qquad [X_2, A_2] = -X_2.$$

Using Propositions 1 and 2 we find that the corresponding pair of cocycles modulo equivalent ones has the following form:

$$U(X_i; A_1, A_2) = \delta_{1,i}\lambda, \qquad V(X_1, X_2; A_j) = \delta_{2,j}\mu, \tag{28}$$

where $i, j = 1, 2$, $\lambda, \mu \in \mathbb{R}$, and δ_{ij} denotes Kronecker's symbol. The corresponding left-invariant forms on K are

$$\omega_u = da \wedge db \wedge dx, \qquad \omega_v = db \wedge dx \wedge dy.$$

Proposition 4. *The functions*

$$\tilde{u}(h; g_1, g_2) = \lambda(x-1)\begin{vmatrix} a_1 & a_2 \\ \ln b_1 & \ln b_2 \end{vmatrix}, \qquad \tilde{v}(h_1, h_2; g) = \mu\left(1 - \frac{1}{b}\right)\begin{vmatrix} \ln x_1 & \ln x_2 \\ y_1 & y_2 \end{vmatrix}, \tag{29}$$

where $\lambda, \mu \in \mathbb{R}$, $h = h(x,y)$, $h_i = h(x_i, y_i)$, $g = g(a,b)$, $g_i = g(a_i, b_i)$, $i = 1, 2$, are solutions of (21), (22), *and* (23), *and define cocycles for the matched pair of the Lie groups (G, H) via* (20).

Proof. Using (25) and (24) we obtain that

$$\tilde{u}(h;g_1,g_2)=\begin{vmatrix} a_1 & a_2 \\ \ln b_1 & \ln b_2 \end{vmatrix}\phi(x), \qquad \tilde{v}(h_1,h_2;g)=\begin{vmatrix} \ln x_1 & \ln x_2 \\ y_1 & y_2 \end{vmatrix}\psi(b),$$

where

$$\phi(x)=\int_{\Delta_3} x^{-s_1+1}\ln x\; ds_1\wedge ds_2\wedge ds_3, \qquad \psi(b)=\int_{\Delta_3}\frac{\ln b}{b}b^{-s_1s_2s_3-1}ds_1\wedge ds_2\wedge ds_3.$$

Instead of finding the functions s_1, s_2, and s_3 we will find the functions $\phi(x)$, $\psi(b)$ such that the corresponding functions $\tilde{u}$ and $\tilde{v}$ satisfy the relations (21), (22), and (23). equations (21) and (22) are satisfied by all ϕ and ψ. Equation (23) will be satisfied if

$$\phi(x_2)+x_2\phi(x_1)=\phi(x_1x_2), \qquad \psi(b_1)+\frac{\psi(b_2)}{b_1}=\psi(b_1b_2).$$

Solving these equations we find that $\phi(x)=(x-1)$, $\psi(b)=\left(1-\frac{1}{b}\right)$ up to a constant. ■

3.3 $G=$ "$ax+b$", $H=\mathbb{R}^2$ with nontrivial actions

Denote

$$g(a,b)=\begin{pmatrix}1 & b & 0\\ 0 & a & 0\\ 0 & 0 & 1\end{pmatrix}, \qquad h(x,y)=\begin{pmatrix}1 & 0 & y\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix}, \qquad k(a,b,x,y)=\begin{pmatrix}1 & b & y\\ 0 & a & x\\ 0 & 0 & 1\end{pmatrix}.$$

Let

$$G=\{g(a,b): a\in\mathbb{R}\setminus\{0\}, b\in\mathbb{R}\}, \qquad H=\{h(x,y): x,y\in\mathbb{R}\},$$
$$K=\{k(a,b,x,y): a\in\mathbb{R}\setminus\{0\}, b,x,y\in\mathbb{R}\}.$$

Then it follows from (2) that

$$h(x,y)\triangleright g(a,b)=g(a,b), \qquad h(x,y)\triangleleft g(a,b)=h\left(\frac{x}{a},-\frac{x}{a}b+y\right). \tag{30}$$

The Lie algebra $\mathfrak{k}$ is then

$$\mathfrak{k}=\mathbb{R}\langle\partial_a,\partial_b,\partial_x,\partial_y\rangle$$

with the following commutation relations:

$$[\partial_a,\partial_b]=-\partial_b, \qquad [\partial_a,\partial_x]=\partial_x, \qquad [\partial_a,\partial_y]=0,$$
$$[\partial_b,\partial_x]=\partial_y, \qquad [\partial_b,\partial_y]=0, \qquad [\partial_x,\partial_y]=0.$$

Using Proposition 1 we find that, up to equivalence as in Proposition 2, the linear space of nonequivalent cocycles for the matched pair $(\mathfrak{g},\mathfrak{h})$ is spanned by V^1 and V^2 that are given by

$$V^1(\partial_a,\partial_1,\partial_2)=0, \qquad V^1(\partial_b,\partial_x,\partial_y)=1,$$
$$V^2(\partial_a,\partial_x,\partial_y)=1, \qquad V^2(\partial_b,\partial_1,\partial_2)=0,$$

where $\partial_1,\partial_2\in\{\partial_x,\partial_y\}$. The corresponding left-invariant forms on K are

$$\omega_v^1=\frac{1}{a}\left(-\frac{b}{a}da+db\right)\wedge dx\wedge dy, \qquad \omega_v^2=\frac{1}{a^2}da\wedge dx\wedge dy.$$

Proposition 5. *Let $\tilde{u}^i = 0$, $i = 1, 2$, and $\tilde{v}^1$, $\tilde{v}^2$ be given by*

$$\tilde{v}^1(h_1, h_2; g) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \frac{b}{a}, \qquad \tilde{v}^2(h_1, h_2; g) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \frac{a-1}{a},$$

where $h_i = h(x_i, y_i)$, $i = 1, 2$, and $g = g(a, b)$. Then (20) defines two pairs of cocycles for the matched pair of the groups (G, H).

Proof. We first consider the connected component of the identity in the group K, $K_0 = \{k(a, b, c, d) : a > 0\}$. Then we find from (24) that

$$\sigma_v = k\left(a^{s_1 s_2 s_3}, \frac{a^{s_1 s_2 s_3 - 1}}{a-1} b,\, s_1 x_1 + s_1 s_2 x_2, s_1 y_1 + s_1 s_2 y_2\right).$$

It follows from (25) that

$$\tilde{v}^1(h_1, h_2, g) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} b \frac{\ln a}{a-1} \int_{\Delta_3} \frac{s_1 s_2^2}{a^{s_1 s_2 s_3}}\, ds_1 \wedge ds_2 \wedge ds_3 \tag{31}$$

and

$$\tilde{v}^2(h_1, h_2; g) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \ln a \int_{\Delta_3} \frac{s_1 s_2^2}{a^{s_1 s_2 s_3}}\, ds_1 \wedge ds_2 \wedge ds_3. \tag{32}$$

Let ϕ denote the function

$$\phi(a) = a \ln a \int_{\Delta_3} \frac{s_1 s_2^2}{a^{s_1 s_2 s_3}}\, ds_1 \wedge ds_2 \wedge ds_3.$$

The equation (22) is satisfied for any ϕ, whereas (23) implies that

$$a_2 \phi(a_1) - \phi(a_1 a_2) + \phi(a_2) = 0.$$

Its unique, up to a constant, differentiable solution satisfying the condition $\phi(1) = 0$ is

$$\phi(a) = a - 1.$$

This, together with (31) and (32), proves the claim for $a > 0$. Extend now $\tilde{v}^1$ and $\tilde{v}^2$ with these formulas over the whole group K. ■

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Permutations in Tensor Products of Factors, and L^2 Betti Numbers of Configuration Spaces

A.Yu. DALETSKII † *and A.A. KALYUZHNYI* ‡

† *School of Computing and Mathematics, The Nottingham Trent University, UK*
E-mail: *alexei.daletskii@ntu.ac.uk*

‡ *Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *kalyuz@imath.kiev.ua*

We prove that the natural action of permutations in a tensor product of type II factors is free, and compute the von Neumann trace of the projection onto the space of symmetric and antisymmetric elements respectively. We apply this result to computation of von Neumann dimensions of spaces of harmonic forms (L^2-Betti numbers) of N-point configuration spaces of infinite coverings of compact manifolds.

1 Introduction

It is difficult to overestimate the role of the theory of von Neumann algebras and their traces in different areas of mathematical and theoretical physics. One of the important applications is the definition of regularized dimensions of certain infinite-dimensional Hilbert modules, in particular of the spaces of harmonic forms over certain non-compact manifolds possessing infinite discrete groups of isometries (L^2-Betti numbers, see [3, 7] and references given there).

Thus an important problem is construction of von Neumann algebras containing particular operators, and computation of the corresponding traces of these operators. In this note, we describe the structure of the von Neumann algebra $\{\mathcal{M}^{\otimes n}, U\}''$, where $\mathcal{M}$ is a von Neumann algebra acting in a separable Hilbert space H and U is the natural action of the symmetric group S_n by permutations in $H^{\otimes n}$, $\otimes n$ denoting the n-th tensor power.

It is clear that the answer to this question is spatial dependent, i.e. it depends on the choice of the concrete $\mathcal{M}$-module H. For example, if $H = \mathbb{C}^m$ is a module over the I_m factor $\mathcal{M} = M_m(\mathbb{C})$, then (for $n = 2$) $\mathcal{M} \otimes \mathcal{M}$ coincides with the space of all linear operators in $\mathbb{C}^m \otimes \mathbb{C}^m$. Therefore the permutation operator U belongs to $\mathcal{M} \otimes \mathcal{M}$ and $\{\mathcal{M} \otimes \mathcal{M}, U\}'' = \mathcal{M} \otimes \mathcal{M}$. On the contrary, if the same I_m factor $\mathcal{M}$ acts on its standard form $H = \mathbb{C}^m \otimes \mathbb{C}^m$ by operators $x(f \otimes g) = xf \otimes g,\ x \in \mathcal{M},\ f, g \in \mathbb{C}^m$, then U does not belong to $\mathcal{M} \otimes \mathcal{M}$. Thus U induces an outer action α of the symmetric group S_2 on the factor $\mathcal{M} \otimes \mathcal{M}$ and the von Neumann algebra $\{\mathcal{M} \otimes \mathcal{M}, U\}''$ is isomorphic to the cross-product $\mathcal{M} \otimes \mathcal{M} \times_\alpha S_2$.

In Section 2, we consider the case where H is a separable module over a type II factor $\mathcal{M}$. We prove that the action α of the group S_n in $\mathcal{M}^{\otimes n}$ generated by the representation U is outer and free and thus the von Neumann algebra $\{\mathcal{M}^{\otimes n}, U\}''$ is isomorphic to the cross-product $\mathcal{M}^{\otimes n} \times_\alpha S_n$. We compute the von Neumann trace of the projection onto the space of symmetric and antisymmetric elements of $H^{\otimes n}$ respectively. In Section 3, we apply this result to computation of the von Neumann dimensions of the spaces of harmonic forms of the spaces of N-point configurations in infinite coverings of compact manifolds (L^2-Betti numbers of configuration spaces).

In the particular case where $\mathcal{M}$ is the commutant of a free action of an infinite discrete group, the results of Section 2 were obtained (by different methods) in [5] ($n = 2$) and [2] (n arbitrary).

In the latter work, the L^2-Betti numbers of infinite configuration spaces equipped with Poisson measures were computed.

In what follows we denote by $\mathcal{L}(H)$ the algebra of all bounded operators in Hilbert space H. We refer to [4, 9] for general notions of the theory of von Neumann algebras.

2 Permutations in a tensor product of type II factors

Let $L^2(\mathcal{M})$ be the standard form of a finite factor $\mathcal{M}$. Denote by Ω the corresponding cyclic and separating vector for $\mathcal{M}$. Let τ be a faithful normal trace on $\mathcal{M}$. Since $(\mathcal{M}\otimes\mathcal{M})' = \mathcal{M}'\otimes\mathcal{M}'$, $\mathcal{M}\otimes\mathcal{M}$ is a finite factor acting in Hilbert space $L^2(\mathcal{M})\otimes L^2(\mathcal{M})$. Let U be the permutation operator in $L^2(\mathcal{M})\otimes L^2(\mathcal{M})$. Denote by α_U the corresponding automorphism of the factor $\mathcal{M}\otimes\mathcal{M}$,

$$\alpha_U(x\otimes y) = U(x\otimes y)U^* = y\otimes x, \qquad x, y\in\mathcal{M}. \tag{1}$$

This automorphism generates a natural action α of the group S_2 on $\mathcal{M}\otimes\mathcal{M}$. Recall that the action of an automorphism β on $\mathcal{M}$ is called free, if each element $x\in\mathcal{M}$ satisfying the equality $xy = \beta(y)x$ for all $y\in\mathcal{M}$ is zero. It is well known that an automorphism of a factor is free iff it is outer. If $\alpha : G\to \mathrm{Aut}(\mathcal{M})$ is a free action of a discrete group G on $\mathcal{M}$ then $\mathcal{M}\times_\alpha G$ is a factor (see, for example, [6, Proposition 1.4.4]).

We have the following statement.

Proposition 1. *(i) The Hilbert space $L^2(\mathcal{M})\otimes L^2(\mathcal{M})$ is the standard form of the factor $\mathcal{M}\otimes\mathcal{M}$;*

(ii) The action α of the group S_2 on $\mathcal{M}\otimes\mathcal{M}$ is free;

(iii) There exists a natural isomorphism of the finite factor $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ and the von Neumann algebra $\{\mathcal{M}\otimes\mathcal{M}, U\}''$.

Proof. Note that the vector $\Omega_1 = \Omega\otimes\Omega$ is cyclic for both $\mathcal{M}\otimes\mathcal{M}$ and $(\mathcal{M}\otimes\mathcal{M})' = \mathcal{M}'\otimes\mathcal{M}'$. Hence Ω_1 is separating for $\mathcal{M}\otimes\mathcal{M}$ (that is, $z\Omega_1 = 0$, $z\in\mathcal{M}\otimes\mathcal{M}$ implies $z = 0$).

Denote $\tau_1 = \tau\otimes\tau$. It is obvious that τ_1 is a trace on $\mathcal{M}\otimes\mathcal{M}$. Moreover the trace τ_1 is faithful on $\mathcal{M}\otimes\mathcal{M}$. Indeed, since Ω_1 is separating for $\mathcal{M}\otimes\mathcal{M}$, for $x = \sum_i x_i\otimes y_i\in\mathcal{M}\otimes\mathcal{M}$ we have

$$\begin{aligned}\tau_1(x^*x) &= \sum_i \tau(x_i^*x_k)\tau(y_i^*y_k) = \sum_i (x_i^*x_k\Omega,\Omega)_{L^2(\mathcal{M})}(y_i^*y_k\Omega,\Omega)_{L^2(\mathcal{M})}\\ &= (x^*x\Omega_1,\Omega_1)_{L^2(\mathcal{M})\otimes L^2(\mathcal{M})} = \|x\Omega_1\|^2\neq 0.\end{aligned} \tag{2}$$

Since

$$\tau_1(x) = (x\Omega_1,\Omega_1)_{L^2(\mathcal{M})\otimes L^2(\mathcal{M})}$$

for any $x\in\mathcal{M}\otimes\mathcal{M}$ we can conclude that $L^2(\mathcal{M})\otimes L^2(\mathcal{M})$ is the standard form of $\mathcal{M}\otimes\mathcal{M}$.

Let us show that α_U given by (1) is a nontrivial outer automorphism of $\mathcal{M}\otimes\mathcal{M}$, i.e. that the operator U does not belong to $(\mathcal{M}\otimes\mathcal{M})\cup(\mathcal{M}\otimes\mathcal{M})'$. Suppose that $U\in\mathcal{M}\otimes\mathcal{M}$. Rewrite the equality $U\Omega\otimes\Omega = \Omega\otimes\Omega$ in the form $(U-1)\Omega_1 = 0$. Moreover Ω_1 is a separating vector for $\mathcal{M}\otimes\mathcal{M}$, which implies that $U = 1$. Thus $U\notin\mathcal{M}\otimes\mathcal{M}$. It can be shown by similar arguments that $U\notin(\mathcal{M}\otimes\mathcal{M})'$.

Since $\mathcal{M}\otimes\mathcal{M}$ is a factor, it follows from Proposition 1.4.4 of [6] that

$$(\mathcal{M}\otimes\mathcal{M})'\cap(\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2) = \mathbb{C}.$$

This implies in particular that the crossed product $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ is also a finite factor. We conclude from the equality $\alpha_U(x)\Omega_1 = UxU^{-1}\Omega_1$ that there exists a natural homomorphism of

$\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ onto $\{\mathcal{M}\otimes\mathcal{M},U\}''$. Since a finite factor does not contain two-sided ideals, any normal homomorphism of it is either identically zero or injective. Hence factors $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ and $\{\mathcal{M}\otimes\mathcal{M},U\}''$ are isomorphic. ■

Now we can consider the case where $\mathcal{M}$ is a type II factor. Let H be a separable $\mathcal{M}$-module. Denote by U the operator of permutation in $H\otimes H$ and let α_U be the corresponding (nontrivial) automorphism of $\mathcal{M}\otimes\mathcal{M}$.

Theorem 1. *The automorphism α_U defines an outer action of the group S_2 on the II-factor $\mathcal{M}\otimes\mathcal{M}$, and there exists an isomorphism of factors*

$$\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2\simeq\{\mathcal{M}\otimes\mathcal{M},U\}''. \tag{3}$$

Proof. *Case II_1.* Let $\mathcal{M}$ be a II_1-factor. Denote by K the standard form of $\mathcal{M}$. Using the theorem on the structure of normal isomorphisms of von Neumann algebras [9] we can conclude that H as $\mathcal{M}$-module is isomorphic to $\mathcal{M}$-module

$$H_d=p(K\otimes l_2) \tag{4}$$

for some $d\in[0,\infty]$, where $p\in\mathcal{M}'\otimes\mathcal{L}(l_2)$ is a projection with $\operatorname{Tr}p=d$. Here Tr denotes the natural trace in $\mathcal{M}'\otimes\mathcal{L}(l_2)$, with the normalization $\operatorname{Tr}(1_{\mathcal{M}}\otimes q)=1$, where q is a projection of rank 1 in $\mathcal{L}(l_2)$. The action of $\mathcal{M}$ on H_d is given by

$$x(p(f\otimes\xi))=p(xf\otimes\xi),\qquad x\in\mathcal{M},\quad f\in K,\quad \xi\in l_2.$$

Let us remark that the Hilbert spaces $K\otimes l_2\otimes K\otimes l_2$ and $K\otimes K\otimes l_2\otimes l_2$ are isomorphic. Thus there exists a projection $\tilde p$ such that $\mathcal{M}\otimes\mathcal{M}$-modules

$$H_d\otimes H_d=(p\otimes p)(K\otimes l_2\otimes K\otimes l_2)$$

and

$$\tilde p(K\otimes K\otimes l_2\otimes l_2)$$

are isomorphic, where the action of $\mathcal{M}\otimes\mathcal{M}$ on the latter space is defined by

$$(x\otimes y)(\tilde p(f\otimes g\otimes\xi\otimes\eta))=\tilde p(xf\otimes yg\otimes\xi\otimes\eta),$$

$x,y\in\mathcal{M}$, $f,g\in K$, $\xi,\eta\in l_2$. The operator U of permutation in $H_d\otimes H_d$ is unitarily isomorphic to the operator $U_1\otimes U_2$ in $\tilde p(K\otimes K\otimes l_2\otimes l_2)$, where U_1 is the operator of permutation in $K\otimes K$ and U_2 is the operator of permutation in $l_2\otimes l_2$. It follows from Proposition 1 that $U_1\notin\mathcal{M}\otimes\mathcal{M}$. Hence the operator U does not belong to $\mathcal{M}\otimes\mathcal{M}$ and thus α_U is outer (and consequently, free) automorphism of $\mathcal{M}\otimes\mathcal{M}$. Repeating the arguments from the proof of Proposition 1 we conclude that the factors $\mathcal{M}\otimes\mathcal{M}\times S_2$ and $\{\mathcal{M}\otimes\mathcal{M},U\}''$ are isomorphic.

Case II_∞. Let $\mathcal{M}$ be a II_∞ factor. Fix an arbitrary finite projection $p\in\mathcal{M}$. Then there exists [9] a spatial isomorphism of $\mathcal{M}$ and the II_∞ factor $\mathcal{M}_p\otimes\mathcal{L}(l_2)$, where $\mathcal{M}_p=p\mathcal{M}p$ (the so-called "corner" of $\mathcal{M}$) is a II_1 factor. Denote $H_p=pH$. Then the II_∞ factor $\mathcal{M}\otimes\mathcal{M}$ is isomorphic to $\mathcal{M}_p\otimes\mathcal{M}_p\otimes\mathcal{L}(l_2\otimes l_2)$ and the permutation operator U in $H\otimes H$ is unitarily equivalent to the operator $U_1\otimes U_2$, where U_1 is the operator of permutation in $H_p\otimes H_p$ and U_2 is the operator of permutation in $l_2\otimes l_2$. Note that the operator U_2 belongs to $\mathcal{L}(l_2\otimes l_2)$.

It follows from the arguments presented above that the operator U_1 does not belong to $\mathcal{M}_p\otimes\mathcal{M}_p$. Thus the operator U does not belong to $\mathcal{M}\otimes\mathcal{M}$ and as above α is free automorphism of $\mathcal{M}\otimes\mathcal{M}$. Therefore $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ is a II_∞ factor. It follows from the arguments presented in the proof of Proposition 1 that there exists a normal homomorphism of $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$ onto $\{\mathcal{M}\otimes\mathcal{M},U\}''$. Since any normal homomorphism of a factor is either identically zero or injective we have that $\{\mathcal{M}\otimes\mathcal{M},U\}''$ is also a II_∞ factor isomorphic to $\mathcal{M}\otimes\mathcal{M}\times_\alpha S_2$. ■

The following result is the extension of the theorem above to the case of the symmetric group S_n acting in $\mathcal{M}^{\otimes n}$, $n \geq 2$.

Theorem 2. *Let $\mathcal{M}$ be a type II factor and H be a separable $\mathcal{M}$-module. Let U_{ij} $(i,j=1,\dots,n)$ be the operator in $H^{\otimes n}$ permuting i-th and j-th components. Then the family of operators $\{U_{ij}\}_{i,j=1,\dots,n}$ defines an outer action of the symmetric group S_n on the factor $\mathcal{M}^{\otimes n}$, and there exists an isomorphism*

$$\mathcal{M}_n \times_\alpha S_n \simeq \{\mathcal{M}_n, \{U_{ij}\}_{i,j=1,\dots,n}\}''. \tag{5}$$

Proof. It follows from Theorem 1 that the operator U_{ij} does not belong to the factor $\mathcal{M}^{\otimes n}$ and therefore determines the outer automorphism $\alpha_{U_{ij}}$ of $\mathcal{M}^{\otimes n}$. Since automorphisms $\alpha_{U_{ij}}$, $i,j=1,\dots,n$ generate the action of the symmetric group S_n on the factor $\mathcal{M}^{\otimes n}$, we conclude that this action is free. Therefore the factors $\mathcal{M}_n \times_\alpha S_n$ and $\{\mathcal{M}_n, \{U_{ij}\}_{i,j=1,\dots,n}\}''$ are isomorphic (see the proof of Theorem 1). ■

Let P_s and P_a be projections in $H^{\otimes n}$ onto the symmetric tensor power $H^{\hat{\otimes} n}$ and the anti-symmetric tensor power $H^{\wedge n}$ respectively,

$$P_s = \frac{1}{n!}\sum_{g\in S_n} U_g \tag{6}$$

and

$$P_a = \frac{1}{n!}\sum_{g\in S_n} (-1)^{\mathrm{sign}\,(g)} U_g. \tag{7}$$

It is obvious that P_s and P_a belong to $\mathcal{M}_n \times_\alpha S_n$.

Denote

$$\mathcal{M}_s = \{\mathcal{M}^{\otimes n}, P_s\}'', \qquad \mathcal{M}_a = \{\mathcal{M}^{\otimes n}, P_a\}'' \tag{8}$$

Proposition 2. *Let $\mathcal{M}$ be a II_∞ factor. Then*

$$\mathcal{M}_s = \mathcal{M}_a = \mathcal{M}^{\otimes n} \times_\alpha S_n. \tag{9}$$

Proof. The inclusion $\mathcal{M}_s \subset \mathcal{M}^{\otimes n} \times_\alpha S_n$ is obvious. For the inverse inclusion it suffices to show that the operators $P_{ij} = \frac{1}{2}(1+U_{ij})$ $i,j=1,\dots,n$ belongs to $\mathcal{M}_s$. Since $\mathcal{M} \simeq \mathcal{M}_p \otimes \mathcal{L}(l_2)$, the factor $\mathcal{M}$ contains an isometry V such that $(V^*)^m \to 0$, $m \to \infty$ strongly (for example $V = 1 \otimes W$ where W is unilateral shift in l_2: $We_k = e_{k+1}$ for a standard basis $\{e_k\}_{k=1}^\infty$ in l_2). Since $(V^*)^m V^m = 1$ we have strong convergence

$$\big(1\otimes 1 \otimes (V^*)^m \otimes \cdots \otimes (V^*)^m\big) P_s \big(1 \otimes 1 \otimes V^m \otimes \cdots \otimes V^m\big) \to \frac{2}{n!} P_{12},$$

$m \to \infty$. Thus $P_{12} \in \mathcal{M}_s$. Similar arguments show that $P_{ij} \in \mathcal{M}_s$ for any $i,j=1,\dots,n$.

The case of $\mathcal{M}_a$ can be treated in a completely similar way. ■

In what follows we denote by $\mathrm{Tr}_\mathcal{N}$ the faithful normal semifinite trace on a II_∞ factor $\mathcal{N}$.

Corollary 1. *For any $A \in \mathcal{M}$ we have*

$$\mathrm{Tr}_{\mathcal{M}_s}(A^{\otimes n} P_s) = \mathrm{Tr}_{\mathcal{M}_a}(A^{\otimes n} P_a) = \frac{(\mathrm{Tr}_\mathcal{M} A)^n}{n!}. \tag{10}$$

Proof. According to the Proposition 2 we will use the trace on a factor $\mathcal{M}^{\otimes n} \times_\alpha S_n$. It is obvious that $\alpha_g(A^{\otimes n}) = A^{\otimes n}$ for any $g \in S_n$. Therefore for any $g \in S_n$ we have $\mathrm{Tr}_{\mathcal{M}^{\otimes n}\times_\alpha S_n}(A^{\otimes n} U_g) = \delta_{e,g}\mathrm{Tr}_{\mathcal{M}^{\otimes n}}(A^{\otimes n})$ (here $\delta_{g,h}$ is the Kronecker symbol). Then (10) follows from (6) and (7). ■

3 L^2-Betti numbers of N-point configuration spaces

In this section, we apply the results described above to computation of L^2-Betti numbers of the spaces of configurations of N points in the universal covering of a compact manifold with infinite fundamental group. We start with the discussion of the structure of the spaces of square-integrable differential forms over configuration spaces.

Let X be a smooth connected Riemannian manifold. Consider the N-point configuration space

$$X^{(N)} := \{\{x_1, \dots, x_N\} \subset X\}, \tag{11}$$

the set of all N-point subsets of X. Clearly,

$$X^{(N)} = \widetilde{X \times \cdots \times X}/S_N, \tag{12}$$

where $\widetilde{X \times \cdots \times X}$ is the Cartesian product of N copies of X without coinciding components. $X^{(N)}$ is a Riemannian manifold equipped with the Riemannian structure induced from X.

For a Riemannian manifold R, we denote by $L^2\Omega^p(R)$ the space of square-integrable (w.r.t. the Riemannian volume) p-forms on R. We let Δ_R^p be the Hodge–deRham Laplacian in $L^2\Omega^p(R)$, and $\mathcal{H}^p(R) := \operatorname{Ker}\Delta_R^p$, the space of square-integrable harmonic p-forms on R.

For a Hilbert space $\mathcal{K}$, we use the notation

$$\mathcal{K}^{\overset{k}{\diamond} s} = \begin{cases} \mathcal{K}^{\widehat{\otimes} s}, & k \text{ is even}, \\ \mathcal{K}^{\wedge s}, & k \text{ is odd}. \end{cases} \tag{13}$$

The following result is a symmetrized version of the Künneth formula.

Theorem 3.

$$\mathcal{H}^p\big(X^{(N)}\big) = \bigoplus_{\substack{s_1, \dots, s_d = 0,1,2,\dots \\ s_1 + s_2 + \cdots + s_d = N \\ s_1 + 2s_2 + \cdots + ds_d = p}} \big(\mathcal{H}^1(X)\big)^{\overset{1}{\diamond} s_1} \otimes \cdots \otimes \big(\mathcal{H}^d(X)\big)^{\overset{d}{\diamond} s_d} \tag{14}$$

where $d = \dim X - 1$.

Proof. Let us first remark that the space $L^2\Omega^p\big(X^{(N)}\big)$ is unitarily isomorphic to $L^2\Omega^p_{\mathrm{sym}}\big(X^N\big)$, the latter being the space of square-integrable p-forms on $X^N := \overbrace{X \times \cdots \times X}^{N}$ which are symmetric w.r.t. the permutations of variables. It is easy to see that there exists a natural unitary isomorphism

$$L^2\Omega^p_{\mathrm{sym}}\big(X^{(N)}\big) = \bigoplus_{\substack{s_0, \dots, s_{d+1} = 0,1,2,\dots \\ s_0 + \cdots + s_{d+1} = N \\ s_1 + 2s_2 + \cdots + (d+1)s_{d+1} = p}} \big(L^2\Omega^0(X)\big)^{\overset{0}{\diamond} s_0} \otimes \cdots \otimes \big(L^2\Omega^{d+1}(X)\big)^{\overset{d+1}{\diamond} s_{d+1}}. \tag{15}$$

It has been proved in [1] that the restriction $(\Delta^p_{X^N})_{\mathrm{sym}}$ of $\Delta^p_{X^N}$ onto $L^2\Omega^p_{\mathrm{sym}}\big(X^N\big)$ is essentially self-adjoint on the space of smooth forms with compact support. Thus $(\Delta^p_{X^N})_{\mathrm{sym}}$ coincides with the Hodge–deRham Laplacian on $X^{(N)}$, and we have

$$\mathcal{H}^p\big(X^{(N)}\big) = \operatorname{Ker}\big(\Delta^p_{X^N}\big)_{\mathrm{sym}} = \operatorname{Ker}\big(\Delta^p_{X^N}\big) \cap L^2\Omega^p_{\mathrm{sym}}\big(X^N\big). \tag{16}$$

By the Künneth formula,

$$\mathrm{Ker}\,(\Delta^p_{X^N}) = \bigoplus_{\substack{1 \le k_1,\dots,k_N \le d \\ k_1+\cdots+k_N=p}} (\mathcal{H}^{k_1}(X)) \otimes \cdots \otimes (\mathcal{H}^{k_N}(X)), \tag{17}$$

(remark that $\mathcal{H}^0(X) = \mathcal{H}^{d+1}(X) = 0$), which together with (15) implies the result. ■

Corollary 2. *If all spaces $\mathcal{H}^k(X)$ are finite-dimensional, then all spaces $\mathcal{H}^p\big(X^{(N)}\big)$ are so. Their dimensions are given by the following formula:*

$$\dim \mathcal{H}^p\big(X^{(N)}\big) = \sum_{\substack{s_1,\dots,s_d = 0,1,2,\dots \\ s_1+s_2+\cdots+s_d=N \\ s_1+2s_2+\cdots+ds_d=p}} \beta_1^{(s_1)} \cdots \beta_d^{(s_d)}, \tag{18}$$

where

$$\beta_k^{(s)} := \begin{cases} \begin{pmatrix} \beta_k \\ s \end{pmatrix}, & k = 1,3,\dots, \\ \begin{pmatrix} \beta_k + s - 1 \\ s \end{pmatrix}, & k = 2,4,\dots, \end{cases} \tag{19}$$

$s \neq 0$, and $\beta_k^{(s)} = 1$ for $s = 0$. Here $\beta_k := \dim \mathcal{H}^k(X)$, $k = 1,\dots,d$. This case occurs for instance when X is compact or has finite number of ends.

An important example of a manifold X with infinite dimensional spaces $\mathcal{H}^{(p)}(X)$ is given by an infinite covering of a compact Riemannian manifold (say M). In this case, the fundamental group $G = \pi_1(M)$ acts by isometries on X and consequently on all spaces $L^2\Omega^p(X)$. The orthogonal projection

$$\mathcal{P}_p : L^2\Omega^p(X) \to \mathcal{H}^{(p)}(X) \tag{20}$$

commutes with the action of G and thus belongs to the commutant $\mathcal{A}_p$ of this action which is a semifinite von Neumann algebra. The corresponding von Neumann trace $b_p := \mathrm{Tr}_{\mathcal{A}}\mathcal{P}_p$ gives a regularized dimension of the space $\mathcal{H}^{(p)}(X)$ and is called the L^2-Betti number of X (or M). L^2-Betti numbers were introduced in [3] studied by many authors (see e.g. [7] and references given there). It is known [3] that (because of the elliptic regularity of Δ^p_X) $b_p < \infty$.

It is natural to ask whether the notion of L^2-Betti numbers can be extended to configuration spaces over infinite coverings. It particular, is formula (19) valid in this case (with β_k replaced by b_k)? In what follows, we use the results of the first section for constructing a von Neumann algebra containing the projection

$$\mathbf{P}_p : L^2\Omega^p\big(X^{(N)}\big) \to \mathcal{H}^p\big(X^{(N)}\big), \tag{21}$$

and computing its von Neumann trace.

In what follows, we assume that G is an ICC group (that is, all non-trivial classes of conjugate elements are infinite). Under this condition we have that the von Neumann algebra $\mathcal{A}_p$ is a II_∞ factor.

Let us define the operator

$$\mathcal{P}_p^{(n)} := \begin{cases} \mathcal{P}_p^{\otimes n} P_s, & p \text{ is even}, \\ \mathcal{P}_p^{\otimes n} P_a, & p \text{ is odd} \end{cases} \tag{22}$$

and the von Neumann algebra

$$\mathcal{A}_p^{(n)} := \begin{cases} \{\mathcal{A}_p^{\otimes n}, P_s\}'', & p \text{ is even}, \\ \{\mathcal{A}_p^{\otimes n}, P_a\}'', & p \text{ is odd}, \end{cases} \tag{23}$$

generated by $\mathcal{A}_p^{\otimes n}$ and projections P_s and P_a respectively. Thus, $\mathcal{P}_p^{(n)}$ is the orthogonal projection $\big(L^2\Omega^p(X)\big)^{\otimes n} \to \big(\mathcal{H}^p(X)\big)^{\overset{p}{\otimes} n}$, $n = 1, 2, \ldots$. Obviously, $\mathcal{P}_p^{(n)} \in \mathcal{A}_p^{(n)}$. It follows from the Proposition 2 that $\mathcal{A}_p^{(n)} = \mathcal{A}_p^{\otimes n} \times_\alpha S_n$. We will use the convention $\mathcal{A}_p^{(0)} = \mathbb{C}^1$.

Further, we introduce the von Neumann algebra

$$\mathbf{A}^{(p)} = \prod_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + s_2 + \cdots + s_d = N \\ s_1 + 2s_2 + \cdots + ds_d = p}} \mathcal{A}_1^{(s_1)} \otimes \cdots \otimes \mathcal{A}_d^{(s_d)}, \tag{24}$$

$d = \dim X - 1$. Since all algebras $\mathcal{A}_k^{(s_k)}$ are II_∞-factors, so is $\mathbf{A}^{(p)}$, with the trace given by the product of the traces in $\mathcal{A}_k^{(s_k)}$.

Theorem 4. *We have* $\mathbf{P}_p \in \mathbf{A}^{(p)}$ *and*

$$\mathrm{Tr}_{\mathbf{A}^{(p)}} \mathbf{P}_p = \sum_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + s_2 + \cdots + s_d = N \\ s_1 + 2s_2 + \cdots + ds_d = p}} \frac{(b_1)^{s_1}}{s_1!} \cdots \frac{(b_d)^{s_d}}{s_d!}, \tag{25}$$

where b_k *are the* L^2*-Betti numbers of* X*,* $k = 1, \ldots, d$*, and* $b_0 = 1$*.*

Proof. It follows from Theorem 3 that

$$\mathbf{P}_p = \sum_{\substack{s_1, \ldots, s_d = 0, 1, 2, \ldots \\ s_1 + s_2 + \cdots + s_d = N \\ s_1 + 2s_2 + \cdots + ds_d = p}} \mathcal{P}_1{}^{(s_1)} \ldots \mathcal{P}_d{}^{(s_d)}, \tag{26}$$

with the convention $\mathcal{P}_k^{(0)} = id$. The result follows now from Corollary 1. ■

We will use the notation $\mathbf{b}_p = \mathrm{Tr}_{\mathbf{A}^{(p)}} \mathbf{P}_p$ and call $\mathbf{b}_p$ the p-th L^2-Betti number of Γ_X.

Remark 1. It is easy to see that formula (25) can be rewritten in the form

$$\mathbf{b}_p = \frac{1}{N!} \sum_{\substack{1 \le k_1, \ldots, k_N \le d \\ k_1 + \cdots + k_N = p}} b_{k_1} \cdots b_{k_N}, \tag{27}$$

or, according to the Künneth formula (17),

$$\mathbf{b}_p = \frac{1}{N!} \mathrm{Tr}_{\mathcal{A}^{\otimes N}} P,$$

where P is the orthogonal projection $L^2\Omega^p\big(X^N\big) \to \mathcal{H}^p\big(X^N\big)$.

Example 1. Let $X = \mathbb{H}^d$, the hyperbolic space of dimension d. It is known that the only non-zero L^2 Betti number of $\mathbb{H}^d$ is $b_{d/2}$ (provided d is even). Then

$$\mathbf{b}_p = \begin{cases} \dfrac{\big(b_{d/2}\big)^k}{k!}, & p = \dfrac{kd}{2}, \\ 0, & p \ne \dfrac{kd}{2}. \end{cases} \tag{28}$$

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Theory of Algebraic Invariants of Vector Spaces of Killing Tensors: Methods for Computing the Fundamental Invariants

Robin J. DEELEY †, *Joshua T. HORWOOD* ‡,
Raymond G. MCLENAGHAN † *and Roman G. SMIRNOV* §

† *Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada*
E-mail: *rjdeeley@sirius.math.uwaterloo.ca, rgmclena@sirius.math.uwaterloo.ca*

‡ *Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, CB3 OWA, United Kingdom*
E-mail: *jh423@cam.ac.uk*

§ *Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada*
E-mail: *smirnov@mathstat.dal.ca*

We review computational techniques that can be used in the determination of fundamental invariants of vector spaces of Killing tensors under the action of the isometry group. In particular, for the first time in this study, we employ for these purposes the method of moving frames introduced by Fels and Olver. Illustrative examples are provided.

1 Introduction

Conceived as an extension of classical invariant theory of homogeneous polynomials [1, 2], the theory of algebraic invariants of Killing tensors has been introduced recently [3–10] as the study of invariant properties of vector spaces of Killing tensors under the action of the isometry group. According to the dictates of the mathematical structure of the classical theory, the theory of algebraic invariants of Killing tensors shares many of its essential features. Thus, the pivotal problem in both theories is the determination of the set of fundamental invariants for a given vector space under the action of a group. Furthermore, in both theories, a solution to the problem is a precursor to solving the intimately related problems of equivalence and finding canonical forms (see [2, 7] for more details).

As is well known, the geometrical properties of Killing two-tensors play a vital role in the Hamilton–Jacobi theory of orthogonal separation of variables [11–15] as well as in the field theory (see, for example [16, 17]). In this spirit, the fundamental invariants of vector spaces of Killing tensors have been effectively employed in classification problems arising in the Hamilton–Jacobi theory [3, 4, 6–10]. It has also been shown that the fundamental invariants of Killing tensors of higher valences can be applied to other classification problems of Hamiltonian mechanics [5].

The main aim of this article is to discuss computational techniques that can be employed within the framework of the theory to determine the fundamental invariants of vector spaces of Killing tensors defined on pseudo-Riemannian spaces of constant curvature.

2 A glimpse into the theory of algebraic invariants of vector spaces of Killing tensors

Assuming only a minimal familiarity with the classical invariant theory (see [2] for an introduction), this section describes how the basic concepts of the classical theory of algebraic invariants of homogeneous polynomials can be naturally adapted to the study of vector spaces of Killing tensors under the action of the isometry group $I(M)$.

Let $(M, \boldsymbol{g})$ be a pseudo-Riemannian manifold of constant curvature. We begin by defining the concept of a Killing tensor (vector).

Definition 1. A *Killing tensor* $\boldsymbol{K}$ *of valence* p *defined in* $(M, \boldsymbol{g})$ is a symmetric $(p, 0)$ tensor satisfying the Killing tensor equation

$$[\boldsymbol{K}, \boldsymbol{g}] = 0, \tag{1}$$

where $[\ ,\ ]$ denotes the Schouten bracket. When $p = 1$, $\boldsymbol{K}$ is said to be a *Killing vector (infinitesimal isometry)* and the equation (1) reads $\mathcal{L}_{\boldsymbol{K}}\boldsymbol{g} = 0$, where $\mathcal{L}$ denotes the Lie derivative operator.

In the case when $(M, \boldsymbol{g})$ is a pseudo-Riemannian manifold of constant curvature, the dimension d of $\mathcal{K}^p(M)$ is determined by the *Delong–Takeuchi–Thompson (DTT) formula* $d = \dim \mathcal{K}^p(M) = \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}$, $p \geq 1$, where $\mathcal{K}^p(M)$ denotes the vector space of Killing tensor of valence p defined on $(M, \boldsymbol{g})$. In this study we treat Killing tensors as elements of their respective vector spaces. This approach differs from the more conventional view of Killing tensors defined on spaces of constant curvature as symmetrized sums of Killing vectors. Accordingly, taking into account the DTT formula, a Killing tensor of valence p is an algebraic object determined by its d parameters, where d is the dimension of the vector space $\mathcal{K}^p(M)$.

We have exploited this idea by making a natural link between the study of Killing tensors and the classical invariant theory. Indeed, whenever the action of a group is defined properly on a vector space, one can pose the question of determining the induced action of the group in the space of the parameters of the vector space under consideration and then proceed to find the functions of the parameters that remain unchanged under the action of the group (i.e. the group invariants). Thus, for a fixed $p \geq 1$, consider the corresponding vector space $\mathcal{K}^p(M)$ on $(M, \boldsymbol{g})$. It is spanned by d arbitrary parameters $a_1, \ldots, a_d$, where d is determined by the DTT formula. Alternatively, this fact can be established by solving the corresponding Killing tensor equation (1) with respect to a fixed coordinate system. Then, the parameters $a_1, \ldots, a_d$ will appear as constants of integration in the general form of the Killing tensor in $\mathcal{K}^p(M)$. The action of the isometry group $I(M)$ on $\mathcal{K}^p(M)$ induces transformation laws for the d parameters $a_1, \ldots, a_d$ of the following form:

$$\begin{aligned}
&\tilde{a}_1 = \tilde{a}_1(a_1, \ldots, a_d, g_1, \ldots, g_r),\\
&\tilde{a}_2 = \tilde{a}_2(a_1, \ldots, a_d, g_1, \ldots, g_r),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\tilde{a}_d = \tilde{a}_d(a_1, \ldots, a_d, g_1, \ldots, g_r),
\end{aligned} \tag{2}$$

where d is given by the DTT formula, $g_1, \ldots, g_r$ are local coordinates on $I(M)$ that parametrize the group and $r = \dim I(M) = \frac{1}{2}n(n+1)$. Let $\Sigma \simeq \mathbb{R}^d$ be the space of parameters $a_1, \ldots, a_d$. By analogy with the classical theory of invariants, we formulate the following problem.

Problem 1. *Let* $\Sigma \simeq \mathbb{R}^d$ *be the space determined by the* d *parameters* $a_1, \ldots, a_d$ *that define the vector space* $\mathcal{K}^p(M)$ *in* $(M, \boldsymbol{g})$ *for some fixed* $p \geq 1$. *In the space of functions in* Σ, *describe the subspace of all functions that remain fixed under the induced action of the corresponding isometry group* $I(M)$.

The functions having this property are called $I(M)$-*invariants of the vector space* $\mathcal{K}^p(M)$. More specifically, they must satisfy the condition $F(\tilde{a}_1,\dots,\tilde{a}_d) = F(a_1,\dots,a_d)$ under the transformation laws (2) induced by the action of the isometry group $I(M)$. Solving Problem 1 then reduces to finding a set of *fundamental invariants* with the property that any other invariant is a (analytic) function of the fundamental invariants. The Fundamental Theorem on invariants of a regular Lie group action [2] determines the number of fundamental invariants required to define the whole of the space of $I(M)$-invariants:

Theorem 1. *Let G be a Lie group acting regularly on an m-dimensional manifold X with s-dimensional orbits. Then, in a neighbourhood N of each point $x_0 \in M$, there exist $m-s$ functionally independent G-invariants $\Delta_1,\dots,\Delta_{m-s}$. Any other G-invariant $\mathcal{I}$ defined near x_0 can be locally uniquely expressed as an analytic function of the fundamental invariants through $\mathcal{I} = F(\Delta_1,\dots,\Delta_{m-s})$.*

In all of the examples studied to this point, the isometry group acts regularly on the corresponding subsets $\Sigma\backslash\{\mathbf{0}\}$, where Σ is the space of the parameters of the vector space of Killing tensors under consideration, with the origin $a_1 = a_2 = \cdots = a_d = 0$, being one of the orbits.

3 Computation of the fundamental $I(M)$-invariants

In this section we briefly describe the methods of computing fundamental invariants of vector spaces of Killing tensors defined in pseudo-Riemannian spaces of constant curvature under the action of the isometry group.

3.1 Method of moving frames

The method of moving frames introduced recently by Fels and Olver [18, 19] is a very useful direct method for computing fundamental invariants. In this paper we use it for the first time within the framework of the theory of algebraic invariants of Killing tensors. As the following example shows, at least in some instances, it can be much more effective than the method of infinitesimal generators (see below) which has been used in the cases already studied. For example, consider the vector space $\mathcal{NK}^2(\mathbb{R}^2)$ of non-trivial Killing tensors of valence two defined in the Euclidean plane $\mathbb{R}^2$ under the action of $I(\mathbb{R}^2)$. Here "non-trivial" means that none of the elements of $\mathcal{NK}^2(\mathbb{R}^2)$ is a multiple of the corresponding metric tensor $\boldsymbol{g}$: $\mathcal{NK}^2(\mathbb{R}^2) = \{\boldsymbol{K} \in \mathcal{K}(\mathbb{R}^2) | \boldsymbol{K} \neq \ell\boldsymbol{g}, \ell \in \mathbb{R}\}$. This is a five-dimensional vector space. Without loss of generality we can assume any element of $\mathcal{NK}^2(\mathbb{R}^2)$ has the following general form:

$$\begin{bmatrix} a_1 + 2a_3y + a_5y^2 & a_2 - a_3x - a_4y - a_5xy \\ a_2 - a_3x - a_4y - a_5xy & 2a_4x + a_5x^2 \end{bmatrix}, \tag{3}$$

where (x, y) is some fixed system of Cartesian coordinates in $\mathbb{R}^2$ and a_i, $i = 1,\dots,5$ are arbitrary parameters that determine the dimension of the vector space $\mathcal{NK}^2(\mathbb{R}^2)$. In view of the standard parametrization of the isometry group $I(\mathbb{R}^2)$, we have the transformation of the coordinates (x, y)

$$\tilde{\boldsymbol{x}} = R_\theta \boldsymbol{x} + \boldsymbol{a}, \tag{4}$$

where $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2)$ and $\boldsymbol{a} = (a, b) \in \mathbb{R}^2$. The tensor transformation laws induce the corresponding transformation laws (2) on the parameters appearing in (3):

$$\begin{aligned} \tilde{a}_1 = {} & a_1(\cos^2\theta - \sin^2\theta) - 4a_2\sin\theta\cos\theta + 2(aa_4 - ba_3)\cos\theta \\ & - 2(ba_4 + aa_3)\sin\theta + a_5(b^2 - a^2), \end{aligned} \tag{5}$$

$$\tilde{a}_2 = a_1 \sin\theta\cos\theta + a_2(\cos^2\theta - \sin^2\theta) + (aa_3 + ba_4)\cos\theta + (aa_4 - ba_3)\sin\theta - a_5 ab, \quad (6)$$

$$\tilde{a}_3 = a_3\cos\theta + a_4\sin\theta - a_5 b, \quad (7)$$

$$\tilde{a}_4 = a_4\cos\theta - a_3\sin\theta - a_5 a, \quad (8)$$

$$\tilde{a}_5 = a_5. \quad (9)$$

We immediately observe that $\Delta_1 = a_5$ is a fundamental $I(\mathbb{R}^2)$-invariant. According to Theorem 1, we must have in total 5 (the dimension of the vector space) $-$ 3 (the dimension of the orbits) $=$ 2 fundamental invariants. (The fact that the orbits of the isometry group are 3-dimensional on the space $\Sigma\backslash\{\mathbf{0}\}$ was established earlier [7, 10].) To find the second fundamental invariant we employ the method of moving frames (see [2] for more details). Consider the following coordinate cross-section K in the space $\Sigma \simeq \mathbb{R}^5$ defined by the parameters a_i, $i = 1, \ldots, 5$, $K = \{a_2 = a_3 = a_4 = 0\}$. Next, we find the solutions to the *normalization equations* $\tilde{a}_2 = \tilde{a}_3 = \tilde{a}_4 = 0$, to determine the *moving frame map*: $\gamma : \Sigma \to I(\mathbb{R}^2)$. Solving (7) and (8) for a and b respectively, we get

$$a = \frac{a_4\cos\theta - a_3\sin\theta}{a_5}, \qquad b = \frac{a_3\cos\theta + a_4\sin\theta}{a_5}. \quad (10)$$

Next, substituting the formulas (10) into (6) we obtain

$$\tan 2\theta = -\frac{2(a_5a_2 + a_3a_4)}{a_1a_5 - a_3^2 + a_4^2}. \quad (11)$$

Therefore we conclude that the equations (10) and (11) define in this case the moving frame map $\gamma : (a_1, \ldots, a_5) \to (a, b, \theta)$. Finally, substituting (10) and (11) into the equation (5) and using elementary trigonometric formulas, we obtain the second fundamental invariant

$$\Delta_2 = \frac{\sqrt{(a_5a_1 - a_3^2 + a_4^2)^2 + 4(a_5a_2 + a_3a_4)^2}}{a_5}. \quad (12)$$

Since $\Delta_1 = a_5$ is also a fundamental invariant, we can conclude that the pair (Δ_1, Δ_2) constitutes a complete system of fundamental invariants in this case. We note that these results are consistent with the results obtained in [10] by the method of infinitesimal generators. Hence, by making use of Theorem 1 and employing the method of moving frames, we have solved Problem 1 for the vector space $\mathcal{NK}^2(\mathbb{R}^2)$ and hence proven the following theorem:

Theorem 2. *Any algebraic $I(\mathbb{R}^2)$-invariant $\mathcal{I}$ of the vector space $\mathcal{NK}^2(\mathbb{R}^2)$ can locally be uniquely expressed as an analytic function $\mathcal{I} = F(\Delta_1, \Delta_2)$ of the fundamental $I(\mathbb{R}^2)$-invariants* (9) *and* (12)*.*

We note that in this case the method of moving frames appears to be more effective in comparison with the method of infinitesimal generators used previously to solve this problem. More specifically, the fundamental invariants have been found by a *direct algebraic* method without having to solve a system of PDEs.

Recall that non-trivial Killing tensors define orthogonal coordinate webs in $\mathbb{R}^2$. From this perspective, the invariant $k^2 = \Delta_2/\Delta_1$ is the square of the (half) distance between the foci in the case when the corresponding Killing tensor gives rise to the elliptic-hyperbolic coordinate web (see [7, 10]) for more details).

3.2 Method of infinitesimal generators

Let $\mathcal{K}^p(M)$ be the vector space of Killing tensors of valence $p \geq 1$ defined on a pseudo-Riemannian manifold M of constant curvature. Consider the action of the isometry group $I(M)$

on $\mathcal{K}^p(M)$. The method of infinitesimal generators for finding fundamental invariants rests on the following idea: One finds the infinitesimal generators of the Lie algebra $i(M)$ of $I(M)$ in the space Σ determined by the parameters $a_1, \ldots, a_d$ of the vector space $\mathcal{K}^p(M)$, $d = \dim \mathcal{K}^p(M)$. Since invariants are annihilated by the generators (Killing vectors) of the Lie algebra of the isometry group, determining the fundamental invariants reduces to solving the corresponding system of PDEs in the space $\Sigma \simeq \mathbb{R}^d$. To find the generators in Σ we employ the following technique. Consider $\mathrm{Diff}\,\Sigma$, the space of all diffeomorphisms of Σ. It yields $\mathrm{Diff}\,\mathcal{K}^p(M)$, the space of Killing tensors of valence p whose parameters are determined by all possible diffeomorphisms of Σ. Thus, an element $\tilde{\boldsymbol{K}} \in \mathrm{Diff}\,\mathcal{K}^p(M)$ is a Killing tensor of valence p, whose d parameters $\tilde{a}_1, \ldots, \tilde{a}_d$ are defined by a diffeomorphism of the coordinates $a_1, \ldots, a_d$ of Σ: $\tilde{a}_i = \tilde{a}_i(a_1, \ldots, a_d)$, $i = 1, \ldots, d$. Define now the following map $\pi : \mathrm{Diff}\,\mathcal{K}^p(M) \to \mathcal{X}(\Sigma)$, where $\mathcal{X}(\Sigma)$ is the space of all vector fields on Σ, as follows:

$$\pi(\tilde{\boldsymbol{K}}) = \sum_{i=1}^{d} \tilde{a}_i(a_1, \ldots, a_d) \frac{\partial}{\partial a_i}, \qquad i = 1, \ldots, d, \tag{13}$$

where $\tilde{\boldsymbol{K}} \in \mathrm{Diff}\,\mathcal{K}^p(M)$, $a_1, \ldots, a_d$ are the coordinates of the space Σ (see Section 2 for more details) and $\tilde{a}_i$, $i = 1, \ldots, d$ determine $\tilde{\boldsymbol{K}}$. Now let $\boldsymbol{K} \in \mathcal{K}^p(M)$ be the general Killing tensor determined by $a_1, \ldots, a_d$ of Σ (in essence, $\boldsymbol{K}$ is the solution of the corresponding Killing tensor equation (1)) and $\boldsymbol{X}_i$, $i = 1, \ldots, r$, $r = \frac{1}{2}n(n+1)$ be the generators of $i(M)$. Define the following vector fields in $\mathcal{X}(\Sigma)$:

$$\boldsymbol{V}_i = \pi(\mathcal{L}_{\boldsymbol{X}_i} \boldsymbol{K}), \qquad i = 1, \ldots, r, \tag{14}$$

where $\mathcal{L}$ denotes the Lie derivative operator. Note that the Killing tensors $\boldsymbol{K}_{\boldsymbol{X}_i} = \mathcal{L}_{\boldsymbol{X}_i} \boldsymbol{K} \in \mathrm{Diff}\,\mathcal{K}^p(M)$, $i = 1, \ldots, r$ and so the vector fields $\boldsymbol{V}_i \in \mathcal{X}(\Sigma)$, $i = 1, \ldots, r$ are well-defined. The vector fields $\boldsymbol{V}_i$ represent the generators $\boldsymbol{X}_i$, $i = 1, \ldots, r$ in $\mathcal{X}(\Sigma)$ in the sense that they satisfy the same commutator relationships. This fact can be established on a case by case basis. It demonstrates that the Lie algebras generated by the vector fields $\boldsymbol{X}_i$ and $\boldsymbol{V}_i$, $i = 1, \ldots, r$ are in fact isomorphic and the vector fields $\boldsymbol{V}_i$, $i = 1, \ldots, r$ can be used in determining the fundamental invariants. Indeed, it follows from this fact that $I(M)$-invariants are annihilated by the generators of the corresponding Lie algebra. The following example will illustrate the procedure. Consider the six-dimensional vector space $\mathcal{K}^1(\mathbb{R}^3)$ which consists of all Killing vectors in Euclidean space $\mathbb{R}^3$. The most general element of $\mathcal{K}^1(\mathbb{R}^3)$ in terms of Cartesian coordinates is given by

$$\boldsymbol{K} = (a_1 + a_5 z - a_6 y)\frac{\partial}{\partial x} + (a_2 + a_6 x - a_4 z)\frac{\partial}{\partial y} + (a_3 + a_4 y - a_5 x)\frac{\partial}{\partial z}. \tag{15}$$

The generators of the Lie algebra $\mathcal{K}^1(\mathbb{R}^3)$ in terms of Cartesian coordinates are given by

$$\begin{aligned} &\boldsymbol{X}_1 = \frac{\partial}{\partial x}, \qquad \boldsymbol{X}_2 = \frac{\partial}{\partial y}, \qquad \boldsymbol{X}_3 = \frac{\partial}{\partial z} \qquad \text{(translations)}, \\ &\boldsymbol{R}_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad \boldsymbol{R}_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad \boldsymbol{R}_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \qquad \text{(rotations)}. \end{aligned} \tag{16}$$

They satisfy the following commutator relations.

$$[\boldsymbol{X}_i, \boldsymbol{X}_j] = 0, \qquad [\boldsymbol{X}_i, \boldsymbol{R}_j] = -\epsilon_{ijk}\boldsymbol{X}_k, \qquad [\boldsymbol{R}_i, \boldsymbol{R}_j] = -\epsilon_{ijk}\boldsymbol{R}_k, \tag{17}$$

where ϵ_{ijk} is the permutation tensor and $i, j, k = 1, 2, 3$. Applying the technique described above (see (14)) we find the corresponding vector fields in $\mathcal{X}(\Sigma)$, where $\Sigma \simeq \mathbb{R}^6$ is determined by the

parameters $a_1, \ldots, a_6$ of (15)

$$
\begin{aligned}
\boldsymbol{U}_1 &= a_6\frac{\partial}{\partial a_2} - a_5\frac{\partial}{\partial a_3},\\
\boldsymbol{U}_2 &= -a_6\frac{\partial}{\partial a_1} + a_4\frac{\partial}{\partial a_3},\\
\boldsymbol{U}_3 &= a_5\frac{\partial}{\partial a_1} - a_4\frac{\partial}{\partial a_2},\\
\boldsymbol{V}_1 &= a_3\frac{\partial}{\partial a_2} - a_2\frac{\partial}{\partial a_3} + a_6\frac{\partial}{\partial a_5} - a_5\frac{\partial}{\partial a_6},\\
\boldsymbol{V}_2 &= -a_3\frac{\partial}{\partial a_1} + a_1\frac{\partial}{\partial a_3} - a_6\frac{\partial}{\partial a_4} + a_4\frac{\partial}{\partial a_6},\\
\boldsymbol{V}_3 &= a_2\frac{\partial}{\partial a_1} - a_1\frac{\partial}{\partial a_2} + a_5\frac{\partial}{\partial a_4} - a_4\frac{\partial}{\partial a_5}.
\end{aligned}
\tag{18}
$$

Note that the vector fields (18) satisfy the same commutator relations (17)

$$
[\boldsymbol{U}_i, \boldsymbol{U}_j] = 0, \qquad [\boldsymbol{U}_i, \boldsymbol{V}_j] = -\epsilon_{ijk}\boldsymbol{U}_k, \qquad [\boldsymbol{V}_i, \boldsymbol{V}_j] = -\epsilon_{ijk}\boldsymbol{V}_k. \tag{19}
$$

Therefore, in view of (17) and (19), we have established the following result:

Lemma 1. *The Lie algebras $i(\mathbb{R}^3) \simeq \mathcal{K}^1(\mathbb{R}^3)$ and $i_\Sigma(\mathbb{R}^3)$ are isomorphic, where the Lie algebra $i_\Sigma(\mathbb{R}^3)$ is spanned by* (18)*.*

Next, we observe that the coefficient matrix of the generators (18)

$$
\begin{bmatrix}
0 & a_6 & -a_5 & 0 & 0 & 0\\
-a_6 & 0 & a_4 & 0 & 0 & 0\\
a_5 & -a_4 & 0 & 0 & 0 & 0\\
0 & a_3 & -a_2 & 0 & a_6 & -a_5\\
-a_3 & 0 & a_1 & -a_6 & 0 & a_4\\
a_2 & -a_1 & 0 & a_5 & -a_4 & 0
\end{bmatrix}
\tag{20}
$$

has rank four almost everywhere. Therefore we conclude that the isometry group has 4-dimensional orbits in the space $\Sigma\backslash\{\mathbf{0}\}$, and so, in view of Theorem 1, we have 6 (the dimension of the group) $-$ 4 (the dimension of the orbits) $=$ 2 fundamental invariants. Indeed, taking into account the result of Lemma 1 and employing the *method of undetermined coefficients* (see Section 3.3) we solve the system of linear PDEs generated by (18)

$$
\boldsymbol{V}_i(F) = 0, \qquad \boldsymbol{U}_i(F) = 0, \qquad i = 1, 2, 3 \tag{21}
$$

and arrive at the following result:

Theorem 3. *Any algebraic $I(\mathbb{R}^3)$-invariant $\mathcal{I}$ of the vector space $\mathcal{K}^1(\mathbb{R}^3)$ can locally be uniquely expressed as an analytic function $\mathcal{I} = F(\Delta_1, \Delta_2)$, where*

$$
\Delta_1 = a_4^2 + a_5^2 + a_6^2, \qquad \Delta_2 = a_1a_4 + a_2a_5 + a_3a_6 \tag{22}
$$

are the corresponding fundamental $I(\mathbb{R}^3)$-invariants.

As expected, we have obtained two fundamental invariants; they can be effectively employed in a classification of the elements of the vector space $\mathcal{K}^1(\mathbb{R}^3)$ [3].

3.3 Method of undetermined coefficients

One typically employs the method of characteristics when solving systems of linear PDEs, such as (21). Unfortunately, this procedure does not always yield a result, and, in particular, becomes intractable when applied to (21). However, an alternate solution method, namely the *method of undetermined coefficients* has helped to alleviate the difficulties. Thus, based on previous problems that we have solved so far, it has been conjectured that there always exists a set of fundamental invariants consisting of homogeneous polynomials in $a_1, \ldots, a_d$. We can take advantage of this by constructing a suitable trial function for the system of PDEs. This is the general idea of the method of undetermined coefficients.

The method works as follows. We construct a general homogeneous polynomial in $a_1, \ldots, a_d$ of some fixed degree with coefficients to be determined. Substituting this trial polynomial into the system of PDEs leads to a system of linear equations in the undetermined coefficients. Thus, any non-trivial solution of this linear system yields a solution of the PDE system, and hence an $I(M)$-invariant of $\mathcal{K}^p(M)$. This process can be repeated with higher degree polynomials until the required number of functionally independent invariants asserted by Theorem 1 is derived. In this way, a set of fundamental $I(M)$-invariants which are functionally independent is obtained.

It turns out that the resulting linear system is overdetermined. There are approximately $r = \frac{1}{2}n(n+1)$ times as many linear equations as there are unknowns. This is understandable since there are r infinitesimal generators in the system. We also emphasize that the linear system is sparse, in the sense that the corresponding coefficient matrix of the system consists primarily of zero entries. As a result, the linear system can easily be solved using a symbolic sparse linear solver.

We have successfully implemented this algorithm in Maple to solve the resulting system of linear equations. In particular, we have computed the complete set of fundamental invariants for a number of vector spaces of Killing tensors, notably for those where the method of characteristics has failed [3]. For the amusement of the reader, we remark that in the space $\mathcal{K}^2(\mathbb{R}^3)$, an arbitrary quintic polynomial in $a_1, \ldots, a_{20}$ must be constructed, which results in a linear system of approximately 250 000 equations in 50 000 unknowns with about 1 000 000 non-zero entries. The computation took approximately ninety hours to complete on a 300 MHz Sun Ultra-5. The exact details of this calculation are discussed in [3]. This paper also examines how a more restrictive trial function can be constructed which further enhances the efficiency of the algorithm.

4 Method of reducing invariants

In some instances, fundamental $I(N)$-invariants of a vector space $\mathcal{K}^p(N)$ can be obtained from fundamental $I(M)$-invariants of $\mathcal{K}^p(M)$, where $N \subset M$. For example, a complete set of fundamental $I(\mathbb{S}^2)$-invariants for $\mathcal{K}^2(\mathbb{S}^2)$ has been obtained [4] from the fundamental $I(\mathbb{R}^3)$-invariants of $\mathcal{K}^2(\mathbb{R}^3)$ derived in [3], based on the result of the Delong theorem [13] describing explicitly the elements of a general space $\mathcal{K}^2(\mathbb{R}^{n+1})$, $n \geq 2$ which are the elements of $\mathcal{K}^2(\mathbb{S}^n)$.

5 Conclusion

We have demonstrated how the method of moving frames [18, 19] can be used to solve the problem of finding the complete sets of fundamental algebraic invariants of vector spaces of Killing tensors under the action of the isometry group. In the study of Killing tensors defined on low-dimension pseudo-Riemannian spaces of constant curvature the method appears to be more effective than the method of infinitesimal generators. However, in finding fundamental invariants of vector spaces of Killing tensors of higher valences defined on spaces of higher

dimensions the above mentioned methods prove to be less effective. This prompts one to use the method of undetermined coefficients combined with a computer algebra system such as Maple to solve the problem of finding complete sets of fundamental $I(M)$-invariants for the vector spaces $\mathcal{K}^p(M)$.

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On Computational Aspects of the Fourier–Mukai Transform

Nikolaj M. GLAZUNOV

Glushkov Institute of Cybernetics, 40 Glushkov Ave., Kyiv-187, Ukraine
E-mail: *glanm@d105.icyb.kiev.ua*

We survey and investigate some computational aspects of the Fourier–Mukai transform.

1 Introduction

Let $f(x)$ be a function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$, $n \in \mathbb{N}$. Then its Fourier transform from the time domain into its frequency domain is given by the discrete Fourier transform

$$\hat{f}(\omega) = \sum_{x=0}^{n-1} f(x) \exp\left(2\pi i \frac{\omega x}{n}\right).$$

Here ωx is the perfect pairing (in the case the scalar product) on the product of the time and of the frequency domains. It is well-known also that the solution of a linear differential equation with constant coefficients is related by the Fourier transform to a solution of the polynomial equation. The Fourier–Mukai transform is a strong generalization of mentioned and some another approaches. Let A be an Abelian variety, $\hat{A}$ its dual Abelian variety and P the Poincaré divisor on $A \times \hat{A}$. Let $D^b(A)$ and $D^b(\hat{A})$ be derived categories of bounded complexes of sheaves on A and $\hat{A}$ respectively. A Fourier–Mukai transform was defined by Mukai as an exact equivalence

$$\mathcal{FM} : \ D^b(\hat{A}) \to D^b(A)$$

between derived categories of above mentioned bounded complexes [1, 2]. For this transform analogies of the Fourier Inversion Theorem and the Parseval Theorem are valid. Works by B. Bartocci, U. Bruzzo, D. Ruipérez [3] and A. Maciocia [4] generalized this approach to another classes of sheaves and varieties. Now by a Fourier–Mukai transform

$$\mathcal{FM} : \ D^b(Y) \to D^b(X)$$

an exact equivalence between bounded derived categories of coherent sheaves on two smooth projective varieties X and Y is understood. It is known that the derived categories of coherent sheaves on some projective varieties are equivalent to the derived categories of representations of n-vertices quivers. For instance, the derived category of coherent sheaves on $\mathbb{P}^2$ is equivalent to the derived category of representations of 3-vertices quivers with relations [5]. There is a very interesting connection between Fourier–Mukai transforms and mirror symmetry [8]. A quiver can be interpreted as a directed graph [6, 7]. We investigate some computational aspects of Fourier–Mukai transforms. This paper is a continuation of [9, 10].

The organization of the article is as follows. In Section 2 we recall from the computational point of view some facts about Abelian varieties. In Section 3 we very shortly consider parameterization of Abelian varieties and some moduli spaces. In Section 4 we recall follow to Mukai the definition of the Fourier–Mukai transform. In Section 5 we present our computer algebraic method for computation of cutsets of some quivers and its implementation.

2 Abelian varieties

Let $\mathbb{C}^g$ be a complex vector space of the dimension g and Λ a lattice in $\mathbb{C}^g$ with a bases $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2g}\}$. For a lattice vector $\boldsymbol{a} = (a_1, \ldots, a_{2g}) \in \Lambda$ its length is denoted by $|\boldsymbol{a}|$. The factor $\mathbb{C}^g/\Lambda = A$ is the commutative compact topological group (complex torus). Recall the case $g = 1$.

Example 1 (Elliptic curves). Let $\Lambda = \{\boldsymbol{a}, \boldsymbol{b}\}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}$, $\boldsymbol{a}/\boldsymbol{b} \notin \mathbb{R}$. Then $E = \mathbb{C}/\Lambda$ is an elliptic curve. In the case every such lattice defines an elliptic curve. What is the simplest representation of Λ?

Lemma 1 (Basis reduction in Λ). *In any such lattice Λ there exists (under the rotations) one and only one basis with conditions:*

(i) $|\boldsymbol{b}| \geq |\boldsymbol{a}|$;

(ii) *length of the projection $\boldsymbol{b}$ on $\boldsymbol{a} \leq \frac{1}{2}|\boldsymbol{a}|$;*

(iii) *the angle $(\boldsymbol{a}, \boldsymbol{b}) < \pi/2$ (acute angle).*

So we can reduce a basis of Λ to the form $\Lambda = \{1, \tau\}$, where $\operatorname{Im}\tau > 0$, $-\frac{1}{2} \leq \operatorname{Re}\tau \leq \frac{1}{2}$, $|\tau| \geq 1$.

Let

$$h = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \qquad \det h = 1, \qquad a, b, c, d \in \mathbb{Z}.$$

Two lattices $\Lambda = \{1, \tau\}$, $\Lambda' = \{1, \tau'\}$ are equivalent if $\tau' = \frac{a\tau+b}{c\tau+d}$. In the case there is a birational isomorphism between elliptic curves $E = \mathbb{C}/\Lambda$ and $E' = \mathbb{C}/\Lambda'$.

If $g > 1$ then there are groups $A = \mathbb{C}^g/\Lambda$ which are not Abelian varieties [11]. The complex torus A is an Abelian variety if and only if there exist $\mathbb{R}$-bilinear antisymmetric form $F(x, y)$ such that the form $F(x, ix)$ is a positive definite Hermitian form that takes integer values at points of Λ. The form $F(x, y)$ is called the *polarization* on Λ and a pair (Λ, F) is called the polarized Abelian variety. For computational purposes a good idea is to reduce the bases of the lattice to a some simple form. The bases reduction of Λ in the case $g = 1$ is well-known (Example 1).

On basis reduction in lattices (the case $g > 1$). Let a lattice $\Lambda = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2g}\}$, $\boldsymbol{a}_i = (a_{i1}, \ldots, a_{i1})$ have coordinates with the condition $a_{ij} \in \mathbb{Q}$.

There are *Minimum Basis problem*, *Shortest Lattice Vector problem* and *LLL Lattice Reduction* [12, 13].

The method of a solution of the Minimum basis problem is based on the theory of successive minima, developed by Minkowski [12]. By this method we try to find short linear independent vectors one after one. The solution of the Shortest Lattice Vector problem is based on Minkowski's theorem on convex body [12]. Probably both the methods are NP-hard.

In some applications the concept "*reduces*" means something like "nearly orthogonal". The L^3 algorithm [14] accept as input any basis $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2g}$ of Λ, and it computes a reduced basis $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2g}$ of Λ. In general, a lattice may have more then one reduced basis. The ordering of the basis vectors is not arbitrary. The algorithm has a very good theoretical complexity (polynomial-time in the length of the input parameters).

3 Parameterization of Abelian varieties and moduli spaces

Let (Λ, F) be a polarized Abelian variety. Denote by M^{tr} the transposition of a matrix M. There is a canonical form of Abelian varieties. In the form the lattice of the Abelian variety has

the representation

$$\Omega = (E_g, Z),$$

where E_g is the identity matrix and $Z = X + iY$ is a complex $g \times g$ matrix with conditions: a) $Z = Z^{\mathrm{tr}}$, $X = X^{\mathrm{tr}}$, $Y = Y^{\mathrm{tr}}$ (X, Y are reals); b) $Y > 0$ is the matrix of a positive quadratic form. The set $\{Z\} = \mathbf{H}_g$ of the matrices is called the *Siegel upper half-space*. Let $J = \begin{vmatrix} 0 & E_g \\ -E_g & 0 \end{vmatrix}$ be the symplectic matrix. Let $M = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $M \in GL(2g, \mathbb{Z})$, $M^{\mathrm{tr}} J M = J$, be the modular group. The modular group acts on $\mathbf{H}_g$ by the formula

$$Z' = (aZ + b)(cZ + d)^{-1}$$

(here a, b, c, d are $g \times g$ matrices with integer coefficients). If matrices X and Y have integer coefficients, then all data with the exception of complex unit are defined over $\mathbb{Z}$. Therefore they and their factors $\mod p$ for prime p are well-adapted for computers. Moduli spaces [15, 16] are used for specification and investigation of classes of objects which could be algebraic curves (or, more generally, schemes), sheaves, vector bundles, morphisms and others. Parameter varieties is a class of moduli spaces. These varieties is a very convenient tool for computer algebra investigation of objects that are parameterized by the parameter varieties. If we investigate a class of algebraic varieties over integers, then the first step of the investigation in many cases is the analysis of the varieties over prime finite fields. We have used the approach for investigation of rational points on families of hyperelliptic curves [17]. Expansion of the group law of an Abelian variety near 0 defines a formal group of the dimension one in the case of elliptic curves [18] and of the dimension $g > 1$ for $g-$dimensional Abelian varieties [19]. Results about formal group over finite fields and over ρ-adic fields [20] can be useful for investigation of the groups over $\mathbb{Z}$. The paper [21] demonstrates computational advantages of the localization under computation of cohomologies.

Let A, B be two Abelian varieties of the dimension g over a field k, $k(A)$, $k(B)$ their fields of functions, $\lambda : A \to B$ some surjective morphism with a finite kernel (an *isogeny* of the varieties). Any isogeny $\lambda : A \to B$ defines a field extension $[k(A) : k(B)]$. The degree of the extension is called the *degree* of the isogeny. The polarization F defines an isogeny of degree d^2. Follow to [22] it is possible to construct for a prime finite field $\mathbf{F}_p$ the moduli space $X_{g,d} \otimes \mathbf{F}_p$ of all Abelian varieties of dimension g with a polarization F of degree d^2.

4 Elements of category theory and the Fourier–Mukai transform

4.1 Equivalence of categories

Let $\mathcal{C}$ be a category [23], $\mathrm{Ob}\,\mathcal{C}$ its class of objects, and for $a, b \in \mathrm{Ob}\,\mathcal{C}$, $\mathcal{C}(a, b)$ the class of morphisms (arrows) from a to b. An arrow $u : a \to b$ is *the equivalence* in $\mathcal{C}$ if there is an arrow $u' : b \to a$ such that $u'u = 1_a$ and $uu' = 1_b$. A *functor* $\mathcal{F}$ from a category $\mathcal{C}$ to a category $\mathcal{K}$ is a function which maps $\mathrm{Ob}\,(\mathcal{C}) \to \mathrm{Ob}\,(\mathcal{K})$, and which for each pair a, b of objects of $\mathcal{C}$ maps $\mathcal{C}(a, b) \to \mathcal{C}(\mathcal{F}(a), \mathcal{F}(b))$, while satisfying the two conditions: $\mathcal{F}\,\mathrm{id}_a = \mathrm{id}_{\mathcal{F}a}$ for every $a \in \mathrm{Ob}\,(\mathcal{C})$, $\mathcal{F}(fg) = \mathcal{F}(f)\mathcal{F}(g)$.

Let $\mathrm{Funct}\,(\mathcal{C}, \mathcal{D})$ be the category of functors from $\mathcal{C}$ to $\mathcal{D}$ with natural transformations as morphisms. An equivalence in $\mathrm{Funct}\,(\mathcal{C}, \mathcal{D})$ is *the equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$. The following theorem is a particular case of the theorem from [23].

Theorem 1. *A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is the equivalence if, and only if*

a) *$\mathcal{F}$ is fully faithful, and*

b) *any $Y \in \mathrm{Ob}\,\mathcal{D}$ is isomorphic to an object of the form $\mathcal{F}(X)$ for an object $X \in \mathrm{Ob}\,\mathcal{C}$.*

4.2 The Fourier–Mukai transform

Let A be an Abelian variety and $\hat{A}$ the dual Abelian variety which is by definition a moduli space of line bundles of degree zero on A [11]. The *Poincaré bundle* $\mathcal{P}$ is a line bundle of degree zero on the product $A \times \hat{A}$, defined in such a way that for all $a \in \hat{A}$ the restriction of $\mathcal{P}$ on $A \times \{a\}$ is isomorphic to the line bundle corresponding to the point $a \in \hat{A}$. This line bundle is also called *the universal bundle*. Let

$$\pi_A : A \times \hat{A} \to A,$$
$$\pi_{\hat{A}} : \hat{A} \times A \to \hat{A},$$

and $\mathcal{C}_A$ be the category of O_A-modules over $A, M \in \mathrm{Ob}\,\mathcal{C}_A$,

$$\hat{S}(M) = \pi_{\hat{A},*}(\mathcal{P} \otimes \pi_A^* M).$$

Then, by definition, *the Fourier–Mukai transform* $\mathcal{FM}$ is the derived functor $R\hat{S}$ of the functor $\hat{S}$. Let $\mathcal{D}(A)$, $\mathcal{D}(\hat{A})$ be bounded derived categories of coherent sheaves on A and $\hat{A}$ respectively.

Theorem 2 (Mukai). *The derived functor $\mathcal{FM} = R\hat{S}$ induces an equivalence of categories between two derived categories $\mathcal{D}(\hat{A})$ and $\mathcal{D}(A)$.*

5 Cutset method and its implementation

A quiver can be interpreted as a directed graph. Here we consider quivers without multiple edges and loops. The solution of some problems requires to cut all cycles or construct a spanning-tree of a directed graph. Rooted graph G is the graph that each node of G is reachable from a node $r \in G$. By the *cutset* of a graph we shall understand an appropriated subset of nodes (called cutpoints) such that any cycle of the graph contains at least one cutpoint. DFS is the abbreviation of Depth First Search method. During DFS we are numbering nodes and label (mark) edges. By Tarjan [24] the DFS method has linear complexity. The main function Cutsetdg of the package CUTSETDG computes Cutset (a subset of vertices which cut all cycles in the graph) of arbitrary rooted directed graph. It uses 3 functions: Adjarcn, Cutpoints and Unicut. For this program the author developed rather simple and efficient algorithm that is based on DFS-method. The function is the base for Cutset methods for systems of procedures. The package is implemented by the Allegro Common Lisp language [25, 26].

Note 1. It is not difficult to modify the Cutset algorithm for processing directed graphs with multiple edges.

Package CUTSETDG
Title: Cutset of a rooted directed graph
Summary: This package implements computation of a cutset of a rooted directed graph. The graph have to defined by adjacency list.
Author: Nikolaj M. Glazunov
Example of Call: (Cutsetdg 'a) where "a" is a root of exploring graph.
The Common Lisp text of the package CUTSETDG

```
;; Allegro CL 3.0.2
;;

(DEFUN Cutsetdg (startnode)

;; startnode is a root of exploring graph
;; DFS - Depth First Search method for connected graph
```

```
;; st - stack
;; v - exploring node
;; sv -  son of the exploring node
;; inl - inverse edges list
;; df -  DFS-numbering of the node (property)
;; Outarcs - adjacency list of Output arcs of the node (property)
;; 1, 2  - lables

    (SETF (GET startnode 'df ) 1)   ;DFS-numbering is equal 1
   (SETF (GET startnode 'Outarcs)
            (Adjarcn  (LIST startnode))) ;;startnode obtained the a-list
                                  ;; property Outarcn (Output arcs of the
                                  ;; node)
    (PROG ( st v sv inl  cuts)
         (PUSH startnode st)
   1   (SETQ v (CAR st)) ;;explored node received value from stack of nodes
   2      (COND ((NOT (EQ  NIL (GET v 'Outarcs)))
                    (SETQ sv (CAAR (GET v 'Outarcs)))
                        (COND ((EQ NIL (GET sv 'df))
                            (SETF (GET sv 'df) (+ 1 (GET v 'df)))
          ;; modification of the edge of son's adj-list
          (SETF (GET sv 'Outarcs)
               (Adjarcn  (LIST sv))) ;; node sv obtained the adj-list property
                                  ;; Outarcs (adjacency Outarcs)
             (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                                :TEST 'EQUAL))
                              (PUSH sv st)
                               (GO 1)
              )
                (T
                   (COND ((AND (> (GET v 'df) (GET sv 'df))
                               (NOT (EQUAL v sv)))
                (SETQ inl (CONS (CAR (GET v 'Outarcs)) inl))
                (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                                :TEST 'EQUAL))
                          (GO 2)
                          )
                (T
                 (SETF (GET v 'Outarcs) (REMOVE (LIST sv v) (GET v 'Outarcs)
                                :TEST 'EQUAL))
                  (GO 2)
                  )
                         )
                  )
                    )
                                (T
                (POP st)
          (COND     ((NOT (NULL st))
                       (GO 1)
               )
                     (T                 ;; end
                    (SETQ cuts (Unicut (Cutpoints inl)))
                (RETURN cuts)
                )
               )
```

```
               ))
     )
   )
(DEFUN Adjarcn (PATH)
      (MAPCAN #'(LAMBDA (E)
              (COND ((MEMBER (CAR E) (CDR E)) NIL)
                    (T (LIST E))))
          (MAPCAR #'(LAMBDA (E)
                            (CONS E PATH))
            (CAR  (GET (CAR PATH) 'NEIGHBORS)))))
(DEFUN Cutpoints (inl)
        (MAPCAR 'CAR inl))
(DEFUN Unicut (cpl)
     (COND ((NULL cpl) NIL)
           (T (CONS (CAR cpl)
                (Unicut (REMOVE (CAR cpl) cpl)))
            )
     )
     )
```

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On the Wave Functions of Ruijsenaars Model Related to q-Analogue of Symmetric Space GL(n)/SO(n)

Nikolai IORGOV

Bogolyubov Institute for Theoretical Physics, 14b Metrologichna Str., 03143 Kyiv, Ukraine
E-mail: *iorgov@aep1.bitp.kiev.ua*

In [1], the wave functions of quantum trigonometric n-particle Ruijsenaars model are defined as matrix elements of operators of representations of Cartan subalgebra between vectors invariant with respect to q-deformation $U_q'(\mathrm{so}_n)$ of Lie algebra so(n). It was shown there, that the wave functions defined in such a way are simultaneous eigenfunctions of commuting set of Macdonald–Ruijsenaars difference operators. Using this information, the expressions for wave functions in terms of Macdonald polynomials were found. In this contribution, these expressions are obtained in a direct manner by using explicit expressions for invariant vectors in representation spaces in terms of Gel'fand–Tsetlin basis.

1 Introduction

In [2], M. Noumi introduced the notion of the quantum analogue $\mathcal{F}_q(\mathrm{GL}(n)/\mathrm{SO}(n))$ of algebra of functions on the homogeneous space GL(n)/SO(n) and zonal spherical functions on it. It turned out that these spherical functions when restricted to Cartan subalgebra are Macdonald polynomials, which are simultaneous eigenfunctions of commuting set of Macdonald–Ruijsenaars difference operators. These difference operators appear as the radial components of Casimir elements of $U_q(\mathrm{gl}_n)$. In [2], the radial components of the "quadratic" Casimir elements were derived. In [1] the radial components of the other basis Casimir elements of $U_q(\mathrm{gl}_n)$ were found. From the other side, Macdonald–Ruijsenaars difference operators (up to a change of variables) coincide with commuting Hamiltonians of quantum trigonometric n-particle Ruijsenaars model. In [1], the wave functions of Ruijsenaars model are defined as matrix elements $\Psi(x_1, x_2, \ldots, x_n) = \langle v|q^{\sum_{k=1}^n x_k \epsilon_k}|v\rangle$, where q^{ϵ_k} belongs to Cartan subalgebra, $\langle v|$ and $|v\rangle$ are invariant vectors with respect to q-deformation $U_q'(\mathrm{so}_n)$ of Lie algebra so(n) introduced in [3]. It was shown that the wave functions defined in such a way are simultaneous eigenfunctions of commuting set of Macdonald–Ruijsenaars difference operators. Using this information, the expressions for wave functions in terms of Macdonald polynomials were found. In this contribution, these expressions are obtained in a direct manner by using explicit expressions for invariant vectors in representation spaces in terms of Gel'fand–Tsetlin basis. In order to express the obtained wave functions in terms of Macdonald polynomials we need the combinatorial formula [4] for such polynomials.

Quantum trigonometric Ruijsenaars model which we discuss in this paper is a q-analogue of quantum Sutherland model connected with symmetric space GL(n)/SO(n). Thus this paper can be considered in the spirit of [5], where different integrable systems are connected with analysis on symmetric spaces.

In [6], the authors used representation theory of GL(n) to obtain the wave functions for open Toda model and Sutherland model. Their Gel'fand–Tsetlin formulas are different from that which are used in this contribution. An important consequence of this difference is that the wave functions obtained in [6] are wave functions in some separated variables.

2 The quantum algebras $U_q(\mathrm{gl}_n)$ and their finite-dimensional representations

By the definition [7], Drinfeld–Jimbo quantum algebra $U_q(\mathrm{gl}_n)$ is a unital associative algebra with the generators e_i, f_i, $i = 1, 2, \ldots, n-1$, and generators q^h, where $h = x_1\epsilon_1 + x_2\epsilon_2 + \cdots + x_n\epsilon_n$, $x_i \in \mathbb{R}$, are elements of a vector space with the basis ϵ_i, $i = 1, 2, \ldots, n$. Defining relations containing complex deformation parameter $q \neq 0, \pm 1$ are:

$$
\begin{aligned}
&q^0 = 1, \qquad q^{h_1} q^{h_2} = q^{h_1+h_2}, \qquad q^h e_i = q^{x_i - x_{i+1}} e_i q^h, \qquad q^h f_i = q^{x_{i+1}-x_i} f_i q^h,\\
&[e_i, f_j] = \delta_{ij} \frac{q^{\epsilon_i - \epsilon_{i+1}} - q^{\epsilon_{i+1} - \epsilon_i}}{q - q^{-1}},\\
&[e_i, e_j] = 0, \quad |i-j| > 1, \qquad e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0, \quad |i-j| = 1,\\
&[f_i, f_j] = 0, \quad |i-j| > 1, \qquad f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0, \quad |i-j| = 1.
\end{aligned}
$$

In the limit $q \to 1$, algebra $U_q(\mathrm{gl}_n)$ becomes the universal enveloping algebra for Lie algebra gl_n.

For every set of n integers $\boldsymbol{m}_n = (m_{1,n}, m_{2,n}, \ldots, m_{n,n})$ such that

$$m_{1,n} \geq m_{2,n} \geq \cdots \geq m_{n,n},$$

there corresponds a simple finite-dimensional left module $\mathcal{V}^{\mathrm{L}}_{\boldsymbol{m}_n}$ over the algebra $U_q(\mathrm{gl}_n)$. Its explicit construction will be given in Gel'fand–Tsetlin formalism. Basis elements of this module are labeled by the sets of integers $\boldsymbol{m}_j = (m_{1,j}, m_{2,j}, \ldots, m_{j,j})$, $j = 1, \ldots, n-1$, such that

$$m_{i,j+1} \geq m_{i,j} \geq m_{i+1,j+1}, \qquad i = 1, 2, \ldots, j, \quad j = 1, 2, \ldots, n-1.$$

It is useful to visualize them by the Gel'fand–Tsetlin tableaux

$$
M = \begin{pmatrix}
m_{1,n} & & m_{2,n} & & \cdots & & m_{n,n}\\
& m_{1,n-1} & & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\
& & \cdots & \cdots & \cdots & & \\
& & & m_{1,1} & & &
\end{pmatrix}. \tag{1}
$$

To the tableau (1), there corresponds basis element denoted by $|M\rangle$.

The generators of $U_q(\mathrm{gl}_n)$ act on the Gel'fand–Tsetlin basis by the formulas [7]

$$q^{e_j}|M\rangle = q^{a_j}|M\rangle, \qquad a_j = \sum_{i=1}^{j} m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1}, \qquad 1 \leq j \leq n, \tag{2}$$

$$e_j|M\rangle = \sum_{i=1}^{j} A_j^i(M)|M_j^{+i}\rangle, \qquad f_j|M\rangle = \sum_{i=1}^{j} B_j^i(M)|M_j^{-i}\rangle, \qquad 1 \leq j \leq n-1. \tag{3}$$

Here $M_j^{\pm i}$ is the Gel'fand–Tsetlin tableau obtained from the tableau (1) by replacement of $m_{i,j}$ by $m_{i,j} \pm 1$, and

$$A_j^i(M) = -\frac{\prod\limits_{s=1}^{j+1} [l_{s,j+1} - l_{i,j}]}{\prod\limits_{s \neq i} [l_{s,j} - l_{i,j}]}, \qquad B_j^i(M) = \frac{\prod\limits_{s=1}^{j-1} [l_{s,j-1} - l_{i,j}]}{\prod\limits_{s \neq i} [l_{s,j} - l_{i,j}]},$$

where

$$l_{s,j} := m_{s,j} - s, \tag{4}$$

and the numbers in square brackets denote q-numbers defined by

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

In complete analogy, it is possible to construct a simple finite-dimensional right module $\mathcal{V}^{\mathrm{R}}_{\boldsymbol{m}_n}$ over the algebra $U_q(\mathrm{gl}_n)$ which is dual to the described above. The basis elements of this module are also parameterized by the same set of Gel'fand–Tsetlin tableaux. The basis element denoted by $\langle M|$ which is dual to $|M\rangle$ corresponds to the tableau (1).

The generators of $U_q(\mathrm{gl}_n)$ act on these basis elements by the formulas

$$\langle M|q^{\epsilon_j} = q^{a_j}\langle M|, \qquad a_j = \sum_{i=1}^{j} m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1}, \qquad 1 \le j \le n, \tag{5}$$

$$\langle M|e_j = \sum_{i=1}^{j} A^i_j(M^{-i}_j)\langle M^{-i}_j|, \qquad \langle M|f_j = \sum_{i=1}^{j} B^i_j(M^{+i}_j)\langle M^{+i}_j|, \qquad 1 \le j \le n-1. \tag{6}$$

3 Some invariant elements in $U_q(\mathrm{gl}_n)$-modules

In this section, we give explicit formulas for elements in left and right modules over $U_q(\mathrm{gl}_n)$ which are annihilating by

$$\theta_k = q^{\epsilon_k} f_k - q q^{\epsilon_{k+1}} e_k, \qquad k = 1, 2, \ldots, n-1.$$

In the limit $q \to 1$, the elements θ_k become generators of the enveloping algebra for the Lie algebra so_n embedded into gl_n. In fact, the elements $q^{-\epsilon_k}\theta_k$ generate [2] the non-standard deformation $U'_q(\mathrm{so}_n)$ of the enveloping algebra $U(\mathrm{so}_n)$ introduced by Gavrilik and Klimyk [3].

The following theorem is due to Noumi [2].

Theorem 1. *The element $|v\rangle \in \mathcal{V}^{\mathrm{L}}_{\boldsymbol{m}_n}$ (resp. $\langle v| \in \mathcal{V}^{\mathrm{R}}_{\boldsymbol{m}_n}$) such that $\theta_k|v\rangle = 0$, (resp. $\langle v|\theta_k = 0$), $k = 1, \ldots, n-1$, exists if and only if $(m_{i,n} - m_{i+1,n})$ are even for $i = 1, \ldots, n-1$. If such element $|v\rangle$ (resp. $\langle v|$) exists, it is unique up to a multiplier.*

The elements $|v\rangle$ and $\langle v|$ which are annihilating by θ_k are called invariant elements with respect to action of θ_k. Now, for the case when conditions of theorem for $\boldsymbol{m}_n$ are satisfied, we present explicit formulas for such elements $|v\rangle$ and $\langle v|$. Let $\mathcal{S}$ be the set of all Gel'fand–Tsetlin tableaux corresponding to $\boldsymbol{m}_n$ and satisfying the additional conditions:

$$(m_{i,j+1} - m_{i,j}) \quad \text{is even for all} \quad i = 1, 2, \ldots, j, \quad j = 1, 2, \ldots, n-1.$$

Then

$$|v\rangle = \sum_{M \in \mathcal{S}} \alpha(M)|M\rangle, \qquad \langle v| = \sum_{M \in \mathcal{S}} \beta(M)\langle M|, \tag{7}$$

where

$$\alpha(M) = \prod_{k=2}^{n} \alpha_k, \qquad \beta(M) = \prod_{k=2}^{n} \beta_k, \tag{8}$$

$$\alpha_k = q^{\gamma_k} \prod_{1 \le i < j \le k} \frac{[l_{i,k-1} - l_{j-1,k-1}]!![l_{i,k-1} - l_{j,k} - 2]!!}{[l_{i,k} - l_{j,k} - 2]!![l_{i,k} - l_{j-1,k-1}]!!}, \tag{9}$$

$$\beta_k = q^{-\gamma_k} \prod_{1 \le i < j \le k} \frac{[l_{i,k} - l_{j,k} - 1]!![l_{i,k} - l_{j-1,k-1} - 1]!!}{[l_{i,k-1} - l_{j-1,k-1} - 1]!![l_{i,k-1} - l_{j,k} - 1]!!}, \tag{10}$$

$$\gamma_k = \frac{1}{2} \sum_{1 \le i < j \le k} (l_{i,k} - l_{i,k-1})(l_{j-1,k-1} - l_{j,k} - 2). \tag{11}$$

We used the notations (4) and

$$[s]!! := [s][s-2] \cdots [2] \text{ (or } [1]), \qquad [0]!! = [-1]!! = 1.$$

These expressions for elements $|v\rangle$ and $\langle v|$ were obtained by straightforward calculation using action formulas (2)–(3) and (5)–(6).

4 Macdonald polynomials

Macdonald polynomials constitute a linear basis in the space of symmetric polynomials of $X = \{X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}\}$ with coefficients being rational functions in two formal variables q and t. Each Macdonald polynomial $P_{\boldsymbol{\lambda}}(X; q, t)$ is labeled by the set of n integers $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n.$$

We introduce a partial ordering among such sets. For two such sets $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, we put

$$\boldsymbol{\lambda} \succeq \boldsymbol{\mu} \Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_n = \mu_1 + \mu_2 + \cdots + \mu_n & \text{and} \\ \lambda_1 + \lambda_2 + \cdots + \lambda_r \ge \mu_1 + \mu_2 + \cdots + \mu_r, & r = 1, 2, \ldots, n-1. \end{cases}$$

The set $\boldsymbol{\lambda}$ defines a monomial $X^{\boldsymbol{\lambda}} = X_1^{\lambda_1} \cdots X_n^{\lambda_n}$. The monomial symmetric function $m_{\boldsymbol{\lambda}}(X)$ is the sum of all *distinct* monomials obtainable from $X^{\boldsymbol{\lambda}}$ by permutations of X's. In particular, if $\boldsymbol{\lambda}$ such that $\lambda_i = 1$ for $i \le r$ and $\lambda_j = 0$ for $j > r$, we have $m_{\boldsymbol{\lambda}}(X) = e_r(X)$, the r-th elementary symmetric polynomial.

To present the definition of Macdonald polynomials we need a commuting family of q-difference operators

$$M_r = t^{-r(n-r)/2} \sum_{\substack{I \subset \{1,2,\ldots,n\} \\ |I| = r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tX_i - X_j}{X_i - X_j} \prod_{i \in I} \tau_i, \qquad r = 1, \ldots, n,$$

where τ_i represents the q-shift operator with respect to the variable X_i. Namely, τ_i is the automorphism of algebra of polynomials of $X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}$ uniquely defined by $\tau_i(X_j) = q^{\delta_{ij}} X_j$. Macdonald proved [4] the following

Theorem 2. *There exists a unique linear basis $\{P_{\boldsymbol{\lambda}}(X)\}$ in the space of symmetric polynomials of $X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_n^{\pm 1}$ satisfying the following two conditions:*

- *For each $\boldsymbol{\lambda}$, $P_{\boldsymbol{\lambda}}$ can be presented as*

  $$P_{\boldsymbol{\lambda}}(X) = m_{\boldsymbol{\lambda}}(X) + \sum_{\boldsymbol{\mu} \prec \boldsymbol{\lambda}} u_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\mu}}(X),$$

  where $u_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ are some rational functions of q and t.

- *For each $\boldsymbol{\lambda}$, $P_{\boldsymbol{\lambda}}(X)$ is a joint eigenfunction of M_r, $r = 1, 2, \ldots, n$:*

  $$M_r P_{\boldsymbol{\lambda}}(X) = e_r(q^{\lambda_1} t^{\rho_1}, \ldots, q^{\lambda_n} t^{\rho_n}) P_{\boldsymbol{\lambda}}(X),$$

  where $\rho_i = (n - 2i + 1)/2$.

We give now some explicit formulas for Macdonald polynomials. For the set $\boldsymbol{\lambda}$ labeling $P_{\boldsymbol{\lambda}}$, it corresponds Gel'fand–Tsetlin tableaux (like (1))

$$\Lambda = \begin{pmatrix} \lambda_{1,n} & & \lambda_{2,n} & & \cdots & & \lambda_{n,n} \\ & \lambda_{1,n-1} & & \lambda_{2,n-1} & \cdots & \lambda_{n-1,n-1} & \\ & & \cdots & \cdots & \cdots & & \\ & & & \lambda_{1,1} & & & \end{pmatrix},$$

where $\lambda_{i,n} = \lambda_i$, $i = 1, \dots, n$, and $\lambda_{i,j}$, $i = 1, \dots, j$, $j = 1, \dots, n-1$, are integers satisfying

$$\lambda_{i,j+1} \geq \lambda_{i,j} \geq \lambda_{i+1,j+1}, \qquad i = 1, 2, \dots, j, \quad j = 1, 2, \dots, n-1.$$

We suppose that Λ runs over all the possible Gel'fand–Tsetlin tableaux corresponding to the fixed $\boldsymbol{\lambda}$. Next theorem is a reformulation in terms of Gel'fand–Tsetlin tableaux of combinatorial formula [4] for Macdonald polynomials. We need the notation:

$$(a;q)_s := (1-a)(1-aq)\cdots(1-aq^{s-1}).$$

Theorem 3. *The expression for Macdonald polynomial $P_{\boldsymbol{\lambda}}$ has form*

$$P_{\boldsymbol{\lambda}}(X_1, X_2, \dots, X_n; q, t) = \sum_{\Lambda} \varphi(\Lambda; q, t) X^{\Lambda}, \tag{12}$$

where

$$X^{\Lambda} = \prod_{k=1}^{n} X_k^{a_k}, \qquad a_k = \sum_{i=1}^{k} \lambda_{i,k} - \sum_{i=1}^{k-1} \lambda_{i,k-1},$$

$$\varphi(\Lambda; q, t) = \prod_{k=2}^{n} \varphi_k(\boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k; q, t), \qquad \boldsymbol{\lambda}_k = (\lambda_{1,k}, \lambda_{2,k}, \dots, \lambda_{k,k}),$$

$$\begin{aligned} \varphi_k(\boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k; q, t) = {} & \prod_{1 \leq i < j \leq k} \frac{(t^{j-i};q)_{\lambda_{i,k}-\lambda_{j-1,k-1}}}{(t^{j-i-1}q;q)_{\lambda_{i,k}-\lambda_{j-1,k-1}}} \cdot \frac{(t^{j-i-1}q;q)_{\lambda_{i,k}-\lambda_{j,k}}}{(t^{j-i};q)_{\lambda_{i,k}-\lambda_{j,k}}} \\ & \times \frac{(t^{j-i};q)_{\lambda_{i,k-1}-\lambda_{j,k}}}{(t^{j-i-1}q;q)_{\lambda_{i,k-1}-\lambda_{j,k}}} \cdot \frac{(t^{j-i-1}q;q)_{\lambda_{i,k-1}-\lambda_{j-1,k-1}}}{(t^{j-i};q)_{\lambda_{i,k-1}-\lambda_{j-1,k-1}}} \\ = {} & \prod_{1 \leq i < j \leq k} \frac{(qt^{j-i-1}q^{\lambda_{i,k}-\lambda_{j-1,k-1}};q)_{\lambda_{j-1,k-1}-\lambda_{j,k}}}{(t^{j-i}q^{\lambda_{i,k}-\lambda_{j-1,k-1}};q)_{\lambda_{j-1,k-1}-\lambda_{j,k}}} \\ & \times \frac{(t^{j-i}q^{\lambda_{i,k-1}-\lambda_{j-1,k-1}};q)_{\lambda_{j-1,k-1}-\lambda_{j,k}}}{(qt^{j-i-1}q^{\lambda_{i,k-1}-\lambda_{j-1,k-1}};q)_{\lambda_{j-1,k-1}-\lambda_{j,k}}}. \end{aligned} \tag{13}$$

Proof. From [4], it follows that

$$\varphi(\Lambda; q, t) = \prod_{k=2}^{n} \frac{b_{\boldsymbol{\lambda}_k}(C_k)}{b_{\boldsymbol{\lambda}_k}} \frac{b_{\boldsymbol{\lambda}_{k-1}}}{b_{\boldsymbol{\lambda}_{k-1}}(C_k)} = \frac{1}{b_{\boldsymbol{\lambda}_n}} \prod_{k=1}^{n} \frac{b_{\boldsymbol{\lambda}_k}(C_k)}{b_{\boldsymbol{\lambda}_{k-1}}(C_k)}.$$

Combinatorial definition of $b_{\boldsymbol{\lambda}_k}$ and $b_{\boldsymbol{\lambda}_k}(C_j)$ is given in [4]. This combinatorics rewritten in terms of Gel'fand–Tsetlin tableaux gives

$$\frac{b_{\boldsymbol{\lambda}_k}(C_k)}{b_{\boldsymbol{\lambda}_k}} = \prod_{1 \leq i < j \leq k} \frac{(t^{j-i};q)_{\lambda_{i,k}-\lambda_{j-1,k-1}}}{(t^{j-i-1}q;q)_{\lambda_{i,k}-\lambda_{j-1,k-1}}} \cdot \frac{(t^{j-i-1}q;q)_{\lambda_{i,k}-\lambda_{j,k}}}{(t^{j-i};q)_{\lambda_{i,k}-\lambda_{j,k}}},$$

$$\frac{b_{\boldsymbol{\lambda}_{k-1}}}{b_{\boldsymbol{\lambda}_{k-1}}(C_k)} = \prod_{1 \leq i < j \leq k} \frac{(t^{j-i};q)_{\lambda_{i,k-1}-\lambda_{j,k}}}{(t^{j-i-1}q;q)_{\lambda_{i,k-1}-\lambda_{j,k}}} \cdot \frac{(t^{j-i-1}q;q)_{\lambda_{i,k-1}-\lambda_{j-1,k-1}}}{(t^{j-i};q)_{\lambda_{i,k-1}-\lambda_{j-1,k-1}}}.$$

■

5 Wave functions of Ruijsenaars model

In [1], the wave function for Schrödinger equation for quantum n-particle Ruijsenaars model was defined as

$$\Psi_{\boldsymbol{m}_n}(x_1,x_2,\dots,x_n) := \langle v|q^h|v\rangle = \langle v|q^{\sum_{k=1}^n x_k\epsilon_k}|v\rangle,$$

where the invariant elements $\langle v|$ and $|v\rangle$ are defined in Section 3 (conditions when they exist are also given there). Here we give a direct proof that $\Psi_{\boldsymbol{m}_n}(x_1,x_2,\dots,x_n)$ can be expressed in terms of Macdonald polynomials.

From (2) and (7)–(8), it follows

$$\Psi_{\boldsymbol{m}_n}(x_1,x_2,\dots,x_n) = \langle v|q^{\sum_{k=1}^n x_k\epsilon_k}|v\rangle = \sum_{M\in\mathcal{S}}\prod_{k=2}^n \alpha_k\beta_k\cdot\prod_{k=1}^n q^{x_k(\sum_{i=1}^k m_{i,k}-\sum_{i=1}^{k-1} m_{i,k-1})},$$

where α_k and β_k are given by (9)–(11). We make change of variables $\lambda_{i,j}=(m_{i,j}-m_{n,n})/2$ for all possible i and j. Since $M\in\mathcal{S}$, all the $\lambda_{i,j}$ are non-negative integers. For $\alpha_k\beta_k$ we have

$$\begin{aligned}
\alpha_k\beta_k &= \prod_{1\le i<j\le k}\frac{[l_{i,k}-l_{j,k}-1]!![l_{i,k}-l_{j-1,k-1}-1]!![l_{i,k-1}-l_{j,k}-2]!![l_{i,k-1}-l_{j-1,k-1}]!!}{[l_{i,k}-l_{j-1,k-1}]!![l_{i,k}-l_{j,k}-2]!![l_{i,k-1}-l_{j-1,k-1}-1]!![l_{i,k-1}-l_{j,k}-1]!!}\\
&= \prod_{1\le i<j\le k}\frac{[2\lambda_{i,k}-2\lambda_{j,k}-i+j-1]!!}{[2\lambda_{i,k}-2\lambda_{j-1,k-1}-i+j-1]!!}\cdot\frac{[2\lambda_{i,k}-2\lambda_{j-1,k-1}-i+j-2]!!}{[2\lambda_{i,k}-2\lambda_{j,k}-i+j-2]!!}\\
&\times\frac{[2\lambda_{i,k-1}-2\lambda_{j,k}-i+j-2]!!}{[2\lambda_{i,k-1}-2\lambda_{j-1,k-1}-i+j-2]!!}\cdot\frac{[2\lambda_{i,k-1}-2\lambda_{j-1,k-1}-i+j-1]!!}{[2\lambda_{i,k-1}-2\lambda_{j,k}-i+j-1]!!}\\
&= \prod_{1\le i<j\le k}\frac{[2\lambda_{i,k}-2\lambda_{j,k}-i+j-1][2\lambda_{i,k}-2\lambda_{j,k}-i+j-3]\cdots[2\lambda_{i,k}-2\lambda_{j-1,k-1}-i+j+1]}{[2\lambda_{i,k}-2\lambda_{j,k}-i+j-2][2\lambda_{i,k}-2\lambda_{j,k}-i+j-4]\cdots[2\lambda_{i,k}-2\lambda_{j-1,k-1}-i+j]}\\
&\times\frac{[2\lambda_{i,k-1}-2\lambda_{j,k}-i+j-2][2\lambda_{i,k-1}-2\lambda_{j,k}-i+j-4]\cdots[2\lambda_{i,k-1}-2\lambda_{j-1,k-1}-i+j]}{[2\lambda_{i,k-1}-2\lambda_{j,k}-i+j-1][2\lambda_{i,k-1}-2\lambda_{j,k}-i+j-3]\cdots[2\lambda_{i,k-1}-2\lambda_{j-1,k-1}-i+j+1]}\\
&= \prod_{1\le i<j\le k}\frac{(q^4q^{2(j-i-1)}q^{4(\lambda_{i,k}-\lambda_{j-1,k-1})};q^4)_{\lambda_{j-1,k-1}-\lambda_{j,k}}}{(q^{2(j-i)}q^{4(\lambda_{i,k}-\lambda_{j-1,k-1})};q^4)_{\lambda_{j-1,k-1}-\lambda_{j,k}}}\\
&\times\frac{(q^{2(j-i)}q^{4(\lambda_{i,k-1}-\lambda_{j-1,k-1})};q^4)_{\lambda_{j-1,k-1}-\lambda_{j,k}}}{(q^4q^{2(j-i-1)}q^{4(\lambda_{i,k-1}-\lambda_{j-1,k-1})};q^4)_{\lambda_{j-1,k-1}-\lambda_{j,k}}} = \varphi_k(\boldsymbol{\lambda}_{k-1},\boldsymbol{\lambda}_k;q^4,q^2),
\end{aligned}$$

where the definition (13) for $\varphi_k(\boldsymbol{\lambda}_{k-1},\boldsymbol{\lambda}_k;q,t)$ is used. Thus

$$\begin{aligned}
\Psi_{\boldsymbol{m}_n}(x_1,x_2,\dots,x_n) &= \sum_\Lambda\varphi(\Lambda;q,t)\prod_{k=1}^n q^{x_k(m_{n,n}+2(\sum_{i=1}^k\lambda_{i,k}-\sum_{i=1}^{k-1}\lambda_{i,k-1}))}\\
&= q^{(x_1+\cdots+x_n)m_{n,n}}\sum_\Lambda\varphi(\Lambda;q,t)\prod_{k=1}^n q^{2x_k(\sum_{i=1}^k\lambda_{i,k}-\sum_{i=1}^{k-1}\lambda_{i,k-1})}.
\end{aligned}$$

Comparing with (12) we obtain

$$\Psi_{\boldsymbol{m}_n}(x_1,x_2,\dots,x_n) = q^{(x_1+\cdots+x_n)m_{n,n}}P_{\boldsymbol{\lambda}}(q^{2x_1},q^{2x_2},\dots,q^{2x_n};q^4,q^2).$$

This wave function satisfies equations

$$\hat{H}_r\Psi_{\boldsymbol{m}_n}(x_1,x_2,\dots,x_n) = E_\Lambda^{(r)}\Psi_{\boldsymbol{m}_n}(x_1,x_2,\dots,x_n),\qquad r=1,2,\dots,n, \tag{14}$$

where $\hat{H}_r$ is r-th Hamiltonian of Ruijsenaars model:

$$\hat{H}_r = q^{r(r-1)} \sum_{\substack{I\subset\{1,2,\dots,n\}\\|I|=r}} \left(\prod_{\substack{i\in I\\ j\notin I}} \frac{q^2 q^{2x_i} - q^{2x_j}}{q^{2x_i} - q^{2x_j}} \right) e^{2\sum_{i\in I}\partial_{x_i}},$$

$$E^{(r)}_{\boldsymbol{m}_n} = e_r(q^{2m_{1,n}+2n-2}, q^{2m_{2,n}+2n-4}, \dots, q^{m_{n,n}}).$$

This statement is easily follows from Theorem 2. Note, the equation (14) at $r = 1$ is precisely Schrödinger equation for quantum trigonometric n-particle Ruijsenaars model.

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Classical Dynamical Yang–Baxter Equations and Quasi-Poisson Homogeneous Spaces

Eugene KAROLINSKY †, *Kolya MUZYKIN* ‡ *and Alexander STOLIN* §

† *Dept. of Math., Kharkiv National University, 4 Svobody Sq., 61077 Kharkiv, Ukraine; Institute for Low Temperature Physics & Engineering, 47 Lenin Ave., 61103 Kharkiv, Ukraine*
E-mail: *eugene.a.karolinsky@univer.kharkov.ua*

‡ *Dept. of Math., Kharkiv National University, 4 Svobody Sq., 61077 Kharkiv, Ukraine*

§ *Department of Mathematics, University of Göteborg, SE-412 96 Göteborg, Sweden*
E-mail: *astolin@math.chalmers.se*

In this paper we provide a connection between the solutions of the classical dynamical Yang–Baxter equation (with not necessary Abelian base) and quasi-Poisson homogeneous spaces of quasi-Poisson Lie groups.

1 Introduction

This paper is a continuation of [6]. Let us recall the main result of [6]. Let G be a Lie group, $\mathfrak{g} = \operatorname{Lie} G$, $U \subset G$ a connected closed Lie subgroup such that the corresponding subalgebra $\mathfrak{u} \subset \mathfrak{g}$ is reductive in $\mathfrak{g}$ (i.e., there exists an $\mathfrak{u}$-invariant subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{m}$), and $\Omega \in (\mathfrak{u} \otimes \mathfrak{u}) \oplus (\mathfrak{m} \otimes \mathfrak{m})$ a symmetric tensor. Take a solution $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ of the classical Yang–Baxter equation such that $\rho + \rho^{21} = \Omega$ and consider the corresponding Poisson Lie group structure π_ρ on G. Assuming additionally that

$$\rho + s \in \frac{\Omega}{2} + \left(\bigwedge^2 \mathfrak{m}\right)^{\mathfrak{u}} \tag{1}$$

for some element $s \in \bigwedge^2 \mathfrak{g}$ that satisfies a certain "twist" equation, we establish a 1-1 correspondence between the moduli space of classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ with the symmetric part $\frac{\Omega}{2}$ and the set of all structures of Poisson homogeneous (G, π_ρ)-spaces on G/U. We emphasize that the first example of such a correspondence was found by Lu in [8].

In this paper we generalize the main result of [6]. We replace Poisson Lie groups (resp. Poisson homogeneous spaces) by quasi-Poisson Lie groups (resp. quasi-Poisson homogeneous spaces), but even in the Poisson case our result (see Theorem 2) is stronger than in [6]: condition (1) is relaxed now. We hope that now we present this result in its natural generality.

The paper is organized as follows. In Section 2 we present the definitions of classical dynamical r-matrices, quasi-Poisson Lie groups and their quasi-Poisson homogeneous spaces, and then formulate and prove the main result of this paper, Theorem 2. In Section 3 we consider an example: the case of quasi-triangular (in the strict sense) classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$, where $\mathfrak{g}$ is a complex semisimple Lie algebra, and $\mathfrak{u}$ is its regular reductive subalgebra.

All Lie algebras in this paper assumed to be finite-dimensional, and the ground field is $\mathbb{C}$.

2 General results

In this section we describe a connection between quasi-Poisson homogeneous spaces and classical dynamical r-matrices (see Theorem 2).

First we recall some definitions. Suppose G is a Lie group, $U \subset G$ its connected Lie subgroup. Let $\mathfrak{g}$ and $\mathfrak{u}$ be the corresponding Lie algebras. Choose a basis $x_1, \dots, x_r$ in $\mathfrak{u}$. Denote by D the formal neighborhood of zero in $\mathfrak{u}^*$. By functions from D to a vector space V we mean the elements of the space $V[[x_1, \dots, x_r]]$, where x_i are regarded as coordinates on D. Further, if $\omega \in \Omega^k(D, V)$ is a k-form on D with values in vector space V, then by $\overline{\omega} : D \to \bigwedge^k \mathfrak{u} \otimes V$ we denote the corresponding function.

Definition 1 (see [5]). *Classical dynamical r-matrix for the pair* $(\mathfrak{g}, \mathfrak{u})$ is an $\mathfrak{u}$-equivariant function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ that satisfies the *classical dynamical Yang–Baxter equation (CDYBE)*:

$$\mathrm{Alt}(\overline{dr}) + \mathrm{CYB}(r) = 0,$$

where $\mathrm{CYB}(r) = \big[r^{12}, r^{13}\big] + \big[r^{12}, r^{23}\big] + \big[r^{13}, r^{23}\big]$, and for $x \in \mathfrak{g}^{\otimes 3}$ we set $\mathrm{Alt}(x) = x^{123} + x^{231} + x^{312}$.

We will also require the *quasi-unitarity property*:

$$r + r^{21} = \Omega \in \big(S^2\mathfrak{g}\big)^{\mathfrak{g}}.$$

It is easy to see that if r satisfies the CDYBE and the quasi-unitarity condition, then Ω is constant.

We denote the set of all classical dynamical r-matrices for the pair $(\mathfrak{g}, \mathfrak{u})$ such that $r + r^{21} = \Omega$ by $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$.

Denote by $\mathbf{Map}(D, G)^{\mathfrak{u}}$ the set of all $\mathfrak{u}$-equivariant maps from D to G. Suppose that $r : D \to \mathfrak{g} \otimes \mathfrak{g}$ is an $\mathfrak{u}$-equivariant function. Then for any $g \in \mathbf{Map}(D, G)^{\mathfrak{u}}$ define a function $r^g : D \to \mathfrak{g} \otimes \mathfrak{g}$ by

$$r^g = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)\big(r - \overline{\eta_g} + \overline{\eta_g}^{21} + \tau_g\big),$$

where $\eta_g = g^{-1}dg$, and $\tau_g(\lambda) = (\lambda \otimes 1 \otimes 1)\big(\big[\overline{\eta_g}^{12}, \overline{\eta_g}^{13}\big](\lambda)\big)$. Then r^g is a classical dynamical r-matrix if and only if r is. The transformation $r \mapsto r^g$ is called a *gauge transformation*. In fact, it is an action of the group $\mathbf{Map}(D, G)^{\mathfrak{u}}$ on $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$.

Following [5], we denote the moduli space $\mathbf{Map}_0(D, G)^{\mathfrak{u}} \backslash \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ by $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ (here $\mathbf{Map}_0(D, G)^{\mathfrak{u}} = \{g \in \mathbf{Map}(D, G)^{\mathfrak{u}} : g(0) = e\}$).

Now we recall the definition of quasi-Poisson Lie groups and their quasi-Poisson homogeneous spaces (for details see [7, 1, 2]).

Definition 2. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, π_G a bivector field on G, and $\varphi \in \bigwedge^3 \mathfrak{g}$. A triple (G, π_G, φ) is called a *quasi-Poisson Lie group* if

$$\pi_G(gg') = (l_g)_* \pi_G(g') + (r_{g'})_* \pi_G(g),$$
$$\frac{1}{2}[\pi_G, \pi_G] = \overleftarrow{\varphi} - \overrightarrow{\varphi},$$
$$[\pi_G, \overleftarrow{\varphi}] = 0,$$

where l_g (resp. r_g) is left (resp. right) multiplication by g, $\overrightarrow{a}$ (resp. $\overleftarrow{a}$) is the left (resp. right) invariant tensor field on G corresponding to a and $[\cdot, \cdot]$ is the Schouten bracket of multivector fields.

Definition 3. Suppose that (G, π_G, φ) is a quasi-Poisson group, X is a homogeneous G-space equipped with a bivector field π_X. Then (X, π_X) is called a *quasi-Poisson homogeneous (G, π_G, φ)-space* if

$$\pi_X(gx) = (l_g)_* \pi_X(x) + (\rho_x)_* \pi_G(g),$$
$$\frac{1}{2}[\pi_X, \pi_X] = \varphi_X$$

(here l_g denotes the mapping $x \mapsto g \cdot x$, ρ_x is the mapping $g \mapsto g \cdot x$, and φ_X is the trivector field on X induced by φ).

Now take $\rho \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\rho + \rho^{21} = \Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$. Let $\Lambda = \rho - \frac{\Omega}{2} \in \bigwedge^2 \mathfrak{g}$. Define a bivector field on G by $\pi_\rho = \overrightarrow{\rho} - \overleftarrow{\rho} = \overrightarrow{\Lambda} - \overleftarrow{\Lambda}$. Set $\varphi = -\mathrm{CYB}(\rho)$. Then (G, π_ρ, φ) is a quasi-Poisson Lie group (such quasi-Poisson Lie groups are called *quasi-triangular*). Denote by $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$ the set of all (G, π_ρ, φ)-homogeneous quasi-Poisson structures on G/U. We will see that, under certain conditions, there is a bijection between $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ and $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$.

Assume that $b \in (\mathfrak{g} \otimes \mathfrak{g})^{\mathfrak{u}}$ is such that $b + b^{21} = \Omega$. Let $B = b - \frac{\Omega}{2}$. Define a bivector field on G by $\tilde{\pi}_b^\rho = \overrightarrow{b} - \overleftarrow{\rho} = \overrightarrow{B} - \overleftarrow{\Lambda}$. Then there is a bivector field on G/U defined by $\pi_b^\rho(\underline{g}) = p_*(\tilde{\pi}_b^\rho(g))$ (here $p : G \to G/U$ is the canonical projection, and $\underline{g} = p(g)$). It is well defined, since b is $\mathfrak{u}$-invariant.

Proposition 1. *In this setting $(G/U, \pi_b^\rho)$ is a (G, π_ρ, φ)-quasi-Poisson homogeneous space iff* $\mathrm{CYB}(b) = 0$ *in* $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$.

Proof. First we check the "multiplicativity" of π_b^ρ. For all $g \in G, u \in U$ we have

$$g \cdot \tilde{\pi}_b^\rho(u) + \pi_\rho(g) \cdot u = gu \cdot b - \rho \cdot gu = \tilde{\pi}_b^\rho(gu).$$

Using p_*, we get the required equality $\pi_b^\rho(\underline{g}) = g \cdot \pi_b^\rho(\underline{e}) + p_*\pi_\rho(g)$.

Now we need to prove that $\frac{1}{2}[\pi_b^\rho, \pi_b^\rho] = \varphi_{G/U}$ iff $\mathrm{CYB}(b) = 0$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$. We check it directly:

$$\frac{1}{2}[\tilde{\pi}_b^\rho, \tilde{\pi}_b^\rho] = \frac{1}{2}\left([\overrightarrow{B}, \overrightarrow{B}] + [\overleftarrow{\Lambda}, \overleftarrow{\Lambda}]\right) = -\overrightarrow{\mathrm{CYB}(B)} + \overleftarrow{\mathrm{CYB}(\Lambda)} = -\overrightarrow{\mathrm{CYB}(b)} + \overleftarrow{\varphi}.$$

Consequently, $\frac{1}{2}[\pi_b^\rho, \pi_b^\rho] = p_*(-\overrightarrow{\mathrm{CYB}(b)} + \overleftarrow{\varphi}) = -p_*(\overrightarrow{\mathrm{CYB}(b)}) + \varphi_{G/U}$. So we see that $\frac{1}{2}[\pi_b^\rho, \pi_b^\rho] = \varphi_{G/U}$ iff $\mathrm{CYB}(b) = 0$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$. ■

Suppose $r \in \mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$.

Proposition 2 (see [8]). $\mathrm{CYB}(r(0)) = 0$ *in* $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$.

Corollary 1. $r \mapsto \pi_{r(0)}^\rho$ *is a map from* $\mathbf{Dynr}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *to* $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$.

Proposition 3 (see [6]). *If* $g \in \mathbf{Map}_0(D, G)^{\mathfrak{u}}$, *then* $\pi_{r(0)}^\rho = \pi_{r^g(0)}^\rho$.

Corollary 2. $r \mapsto \pi_{r(0)}^\rho$ *defines a map from* $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ *to* $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$.

From now on we will assume that the following conditions are satisfied:

$$\mathfrak{u} \text{ has an } \mathfrak{u}\text{-invariant complement } \mathfrak{m} \text{ in } \mathfrak{g}; \tag{2a}$$

$$\Omega \in (\mathfrak{u} \otimes \mathfrak{u}) \oplus (\mathfrak{m} \otimes \mathfrak{m}). \tag{2b}$$

Consider the algebraic variety

$$\mathcal{M}_\Omega = \left\{ x \in \frac{\Omega}{2} + \left(\bigwedge^2 \mathfrak{m}\right)^{\mathfrak{u}} \,\middle|\, \mathrm{CYB}(x) = 0 \text{ in } \bigwedge^3(\mathfrak{g}/\mathfrak{u}) \right\}.$$

Theorem 1 (Etingof, Schiffman; see [5]). (1) *Any class $\mathcal{C} \in \mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega)$ has a representative $r \in \mathcal{C}$ such that $r(0) \in \mathcal{M}_\Omega$. Moreover, this defines an embedding $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \to \mathcal{M}_\Omega$.*

(2) *Assume that* (2b) *holds. Then the map $\mathcal{M}(\mathfrak{g}, \mathfrak{u}, \Omega) \to \mathcal{M}_\Omega$ defined above is a bijection.*

Proposition 4. *The mapping $b \mapsto \pi_b^\rho$ from $\mathcal{M}_\Omega$ to* $\mathbf{Homsp}(G, \pi_\rho, \varphi, U)$ *is a bijection.*

Proof. Let us construct the inverse mapping. Assume that π is a bivector field on G/U defining a structure of a (G,π_ρ,φ)-quasi-Poisson homogeneous space. Then $\pi(\underline{e})\in\bigwedge^2(\mathfrak{g}/\mathfrak{u})=\bigwedge^2\mathfrak{m}$. Consider $b=\frac{\Omega}{2}+\pi(\underline{e})+p_*(\Lambda)$. We will prove that $b\in\mathcal{M}_\Omega$ and the mapping $\pi\mapsto b$ is inverse to the mapping $g\mapsto\pi_b^\rho$.

First we prove that $b\in\left(\bigwedge^2\mathfrak{m}\right)^{\mathfrak{u}}+\frac{\Omega}{2}$. For all $u\in U$ we have $\pi(\underline{e})+p_*(\Lambda)=\pi(u\cdot\underline{e})+p_*(\Lambda\cdot u)=u\cdot\pi(\underline{e})+p_*(\pi_\rho(u))+p_*(\Lambda\cdot u)=u\cdot\pi(\underline{e})+p_*(u\cdot\rho-u\cdot\frac{\Omega}{2})=u\cdot(\pi(\underline{e})+p_*(\Lambda))$. It means that $\pi(\underline{e})+p_*(\Lambda)\in\left(\bigwedge^2\mathfrak{m}\right)^{\mathfrak{u}}$.

Now we prove that $\pi=\pi_b^\rho$. By definition, $\pi_b^\rho(\underline{g})=p_*(g\cdot\pi(\underline{e})+g\cdot p_*\Lambda-\Lambda\cdot g)=\pi(\underline{g})+p_*(g\cdot p_*\Lambda-\Lambda\cdot g-g\cdot\Lambda+\Lambda\cdot g)=\pi(\underline{g})$. So π_b^ρ defines a structure of (G,π_ρ,φ)-quasi-Poisson homogeneous space. By Proposition 1, it means that $b\in\mathcal{M}_\Omega$. ■

Theorem 2. *Suppose* (2a) *and* (2b) *are satisfied. Then the map* $r\mapsto\pi^\rho_{r(0)}$ *from* $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ *to* $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$ *is a bijection.*

Proof. This theorem follows from Theorem 1 and Proposition 4. ■

Remark 1. If $\varphi=-\mathrm{CYB}(\rho)=0$, then (G,π_ρ) is a Poisson Lie group. In this case we get a bijection between $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ and the set of all Poisson (G,π_ρ)-homogeneous structures on G/U.

Remark 2. Assume that only (2a) holds. Clearly, in this case the map $r\mapsto\pi^\rho_{r(0)}$ defines an embedding $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)\hookrightarrow\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$.

Remark 3. If (2a) fails, then the space $\mathcal{M}(\mathfrak{g},\mathfrak{u},\Omega)$ may be infinite-dimensional (see [9]), while $\mathbf{Homsp}(G,\pi_\rho,\varphi,U)$ is always finite-dimensional.

3 Example: the semisimple case

Assume that $\mathfrak{g}$ is a semisimple Lie algebra. Choose a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ and denote by $\mathbf{R}$ the corresponding root system. Suppose $\langle\cdot,\cdot\rangle$ is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$, and $\Omega\in(S^2\mathfrak{g})^{\mathfrak{g}}$ is the corresponding tensor. We will describe $\mathcal{M}_\Omega$ for a reductive Lie subalgebra $\mathfrak{u}\subset\mathfrak{g}$ containing $\mathfrak{h}$.

Precisely, consider a set $\mathbf{U}\subset\mathbf{R}$ such that $\mathfrak{u}=\mathfrak{h}\oplus\sum\limits_{\alpha\in\mathbf{U}}\mathfrak{g}_\alpha$ is a reductive Lie subalgebra. In this case we will call $\mathbf{U}$ *reductive* (in other words, a set $\mathbf{U}\subset\mathbf{R}$ is reductive iff $(\mathbf{U}+\mathbf{U})\cap\mathbf{R}\subset\mathbf{U}$ and $-\mathbf{U}=\mathbf{U}$). Note that in this situation condition (2a) is satisfied, since $\mathfrak{m}=\sum\limits_{\alpha\in\mathbf{R}\setminus\mathbf{U}}\mathfrak{g}_\alpha$ is an $\mathfrak{u}$-invariant complement to $\mathfrak{u}$ in $\mathfrak{g}$.

Fix $E_\alpha\in\mathfrak{g}_\alpha$ such that $\langle E_\alpha,E_{-\alpha}\rangle=1$ for all $\alpha\in\mathbf{R}$. Then $\Omega=\Omega_{\mathfrak{h}}+\sum\limits_{\alpha\in\mathbf{R}}E_\alpha\otimes E_{-\alpha}$, where $\Omega_{\mathfrak{h}}\in S^2\mathfrak{h}$. Notice that (2b) is also satisfied.

Proposition 5. *Suppose that* $x=\sum\limits_{\alpha\in\mathbf{R}}x_\alpha E_\alpha\otimes E_{-\alpha}$. *Then* $x+\frac{\Omega}{2}\in\mathcal{M}_\Omega$ *iff*

$$x_\alpha=0 \text{ for } \alpha\in\mathbf{U}; \tag{3a}$$

$$x_{-\alpha}=-x_\alpha \text{ for } \alpha\in\mathbf{R}; \tag{3b}$$

$$\text{if } \alpha,\beta\in\mathbf{R}\setminus\mathbf{U},\gamma\in\mathbf{U},\alpha+\beta+\gamma=0, \text{then } x_\alpha+x_\beta=0; \tag{3c}$$

$$\text{if } \alpha,\beta,\gamma\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0, \text{then } x_\alpha x_\beta+x_\beta x_\gamma+x_\gamma x_\alpha=-1/4. \tag{3d}$$

Note that (3c) is equivalent to the following condition:

$$\text{if } \alpha\in\mathbf{R}\setminus\mathbf{U},\beta\in\mathbf{U}, \text{then } x_{\alpha+\beta}=x_\alpha.$$

Proof. It is easy to see that $x \in \left(\bigwedge^2 \mathfrak{m}\right)^{\mathfrak{h}}$ iff (3a) and (3b) are satisfied.

Suppose that $c_{\alpha\beta}$ are defined by $[E_\alpha, E_\beta] = c_{\alpha\beta}E_{\alpha+\beta}$.

For any $\gamma \in \mathbf{U}$ we have

$$\begin{aligned}
[E_\gamma, x] &= \sum_{\alpha\in\mathbf{R}\setminus\mathbf{U}} x_\alpha([E_\gamma, E_\alpha]\otimes E_{-\alpha} + E_\alpha \otimes [E_\gamma, E_{-\alpha}]) = \\
&= \sum_{\alpha,\beta\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} (x_\alpha c_{\gamma\alpha}E_{-\beta}\otimes E_{-\alpha} - x_\alpha c_{\gamma\alpha}E_{-\alpha}\otimes E_{-\beta}) \\
&= \sum_{\alpha,\beta\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} (x_\beta c_{\gamma\alpha} - x_\alpha c_{\gamma\beta})\, E_{-\alpha}\otimes E_{-\beta} \\
&= \sum_{\alpha,\beta\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} (x_\alpha + x_\beta)c_{\gamma\alpha}E_{-\alpha}\otimes E_{-\beta}.
\end{aligned}$$

Thus x is $\mathfrak{u}$-invariant if and only if $x_\alpha + x_\beta = 0$ for all $\alpha, \beta \in \mathbf{R}\setminus\mathbf{U}$ such that $\alpha+\beta \in \mathbf{U}$.

Finally, we calculate $\mathrm{CYB}\left(x + \frac{\Omega}{2}\right) = \mathrm{CYB}(x) + \mathrm{CYB}\left(\frac{\Omega}{2}\right)$ (see [1]):

$$\begin{aligned}
\mathrm{CYB}(x) &= \sum_{\alpha,\beta\in\mathbf{R}} x_\alpha x_\beta\big([E_\alpha, E_\beta]\otimes E_{-\alpha}\otimes E_{-\beta} + E_\alpha\otimes[E_{-\alpha}, E_\beta]\otimes E_{-\beta} \\
&\quad + E_\alpha\otimes E_\beta\otimes[E_{-\alpha}, E_{-\beta}]\big) \\
&= \sum_{\alpha,\beta,\gamma\in\mathbf{R},\alpha+\beta+\gamma=0} \big(x_\alpha x_\beta c_{\alpha\beta}E_{-\gamma}\otimes E_{-\alpha}\otimes E_{-\beta} \\
&\quad - x_\alpha x_\beta c_{\alpha\beta}E_{-\alpha}\otimes E_{-\gamma}\otimes E_{-\beta} + x_\alpha x_\beta c_{\alpha\beta}E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma}\big) \\
&= \sum_{\alpha,\beta,\gamma\in\mathbf{R},\alpha+\beta+\gamma=0} c_{\alpha\beta}(x_\alpha x_\beta + x_\alpha x_\gamma + x_\beta x_\gamma)E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma}, \\
\mathrm{CYB}\left(\frac{\Omega}{2}\right) &\equiv \frac{1}{4}\sum_{\alpha,\beta,\gamma\in\mathbf{R}\setminus\mathbf{U},\alpha+\beta+\gamma=0} c_{\alpha\beta}E_{-\alpha}\otimes E_{-\beta}\otimes E_{-\gamma} \\
&\quad (\mathrm{mod}\ \mathfrak{u}\otimes\mathfrak{g}\otimes\mathfrak{g} + \mathfrak{g}\otimes\mathfrak{u}\otimes\mathfrak{g} + \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{u}).
\end{aligned}$$

So the image of $\mathrm{CYB}\left(x + \frac{\Omega}{2}\right)$ in $\bigwedge^3(\mathfrak{g}/\mathfrak{u})$ vanishes if and only if the condition (3d) is satisfied. ■

Proposition 6. *Suppose $\Pi \subset \mathbf{R}$ is a set of simple roots, $\mathbf{R}_+$ is the corresponding set of positive roots. Choose a subset $\Delta \subset \Pi$ such that $\mathbf{N} = (\mathrm{span}\Delta) \cap \mathbf{R}$ contains $\mathbf{U}$. Find $h \in \mathfrak{h}$ such that $\alpha(h) \notin \pi i\mathbb{Z}$ for $\alpha \in \mathbf{N}\setminus\mathbf{U}$ and $\alpha(h) \in \pi i\mathbb{Z}$ for $\alpha \in \mathbf{U}$. Then x_α defined by*

$$x_\alpha = \begin{cases} 0, & \alpha \in \mathbf{U}, \\ \frac{1}{2}\coth\alpha(h), & \alpha \in \mathbf{N}\setminus\mathbf{U}, \\ \pm 1/2, & \alpha \in \pm\mathbf{R}_+\setminus\mathbf{N} \end{cases}$$

satisfies (3a)–(3d). *Moreover, any function satisfying* (3a)–(3d) *is of this form.*

First, we prove the second part of the proposition. Set

$$\mathbf{P} = \{\alpha \,|\, x_\alpha \neq -1/2\}.$$

It is obvious that $\mathbf{U} \subset \mathbf{P}$.

Lemma 1. $\mathbf{P}$ *is parabolic.*

Proof. Obviously, $\mathbf{P} \cup (-\mathbf{P}) = \mathbf{R}$.

We have to prove that if $\alpha, \beta \in \mathbf{P}$ and $\alpha + \beta \in \mathbf{R}$, then $\alpha + \beta \in \mathbf{P}$. We do it by considering several cases. If $\alpha, \beta \in \mathbf{U}$, then $\alpha+\beta \in \mathbf{U} \subset \mathbf{P}$. If $\alpha \in \mathbf{P}\backslash\mathbf{U}$ and $\beta \in \mathbf{U}$, then $x_{\alpha+\beta} = x_\alpha \neq -1/2$ by (3c) and $\alpha + \beta \in \mathbf{P}$. If $\alpha, \beta \in \mathbf{P}\backslash\mathbf{U}$, there are two possibilities. If $\alpha + \beta \in \mathbf{U}$, then there is nothing to prove. If $\alpha + \beta \notin \mathbf{U}$, then, by (3d), $x_\alpha x_\beta - x_{\alpha+\beta}(x_\alpha + x_\beta) = -1/4$. If $x_{\alpha+\beta} = -1/2$, then from this equation it follows that $x_\alpha = -1/2$. Consequently, $\alpha + \beta \in \mathbf{P}$. ■

Since $\mathbf{P}$ is parabolic, there exists a set of positive roots $\Pi \subset \mathbf{R}$ and a subset $\Delta \subset \Pi$ such that $\mathbf{P} = \mathbf{R}_+ \cup \mathbf{N}$ (see [4], chapter VI, § 1, proposition 20); here $\mathbf{R}_+$ is the set of positive roots corresponding to Π, and $\mathbf{N} = (\mathrm{span}\Delta) \cap \mathbf{R}$ is the Levi subset corresponding to Δ.

Let $\mathbf{N}_+ = \mathbf{N} \cap \mathbf{R}_+$ be the set of positive roots in $\mathbf{N}$ corresponding to Δ. For all $\alpha \in \Delta\backslash\mathbf{U}$ let $y_\alpha = \mathrm{arccoth}\, 2x_\alpha$, for $\alpha \in \Delta \cap \mathbf{U}$ let $y_\alpha = 0$. Find $h \in \mathfrak{h}$ such that $y_\alpha = \alpha(h)$. Now we prove that h satisfies Proposition 6.

Lemma 2. *$\alpha(h) \notin \pi i\mathbb{Z}$ and $x_\alpha = \frac{1}{2}\coth\alpha(h)$ for all $\alpha \in \mathbf{N}\backslash\mathbf{U}$; $\alpha(h) \in \pi i\mathbb{Z}$ for $\alpha \in \mathbf{U}$.*

Proof. It is enough to prove this for α positive, so that we can use the induction on the length $l(\alpha)$. The case $l(\alpha) = 1$ is trivial. Suppose that $l(\alpha) = k$. Then we can find $\alpha' \in \mathbf{N}_+$ and $\alpha_k \in \Delta$ such that $l(\alpha') = k - 1$ and $\alpha = \alpha' + \alpha_k$. Consider two cases.

First, suppose that $\alpha \in \mathbf{U}$.

If $\alpha_k \in \mathbf{U}$, then $\alpha' \in \mathbf{U}$. By induction, $\alpha(h) = \alpha'(h) \in \pi i\mathbb{Z}$.

If $\alpha_k \notin \mathbf{U}$, then $\alpha' \notin \mathbf{U}$. By induction assumption, $x_{\alpha'} = \frac{1}{2}\coth\alpha'(h)$. From (3c) it follows that $0 = x_{\alpha'} + x_{\alpha_k} = \frac{1}{2}(\coth\alpha'(h) + \coth\alpha_k(h))$ and, consequently, $\alpha(h) \in \pi i\mathbb{Z}$.

Now suppose that $\alpha \notin \mathbf{U}$.

If $\alpha_k \in \mathbf{U}$, then $\alpha' \notin \mathbf{U}$. Since $\alpha_k(h) = 0$, by (3c) we have $x_\alpha = x_{\alpha'+\alpha_k} = x_{\alpha'} = \frac{1}{2}\coth\alpha'(h) = \frac{1}{2}\coth\alpha(h)$.

When $\alpha_k \notin \mathbf{U}$, then there are two possibilities again. If $\alpha' \in \mathbf{U}$, then by induction $\alpha'(h) \in \pi i\mathbb{Z}$. By (3c), $0 = x_\alpha + x_{-\alpha_k}$. Consequently, $x_\alpha = x_{\alpha_k} = \frac{1}{2}\coth\alpha_k(h) = \frac{1}{2}\coth\alpha(h)$. If $\alpha' \notin \mathbf{U}$, then, by (3d), $x_\alpha x_{-\alpha'} + x_{-\alpha'}x_{-\alpha_k} + x_{-\alpha_k}x_\alpha = -1/4$. This equation can be rewritten as

$$x_\alpha = \frac{1/4 + x_{\alpha'}x_{\alpha_k}}{x_{\alpha'} + x_{\alpha_k}} = \frac{1}{2}\,\frac{1 + \coth\alpha'(h)\coth\alpha_k(h)}{\coth\alpha'(h) + \coth\alpha_k(h)} = \frac{1}{2}\coth\alpha(h),$$

and the lemma is proved. ■

To prove the first part of the proposition we need the following root theory lemma.

Lemma 3. *Suppose $\mathbf{P} \subset \mathbf{R}$ is parabolic. Then $\mathbf{Y} = \mathbf{R}\backslash\mathbf{P}$ has the following properties:*

$$(-\mathbf{Y}) \cap \mathbf{Y} = \varnothing; \tag{4a}$$

$$(\mathbf{Y} + \mathbf{Y}) \cap \mathbf{R} \subset \mathbf{Y}; \tag{4b}$$

if $\alpha \in \mathbf{Y}, \beta \in \mathbf{R}\backslash\mathbf{Y}$ and $\alpha - \beta \in \mathbf{R}$, then $\alpha - \beta \in \mathbf{Y}$. (4c)

Proof. Since (4a) is obvious and (4b) follows from (4a) and (4c), we prove only the latter property: if $\alpha \in \mathbf{Y}$ and $\beta \in \mathbf{P}$ are such that $\alpha - \beta \in \mathbf{P}$, then, since $\mathbf{P}$ is parabolic, we would have $\alpha = (\alpha - \beta) + \beta \in \mathbf{P}$. So $\alpha - \beta \in \mathbf{Y}$. ■

Now we just check (3a)–(3d) directly. Suppose that $\mathbf{N}$ is defined as in the proposition. Let $\mathbf{Y} = \mathbf{R}_+\backslash\mathbf{N}$. Then $\mathbf{P} = \mathbf{R}\backslash\mathbf{Y} = -\mathbf{R}_+ \cup \mathbf{N}$ is a parabolic set, and $\mathbf{Y}$ satisfies (4a)–(4c).

Lemma 4. *Suppose x_α is as defined in Proposition 6. Then x_α satisfies* (3a)–(3d).

Proof. (3a) we have already, (3b) is trivial.

To prove (3c), consider the following cases. First, take $\alpha, \beta \in \mathbf{N}\backslash\mathbf{U}$, $\gamma \in \mathbf{U}$, $\alpha+\beta+\gamma=0$. Then $x_\alpha + x_\beta = \frac{1}{2}(\coth\alpha(h) + \coth\beta(h)) = 0$ as $\alpha+\beta = -\gamma \in \mathbf{U}$. The case $\alpha \in \mathbf{N}\backslash\mathbf{U}$, $\beta \in \mathbf{R}\backslash\mathbf{N}$, $\gamma \in \mathbf{U}$ is impossible, because then we would have $\beta = -\alpha - \gamma \in \mathbf{N}$. The case $\alpha, \beta \in \pm\mathbf{Y}$, $\gamma \in \mathbf{U}$ is also impossible, because $-\gamma = \alpha + \beta \in \pm\mathbf{Y}$. Finally, if $\alpha \in \pm\mathbf{Y}$, $\beta \in \mp\mathbf{Y}$, $\gamma \in \mathbf{U}$, then $x_\alpha + x_\beta = \pm\frac{1}{2} \mp \frac{1}{2} = 0$.

Condition (3d) can be proved in a similar way. ■

Now to summarize:

Theorem 3. *Suppose $U \subset G$ is the connected Lie subgroup corresponding to $\mathfrak{u} \subset \mathfrak{g}$. Take $\rho \in \mathfrak{g}\otimes\mathfrak{g}$ such that $\rho + \rho^{21} = \Omega$ and set $\varphi = -\mathrm{CYB}(\rho)$. Then any (G, π_ρ, φ)-homogeneous quasi-Poisson space structure on G/U is exactly of the form $\pi = \pi^\rho_{x+\Omega/2}$ for some $x = \sum\limits_{\alpha\in R} x_\alpha E_\alpha \otimes E_{-\alpha}$, where x_α is defined in Proposition 6.*

Remark 4. Let ρ be any solution of the classical Yang–Baxter equation such that $\rho + \rho^{21} = \Omega$ (see [3]). Then (G, π_ρ) is a Poisson Lie group and therefore Theorem 3 provides the list of all (G, π_ρ)-homogeneous Poisson space structures on G/U.

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p-Mechanics and De Donder–Weyl Theory

Vladimir V. KISIL

School of Mathematics, University of Leeds, Leeds LS2 9JT, UK
On leave from the Odesa University (Ukraine)
E-mail: *kisilv@maths.leeds.ac.uk*
URL: http://amsta.leeds.ac.uk/~kisilv/

The orbit method of Kirillov is used to derive the p-mechanical brackets [26]. They generate the quantum (Moyal) and classic (Poisson) brackets on respective orbits corresponding to representations of the Heisenberg group. The extension of p-mechanics to field theory is made through the De Donder–Weyl Hamiltonian formulation. The principal step is the substitution of the Heisenberg group with Galilean.

1 Introduction

The purpose of this paper is to extended the p-mechanical approach [23,26,25,4,5] from particle mechanics to the field theory. We start in Section 2.1 from the Heisenberg group and its representations derived through the orbit method of Kirillov. In Section 2.2 we define p-mechanical observables as convolutions on the Heisenberg group $\mathbb{H}^n$ and study their commutators. We modify the commutator of two p-observables by the antiderivative to the central vector field in the Heisenberg Lie algebra in Section 2.3, this produces p-mechanical brackets and corresponding dynamic equation. Then the p-mechanical construction is extended to the De Donder–Weyl Hamiltonian formulation of the field theory [12–20] in Section 3.1. To this end we replace the Heisenberg group by the Galilean group in Section 3.2. Expanded presentation of Section 2 could be found in [25]. Development of material from Section 3 will follow in subsequent papers.

2 Elements of *p*-mechanics

2.1 The Heisenberg group and its representations

Let (s,x,y), where x, $y\in\mathbb{R}^n$ and $s\in\mathbb{R}$, be an element of the Heisenberg group $\mathbb{H}^n$ [9,11]. The group law on $\mathbb{H}^n$ is given as follows:

$$(s,x,y)*(s',x',y')=\left(s+s'+\frac{1}{2}\omega(x,y;x',y'),x+x',y+y'\right), \tag{1}$$

where the non-commutativity is made by ω – the *symplectic form* [1, § 37] on $\mathbb{R}^{2n}$:

$$\omega(x,y;x',y')=xy'-x'y. \tag{2}$$

The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by left-invariant vector fields

$$S=\partial_s,\qquad X_j=\partial_{x_j}-y_j/2\partial s,\qquad Y_j=\partial_{y_j}+x_j/2\partial s \tag{3}$$

on $\mathbb{H}^n$ with the Heisenberg *commutator relations*

$$[X_i,Y_j]=\delta_{i,j}S \tag{4}$$

and all other commutators vanishing. The exponential map $\exp : \mathfrak{h}^n \to \mathbb{H}^n$ respecting the multiplication (1) and Heisenberg commutators is

$$\exp : sS + \sum_{k=1}^{n}(x_k X_k + y_k Y_k) \mapsto (s, x_1, \ldots, x_n, y_1, \ldots, y_n).$$

As any group $\mathbb{H}^n$ acts on itself by the conjugation automorphisms $\mathrm{A}\,(g)h = g^{-1}hg$, which fix the unit $e \in \mathbb{H}^n$. The differential $\mathrm{Ad} : \mathfrak{h}^n \to \mathfrak{h}^n$ of A at e is a linear map which could be differentiated again to the representation ad of the Lie algebra $\mathfrak{h}^n$ by the commutator: $\mathrm{ad}\,(A) : B \mapsto [B, A]$. The dual space $\mathfrak{h}_n^*$ to the Lie algebra $\mathfrak{h}^n$ is realised by the left invariant first order differential forms on $\mathbb{H}^n$. By the duality between $\mathfrak{h}^n$ and $\mathfrak{h}_n^*$ the map Ad generates the *co-adjoint representation* [22, § 15.1] $\mathrm{Ad}^* : \mathfrak{h}_n^* \to \mathfrak{h}_n^*$:

$$\mathrm{ad}^*(s, x, y) : (h, q, p) \mapsto (h, q + hy, p - hx), \qquad \text{where} \quad (s, x, y) \in \mathbb{H}^n \tag{5}$$

and $(h, q, p) \in \mathfrak{h}_n^*$ in bi-orthonormal coordinates to the exponential ones on $\mathfrak{h}^n$. There are two types of orbits in (5) for Ad^* – Euclidean spaces $\mathbb{R}^{2n}$ and single points:

$$\mathcal{O}_h = \{(h, q, p) : \text{ for a fixed } h \neq 0 \text{ and all } (q, p) \in \mathbb{R}^{2n}\}, \tag{6}$$

$$\mathcal{O}_{(q,p)} = \{(0, q, p) : \text{ for a fixed } (q, p) \in \mathbb{R}^{2n}\}. \tag{7}$$

The *orbit method* of Kirillov [22, § 15] starts from the observation that the above orbits parametrise irreducible unitary representations of $\mathbb{H}^n$. All representations are *induced* [22, § 13] by a character $\chi_h(s, 0, 0) = e^{2\pi \mathrm{i} hs}$ of the centre of $\mathbb{H}^n$ generated by $(h, 0, 0) \in \mathfrak{h}_n^*$ and shifts (5) from the *left* on orbits. Using [22, § 13.2, Prob. 5] we get a neat formula, which (unlike many other in literature) respects all *physical units* [25]:

$$\rho_h(s, x, y) : f_h(q, p) \mapsto e^{-2\pi \mathrm{i}(hs + qx + py)} f_h\left(q - \frac{h}{2}y, p + \frac{h}{2}x\right). \tag{8}$$

The derived representation $d\rho_h$ of the Lie algebra $\mathfrak{h}^n$ defined on the vector fields (3) is:

$$d\rho_h(S) = -2\pi \mathrm{i} h I, \qquad d\rho_h(X_j) = h\partial_{p_j} + \frac{\mathrm{i}}{2} q_j I, \qquad d\rho_h(Y_j) = -h\partial_{q_j} + \frac{\mathrm{i}}{2} p_j I. \tag{9}$$

Operators D_h^j, $1 \leq j \leq n$ representing vectors from the complexification of $\mathfrak{h}^n$:

$$D_h^j = d\rho_h(-X_j + \mathrm{i}Y_j) = \frac{h}{2}(\partial_{p_j} + \mathrm{i}\partial_{q_j}) + 2\pi(p_j + \mathrm{i}q_j)I = h\partial_{\bar{z}_j} + 2\pi z_j I, \tag{10}$$

where $z_j = p_j + \mathrm{i}q_j$ are used to give the following classic result in terms of orbits:

Theorem 1 (Stone–von Neumann, cf. [22, § 18.4], [9, Chap. 1, § 5]). *All unitary irreducible representations of $\mathbb{H}^n$ are parametrised up to equivalence by two classes of orbits* (6) *and* (7) *of co-adjoint representation* (5) *in $\mathfrak{h}_n^*$:*

1. *The infinite dimensional representations by transformation ρ_h* (8) *for $h \neq 0$ in Fock* [9,11] *space $F_2(\mathcal{O}_h) \subset L_2(\mathcal{O}_h)$ of null solutions to the operators D_h^j* (10)*:*

$$F_2(\mathcal{O}_h) = \{f_h(p, q) \in L_2(\mathcal{O}_h) \mid D_h^j f_h = 0,\ 1 \leq j \leq n\}. \tag{11}$$

2. *The one-dimensional representations as multiplication by a constant on $\mathbb{C} = L_2(\mathcal{O}_{(q,p)})$ which drops out from* (8) *for $h = 0$:*

$$\rho_{(q,p)}(s, x, y) : c \mapsto e^{-2\pi \mathrm{i}(qx + py)} c. \tag{12}$$

Note that $f_h(p,q)$ is in $F_2(\mathcal{O}_h)$ if and only if the function $f_h(z)e^{-|z|^2/h}$, $z=p+\mathrm{i}q$ is in the classical Segal–Bargmann space [9, 11], particularly is analytical in z. Furthermore the space $F_2(\mathcal{O}_h)$ is spanned by the Gaussian *vacuum vector* $e^{-2\pi(q^2+p^2)/h}$ and all *coherent states*, which are "shifts" of the vacuum vector by operators (8).

Commutative representations (12) correspond to the case $h=0$ in the formula (8). They are always neglected, however their union naturally (see the appearance of Poisson brackets in (21)) act as the classic *phase space*:

$$\mathcal{O}_0 = \bigcup_{(q,p)\in\mathbb{R}^{2n}} \mathcal{O}_{(q,p)}. \tag{13}$$

Furthermore the structure of orbits of $\mathfrak{h}_n^*$ echoes in equation (22) and its symplectic invariance [25].

2.2 Convolution algebra of $\mathbb{H}^n$ and commutator

Using a left invariant measure dg on $\mathbb{H}^n$ the linear space $L_1(\mathbb{H}^n, dg)$ can be upgraded to an algebra with the convolution multiplication:

$$(k_1 * k_2)(g) = \int_{\mathbb{H}^n} k_1(g_1)\, k_2(g_1^{-1}g)\, dg_1 = \int_{\mathbb{H}^n} k_1(gg_1^{-1})\, k_2(g_1)\, dg_1. \tag{14}$$

Inner *derivations* D_k, $k\in L_1(\mathbb{H}^n)$ of $L_1(\mathbb{H}^n)$ are given by the *commutator* for $f\in L_1(\mathbb{H}^n)$:

$$D_k : f \mapsto [k,f] = k*f - f*k = \int_{\mathbb{H}^n} k(g_1)\left(f\left(g_1^{-1}g\right) - f\left(gg_1^{-1}\right)\right) dg_1. \tag{15}$$

A unitary representation ρ_h of $\mathbb{H}^n$ extends to $L_1(\mathbb{H}^n, dg)$ by the formula:

$$\begin{aligned}[\rho_h(k)f](q,p) &= \int_{\mathbb{H}^n} k(g)\rho_h(g)f(q,p)\, dg \\ &= \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}} k(s,x,y)e^{-2\pi\mathrm{i}hs}\, ds\right) e^{-2\pi\mathrm{i}(qx+py)} f(q-hy, p+hx)\, dx\, dy,\end{aligned} \tag{16}$$

thus $\rho_h(k)$ for a fixed $h\neq 0$ depends only from $\hat{k}_s(h,x,y)$ – the partial Fourier transform $s\to h$ of $k(s,x,y)$. Then the representation of the composition of two convolutions depends only from

$$(k'*k)\hat{}_s(h,x,y) = \int_{\mathbb{R}^{2n}} e^{\pi\mathrm{i}h(xy'-yx')}\, \hat{k}'_s(h,x',y')\, \hat{k}_s(h,x-x',y-y')\, dx'dy'.$$

The last expression for the full Fourier transforms of k' and k turn to be the *star product* known in deformation quantisation, cf. [30, (9)–(13)]. Consequently the representation of commutator (15) depends only from [25]:

$$[k',k]\hat{}_s = 2\mathrm{i}\int_{\mathbb{R}^{2n}} \sin(\pi h(xy'-yx'))\, \hat{k}'_s(h,x',y')\hat{k}_s(h,x-x',y-y')\, dx'dy', \tag{17}$$

which turn to be exactly the "Moyal brackets" [30] for the full Fourier transforms of k' and k. Also the expression (17) vanishes for $h=0$ as can be expected from the commutativity of representations (12).

2.3 p-mechanical brackets on $\mathbb{H}^n$

A multiple $\mathcal{A}$ of a right inverse operator to the vector field S (3) on $\mathbb{H}^n$ is defined by:

$$S\mathcal{A} = 4\pi^2 I, \qquad \text{where} \quad \mathcal{A}e^{2\pi \mathrm{i} hs} = \begin{cases} \frac{2\pi}{\mathrm{i}h} e^{2\pi \mathrm{i} hs}, & \text{if } h \neq 0, \\ 4\pi^2 s, & \text{if } h = 0. \end{cases} \tag{18}$$

It can be extended by the linearity to $L_1(\mathbb{H}^n)$. We introduce [26] a modified convolution operation $\star$ on $L_1(\mathbb{H}^n)$ and the associated modified commutator:

$$k_1 \star k_2 = k_1 * (\mathcal{A}k_2), \qquad \{\![k_1, k_2]\!\} = k_1 \star k_2 - k_2 \star k_1. \tag{19}$$

Then from (16) one gets $\rho_h(\mathcal{A}k) = (ih)^{-1}\rho_h(k)$ for $h \neq 0$. Consequently the modification of (17) for $h \neq 0$ is only slightly different from the original one:

$$\big\{\!\big[k', k\big]\!\big\}\hat{}_s = \int_{\mathbb{R}^{2n}} \frac{2\pi}{h} \sin(\pi h(xy' - yx'))\, \hat{k}'_s(h, x', y')\, \hat{k}_s(h, x - x', y - y')\, dx'dy', \tag{20}$$

However the last expression for $h = 0$ is significantly distinct from the vanishing (17). From the natural assignment $\frac{4\pi}{h}\sin(\pi h(xy' - yx')) = 4\pi^2(xy' - yx')$ for $h = 0$ we get the Poisson brackets for the Fourier transforms of k' and k defined on $\mathcal{O}_0$ (13):

$$\rho_{(q,p)}\, \big\{\!\big[k', k\big]\!\big\} = \frac{\partial \hat{k}'}{\partial q}\frac{\partial \hat{k}}{\partial p} - \frac{\partial \hat{k}'}{\partial p}\frac{\partial \hat{k}}{\partial q}. \tag{21}$$

Furthermore the dynamical equation based on the modified commutator (19) with a suitable Hamilton type function $H(s, x, y)$ for an observable $f(s, x, y)$ on $\mathbb{H}^n$

$$\dot{f} = \{\![H, f]\!\} \text{ is reduced } \begin{cases} \text{by } \rho_h,\ h \neq 0 \text{ on } \mathcal{O}_h \text{ (6) to Moyal's equation [30, (8)]}; \\ \text{by } \rho_{(q,p)} \text{ on } \mathcal{O}_0 \text{ (13) to Poisson's equation [1, § 39]}. \end{cases} \tag{22}$$

The same connections are true for the solutions of these three equations, see [26] for the harmonic oscillator and [4, 5] for forced oscillator examples.

3 De Donder–Weyl field theory

We extend p-mechanics to the De Donder–Weyl field theory, see [12–20] for detailed exposition and further references. We will be limited here to the preliminary discussion which extends the comment 5.2.(1) from the earlier paper [25]. Our notations will slightly different from the used in the papers [12–20] to make it consistent with the used above and avoid clashes.

3.1 Hamiltonian form of field equation

Let the underlying space-time have dimension and $n + 1$ parametrised by coordinates u^μ, $\mu = 0, 1, \ldots, n$ (with u^0 parameter traditionally associated with a time-like direction). Let a field be described by m component tensor q^a, $a = 1, \ldots, m$. For a system defined by a Lagrangian density $L(q^a, \partial_\mu q^a, u^\mu)$ De Donder–Weyl theory suggests new set of *polymomenta* p^μ_a and *DW Hamiltonian function* $H(q^a, p^\mu_a, u^\mu)$ defined as follows:

$$p^\mu_a := \frac{\partial L(q^a, \partial_\mu q^a, u^\mu)}{\partial(\partial_\mu y^a)} \quad \text{and} \qquad H(q^a, p^\mu_a, u^\mu) = p^\mu_a\, \partial_\mu q^a - L(q^a, \partial_\mu q^a, u^\mu). \tag{23}$$

Consequently the Euler–Lagrange field equations could be transformed to the Hamilton form:

$$\frac{\partial q^a}{\partial u^\mu} = \frac{\partial H}{\partial p_a^\mu}, \qquad \frac{\partial p_a^\mu}{\partial u^\mu} = -\frac{\partial H}{\partial q^a}, \tag{24}$$

with the standard summation (over repeating indexes) agreement. The main distinction from a particle mechanics is the existence of $n+1$ different polymomenta p_a^μ associated to each field variable q^a. Correspondingly particle mechanics could be considered as a particular case when $n+1$ dimensional space-time degenerates for $n=0$ to "time only".

The next two natural steps [12–20] inspired by particle mechanics are:

1. Introduce an appropriate Poisson structure, such that the Hamilton equations (24) will represent the Poisson brackets.
2. Quantise the above Poisson structure by some means, e.g. Dirac–Heisenberg–Shrödinger–Weyl technique or geometric quantisation.

We use here another path: first to construct a p-mechanical model for equations (24) and then deduce its quantum and classical derivatives as was done for the particle mechanics above. To simplify presentation we will start from the scalar field, i.e. $m=1$. Thus we drop index a in q^a and p_a^μ and simply write q and p^μ instead.

We also assume that underlying space-time is flat with a constant metric tensor $\eta^{\mu\nu}$. This metric define a related *Clifford algebra* [3, 6, 8] with generators e^μ satisfying the relations

$$e^\mu e^\nu + e^\nu e^\mu = \eta^{\mu\nu}. \tag{25}$$

Remark 1. For the Minkowski space-time (i.e. in the context of special relativity) a preferable choice is *quaternions* [29] with generators i, j, k instead the general Clifford algebra.

Since q and p^μ look like conjugated variables p-mechanics suggests that they should generate a Lie algebra with relations similar to (4). The first natural assumption is the $n{+}3(=1{+}(n{+}1){+}1)$-dimensional Lie algebra spanned by X, Y_μ, and S with the only non-trivial commutators $[X, Y_\mu] = S$. However as follows from the Kirillov theory [21] any its unitary irreducible representation is limited to a representation of $\mathbb{H}^1$ listed by the Stone–von Neumann Theorem 1. Consequently there is little chances that we could obtain the field equations (24) in this way.

3.2 *p*-mechanical approach to the field theory

The next natural candidate is the Galilean group $\mathbb{G}^{n+1}$, i.e. a nilpotent step 2 Lie group with the $2n+3(=1+(n+1)+(n+1))$-dimensional Lie algebra. It has a basis X, Y_μ, and S_μ with $n+1$-dimensional centre spanned by S_μ. The only non-trivial commutators are

$$[X, Y_\mu] = S_\mu, \qquad \text{where} \quad \mu = 0, 1, \ldots, n. \tag{26}$$

Again the Kirillov theory [21] assures that any its *complex valued* irreducible representation is a representation of $\mathbb{H}^1$, but multidimensionality of the centre offers an option [7, 24] to consider *Clifford valued* representations of $\mathbb{G}^{n+1}$. Thus we proceed with this group.

Remark 2. The appearance of Clifford algebra in connection with field theory and space-time geometry is natural. For example, the conformal invariance of space-time has profound consequences in astrophysics [28] and, in their turn, conformal (Möbius) transformations are most naturally represented by linear-fractional transformations in Clifford algebras [6]. Some other links between nilpotent Lie groups and Clifford algebras are listed in [24].

The Lie group $\mathbb{G}^{n+1}$ as manifold homeomorphic to $\mathbb{R}^{2n+3}$ with coordinates (s, x, y), where $x \in \mathbb{R}$ and $s, y \in \mathbb{R}^{n+1}$. The group multiplication in this coordinates defined by, cf. (1):

$$\begin{aligned}(s, x, y) * (s', x', y') \\ = \left(s_0 + s'_0 + \frac{1}{2}\omega(x, y_0; x', y'_0), \ldots, s_n + s'_n + \frac{1}{2}\omega(x, y_n; x', y'_n), x + x', y + y'\right).\end{aligned} \tag{27}$$

Observables are again defined as convolution operators on $L_2(\mathbb{G}^{n+1})$. To define an appropriate brackets of two observables k_1 and k_2 we will again modify their commutator $[k_1, k_2]$ by antiderivative operators $\mathcal{A}_0$, $\mathcal{A}_1$, …, $\mathcal{A}_n$ which are multiples of right inverse to the vector fields S_0, S_1, …, S_n, cf. (18):

$$S_\mu \mathcal{A}_\mu = 4\pi^2 I, \quad \text{where} \quad \mathcal{A}_\mu e^{2\pi \mathrm{i} h s_\mu} = \begin{cases} \frac{2\pi}{\mathrm{i}h} e^{2\pi \mathrm{i} h s_\mu}, & \text{if } h \neq 0, \\ 4\pi^2 s_\mu, & \text{if } h = 0, \end{cases} \quad \text{and} \quad \mu = 0, 1, \ldots, n. \tag{28}$$

The definition of the brackets follows the ideas of [7, § 3.3]: to each vector field S_μ should be associated a generator e^μ of Clifford algebra (25). Thus our brackets are as follows, cf. (19):

$$\{\![B_1, B_2]\!\} = B_1 * \mathcal{A} B_2 - B_2 * \mathcal{A} B_1, \qquad \text{where} \quad \mathcal{A} = e^\mu \mathcal{A}_\mu. \tag{29}$$

These brackets will be used in the right-hand side of the p-dynamic equation. Its left-hand side should contain a replacement for the time derivative. As was already mentioned in [12–20] the space-time play a rôle of multidimensional time in the De Donder–Weyl construction. Thus we replace time derivative by the symmetric pairing $D\circ$ with the *Dirac operator* [3, 6, 8] $D = e^\mu \partial_\mu$ as follows:

$$D \circ f = -\frac{1}{2}\left(e^\mu \frac{\partial f}{\partial u^\mu} + \frac{\partial f}{\partial u^\mu} e^\mu\right), \qquad \text{where} \quad D = e^\mu \partial_\mu. \tag{30}$$

Finally the p-mechanical dynamic equation, cf. (22):

$$D \circ f = \{\![H, f]\!\}, \tag{31}$$

is defined through the brackets (29) and the Dirac operator (30).

To "verify" the equation (31) we will find its classical representation and compare it with (24). Similarly to calculations in section 2.2 we find, cf. (21):

$$\rho_{(q,p^\mu)}\left\{\!\left[k', k\right]\!\right\} = \frac{\partial \hat{k}'}{\partial q} e^\mu \frac{\partial \hat{k}}{\partial p^\mu} - \frac{\partial \hat{k}'}{\partial p^\mu} e^\mu \frac{\partial \hat{k}}{\partial q}. \tag{32}$$

Consequently the dynamic of field observable q from the equation (31) with a scalar-valued Hamiltonian H is given by:

$$D \circ q = \left(\frac{\partial H}{\partial q} e^\mu \frac{\partial}{\partial p^\mu} - \frac{\partial H}{\partial p^\mu} e^\mu \frac{\partial}{\partial q}\right) q \qquad \Longleftrightarrow \qquad \frac{\partial q}{\partial u^\mu} e^\mu = \frac{\partial H}{\partial p^\mu} e^\mu, \tag{33}$$

i.e. after separation of components with different generators e^μ we get first $n + 1$ equations from (24).

To get the last equation for polymomenta (24) we again use the Clifford algebra generators to construct the *combined polymomenta* $p = e_\nu p^\nu$. For them:

$$\begin{aligned} D \circ p &= -\frac{1}{2}\left(e^\mu \frac{\partial e_\nu p^\nu}{\partial u^\mu} + \frac{\partial e_\nu p^\nu}{\partial u^\mu} e^\mu\right) = -\frac{\partial p^\mu}{\partial u^\mu} e^\mu e_\mu, \\ \{\![H, f]\!\} &= \frac{\partial H}{\partial q} e^\mu \frac{\partial e_\nu p^\nu}{\partial p^\mu} - \frac{\partial H}{\partial p^\mu} e^\mu \frac{\partial e_\nu p^\nu}{\partial q} = \frac{\partial H}{\partial q} e^\mu e_\mu. \end{aligned}$$

Thus the equation (31) for the combined polymomenta $p = e_\nu p^\nu$ becomes:

$$\frac{\partial p^\mu}{\partial u^\mu} e^\mu e_\mu = -\frac{\partial H}{\partial q} e^\mu e_\mu, \tag{34}$$

i.e. coincides with the last equation in (24) up to a constant factor $e^\mu e_\mu$.

Consequently images of the equation (31) under the infinite dimensional representation of the group $\mathbb{G}^{n+1}$ could stand for quantisations of its classical images in (24), (32). A further study of quantum images of the equation (31) as well as extension to vector or spinor fields should follow in subsequent papers [27].

Remark 3. To consider vector or spinor fields with components q_a, $a = 1, \dots, m$ it worths to introduce another Clifford algebra with generators c^a and consider a composite field $q = c^a q_a$. There are different ways to link Clifford and Grassmann algebras, see e.g. [2, 10]. Through such a link the Clifford algebra with generators e_μ corresponds to horizontal differential forms in the sense of [12–20] and the Clifford algebra generated by c_a – to the vertical.

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Poisson Algebras on Elliptic Curves

A. KOROVNICHENKO [†], *V.P. SPIRIDONOV* [‡] *and A.S. ZHEDANOV* [§]

[†] *University of Notre-Dame, Notre-Dame, USA*
E-mail: *okorovni@nd.edu*

[‡] *Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow Region, Russia*
E-mail: *spiridon@thsun1.jinr.ru*

[§] *Donetsk Institute for Physics and Technology, 83114 Donetsk, Ukraine*
E-mail: *zhedanov@kinetic.ac.donetsk.ua*

We describe Poisson algebras associated with classical analogs of some self-dual generalized eigenvalue problems. These algebras are related in a natural way to various elliptic curves.

1 Introduction

We work with the classical mechanical systems of one degree of freedom. Let $X(q,p)$ and $Y(q,p)$ are two independent dynamical variables of canonical variables q, p with the standard Poisson bracket $\{q,p\}=1$. Functions X and Y are called independent if they satisfy the condition

$$\frac{\partial(X,Y)}{\partial(q,p)} \equiv \frac{\partial X}{\partial q}\frac{\partial Y}{\partial p} - \frac{\partial X}{\partial p}\frac{\partial Y}{\partial q} = \{X,Y\} \neq 0, \tag{1}$$

in some domain of interest of the phase space (q,p). Here $\partial(X,Y)/\partial(q,p)$ denotes the Jacobian of a change of variables.

According to the definition proposed in [9], two independent variables X and Y form a *classical Leonard pair* (CLP) if there exist two different canonical transformations $(q,p)\to(x,y)$ and $(q,p)\to(\xi,\eta)$ such that the first transformation brings X and Y to the form

$$X=\varphi(x), \qquad Y=A_1(x)e^y + A_2(x)e^{-y} + A_3(x) \tag{2}$$

and in the second case we have the representation

$$X=B_1(\xi)e^\eta + B_2(\xi)e^{-\eta} + B_3(\xi), \qquad Y=\psi(\xi), \tag{3}$$

where (x,y) and (ξ,η) are canonical pairs (i.e., $\{x,y\}=\{\xi,\eta\}=1$) and $\varphi(x)$, $A_i(x)$, $\psi(\xi)$, $B_i(\xi)$ are some functions. Using canonical transformations $y\to\kappa y$, $x\to x/\kappa$ and taking the limit $\kappa\to 0$ one can obtain from (2) the limiting form $Y=a_1(x)y^2+a_2(x)y+a_3(x)$. Therefore we shall assume that CLP admit such degenerate forms of Y in (2) (or of X in (3)) without further reservations.

It is convenient to introduce the variable

$$Z=\{X,Y\}. \tag{4}$$

We assume that there exists a region of values of X, Y where X and Y form independent variables, that is $Z\neq 0$. The latter means that in this domain we can invert the changes of variables and find $x=x(X,Y)$, $y=y(X,Y)$. As a result, we can consider Z as a function of X and Y, $Z=Z(X,Y)$. The condition that X and Y form a CLP allows us to establish the explicit

form of this function $Z(X,Y)$. Namely, as shown in [9] there exist 9 arbitrary constants α_{ik}, $i,k=0,1,2$, such that

$$Z^2=\sum_{i,k=0}^{2}\alpha_{ik}X^iY^k\equiv -F(X,Y). \tag{5}$$

Vice versa, it can be shown that starting from the condition (5) for arbitrary α_{ik} one arrives at a CLP (including its degenerate form mentioned above). The condition $F=0$ determines the region of the phase space with complex values of q, p where such a consideration breaks down.

From (5) it follows that the dynamical variables X, Y and $Z=\{X,Y\}$ form a Poisson algebra with the defining relations (4) and

$$\{Z,X\}=\frac{1}{2}\frac{\partial F(X,Y)}{\partial Y},\qquad \{Y,Z\}=\frac{1}{2}\frac{\partial F(X,Y)}{\partial X}, \tag{6}$$

which are known as the classical Askey–Wilson algebra relations [6]. This Poisson algebra generates relation (5) with the interpretation of the constant α_{00} as a value of the corresponding Casimir element [9]. In this way we get a particular example of the quadratic algebras, the most popular representative of which is given by the Sklyanin algebra [13]. In a more general setting, algebraic relations between dynamical variables involve polynomials of generators (see, e.g., particular polynomial quantum algebras in [14]).

Suppose that X is the Hamiltonian of some physical system. Then the first canonical transformation $(q,p)\to(x,y)$ is, in fact, an action-angle transformation: it maps X into a function depending on only one canonical variable x. Similarly, canonical transformation $(q,p)\to(\xi,\eta)$ is an action-angle variables transformation for a system with the Hamiltonian Y. Existence of a CLP can be considered as some duality property of two Hamiltonians with respect to prescribed dependence on the momenta y and η of the "conjugated" Hamiltonians (i.e., Y and X, respectively). From this point of view, the CLP property is equivalent to the notion of duality discussed in the theory of integrable systems, see, e.g., [12, 5].

After quantization, functions $e^{\pm p}$ become shift operators and the quantum analogue of the CLP property coincides with the standard Leonard duality [10] or the bispectrality condition [4] for two tridiagonal $N\times N$ matrices L,M. In this case, the matrix M is tridiagonal in the basis formed by eigenvectors ϕ_k of the matrix L, whereas M is tridiagonal in the basis formed by eigenvectors ψ_k of M:

$$L\phi_k=\lambda_k\phi_k,\qquad M\phi_k=\alpha_{k+1}\phi_{k+1}+\beta_k\phi_k+\gamma_k\phi_{k-1}, \tag{7}$$

and

$$M\psi_k=\mu_k\psi_k,\qquad L\psi_k=\xi_{k+1}\psi_{k+1}+\eta_k\psi_k+\zeta_k\psi_{k-1}, \tag{8}$$

where $k=1,2,\dots,N$. It is assumed that eigenvalues λ_k, μ_k are nondegenerate, so that the vectors ϕ_k and ψ_k form two independent complete bases. In such a form the problem of classifying all ("quantum") Leonard pairs L,M was investigated by Terwilliger [18]. It is equivalent to the original Leonard problem [10] in the following sense. Let us decompose the vectors ψ_k in the basis of vectors ϕ_k

$$\psi_k=\sum_{s=1}^{N}P_{ks}\phi_s, \tag{9}$$

with some expansion coefficients P_{ks}. It follows from (7), (8) that the coefficients P_{ks} satisfy simultaneously two three-term recurrence relations

$$\zeta_{s+1}P_{k,s+1}+\eta_sP_{ks}+\xi_{s-1}P_{k,s-1}=\mu_kP_{ks} \tag{10}$$

and

$$\gamma_{k+1}P_{k+1,s} + \beta_k P_{ks} + \alpha_{k-1}P_{k-1,s} = \lambda_s P_{ks}, \tag{11}$$

which mean that P_{ks} can be expressed in terms of some orthogonal polynomials of the argument λ_s or μ_k. It appears that these polynomials are self-dual: permutation of the discrete variables k and s is equivalent to some permutation of parameters entering the recurrence coefficients η_s, ξ_s (for details, see [10, 18, 19]).

Leonard theorem [10] states that the q-Racah polynomials, discovered by Askey and Wilson [1], are the most general self-dual orthogonal polynomials. As shown in [19], the quantum analog of the algebra (6) with the generators L, M and $N = [L, M] \equiv LM - ML$ describes these polynomials via the representation theory (see also [18] for similar algebraic treatments). Relations of this algebra to the standard $sl_q(2)$ quantum algebra have been established in [7, 8].

2 Duality for a generalized eigenvalue problem

In [15–17], a new family of discrete biorthogonal rational functions $R_n(z)$ and $T_n(z)$, $n = 0, \ldots, N-1$, has been found. These functions satisfy the property

$$\sum_{s=0}^{N-1} w_s R_n(z_s) T_m(z_s) = h_n \delta_{nm}, \tag{12}$$

with some weight function w_s and normalization constants h_n. The sequence z_s is called the "grid" and it is expressed in terms of the Jacobi theta functions. Both $R_n(z_s)$ and $T_n(z_s)$ satisfy three term recurrence relations in the variable n and second order difference equations in s.

Let us introduce a set of N vectors $\Phi_s = (R_0(z_s), R_1(z_s), \ldots, R_{N-1}(z_s))^t$, $s = 0, \ldots, N-1$. Then there exist two tridiagonal matrices L_1, L_2 such that

$$L_1 \Phi_s = z_s L_2 \Phi_s, \tag{13}$$

where it is assumed that the matrices $L_{1,2}$ act on the vector Φ_s in the standard manner. Similarly, we introduce a set of N vectors $\Psi_n = (R_n(z_0), R_n(z_1), \ldots, R_n(z_N))^t$, $n = 0, \ldots, N-1$. Then there exist two tridiagonal matrices M_1, M_2 such that

$$M_1 \Psi_n = \lambda_n M_2 \Psi_n, \tag{14}$$

for some sequence of numbers λ_n (the dual "grid"). Since the functions $R_n(z_s)$ satisfy simultaneously two generalized eigenvalue problems (13) and (14), it is natural to consider the following problem.

Let X and Y be two invertible $N \times N$ matrices with different eigenvalues λ_k and μ_k, $k = 0, \ldots, N-1$. We denote as ϕ_k and ψ_k linearly independent eigenvectors of X and Y, respectively:

$$X\phi_k = \lambda_k \phi_k, \qquad Y\psi_k = \mu_k \psi_k. \tag{15}$$

Now we assume that in the basis of vectors ψ_k the matrix X takes the form

$$X\psi_k = X_2^{-1} X_1 \psi_k, \tag{16}$$

where X_1, X_2 are two tridiagonal matrices, that is

$$\begin{aligned} X_1\psi_k &= \alpha_{k+1}^{(1)}\psi_{k+1} + \beta_k^{(1)}\psi_k + \gamma_k^{(1)}\psi_{k-1}, \\ X_2\psi_k &= \alpha_{k+1}^{(2)}\psi_{k+1} + \beta_k^{(2)}\psi_k + \gamma_k^{(2)}\psi_{k-1}. \end{aligned} \tag{17}$$

In the same way, we assume that there exist two tridiagonal matrices Y_1, Y_2 such that

$$Y\phi_k = Y_2^{-1}Y_1\phi_k, \tag{18}$$

with the properties

$$\begin{aligned} Y_1\phi_k &= \xi_{k+1}^{(1)}\phi_{k+1} + \eta_k^{(1)}\phi_k + \zeta_k^{(1)}\phi_{k-1}, \\ Y_2\phi_k &= \xi_{k+1}^{(2)}\phi_{k+1} + \eta_k^{(2)}\phi_k + \zeta_k^{(2)}\phi_{k-1}. \end{aligned} \tag{19}$$

The main problem of interest consists in the classification of all matrices X and Y admitting such a representation. It implies an explicit description of algebraic structures behind this construction that generalize the Askey–Wilson algebra. We conjecture that complete solution of this problem yields discrete biorthogonal rational functions of [15–17]. Here we make a step towards solution of this problem and announce some results for its classical mechanics analog.

We take two independent dynamical variables $X(q,p)$ and $Y(q,p)$ depending on canonical variables q and p. Now we suppose that there exist two canonical transformations $(q,p) \to (x,y)$ and $(q,p) \to (\xi,\eta)$ such that the first transformation leads to

$$X = \varphi(x), \qquad Y = \frac{Y_1(x,y)}{Y_2(x,y)} \tag{20}$$

and in the second case we have

$$Y = \psi(\xi), \qquad X = \frac{X_1(\xi,\eta)}{X_2(\xi,\eta)}, \tag{21}$$

where X_r, Y_r, $r = 1,2$, are some classical "tridiagonal functions", that is

$$\begin{aligned} X_r(\xi,\eta) &= A_1^{(r)}(\xi)e^{\eta} + A_2^{(r)}(\xi)e^{-\eta} + A_0^{(r)}(\xi), \\ Y_r(x,y) &= B_1^{(r)}(x)e^{y} + B_2^{(r)}(x)e^{-y} + B_0^{(r)}(x). \end{aligned} \tag{22}$$

We shall call the pair (X,Y) satisfying such a property as generalized CLP.

3 Main results

We consider the general situation when none of the coefficients $A_i^{(r)}$, $B_i^{(r)}$ in (22) vanishes identically. Note, however, that some of the potentials can be set constant by changes of variables.

Theorem 1. *Suppose that independent variables X and Y admit representations* (20) *and* (21) *via two canonical transformations. Then they necessarily satisfy the following quadratic equation determining a particular elliptic curve:*

$$Z^2 + F_1(X,Y)Z + F_2(X,Y) = 0, \tag{23}$$

where $Z \equiv \{X,Y\}$ is the Poisson bracket of X and Y,

$$F_2(X,Y) = \frac{1}{4}F_1(X,Y)\left(F_1(X,Y) - q^2(X,Y)\right), \tag{24}$$

and

$$F_1(X,Y) = \sum_{i,k=0}^{2} \alpha_{ik}X^iY^k, \qquad q(X,Y) = \sum_{i,k=0}^{1} \beta_{ik}X^iY^k \tag{25}$$

for some 13 constants α_{ik} and β_{ik}. Condition (23) *is also sufficient.*

For the "grid" function $\varphi(x)$ we obtain equation

$$(\varphi'(x))^2 = \pi_4(\varphi(x)), \tag{26}$$

where $\pi_4(y)$ is a polynomial of fourth degree depending on parameters α_{ik} and β_{ik}. In the general situation, the polynomial $\pi_4(y)$ is not degenerate. It is known that solution of equation (26) is given by general elliptic function of the second order, which can be represented in the form [3]:

$$\varphi(x) = \gamma\frac{\theta_1(\beta x+u_1)\theta_1(\beta x+u_2)}{\theta_1(\beta x+v_1)\theta_1(\beta x+v_2)}, \qquad u_1+u_2 = v_1+v_2, \tag{27}$$

where $\theta_1(u)$ is the standard Jacobi theta function. Recall that the order of the elliptic function is defined as the number of zeros or poles (counting their multiplicity) inside the fundamental parallelogram of periods [3]. Elliptic functions of the second order are the simplest non-trivial double-periodic meromorphic functions. Well-known examples are given by the Weierstrass function $\wp(x)$ and Jacobi functions $\mathrm{sn}\,(x)$, $\mathrm{cn}\,(x)$, $\mathrm{dn}\,(x)$.

The potentials $A_i^{(r)}$, $B_i^{(r)}$ are also given by elliptic functions, but their expressions are rather complicated and we omit them here. A proof of the above theorem will be given elsewhere.

There is an important subcase when equation (23) is reduced to the complete square:

$$Z = \sum_{i,k=0}^{2}\alpha_{ik}X^iY^k \tag{28}$$

for some coefficients α_{ik}. This situation corresponds to symmetric, or two-diagonal representation for the variables X, Y. More exactly, we have

Theorem 2. *Dynamical variables X and Y form a generalized symmetric CLP, that is they admit representations* (20) *and* (21) *with $A_0^{(r)} = B_0^{(r)} = 0$, $r = 1, 2$, if and only if their Poisson bracket $Z = \{X, Y\}$ takes the form* (28) *with arbitrary 9 coefficients α_{ik}.*

We have shown that condition (23) is necessary and sufficient for the dynamical variables X and Y to satisfy the classical analog of the generalized eigenvalue problem for two tridiagonal matrices. The corresponding Poisson algebra takes the form

$$\{X,Y\} = Z, \qquad \{Z,X\} = -Z\frac{\partial Z}{\partial Y}, \qquad \{Y,Z\} = -Z\frac{\partial Z}{\partial X}. \tag{29}$$

However, the variable Z considered as a function of X, Y is given by a root of the quadratic equation (23). The algebra obtained after the substitution $Z = (-F_1 \pm q\sqrt{F_1})/2$ into (29) has much less attractive form with respect to the Askey–Wilson case (6). Therefore we need to find a simpler approach to this problem. For instance, we can try to express X, Y, Z in terms of some other dynamical variables U_i, $i = 1, 2, \ldots$, so that the algebra (29) is reproduced by relatively simple Poisson algebraic relations between generators U_i.

This idea can be explained as follows. From the very beginning we have demanded that in the picture (20) the variable Y is presented as the ratio $Y = U_3/U_4$ where both U_3 and U_4 are "tridiagonal" in y. Similarly, in the dual picture (21) one should be able to represent $X = U_1/U_2$, where U_1, U_2 have the same tridiagonality property with respect to η. It is therefore natural to set

$$X = \frac{U_1(q,p)}{U_2(q,p)}, \qquad Y = \frac{U_3(q,p)}{U_4(q,p)}$$

and seek 4 dynamical variables U_1, U_2, U_3, U_4 such that for some canonical transformation $(q,p) \to (x,y)$ all U_i are reduced to the tridiagonal form and, additionally, $U_1 = \varphi(x)U_2$ for

some function $\varphi(x)$. Similarly, there should exist a dual canonical transformation $(q,p)\to(\xi,\eta)$ which reduces again all U_i to tridiagonal form and, additionally, guarantees that $U_3=\psi(\xi)U_4$ for some function $\psi(\xi)$. All these requirements do not bring anything new to the picture we have considered so far. The crucial additional requirements look as follows:

i) pairwise Poisson brackets of $\{U_i,U_j\}$ should be quadratic polynomials in all U_1, U_2, U_3, U_4;

ii) the linear transformations $\tilde U_1=m_{11}U_1+m_{12}U_2$, $\tilde U_2=m_{21}U_1+m_{22}U_2$ and $\tilde U_3=\ell_{11}U_3+\ell_{12}U_4$, $\tilde U_4=\ell_{21}U_3+\ell_{22}U_4$ with two arbitrary nonsingular matrices ℓ_{ij}, m_{ij} do not change the form of the Poisson algebra for U_i, that is it should be covariant with respect to such linear fractional transformations.

In [11], some generalizations of the Sklyanin algebra were discussed. They are generated by two polynomials $Q_1(U)$ and $Q_2(U)$ depending on four dynamical variables U_i, $i=1,\dots,4$, whose Poisson brackets are defined in the following nice way

$$\{U_i,U_j\}=(-1)^{i+j}\det\left(\frac{\partial Q_k}{\partial U_l}\right),\qquad l\neq i,j,\quad i>j. \tag{30}$$

The functions Q_1,Q_2 serve as Casimir elements of these algebraic relations, that is $\{U_i,Q_k\}=0$, $k=1,2$.

In order to get quadratic Poisson algebra it is necessary to fix $Q_1(U)$ and $Q_2(U)$ as quadratic polynomials. For example, the standard Sklyanin algebra is obtained from

$$\begin{aligned}&Q_1(U)=U_1^2+U_2^2+U_3^2,\\&Q_2(U)=U_4^2+J_1U_1^2+J_2U_2^2+J_3U_3^2.\end{aligned}$$

We shall use this construction in order to model our classical Poisson algebraic relation (23).

We consider the case of two-diagonal representation, when

$$Z=\{X,Y\}=F(X,Y)=\sum_{i,k=0}^{2}\alpha_{ik}X^iY^k \tag{31}$$

is an arbitrary polynomial of the second degree in each variable X and Y.

Calculating Poisson bracket of $X=U_1/U_2, Y=U_3/U_4$ we get

$$Z=\{X,Y\}=\frac{\{U_1,U_3\}}{U_2U_4}+\frac{U_1U_3\{U_2,U_4\}}{U_2^2U_4^2}-\frac{U_3\{U_1,U_4\}}{U_2U_4^2}-\frac{U_1\{U_2,U_3\}}{U_2^2U_4}. \tag{32}$$

It is seen that in order to get (31) it is necessary to demand that each Poisson bracket $\{U_i,U_j\}$ in (32) contains the same variables U_iU_j, their adjacent pair U_kU_l (where k or l are not equal to i or j) and two adjacent pairs U_iU_k and U_lU_j. These conditions can be satisfied if we choose Casimir elements in the form

$$\begin{aligned}&Q_1(U)=\sum_{i=1}^{4}a_iU_i^2+\xi_1U_1U_2+\eta_1U_3U_4,\\&Q_2(U)=\sum_{i=1}^{4}b_iU_i^2+\xi_2U_1U_2+\eta_2U_3U_4\end{aligned} \tag{33}$$

with arbitrary parameters a_i, b_i, ξ_m, η_m. Direct calculations show that the equality (31) is indeed satisfied with 9 arbitrary parameters α_{ik} which are expressed through 12 parameters a_i, b_i, ξ_m, η_m. Note that the property ii) is fulfilled, e.g. the permutation $U_1\leftrightarrow U_2$ leads only to the permutation of parameters a_1, a_2 and b_1, b_2 in Q_1, Q_2. We thus have the following statement.

Theorem 3. *Any two-diagonal generalized CLP with $X=U_1/U_2$ and $Y=U_3/U_4$ can be realized in terms of the quadratic Poisson algebra* (30) *with two Casimir elements given by* (33).

4 Further generalizations and perspectives

We consider now condition (23) without restriction (24), that is we assume that

$$F_1(X,Y)=\sum_{i,k=0}^{2}\alpha_{ik}X^iY^k, \qquad F_2(X,Y)=\sum_{i,k=0}^{4}\gamma_{ik}X^iY^k \tag{34}$$

are arbitrary polynomials of degrees at most two and four with respect to both variables X and Y. In this case, obviously, the basic properties (20) and (21) are not valid any more. Still, relation (23) in this case generates many interesting special systems.

For example, if we set $F_1(X,Y)=0$ then we have the condition

$$Z^2+F_2(X,Y)=0, \tag{35}$$

which generates some Poisson algebra generalizing the Askey–Wilson algebra (the latter algebra is obtained when $\deg(F_2)=2$). Some special cases of dynamical systems described by this algebra (e.g., the Euler and Lagrange tops) were presented in [9]. It was observed that in all these cases one of the dynamical variables X or Y is an elliptic function of time.

Here we present the general result.

Theorem 4. *Suppose that independent dynamical variables X and Y have the following property: if X is chosen as the Hamiltonian then $Y(t)$ is described by an elliptic function of the second order. Conversely, if Y is chosen as the Hamiltonian then $X(t)$ is also an elliptic function of the second order. Then the dynamical variables X, Y, and Z satisfy the key relation* (35) *with $F_2(X,Y)$ being an arbitrary polynomial of degree at most four with respect to each variable X and Y.*

On the one hand, this theorem can be considered as a generalization of the previous result concerning the classical Leonard pairs [20]. On the other hand, this result provides a characterization of the so-called "double-elliptic systems" [2, 11]. Examples of double-elliptic systems described by relation (35) will be considered separately.

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Relation of Majorization for $*$-Categories and $*$-Wild Categories

Stanislav A. KRUGLYAK

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine

In the present paper notions of majorization and $*$-wildness are defined for the class of $*$-categories.

1 Introduction

In the articles [1, 2] a relation of majorization in the class of $*$-algebras was defined and a definition of $*$-wildness of algebras was given: $*$-algebra is called $*$-*wild* if it majorizes $*$-algebra $C^*(\mathcal{F}_2)$. It is possible (and, seemingly, necessary) to extend these definitions to the class of $*$-categories [3], which is done in the present article.

2 $*$-categories and $*$-quivers

Let $\mathcal{K}$ be a category with involution $*$ over the field $\mathbb{C}$ of complex numbers, in which $a = a^*$ for each $a \in \operatorname{Ob}\mathcal{K}$. Thus, to each morphism $\alpha : a \to b$ a morphism $\alpha^* : b \to a$ is associated, such that:

1) $\alpha^{**} = \alpha$;

2) $(\alpha\beta)^* = \beta^*\alpha^*$;

3) $(z_1\alpha_1 + z_2\alpha_2)^* = \overline{z}_1\alpha_1^* + \overline{z}_2\alpha_2^* \quad (z_1, z_2 \in \mathbb{C})$.

We will assume the presence of zero object in $\mathcal{K}$. Hereinafter we will call such categories $*$-*categories*.

Along with category $\mathcal{K}$ we will consider as a system of generators an involutive quiver Q (see [3, 4]), for any point a of which $a^* = a$ and for any arrow $\alpha : a \to b$ of which an arrow α^* is associated such that $\alpha^{**} = \alpha$. Further such quivers are said to be $*$-*quivers*. Category $\mathcal{K}$ is obtained from the category of paths of the quiver Q by factorization.

The category $\mathcal{K}$ is said to be *finitely generated* if it can be defined in such a way by a finite $*$-quiver and a finite set of interrelations (linear combinations of quiver paths with common beginning and end are declared to be equal to zero). The set of interrelations we always consider to be closed under the involution.

Morphism $\varphi : a \to b$ of the category $\mathcal{K}$ will be called *isomorphism* if there exists a morphism $\varphi^{-1} : b \to a$ such that $\varphi^{-1}\varphi = \varepsilon_a$, $\varphi\varphi^{-1} = \varepsilon_b$; isomorphism φ is called *congruence* if $\varphi^{-1} = \varphi^*$ [3].

Representation of the category $\mathcal{K}$ is a functor concerted with involution (involutive functor) π over the field $\mathbb{C}$ from the category $\mathcal{K}$ to the category $\mathcal{H}$ of Hilbert spaces whose objects are separable Hilbert spaces, and morphisms are bounded linear operators from one space to another; the involution on objects is identical and on morphisms it is a transition to the adjoint operator.

Representations of the category $\mathcal{K}$ themselves form a $*$-category $\operatorname{Rep}\mathcal{K}$, whose objects are involutive functors π (representations) and morphisms are families of morphisms of the category $\mathcal{H}$ intertwining these functors (natural transformations of functors). Two representations are said to be *equivalent* if they are isomorphic in the category of representations, and *unitary equivalent* if they are congruent in the category of representations.

As it is well known, the category of representations $\operatorname{Rep}\mathcal{K}$ is quadratically closed (for any morphism α there exists a self-adjoint morphism $\beta = \beta^*$ such that $\beta^2 = \alpha^*\alpha$), therefore the equivalence of two representations implies their unitary equivalence (see [3]).

Let us define a category of matrices $\mathcal{M}(\mathcal{K})$ over the category $\mathcal{K}$. Its objects are ordered collections $(a_1, a_2, \ldots, a_n)$ of objects from $\mathcal{K}$ (not excluding the matching of a_i and a_j when $i \neq j$). Morphism from $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_m)$ is a matrix of dimension $m \times n$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{bmatrix},$$

where $\alpha_{ij} : a_j \to b_i$ is a morphism of the category $\mathcal{K}$. We will indicate over the j-th column an object a_j, on the left of i-th row we will indicate an object b_i such that α_{ij} "leads" from the object a_j to the object b_i. Final notation for the matrices-morphisms of the category $\mathcal{M}(\mathcal{K})$ has a form:

$$A = \begin{array}{c|cccc} & a_1 & a_2 & \ldots & a_n \\ \hline b_1 & \alpha_{11} & \alpha_{12} & \ldots & \alpha_{1n} \\ b_2 & \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_m & \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{array}.$$

In this case it is possible to construct a composition and add the morphisms of the category $\mathcal{M}(\mathcal{K})$ by usual rules of multiplication and adding of matrices.

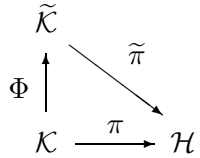
The involution on $\mathcal{M}(\mathcal{K})$ is defined naturally:

$$A^* = \begin{array}{c|cccc} & b_1 & b_2 & \ldots & b_m \\ \hline a_1 & \alpha_{11}^* & \alpha_{21}^* & \ldots & \alpha_{m1}^* \\ a_2 & \alpha_{12}^* & \alpha_{22}^* & \ldots & \alpha_{m2}^* \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & \alpha_{1n}^* & \alpha_{2n}^* & \ldots & \alpha_{mn}^* \end{array}.$$

3 Relation of majorization for ∗-categories

Let us define a wrapping category.

Definition 1. Let $\mathcal{K}$, $\widetilde{\mathcal{K}}$ be ∗-categories. A pair $(\widetilde{\mathcal{K}}, \Phi : \mathcal{K} \to \widetilde{\mathcal{K}})$, where Φ is involutive functor from ∗-category $\mathcal{K}$ to ∗-category $\widetilde{\mathcal{K}}$, is called a *wrapping category* of the category $\mathcal{K}$ if for any ∗-representation $\pi : \mathcal{K} \to \mathcal{H}$ there exists a unique representation $\widetilde{\pi} : \widetilde{\mathcal{K}} \to \mathcal{H}$ such that the diagram



is commutative, and any morphism of the category $\operatorname{Rep}\mathcal{K}$ intertwining two representations π_1 and π_2 of the category $\mathcal{K}$ also intertwines representations $\widetilde{\pi}_1$ and $\widetilde{\pi}_2$ of the category $\widetilde{\mathcal{K}}$.

Let $\mathcal{M}(\mathcal{K})$ be a category of matrices over ∗-category $\mathcal{K}$, $(\widetilde{\mathcal{M}(\mathcal{K})}, \Phi : \mathcal{M}(\mathcal{K}) \to \widetilde{\mathcal{M}(\mathcal{K})})$ is its wrapping category.

Any ∗-representation $\pi : \mathcal{K} \to \mathcal{H}$ induces a representation $\mathcal{M}(\mathcal{K}) \to \mathcal{H}$ and, therefore, representation $\widetilde{\pi} : \widetilde{\mathcal{M}(\mathcal{K})} \to \mathcal{H}$.

If $\Psi : \mathcal{L} \to \widetilde{\mathcal{M}(\mathcal{K})}$ is involutive functor, then in a natural way a functor

$$F_\Psi : \operatorname{Rep}\mathcal{K} \to \operatorname{Rep}\mathcal{L}$$

can be constructed.

By definition $F_\Psi(\pi) = \widetilde{\pi} \circ \Psi$, and if $C = (C_a)_{a \in \operatorname{Ob}\mathcal{K}}$ is a morphism of representations from $\operatorname{Rep}\mathcal{K}$, $C_a : \pi(a) \to \pi_1(a)$, $(a_1, a_2, \dots, a_k) \in \operatorname{Ob}\mathcal{M}(\mathcal{K})$, then

$$F_\Psi(C) = \operatorname{diag}(C_{a_1}, C_{a_2}, \dots, C_{a_k}) \in \operatorname{Hom}_{\mathcal{M}(\mathcal{K})}((\pi(a_1), \pi(a_2), \dots, \pi(a_k)); (\pi_1(a_1), \pi_1(a_2), \dots, \pi_1(a_k))).$$

Definition of the faithful and complete functor see in [5].

Definition 2. $*$-category $\mathcal{L}$ *directly majorizes* a $*$-category $\mathcal{K}$ ($\mathcal{L} \rhd \mathcal{K}$) if at the category of matrices $\mathcal{M}(\mathcal{K})$ exist a wrapping category $(\widetilde{\mathcal{M}(\mathcal{K})},\ \Phi)$ and $*$-functor $\Psi : \mathcal{L} \to \widetilde{\mathcal{M}(\mathcal{K})}$ such that the functor $F_\Psi : \operatorname{Rep}\mathcal{K} \to \operatorname{Rep}\mathcal{L}$ (faithful by definition) will be complete.

$*$-category *majorizes* a $*$-category $\mathcal{K}$ ($\mathcal{L} \succ \mathcal{K}$) if there exist $*$-categories $\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_n$ such that $\mathcal{L} \equiv \mathcal{K}_0 \rhd \mathcal{K}_1 \rhd \mathcal{K}_2 \rhd \cdots \rhd \mathcal{K} \equiv \mathcal{K}_{n+1}$.

Remark 1. Let $F_{\Psi_i} : \operatorname{Rep}\mathcal{K}_i \to \operatorname{Rep}\mathcal{K}_{i-1}$ be a faithful and complete functor existing due to direct majorization $\mathcal{K}_i \rhd \mathcal{K}_{i-1}$. Then functor

$$F = F_{\Psi_1} \circ F_{\Psi_2} \circ \cdots \circ F_{\Psi_{n+1}} : \operatorname{Rep}\mathcal{K} \to \operatorname{Rep}\mathcal{L}$$

is faithful and complete.

We will say that a $*$-quiver Q_1 *majorizes* a $*$-quiver Q_2 ($Q_1 \succ Q_2$) if for categories corresponding to these quivers $\mathcal{K}(Q_1) \succ \mathcal{K}(Q_2)$.

Absolutely evident is

Proposition 1. *Relation of majorization is relation of quasiorder in the class of $*$-categories ($*$-quivers).*

4 On $*$-wild categories

Let us consider $*$-algebras:

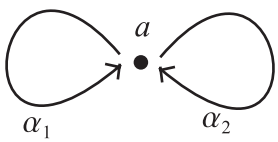
$$\mathcal{F}_2 = \mathbb{C}\langle u_1, u_2 \,|\, u_i^* = u_i^{-1},\ i = 1, 2\rangle,$$
$$\mathcal{A}_n = \mathbb{C}\langle \alpha_1, \alpha_2\ \dots, \alpha_n \,|\, \alpha_i^* = \alpha_i,\ i = \overline{1, n}\rangle.$$

In article [6], in fact, it was shown that $\mathcal{F}_2 \succ \mathcal{A}_2$, $\mathcal{A}_2 \succ \mathcal{F}_2$ and $\mathcal{A}_2 \succ \mathcal{A}_n$ when $n \geq 2$.

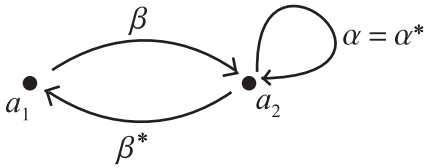
Definition 3. We will call $*$-category $\mathcal{K}$ ($*$-quiver $\mathcal{Q}$) $*$-*wild* if $\mathcal{K} \succ \mathcal{F}_2$ (category corresponding to the quiver $\mathcal{K}(Q) \succ \mathcal{F}_2$).

In [4, 7] it was proved that $*$-quiver Q (without interrelations) is $*$-wild iff Q (as a quiver without involution $*$) is wild.

Example 1. Let Q_1 be a $*$-quiver



with involution $\alpha_1^* = \alpha_1$, $\alpha_2^* = \alpha_2$ (without additional interrelations) and Q_2 be a $*$-quiver



with interrelation $\beta^*\beta = \varepsilon_{a_1}$.

Let us show that $Q_1 \prec Q_2$, i.e. quiver Q_2 is $*$-wild.

Let us construct a functor $\Psi : \mathcal{K}(Q_2) \to \mathcal{M}(\mathcal{K}(Q_1))$ putting $\Psi(a_1) = a$, $\Psi(a_2) = (a, a)$,

$$\Psi(\beta) = \begin{array}{c} \\ a \\ a \end{array} \begin{array}{c} \begin{array}{c} a \end{array} \\ \boxed{\begin{array}{c} \varepsilon_a \\ 0_a \end{array}} \end{array}, \quad \Psi(\alpha) = \begin{array}{c} \\ a \end{array} \begin{array}{c} \begin{array}{cc} a & a \end{array} \\ \boxed{\begin{array}{cc} \alpha_1 & \varepsilon_a \\ \varepsilon_a & \alpha_2 \end{array}} \end{array}.$$

It is immediately checked on that functor F_Ψ is faithful and complete.

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The Structure of Algebras $Q_{n,\vec{\alpha}}$ Generated by Linear Connected Idempotents

Anatolii A. KYRYCHENKO

Kyiv National University of Building and Architecture, 31 Povitroflotsky Ave., Kyiv, Ukraine
E-mail: *akyrych@imath.kiev.ua*

In the article we investigate the structure of algebras generated by linear connected idempotents. In particular, we proof the existence of nonsemisimple ones among such algebras.

Algebras generated by linear connected idempotents were studied in [1–3] and others. In particular, a problem of polynomial relations existence, description of algebras growth $Q_{n,\vec{\alpha}} = \mathbb{C}\langle q_1, q_2, \dots, q_n \,|\, q_k^2 = q_k;\ \sum\limits_{k=1}^{n} \alpha_k q_k = e\rangle$, $n \in \mathbb{N}$, $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, $\forall\ i\colon\ \alpha_i \neq 0$, etc. were considered in [2, 3]. All of algebras $Q_{n,\vec{\alpha}}$ are finite-dimensional when $n \leq 3$. Algebras $Q_{n,\vec{\alpha}}$, where $n \geq 4$, are infinite-dimensional for all $\vec{\alpha}$.

But the structure of algebras was studied insufficiently even in the finite-dimensional case (for $n = 3$). In particular, the problem of the existence among such algebras of nonsemisimple ones was not investigated sufficiently.

In this article the algebra

$$Q_{3,\vec{\alpha}} = \mathbb{C}\langle q_1, q_2, q_3 \,|\, q_i^2 = q_i, \alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = e\rangle,$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i > 0$ is studied in detail and here is the proof of the fact that algebras $Q_{3,\vec{\alpha}}$ are nonsemisimple for $\alpha_i = 1$, $\alpha_j + \alpha_k = 1$.

Denote $A = \sum\limits_{i=1}^{3} \alpha_i$ and $M = \{\vec{\alpha} : A = 1,\ A = 2,\ \exists\, i : \alpha_i = 1 \text{ or } \exists\, i, j : \alpha_i + \alpha_j = 1\}$.

Proposition 1. *If $\vec{\alpha} \notin M$ then $Q_{3,\vec{\alpha}} = 0$.*

Proof. Since $\alpha_k q_k = e - (\alpha_i q_i + \alpha_j q_j)$ then $\alpha_k^2 q_k = e + \alpha_i^2 q_i + \alpha_j^2 q_j - 2\alpha_i q_i - 2\alpha_j q_j + \alpha_i\alpha_j\{q_i, q_j\}$. After elimination q_k we obtain $\alpha_k(e - \alpha_i q_i - \alpha_j q_j) = e + \alpha_i^2 q_i + \alpha_j^2 q_j - 2\alpha_i q_i - 2\alpha_j q_j + \alpha_i\alpha_j\{q_i, q_j\}$. That is why $q_j q_i = \frac{1}{\alpha_i\alpha_j}((\alpha_k - 1)e + (2\alpha_i - \alpha_i^2 - \alpha_i\alpha_k)q_i + (2\alpha_j - \alpha_j^2 - \alpha_j\alpha_k)q_j - \alpha_i\alpha_j q_i q_j)$. Since $q_j q_i = q_j(q_j q_i)$ then

$$\begin{aligned}
q_j q_i &= \frac{1}{\alpha_i\alpha_j}((\alpha_k - 1 + 2\alpha_j - \alpha_j^2 - \alpha_j\alpha_k)q_j + (2\alpha_i - \alpha_i^2 - \alpha_i\alpha_k)q_j q_i - \alpha_i\alpha_j(q_j q_i)q_j),\\
q_j q_i &= \frac{1}{\alpha_i\alpha_j}((\alpha_k - 1 + 2\alpha_j - \alpha_j^2 - \alpha_j\alpha_k)q_j + (2\alpha_i - \alpha_i^2 - \alpha_i\alpha_k)q_j q_i\\
&\quad - ((\alpha_k - 1 + 2\alpha_j - \alpha_j^2 - \alpha_j\alpha_k)q_j + (2\alpha_i - \alpha_i^2 - \alpha_i\alpha_k - \alpha_i\alpha_j)q_i q_j)). \qquad (1)
\end{aligned}$$

From here $(2\alpha_i - \alpha_i^2 - \alpha_i\alpha_k - \alpha_i\alpha_j)[q_i, q_j] = 0$ or $-\alpha_i(\alpha_i + \alpha_j + \alpha_k - 2)[q_i, q_j] = 0$. Since $\vec{\alpha} \notin M$ then $[q_i, q_j] = 0$ and then from (1) it follows $(2\alpha_j - \alpha_j^2 - \alpha_j\alpha_k + \alpha_k - 1 + 2\alpha_i - \alpha_i^2 - \alpha_i\alpha_k - 2\alpha_i\alpha_j)q_i q_j = 0$.

Since $2\alpha_j - \alpha_j^2 - \alpha_j\alpha_k + \alpha_k - 1 + 2\alpha_i - \alpha_i^2 - \alpha_i\alpha_k - 2\alpha_i\alpha_j = -(\alpha_i + \alpha_j - 1)(\alpha_i + \alpha_j + \alpha_k - 1) \neq 0$, when $\alpha \notin M$, then $\forall\ i, j\colon\ q_i q_j = 0$.

After multiplying $\alpha_1 q_1 + \alpha_2 q_2 + \alpha_3 q_3 = e$ by q_i we obtain $\alpha_i q_i = q_i$ or $(\alpha_i - 1)q_i = 0$. Since $\vec{\alpha} \notin M$ then $q_i = 0$, $i = 1, 2, 3$. So the algebra is trivial. ■

From the proof follows that the dimension of the algebra $Q_{3,\vec{\alpha}}$ is not more than 4, and if $\alpha_1+\alpha_2+\alpha_3 \neq 2$ then the algebra $Q_{3,\vec{\alpha}}$ is commutative.

Let us prove that $Q_{3,\vec{\alpha}} \neq 0$ if and only if $\vec{\alpha} \in M$ and investigate the structure of algebras $Q_{3,\vec{\alpha}}$, where $\alpha \in M$.

Theorem 1. *Algebra $Q_{3,\vec{\alpha}} \neq 0$ if and only if $\vec{\alpha} \in M$ i.e. one of the conditions 1–9 is satisfied and then*

1) $\alpha_1+\alpha_2+\alpha_3=1$ *and the algebra $Q_{3,\vec{\alpha}}$ is one-dimensional algebra;*
2) $\alpha_i=1$, $\alpha_j \neq 1$, $\alpha_k \neq 1$, $\alpha_j+\alpha_k \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is one-dimensional algebra;*
3) $\alpha_i=\alpha_j=1$, $\alpha_k \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is commutative two-dimensional semisimple algebra;*
4) $\alpha_1=\alpha_2=\alpha_3=1$ *and the algebra $Q_{3,\vec{\alpha}}$ is a commutative three-dimensional semisimple algebra;*
5) $\alpha_j+a_k=1$, $\alpha_i+\alpha_j \neq 1$, $\alpha_i+\alpha_k \neq 1$, $\alpha_i \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is commutative two-dimensional semisimple algebra;*
6) $\alpha_i+\alpha_j=1$, $\alpha_i+\alpha_k=1$, $\alpha_j+\alpha_k \neq 1$ *and the algebra $Q_{3,\vec{\alpha}}$ is commutative two-dimensional semisimple algebra;*
7) $\alpha_i+\alpha_j=1$, $\alpha_i+\alpha_k=1$, $\alpha_j+\alpha_k=1$ *that is $\alpha_i=\alpha_2=\alpha_3=\frac{1}{2}$ and the algebra $Q_{3,\vec{\alpha}}$ is commutative three-dimensional semisimple algebra;*
8) $\alpha_1+\alpha_2+\alpha_3=2$ *and $\forall\ \alpha_i \neq 1$ and the algebra $Q_{3,\vec{\alpha}}$ is noncommutative four-dimensional semisimple algebra;*
9) $\alpha_i=1$, $\alpha_j+\alpha_k=1$ *and the algebra $Q_{3,\vec{\alpha}}$ is noncommutative four-dimensional nonsemisimple algebra, the radical of the algebra is generated by $q_jq_k-q_j, q_jq_k-q_k$.*

Proof. Consider only cases 8 and 9 when algebra is noncommutative.

In the case $\alpha_1+\alpha_2+\alpha_3=2$, where $\forall\ \alpha_l \neq 1$, the algebra $Q_{3,\vec{\alpha}}$ is generated by two idempotents q_1 and q_2, so all its irreducible representations are not more than two-dimensional. Since an idempotent in one-dimensional space is equal to 0 or 1 then it is easy to prove that there are no one-dimensional representations of the algebra. Consider an irreducible representation of algebra

$$\pi(q_1)=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi(q_2)=\begin{pmatrix} z & 1 \\ z-z^2 & 1-z \end{pmatrix}, \quad \pi(q_3)=\begin{pmatrix} \frac{1-\alpha_1-\alpha_2 z}{\alpha_3} & -\frac{\alpha_2}{\alpha_3} \\ \frac{\alpha_2 z^2-\alpha_2 z}{\alpha_3} & \frac{1+\alpha_2 z-\alpha_2}{\alpha_3} \end{pmatrix},$$

where $z=\frac{(1-\alpha_1)(1-\alpha_2)}{\alpha_2\alpha_1}$.

It is easy to check that from the equation $\pi(x_0e+x_1q_1+x_2q_2+x_3q_1q_2)=0$ follows $\forall\ l\colon\ x_l=0$. Then 1) since the dimension of the algebra is not more than 4 then elements e, q_1, q_2, q_1q_2 form a basis of the algebra; 2) the representation π is faithful irreducible two-dimensional representation of the algebra. That is why $Q_{3,\vec{\alpha}}$ is a noncommutative four-dimensional semisimple algebra.

Consider now $\alpha_i=1$, $\alpha_j+\alpha_k=1$. Take the representation π_1 of algebra:

$$\pi_1(q_i)=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \pi_1(q_j)=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \pi_1(q_k)=\begin{pmatrix} 0 & -\frac{\alpha_j}{\alpha_k} \\ 0 & 1 \end{pmatrix}.$$

Since π_1 has only one nontrivial invariant subspace then π_1 is not irreducible but indecomposed representation of algebra. A finite-dimensional semisimple algebra has no indecomposed representations which are not irreducible. Then in this case the algebra $Q_{3,\vec{\alpha}}$ is not semisimple.

If $\alpha_i = 1$, $\alpha_j+\alpha_k = 1$ then $q_i+\alpha_j q_j+\alpha_k q_k = e$. After squaring the equality $\alpha_j q_j+\alpha_k q_k = e-q_i$, we obtain

$$\alpha_j^2 q_j + \alpha_k^2 q_k + \alpha_j\alpha_k\{q_j, q_k\} = e - q_i = \alpha_j q_j + \alpha_k q_k.$$

So $(\alpha_j^2 - \alpha_j)q_j + (\alpha_k^2 - \alpha_k)q_k = -\alpha_j\alpha_k\{q_j, q_k\}$.

Since $\alpha_j + \alpha_k = 1$ then $a_j \neq 1$ and $\alpha_k \neq 1$ and $\alpha_j^2 - \alpha_j = -\alpha_j(1 - \alpha_j) = -\alpha_j\alpha_k = -(1-\alpha_k)\alpha_k = \alpha_k^2-\alpha_k \neq 0$, so $q_j+q_k = \{q_j, q_k\}$ and $q_kq_j = q_j+q_k-q_jq_k$. We prove that elements e, q_j, q_k, q_jq_k form a basis of the algebra. Suppose in a converse way that there exists some nontrivial linear combination of the elements that is equal to zero: $x_0e+x_1q_j+x_2q_k+x_3q_jq_k = 0$.

Consider the one-dimensional representation π_0 of the algebra: $\pi_0(q_j) = \pi_0(q_k) = 0$ then $0 = \pi_0(0) = \pi_0(x_0e + x_1q_j + x_2q_k + x_3q_jq_k) = x_0\pi_0(e) = x_0$ (hereinafter $x_0 = 0$).

Since $\pi_1(q_jq_k) = \pi_1(q_j)$ we have

$$\pi_1(x_0e + x_1q_j + x_2q_k + x_3q_jq_k) = \begin{pmatrix} 0 & x_1 - \frac{\alpha_j}{\alpha_k}x_2 + x_3 \\ 0 & x_1 + x_2 + x_3 \end{pmatrix} = 0_{2\times 2}. \tag{2}$$

Define the representation π_2:

$$\pi_2(q_j) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \pi_2(q_k) = \begin{pmatrix} 0 & 0 \\ -\frac{\alpha_j}{\alpha_k} & 1 \end{pmatrix}.$$

Then $\pi_2(q_jq_k) = \pi_2(q_k)$ and

$$\pi_2(x_0e + x_1q_j + x_2q_k + x_3q_jq_k) = \begin{pmatrix} 0 & 0 \\ x_1 - \frac{\alpha_j}{\alpha_k}(x_2 + x_3) & x_1 + x_2 + x_3 \end{pmatrix} = 0_{2\times 2}. \tag{3}$$

From (2) and (3) follows that

$$x_1 - \frac{\alpha_j}{\alpha_k}x_2 + x_3 = 0, \qquad x_1 + x_2 + x_3 = 0,$$
$$x_1 - \frac{\alpha_j}{\alpha_k}x_3 = 0, \qquad x_1 - x_3 = 0.$$

From here $\forall\, i$: $x_i = 0$. We have obtained the contradiction. So e, q_j, q_k, q_jq_k form the basis and the dimension of the algebra is equal to 4.

Since the algebra is generated by two idempotents q_j, q_k then dimensions of its irreducible representations are not more than 2. Since irreducible pair of idempotents is equivalent to the pair

$$Q_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q_k = \begin{pmatrix} z & 1 \\ z - z^2 & 1 - z \end{pmatrix},$$

for some $z \in \mathbb{C} \setminus \{0, 1\}$, then one can verify that the algebra has no two-dimensional irreducible representations. So all its irreducible representations are one-dimensional. The algebra has two one-dimensional representations: $\pi_3(q_j) = \pi_3(q_k) = 0$ and $\pi_4(q_j) = \pi_4(q_k) = 1$.

Let $x = x_0e + x_1q_j + x_2q_k + x_3q_jq_k$ belong to the radical of the algebra then $\pi_3(x) = 0$, $\pi_4(x) = 0$ or $x_0 = 0$, $x_1 + x_2 + x_3 = 0$. The system has a fundamental system of solutions $(0, 1, 0, -1)$, $(0, 0, 1, -1)$. So elements $q_j - q_jq_k$, $q_k - q_jq_k$ generate the radical of algebras. ■

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On the Algebra of Unharmonic Quantum Oscillator

Tatyana MAISTRENKO, Stanislav POPOVYCH and Yurii SAMOĬLENKO

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *maistrenko@imath.kiev.ua, stascon@i.com.ua, yurii_sam@imath.kiev.ua*

In present work we consider C^*-algebras $C^*(\mathcal{A}_f)$ associated with simple unimodal non-bijective dynamical system $(f, \mathbb{R})$ with special requirements. In the case when f is polynomial, $\mathcal{A}_f = \mathbb{C}\langle X, X^* |\ XX^* = f(X^*X)\rangle$ and $C^*(\mathcal{A}_f)$ is its enveloping C^*-algebra. As typical examples we consider one-parameter family $f_\mu(x) = \mu x(1-x)$ and two-parameter family called Unharmonic Quantum Oscillator $f_{p,q}(x) = 1 + px - qx^2$. The crossed product structure of $C^*(\mathcal{A}_f)$ is investigated. As a consequence we describe complete isomorphism invariant in terms of corresponding dynamical systems.

1 Introduction

C^*-algebras associated with dynamical systems arise naturally in pure mathematics as well as in applications to physics (see [1] and bibliography for more details) in particular to quantum optics (see [4]). For example Heisenberg algebra generated by operator X such that $XX^* - X^*X = \hbar I$ associated with linear dynamical system $x \to \hbar - x$ on $\mathbb{R}$, q-CCR algebra also associated with linear dynamics $x \to \hbar - qx$. More complicated dynamics appear in algebra of Quantum unit Disk (see [5]), more precisely this algebra is associated with one dimensional dynamical system $x \to \frac{(q+\mu)x+1-q-\mu}{\mu x+1-\mu}$ where μ and q are parameters of deformation. In this article we are concerned with non-linear deformation of q-CCR which we call algebra of Unharmonic Quantum Oscillator. It is given by generator X obeying the following relation $XX^* = \hbar + pX^*X - q(X^*X)^2$ where $q > 0,\ p > 0$.

The representation theory of C^*-algebras given by "dynamical relations" is extensively studied and well known (see [1]). Its connection with many concurrent approaches to associate C^*-algebra to a dynamical systems, for example groupoid approach and cross-product by partial actions of a group or semigroup (see [13, 12]) is very intriguing. In the paper we use recent work [3] to establish connection of algebra of unharmonic quantum oscillator with cross product like algebras.

2 Cross-product like structure of C^*-algebras associated with dynamical systems

Here we present some recent results on cross-product like structure of C^*-algebras associated with dynamical systems developed in [3] which are necessary for the last section of the paper.

Let $\mathcal{A}$ be some unital C^*-subalgebra of $B(H)$ and $U \in B(H)$ be a partial isometry such that the mapping $\mathcal{A} \ni a \mapsto UaU^*$ is an endomorphism of $\mathcal{A}$. If in addition pair $\mathcal{A}$ and U satisfies $Ua = UaU^*U$ and $U^*aU \in \mathcal{A}$ for all $a \in \mathcal{A}$ then $\mathcal{A}$ is called coefficient algebra for the C^*-subalgebra $\mathcal{B}$ generated by $\mathcal{A}$ and U.

Let us fix some notations: $d(x) = UxU^*$, $d_*(x) = U^*xU$. Then the condition that $\mathcal{A}$ is an algebra of coefficients for $\mathcal{B}$ will be reformulated in the following form

$$Ua = d(a)U^*, \quad a \in \mathcal{A}, \qquad d : \mathcal{A} \to \mathcal{A}, \qquad d_* : \mathcal{A} \to \mathcal{A}.$$

If $\mathcal{A}$ is an algebra of coefficients for $\mathcal{B}$ then $\mathcal{B}$ is a uniform closure of the finite combinations of the form

$$x = U^{*N} a_{\overline{N}} + \cdots + U^* a_{\overline{1}} + a_0 + a_1 U + \cdots + a_N U^N. \tag{1}$$

Where $a_{\overline{j}}$, $a_j \in \mathcal{A}$ and satisfy the following property for all k:

$$a_k U^k U^{*k} = a_k, \qquad a_{\overline{k}} U^k U^{*k} = a_{\overline{k}}.$$

In order to guarantee the very important property of uniqueness of representation in the form (1) one needs to impose the following $(*)$-property for all x of the form (1):

$$||a_0|| \le ||x||. \tag{$*$}$$

We will need the following central result from [3, 2.13]:

Theorem 1. *Let $\mathcal{A}_j$ be an algebra of coefficients for $\mathcal{B}_j$ generated by $\mathcal{A}_j$ and U_j where $j = 1, 2$. Assume that for both algebras property $(*)$ is satisfied. And assume that a mapping $\vartheta : \mathcal{A}_1 \to \mathcal{A}_2$ is an isomorphism such that $\vartheta \circ d_1 = d_2 \circ \vartheta$. Then the mapping $\Psi(x) = \vartheta(x)$ for $x \in \mathcal{A}_1$ and $\Psi(U_1) = U_2$ can be extended to isomorphism of C^*-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$.*

In order to construct an algebra of coefficients we need an additional piece of notations: if $X \subset \mathcal{B}$ then $E(X)$ will denote the C^*-algebra generated by $\{X, d(X), d^2(X), \ldots, d^n(x), \ldots\}$ and analogously $E_*(X)$ will denote the C^*-algebra generated by $\{X, d_*(X), d_*^2(X), \ldots, d_*^n(x), \ldots\}$. The following theorem (see [3, Theorem 3.11]) gives conditions for existence of algebra of coefficients:

Theorem 2. *Let $d : \mathcal{A}_0 \to B(H)$ is a morphism.*

1. *The following statements are equivalent:*
   a) There exists an algebra of coefficients $\mathcal{A} \supseteq \mathcal{A}_0$.
   b) $U^*U \in \cap_{n=0}^{\infty} d^n(\mathcal{A}_0)'$.
2. *If the above condition is satisfied then $E_*(E(\mathcal{A}_0))$ is the minimal algebra of coefficients containing $\mathcal{A}_0$ and d is an endomorphism of $E_*(E(\mathcal{A}_0))$. Moreover, each element $\beta \in E_*(E(\mathcal{A}_0))$ can be written as*

$$\beta = \alpha_0 + d_8(\alpha_1) + \cdots + d_*^N(\alpha_N).$$

$U^{*k}U^k$ and U^kU^{*k} are the decreasing sequences of commuting projections.

3 One-dimensional dynamical systems

For convenience of the reader we repeat the relevant material from [2, 1] without proofs, thus making our exposition self-contained. By the dynamical system we mean a continuous map $f : \mathbb{R} \to \mathbb{R}$ or $f : I \to I$, where $I \subset \mathbb{R}$ is a closed bounded interval. By the orbit of dynamical system $(f, \mathbb{R})$ we mean a sequence $\delta = (x_k)_{k \in P}$, where P is one of the sets $\mathbb{Z}$, $\mathbb{N}$ or $-\mathbb{N} = \{-1, -2, \ldots\}$ such that $f(x_k) = x_{k+1}$. But sometimes we will consider orbit as the set $\{x_k |\ k \in P\}$. The set of all orbits will be denoted by $\mathrm{Orb}\,(f)$. For $x \in \mathbb{R}$ denote by $\mathcal{O}_+(x)$ the forward orbit, i.e. $(f^k(x))_{k \ge 0}$. For every orbit $\delta \in \mathrm{Orb}\,(f)$ define $\omega(\delta)$ be the set of accumulation points of forward half-orbit and $\alpha(\delta)$ be the set of accumulation points of backward half-orbit.

By the positive orbit of a dynamical system $(f(), \mathbb{R})$ we mean a sequence $\omega = (x_k)_{k \in \mathbb{Z}}$ such that $f(x_k) = x_{k+1}$ and $x_k > 0$ for all integer k. Unilateral positive orbit is a sequence $\omega = (x_k)_{k \in \mathbb{N}}$ (Fock-orbit) such that $x_1 = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k > 1$ or $\omega = (x_{-k})_{k \in \mathbb{N}}$ (anti-Fock-orbit) such that $x_{-1} = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k < -1$. Define $\mathrm{Orb}_+(f)$

be the set of all positive non-cyclic orbits. Note that $\omega(\delta) = \varnothing$ for any anti-Fock orbit δ and $\alpha(\delta_1) = \varnothing$ for the Fock orbit δ_1.

Cycle $\beta = \{\beta_1, \ldots, \beta_m\}$ is called attractive if there is a neighborhood U of β such that $f(U) \subseteq U$ and $\cap_{i>0} f^i(U) = \beta$.

Point $x \in \mathbb{R}$ is called non-wandering if for every its neighborhood U there exists a positive integer m such that $f^m(U) \cap U \neq \varnothing$.

Since we will consider only bounded from above functions f and positive orbits we can always consider our dynamical system defined on a closed interval $[0, \sup f]$.

In this article we will deal with simple dynamical system, which possesses one of the equivalent properties listed in the following theorem (see [2, Theorem 3.14]):

Theorem 3. *Let $(f(), I)$ be continuous dynamical system, ($I \subset \mathbb{R}$ is closed bounded interval). The following conditions are equivalent:*

1. *For every $x \in I$ $\omega(x) = \omega(\mathcal{O}_+(x))$ is cycle.*
2. *$\mathrm{Per}\,(f)$ is closed.*
3. *Every non-wandering point is periodic.*

f is called partially monotone if I decomposes into a finite union of sub intervals, on which f is monotone.

For a simple dynamical system (f, I) for some positive integer m the relation $\mathrm{Fix}\left(f^{2^{m+1}}\right) = \mathrm{Fix}\left(f^{2^m}\right)$ holds (see [1]).

The class of such dynamical system is denoted by $\mathcal{F}_{2^m}$. Let us note that when $\mathrm{Per}\,(f)$ is closed [2, Theorem 3.12] implies that the length of every cycle is a power of 2 and They're no homoclinical orbits (i.e. orbit δ such that $\alpha(\delta) = \omega(\delta)$ is a cycle).

We will need the following lemma (see [10]):

Lemma 1. *Let $(f, \mathbb{R})$ be dynamical system with bounded from above f such that $(f(\cdot), [0, \sup f])$ is simple d.s. And let the set of periodic points which are not the points of attractive cycles, i.e. the set $[0, \sup f] \cap \mathrm{Per}\,(f) \setminus \cup_{\beta \text{ is attractive cycle}} \beta$ be finite then for every orbit $\delta \in \mathrm{Orb}_+(f)$ the α-boundary $\alpha(\delta)$ is cycle, which is not attractive.*

4 Simple unimodal mapping

Definition 1. Let $f \in C^0(I, I)$ where $I = [0, 1]$. Then f is called unimodal mapping if it satisfies the following conditions:

1. $f(0) = f(1) = 0$.
2. There is unique extreme point $c \in \mathrm{int}\, I$ and f is monotonously increasing on $[0, c]$ and is monotonously decreasing on $[c, 1]$.

Definition 2. Let dynamical system (f, I) be as in theorem 4 with minimal possible n.

1. Then $B_0 = s_0$ and $B_1 = s_1$ are two one-dimensional cycles. $B_{2^k} = \{\beta_1, \beta_2, \ldots, \beta_{2^k}\}$ denote the unique cycle of period 2^k where $\beta_i < \beta_j$ whereas $i < j$. $B_{2^k} = B^-_{2^k} \dot\cup B^+_{2^k}$ where $B^-_{2^k} = \{\beta_1, \ldots, \beta_{2^{k-1}}\}$ and $B^-_{2^k} = \{\beta_{2^{k-1}}, \ldots, \beta_{2^k}\}$. Denote by $B_{2^k}(f^2)$ the cycle of period 2^k of dynamical system (f^2, I_2).
2. We will say that orbit $\delta = (x_k)_{k \in \mathbb{Z}}$ is glued to point β_i of cycle B_{2^k} if there exists integer k_0 such that $x_{k_0} = \beta_i$ and $x_k \notin B_{2^k}$ for all $k < k_0$. An orbit is glued to cycle if it is glued to some point of this cycle.
3. We will say that an orbit is degenerate if it is glued to a cycle of period less then 2^n.

Definition 3. Let $\beta_i \in B_{2^m}$ and denote $D^{\beta_i}_{B_{2^k}} = \{\delta \in P_{B_{2^k}} | \delta \text{ is glued to } \beta_i\}$. Denote by $D^{\beta_i}_{B_{2^k}}(f^2)$ the set $D^{\beta_j}_{B_{2^{k-1}}(f^2)}$ where $j = i - 2^{m-1}$ and $\beta_j \in B_{2^{k-1}}(f^2)$.

In the following theorem from [11] an analog of measurable section for dynamical system has been constructed.

Theorem 4. *Let (f, I) be $\mathcal{F}_{2^n}$ dynamical system with unimodal mapping f, which has only two fix points $s_0 = 0$, $0 < s_1 < 1$, and assume that for every $m \leq n$ there is only one cycle of period 2^m which is repellent for $m < n$ and attractive for $m = n$. Define $P_B = \{\delta \,|\, \delta \in \mathrm{Orb}_+(f), \alpha(\delta) = B\}$ for every cycle B of period $m < n$. Then*

1. *$\mathrm{Orb}_+(f) = \dot\cup_B P_B$, where union is taken over all repellent cycles.*
2. *for each B there is $I_B = [t_1, t_2)$ and one-to-one mapping $\phi : I_B \to P_B$ such that $t \in \phi(t)$ for every $t \in I_B$. Moreover I_B can be chosen to lie in arbitrary neighborhood of B.*
3. *$I_{B_1} \cap I_{B_2} = \varnothing$ for $B_1 \neq B_2$.*

Corollary 1. *Mapping $\cup_\beta I_\beta \ni t \to \delta_t$ constructed in the proof is bijective correspondence between n-copies of $[0, 1)$ and the set of non-cyclic positive orbits.*

5 Enveloping C^*-algebra

By $C^*(A_f)$ we mean a C^*-algebra obtained from free $*$-algebra $\mathcal{F}(X, X^*)$ generated by X with sub-norm $\|b\| = \sup_\pi \|\pi(b)\|$ where supremum is taken over all $\pi \in \mathrm{Rep}\,(\mathcal{F}(X, X^*))$ such that $\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure.

As shown in (see [1]) there is a bijective correspondence between representations of C^*-algebras

$$A = C^*\langle X, X^* \,|\, XX^* = f(X^*X)\rangle$$

with certain orbits of dynamical systems $(f, \mathbb{R}_+)$. In particular, if f partially monotone continuous map and $(f, \mathbb{R})$ is $\mathcal{F}_{2^m}$ dynamical system. Then every positive non-cyclic orbit $\omega(x_k)_{k\in Z}$ corresponds to an irreducible representation π_ω in Hilbert space $l_2(Z)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k \in Z$ and $X = UC$ is a polar decomposition. For the Fock and anti-Fock representations the similar formulae hold with the exception that space is $l_2(N)(l_2(-N))$ and $Ue_1 = 0$ for the Fock representation. To cyclic positive orbit $\omega = (x_k)_{k\in N}$ of length m there corresponds a family of m-dimensional irreducible representation $\pi_{\omega,\phi}$ in Hilbert space $l_2(\{1, \ldots, m\})$ given by the formulae: $Ue_0 = e^{i\phi}e_{m-1}$, $Ue_k = e_{k-1}$ $Ce_k = \sqrt{x_k}e_k$ for $k = 1, \ldots, m$; $0 \leq \phi \leq 2\pi$ and $X = UC$

Let f be bounded from above Hermitian polynomial (hence f is always partially monotone and continuous). Let $A_f = \mathbb{C}\langle X, X^* \,|\, XX^* = f(X^*X)\rangle$ be $*$-algebra given by generators and relations which has at least one representation. Let $C = \sup f$. Then for any representation π of $*$-algebra A_f we have $\|X\| \leq \sqrt{C}$. Thus there is (exists) enveloping C^*-algebra, which we denote by $C^*(A_f)$. Let us note that by Theorem 3.3 [2] for $f \in C^1(I, I)$ simplicity of dynamical system is equivalent to $(f, I) \in \mathcal{F}_{2^m}$ for some integer m.

6 Description of the dual space of $C^*(\mathcal{A}_f)$

Let A be C^*-algebra by its spectrum (sometimes called dual space), denoted by $\hat A$ we understand the set of unitary equivalence classes in the set $\mathrm{Irr}\,(A)$ of irreducible representations of A with the Jacobson topology (see [7, Chapter 3] about several equivalent definitions). The closure of the set $S \subseteq \hat A$ is $[S] = \{\pi \in \hat A \,|\, \mathrm{Ker}\,\pi \supset \cap_{\rho\in S}\mathrm{Ker}\,\rho\}$ or equivalently $[S] = \{\pi \in \hat A \,|$ for all $y \in A$ $\|\pi(y)\| \leq \sup_{\rho\in S} \|\rho(y)\|\}$ obviously it is enough to verify last inequality only for elements of a dense subspace of A.

In the following and consequent theorems, in case f is not a polynomial, by $C^*(A_f)$ we mean a C^*-algebra obtained from free $*$-algebra $\mathcal{F}(X, X^*)$ generated by X with prenorm $\|b\| =$

$\sup_\pi \|\pi(b)\|$ where supremum is taken over all $\pi \in \mathrm{Rep}\,(\mathcal{F}(X, X^*))$ such that $\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure. This C^*-algebra has obvious universal properties similar to those in case of polynomial map f. Theorem 4 describes the set $\mathrm{Orb}_+(f)$ but in order to describe spectrum of $C^*(\mathcal{A}_f)$ we need finer description. The reason is that some orbit $\delta \in P_\beta$ may be eventually periodic, hence $\omega(\delta)$ could be a cycle of length 2^m for $m < n$ and so $C^*(\pi_\delta)$ would not be isomorphic to $C^*(\pi_\gamma)$ for non-eventually periodic ('generic') orbit γ.

Let us give some definitions.

Definition 4. 1. Let $\delta = (x_k)_{k\in\mathbb{Z}} \in P_{B_{2^k}}$ where $k \geq 0$ be such that $x_0 \in I_2$ then $r(\delta) = (x_{2k})_{k\in\mathbb{Z}}$ is an orbit of (f^2, I_2). If $\delta = (y_k)_{k\in\mathbb{Z}}$ is in $\mathrm{Orb}_+(f^2, I_2)$ then $r^{-1}(\delta)$ where $(r^{-1}(\delta))_{2k} = y_k$, $(r^{-1}(\delta))_{2k+1} = f(y_k)$ is an orbit of (f, I), moreover r^{-1} is inverse to r.

2. Let $x \in [0, M]$ define $\mu_-(x) = (y_k)_{k\in\mathbb{Z}}$ be in P_{B_0} where $y_k = f^k(x)$ for $k \geq 0$ and $y_{-k} = f_-^{-1}(y_{-k+1})$ for $k > 0$. If $x \in [s_1, M]$ define $\mu_+(x) = (y_k)_{k\in\mathbb{Z}}$ be in P_{B_0} where $y_k = f^k(x)$ for $k \geq 0$ and $y_{-1} = f_+^{-1}(x)$ and $y_{-k} = f_-^{-1}(y_{-k+1})$ for $k > 1$.

Denote $R_{B_{2^k}} = P_{B_{2^k}} \setminus \cup_{k<m<n} D_{B_{2^k}}^{B_{2^m}}$.

Let H be Hilbert space with orthonormal basis $(e_k)_{k\in\mathbb{Z}}$. Let U be unitary operator defined by $Ue_k = e_{k+1}$. For every orbit $\delta = (x_k)_{k\in\mathbb{Z}} \in Orb_+(f)$ there is repellent cycle B such that $\delta \in P_B$ further on we will always assume that $x_0 \in I_B$. Let us define operator C_δ via the rule $C_\delta e_k = x_k e_k$. Let Z denote the set of non-periodic orbit. Define $(\Psi(X))(\delta) = U\sqrt{C_\delta}$ and extend it to $C^*(\mathcal{A}_f)$. We have presentation $\Psi : C^*(\mathcal{A}_f) \to B(H)^Z$ of elements of enveloping algebra as a operator-valid functions on Z. Later on we will see that if Z endowed with topology induced from dual space, $\hat{C}^*(\mathcal{A}_f)$, and R is a subspace of non-degenerate orbits then for all $y \in C^*(\mathcal{A}_f)$ $\Psi(y)$ is continuous on R in norm topology on $B(H)$ and continuous on Z in strong topology.

In the following theorem we denote by $[X]$ the closure of X in the topology of $\hat{C}^*(\mathcal{A}_f)$ where subset $X \subset \mathrm{Orb}_+(f)$ is identified with the corresponding set of irreducible representations. If $Y \subset \mathbb{R}$ then $\overline{Y}$ denote closure in topology of $\mathbb{R}$. The set of cyclic orbits is $\mathrm{Per}(f)/_\sim$ where $x \sim y$ iff x and y belong to the same orbit. The following theorem from [11] gives the complete description of the dual space.

Theorem 5. *Let dynamical system (f, I) be as in Theorem 4 with minimal possible n. The dual space (spectrum) of $C^*(\mathcal{A}_f)$ is homeomorphic to $\mathrm{Orb}_+(f) \sqcup_\theta (\mathrm{Per}\,(f)/_\sim \times \mathbb{T})$ where $\theta : \mathrm{Per}\,(f)/_\sim \times \mathbb{T} \to \mathrm{Orb}_+(f)$ via the rule $\theta((x, \phi))$ is the cyclic orbit containing x. Topology on $\mathrm{Orb}_+(f)$ is given by the following family of closed sets $\{\overline{\Sigma} = \{\gamma \in \mathrm{Orb}_+(f) \,|\, \gamma \in \overline{\cup_{\delta\in\Sigma}\delta}\};\ \Sigma \subseteq \mathrm{Orb}_+(f)\}$ and the space $\mathrm{Orb}_+(f) \bigsqcup_\theta (\mathrm{Per}\,(f)/_\sim \times \mathbb{T})$ is factor-set of disjoint union $\mathrm{Orb}_+(f) \bigsqcup (\mathrm{Per}\,(f)/_\sim \times \mathbb{T})$ with equivalence relation which identifies cyclic orbit containing x with $(x, 1) \in \mathrm{Per}\,(f)/_\sim \times \mathbb{T}$ where topology is given by the following family of closed sets: S is closed in $\mathrm{Per}\,(f)/_\sim \times \mathbb{T}$ and $\overline{\Sigma} \cup \theta^{-1}(\overline{\Sigma})$ where $\Sigma \subset \mathrm{Orb}_+(f)$. Moreover it is determined up to homeomorphism by integer n.*

Corollary 2. *Representation $\oplus_{\delta\in R_{B_0}} \pi_\delta$ of $C^*(\mathcal{A}_f)$ is faithful. Let $\delta \in R_{B_0}$. Then $X = \overline{\{x | x \in \delta\}}$ is a compact subspace in $\mathbb{R}$ and R_{B_0} is locally compact space. $C^*(\mathcal{A}_f)$ is a C^*-subalgebra in the C^*-algebra $C(R_{B_0}, \mathbb{Z} \times_\psi C(X))$ of all continuous mappings from R_{B_0} to cross-product C^*-algebra $\mathbb{Z} \times_\psi C(X)$.*

Note that in case $n = 0$ there systems that are not orbit-equivalent but as we saw they have isomorphic enveloping C^*-algebras.

Before considering examples let us make some remarks

Remark 1. Then class of $\mathcal{F}_{2^n}$ unimodal dynamical systems with negative Swartzian, i.e. $\mathcal{F}_{2^n} \cap SU$ gives examples of d.s. satisfying conditions imposed in this section.

Now let us consider some examples. Consider the family of quadratic maps $f_\mu(x) = \mu x(1-x)$ where $\mu > 0$. It is known that $f_\mu \in SU$. Then for $\mu < \mu^* \approx 3.57$ dynamical system $(f_\mu, \mathbb{R})$ is $\mathcal{F}_{2^n}$ for some n. Denote by $\mu_n > 0$ the greatest value for which $(f_\mu, \mathbb{R})$ is $\mathcal{F}_{2^n}$. Then for $0 < \mu \le \mu_1$ dynamical system $(f_\mu, \mathbb{R})$ has two stable points and has no other cycles, for $\mu_1 < \mu \le \mu_2$ dynamical system $(f_\mu, \mathbb{R})$ has two stable points and one cycle of period 2, for $\mu_n < \mu \le \mu_{n+1}$ dynamical system $(f_\mu, \mathbb{R})$ has two stable points and one repellent cycle of period 2^m for $m < n+1$ one attractive cycle of period 2^{n+1} and no other cycles. The following proposition is a consequence of more general result from [14].

Proposition 1. *There is $\nu_{n+1} : \mu_n < \nu_{n+1} \le \mu_{n+1}$ such that dynamical systems (f_{μ_1}, I) and (f_{μ_2}, I) are conjugate iff μ_1 and μ_2 belongs to the same set (μ_n, ν_{n+1}) or $\{\nu_{n+1}\}$ or $(\nu_{n+1}, \mu_{n+1}]$.*

7 Cross-product structure of $C^*(\mathcal{A}_f)$

Let X be a compact topological Hausdorff space and (f, X) a continuous dynamical system. Let $\Xi = \{\delta \subseteq X | \delta - \text{minimal invariant subset such that for all } x \in \delta \text{ exists } y \in \delta : \ f(y) = x\}$. For each non eventually periodic orbit $\delta \in \Xi$ for all $x \in \delta$ exists unique $y \in \delta$ such that $f(y) = x$. Denote such y by $f_\delta^{(-1)}(x)$. And for arbitrary integer $l > 0$ define $f_\delta^{(-l)}(x)$ recursively as $f_\delta^{(-l)}(x) = f_\delta^{(-1)}\big(f_\delta^{(-l+1)}(x)\big)$. If δ is eventually periodic then $f_\delta^{(-1)}(x)$ is defined as such an $y \in \delta$ that $f(y) = x$ and y is non-cyclic. Define topological space $\Omega = \{(\delta, y) \,|\, \delta \in \Xi,\ y \in \delta\}$. The family of sets of the form $B_{n,U}(\delta, y) = \{(\tau, z) \in \Omega \,|\, f_\tau^n(z) \in U\}$ where $n \in \mathbb{Z}$, U is an open neighborhood of $f_\delta^n(y)$ constitute a basis of open neighborhoods of $(\delta, y) \in \Omega$.

In terms of convergent sequences the topology on Ω could be defined as follows (δ_k, y_k) converges to (δ, y) if and only if for any integer $n \in \mathbb{Z}$

$$\lim_{k \to +\infty} f_{\delta_k}^{(n)}(y_k) = f_\delta^{(n)}(y).$$

Proposition 2. *Ω is a compact Hausdorff space. And mapping $\sigma : \Omega \to \Omega$, $\sigma((\delta, y)) = (\delta, f(y))$ is a homeomorphism of Ω.*

Theorem 6. *Let (f, I) be $\mathcal{F}_{2^n}$ and (g, I) be $\mathcal{F}_{2^k}$ dynamical systems with unimodal mapping f, which has only two fixed points $s_0 = 0$, $0 < s_1 < 1$, and assume that for every $m \le n$ ($m \le k$ for g correspondingly) there is only one cycle of period 2^m which is repellent for $m < n$ ($m < k$ for g correspondingly) and attractive for $m = n$ ($m = k$ for g correspondingly). Then $\Omega(f)$ is homeomorphic to $\Omega(g)$ if and only if $m = n$. In this case dynamical systems $(\Omega(f), \sigma)$ and $(\Omega(g), \sigma)$ are conjugated.*

Proof. We give an internal description of Ω from which this theorem follows. Consider in more detail the case $n = 1$. The general case could be dealt with by induction in n which is a common situation for $\mathcal{F}_{2^n}$ dynamical systems (see [11]). In this case $\Xi = I_0 \dot\cup I_1 \dot\cup \{0, s_1, \beta_1, \beta_2\}$ where 0, s_1 are fixed points and $B_2 = \{\beta_1, \beta_2\}$ is a cycle of period 2. (I_0 and I_1 are I_{B_0} and I_{B_1} correspondingly in our previous notations). Then Ω could be identified with $(I_0 \times \mathbb{Z}) \dot\cup (I_1 \times \mathbb{Z}) \dot\cup \{0, s_1, \beta_1, \beta_2\}$. We use the homeomorphism $\Phi : I_B \to P_B$ to identify the subspace of orbits with α-boundary B with semi-interval I_B. Then point $(x, m) \in I_B \times \mathbb{Z}$ is identified with $(\Phi(x), f_{\Phi(x)}^{(m)}(x)) \in \Omega$. Periodic points 0, s_1, β_1, β_2 correspond to $(\{0\}, 0)$, $(\{0\}, 0)$, (B_2, β_1), (B_2, β_2) respectively. In order to distinguish point $(x, m) \in I_0 \times \mathbb{Z}$ from the point $(x, m) \in I_1 \times \mathbb{Z}$ we will write subscript $(x, m)_0$ and $(x, m)_1$ respectively. From the definition of the topology on Ω one can easily obtain the following *homogeneity* condition: sequence (x_k, m_k) converges to (x, m) if and only if $(x_k, m_k - m)$ converge to $(x, 0)$. We will need some notation, definitions and facts from the proof of Theorem 3 [11]. Let $I_0 = (a, b]$ and $I_1 = (t_1, t_2]$. Then there is

a family of semi-intervals $J^k_{B_0}$ where $k \in \mathbb{Z}$ such that $\dot\cup_{k\geq 1} J^k_{B_0} = [b_1, b)$ and $\dot\cup_{k\leq 1} J^k_{B_0} = (a, a_1]$ for some $a_1 < b_1$ and such that $f^{2k} : J^k_{B_0} \to I_1$ is a homeomorphism for each $k \in \mathbb{Z}$. Denote $J^\infty_{B_0} = (a_1, b_1)$. Further on we will identify $J^k_{B_0}$ with $(0, 1]$ by means of a homeomorphism $\phi_k : J^k_{B_0} \to (0, 1]$ and $\phi_\infty : J^\infty_{B_0} \to (0, 1)$. Thus each point $(x, m)_0$ will be coded by a triple $(y, k, m) \in (0, 1] \times (\mathbb{Z} \cup \{\infty\}) \times \mathbb{Z}$ where $x \in J^k_{B_0}$ and $y = \phi_k(x)$. Then using nice descriptions of $\mathcal{F}_{2^n}$ dynamics (see Theorems 1, 2, 3 [11]) the convergent sequences in Ω could be completely described, for example $(x_k, m)_1$ converges to $(x, t)_1$ if and only if x_k converge to $x < 1$ and $m = t$ or x_k converge to 1 and $x = 0$, $t = m + 1$; $(x_m, k, 2 + 2k) \to (x, 0)_1$ if $k \to \infty$, $x_m \to x$; $(x, 2m)_i \to (B_2, \beta_2)$ whenever $m \to +\infty$ and $i = 0, 1$; $(x, m)_0 \to (B_0, 0)$ and $(x, m)_1 \to (B_1, s_1)$ whenever $m \to -\infty$. This description of Ω does not depend on concrete function f but only on the integer n. The rest of the claims are consequences of this fact. ■

Let us proceed with the description of the $C^*(A_f)$. In the case if f is unimodal $C^*(A_f)$ is generated by unitary U and Hermitian C such that $UC^2U^* = f(C^2)$. Let $\mathcal{A}_0$ be unital C^*-algebra generated by C. Obviously, $\mathcal{A}_0 \subset C^*(A_f)$ and letter is generated by $\mathcal{A}_0$ and U. Since each irreducible representation π of $C^*(A_f)$ is associated to some orbit $\delta \in \mathrm{Orb}\,(f)$ and $\pi(C^2)$ is the diagonal operator with points of δ on its main diagonal the universal representation π_u of $C^*(A_f)$ acts on $H = \oplus_{\delta\in \mathrm{Orb}\,(f)} l_2(\mathbb{Z})$ and $\pi_u(C^2)$ is the multiplication operator $\pi_u(C^2)\xi(\delta) = \delta\xi(\delta)$ for $\xi(\delta) \in l_2(\mathbb{Z})$ and $\pi_u(U)\xi(\delta) = \xi(f(\delta))$, $\pi_u(U^*)\xi(\delta) = \xi(f^{(-1)}_\delta(\delta))$. Then $\mathcal{A} = E_*(E(\mathcal{A}_0))$ is a commutative algebra of diagonal operators generated by $\pi_u(C^2)$ and $\pi_u(U)$ and thus is isomorphic to some $C(Y)$ where Y is a space of multiplicative linear functionals on $\mathcal{A}$. From the above description of $\mathcal{A}$ follows that a multiplicative linear functional $\rho : \mathcal{A} \to \mathbb{C}$ is of the form $\rho(g) = (g(\delta))_k$ where $g \in \mathcal{A}$ (considered as a diagonal operator in H), $\delta \in \mathrm{Orb}_+(f)$ and $k \in \mathbb{Z}$. Hence $\hat{\mathcal{A}}$ can be identified with Ω. One can check that the weak topology on $\hat{\mathcal{A}}$ coincides with the topology on Ω under this identification. Thus $\mathcal{A}$ is an algebra of coefficients for $C^*(A_f)$. Consider an element

$$x = U^{*N} a_{\overline{N}} + \cdots + U^* a_{\overline{1}} + a_0 + a_1 U + \cdots + a_N U^N.$$

Since $\langle x e_k(\delta), e_k(\delta)\rangle = \langle a_0 e_k(\delta), e_k(\delta)\rangle$ we get that $||x|| \leq ||a_0||$. Thus property $(*)$ is satisfied also. Applying theorem (isomorphism) we get the following

Theorem 7. *Let (f, I) be $\mathcal{F}_{2^n}$ and (g, I) be $\mathcal{F}_{2^k}$ dynamical systems with unimodal mapping f, which has only two fixed points $s_0 = 0$, $0 < s_1 < 1$, and assume that for every $m \leq n$ ($m \leq k$ for g correspondingly) there is only one cycle of period 2^m which is repellent for $m < n$ ($m < k$ for g correspondingly) and attractive for $m = n$ ($m = k$ for g correspondingly). (In particular, if $f \in \mathcal{F}_{2^n} \cap U$ and $g \in \mathcal{F}_{2^k} \cap U$). Then*

1. *$C^*(A_f) \cong C(\Omega) \times_\sigma \mathbb{Z}$.*
2. *$C^*(A_f) \cong C^*(A_g)$ if and only if $n = k$.*

Let $f_{p,q}(x) = 1 + px - qx^2$ with $\{p, q\} \subset \mathbb{R}$ and $q > 0$ to provide boundedness. Since when $p < 0$ dynamical system is one-to-one on $\mathbb{R}_+$ (and so all irreducible representations are one-dimensional) we assume that $p > 0$. This dynamical system is conjugated to $f_\mu(x) = \mu x(1 - x)$ where $\mu = 1 + \sqrt{p^2 - 2p + 1 + 4q}$. The values of parameter μ when bifurcations of cycles of one parametric family $\{f_\mu\}$ occurs are given in [2]. However, conjugacy relation does not preserve positiveness, i.e. $\mathrm{Orb}_+(f_{p,q})$ may not map into $\mathrm{Orb}_+(f_\mu)$. This two-parameter deformation unlike previously considered f_μ give rise to Fock and anti-Fock representations. If (p, q) belong to domain $D = \left\{(p, q) \,|\, q < \frac{1}{2} - \frac{p^2}{4} + \frac{p}{2} + \frac{\sqrt{1+2p}}{2}\right\}$ then for every $x \in [0; \sup f_{p,q}]$ $\mathcal{O}_+(x) \subset [0; \sup f_{p,q}]$. Thus for such (p, q) algebra $C^*(\mathcal{A}_{f_{p,q}})$ has Fock representation and as it easily can be shown has no anti-Fock representations. In the complement of D algebra $C^*(\mathcal{A}_{f_{p,q}})$ has anti-Fock representations. In present paper we consider only the case when there is no

anti-Fock representations. Let (p,q) be such that $\mu_{n-1} < 1+\sqrt{p^2-2p+1+4q} \le \mu_n$ and $(p,q) \in D = \left\{(p,q) \,|\, q < \frac{1}{2} - \frac{p^2}{4} + \frac{p}{2} + \frac{\sqrt{1+2p}}{2}\right\}$ then $f_{p,q} \in \mathcal{F}_{2^n}$. Using results cited in Section 1 and above theorem one can prove the following

Theorem 8. *Let $(p,q) \in D$ and $\mu_{n-1} < 1+\sqrt{p^2-2p+1+4q} \le \mu_n$. Let $C^*(\mathcal{A}_{f_{p,q}})'$ is a C^*-algebra generated by elements of polar decomposition of operator X in universal representation of $C^*(\mathcal{A}_{f_{p,q}})$ that is by partial isometry U and positive operator C such that $X = UC$. Then U is an isometry and $C^*(\mathcal{A}_{f_{p,q}})'$ has an algebra of coefficients extending $C^*(C)$. The isomorphism class of $C^*(\mathcal{A}_{f_{p,q}})'$ does not depend on (p,q).*

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On $*$-Representations of Algebras of Temperley–Lieb Type and Algebras Generated by Linearly Dependent Generators with Given Spectra

Anton S. MELLIT, Yurii S. SAMOĬLENKO and Maxim V. ZAVODOVSKY

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *mellit@imath.kiev.ua, yurii_sam@imath.kiev.ua, mzv@imath.kiev.ua*

Algebras of Temperley–Lieb type and algebras generated by linearly dependent generators with a given spectra are presented in this paper. We consider their $*$-representations and sets of parameters, for which $*$-representations exist. Examples of algebras are considered.

1 Introduction

Let $\{A_k\}_{k=1}^n$ be a set of linear operators in separable complex Hilbert space H with scalar sum $\sum\limits_{k=1}^{n} A_k = \lambda I_H$ and a restriction that spectrum of each A_k belongs to a certain finite set $M_k \subset \mathbb{C}$. Such sets of operators play an important role in analysis, algebraic geometry, operator theory and mathematical physics [1–3].

Algebras, generated by linearly connected generators with given spectra were introduced and studied in [5–10]. Particularly, results on their growth, existence of polynomial identities, representations etc. where obtained. Following scheme presented in [11, 12]. In Sections 2, 3 we introduce connections of these algebras with Temperley–Lieb algebras. We study their representations and sets of parameters for which representations exist. In Sections 4–6 we consider examples of algebras including algebras connected with extended Dynkin diagrams.

2 Homomorphisms of algebras $\mathcal{P}_{M_1,\dots,M_n;\lambda}$ and $\mathcal{P}_{N_1,\dots,N_n,abo}$

Consider $*$-algebra, generated by n self-adjoint elements a_i for $i = 1, \dots, n$ which satisfy corresponding restrictions on spectra $\sigma(a_i) \subseteq \{x_0^{(i)}, \dots, x_{m_i}^{(i)}\} = M_i$ for given finite sets of real numbers $x_0^{(i)} < \dots < x_{m_i}^{(i)}$ and relation $\sum\limits_{i=1}^{n} a_i = \lambda e$, $\lambda \in \mathbb{R}$.

Without loss of generality we shall think, that $x_k^{(i)} > 0$ and $x_0^{(i)} = 0$ where $i = 1, \dots, n$, $k = 1, \dots, m_i$. We consider $\lambda > 0$ (for if $\lambda < 0$ representations of such algebra do not exist and if $\lambda = 0$ there are only trivial ones).

Each element a_i can be presented in a form $a_i = \sum\limits_{k=1}^{m_i} x_k^{(i)} p_k^{(i)}$, where $p_k^{(i)}$-projection, $p_k^{(i)} p_l^{(i)} = 0$, $k \neq l$, $k, l = 1, \dots, m_i$.

In [5] the following algebra generated by projections was introduced:

$$\mathcal{P}_{M_1,\dots,M_n;\lambda} = \mathbb{C}\langle p_1^{(1)}, \dots, p_{m_1}^{(1)}, p_1^{(2)}, \dots, p_{m_2}^{(2)}, \dots, p_1^{(n)}, \dots, p_{m_n}^{(n)} \mid \sum_{i=1}^{n} \sum_{k=1}^{m_i} x_k^{(i)} p_k^{(i)} = \lambda e,$$

$$p_k^{(i)} p_l^{(i)} = 0 \ (l \neq k),\ p_k^{(i)2} = p_k^{(i)},\ l, k = 1, \dots, m_i,\ i = 1, \dots, n\rangle.$$

With every such algebra we associate a graph (or diagram) $G = G(\mathcal{P}_{M_1,\dots,M_n;\lambda})$ which consists of one root vertex and n branches, i-th branch is a sequence of $\mathrm{Card}\,(M_i) - 1$ connected vertices.

We mark vertices of i-th branch with nonzero real numbers of set M_i starting from root in descending order. Also we mark root vertex with λ.

Define $N_j = \frac{1}{\lambda}M_j = \left\{0 < \frac{x_1^{(j)}}{\lambda} < \cdots < \frac{x_{m_j}^{(j)}}{\lambda}\right\}$, $j = 1, \ldots, n$. Consider algebra $\mathcal{P}_{N_1,\ldots,N_n,abo}$ generated by projections p and $q_k^{(i)}$, where $k = 1, \ldots, m_i$, $i = 1, \ldots, n$

$$\mathcal{P}_{N_1,\ldots,N_n,abo} = \mathbb{C}\langle q_1^{(1)}, \ldots, q_{m_1}^{(1)}, q_1^{(2)}, \ldots, q_{m_2}^{(2)}, \ldots, q_1^{(n)}, \ldots, q_{m_n}^{(n)}, p \mid \sum_{i=1}^{n}\sum_{k=1}^{m_i} q_k^{(i)} = e,$$

$$q_k^{(i)} p q_k^{(i)} = \frac{x_k^{(i)}}{\lambda} q_k^{(i)},\ q_k^{(i)} p q_l^{(i)} = 0\ (l \neq k),\ p^2 = p,\ q_k^{(i)2} = q_k^{(i)},$$

$$q_k^{(i)} q_s^{(j)} = 0\ ((i,k) \neq ((j,s)),\ k, l = 1, \ldots, m_i,\ s = 1, \ldots, m_j,\ i, j = 1, \ldots, n\rangle.$$

These algebras are called *abo*-analogs ("*all but one*").

Proposition 1. *There exists a homomorphism of algebras*

$$\varphi_1 : \mathcal{P}_{M_1,\ldots,M_n;\lambda} \longrightarrow p\mathcal{P}_{N_1,\ldots,N_n,abo}p,$$

which is defined on generators in the following way

$$\varphi_1\big(p_k^{(i)}\big) = \frac{\lambda}{x_k^{(i)}} p q_k^{(i)} p.$$

Assume $\mathrm{q} = \operatorname{diag}\big(p_1^{(1)}, \ldots, p_{m_1}^{(1)}, \ldots, p_1^{(n)}, \ldots, p_{m_n}^{(n)}\big) \in \mathrm{M}_m(\mathcal{P}_{M_1,\ldots,M_n;\lambda})$, $m = \sum_{i=1}^{n} m_i$ and $e_{i,j}$ is matrix unity in $\mathrm{M}_m(\mathcal{P}_{M_1,\ldots,M_n;\lambda})$.

Proposition 2. *There exists a homomorphism of algebras*

$$\varphi_2 : \mathcal{P}_{N_1,\ldots,N_n,abo} \longrightarrow q\mathrm{M}_m(\mathcal{P}_{M_1,\ldots,M_n;\lambda})q,$$

which is defined on generators in the following way

$$\varphi_2\big(q_k^{(i)}\big) = p_k^{(i)} \otimes \mathrm{e}_{k+t_i,k+t_i},$$

$$\varphi_2(p) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m_i}\sum_{l=1}^{m_j} \frac{\sqrt{x_k^{(i)} x_l^{(j)}}}{\lambda} p_k^{(i)} p_l^{(j)} \otimes \mathrm{e}_{k+t_i,l+t_j},$$

where $t_i = \sum_{j<i} m_j$.

Remark 1. If we introduce involution "$*$" on algebras $\mathcal{P}_{M_1,\ldots,M_n;\lambda}$ and $\mathcal{P}_{N_1,\ldots,N_n,abo}$ in the obvious way (by assuming all generators to be self-adjoint), then homomorphisms φ_1 and φ_2 appear to be $*$-homomorphisms of $*$-algebras.

3 Equivalence of categories $\operatorname{Rep}\mathcal{P}_{M_1,\ldots,M_n;\lambda}$ and $\operatorname{Rep}\mathcal{P}_{N_1,\ldots,N_n,abo}$

With the help of homomorphisms φ_1 and φ_2 it is possible to build corresponding functors on categories of representations [11].

Given a homomorphism of algebras $\varphi : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$, there exists a functor $F : \operatorname{Rep}\mathcal{A}_2 \longrightarrow \operatorname{Rep}\mathcal{A}_1$ defined by rules $F(\pi) = \pi \circ \varphi$ and $F(K) = K$, where $\pi \in \operatorname{Ob}(\operatorname{Rep}\mathcal{A}_2)$ and K is a morphism in the category $\operatorname{Rep}\mathcal{A}_2$.

Assume $\mathcal{A}$ to be an algebra and q to be an idempotent in $\mathcal{A}$. Then $\mathcal{B} = q\mathcal{A}q$ is algebra with unit q. For $\pi \in \operatorname{Rep}\mathcal{A}$ we can define representation $\widehat{\pi} \in \operatorname{Rep}\mathcal{B}$ in space $\operatorname{Im}\pi(q)$ by $\widehat{\pi}(x) = \pi(x)\,|_{\operatorname{Im}\pi(q)}$ for all $x \in \mathcal{B}$. If K is the intertwining operator between representations π_1 and π_2, then $K\,|_{\operatorname{Im}\pi(q)}$ is the intertwining operator between representations $\widehat{\pi_1}$ and $\widehat{\pi_2}$. Thus we constructed a functor from the category $\operatorname{Rep}\mathcal{A}$ to $\operatorname{Rep}\mathcal{B}$.

Let $\mathcal{A}$ be an algebra, $\pi : \mathcal{A} \longrightarrow L(H)$ be a representation in space H. Construct a functor $F : \operatorname{Rep}\mathcal{A} \longrightarrow \operatorname{Rep}M_n(\mathcal{A})$ (where $M_n(\mathcal{A}) = \mathcal{A} \otimes M_n(\mathbb{C})$ is the algebra of matrices over $\mathcal{A}$). Define a representation $\widetilde{\pi} : M_n(\mathcal{A}) \longrightarrow L(H \oplus \cdots \oplus H)$ by $\widetilde{\pi}(a \otimes b) = \pi(a) \otimes b$, where $a \in \mathcal{A}$ and $b \in M_n(\mathbb{C})$. Put $F(\pi) = \widetilde{\pi}$ and $F(K) = K \otimes I_n$, where K is a morphism in the category $\mathcal{A}$ and I_n is the identity operator in $M_n(\mathcal{A})$.

Given two algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, let q be an idempotent in $M_n(\mathcal{A}_2)$. For a homomorphism of algebras $\varphi : \mathcal{A}_1 \longrightarrow qM_n(\mathcal{A}_2)q$, with the previous techniques a functor $F_\varphi : \operatorname{Rep}\mathcal{A}_2 \longrightarrow \operatorname{Rep}\mathcal{A}_1$ can be easily constructed. If $\pi : \mathcal{A}_2 \longrightarrow L(H)$, then $F_\varphi(\pi) : \mathcal{A}_1 \longrightarrow L(\mathcal{H})$, where $\mathcal{H} = \widetilde{\pi}(q)(H \oplus \cdots \oplus H)$. We will identify the algebra $L(\mathcal{H})$ with an algebra of operators A in $L(H \oplus \cdots \oplus H)$, such that $\widetilde{\pi}(q)A = A\widetilde{\pi}(q) = A$. If K is a morphism in the category $\operatorname{Rep}\mathcal{A}_2$, then $F_\varphi(K) = (K \otimes I_n)|_{\mathcal{H}}$.

Proposition 3. *Functors F_{φ_1} and F_{φ_2} give equivalence of the categories* $\operatorname{Rep}\mathcal{P}_{M_1,\ldots,M_n;\lambda}$ *and* $\operatorname{Rep}\mathcal{P}_{N_1,\ldots,N_n,abo}$.

Remark 2. Proposition 3 remains true if we replace categories of representations with categories of $*$-representations of corresponding $*$-algebras with involution defined as in Remark 1.

Let

$$\mathrm{W} = \{\lambda \in \mathbb{R} \mid *\text{-}\operatorname{Rep}\mathcal{P}_{M_1,\ldots,M_n;\lambda} \neq \varnothing\}, \qquad \widetilde{\mathrm{W}} = \left\{\lambda > 0 \mid *\text{-}\operatorname{Rep}\mathcal{P}_{\frac{M_1}{\lambda},\ldots,\frac{M_n}{\lambda},abo} \neq \varnothing\right\}.$$

Theorem 1. $\mathrm{W} = \widetilde{\mathrm{W}} \cup \{0\}$.

4 Sums of projections

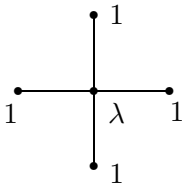
Let M_i for $i = 1, \ldots, n$ be a sequence of sets of nonnegative real numbers containing zero. We denote by $W_{M_1,\ldots,M_n}$ the set of all real numbers λ such that there exists a separable Hilbert space H and a sequence A_i, $i = 1, \ldots, n$ of self-adjoint operators in H such that $\sum\limits_{i=1}^{n} A_i = \lambda I_H$ and $\sigma(A_i) \subset M_i$, i.e. there exist $*$-representations of $*$-algebra $\mathcal{P}_{M_1,\ldots,M_n;\lambda}$. Let $\widetilde{W}_{M_1,\ldots,M_n}$ be the corresponding set of all $\lambda > 0$ for which $*$-representations of $\mathcal{P}_{\frac{M_1}{\lambda},\ldots,\frac{M_n}{\lambda},abo}$ exist together with a single element 0. Theorem 1 states that $\widetilde{W}_{M_1,\ldots,M_n} = W_{M_1,\ldots,M_n}$

Consider several examples of description of sets $W_{M_1,\ldots,M_n}$ and $\widetilde{W}_{N_1,\ldots,N_n}$ for some sequences (M_i), $i = 1, \ldots, n$.

1. Let $M_1 = \cdots = M_4 = \{0, 1\}$, then

$$\mathcal{P}_{M_1,M_2,M_3,M_4;\lambda} = \mathbb{C}\langle p_1, p_2, p_3, p_4 \mid p_1 + p_2 + p_3 + p_4 = \lambda e,\ p_i^2 = p_i^* = p_i\rangle,$$

and $*$-representations of such algebras are orthogonal projections with sum λI_H. The algebra $\mathcal{P}_{M_1,M_2,M_3,M_4;\lambda}$ corresponds to the diagram $\widetilde{D}_4$.



The *abo*-analog for this algebra is

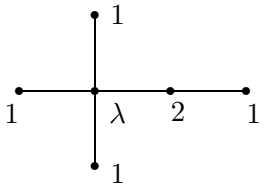
$$\mathcal{P}_{N_1,N_2,N_3,N_4,abo} = \mathbb{C}\langle q_1, q_2, q_3, q_4, p \mid q_1+q_2+q_3+q_4 = e,\ q_i p q_i = \frac{1}{\lambda} q_i,$$
$$q_i^2 = q_i^* = q_i,\ i = 1,2,3,4,\ p^2 = p^* = p,\ q_i q_j = 0\ (i \neq j)\rangle.$$

Description of set W_{M_1,M_2,M_3,M_4} was presented in [4]. We present it in our terms in the following proposition.

Proposition 4. *Assume* $M_1 = M_2 = M_3 = M_4 = \{0,1\}$ *and* $S = \left\{\frac{1}{2},\ 1\right\}$. *Then*

$$\widetilde{W}_{M_1,M_2,M_3,M_4} = W_{M_1,M_2,M_3,M_4} = \left\{2 \pm \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in S\right\} \cup \{2\}.$$

2. Let $M_1 = M_2 = M_3 = \{0,1\}$, $M_4 = \{0,1,2\}$. Algebra $\mathcal{P}_{M_1,M_2,M_3,M_4;\lambda}$ corresponds to the diagram



Description of W_{M_1,M_2,M_3,M_4} can be found in [9]. In our terms:

Proposition 5. *If* $M_1 = M_2 = M_3 = \{0,1\}$, $M_4 = \{0,1,2\}$ *then*

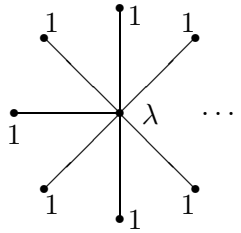
$$\widetilde{W}_{M_1,M_2,M_3,M_4} = W_{M_1,M_2,M_3,M_4} = \Lambda_4 \cup [2,3] \cup (5 - \Lambda_4),$$

where $\Lambda_4 = \left\{2 - \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in \left\{\frac{1}{2}, 1\right\}\right\}$.

3. Consider $*$-algebra generated by n projections with scalar sum. Let $M_1 = \cdots = M_n = \{0,1\}$, then

$$\mathcal{P}_{M_1,\ldots,M_n;\lambda} = \mathbb{C}\langle p_1, \ldots, p_n \mid \sum_{i=1}^{n} p_i = \lambda e,\ p_i^2 = p_i^* = p_i\rangle.$$

Its corresponding diagram is:



From [6] we conclude the following:

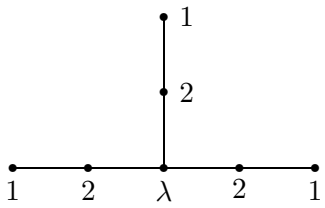
Proposition 6. *If* $M_1 = \cdots = M_n = \{0,1\}$ *then*

$$\widetilde{W}_{M_1,\ldots,M_n} = W_{M_1,\ldots,M_n} = \left\{\Lambda_n^{(0)}, \Lambda_n^{(1)}, \left[\frac{n-\sqrt{n^2-4n}}{2}, \frac{n+\sqrt{n^2-4n}}{2}\right], n - \Lambda_n^{(0)}, n - \Lambda_n^{(1)}\right\},$$

where $\Lambda_n^{(i)} = \left\{i, 1 + \frac{1}{n-1-i}, 1 + \frac{1}{n-2-\frac{1}{n-1-i}}, 1 + \frac{1}{n-2-\frac{1}{n-2-\frac{1}{n-1-i}}}, \ldots\right\}$ *for* $i = 0,1$, $n \in \mathbb{N}$.

5 Sums of operators with spectrum $\{0,1,2\}$

1. Let $M_1 = M_2 = M_3 = \{0,1,2\}$. Consider algebra $\mathcal{P}_{M_1,M_2,M_3;\lambda}$. Corresponding diagram is $\widetilde{E}_6$:

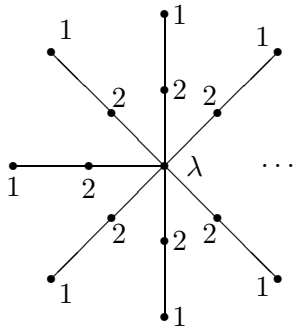


Description of W_{M_1,M_2,M_3} was given in [9]:

Proposition 7. *If $M_1 = M_2 = M_3 = \{0,1,2\}$, let $S = \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$. Then*

$$\widetilde{W}_{M_1,M_2,M_3} = W_{M_1,M_2,M_3} = \left\{3 \pm \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in S\right\} \cup \{3\}.$$

2. Consider $*$-algebra $\mathcal{P}_{M_1,\dots,M_n;\lambda}$ for $M_1 = \dots = M_n = \{0,1,2\}$, it corresponds to the diagram:

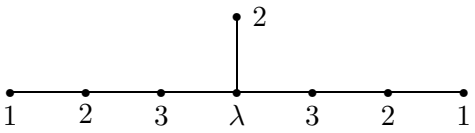


Proposition 8 ([9]). *If $M_1 = \dots = M_n = \{0,1,2\}$, then*

$$\widetilde{W}_{M_1,\dots,M_n} = W_{M_1,\dots,M_n} = ([0,2] \cap \Sigma_n) \cup [2, 2n-2] \cup (2n - ([0,2] \cap \Sigma_n))$$
$$= \Lambda_n^{(0)} \cup \Lambda_n^{(1)} \cup \left[n - \frac{n+\sqrt{n^2-4n}}{2}, n + \frac{n+\sqrt{n^2-4n}}{2}\right] \cup \left(2n - \Lambda_n^{(0)}\right) \cup \left(2n - \Lambda_n^{(1)}\right).$$

6 Extended Dynkin diagrams E_7 and E_8

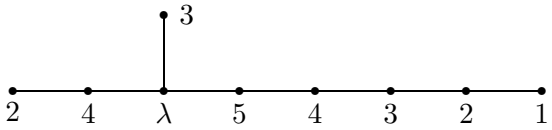
1. Let $M_1 = M_2 = \{0,1,2,3\}$, $M_3 = \{0,2\}$. Considering algebra $\mathcal{P}_{M_1,M_2,M_3;\lambda}$ we get diagram $\widetilde{E}_7$:



Proposition 9. *If $M_1 = M_2 = \{0,1,2,3\}$, $M_3 = \{0,2\}$ and $S = \left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\}$ then*

$$\widetilde{W}_{M_1,M_2,M_3} = W_{M_1,M_2,M_3} = \left\{4 \pm \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in S\right\} \cup \{4\}.$$

2. Let $M_1 = \{0,1,2,3,4,5\}$, $M_2 = \{0,2,4\}$ and $M_3 = \{0,3\}$. Consider $\mathcal{P}_{M_1,M_2,M_3;\lambda}$ Its corresponding diagram is $\widetilde{E}_8$



Proposition 10. *If* $M_1 = \{0,1,2,3,4,5\}$, $M_2 = \{0,2,4\}$, $M_3 = \{0,3\}$ *and* $S = \left\{\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, 1\right\}$ *then*

$$\widetilde{W}_{M_1,M_2,M_3} = W_{M_1,M_2,M_3} = \left\{6 \pm \frac{1}{k+s} \mid k \in \mathbb{N} \cup \{0\},\ s \in S\right\} \cup \{6\}.$$

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(Pseudo-)Trace Functions and Modular Invariance of Vertex Operator Algebra

Masahiko MIYAMOTO

Institute of Mathematics, University of Tsukuba, Tsukuba 305, Japan
E-mail: *miyamoto@math.tsukuba.ac.jp*

By including interlocked modules, we showed a modular invariance property of (pseudo) trace functions of modules for a 2-dimensional conformal field theory satisfying C_2-finiteness condition.

1 Introduction

When we encounter a new model of 2-dimensional conformal field theory, we usually try to classify all simple modules. As one of the most popular ways, we define an S-transformation $\mathrm{ch}_{W_i}(-1/\tau)$ of a character

$$\mathrm{ch}_{W_i}(\tau) = \sum_{n\in\mathbb{C}} \dim W_i(n) q^{n-c/24}, \qquad q = \exp\left(2\pi\sqrt{-1}\tau\right)$$

of some known module $W_i = \oplus_{n\in\mathbb{C}} W_i(n)$ and expand it into a power sum of q, where c is a central charge of the model. In the classical cases with only finitely many simple modules, the expansion became a linear sum of other characters, that is,

$$\mathrm{ch}_{W_i}(-1/\tau) = \sum_j a_{ij}\,\mathrm{ch}_{W_j}(\tau), \qquad a_{ij} \in \mathbb{C}.$$

Furthermore, if W_i is a module identified with a chiral algebra, we call such a module "**vertex operator algebra**", then every simple module has appeared in this expansion (as a trace function) and the fusion rules, which are multiplicities N_{jk}^s of modules W_s in the tensor product $W_j \boxtimes W_k$ (or operator product expansion) of two simple modules W_j, W_k,

$$W_j \boxtimes W_k = \sum N_{jk}^s W_s,$$

were determined by these coefficients a_{ij} (Verlinde formula).

These are just phenomena observed in the known models and there is no guarantee for the future, that is, there is no mathematical proof.

Actually, we have faced some models, which were discovered recently, in which an S-transformation

$$\mathrm{ch}_{W_i}(-1/\tau)$$

of character of some module is not a sum of q-powers and it has $\log q$-terms, (for example, see [3]). For example, a triplet model with central charge -2 has eight simple modules $W_1, \ldots, W_8$ and one of their characters is

$$S^1(\tau) = \frac{1}{2}\left(\eta(\tau)^{-1}\theta_{1,2}(\tau) + \eta(\tau)^2\right)$$

and its S-transformation becomes

$$S^1(-1/\tau) = \frac{1}{4}S^3(\tau) - \frac{1}{4}S^4(\tau) - \frac{\sqrt{-1}\tau}{2}\left(S^1(\tau) + S^2(\tau)\right),$$

which has a term with $\log(q) = 2\pi\sqrt{-1}\tau$. Flohr called such a logarithmic term "a generalized character" in [F].

The purpose of my paper is to show that if a chiral algebra (or *vertex operator algebra*) V of 2-dimensional conformal field theory satisfies the following C_2-cofinite condition

$$\dim V/C_2(V) < \infty,$$

where $C_2(V) = \langle v_{-2}u \,|\, v, u \in V\rangle$, then the above logarithmic terms have natural meaning. Namely, every logarithmic term implies an existence of some kind of a module (of course, it is not a direct sum of simple modules) and conversely, the existence of such a module gives a logarithmic term in an S-transformation of some character.

2 Definition of vertex operator algebra

In this section, we will explain a vertex operator algebra which is identified with a chiral algebra. We are not talking about the known models, but the models we will encounter in future. Therefore, we have to give the precise description of the condition we will use.

Let us check the situation. For a given set of fields $\{f_a(z) : a \in A\}$, define a vector space V spanned by the fields. By adding the fields Ta defined by the differential $f_{Ta}(z) = \frac{d}{dz}f_a(z)$, we may assume that V has an action of derivation T. We also assume that one of $f_a(z)$ is a vacuum, that is, $f_{\mathbf{1}}(z) = 1$. With account of the weights of the fields, V becomes a $\mathbb{Z}$-graded vector space

$$V = \oplus_{n\in\mathbb{Z}}V(n).$$

Consider the normal product

$$f_a(z) * f_b(z) = f_a(z)^- f_b(z) + f_b(z)f_a(z)^+,$$

where $f_a(z)^- = \sum_{m\le -1} a_m z^{-m-1}$ is the generating operator of $f_a(z) = \sum_{m\in\mathbb{Z}} a_m z^{-m-1}$ and $f_a(z)^+ = \sum_{m\ge 0} a_m z^{-m-1}$ denotes the annihilating operator of $f(z)$. Using the normal product and derivation T, we can define infinitely many products $\{*_n* : n \in \mathbb{Z}\}$ on V as follows:

$$\begin{aligned}
&f_a(z)_n f_b(z) = \frac{1}{2\pi\sqrt{-1}}\oint_{x=0}(x-z)^n[f_a(x), f_b(z)] \qquad \text{for} \quad n \ge 0,\\
&f_a(z)_{-1} f_b(z) = f_a(z) * f_b(z), \qquad \text{and}\\
&f_a(z)_{-k} f_b(z) = \frac{1}{k!}\left(\left(\frac{d}{dz}\right)^k f_a(z)^-\right) f_b(z) + f_b(z)\left(\left(\frac{d}{dz}\right)^k f_a(z)^+\right), \qquad \text{for} \quad k \ge 2.
\end{aligned}$$

We will formulate the above setting as follows. We will treat only local fields in this paper, but we can define a similar setting for super-local fields.

A vertex algebra is a triple $V = (V, Y, \mathbf{1})$ consisting of a $\mathbb{Z}$-graded vector space $V = \oplus_{n\in\mathbb{Z}}V(n)$ and a *vertex operator* Y, which assigns an $\mathrm{End}(V)$-valued function in z for each $v \in V$ whose Lorentz expansion has a form

$$Y(v,z) = \sum_{m\in\mathbb{Z}} v_m z^{-m-1}, \qquad v_m \in \mathrm{End}(V)$$

and a special element $\mathbf{1}$ called *vacuum* satisfying the following conditions: $Y(\mathbf{1}, z)$ is an identity on V, $\lim\limits_{z\to 0} Y(v, z)\mathbf{1} = v$ and $Y(v, z)w$ has a lower bound in the power of z for any $w \in W$.

Moreover, for any $w_0 \in V^* = \mathrm{Hom}\,(V, \mathbb{C})$, $w \in V$, and the natural paring $\langle V^*, V\rangle$,

$$\begin{aligned}
&\langle w_0, Y(v_1, z_1)Y(v_2, z_2)w\rangle, \qquad |z_1| > |z_2| > 0,\\
&\langle w_0, Y(v_2, z_2)Y(v_1, z_1)w\rangle, \qquad |z_2| > |z_1| > 0,\\
&\langle w_0, Y(Y(v_1, z_1 - z_2)v_2, z_2)w\rangle, \qquad |z_2| > |z_1 - z_2| > 0
\end{aligned}$$

are expansions of the same rational function with singularities at z_1, z_2, $z_1 - z_2$ at most in the defined regions, respectively. Here $\mathrm{End}\,(V)$ is the set of linear transformations on V.

It is known that the latter condition is equivalent to

$$(z_1 - z_2)^N [Y(v_1, z_1), Y(v_2, z_2)] = 0$$

for some integer N by viewing them as formal power series.

Among many vertex algebras, we are interested in the one which has a relation with conformal field theory, that is, it has a representation of Virasoro algebra and some finiteness conditions. We will call it *a vertex operator algebra*.

Namely, it is a vertex algebra $(V, Y, \mathbf{1})$ such that V has no negative weights and all homogeneous spaces are finite-dimensional, that is,

$$V = \oplus_{n=0}^{\infty} V(n) \qquad \text{and} \qquad \dim V(n) < \infty.$$

It also has a special element $\omega \in V(2)$ called *conformal vector* such that a vertex operator $Y(\omega, z) = \sum\limits_{n\in\mathbb{Z}} L(n)z^{-n-2}$ of ω satisfies

(1) Virasoro algebra relation:

$$[L(m), L(n)] = (n - m)L(n + m) + \delta_{n+m,0}\binom{m+1}{3} \cdot \frac{c}{2},$$

where $c \in \mathbb{C}$ is called central charge of V;

(2) $L(-1)$-derivativity: $Y(L(-1)v, z) = \frac{d}{dz}Y(v, z)$ and

(3) the grading operator $L(0)$: $L(0) = n$ on $V(n)$.

Of course, we also consider modules. The definition of a module is a pair (W, Y^W) of a vector space W and a vertex operator Y^W such that for each $v \in V$, a vertex operator of v on W is an $\mathrm{End}\,(W)$-valued function $Y^W(v, z)$ whose Lorentz expansion has a form

$$Y^W(v, z) = \sum_{n\in\mathbb{Z}} v_n^W z^{-n-1}, \qquad v_n^W \in \mathrm{End}\,(W)$$

satisfying

$$Y^W(v, z)w \in W((z)) \qquad \text{for all} \quad w \in W$$

such that the set of vertex operators has a structure of vertex operator algebra isomorphic to V (or its homomorphic image) by the normal products. Namely, we can identify $Y^W(v, z)$ as an original field $f_v(z)$ of V.

The best way to classify all simple $\mathbb{Z}_+$-modules is to determine a Zhu algebra $A(V) = V/O(V)$ which is given as a factor space of V. Namely, following [6], V has a product

$$v * u = \mathrm{Res}_x \frac{(1 + x)^{\mathrm{wt}\,(v)}}{x} Y(v, x)u$$

for $v \in V_{\mathrm{wt}\, v}$ and $u \in V$, where $\mathrm{wt}\, v$ denotes the weight of v. Set

$$O(V) = \langle \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,(v)}}{x^2} Y(v,x)u \mid v, u \in V \rangle$$

and $A(V) = V/O(V)$. Then Zhu algebra $A(V)$ is an associative algebra with a product $*$. The essential property of the Zhu algebra is that a top module $W(0)$ of an $\mathbb{Z}_+$-graded module $W = \oplus_{m=0}^{\infty} W(m)$ is an $A(V)$-module, and every $A(V)$-module is a top module of some $\mathbb{N}$-graded module.

If we want to study nonsimple modules, the Zhu algebra is not enough. The above concept was naturally extended to n-th graded piece of $\mathbb{N}$-graded modules in [1]. Set

$$O_n(V) = \langle \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,(v)+n}}{x^{2+2n}} Y(v,x)u \mid v, u \in V \rangle$$

and $A_n(V) = V/O_n(V)$. Like $A(V)$, $A_n(V)$ is an associative algebra with a product

$$v *_n u = \sum_{m=0}^{n} \binom{-n}{m} \mathrm{Res}_x \frac{(1+x)^{\mathrm{wt}\,(v)+n}}{x^{n+m+1}} Y(v,x)u$$

and has a property that an n-th (and less) graded piece $W(n)$ of an $\mathbb{N}$-graded module is an $A_n(V)$-module and every $A_n(V)$-module is an n-th (or less) graded piece of a module. $A_n(V)$ is called an n-th Zhu algebra. Another method is to study a Poisson algebra $V/C_2(V)$ which is also given as a factor space of V, where

$$C_2(V) = \langle v_{-2}u \mid v, u \in V \rangle$$

is a subspace of V spanned by elements of the form $v_{-2}u$ with $v, u \in V$. If $\dim V/C_2(V)$ is finite, then V is called C_2-cofinite. Zhu introduced the C_2-cofinite condition in his paper [6] as a technical condition, in order to make the trace function satisfy a differential equations, he called it Condition C.

It was proved by [1] that as long as V is C_2-cofinite, V has only finitely many simple modules. We note that the known models with finitely many simple modules seem to satisfy this condition.

Classically, we have treated only modules $W = \oplus_{m \in \mathbb{C}} W(m)$ on which the grading operator $L(0)$ acts on $W(m)$ as a scalar m (that is, W is a direct sum of eigenspaces of $L(0)$), but we propose to include modules on which W is a direct sum of generalized eigenspaces

$$W(m) = \left\{ w \in W \mid (L(0) - m)^N w = 0 \text{ for some } N \right\}.$$

We note that if it is a simple module, it is a direct sum of eigenspaces of $L(0)$.

When we cover modules on which the grading operator $L(0)$ does not act semisimply (we call such a module "logarithmic"), we have to change the setting a little. For example, a character is usually given by

$$\mathrm{ch}_W(\tau) = \sum_{m \in \mathbb{C}} \dim W(m) q^{m - c/24},$$

which coincides with

$$\mathrm{ch}_W(\tau) = \sum_{m \in \mathbb{C}} \mathrm{tr}_{W(m)} \left(q^{L(0) - c/24} \right).$$

Since we treat a grading operator $L(0)$ which may not be a scalar on $W(m)$, we denote it by a Jordan decomposition

$$L(0) = L^{\mathrm{semi}}(0) + L^{\mathrm{nil}}(0),$$

where $L^{\mathrm{semi}}(0)$ is the semisimple part of $L(0)$, that is, $L^{\mathrm{semi}}(0)$ acts on a generalized eigenspace $W(m)$ as a scalar m and $L^{\mathrm{nil}}(0)$ is the nilpotent part of $L(0)$ and $L^{\mathrm{semi}}(0)L^{\mathrm{nil}}(0)=L^{\mathrm{nil}}(0)L^{\mathrm{semi}}(0)$. In this case, we have

$$q^{L(0)} = q^{L^{\mathrm{semi}}(0)}q^{L^{\mathrm{nil}}(0)} = q^m \sum_{n=0}^{\infty} \frac{1}{n!}\left(2\pi\sqrt{-1}\tau L^{\mathrm{nil}}(0)\right)^n \qquad \text{on} \quad W(m)$$

and so we can see a logarithmic term. We note that since $L^{\mathrm{nil}}(0)$ is nilpotent, the above sum is a finite sum.

As we showed, if W is a direct sum of generalized eigenspaces of $L(0)$, $q^{L(0)}$ is well-defined. The C_2-cofinite condition has a close connection with this fact.

Theorem 1 ([4]). *The following are equivalent:*
(1) *V is C_2-cofinite.*
(2) *All modules are $\mathbb{Z}_+$-graded.*
(3) *Every module is a direct sum of generalized eigenspaces of $L(0)$.*
(4) *$\mathrm{ch}_W(\tau)$ are well-defined for all modules W.*

Moreover, if V is C_2-cofinite, then all conformal weights are rational numbers [4].

3 Interlocked module

In the previous section we showed that if $L(0)$ has a nonzero nilpotent part $L^{\mathrm{nil}}(0)$, then $q^{L(0)}$ has a $\log(q)$-term. On the other hand, it is natural to consider a character of W

$$\mathrm{ch}_W(\tau) = \sum \dim W(m) q^{L(0)-c/24}$$

as one of trace functions

$$\mathrm{tr}_W(v,\tau) = \sum_m \left(\mathrm{tr}_{W(m)} o(v) q^{L(0)-c/24}\right), \qquad v \in V$$

at $v = \mathbf{1}$, where $o(v)$ is the grade-preserving operator of v, for example, if $v \in V_n$, then $o(v) = v_{n-1}$. Actually, from this point of view, Zhu proved that if V satisfies three conditions: C_2-cofinite condition, V has only finitely many simple modules $\{W_1, \dots, W_k\}$ and all modules are completely reducible, then an S-transformation

$$\left(\frac{1}{-\tau}\right)^{\mathrm{wt}\,[v]} \mathrm{tr}_{W_i}(v, -1/\tau)$$

of trace function $\mathrm{tr}_{W_i}(v,\tau)$ is a linear sum of trace functions $\mathrm{tr}_{W_j}(v,\tau)$, that is,

$$\left(\frac{1}{-\tau}\right)^{\mathrm{wt}\,[v]} \mathrm{tr}_{W_i}(v, -1/\tau) = \sum_{j=1}^{k} a_{ij}\, \mathrm{tr}_{W_j}(v,\tau)$$

for some $a_{ij} \in \mathbb{C}$. For the definition of $\mathrm{wt}[v]$, see [6]. We note that if $L(n)v = 0$ for $n > 0$, then $\mathrm{wt}\,[v] = \mathrm{wt}\, v$. The important thing is that the coefficients a_{ij} do not depend on v.

Unfortunately, if we use only ordinary trace functions, since $L^{\mathrm{nil}}(0)$ is a nilpotent operator, $\mathrm{tr}\, o(v)\big(L^{\mathrm{nil}}(0)\big)^m = 0$ for $m \geq 1$ and so we do not have a logarithmic form in a trace function. In order to preserve a logarithmic form, we have to extend the concept of trace function. So, the first question is:

What is a trace function, or trace?

The answer is that it is just a symmetric function on the algebra consisting of all grade-preserving operators. So, the next question is whether there is another symmetric function or not. The finite dimensional algebras with faithful symmetric functions are called "symmetric algebras" in mathematics, and this concept has a long history in the ring theory, see [5]. The key points are that every trace function and its S-transformed form offer symmetric functions of n-th Zhu algebra and, conversely, we can construct some modules (we will call "interlocked" with a symmetric function) from symmetric functions of n-th Zhu algebra.

Theorem 2 ([4]). *If we have a logarithmic form in an S-transformed trace function, then there is such a symmetric function. Conversely, if there is such a symmetric function, then it defines a new trace function with logarithmic form.*

Let us show an example. The ordinary trace is a symmetric linear function of $\mathrm{End}\,(W(n))$ for a finite-dimensional vector space $W(n)$. It is uniquely determined up to scalar multiple. However, if we consider a subring R of $\mathrm{End}\,(W(n))$, it is possible to have a new symmetric function of R. We will show an example of such rings.

Definition 1. Let W be a V-module. We call W an *interlocked module* if it satisfies the following condition: For any submodule W_1 of W, there is a submodule W_2 such that $W_1 \cong W/W_2$ and $W_2 \cong W/W_1$ as V-modules.

In particular, since $W/J(W) \cong \mathrm{soc}\,(W)$, every grade-preserving operator α of V on $W(n)$ has a form

$$\alpha = \begin{pmatrix} A & C & B \\ O & E & {}^*C \\ O & O & A \end{pmatrix},$$

where we have assumed $J(W) \supset \mathrm{soc}\,(W)$ for simplicity and B is a square matrix corresponding to the part $W/J(W) \to \mathrm{soc}\,(W)$. Here a *Jacobson radical* $J(W)$ denotes the intersection of all maximal submodules of W and a *socle* $\mathrm{soc}\,(W)$ of W is the direct sum of all simple submodules of W. For such an endomorphism α, we define a pseudo-trace by

$$\mathrm{pstr}\,(\alpha) = \mathrm{tr}\,B.$$

It is not difficult to check that the above pseudo-trace is a symmetric linear function and we will use it as well as the ordinary trace.

We note that the ordinary trace is one of pseudo-traces. Moreover, as we expected, it leaves a logarithmic term of $L^{\mathrm{nil}}(0)$ and then we can define pseudo-trace function

$$\mathrm{pstr}_W(v,\tau) = \sum_{m\in\mathbb{C}} \mathrm{pstr}_{W(m)}\left(o(v)q^{L^{\mathrm{nil}}(0)}\right) q^{m-c/24}$$

with a logarithmic form.

Furthermore, we have:

Theorem 3 ([4]). *If V is C_2-finite, then the space of pseudo-trace functions:*

$$\langle \mathrm{pstr}_W(v,\tau) \mid W \textit{ interlocked mods.}\rangle$$

is $SL_2(\mathbb{Z})$-invariant. Namely, an S-transformation of pseudo-trace function is a linear sum of pseudo-trace functions.

Note that the above space contains all ordinary trace functions and so an S-transformation of ordinary trace function is a linear sum of pseudo-trace functions.

Corollary 1 ([4]). *If $V = \oplus_{m=0}^{\infty} V_m$ is C_2-cofinite and there is no logarithmic modules, then the space spanned by the set of all (ordinary) characters is $SL_2(\mathbb{Z})$-invariant.*

We will also show a bound of the effective central charge $\tilde{c} = c - 24h_{\min}$, where c is the central charge and $h_{\min}$ is the smallest conformal weight.

Corollary 2 ([4]). *If $V = \oplus_{m=0}^{\infty} V_m$ is C_2-cofinite, then the*

$$\tilde{c} \leq \frac{\dim(V/C_2(V)) - 1}{2}.$$

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Orbit Functions of Compact Semisimple Lie Groups as Special Functions

Jiri PATERA

Centre de recherches mathématiques, Université de Montréal,
C.P.6128-Centre ville, Montréal, H3C 3J7, Québec, Canada
E-mail: *patera@crm.umontreal.ca*

An orbit function is the contribution to an irreducible character of a compact semisimple Lie group G of rank n from one of its Weyl group orbits. Properties of such functions will be described for compact simple Lie groups of all types. In particular, products of the functions decompose into their sums, the functions are periodic on copies of the fundamental region F of G, they are solutions of corresponding Laplace equation in n dimensions, satisfying Neumann condition at the boundary of F, ... Uncommon applications to image enhancements and a new approach to n-dimensional data compression are shown, when orbit functions are evaluated at suitable sets of conjugacy classes of elements of finite order in G.

1 Introduction

In this talk I will make some general claims, which are to be justified elsewhere [1]. Here the claims are illustrated by examples. More precisely, my aim is twofold: First I want to introduce some mathematical background to [2–4], and then I want to show that the orbit functions, the main subject here, are eigenfunctions of corresponding Laplace operators in $\mathbb{R}^n$ and that they satisfy Neumann condition at the boundary of the fundamental region of the corresponding compact semisimple Lie group.

An orbit function is the contribution to the character of an irreducible representation from a single Weyl group orbit [5–9]. Thus in the term 'orbit function of the compact semisimple Lie group G' the adjective 'orbit' refers to the orbit of the Weyl group W of G rather than to the orbit of G. The term was introduced in [10].

Generally one may argue in favor of the fact that a large family of functions, defined on conjugacy classes of any compact semisimple Lie group G, called 'orbit functions', belong among the special functions of mathematical physics. The family is quite rich in the sense that it contains not only a basis of the space of class functions on G, but it contains such bases for semisimple Lie groups of all types and ranks.

Although there is no generally accepted definition of special function [11–13], most of the practitioners would probably agree that there are two related requirements which need to be satisfied before a family of functions would be called special.

Firstly, the functions should find an extensive use in some part of science and engineering. Secondly, the functions should have some 'nice' mathematical properties. The later often means that they should form a basis in a functional space with some kind of orthogonality property, which almost always implies that they are solutions of a meaningful differential equations, and that products of two functions should simply decompose into the sums of them. In most cases it is assured by the presence of a group underlying the special functions.

So far a claim of practical applicability of orbit functions, would find only a feeble justification, and that only in the work related to us in some way and dealing with 'abstract' although rather challenging problems [10, 14–17].

Actual motivation for this article is the recent recognition of the fact that the orbit functions are indispensable for a vast generalization of the discrete cosine transform [2–4], which turned up to be extremely useful in recent years [18]. In addition, it appears that it is difficult to overestimate the future role of orbit functions in some applications to image enhancements and/or requiring data compression [2–4].

The goal in here is to bring together diverse facts about orbit functions, many of which are not found in the literature, although they often are straightforward consequences of known facts. In general, for a given compact simple Lie group G of rank n, most of the properties of orbit functions, which are described in this paper, are implications of properties of either the orbits of the Weyl group, or of the irreducible characters [5–9, 11].

Only examples of dimension 2 and 3 are shown here because they are likely to be used more often. Hopefully they can be understood without general arguments by directly verifying their properties through explicit calculation. Nevertheless, since our conclusions are to be general as to the type of the simple Lie group G, it is imperative to profit from the uniformity of the pertinent parts of Lie theory: Many general facts of the theory are recalled without explanation and with only a few references to the literature.

2 Preliminaries

We are interested in complex valued functions $f(g)$ on any semisimple compact Lie group G, i.e. $g \in G$, which are constant on the conjugacy classes of G (called *class functions*)

$$f(g) = f\big(hgh^{-1}\big) \qquad \text{for all} \quad g, h \in G.$$

Suppose that a maximal torus $T \subset G$ has been chosen and fixed. Every element of G has conjugates in T. One can choose in T a region $\tilde{F}$ in which every conjugacy class of elements of G is represented by precisely one element of G. For any group, $\tilde{F}$ would be called the fundamental region. In Lie theory it is conventional to call the fundamental region F of G, the region where the parameters x of the elements $g(x) \in \tilde{F}$ are found in a standard parametrization of G. Thus for a group of rank n, the parameters of $g(x)$ are components of a vector $x \in \mathbb{R}^n$.

A well known basis in the space of class functions on G consists of the characters $\chi_\Lambda(g(x))$, of finite dimensional irreducible representations Λ of G. To simplify the notation, we write $\chi(\Lambda, x)$ for $\chi_\Lambda(g(x))$. A standard problem of Fourier analysis on G is to find the coefficients of the expansion of a given function $f(x)$ into the series of irreducible characters,

$$f(x) = \sum_\Lambda C_\Lambda \chi(\Lambda, x), \qquad x \in F. \tag{1}$$

In this talk I am interested in the properties of the orbit functions $\Omega(d, x)$, which form another basis in the functional space spanned by the irreducible characters.

$$f(x) = \sum_d D_d \, \Omega(d, x), \qquad x \in F, \tag{2}$$

where the summation extends over the dominant weights d of G.

Orbit functions are closely related to characters $\chi(\Lambda, x)$ of G. Let Λ be the highest weight labelling an irreducible representation of G of finite dimension, and let $DV(\Lambda)$ be the set of its *distinct* dominant weights in the weight system of Λ. Then

$$\chi(\Lambda, x) = \sum_{d \in DV(\Lambda)} c_{\Lambda,d} \Omega(d, x). \tag{3}$$

Here the coefficients $c_{\Lambda,d} \in \mathbb{N}$ are the multiplicities of the dominant weights d in the weight system of the representation Λ. Hence the sum in (3) is finite.

As one of the reasons, why characters are rarely used in extensive applications, one may bring forward the need to know the multiplicities $c_{\Lambda,d}$, and the fact that the larger a representation Λ is, the larger is the sum (3). The multiplicities are calculated using a laborious recursive algorithm, starting from the highest weight Λ. In many situations it is practical to read off their values from the tables [20]. Standard bases of simple roots (α-basis) and of dominant weights (ω-basis) are used here [9].

Due to (3) a number of important properties of characters carry over to properties of orbit functions.

The sets P and P_+,

$$P = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n \ \supset \ P_+ = \mathbb{Z}^{\geq 0}\,\omega_1 + \cdots + \mathbb{Z}^{\geq 0}\,\omega_n$$

are called respectively the weight lattice and the cone of dominant weights. The smallest dominant weights, different from zero, are the fundamental weights ω_j.

The fundamental region $F \in \mathbb{R}^n$ of the simple Lie group G of rank n is the simplex which has for its vertices the origin of $\mathbb{R}^n$ and the fundamental weights appropriately scaled. More precisely, the following $n+1$ points are the vertices of the simplex F.

$$F: \quad \left\{0, \frac{\omega_1}{q_1}, \ldots, \frac{\omega_n}{q_n}\right\}, \tag{4}$$

where q_k are the coefficients of the *dual of the highest short root* in α-basis. For A_2 we have $q_1 = q_2 = 1$. In particular, F is an equilateral triangle in the case of A_2. See Fig. 1 of [21] for instructive examples of F.

3 Orbit functions

An orbit function $\Omega(d,x)$ of a compact simple Lie group G (equivalently of the Lie algebra L) can be written as a sum of exponential functions $e^{2\pi i\langle v\,|\,x\rangle}$. Each term of the sum is determined by a weight v of a Weyl group orbit $O(d)$ as function of $x \in \mathbb{R}^n$. The symbol $\langle v\,|\,x\rangle$ denotes scalar product in the Euclidean space $\mathbb{R}^n$.

Let $O(d)$ be the Weyl group orbit containing the (unique) dominant weight d. The orbit function $\Omega(d,x)$, referring to $O(d)$, is the following finite sum of exponential functions,

$$\Omega(d,x) = \sum_{v\in O(d)} e^{2\pi i\langle v\,|\,x\rangle}, \qquad x \in \mathbb{R}^n. \tag{5}$$

Note that for any $v, d \in O(d)$, we have the equality $\Omega(d,x) = \Omega(v,x)$. It is an convention to label the orbit functions by their dominant weights because those weights are easily recognized in ω-basis. The number of summands is equal to $|O(d)|$, the number of weights in the orbit $O(d)$.

3.1 Examples

Clearly the orbit functions of A_1, where $x = \theta\omega$, and $d = m\omega$, $m \in \mathbb{Z}^{\geq 0}$, with $\theta \in \mathbb{R}$, are $\Omega(d,x) = 2\cos(\pi m\theta)$ if $m \in \mathbb{Z}^{>0}$, because $\langle\omega\,|\,\omega\rangle = \frac{1}{2}$, and $\Omega(0,x) = 1$.

Consider the example of the generic orbit functions of A_2. Putting $d = (a\,b) := a\omega_1 + b\omega_2$, with $a, b > 0$, we have,

$$\begin{aligned}\Omega((a\,b),x) = {} & e^{2\pi i\langle(a\ b)\,|\,x\rangle} + e^{2\pi i\langle(-a\ a+b)\,|\,x\rangle} + e^{2\pi i\langle(a+b\ -b)\,|\,x\rangle} \\ & + e^{2\pi i\langle(b\ -a-b)\,|\,x\rangle} + e^{2\pi i\langle(-a-b\ a)\,|\,x\rangle} + e^{2\pi i\langle(-b\ -a)\,|\,x\rangle}.\end{aligned}$$

Using $x = \theta_1\omega_1 + \theta_2\omega_2$, where $\theta_1, \theta_2 \in \mathbb{R}$, and $(a\,b) = a\omega_1 + b\omega_2 = \frac{1}{3}(2a+b)\alpha_1 + \frac{1}{3}(a+2b)\alpha_2$, where $a, b \in \mathbb{Z}$, we have

$$\begin{aligned}\Omega((a\,b),x) = {} & e^{2\pi i\frac{1}{3}((2a+b)\theta_1+(a+2b)\theta_2)} + e^{2\pi i\frac{1}{3}((-a+b)\theta_1+(a+2b)\theta_2)} \\ & + e^{2\pi i\frac{1}{3}((2a+b)\theta_1+(a-b)\theta_2)} + e^{2\pi i\frac{1}{3}((b-a)\theta_1+(-2a-b)\theta_2)} \\ & + e^{2\pi i\frac{1}{3}((-a-2b)\theta_1+(a-b)\theta_2} + e^{2\pi i\frac{1}{3}((-a-2b)\theta_1+(-2a-b)\theta_2)}. \end{aligned} \tag{6}$$

Similarly one finds $O((a\,0),x)$ and $O((0\,b),x)$ using (5):

$$\Omega((a\,0),x) = e^{\frac{2\pi i}{3}a(2\theta_1+\theta_2)} + e^{\frac{2\pi i}{3}a(-\theta_1+\theta_2)} + e^{\frac{2\pi i}{3}a(-\theta_1-2\theta_2)}, \tag{7}$$

$$\Omega((0\,b),x) = e^{\frac{2\pi i}{3}b(\theta_1+2\theta_2)} + e^{\frac{2\pi i}{3}b(\theta_1-\theta_2)} + e^{\frac{2\pi i}{3}b(-2\theta_1-\theta_2)}. \tag{8}$$

Note that $\Omega((a\,a),x)$ are real valued for all $a \in \mathbb{Z}^{>0}$:

$$\begin{aligned}\Omega((a\,a),x) &= 2\left\{\cos 2\pi a(\phi_1+\phi_2) + \cos 2\pi a(2\phi_2-\phi_1) + \cos 2\pi a(2\phi_1-\phi_2)\right\} \\ &= 2\left\{\cos 2\pi a(\theta_1+\theta_2) + \cos 2\pi a\theta_1 + \cos 2\pi a\theta_2\right\}. \end{aligned} \tag{9}$$

Note also that the pairs $\Omega((a\,0),x) + \Omega((0\,a),x)$ or, more generally, $\Omega((a\,b),x) + \Omega((b\,a),x)$ are always real.

3.2 Other properties of orbit functions

A number of properties of the orbit functions can be either inferred directly from (5) or from the properties of characters.

1. An orbit function is a finite sum of exponential functions. Therefore it is continuous and has continuous derivatives of all orders in $\mathbb{R}^n$.
2. All orbit functions of the following groups are real:

   $$A_1,\ B_n,\ C_n,\ D_{2k},\ E_7,\ E_8,\ F_4,\ G_2.$$

   The orbit functions of the remaining groups are real provided the dominant weights d are invariant with respect to the symmetry transformation of the Dynkin diagram.
3. Matrix $C = (c_{\Lambda,z})$ of the multiplicities of the dominant weights in irreducible representation Λ is triangular with diagonal elements being all equal to 1 (assuming a suitable ordering of the weights). Hence it can be easily inverted. Consequently, every orbit function can be expressed as an integer linear combination of a finite number of irreducible characters. Many examples of C are found in the tables of [20].
4. Symmetries of the orbit functions.

   By definition, $\Omega(d,x)$ is invariant under the action of the Weyl group:

   $$\Omega(d,x) = \Omega(wd,x), \qquad \text{for all} \quad w \in W. \tag{10}$$

   Indeed, the weights d and wd belong to the same orbit and the expression for $\Omega(d,x)$ contains summation over all the weights of the orbit.

   From the equalities of scalar products, $\langle wa\,|\,x\rangle = \langle a\,|\,w^{-1}x\rangle$ and $\langle a\,|\,x\rangle = \langle wa\,|\,wx\rangle$ for all $a \in O(d)$ and $w \in W$, follow the corresponding equalities of orbit functions.

   Multidimensional periodicity of $\Omega(d,x)$ follows from such a periodicity of $\chi(\Lambda,x)$. Copies of F tile $\mathbb{R}^n$. Character values on each tile are the same. Such a symmetry of characters

and orbit functions is naturally expressed as the invariance under the affine Weyl group as illustrated in the last section below. Values of orbit functions are thus repeated on each tile. More precisely, characters, and thus also orbit functions, are symmetric with respect to any common $(n-1)$-dimensional face of two adjacent copies of F.

5. The normal derivative of $\Omega(d,x)$ to the boundary of F equals zero. It follows from the continuity of $\Omega(d,x)$ and of its derivatives, together with the symmetry with respect to the reflections in sides of F.
6. Orthogonality of orbit functions on F follows from the orthogonality of characters, and from the fact that a given weight belongs to precisely one orbit function.
7. There are other aspects of orbit functions which could be rather useful to study. Let us just point out the following ones:

   Decomposition of products of orbit functions into their sums [17].

   Branching rules for orbit functions [22–24].

   Recursive construction of orbit functions from a few smallest ones.

   Scaling symmetries of orbit functions.

   Derivatives of orbit functions.

4 Orbit functions and the Neumann boundary value problem on F

It is shown here that an orbit function is an eigenfunction of the Laplace operator Δ, acting on the class functions of G, and that its eigenvalue is, up to a constant, given by the square length of the weight of the corresponding Weyl group orbit. It was already explained that the normal derivative of an orbit function at the boundary of F equals zero (Neumann condition). Hence any function, expanded into a series of orbit functions, has to satisfy the same condition at the boundary of F.

Let $x=\theta_1\omega_1+\cdots+\theta_2\omega_2$. Denoting by ∂_k partial derivative with respect to θ_k, we have the Laplace operator Δ,

$$\Delta=\sum_{i,j=1}^{n} M_{ij}\partial_i\partial_j \tag{11}$$

acting on the orbit functions and, more generally, on the class functions $f(x)$. For simple Lie groups of types A_n, D_n, and E_n, the matrix M coincides with the Cartan matrix. For the other cases some columns of the Cartan matrix need to be rescaled.

Next let us show the following properties of orbit functions $\Omega(d,x)$ of A_2, equivalently of $SU(3)$:

$$\Delta\Omega(d,x)=\lambda\Omega(d,x), \qquad \left.\frac{\partial\Omega(d,x)}{\partial m}\right|_{\partial F}=0, \qquad x\in F\subset\mathbb{R}^n, \qquad d\in P_+, \tag{12}$$

where ∂F denotes the boundary of F from (4), formed by its $(n-1)$-dimensional faces, and m denotes the normal to the boundary, and where

$$\lambda=-4\pi^2\langle d\,|\,d\rangle. \tag{13}$$

Above it was already explained that the orbit functions satisfy the Neumann condition on the boundary of F, and that F is an equilateral triangle in the case of A_2.

Consider the $SU(3)$-Laplace operator Δ acting on a single term $e^{2\pi i\langle z\,|\,x\rangle}$, and suppose that both, the fixed vector $z=(a,b)$ and the variable vector $x=(\theta_1,\theta_2)$, are given in ω-basis. Their scalar product is calculated using the inverse of the Cartan matrix of A_2.

$$\langle z\,|\,x\rangle = z^T Q\,x = \tfrac{1}{3}(a\ b)\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}\theta_1\\\theta_2\end{pmatrix} = \tfrac{1}{3}(2a+b)\theta_1+\tfrac{1}{3}(a+2b)\theta_2. \tag{14}$$

In particular, for a general $z\in P$ and $\xi=\omega_1+\omega_2$,

$$\langle z\,|\,z\rangle = \tfrac{2}{3}(a^2+ab+b^2), \qquad \text{and} \qquad \langle z\,|\,\xi\rangle = a+b.$$

Application of the Laplace operator to the exponential function is straightforward,

$$\begin{aligned}\Delta e^{2\pi i\langle z\,|\,x\rangle} &= (\partial_1\ \partial_2)\begin{pmatrix}2&-1\\-1&2\end{pmatrix}\begin{pmatrix}\partial_1\\\partial_2\end{pmatrix} e^{2\pi i\langle z\,|\,x\rangle} = 2\left(\partial_1^2-\partial_1\partial_2+\partial_2^2\right)e^{\frac{2\pi i}{3}\{(2a+b)\theta_1+(a+2b)\theta_2\}}\\ &= 6\left(\frac{2\pi i}{3}\right)^2(a^2+ab+b^2)e^{\frac{2\pi i}{3}\{(2a+b)\theta_1+(a+2b)\theta_2\}} = -4\pi^2\langle z\,|\,z\rangle e^{2\pi i\langle z\,|\,x\rangle}.\end{aligned}$$

Weights of an orbit $O(d)$ are equidistant from the origin, i.e. for any $z\in O(d)$ we have $\langle z\,|\,z\rangle=\langle d\,|\,d\rangle$. Therefore the eigenvalues of Δ are the same on all the exponential terms of an orbit function. Consequently, the eigenvalues of the orbit functions and of each of its exponential terms are the same.

4.1 Symmetry under the affine Weyl group

The invariance of the orbit functions under the Weyl group is built into their definition through the summation over the elements of an W-orbit. Indeed, W merely permutes the terms in that sum. More spectacular is the symmetry of orbit functions under the affine Weyl group $W^{\rm aff}$. It is a larger group than W. It has one more generator, namely the affine reflection r_0,

$$r_0 x = x+\left(1-\frac{2\langle x\,|\,\xi\rangle}{\langle\xi\,|\,\xi\rangle}\right)\xi = \xi + r_\xi x.$$

Here $r_\xi\in W$ is the reflection in the plane orthogonal to the highest short root ξ. In case of A_2 one has $\xi=\alpha_1+\alpha_2=\omega_1+\omega_2$, and $\langle\xi\mid\xi\rangle=2$. The action of $W^{\rm aff}$ on F results in tiling of $\mathbb{R}^n$ by copies of F. In particular,

$$r_0F = \{r_0, r_0\omega_1, r_0\omega_2\} = \{\xi,\omega_1,\omega_2\}.$$

Note the translations $r_0r_\xi x = x-\xi$, and $r_\xi r_0 = x+\xi$. Putting $z=(a\,b)$, $a,b\in\mathbb{Z}$ and $x=(\theta_1\,\theta_2)$, $\theta_1,\theta_2\in\mathbb{R}$,

$$\langle z\,|\,r_0x\rangle = \langle z\,|\,\xi\rangle + (1-\langle\xi\,|\,x\rangle)\langle z\,|\,\xi\rangle = \langle z\,|\,\xi\rangle + \langle z\,|\,r_\xi x\rangle = a+b+\langle r_\xi z\,|\,x\rangle.$$

Consequently,

$$e^{2\pi i\langle z\,|\,r_0x\rangle} = e^{2\pi i\langle r_\xi z\,|\,x\rangle}.$$

Here $r_\xi z=-b\omega_1-a\omega_2$ is another weight from the orbit of z. Therefore $\Omega(z,x)=\Omega(z,r_0x)$. Thus the orbit functions are invariant with respect to the affine Weyl group $W^{\rm aff}$.

5 Discretization of the orbit functions

Here I want to recall the main result of [10], which is a discrete orthogonality of a finite (arbitrarily large) set of orbit functions of G, on a grid S of discrete points in F (hence in all $\mathbb{R}^n$). Equipped with such an orthogonality, one can invert finite (arbitrarily long) Fourier expansions of class functions $f(x)$ on G. This setting of the expansion problem is particularly suitable when the function $f(x)$ is provided digitally, i.e. by its values on the points of a grid.

Start by fixing a natural number M. Then $S^{[M]}$, the set of points of the grid, consists of all distinct points $s \in S^{[M]} \subset F$ which, in the ω-basis, have rational coordinates with the common denominator equal to M, subjects to the additional restriction $\langle s \,|\, \xi \rangle \leq 1$.

Consider $\Omega(d,s)$ for $s \in S^{[M]}$. Then for certain set of dominant weights d, we have

$$\langle \Omega(d) \,|\, \Omega(d') \rangle_M := \sum_{s \in S^{[M]}} C_s^{[M]} \Omega(d,s) \overline{\Omega(d',s)} = \delta_{d,d'} \cdot K. \tag{15}$$

Here $C_s^{[M]}$, K are known constants [10], overbar indicates complex conjugation, and the dominant weights d, d' belong to a set which contains comparable number of distinct dominant weights as is the number of elements in $S^{[M]}$.

5.1 Examples

Let $G = A_2$, equivalently $G = SU(3)$. In this case F is an equilateral triangle. Put $M = 2$. The grid $S^{[2]}$ in F contains 6 points s. In ω-basis these are the following ones:

Three vertices of F: $(0\,0)$, $(0\,1)$, $(1\,0)$, and the middle points of the edges of F: $(\frac{1}{2}\,\frac{1}{2})$, $(0\,\frac{1}{2})$, $(\frac{1}{2}\,0)$.

Values of a few lowest orbit functions $\Omega(d,s)$ at the points $s \in S^{[2]}$, are shown in Table 1. To simplify the notation, we also use $\omega = e^{2\pi i/6}$. Note that the orbit functions of the first 6 rows of the table are orthogonal on the $S^{[2]}$ according to (15). Orbit functions of the remaining rows (and any higher one we might have shown) are multiples of one of the first six ones.

Table 1. Values of several A_2-orbit functions at the six points of $S^{[2]}$, and the coefficients $C_s^{[2]}$. Notation: $\omega = e^{2\pi i/6}$.

s	$(0\,0)$	$\left(\frac{1}{2}\,0\right)$	$\left(0\,\frac{1}{2}\right)$	$(1\,0)$	$\left(\frac{1}{2}\,\frac{1}{2}\right)$	$(0\,1)$
$\Omega((0\,0),s)$	1	1	1	1	1	1
$\Omega((1\,0),s)$	3	ω^5	ω	$3\omega^4$	-1	$3\omega^2$
$\Omega((0\,1),s)$	3	ω	ω^5	$3\omega^2$	-1	$3\omega^4$
$\Omega((2\,0),s)$	3	$3\omega^4$	$3\omega^2$	$3\omega^2$	3	$3\omega^4$
$\Omega((1\,1),s)$	6	-2	-2	6	-2	6
$\Omega((0\,2),s)$	3	$3\omega^2$	$3\omega^4$	$3\omega^4$	3	$3\omega^2$
$\Omega((3\,0),s)$	3	-1	-1	3	-1	3
$\Omega((2\,1),s)$	6	$2\omega^5$	2ω	$6\omega^4$	-2	$6\omega^2$
$\Omega((1\,2),s)$	6	2ω	$2\omega^5$	$6\omega^2$	-2	$6\omega^4$
$\Omega((0\,3),s)$	3	-1	-1	3	-1	3
$\Omega((2\,2),s)$	6	6	6	6	6	6
$C_s^{[2]}$	1	3	3	1	3	1

In particular, one finds

$$\langle \Omega(0\,0) \,|\, \Omega(0\,0) \rangle_2 = 12, \qquad \langle \Omega(1\,0) \,|\, \Omega(1\,0) \rangle_2 = 36, \qquad \langle \Omega(1\,1) \,|\, \Omega(1\,1) \rangle_2 = 144.$$

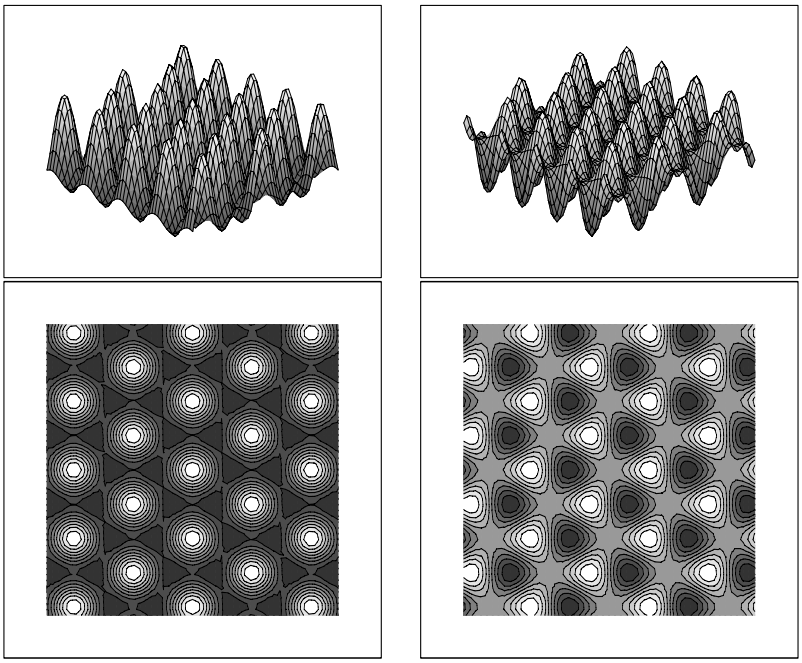

Figure 1. A_2 orbit function $\Omega((2\,0),x)$ real (left) and imaginary (right) parts.

Finally let us illustrate how to calculate the entries of Table 1. Take $\Omega((2\,1),(\frac{1}{2}\,0))$. Its value, $2\omega^5$, is found from (6) by putting there $a=2$, $b=1$, and $\theta_1=\frac{1}{2}$, $\theta_2=0$, followed by obvious simplifications.

Crystallographic symmetries of orbit functions are illustrated on Fig. 1 and 2. Fig. 1 shows the real and imaginary parts of the A_2-orbit function $\Omega((2\,0),x)$. Examples of the real orbit functions $\Omega((2\,1),x)$ and $\Omega((1\,1),x)$ of C_2 and G_2 respectively are shown on Fig. 2.

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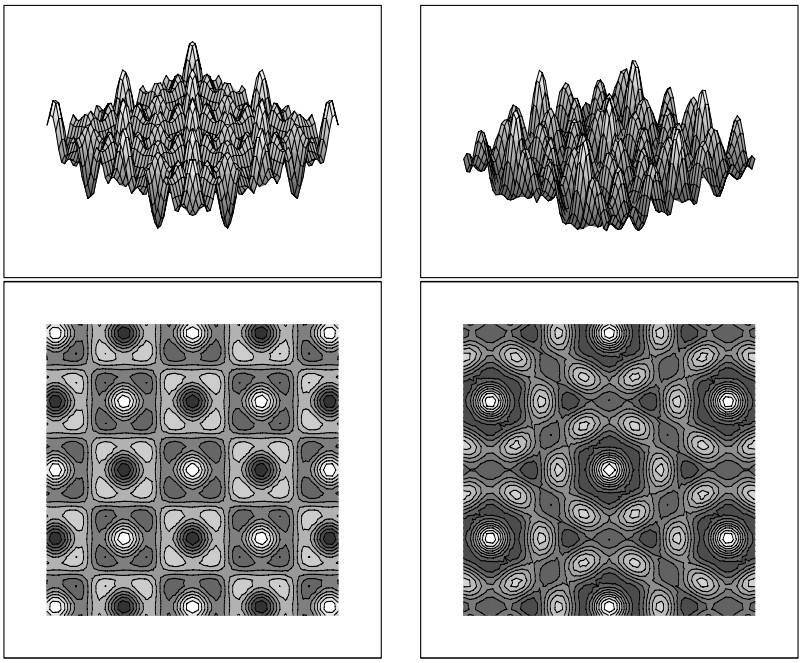

Figure 2. $\Omega((2\,1),x)$ of C_2 (left) and $\Omega((1\,1),x)$ of G_2 (right) orbit functions.

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Modified Hypergeometric Equations Arising from the Markoff Theory

Serge PERRINE

FTRD, 38-40 rue du General Leclerc, 92794 Issy les Moulineaux Cedex 9, France
E-mail: *serge.perrine@francetelecom.com*

After recalling what the Markoff theory is, this article summarizes some links which exist with the group $GL(2,\mathbb{Z})$ of 2×2 matrices with integer coefficients and determinant ± 1 and with its subgroups $SL(2,\mathbb{Z})$ and the triangle group $\mathbf{T}_3$. Then we consider the relation with conformal punctured tori. The main part of the article is about the monodromy representation of the Poincaré group of such a torus. We give the corresponding solution of the associated Riemann–Hilbert problem and the corresponding differential operator whose spectral analysis remains to be done. We conclude quoting the Hilbert's 22^{nd} problem and some information about the accessory parameter problem.

1 Introduction

For a real quadratic form $f(x,y)=ax^2+bxy+cy^2\in\mathbb{R}[x,y]$, the issue of knowing the minimal value of $|f(x,y)|$ when x and y are non-zero integers is classical. When $f(x,y)$ is a definite form, i.e. $\Delta(f)=b^2-4ac<0$, the problem was solved by J.L. Lagrange and C. Hermite [18]:

$$C(f)=\frac{\inf_{(x,y)\in\mathbb{Z}^2/\{(0,0)\}}|f(x,y)|}{\sqrt{|\Delta(f)|}}\leq\frac{1}{\sqrt{3}}=C(x^2+xy+y^2).$$

It has been shown [3, p. 33] that for any $\rho\in]0,(1/\sqrt{3})]$, we can find $f(x,y)\in\mathbb{R}[x,y]$ a quadratic form verifying $\rho=C(f)$. When $f(x,y)$ is a indefinite form, i.e. $\Delta(f)=b^2-4ac>0$, A. Korkine and G. Zolotareff [23] demonstrated:

$$C(f)\leq\frac{1}{\sqrt{5}}=C(x^2-xy-y^2)=C(f_0),$$

an isolated value giving also $C(f)\leq 1/\sqrt{8}$ for any other form f not $GL(2,\mathbb{Z})$-equivalent to f_0. Trying to understand this phenomenon motivated A.A. Markoff to write [27]. He described an infinity of values $C(f_i)_{i\in\mathbb{N}}$ comprised between $(1/\sqrt{5})$ and $(1/3)$ and having the same properties as $C(f_0)$. These values are isolated and convergent towards $(1/3)$. They can be built thanks to the tree of solutions of the Diophantine equation, so-called Markoff equation [9]:

$$m^2+m_1^2+m_2^2=3mm_1m_2.$$

For values $C(f)$ less than $(1/3)$, the author has shown that more general diophantine equations give an insight, sometimes with theories similar to the Markoff one [32], but with some complication. Moreover, a geometrical interpretation of such results has been found, similar to what was done by H. Cohn [8] for the classical Markoff theory. The general situation can be understood by the Teichmüller theory on the topological punctured torus $\mathcal{T}^\bullet$ (see for example [20]). This topological object is quite frequent in physical problems, for example when linked to the K.A.M. theorem [1], and some work has been done after the observation that the Markoff theory could be useful in order to understand the behavior of some oscillators [33]. It was possible to realize

that two types of geometric punctured tori exist, we called them hyperbolic and parabolic. The Markoff theory is then linked to the parabolic case, and geometrically to special fuchsian groups, the Fricke groups Γ as defined in [37]:

(1): Γ is isomorphic to a free group with two generators $\mathbf{F}_2 = \mathbb{Z} * \mathbb{Z}$.

(2): The Riemann surface $\mathcal{H}/\Gamma$ (where $\mathcal{H}$ is the Poincaré half-plane) is homeomorphic to a punctured torus.

The closed geodesics on such Riemann surfaces are linked to indefinite quadratic forms f and to the associated Markoff constant $C(f)$, which can be seen as shorter length of such geodesics [42]. The Markoff theory gives an explanation [38] to the quantification which appears when changing from such a geodesic to another: no continuous deformation is possible on the torus because of the puncture. Such remarks are the basis of the interest of physicists in this subject [17]. In another direction, it is known that the Markoff theory has links with the study of exceptional bundles and helices of the projective plane $P_2(\mathbb{C})$ (see [36, 30, 31, 14, 15, 11]), and also with the spectrum of Hermitian operators [22], and this is also important for physics [39]. A project that we had a long time ago was to build a common interpretation of such remarks in order to get the set of all Markoff constants of indefinite quadratic forms, the Markoff spectrum, as the spectrum of some operator on a Hilbert space. The reason for it is that the Markoff spectrum seems like the spectrum of some operators. It has a discrete part from $(1/\sqrt{5})$ to $(1/3)$, then a Cantorian part from $(1/3)$ to the Freiman number β, which is:

$$\beta^{-1} = 4 + \frac{253589820 + 283748\sqrt{462}}{491993569}.$$

From β to 0 the spectrum is continuous, any real number is a Markoff constant [9]. The present article gives hints of the possibility to implement such a project, building a possible operator to consider.

Looking at the link with the Fuchsian groups that we mentioned, the Markoff theory can be described [32] thanks to the two following matrices generating in $SL(2,\mathbb{Z})$ a free group isomorphic to $\mathbf{F}_2$ which is $[SL(2,\mathbb{Z}), SL(2,\mathbb{Z})]$:

$$A_0 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

These matrices give rise to a representation $\rho : \mathrm{Aut}(\mathbf{F}_2) \longrightarrow GL(2,\mathbb{Z})$, and we have described the algebraic importance of this situation in [32]. In this article, the main goal is to give a differential equation whose former representation ρ is the monodromy representation. With such a construction, we hope to understand the Lamé equations appearing for the accessory parameters of punctured tori [21,34]. Such equations have similarities with hypergeometric equations and also with some Schrödinger equations whose monodromy group has recently been studied [44]. Also we could soon build an Hamiltonian interpretation in the spirit of L.D. Fadeev [12] and others enabling us to realize our above mentionned project.

2 Considering the triangle group

The Markoff equation gives a complete tree of integer solutions thanks to the solution $(1,1,1)$ and to the three transformations which are involutions:

$$X : (m, m_1, m_2) \longmapsto ((3m_1m_2 - m, m_1, m_2),$$
$$Y : (m, m_1, m_2) \longmapsto (m, 3mm_2 - m_1, m_2),$$
$$Z : (m, m_1, m_2) \longmapsto (m, m_1, 3mm_1 - m_2).$$

The involutions X, Y et Z, give rise to the triangle group $\mathbf{T}_3 = \mathbf{C}_2 * \mathbf{C}_2 * \mathbf{C}_2$, which is the free product of three cyclic groups $\mathbf{C}_2$ with two elements. In [32], we showed how this group $\mathbf{T}_3$ is linked to the group of 2×2 matrices $GL(2,\mathbb{Z})$. For this we used an abelianisation morphism π' from the automorphism group $\mathrm{Aut}(\mathbf{F}_2)$ to $GL(2,\mathbb{Z})$, and two matrices generating the dihedral group $\mathbf{D}_6$ with 12 elements inside $GL(2,\mathbb{Z})$:

$$\pi'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \pi'(o) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

We also defined:

$$\pi'(X) = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}, \qquad \pi'(Y) = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}, \qquad \pi'(Z) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The group $\mathbf{T}_3$ acts in $GL(2,\mathbb{Z})$ defining with $ch = ch(X,Y,Z) \in \mathbf{T}_3$:

$$ch(\pi'(X),\pi'(Y),\pi'(Z)) = \pi'(ch(X,Y,Z)) \in \pi'(\mathbf{T}_3).$$

It gives a ternary decomposition in $GL(2,\mathbb{Z})$ using the triangle group (see [32] for demonstration):

Proposition 1. *Every element $V \in GL(2,\mathbb{Z})$ has a unique decomposition*

$$\pi'(o)^h \pi'(t)^k ch(\pi'(X),\pi'(Y),\pi'(Z)), \qquad \textit{where} \quad h = 0,1; \quad k = 0,1,\ldots,5; \quad ch \in \mathbf{T}_3.$$

The elements of $\pi'(\mathbf{T}_3)$ are characterized by the conditions $h = 0$ et $k = 0$. The group $\pi'(\mathbf{T}_3)$ is not normal inside $GL(2,\mathbb{Z})$. It is isomorphic by π' to the group $\mathbf{T}_3$. The elements of the group $\mathbf{D}_6$, not normal in $GL(2,\mathbb{Z})$, are characterized by the condition $ch(\pi'(X),\pi'(Y),\pi'(Z)) = \mathbf{1}_2$.

The group $\mathbf{D}_6$ introduces two equivalence relations in $GL(2,\mathbb{Z})$, which are defined with $V_1 \ \Re_{\mathbf{D}_6} \ V_2 \quad \Leftrightarrow \quad V_1 V_2^{-1} \in \mathbf{D}_6$ and $V_1 \ {}_{\mathbf{D}_6}\Re \ V_2 \quad \Leftrightarrow \quad V_1^{-1} V_2 \in \mathbf{D}_6$. The quotients $GL(2,\mathbb{Z})/\Re_{\mathbf{D}_6}$ and $GL(2,\mathbb{Z})/_{\mathbf{D}_6}\Re$ are equipotent, but different because $\mathbf{D}_6$ is not a normal subgroup of $GL(2,\mathbb{Z})$. Each $V \in GL(2,\mathbb{Z})$ defines a unique $ch(\pi'(X),\pi'(Y),\pi'(Z)) \in \pi'(\mathbf{T}_3)$, such that $V \ \Re_{\mathbf{D}_6} \ ch(\pi'(X),\pi'(Y),\pi'(Z))$. Hence, we get a description of the complete tree of the Markoff theory:

Proposition 2. *The group $\mathbf{T}_3$ is equipotent to the quotient (right or left) of the group $GL(2,\mathbb{Z})$ by its non-normal subgroup $\mathbf{D}_6$. It is an homogeneous $GL(2,\mathbb{Z})$-space. But also it can be considered a subgroup of $GL(2,\mathbb{Z})$, thanks to the former proposition.*

We find a decomposition using the free group $\mathbf{F}_2 \simeq [SL(2,\mathbb{Z}), SL(2,\mathbb{Z})]$:

Proposition 3. *Any element $V \in GL(2,\mathbb{Z})$ has a unique decomposition*

$$\begin{aligned} &\pm W(A_0,B_0) O^h W_k(S,T), \\ &W(A_0,B_0) \in \mathbf{F}_2 \simeq [SL(2,\mathbb{Z}), SL(2,\mathbb{Z})], \qquad h \in \{0,1\}, \\ &W_k(S,T) \in \{\mathbf{1}_2, S, ST, STS, STST, STSTS\} \qquad \textit{with} \quad k = 0,1,\ldots,5. \end{aligned}$$

The elements of the normal subgroup $SL(2,\mathbb{Z})$ in $GL(2,\mathbb{Z})$ are characterized by $h = 0$.

For the last proposition, we defined in $GL(2,\mathbb{Z})$:

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad O = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The words $W(A_0, B_0)$ are written in a multiplicative way with two generators $A_0 = [(TS)^{-1}, S^{-1}]$ and $B_0 = [S^{-1}, (TS)^{-2}]$ of $\mathbf{F}_2$, thanks to [26, p. 97–98]. In fact, we have a presentation with two generators T and $I = OS$ (see [2]):

$$GL(2,\mathbb{Z}) = \langle I, T^{-1} \mid I^2 = ([T^{-1}, I]T^{-1})^4 = ([T^{-1}, I]T^{-1}I)^2 = \mathbf{1}_2 \rangle.$$

The subgroup $\pi'(\mathbf{T}_3)$ is generated by $\pi'(X_0) = T^{-1}IOT^{-1}IOIT^{-1}B_0^{-1}$, $\pi'(Y_0) = IOIOA_0^{-1}TS$, $\pi'(Z_0) = IS$. Moreover [2], the triangle group $\mathbf{T}_3 \simeq \pi'(\mathbf{T}_3)$ is isomorphic to the projective

$$PGL(2,\mathbb{Z}) = \langle \overline{I}, \overline{T}^{-1} \mid \overline{I}^2 = ([\overline{T}^{-1}, \overline{I}]\overline{T}^{-1})^2 = ([\overline{T}^{-1}, \overline{I}]\overline{T}^{-1}\overline{I})^2 = \mathbf{1} \rangle.$$

We can verify that $\mathbf{F}_2 \simeq [PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]$ has an index 2 in this last group where we have, with $\mathbf{C}_3$ the cyclic group containing three elements, $\overline{V_1} = [\overline{I}, \overline{T}^{-1}]$ and $\overline{V_2} = [\overline{I}, \overline{T}]$:

$$[PGL(2,\mathbb{Z}), PGL(2,\mathbb{Z})] = \langle \overline{V_1}, \overline{V_2} \mid \overline{V_1}^3 = \overline{V_2}^3 = \mathbf{1} \rangle \simeq \mathbf{C}_3 * \mathbf{C}_3.$$

3 Conformal punctured tori

The conformal punctured tori are easily built with the Poincaré $\mathcal{H}$ half-plane. We use four geodesics of $\mathcal{H}$ referred to as αs, $s\beta$, βp, $p\alpha$, not crossing each other, and with α, s, β, p, on the border of $\mathcal{H}$. Any torus is given by transformations $t_A : \alpha p \to s\beta$, $t_B : \alpha s \to p\beta$. These transformations being given by matrices A and B of $SL(2,\mathbb{R})$ acting on $\mathcal{H}$ as conformal transformations, we can compute with $\alpha < 0$, $\beta > 0$, $c \neq 0$, $c' \neq 0$:

$$A = \begin{bmatrix} c\beta & -c\alpha\beta \\ c & (1/c\beta) - c\alpha \end{bmatrix}, \qquad \text{where} \quad c \neq 0,$$

$$B = \begin{bmatrix} c'\alpha & -c'\alpha\beta \\ c' & (1/c'\alpha) - c'\beta \end{bmatrix}, \qquad \text{where} \quad c' \neq 0,$$

$$A(\alpha) = s, \qquad A(p) = \beta, \qquad B(\beta) = s, \qquad B(p) = \alpha.$$

In $SL(2,\mathbb{R})$, the two matrices A and B generate $G = gp(A, B)$ and define a fuchsian group acting on $\mathcal{H}$, where P is the canonical projection from $SL(2,\mathbb{R})$ to $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm\mathbf{1}_2\}$:

$$\Gamma = PG = G/G \cap \{\pm\mathbf{1}_2\} = gp(P(A), P(B)).$$

The Markoff theory with $A = A_0$, $B = B_0$ is given by $c = \beta = -c' = -\alpha = 1$. For more general cases, we consider the commutator $L = [A, B] = ABA^{-1}B^{-1}$. It contents all the necessary information concerning the associated punctured torus because

$$L(s) = ABA^{-1}B^{-1}(s) = ABA^{-1}(\beta) = AB(p) = A(\alpha) = s.$$

Also $\operatorname{tr}(L) = \operatorname{tr}([A, B]) \leq -2$ and we get the condition due to Fricke:

$$\operatorname{tr}(L) + 2 = \operatorname{tr}(A)^2 + \operatorname{tr}(B)^2 + \operatorname{tr}(AB)^2 - \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(AB) \leq 0.$$

The parabolic case defined by the condition $\operatorname{tr}(L) = -2$ gives the Markoff equation again, thanks to a factor 3 in the traces which are related by:

$$\operatorname{tr}(A)^2 + \operatorname{tr}(B)^2 + \operatorname{tr}(AB)^2 = \operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(AB).$$

We now have a parametric representation with $(\lambda, \mu) \in \mathbb{R}^2 \backslash \{(0,0)\}$, also due to Fricke:

$$\operatorname{tr}(A) = \frac{1 + \lambda^2 + \mu^2}{\mu}, \qquad \operatorname{tr}(B) = \frac{1 + \lambda^2 + \mu^2}{\lambda}, \qquad \operatorname{tr}(AB) = \frac{1 + \lambda^2 + \mu^2}{\lambda\mu},$$

The Markoff theory is obtained with $\lambda = \mu = 1$. Easily:

Proposition 4. *Let (A,B) and (A',B') generate $\Gamma = Pgp(A,B)$ and $\Gamma' = Pgp(A',B')$, two Fricke groups associated to conformal punctured tori, we then have the following equivalence:*

1. *(A,B) et (A',B') are equivalent thanks to an interior automorphism of $GL(2,\mathbb{R})$:*

$$A = DA'D^{-1}, \qquad B = DB'D^{-1}, \qquad \text{where} \quad D \in GL(2,\mathbb{R}).$$

2. *The following two triples are equal:*

$$\Pi(A,B) = (\mathrm{tr}\,(B^{-1}), \mathrm{tr}\,(A), \mathrm{tr}\,(B^{-1}A^{-1})), \quad \Pi(A',B') = (\mathrm{tr}\,(B'^{-1}), \mathrm{tr}\,(A'), \mathrm{tr}\,(B'^{-1}A'^{-1})).$$

3. *The couples (A,B) and (A',B') give the same parameters λ, $\mu \in \mathbb{R}^+$*

$$\lambda = (\mathrm{tr}\,(A)/\mathrm{tr}\,(AB)) = (\mathrm{tr}\,(A')/\mathrm{tr}\,(A'B')), \qquad \mu = (\mathrm{tr}\,(B)/\mathrm{tr}\,(AB)) = (\mathrm{tr}\,(B')/\mathrm{tr}\,(A'B')).$$

It is easy to develop a theory of reduction for parabolic tori and to find a link with quaternions. It gives:

Proposition 5. *Any conformal equivalence from a parabolic punctured torus $\mathcal{T}^\bullet_\Gamma$ to itself given by an interior automorphism of $GL(2,\mathbb{R})$ is equal to identity.*

The study of the Laplacian on such surfaces is not so easy [43], though important for physics [29]. The parabolic punctured tori are of the form $\mathcal{H}/\langle A,B,L \mid [A,B]L^{-1} = 1\rangle$. Now if $\mathcal{T}^\bullet$ is the associated topological punctured torus, the conformal structure built on it thanks to A and B is only given [40] by a representation $\rho : \pi_1(\mathcal{T}^\bullet,*) \to SL(2,\mathbb{R})$, where $\pi_1(\mathcal{T}^\bullet,*) \simeq \mathbf{F}_2$ is the Poincaré group of the punctured torus. Introducing the space of all the deformations $\mathcal{R} = \mathcal{R}(\pi_1(\mathcal{T}^\bullet,*), PSL(2,\mathbb{R}))$, and the morphism $\overline{\rho} = P \circ \rho$, we find by this construction all the possible parabolic conformal punctured tori $\mathcal{H}/\overline{\rho}(\pi_1(\mathcal{T}^\bullet,*))$. This approach corresponds to the Teichmüller theory, here specialized to punctured tori. Replacing $PSL(2,\mathbb{R})$ by $PSL(2,\mathbb{C})$, under the former proposition 4, we also have a link with the variety of representations [25] of the group of Poincaré $\pi_1(\mathcal{T}^\bullet,*)$:

$$\rho \in \mathcal{R}(\pi_1(\mathcal{T}^\bullet,*), PSL(2,\mathbb{C})) \to (\mathrm{tr}\,\rho(g_1), \mathrm{tr}\,\rho(g_2), \mathrm{tr}\,\rho(g_3)) \in \mathbb{C}^3.$$

4 Monodromy

A monodromy representation of the group $\pi_1(\mathcal{T}^\bullet,*)$ is a morphism $\rho : \pi_1(\mathcal{T}^\bullet,*) \longrightarrow GL(n,\mathbb{C})$. Its image is the group of monodromy. Such representations are classified with interior automorphisms of $GL(n,\mathbb{C})$. They are considered in Fuchs differential equations (see [46, p. 75] and [16, 24]) as symmetries letting such an equation invariant:

$$\frac{d^n f}{dz^n} + a_1(z)\frac{d^{n-1}f}{dz^{n-1}} + \cdots + a_n(z)f = 0.$$

With $n = 2$ and $\pi_1(\mathcal{T}^\bullet,*) \simeq \mathbf{F}_2$ generated by A et B, the monodromy representations are completely described in [46, p. 80]. The irreducible ones are given thanks to an interior automorphism of $GL(2,\mathbb{C})$ with expressions

$$\rho(A) = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix}, \qquad \rho(B) = \begin{bmatrix} \mu_1 & 0 \\ (\nu_1+\nu_2) - (\lambda_1\mu_1 + \lambda_2\mu_2) & \mu_2 \end{bmatrix}, \qquad \lambda_i\mu_j \neq \nu_k.$$

They are uniquely determined by the couples (λ_1,λ_2), (μ_1,μ_2), (ν_1,ν_2) of eigenvalues of A, B and AB, with the former constraints. Diagonalizing the matrices A_0 et B_0 of the Markoff theory, we can consider:

$$\rho(A_0) = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 1 \\ 0 & \frac{3+\sqrt{5}}{2} \end{bmatrix}, \qquad \rho(B_0) = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 0 \\ -4 & \frac{3+\sqrt{5}}{2} \end{bmatrix}.$$

We now give a solution of the corresponding problem of Riemann–Hilbert, which consists in finding a differential equation having ρ as a monodromy representation. For this we use [46, theorem 4.3.2, p. 85] in order to compute the associated Riemann scheme. This way we find the following fuchsian equation (modified hypergeometric), given with $\sigma_3 = 1-\tau_3 = (\log(\frac{3\pm\sqrt{5}}{2}))/2i\pi$ under the following form:

$$z(1-z)\frac{d^2f}{dz^2} + (1-2z)\frac{df}{dz} - (\sigma_3\tau_3)f = \frac{\log(\frac{3+\sqrt{5}}{2})\log(\frac{3-\sqrt{5}}{2})}{4\pi^2 z(1-z)}f.$$

This equation which constitutes the main innovation of this article can be studied with the methods of [6]. Also, we get a differential operator whose spectral analysis is now to be made:

$$\mathfrak{L} = D^2 + \frac{(1-2z)}{z(1-z)}D - \frac{(\sigma_3\tau_3)4\pi^2 z(1-z) + \log\left(\frac{3+\sqrt{5}}{2}\right)\log\left(\frac{3-\sqrt{5}}{2}\right)}{4\pi^2 z^2(1-z)^2}.$$

The comparison of the corresponding spectrum with the Markoff spectrum must now be made and it will be detailed in a future article. In fact, there are two possibilities for $\mathfrak{L}$ due to the chosen value of σ_3 corresponding to the geometrical phenomenon of Schröder pairs for the same punctured torus, and showing that the two possibilities are linked by an easy transformation. A Hamiltonian interpretation could be important in the present case. It is effective for very important physical equation appearing in Physics (Lamé – that is to say periodical Schrödinger in one dimension [13], sine-Gordon, nonlinear Schrödinger, Korteweg–de Vries, ..., solitons) admitting a Hamiltonian representation, with states in an Hilbert space. It could give a solution for our above project.

5 Conclusion

The comparison with the hypergeometrical approach of Harvey Cohn [7] of the Markoff theory needs to be made. He discovered the link with the following relation between the classical modular function J automorphic for $PSL(2,\mathbb{Z})$ and the Weierstrass function $\wp$:

$$1 - J(\tau) = \wp'^2(z) = 4\wp^3(z) + 1.$$

He explained the link with triples of matrices (A, B, C) associated to the Markoff theory and gave the opportunity to look at a formula assuming an hexagonal symmetry

$$dz = \text{const} \times \frac{dJ}{J^{2/3}(J-1)^{1/2}}.$$

It does not seem to the author of this article that the way between these two formulas has been detailed. The problem is known to be linked to an accessory parameter [5, 21] verifying a Lamé differential equation [47, p. 110]. This question also has a link with the Hilbert's 22^{nd} problem [19]. This famous problem is not yet completely solved [41], even if the Lamé equations are much more studied today [44]. We suggest to get insight in this problem for punctured tori through the former developments. Considering the first of the last two equations and differentiating, we get the former differential relations:

$$-J'(\tau)d\tau = 12\wp^2(z)\wp'(z)dz, \qquad \wp'(z) = (1-J(\tau))^{1/2}, \qquad \wp^2(z) = (J(\tau)/4)^{2/3}.$$

The difficulties for integrating the differential relation between dz and dJ are known [45, p. 85–90], together with the links with the hypergeometric function $F(a,b,c,z)$ solution of the differential equation with two singularities $z = 0$ and $z = 1$, where $z \in \mathbb{C}$:

$$E(a,b,c): \quad z(1-z)\frac{d^2F}{dx^2} + (c-(a+b+1)z)\frac{dF}{dx} - abF = 0.$$

When the parameters a, b, c, are real and c, $c-a-b$, $a-b$, non integers, we find the Schwarz application on $\mathcal{D} = \mathbb{C}\backslash\{]-\infty, 0] \cup [1, \infty[\}$:

$$Sch : J \in \mathcal{D} \longrightarrow (F(a,b,c,J) : J^{1-c}F(a+1-c, b+1-c, 2-c, J)) \in \mathbf{P}^1(C).$$

The expression of H. Cohn between dz et dJ leads to consider the case $a = (1/3)$, $b = 0$, $c = (5/6)$ giving $|1-c| = (1/6)$, $|c-a-b| = (1/2)$, $|a-b| = (1/3)$. These values give confirmation that we are in an euclidian hexagonal crystal case. Also we get the known link with the work of R. Dedekind [10] and his function η. Indeed, we get $dz = w(\tau)^2 d\tau$ with:

$$w(\tau) = \text{const}\, \frac{J'(\tau)^{1/2}}{J(\tau)^{1/3}(1-J(\tau))^{1/4}}.$$

A new hypergeometric equation $E((1/12), (1/12), (2/3))$ appears between w and J. The function η is a square root of w (see [4, p.135] or [28, p.180]), which is known to precisely verify:

$$\eta(\tau)^{24} = \frac{1}{(48\pi^2)^3} \frac{J'(\tau)^6}{J(\tau)^4(1-J(\tau))^3}.$$

The function η has indeed a tight link with the Markoff equation [35].

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On ∗-Representations of One Deformed Quotient of Affine Temperley–Lieb Algebra

Nataly POPOVA

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *popova_n@yahoo.com*

We consider ∗-algebra generated by orthogonal projections with relations of Temperley–Lieb type. In this article we study all irreducible ∗-representations of this algebra and obtain the set of values of parameters when these representations exist.

1 Introduction

Temperley–Lieb algebras generated by n projections $p_1, \dots, p_n$ with relations

$$p_i p_j = p_j p_i, \qquad |i-j| > 1, \qquad p_i p_{i\pm1} p_i = \tau p_i, \qquad \tau \in \mathbb{R}, \tag{1}$$

appeared in [3, 4] in the context of ice-type models but they also play an important role in the analysis of subfactors of II_1 factor and in the knot theory (see, e.g., [5–7]). Jones proved that the chain (1) of orthogonal projections in Hilbert space with adding condition involving the trace can be infinite one if $\tau \in [0; 1/4] \cup \left\{ \frac{1}{4\cos^2 \frac{\pi}{n}} \,|\, n \geq 3 \right\}$.

In the present paper we consider ∗-algebra $TL_{\vec{\tau},n}$ generated by orthogonal projections $p_0, \dots, p_{n-1}$ with relations

$$p_i p_j = 0, \quad |i-j| > 1, \quad (i,j) \neq (0, n-1) \quad \text{and} \quad p_i p_{\overline{i+1}} p_i = \tau_i p_i, \quad p_i p_{\overline{i-1}} p_i = \tau_{\overline{i-1}} p_i. \tag{2}$$

In [2] we studied such ∗-algebra for $\tau_i = \tau$. For this more general algebra (2) we have found all irreducible ∗-representations and described the set of values of the parameters when these representations exist.

2 Description of all irreducible ∗-representations of algebra $TL_{\vec{\tau},n}$, their existence in depending on values of parameter $\vec{\tau}$

We study ∗-algebra over complex field generated by n ($n \geqslant 3$) orthogonal projections $p_0, \dots, p_{n-1}$ with relations of Temperley–Lieb type or orthogonality between any two projections. In other words, $p_i^2 = p_i^* = p_i$ and any projections p_i and p_j fulfil condition $p_i p_j = 0$ or for some $0 < \tau_{i,j} < 1$ relations $p_i p_j p_i = \tau_{i,j} p_i$ and $p_j p_i p_j = \tau_{i,j} p_j$ are correct. Such algebra can be described by a marked graph G with n vertices, where two vertices i, j are joined by a line marked with $\tau_{i,j}$ if and only if orthogonal projections p_i, p_j satisfy relations of Temperley–Lieb type. If $\vec{\tau} = (\tau_0, \dots, \tau_{n-1})$ with $0 < \tau_i < 1$ is fixed vector we may consider ∗-algebra $TL_{\vec{\tau},n}$ described by a graph (see Fig. 1).

In [1] there were proved that ∗-algebra $TL_{\vec{\tau},n}$ has only finite-dimensional irreducible ∗-representations, so in the following we consider nontrivial irreducible finite-dimensional ∗-representations of this algebra and name them simply 'representations'. If π is a ∗-representation of algebra $TL_{\vec{\tau},n}$ in unitary space H we write P_i for $\pi(p_i)$. Next theorem give a description of ∗-representations of algebra $TL_{\vec{\tau},n}$.

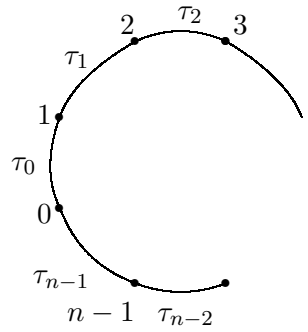

Figure 1.

Theorem 1. *Let irreducible $*$-representation of algebra $TL_{\vec{\tau},n}$ exists in unitary space H. Then we can find the orthonormal basis of H such that in this basis matrices of operators $P_0, \dots, P_{n-1}$ are as follows:*

$$P_0 = \operatorname{diag}(1, 0, \dots, 0),$$

$$P_i = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & t_{i-1} & \sqrt{t_{i-1} - t_{i-1}^2} & 0 & \cdots \\ 0 & \cdots & 0 & \sqrt{t_{i-1} - t_{i-1}^2} & 1 - t_{i-1} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad i = 1, \dots, n-2,$$

where $t_{i-1} = \frac{\tau_{i-1}}{1-t_{i-2}}$, $t_0 = \tau_0$ and the number of zeroes on the top of diagonal is equal to $i-1$.

$$P_{n-1} = \begin{pmatrix} \tau_{n-1} & b_1 & \cdots & b_{n-3} & \lambda & \mu \\ b_1 & \frac{b_1^2}{\tau_{n-1}} & \cdots & \frac{b_1 b_{n-3}}{\tau_{n-1}} & \frac{b_1 \lambda}{\tau_{n-1}} & \frac{b_1 \mu}{\tau_{n-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_{n-3} & \frac{b_1 b_{n-3}}{\tau_{n-1}} & \cdots & \frac{b_{n-3}^2}{\tau_{n-1}} & \frac{b_{n-3}\lambda}{\tau_{n-1}} & \frac{b_{n-3}\mu}{\tau_{n-1}} \\ \bar{\lambda} & \frac{b_1 \bar{\lambda}}{\tau_{n-1}} & \cdots & \frac{b_{n-3}\bar{\lambda}}{\tau_{n-1}} & \frac{|\lambda|^2}{\tau_{n-1}} & \frac{\bar{\lambda}\mu}{\tau_{n-1}} \\ \mu & \frac{b_1 \mu}{\tau_{n-1}} & \cdots & \frac{b_{n-3}\mu}{\tau_{n-1}} & \frac{\mu\lambda}{\tau_{n-1}} & \frac{\mu^2}{\tau_{n-1}} \end{pmatrix},$$

where $b_i = (-1)^i \tau_{n-1} \prod\limits_{j=0}^{i-1} \frac{t_j}{\sqrt{t_j - t_j^2}}$. Entry $\lambda \in \mathbb{C}$ that 'number' the representations is such that

$$\left| t_{n-3} b_{n-3} + \lambda \sqrt{t_{n-3} - t_{n-3}^2} \right|^2 = \tau_{n-2} \tau_{n-1} t_{n-3}$$

and $\mu = \sqrt{\tau_{n-1} - \tau_{n-1}^2 - \sum\limits_{j=1}^{n-3} b_j^2 - |\lambda|^2}$.

Remark 1. If parameter $\vec{\tau}$ is such that $t_{n-3} = 1$ the matrix of operator P_{n-1} differs from the one pointed out in the Theorem 1, more precisely, first $n-2$-nd rows and columns are the same but $n-1$-st (or even $n-1$-st and n-th) row and column are absent, b_{n-3} satisfies additional condition $b_{n-3}^2 = \tau_{n-2}\tau_{n-1}$ and $\mu^2 = \tau_{n-1} - \tau_{n-1}^2 - \sum\limits_{i=1}^{n-3} b_i^2$, $\left(\tau_{n-1} - \tau_{n-1}^2 - \sum\limits_{i=1}^{n-3} b_i^2 = 0\right)$.

Remark 2. The rank of all orthogonal projections P_i is 1 and dimension of irreducible $*$-representation may be equal to n, $n-1$, or to $n-2$.

Remark 3. If parameter $\vec{\tau}$ is fixed then different permissible λ's define inequivalent irreducible $*$-representations. So, we may say that each irreducible $*$-representation of algebra $TL_{\vec{\tau},n,}$ is given by the number λ.

Now our goal is to produce the set of values of parameter $\vec{\tau}$ for that the $*$-representations exist. Let $F_i^{(k)}$, $i \geq 0$, $0 \leq k \leq n-1$ be the collection of numbers given by recurrent formulas

$$F_0^{(k)} = F_1^{(k)} = 1, \qquad F_{i+2}^{(k)} = F_{i+1}^{(k)} - \tau_{\overline{i+k}} F_i^{(k)}.$$

Proposition 1. *The irreducible $*$-representations of algebra $TL_{\vec{\tau},n,}$ exist if and only if one of following two cases takes place:*

1) $F_i^{(0)} > 0$, $i = 2, \ldots, n-1$ *and at least one of the following inequalities is true*

$$\frac{\left|(-1)^n \sqrt{\tau_0 \cdots \tau_{n-3}\tau_{n-1}} \pm \sqrt{\tau_{n-2}} F_{n-2}^{(0)}\right|}{\sqrt{F_{n-1}^{(0)}}} \leq \sqrt{(1-\tau_{n-1})\, F_{n-2}^{(0)} - \tau_0 \tau_{n-1} F_{n-4}^{(2)}},$$

2) $F_i^{(0)} > 0$, $i = 2, \ldots, n-2$, $F_{n-1}^{(0)} = 0$, $F_{n-2}^{(0)} = \sqrt{\frac{\tau_0 \cdots \tau_{n-3}\tau_{n-1}}{\tau_{n-2}}}$ *and*

$$1 - \tau_{n-1} - \tau_0 \tau_{n-1} \frac{F_{n-4}^{(2)}}{F_{n-2}^{(0)}} \geq 0.$$

Note that for $n = 3$ the expressions in the proposition 1 will be correct if $P_{-1}^{(2)} := 0$.

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Twisted Product of Fock Spaces

Daniil P. PROSKURIN

Department of Cybernetics, Kyiv Taras Shevchenko University,
60 Volodymyrs'ka Str., 01033 Kyiv, Ukraine
E-mail: *prosk@unicyb.kiev.ua*

We discuss the construction of the twisted product of the interacting Fock spaces.

1 Introduction

Recall that the C^*-probability space is the pair $(\mathcal{A},\phi)$, where $\mathcal{A}$ is a C^*-algebra and ϕ is a state. The self-adjoint element $a\in\mathcal{A}$ is called a self-adjoint quantum random variable. The distribution of a with respect to ϕ is a probability measure μ on $\mathbb{R}$ such that

$$\phi(a^k)=\int_{\mathbb{R}} t^k d\mu(t).$$

Remark 1. If $\mathcal{A}$ is a $*$-algebra or W^*-algebra then the corresponding spaces are called the $*$-algebraic and W^*-algebraic quantum probability spaces respectively.

The notion of monotone independence was introduced by N. Muraki in [10].

Definition 1. The family $\{Y_i,\ i=1,\dots,n\}$ of random variables in $(\mathcal{A},\phi)$ is called anti-monotone independent if the following two conditions hold

1. $Y_i^pY_j^rY_k^s=\phi(Y_j^r)Y_i^pY_k^s$ for any $i>j<k$ and $p,s\in\mathbb{N}$.
2. For any $i_1<i_2<\cdots<i_s<j>j_t>\cdots>j_2>j_1$

$$\phi\big(Y_{i_1}^{k_1}\cdots Y_{i_s}^{k_s}Y_j^lY_{j_t}^{r_t}\cdots Y_{j_1}^{r_1}\big)=\prod_{\nu=1}^{s}\phi\big(Y_{i_\nu}^{k_\nu}\big)\phi\big(Y_j^l\big)\prod_{\omega=1}^{t}\phi\big(Y_{i_\omega}^{k_\omega}\big).$$

The notation $i<j>k$ means $i<j$, $k<j$ and $i>j<k$ means $i>j$, $k>j$.
The notion of boolean independence was studied by many authors, see for example [11].

Definition 2. Let $(\mathcal{B},\phi)$ be the $*$-algebraic probability space. The family of elements $\{X_i,\ i\in\mathcal{I}\}$ is called boolean independent if one has

$$\phi\big(X_{i_1}^{k_1}X_{i_2}^{k_2}\cdots X_{i_n}^{k_n}\big)=\phi\big(X_{i_1}^{k_1}\big)\phi\big(X_{i_2}^{k_2}\big)\cdots\phi\big(X_{i_n}^{k_n}\big)$$

for any $i_1\neq i_2\neq\cdots\neq i_n\in\mathcal{I}$.

The notion of the interacting Fock space was introduced by L. Accardi, Y. Lu and I. Volovich, see [2]. Let $\mathcal{H}$ be the Hilbert space. Denote by $\mathcal{T}(\mathcal{H})$ the full tensor space over $\mathcal{H}$ and by ω the vacuum vector. Then construct operators of the creation $a(f)$, $f\in\mathcal{H}$,

$$a(f)f_1\otimes f_2\otimes\cdots\otimes f_n=f\otimes f_1\otimes\cdots\otimes f_n,\qquad n\in\mathbb{N},\quad a(f)\omega=f.$$

Finally for any $n\in\mathbb{N}$ we supply $\mathcal{H}^{\otimes n}$ with some scalar product $\langle\cdot\mid\cdot\rangle_n$ and define the scalar product $\langle\cdot\mid\cdot\rangle$ on the $\mathcal{T}(\mathcal{H})$ by the rule

$$\mathcal{H}^{\otimes n}\perp\mathcal{H}^{\otimes m},\quad m\neq n,\qquad \langle x\mid y\rangle=\langle x\mid y\rangle_n,\quad x,y\in\mathcal{H}^{\otimes n}.$$

The operators $a^*(f)$, $f \in \mathcal{H}$ are called the annihilation operators. The following properties of the annihilation operators are obvious

$$a^*(f)\omega = 0, \qquad a^*(f)\colon \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n-1}, \qquad n \in \mathbb{N}.$$

Definition 3. The system $\mathcal{F} = \Big(\mathcal{T}(H), \langle\cdot \mid \cdot\rangle, a(f), a^*(f),\ f \in \mathcal{H}\Big)$ is called the interacting Fock space.

The operators $X(f) = a(f) + a^*(f)$, $f \in \mathcal{H}$ are called the field operators and the operator N, defined by the rule

$$N\omega = 0, \qquad N(f_1 \otimes \cdots \otimes f_k) = kf_1 \otimes \cdots \otimes f_k, \qquad k \in \mathbb{N}$$

is called the number operator. The vector state associated with vacuum vector ω is called Fock state. If $\dim \mathcal{H} = 1$ then the corresponding interacting Fock space is called a one-mode interacting Fock space.

The role of the one-mode interacting Fock spaces in quantum probability was clarified by L. Accardi and M. Bożejko in [1]. Namely, it was shown that any self-adjoint quantum variable can be realized in the form $a + a^* + f(N)$, where a, a^*, N are the creation, annihilation and the number operators acting on some one-mode interacting Fock space.

Let us discuss some type of general central limit theorem of the type considered by R. Speicher and W. von Waldenfels, see [15].

Theorem 1. *Consider the $*$-algebra $\mathcal{A}$ and state ϕ and the sequence of elements $a_i, a_i^* \in \mathcal{A}$, $i \in \mathbb{N}$. Denote by $b_{2i-1} := a_i$, $b_{2i} := a_i^*$ and $X_i := a_i + a_i^*$. Put S_N to be*

$$S_N = \frac{X_1 + \cdots + X_N}{\sqrt{N}}.$$

Suppose that the following assumptions are satisfied.

(i) For any odd $n \in \mathbb{N}$ one has $\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) = 0$.

(ii) For even n the mixed moment $\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) \neq 0$ only if $(\sigma(1), \ldots, \sigma(n))$ is the permutation (with replications) of the collection

$$\{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \ldots, 2i_k - 1, 2i_k,\ k \leq n/2\}.$$

(iii) Let $(\sigma(1), \ldots, \sigma(n))$ be the permutation of the collection

$$\{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \ldots, 2i_k - 1, 2i_k,\ k \leq n/2\},$$

where $i_1 < i_2 < \cdots < i_k$. Then for any $\{j_1 < j_2 < \cdots < j_k\}$ construct $(\widetilde{\sigma}(1), \cdots, \widetilde{\sigma}(n))$ by the rule

$$\widetilde{\sigma}(i) = 2j_s, \quad if \quad \sigma(i) = 2i_s, \qquad \widetilde{\sigma}(i) = 2j_s - 1, \quad if \quad \sigma(i) = 2i_s - 1.$$

Obviously the relation $\sigma \sim \widetilde{\sigma}$ is equivalence. Then if σ and $\widetilde{\sigma}$ are equivalent permutations with property $\sharp\sigma^{-1}(i) = 1$, we have the equality

$$\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) = \phi(b_{\widetilde{\sigma}(1)} \cdots b_{\widetilde{\sigma}(n)}).$$

(iv) For any even $n \in \mathbb{N}$ there exists the constant $C_n > 0$ such that

$$\forall\, \sigma \quad |\phi(b_{\sigma(1)} \cdots b_{\sigma(n)}| < C_n.$$

Then for even n

$$\lim_{N\to\infty} \phi(S_N^n) = \frac{1}{(n/2)!} \sum_{\sigma\in S_n} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}).$$

For odd n

$$\phi(S_N^n) = 0 \quad \forall\, N.$$

Proof. We use the standard arguments, see, for example [15]. For any even n on has

$$\phi(S_N^n) = N^{-\frac{n}{2}} \sum_{\sigma\colon \{1,\dots,n\}\to\{1,\dots 2N\}} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}).$$

By the condition (ii) we have consider only σ which are the permutations of the collections

$$\{2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \dots, 2i_k - 1, 2i_k,\ k \le n/2\}.$$

Denote by $\mathcal{M}$ the union of the classes of equivalence of σ. Let $\mathcal{M}_k$ be the set of classes of equivalence consisting of σ with $\sharp\{\sigma(i)\} = 2k$, $k = 1, \dots, n/2$. For any $\mathfrak{m} \in \mathcal{M}_k$ there exists a unique $\sigma_{\mathfrak{m}} \in \mathfrak{m}$, which is the permutation with replications of the collection $\{1, 2, \dots, 2k\}$. Obviously, for any $\mathfrak{m} \in \mathcal{M}_k$ one has $\sharp\mathfrak{m} = C_N^k$, since any $\sigma \in \mathfrak{m}$ is uniquely determined by the ordered collection $1 \le i_1 < \cdots < i_k \le N$. Then

$$\begin{aligned}\phi(S_N^n) &= N^{-\frac{n}{2}} \sum_{k=1}^{\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} \sum_{\sigma\in\mathfrak{m}} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}) \\ &= N^{-\frac{n}{2}} \sum_{k<\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} \sum_{\sigma\in\mathfrak{m}} \phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)}) + N^{-\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_{n/2}} C_N^{n/2} \phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)}).\end{aligned}$$

Since the number of summands is finite and independent on N and

$$\lim_{N\to\infty} N^{-\frac{n}{2}} C_N^k = 0, \quad k < \frac{n}{2}, \qquad \lim_{N\to\infty} N^{-\frac{n}{2}} C_N^{n/2} = \frac{1}{(n/2)!}$$

and

$$\begin{aligned}\left| N^{-\frac{n}{2}} \sum_{k<\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} \sum_{\sigma\in\mathfrak{m}} \phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)}) \right| &\le N^{-\frac{n}{2}} \sum_{k<\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} \sum_{\sigma\in\mathfrak{m}} |\phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)})| \\ &\le C_n \sum_{k<\frac{n}{2}} \sum_{\mathfrak{m}\in\mathcal{M}_k} N^{-\frac{n}{2}} C_N^k \to 0, \qquad N \to \infty.\end{aligned}$$

Thus

$$\lim_{N\to\infty} \phi(S_N^n) = \frac{1}{(n/2)!} \sum_{\mathfrak{m}\in\mathcal{M}_{n/2}} \phi(b_{\sigma_{\mathfrak{m}}(1)} \cdots b_{\sigma_{\mathfrak{m}}(n)}).$$

Evidently, the canonical representatives $\sigma_{\mathfrak{m}}$ of classes $\mathfrak{m} \in \mathcal{M}_{n/2}$ are the elements $\sigma_{\mathfrak{m}} \in S_n$, where we denote by S_n the group of permutations on n symbols; please do not confuse with S_N. Finally

$$\lim_{N\to\infty} \phi(S_N^n) = \frac{1}{(n/2)!} \sum_{\sigma\in S_n} \phi(b_{\sigma(1)} \cdots b_{\sigma(n)}).$$

The equality $\phi(S_N^n) = 0$ for odd n is evident. ■

Example 1. Consider the family of centered identically distributed elements $\{a_i, a_i^*,\ i \in \mathbb{N}\}$ satisfying the following conditions

1. $a_i^* a_j = q a_j a_i^*$, $a_j a_i = q a_i a_j$, $i < j$, $q \neq 0$.
2. If $\sigma(1) < \cdots < \sigma(k)$, one has

$$\phi(y_{\sigma(1)} \cdots y_{\sigma(k)}) = \phi(y_{\sigma(1)}) \cdots \phi(y_{\sigma(k)}),$$

where $y_s \in \{a_s, a_s^*, a_s^* a_s, a_s a_s^*\}$.

Then the assumptions of the central limit theorem hold. Moreover, if we suppose that operators a_i, a_i^*, $i \in \mathbb{N}$ are the creation and annihilation operators acting on the interacting Fock space and ϕ is the Fock functional, then we additionally have the property $\phi(b_{\sigma(1)} \cdots b_{\sigma(2k)}) \neq 0$, where $\sigma \in S_{2k}$, only if $\sigma^{-1}(2i) < \sigma^{-1}(2i-1)$ for any $i = 1, \ldots, k$.

In particular, if a_i, a_i^* are the creation and annihilation operators on the twisted product of copies of the one-mode interacting Fock space, see next Section, and ϕ is the Fock state, the conditions above are satisfied.

Example 2. Suppose that X_i, $i \in \mathbb{N}$ are anti-monotone independent centered symmetric identically distributed random variables with variance 1. Then one can realize them as the field operators acting on the monotone product of the one-mode interacting Fock spaces, i.e. suppose that $X_i = a_i + a_i^*$, a_i the creation and a_i^* the annihilation operators and ϕ is the Fock state. Let us find the measure given by the central limit theorem. To do it we note that $\phi(b_{\sigma(1)} \cdots b_{\sigma(2m)})$ is either 1 or 0. Let us find the number of the non-zero summands in the sum

$$\sum_{\sigma \in S_{2m}} \phi(b_{\sigma(1)} \cdots b_{\sigma(2m)}) \tag{1}$$

Firstly note that $\phi(Y_1 a_i Y_2 a_i^* Y_3) = 0$ if $Y_2 \neq 1$, here

$$Y_1 = \prod_{j < \sigma^{-1}(2i-1)} b_{\sigma(j)}, \qquad Y_2 = \prod_{\sigma^{-1}(2i-1) < j < \sigma^{-1}(2i)} b_{\sigma(j)}, \qquad Y_3 = \prod_{\sigma^{-1}(2i) < j} b_{\sigma(j)}.$$

Further, by definition of the anti-monotone independence if $Y_2 \neq 1$ we have $\phi(Y_1 a_1^* Y_2 a_1 Y_3) = \phi(a_1^*)\phi(Y_1 Y_2 a_1^* Y_3) = 0$, here

$$Y_1 = \prod_{j < \sigma^{-1}(2)} b_{\sigma(j)}, \qquad Y_2 = \prod_{\sigma^{-1}(2) < j < \sigma^{-1}(1)} b_{\sigma(j)}, \qquad Y_3 = \prod_{\sigma^{-1}(2i) < j} b_{\sigma(j)}.$$

Hence if $\phi(b_{\sigma(1)} \cdots \sigma(n)) \neq 0$ we have $Y_2 = 1$ and

$$\phi(b_{\sigma(1)} \cdots \sigma(n)) = \phi(Y_1 a_1^* a_1 Y_3) = \phi(a_1^* a_1)\phi(Y_1 Y_3),$$

where $Y_1 Y_3$ is any product of $a_2, a_2^*, \ldots, a_m, a_m^*$ where each term appears only once. Let k_m be the number of the non-zero summands in (1), then arguments presented above imply the following recurrent formula

$$k_m = (2m-1) k_{m-1}.$$

Indeed, we have $2m-1$ different positions for $a_1^* a_1$ in our permutation. Evidently, $k_1 = 1$, hence

$$k_m = \lim_{N \to \infty} \frac{1}{m!} \phi(S_N^{2m}) = \frac{(2m-1)!!}{m!} = \frac{C_{2m}^m}{2^m},$$

and these moments correspond to the arcsin distribution with density

$$d\mu(x) = 1/\pi\ \chi\big(-\sqrt{2}, \sqrt{2}\big) \frac{dx}{\sqrt{2 - x^2}}.$$

Example 3. Boolean central limit theorem. Let $\{a_i, a_i^*\}$ are boolean independent family of the creation and annihilation operators acting on the boolean Fock space. As above suppose that ϕ is the Fock state and $\phi(a_i^* a_i) = 1$, $i = 1, \dots, d$. Then all conditions of the central limit theorem are satisfied. Since for any i

$$\phi\left(Y_1 a_i^{(\varepsilon_i)} Y_2 a_i^{(-\varepsilon_i)} Y_3\right) = \phi(Y_1)\phi\left(a_i^{(\varepsilon_i)}\right)\phi(Y_2)\phi\left(a_i^{(-\varepsilon_i)}\right)\phi(Y_3) = 0, \quad \text{if } Y_2 \neq 1 \text{ or } \varepsilon_i = 1,$$

where $\varepsilon_i \in \{1, -1\}$ and $a_s^{(1)} = a_s$, $a_s^{(-1)} = a_s^*$. Hence, as in the monotone case, we have $\phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) \neq 0$ only if

$$b_{\sigma(1)} \cdots b_{\sigma(2n)} = Y_1 a_1^* a_1 Y_3$$

and

$$\phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) = \phi(a_1^* a_1)\phi(Y_1)\phi(Y_3),$$

where Y_1 is the word obtained by some permutation of the word $a_{i_1}^* a_{i_1} \cdots a_{i_k}^* a_{i_k}$, $i_1 < i_2 < \cdots < i_k \in \{2, \dots, n\}$ and analogously for Y_3. The arguments presented above shows that $\phi(b_{\sigma(1)} \cdots b_{\sigma(2n)}) \neq 0$ if and only if

$$b_{\sigma(1)} \cdots b_{\sigma(2n)} = a_{\pi(1)}^* a_{\pi(1)} \cdots a_{\pi(n)}^* a_{\pi(n)} = 1,$$

where $\pi \in S_n$ is any permutation. Hence we have

$$\lim_{N\to\infty} \phi(S_n^{2n}) = \frac{1}{n!} n! = 1$$

so, $m_{2n-1} = 0$ and $m_{2n} = 1$ – the moments of the discrete measure concentrated on $\{-1, 1\}$.

2 Twisted product

In this Section we discuss the construction of the twisted product Fock space. This is the special kind of the interacting Fock space.

Let $\mathcal{I}$ be totally ordered set. Consider the collection of the one-mode interacting Fock spaces

$$\{(\mathcal{T}(\mathcal{H}_i), a_i, a_i^* \mid \ i \in \mathcal{I}\}.$$

Let $\Omega_i \in \mathcal{T}(\mathcal{H}_i)$ be vacuum vector, consider the orthonormal system

$$\left\{e_i^{(n)}, \ n \in \mathbb{Z}_+\right\},$$

such that $e_i^{(0)} := \Omega_i$ and $a_i e_i^{(n)} = \alpha_i^{(n)} e_{n+1}$, $n \in \mathbb{Z}_+$. Denote by ϕ_i the Fock state on $\mathcal{T}(\mathcal{H}_i)$. Then consider the Hilbert space $\mathcal{T}$ with orthonormal basis

$$\Omega, \quad e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad i_1 < \cdots < i_k, \quad k \in \mathbb{N}, \quad i_s \in \mathcal{I}, \quad n_s \in \mathbb{N}, \quad s = 1, \dots, k.$$

Further define the creation operators $\widetilde{a}_j$, $j \in \mathcal{I}$

$$\begin{aligned}
&\widetilde{a}_j \Omega = \alpha_j^{(0)} e_j^{(1)},\\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)} = \mu^k \alpha_j^{(0)} e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)} \otimes e_j^{(1)}, \quad j > i_k,\\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}\\
&\qquad = \mu^{n_1+\cdots+n_s} \alpha_j^{(0)} e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s)} \otimes e_j^{(1)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad i_s < j < i_{s+1},\\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}\\
&\qquad = \mu^{n_1+\cdots+n_{s-1}} \alpha_j^{(n_s)} e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s+1)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad j = i_s,\\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)} = \alpha_j^{(0)} e_j^{(1)} \otimes e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad j < i_1.
\end{aligned}$$

For the adjoint (annihilation) operators $\widetilde{a}_j^*$ one has the following

$$\begin{aligned}
&\widetilde{a}_j^*\Omega = 0,\\
&\widetilde{a}_j e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_k}^{(n_k)} = 0, \quad j \neq i_s, \quad s = 1, \ldots, k,\\
&\widetilde{a}_j^* e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}\\
&\qquad = (\overline{\mu})^{n_1+\cdots+n_{s-1}} (\overline{\alpha}_j)^{(n_s-1)} e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(n_s-1)} \otimes e_{i_{s+1}}^{(n_{s+1})} \otimes \cdots \otimes e_{i_k}^{(n_k)}, \quad j = i_s,
\end{aligned}$$

where we identify $e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_s}^{(0)} \otimes \cdots \otimes e_{i_k}^{(n_k)}$ with $e_{i_1}^{(n_1)} \otimes \cdots \otimes e_{i_{s-1}}^{(n_{s-1})} \otimes e_{i_{s+1}}^{(n_{s+1})}$.

We call the interacting Fock space $(\mathcal{T}, \widetilde{a}_i, \widetilde{a}_i^*, i \in \mathcal{I})$ the **twisted product** of the one-mode interacting Fock spaces $(\mathcal{T}(\mathcal{H}_i), a_i, a_i^*)$, $i \in \mathcal{I}$ with the twist parameter $\mu \in \mathbb{C}$. Below we denote by ϕ the Fock state, i.e. the vector state defined by Ω, on $\mathcal{T}$.

It is easy to verify that the operators $\widetilde{a}_i$, $\widetilde{a}_j$ and $\widetilde{a}_i^*$, $\widetilde{a}_j$, $i > j$ satisfy the μ-commutation relations, i.e.

$$\widetilde{a}_i^*\widetilde{a}_j = \mu\widetilde{a}_j\widetilde{a}_i^*, \qquad \widetilde{a}_i\widetilde{a}_j = \mu\widetilde{a}_j\widetilde{a}_j.$$

One can verify also that the joint distributions of a_i, a_i^* with respect to ϕ_i and $\widetilde{a}_i$, $\widetilde{a}_i^*$ with respect to ϕ coincide, i.e. for any non-commutative polynomial $p(x, y)$ one has

$$\phi_i(p(a_i, a_i^*)) = \phi(p(\widetilde{a}_i, \widetilde{a}_i^*)).$$

Finally note that for $\mu = 1$ one has the usual tensor product and for $\mu = 0$ the monotone product of interacting Fock spaces considered by N. Muraki, see [12].

When we have the finite set $\mathcal{I} = \{1, 2, \ldots, d\}$ the twisted product is just the twisted Fock space constructed by W. Pusz and S.L. Woronowicz, see [13]. In this case the orthonormal basis of $\mathcal{T}$ has the form

$$e_1^{(n_1)} \otimes \cdots \otimes e_d^{(n_d)}, \qquad n_s \in \mathbb{Z}_+, \quad s = 1, \ldots, d,$$

where $\Omega := e_1^{(0)} \otimes \cdots \otimes e_d^{(0)}$, i.e. $\mathcal{T} = \bigotimes_{i=1}^d \mathcal{T}(\mathcal{H}_i)$. In this case the operators $\widetilde{a}_i$, $i = 1, \ldots, d$ can be presented as tensor products

$$\widetilde{a}_i = \bigotimes_{j=1}^{i-1} d(\mu) \otimes a_i \otimes \bigotimes_{j=i+1}^{d} 1,$$

where $d(\mu)e_j^{(n)} = \mu^n e_j^{(n)}$, $n \in \mathbb{Z}_+$, $j = 1, \ldots, d$. If we consider the special case $\mu = 0$, we get the realization of the monotone independent non-commutative random variables constructed by U. Franz, see [6]

$$\widetilde{a}_i = \bigotimes_{j=1}^{i-1} P_j \otimes a_i \otimes \bigotimes_{j=i+1}^{d} 1.$$

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On Polynomial Identities in Algebras Generated by Idempotents and Their $*$-Representations

Vyacheslav I. RABANOVICH and Alexander V. STRELETS

Institute of Mathematics of NAS Ukraine, 3 Tereschenkivska Str., 01601 Kyiv-4, Ukraine
E-mail: *slavik@imath.kiev.ua, sav@imath.kiev.ua*

Among algebras generated by linearly dependent idempotents we look for algebras with polynomial identities. One of the way to find a polynomial identity in an algebra is to calculate a linear basis for this algebra, build a "big" family of representations and prove using the basis that it is a residual family. In the paper was found a linear basis for some algebra generated by idempotents.

1 Introduction

Algebras generated by linearly dependent idempotents are investigated in the paper. Among these algebras we look for algebras with polynomial identities, so called PI-algebras (see, for example, [1]). The theory of PI-algebras is well developed and gives additional information about algebras. So for applications it is important to determine that an algebra has a polynomial identity. One of the way to find a polynomial identity in an algebra is to calculate a linear basis for this algebra, build a "big" family of representations such that the supremum of dimensions of representations from the family is finite and prove that it is a residual family (see [2, Theorem 2] or [3]).

Algebras which we are interested in are

$$\mathbf{Q}_{n,\overline{\lambda}} = \mathbb{C}\langle q_1, \dots, q_n \,|\, q_k^2 = q_k, \sum_{k=1}^{n} \lambda_k q_k = e\rangle,$$

$n \in \mathbb{N}$, $\overline{\lambda} \in \mathbb{C}^n$, and its factor-algebras $\mathbf{Q}_{n,\overline{\lambda}}/\{q_{i_l} q_{j_l} = q_{j_l} q_{i_l} = 0\}_{l=1}^{m}$ where $i_l \neq j_l \in \{1, \dots, n\}$.

In paper [4] a criterion was given when algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are PI-algebras (note that the case $\lambda_i = \frac{1}{\lambda}$ was studied in the paper [5]). We remind the respective results.

In the case $n \leqslant 3$ all algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are finite-dimensional and so they are PI-algebras. In the case $n \geqslant 4$ all algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are infinite-dimensional. Let $\hat{\delta}(\overline{\lambda}) = 1 - \frac{1}{2}\sum\limits_{j=1}^{n} \lambda_j$. Then we have the following results.

Theorem 1. *If $\hat{\delta}(\overline{\lambda}) \neq 0$ then the algebra $\mathbf{Q}_{4,\overline{\lambda}}$ is not a PI-algebra.*

Corollary 1. *When $n \geqslant 5$ all algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are not PI-algebras.*

Theorem 2. *If $\hat{\delta}(\overline{\lambda}) = 0$ then the algebra $\mathbf{Q}_{4,\overline{\lambda}}$ is an F_4-algebra, i.e.*

$$\sum_{\sigma \in S_4} (-1)^{p(\sigma)} v_{\sigma(1)} v_{\sigma(2)} v_{\sigma(3)} v_{\sigma(4)} = 0,$$

for any elements $v_1, v_2, v_3, v_4 \in \mathbf{Q}_{n,\overline{\lambda}}$, where S_4 is a symmetric group.

The Corollary 1 shows that when $n \geqslant 5$ algebras $\mathbf{Q}_{n,\overline{\lambda}}$ are not PI-algebras. But if we add some relations, for example, relations of commuting ($q_iq_j = q_jq_i$) or orthogonality ($q_iq_j = 0$) for some generators, then some of obtained factor-algebras are PI-algebras and are infinite-dimensional.

In paper [6] for the algebra

$$\mathcal{R} = \mathbb{C}\langle p_1, p_2, p_3, q_1, q_2, q_3 \,|\, q_k^2 = q_k, p_k^2 = p_k, p_kq_k = 0, \sum_{k=1}^{3}(p_k + 2q_k) = 3e\rangle,$$

the following results were obtained:

Lemma 1. *The algebra $\mathcal{R}$ is infinite-dimensional and has the quadratic growth.*

Theorem 3. *The algebra $\mathcal{R}$ is an F_6-algebra.*

In both cases to prove Theorem 2 and Theorem 3 a residual family for corresponding algebras was found.

In this paper we will find a linear basis of an algebra

$$\mathbf{A} = \mathbb{C}\langle a, b \,|\, a^3 + \alpha_1 a + \alpha_0 = 0,\ b^3 + \beta_1 a + \beta_0 = 0,\ (a+b)^3 + \gamma_1(a+b) + \gamma_0 = 0\rangle,$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$. But first we give some argumentation to solve this problem. Consider algebras

$$\mathbf{A}' = \mathbb{C}\langle p_1, p_2, p_3, q_1, q_2, q_3 \,|\, q_k^2 = q_k, p_k^2 = p_k, p_kq_k = 0, \sum_{k=1}^{3}(\mu_k p_k + \nu_k q_k) = e\rangle,$$

where $\mu_k, \nu_k \in \mathbb{C}$, $\mu_k \neq \nu_k$, $\sum\limits_{k=1}^{3}(\mu_k + \nu_k) = 3$ (in the case of $\mu_k = 1/3$, $\nu_k = 2/3$ we have the algebra $\mathcal{R}$). To find an answer the question whether some family of representations is a residual family we need a linear basis in this algebra or in some algebra which is isomorphic with this one.

The algebra $\mathbf{A}'$ and an algebra

$$\mathbf{A}'' = \mathbb{C}\langle x_1, x_2, x_3 \,|\, x_1 + x_2 + x_3 = e,\ x_k(x_k - \mu_k)(x_k - \nu_k) = 0\rangle$$

are isomorphic. A map $x_k \mapsto \mu_k p_k + \nu_k q_k$ is a corresponding isomorphism. A map $x_k \mapsto x'_k + (\mu_k + \nu_k)/3$ gives isomorphism between the algebra $\mathbf{A}''$ and an algebra

$$\mathbf{A}''' = \mathbb{C}\langle x'_1, x'_2, x'_3 \,|\, x'_1 + x'_2 + x'_3 = 0,\ f_k(x_k) = 0\rangle,$$

where f_k are polynomials such that $\deg f_k = 3$ and sum of roots is zero. So this algebra and the algebra $\mathbf{A}$ are isomorphic.

2 A linear basis in the algebra A

We introduce a homogeneous lexicographical order on words in alphabet $\{a, b\}$: $a < b$. Using the Diamond Lemma (for definitions of the notions *Gröbner basis, reductions, compositions, growth of an algebra* etc. see review [7] and references therein) we will prove the following theorem which describes a linear basis in the algebra $\mathbf{A}$. We will denote reductions by symbol $\to$.

Theorem 4. *The set of words*

$$\left\{a^{\sigma_1}(ba^2)^n(ba)^m b^{\sigma_2} \,|\, \sigma_1, \sigma_2 \in \{0,1,2\}, n, m \in \mathbb{N} \cup \{0\}\right\}$$

is a linear basis of the algebra $\mathbf{A}$.

Proof. Let us introduce notations

$$
\begin{aligned}
&p = -\alpha_1 a - \alpha_0,\\
&q = -\beta_1 b - \beta_0,\\
&s = \gamma_1(a+b) + \gamma_0,\\
&r = s + p + q,\\
&\Lambda = bab + ba^2 + ab^2 + aba + a^2 b.
\end{aligned}
$$

Then $(a+b)^3 + \gamma_1(a+b) + \gamma_0 \to b^2a + \Lambda + r$. So we have an ideal I generated by elements $a^3 - p$, $b^3 - q$, $b^2a + \Lambda + r$. Let G be the set of this elements. The main words of the set G are b^3, b^2a and a^3. So we have 7 compositions:

$$b\cdot b^2\cdot b,\ b^2\cdot b\cdot b^2,\ b\cdot b^2\cdot a,\ b^2\cdot b\cdot ba,\ a\cdot a^2\cdot a,\ a^2\cdot a\cdot a^2 \text{ and } b\cdot ba\cdot a^2.$$

The first composition is a sub-word of the second, the third is a sub-word of the fourth and the fifth is a sub-word of the sixth, so by the Triangle Lemma (see for example [7, sec. 2.10]) it is enough to calculate only the first, the third, the fifth and the seventh compositions. But the first and the fifth compositions are reduced to 0 because $[b,q]=0$ and $[a,p]=0$.

By reductions of the third composition we get 0:

$$
\begin{aligned}
b\cdot b^2\cdot a:\ b\Lambda + br + qa^2 &= \underline{\underline{b^2ab}} + \underline{b^2a^2} + \underline{\underline{bab^2}} + \underline{baba} + \underline{\underline{ba^2b}} + br + qa\\
&= (b^2a + bab + ba^2)b + (b^2a + bab)a + br + qa\\
&\to -(ab^2 + \underline{aba} + \underline{a^2b} + r)b - (ba^2 + \underline{ab^2} + \underline{aba} + \underline{a^2b} + r)a + br + qa\\
&\to -a(b^2a + bab + ba^2 + ab^2 + aba) - bp + [q,a] + [b,r] - ra\\
&\to a(a^2b + r) - bp + [q,a] + [b,r] - ra\\
&\to [b+a, r] - [a,q] - [b,p] = [b+a, p+q] - [a,q] - [b,p] = 0.
\end{aligned}
$$

The seventh composition gives a new element:

$$
\begin{aligned}
b^2\cdot a\cdot a^2:\ \Lambda a^2 + ra^2 + b^2p &\to baba^2 + \underline{ab^2a^2} + \underline{a^2ba^2} + bap + abp + ra^2 + b^2p\\
&= baba^2 + a(b^2a + aba)a + \{bp, a\} + ra^2 + b^2p\\
&\to baba^2 - ababa - abp - a^2b^2a - pba - ara + \{bp,a\} + ra^2 + b^2p\\
&\to baba^2 - ababa + a^2(\Lambda + r) - ara + [ba, p] + ra^2 + b^2p\\
&\to baba^2 - ababa + a^2bab + a^2ba^2\\
&+ pb^2 + pba + pab + a^2r - ara + [ba,p] + ra^2 + b^2p\\
&= baba^2 - ababa + a^2bab + a^2ba^2 + \{b^2, p\} + \{b, ap\} + \{a^2, r\} - ara.
\end{aligned}
$$

We introduce notations $\Sigma = \{a^2, r\} - ara$ and $\Omega = \{b^2, p\} + \{b, ap\}$ and add the element

$$baba^2 - ababa + a^2bab + a^2ba^2 + \Omega + \Sigma$$

to the set G.

So we obtain new compositions: $b\cdot ba\cdot ba^2$, $b^2\cdot ba\cdot ba^2$, $bab\cdot a^2\cdot a$ and $baba\cdot a\cdot a^2$. Again the first composition is a sub-word of the second and the third composition is a sub-word of the forth. And we need to calculate only first and third ones.

To calculate the first composition we use that

$$\Omega = \{b^2, p\} + \{b, ap\} = \alpha_1(\{b^2, a\} + \{b, a^2\}) + 2\alpha_0(b^2 + ba + ab)$$

$$\to -\alpha_1(bab + aba + r) + \alpha_0(2b^2 + ba + ab).$$

Thus we have

$$\begin{aligned}
bab\cdot a^2\cdot a:\ & -\underline{ababa^2} + \underline{a^2baba} + a^2bp + \Omega a + \Sigma a + babp\\
&= -a(baba^2 - ababa) + a^2bp + \Omega a + \Sigma a + babp\\
&\to a(a^2bab + a^2ba^2 + \Omega + \Sigma) + a^2bp + \Omega a + \Sigma a + babp\\
&\to pba^2 + pbab + a^2bp + babp + \{\Omega, a\} + \{\Sigma, a\}\\
&= \alpha_1(aba^2 + abab + a^2ba + baba - \{bab + aba + r, a\})\\
&+ \alpha_0(ba^2 + 2bab + a^2b + \{2b^2 + ba + ab, a\}) + \{a^2r + ra^2 - ara, a\}\\
&\to -\alpha_1\{r, a\} - 2\alpha_0(b^2a + \Lambda) + a^2ra + rp - ara^2 + pr + ara^2 - a^2ra\\
&\to -\alpha_1\{r, a\} - 2\alpha_0 r + \{r, p\} = 0.
\end{aligned}$$

The third composition also does not give a new element into the set G:

$$\begin{aligned}
b\cdot ba\cdot ba^2:\ & (\underline{\underline{bab}} + \underline{\underline{ba^2}} + ab^2 + aba + a^2b + r)ba^2 - b(-\underline{\underline{ababa}} + \underline{a^2bab} + \underline{a^2ba^2} + \Omega + \Sigma)\\
&= ba(b^2a + bab + ab^2 + aba + a^2b)a + ababa^2 + a^2b^2a^2\\
&- ba^2(b^2a + bab + ba^2 + aba) + (aqa^2 + rba^2 - b\Omega - b\Sigma)\\
&\to -ba(\underline{\underline{ba^2}} + r)a + \underline{\underline{ababa^2}} + a^2b^2a^2 + ba^2(ab^2 + a^2b + r)\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma)\\
&\to -(baba^2 - ababa)a + a^2b^2a^2\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r)\\
&\to (\underline{\underline{a^2bab}} + \underline{\underline{a^2ba^2}} + \Omega + \Sigma)a + \underline{\underline{a^2b^2a^2}}\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r)\\
&= a^2(b^2a + bab + ba^2)a\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r + \Omega a + \Sigma a)\\
&\to -a^2(ab^2 + aba + a^2b + r)a\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r + \Omega a + \Sigma a)\\
&\to -p(b^2a + ba^2 + aba)a\\
&+ (aqa^2 + rba^2 - b\Omega - b\Sigma - bara + bpb^2 + bpab + ba^2r + \Omega a + \Sigma a - a^2ra)\\
&\to aqa^2 + rba^2 - \underline{\underline{b\Omega}} - \underline{b\Sigma} - bara + bpb^2 + bpab + ba^2r + \underline{\underline{\Omega a}} + \underline{\Sigma a} - a^2ra\\
&+ pbab + pab^2 + pa^2b + pr\\
&= aqa^2 + rba^2 - bara + bpb^2 + bpab + ba^2r - a^2ra + \underline{\underline{pbab}} + \underline{\underline{pab^2}} + \underline{\underline{pa^2b}} + \underline{\underline{pr}}\\
&+ (b^2pa + \underline{\underline{pb^2a}} + bapa + \underline{\underline{apba}} - qp - bpb^2 - b^2ap - bapb)\\
&+ (a^2ra + rp - ara^2 - ba^2r - bra^2 + bara)\\
&= p(b^2a + bab + ab^2 + aba + a^2b + r)\\
&+ aqa^2 + rba^2 + bapa - qp + rp - ara^2 - bra^2\\
&\to -pba^2 + aqa^2 + rba^2 + bapa - qp + rp - ara^2 - bra^2\\
&= ([b, p] + [a, q] + [r, b])a^2 + qa^3 - qp + rp - ara^2\\
&\to ([s, b] + [s, a] - [q, a] - [s, a])a^2 - ara^2 + rp = -([r, a] + ar)a^2 + rp \to 0.
\end{aligned}$$

Then by the Diamond Lemma the main words of the Gröbner bases G are b^3, a^3, b^2a, $baba^2$, so they are disallowed and the theorem is proved. ■

Corollary 2. *Algebras* $\mathbf{A}$ *are infinite-dimensional and have the quadratic growth.*

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On the Liouville–Arnold Integrable Flows Related with Quantum Algebras and Their Poissonian Representations

Anatoliy SAMOILENKO [†1], *Yarema PRYKARPATSKY* [†2], *Denis BLACKMORE* [†3] *and Anatoliy PRYKARPATSKY* [†4]

[†1] *Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *sam@imath.kiev.ua*

[†2] *Dept. of Applied Mathematics at the AGH University of Science and Technology, 30 Mickiewicz Al. Bl. A4, 30059 Kraków, Poland;*
Brookhaven Nat. Lab., SDIC, SUNY, Upton, NY 11973, USA
E-mail: *yarpry@bnl.gov*

[†3] *Dept. of Math. Studies at the NJIT, Newark, University Heights, New Jersey 07102, USA*
E-mail: *deblac@m.njit.edu*

[†4] *Dept. of Applied Mathematics at the AGH University of Science and Technology, 30 Mickiewicz Al. Bl. A4, 30059 Kraków, Poland;*
Dept. of Nonlinear Math. Analysis at the Institute of APMM of NAS of Ukraine, 3b Naukova Str., 79601 Lviv, Ukraine
E-mail: *prykanat@cyberagl.com, pryk.anat@ua.fm*

Based on the structure of Casimir elements associated with general Hopf algebras there Liouville–Arnold integrable flows, related with naturally induced Poisson structures on arbitrary co-algebra and their deformations, are constructed. Some interesting special cases including the oscillatory Heisenberg–Weil algebra, related co-algebra structures and adjoint with them integrable Hamiltonian systems are considered.

1 Hopf algebras and co-algebras: main definitions

Consider a Hopf algebra $\mathcal{A}$ over $\mathbb{C}$ endowed with two special homomorphisms called coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and counit $\varepsilon : \mathcal{A} \to \mathbb{C}$, as well an antihomomorphism (antipode) $\nu : \mathcal{A} \to \mathcal{A}$, such that for any $a \in \mathcal{A}$

$$(\mathrm{id} \otimes \Delta)\Delta(a) = (\Delta \otimes \mathrm{id})\Delta(a), \qquad (\mathrm{id} \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes \mathrm{id})\Delta(a) = a,$$
$$m((\mathrm{id} \otimes \nu)\Delta(a)) = m((\nu \otimes \mathrm{id})\Delta(a)) = \varepsilon(a)I, \tag{1}$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the usual multiplication mapping, that is for any $a, b \in \mathcal{A}$ $m(a \otimes b) = ab$. The conditions (1) were introduced by Hopf [1] in a cohomological context. Since most of the Hopf algebras properties depend on the coproduct operation $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and related with it Casimir elements, below we shall dwell mainly on the objects called co-algebras endowed with this coproduct.

The most interesting examples of co-algebras are provided by the universal enveloping algebras $U(\mathcal{G})$ of Lie algebras $\mathcal{G}$. If, for instance, a Lie algebra $\mathcal{G}$ possesses generators $X_i \in \mathcal{G}$, $i = \overline{1, n}$, $n = \dim \mathcal{G}$, the corresponding enveloping algebra $U(\mathcal{G})$ can be naturally endowed with a Hopf algebra structure by defining

$$\Delta(X_i) = I \otimes X_i + X_i \otimes I, \qquad \Delta(I) = I \otimes I, \qquad \varepsilon(X_i) = -X_i, \qquad \nu(I) = -I. \tag{2}$$

These mappings acting only on the generators of $\mathcal{G}$ are straightforwardly extended to any monomial in $U(\mathcal{G})$ by means of the homomorphism condition $\Delta(XY) = \Delta(X)\Delta(Y)$ for any $X, Y \in \mathcal{G} \subset U(\mathcal{G})$. In general an element $Y \in U(\mathcal{G})$ of a Hopf algebra such that $\Delta(Y) = I \otimes Y + Y \otimes I$ is called primitive, and the known Friedrichs theorem [2] ensures that in $U(\mathcal{G})$ the only primitive elements are exactly generators $X_i \in \mathcal{G}$, $i = \overline{1,n}$.

On the other hand, the homomorphism condition for the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ implies the compatibility of the coproduct with the Lie algebra commutator structure:

$$[\Delta(X_i), \Delta(X_j)]_{\mathcal{A}\otimes\mathcal{A}} = \Delta([X_i, X_j]_{\mathcal{A}}) \tag{3}$$

for any $X_i, X_j \in \mathcal{G}$, $i, j = \overline{1,n}$. Since the Drinfeld report [3] the co-algebras defined above are also often called "quantum" groups due to their importance [4] in studying many two-dimensional quantum models of modern field theory and statistical physics.

It was also observed (see for instance [4]) that the standard co-algebra structure (2) of the universal enveloping algebra $U(\mathcal{G})$ can be nontrivially extended by means of some of its infinitesimal deformations saving the co-associativity (3) of the deformed coproduct $\Delta : U_z(\mathcal{G}) \to U_z(\mathcal{G}) \otimes U_z(\mathcal{G})$ with $U_z(\mathcal{G})$ being the corresponding universal enveloping algebra deformation by means of a parameter $z \in \mathbb{C}$, such that $\lim\limits_{z\to 0} U_z(\mathcal{G}) = U(\mathcal{G})$ is subject to some natural topology on $U_z(\mathcal{G})$.

2 Casimir elements and their special properties

Take any Casimir element $C \in U_z(\mathcal{G})$ that is an element satisfying the condition $[C, U_z(\mathcal{G})] = 0$, and consider the action on it of the coproduct mapping Δ:

$$\Delta(C) = C(\{\Delta(X)\}), \tag{4}$$

where we put, by definition, $C := C(\{X\})$ with a set $\{X\} \subset \mathcal{G}$. It is a trivial consequence that for $\mathcal{A} := U_z(\mathcal{G})$

$$[\Delta(C), \Delta(X_i)]_{\mathcal{A}\otimes\mathcal{A}} = \Delta([C, X_i]_{\mathcal{A}}) = 0 \tag{5}$$

for any $X_i \in \mathcal{G}$, $i = \overline{1,n}$.

Define now recurrently the following N-th coproduct $\Delta^{(N)} : \mathcal{A} \to \overset{(N+1)}{\otimes}\mathcal{A}$ for any $N \in \mathbb{Z}_+$, where $\Delta^{(2)} := \Delta$ and $\Delta^{(1)} := \mathrm{id}$ and

$$\Delta^{(N)} := \big((\mathrm{id}\otimes)^{N-2} \otimes \Delta\big) \cdot \Delta^{(N-1)}, \tag{6}$$

or as

$$\Delta^{(N)} := \big(\Delta \otimes (\mathrm{id}\otimes)^{N-2} \otimes \mathrm{id} \otimes \mathrm{id}\big) \cdot \Delta^{(N-1)}. \tag{7}$$

One can straightforwardly verify that

$$\Delta^{(N)} := \big(\Delta^{(m)} \otimes \Delta^{(N-m)}\big) \cdot \Delta \tag{8}$$

for any $m = \overline{0,N}$, and the mapping $\Delta^{(N)} : \mathcal{A} \to \overset{(N+1)}{\otimes}\mathcal{A}$ is an algebras homomorphism, that is

$$\big[\Delta^{(N)}(X), \Delta^{(N)}(Y)\big]_{\overset{(N+1)}{\otimes}\mathcal{A}} = \Delta^{(N)}([X, Y]_{\mathcal{A}}) \tag{9}$$

for any $X, Y \in \mathcal{A}$. In a particular case if $\mathcal{A} = U(\mathcal{G})$, the following exact expression

$$\Delta^{(N)}(X) = X(\otimes\,\mathrm{id})^{N-1} \otimes \mathrm{id} + \mathrm{id} \otimes X(\otimes\,\mathrm{id})^{N-1} \otimes \mathrm{id} + \cdots + (\otimes\,\mathrm{id})^{N-1} \otimes \mathrm{id} \otimes X \tag{10}$$

holds for any $X \in \mathcal{G}$.

3 Poisson co-algebras and their realizations

As is well known [5, 6], a Poisson algebra $\mathcal{P}$ is a vector space endowed with a commutative multiplication and a Lie bracket $\{\cdot,\cdot\}$ including a derivation on $\mathcal{P}$ in the form

$$\{a,bc\} = b\{a,c\} + \{a,b\}c \tag{11}$$

for any a, b and $c \in \mathcal{P}$. If $\mathcal{P}$ and $\mathcal{Q}$ are Poisson algebras one can naturally define the following Poisson structure on $\mathcal{P}\otimes\mathcal{Q}$:

$$\{a\otimes b, c\otimes d\}_{\mathcal{P}\otimes\mathcal{Q}} = \{a,c\}_{\mathcal{P}}\otimes(bd) + (ac)\otimes\{b,d\}_{\mathcal{Q}} \tag{12}$$

for any $a,c\in\mathcal{P}$ and $b,d\in\mathcal{Q}$. We shall also say that $(\mathcal{P};\Delta)$ is a Poisson co-algebra if $\mathcal{P}$ is a Poisson algebra and $\Delta:\mathcal{P}\to\mathcal{P}\otimes\mathcal{P}$ is a Poisson algebras homomorphism, that is

$$\{\Delta(a),\Delta(b)\}_{\mathcal{P}\otimes\mathcal{P}} = \Delta(\{a,b\}_{\mathcal{P}}) \tag{13}$$

for any $a,b\in\mathcal{P}$.

It is useful to note here that any Lie algebra $\mathcal{G}$ generates naturally a Poisson co-algebra $(\mathcal{P};\Delta)$ by defining a Poisson bracket on $\mathcal{P}$ by means of the following expression: for any $a,b\in\mathcal{P}$

$$\{a,b\}_{\mathcal{P}} := \langle \operatorname{grad}, \vartheta \operatorname{grad} b\rangle. \tag{14}$$

Here $\mathcal{P}\simeq C^\infty(\mathbb{R}^n;\mathbb{R})$ is a space of smooth mappings linked with base variables of the Lie algebra $\mathcal{G}$, $n=\dim\mathcal{G}$, and the implectic [6] matrix $\vartheta: T^*(\mathcal{P})\to T(\mathcal{P})$ is given as

$$\vartheta(x) = \left\{ \sum_{k=1}^{n} c_{ij}^k x_k :\ i,j=\overline{1,n} \right\}, \tag{15}$$

where c_{ij}^k, $i,j,k=\overline{1,n}$, are the corresponding structure constants of the Lie algebra $\mathcal{G}$ and $x\in\mathbb{R}^n$ are the corresponding linked coordinates. It is easy to check that the coproduct (1.2) is a Poisson algebras homomorphism between $\mathcal{P}$ and $\mathcal{P}\otimes\mathcal{P}$. If one can find a "quantum" deformation $U_z(\mathcal{G})$, then the corresponding Poisson co-algebra $\mathcal{P}_z$ can be constructed with the naturally deformed implectic matrix $\vartheta_z: T^*(\mathcal{P}_z)\to T(\mathcal{P}_z)$. For instance, if $\mathcal{G}=so(2,1)$, there exists a deformation $U_z(so(2,1))$ defined by the following deformed commutator relations with a parameter $z\in\mathbb{C}$:

$$[\tilde{X}_2,\tilde{X}_1]=\tilde{X}_3, [\tilde{X}_2,\tilde{X}_3]=-\tilde{X}_1, \qquad [\tilde{X}_3,\tilde{X}_1]=\frac{1}{z}\sinh(z\tilde{X}_2), \tag{16}$$

where at $z=0$ elements $\tilde{X}_i\big|_{z=0} = X_i\in so(2,1)$, $i=\overline{1,3}$, compile a base of generators of the Lie algebra $so(2,1)$. Then, based on expressions (16) one can easily construct the corresponding Poisson co-algebra $\mathcal{P}_z$, endowed with the implectic matrix

$$\vartheta_z(\tilde{x}) = \begin{pmatrix} 0 & -\tilde{x}_3 & -\frac{1}{z}\sinh(z\tilde{x}_2) \\ \tilde{x}_3 & 0 & -\tilde{x}_1 \\ \frac{1}{z}\sinh(z\tilde{x}_2) & \tilde{x}_1 & 0 \end{pmatrix} \tag{17}$$

for any point $\tilde{x}\in\mathbb{R}^3$, linked naturally with the deformed generators $\tilde{X}_i$, $i=\overline{1,3}$, taken above. Since the corresponding coproduct on $U_z(so(2,1))$ acts on this deformed base of generators as

$$\begin{aligned} &\Delta(\tilde{X}_2) = I\otimes\tilde{X}_2 + \tilde{X}_2\otimes I, \qquad \Delta(\tilde{X}_1) = \exp\left(-\frac{z}{2}\tilde{X}_2\right)\otimes\tilde{X}_1 + \tilde{X}_1\otimes\exp\left(\frac{z}{2}\tilde{X}_2\right), \\ &\Delta(\tilde{X}_2) = \exp\left(-\frac{z}{2}\tilde{X}_2\right)\otimes\tilde{X}_3 + \tilde{X}_3\otimes\exp\left(\frac{z}{2}\tilde{X}_2\right), \end{aligned} \tag{18}$$

satisfying the main homomorphism property for the whole deformed universal enveloping algebra $U_z(so(2,1))$.

Consider now some realization of the deformed generators $\tilde{X}_i \in U_z(\mathcal{G})$, $i = \overline{1,n}$, that is a homomorphism mapping $D_z : U_z(\mathcal{G}) \to \mathcal{P}(M)$, such that

$$D_z(\tilde{X}_i) = \tilde{e}_i, \qquad i = \overline{1,n}, \tag{19}$$

are some elements of a Poisson manifold $\mathcal{P}(M)$ realized as a space of functions on a finite-dimensional manifold M, satisfying the deformed commutator relationships $\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}(M)} = \vartheta_{z,ij}(\tilde{e})$, where, by definition, expressions $[\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X})$, $i,j = \overline{1,n}$, generate a Poisson co-algebra structure on the function space $\mathcal{P}_z := \mathcal{P}_z(\mathcal{G})$ linked with a given Lie algebra $\mathcal{G}$. Making use of the homomorphism property (13) for the coproduct mapping $\Delta : \mathcal{P}_z(\mathcal{G}) \to \mathcal{P}_z(\mathcal{G}) \otimes \mathcal{P}_z(\mathcal{G})$, one finds that for all $i,j = \overline{1,n}$

$$\{\Delta(\tilde{x}_i), \Delta(\tilde{x}_j)\}_{\mathcal{P}_z(\mathcal{G})\otimes\mathcal{P}_z(\mathcal{G})} = \Delta(\{\tilde{x}_i, \tilde{x}_j\}_{\mathcal{P}_z(\mathcal{G})}) = \vartheta_{z,ij}(\Delta(\tilde{x})) \tag{20}$$

and for the corresponding coproduct $\Delta : \mathcal{P}(M) \to \mathcal{P}(M) \otimes \mathcal{P}(M)$ one gets similarly

$$\{\Delta(\tilde{e}_i), \Delta(\tilde{e}_j)\}_{\mathcal{P}(M)\otimes\mathcal{P}(M)} = \Delta(\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}(M)}) = \vartheta_{z,ij}(\Delta(\tilde{e})), \tag{21}$$

where $\{\cdot,\cdot\}_{\mathcal{P}(M)}$ is some, eventually, canonical Poisson structure on a finite-dimensional manifold M.

Let $q \in M$ be a point of M and consider its coordinates as elements of $\mathcal{P}(M)$. Then one can define the following elements

$$q_j := (I\otimes)^{j-1} q (\otimes I)^{N-j} \in \overset{(N)}{\otimes} \mathcal{P}(M), \tag{22}$$

where $j = \overline{1,N}$ by means of which one can construct the corresponding N-tuple realization of the Poisson co-algebra structure (21) as follows:

$$\big\{\tilde{e}_i^{(N)}, \tilde{e}_j^{(N)}\big\}_{\overset{(N)}{\otimes}\mathcal{P}(M)} = \vartheta_{z,ij}\big(\tilde{e}^{(N)}\big), \tag{23}$$

with $i,j = \overline{1,n}$ and

$$\overset{(N)}{\otimes} D_z\big(\Delta^{(N-1)}(\tilde{e}_i)\big) := \tilde{e}_i^{(N)}(q_1, q_2, \ldots, q_N). \tag{24}$$

For instance, for the $U_z(so(2,1))$ case (16), one can take [7] the realization Poisson manifold $\mathcal{P}(M) = \mathcal{P}(\mathbb{R}^2)$ with the standard canonical Heisenberg–Weil Poissonian structure on it:

$$\{q,q\}_{\mathcal{P}(\mathbb{R}^2)} = 0 = \{p,p\}_{\mathcal{P}(\mathbb{R}^2)}, \qquad \{p,q\}_{\mathcal{P}(\mathbb{R}^2)} = 1, \tag{25}$$

where $(q,p) \in \mathbb{R}^2$. Then expressions (24) for $N=2$ give rise to the following relationships

$$\begin{aligned}
&\tilde{e}_1^{(2)}(q_1,q_2,p_1,p_2) := (D_z \otimes D_z)\Delta(\tilde{X}_1)\\
&\qquad = 2\frac{\sinh\left(\frac{z}{2}p_1\right)}{z}\cos q_1 \exp\left(\frac{z}{2}p_1\right) + 2\exp\left(-\frac{z}{2}p_1\right)\frac{\sinh\left(\frac{z}{2}p_2\right)}{z}\cos q_2,\\
&\tilde{e}_2^{(2)}(q_1,q_2,p_1,p_2) := (D_z \otimes D_z)\Delta(\tilde{X}_2) = p_1 + p_2,\\
&\tilde{e}_3^{(2)}(q_1,q_2,p_1,p_2) := (D_z \otimes D_z)\Delta(\tilde{X}_3)\\
&\qquad = 2\frac{\sinh\left(\frac{z}{2}p_1\right)}{z}\sin q_1 \exp\left(\frac{z}{2}p_2\right) + 2\exp\left(-\frac{z}{2}p_1\right)\frac{\sinh\left(\frac{z}{2}p_2\right)}{z}\sin q_2,
\end{aligned} \tag{26}$$

where elements $(q_1,q_2,p_1,p_2) \in \mathcal{P}(\mathbb{R}^2) \otimes \mathcal{P}(\mathbb{R}^2)$ satisfy the induced by (25) Heisenberg–Weil commutator relations:

$$\{q_i,q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = 0 = \{p_i,p_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)}, \qquad \{p_i,q_j\}_{\mathcal{P}(\mathbb{R}^2)\otimes\mathcal{P}(\mathbb{R}^2)} = \delta_{ij} \tag{27}$$

for any $i,j = \overline{1,2}$.

4 Casimir elements and the Heisenberg–Weil algebra related algebraic structures

Consider any Casimir element $\tilde{C} \in U_z(\mathcal{G})$ related with an $\mathbb{R} \ni z$-deformed Lie algebra $\mathcal{G}$ structure in the form

$$[\tilde{X}_i, \tilde{X}_j] = \vartheta_{z,ij}(\tilde{X}), \tag{28}$$

where $i, j = \overline{1,n}$, $n = \dim \mathcal{G}$, and, by definition, $[\tilde{C}, \tilde{X}_i] = 0$. The following general lemma holds.

Lemma 1. *Let $(U_z(\mathcal{G}); \Delta)$ be a co-algebra with generators satisfying (28) and $\tilde{C} \in U_z(\mathcal{G})$ be its Casimir element; then*

$$[\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{X}_i)]_{\substack{(N+1)\\ \otimes \\ U_z(\mathcal{G})}} = 0 \tag{29}$$

for any $i = \overline{1,n}$ and $m = \overline{1,N}$.

As a simple corollary of this Lemma one finds from (29) that

$$[\Delta^{(m)}(\tilde{C}), \Delta^{(N)}(\tilde{C})]_{\substack{(N+1)\\ \otimes \\ U_z(\mathcal{G})}} = 0$$

for any $k, m \in \mathbb{Z}_+$.

Consider now some realization (19) of our deformed Poisson co-algebra structure (28) and check that the expression

$$\left[\Delta^{(m)}(C(\tilde{e})), \Delta^{(N)}(\mathcal{H}(\tilde{e}))\right]_{\substack{(N+1)\\ \otimes \\ \mathcal{P}(M)}} = 0 \tag{30}$$

too for any $m = \overline{1,N}$, $N \in \mathbb{Z}_+$, if $C(\tilde{e}) \in I(\mathcal{P}(M))$, that is $\{C(\tilde{e}), q\}_{\mathcal{P}(M)} = 0$ for any $q \in M$. Since

$$\mathcal{H}^{(N)}(q) := \Delta^{(N-1)}(\mathcal{H}(\tilde{e})) \tag{31}$$

are in general, smooth functions on $\overset{(N+1)}{\otimes} M$, which can be used as Hamilton ones subject to the Poisson structure on $\overset{(N+1)}{\otimes} \mathcal{P}(M)$, the expressions (31) mean nothing else that functions

$$\gamma^{(m)}(q) := \Delta^{(N)}(C(\tilde{e})) \tag{32}$$

are their invariants, that is

$$\left\{\gamma^{(m)}(q), \mathcal{H}^{(N)}(q)\right\}_{\substack{(N+1)\\ \otimes \\ \mathcal{P}(M)}} = 0 \tag{33}$$

for any $m = \overline{1,N}$. Thereby, the functions (31) and (32) generate under some additional but natural conditions a hierarchy of a priori Liouville–Arnold integrable Hamiltonian flows on the Poisson manifold $\overset{(N+1)}{\otimes} \mathcal{P}(M)$.

Consider now a case of a Poisson manifold $\mathcal{P}(M)$ and its co-algebra deformation $\mathcal{P}_z(\mathcal{G})$. Thus for any coordinate points $x_i \in \mathcal{P}(\mathcal{G})$, $i = \overline{1,n}$, the following relationships

$$\{x_i, x_j\} = \sum_{k=1}^{n} c_{ij}^k x_k := \vartheta_{ij}(x) \tag{34}$$

define a Poisson structure on $\mathcal{P}(\mathcal{G})$, related with the corresponding Lie algebra structure of $\mathcal{G}$, and there exists a representation (19), such that elements $\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x)$ satisfy the relationships $\{\tilde{e}_i, \tilde{e}_j\}_{\mathcal{P}_z(\mathcal{G})} = \vartheta_{z,ij}(\tilde{e})$ for any $i = \overline{1,n}$, with the limiting conditions

$$\lim_{z\to 0} \vartheta_{z,ij}(\tilde{e}) = \sum_{k=1}^{n} c_{ij}^k x_k, \qquad \lim_{z\to 0} \tilde{e}_i(x) = x_i \tag{35}$$

for any $i,j = \overline{1,n}$ being held. For instance, take the Poisson co-algebra $\mathcal{P}_z(so(2,1))$ for which there exists a realization (19) in the following form:

$$\tilde{e}_1 := D_z(\tilde{X}_1) = \frac{\sinh\left(\frac{z}{2}x_2\right)}{zx_2}x_1, \qquad \tilde{e}_2 := D_z(\tilde{X}_2) = x_2, \qquad \tilde{e}_3 := D_z(\tilde{X}_3) = \frac{\sinh\left(\frac{z}{2}x_2\right)}{zx_2}x_3,$$

where $x_i \in \mathcal{P}(so(2,1))$, $i = \overline{1,3}$, satisfy the $so(2,1)$-commutator relationships

$$\{x_2, x_1\}_{\mathcal{P}(so(2,1))} = x_3, \qquad \{x_2, x_3\}_{\mathcal{P}(so(2,1))} = -x_1, \qquad \{x_3, x_1\}_{\mathcal{P}(so(2,1))} = x_2, \tag{36}$$

with the coproduct operator $\Delta : \mathcal{U}_z(so(2,1)) \to \mathcal{U}_z(so(2,1)) \otimes \mathcal{U}_z(so(2,1))$ being given by (18). It is easy to check that conditions (34) and (35) hold.

The next example is related with the co-algebra $\mathcal{U}_z(\pi(1,1))$ of the Poincaré algebra $\pi(1,1)$ for which the following non-deformed relationships

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_2, \qquad [X_3, X_2] = 0 \tag{37}$$

hold. The corresponding coproduct $\Delta : \mathcal{U}_z(\pi(1,1)) \to \mathcal{U}_z(\pi(1,1)) \otimes \mathcal{U}_z(\pi(1,1))$ is given by the Woronowicz [8] expressions

$$\begin{aligned}
&\Delta(\tilde{X}_1) = I \otimes \tilde{X}_1 + \tilde{X}_1 \otimes I, \qquad \Delta(\tilde{X}_2) = \exp\left(-\frac{z}{2}\tilde{X}_1\right) \otimes \tilde{X}_1 + \tilde{X}_1 \otimes \exp\left(\frac{z}{2}\tilde{X}_1\right),\\
&\Delta(\tilde{X}_3) = \exp\left(-\frac{z}{2}\tilde{X}_1\right) \otimes \tilde{X}_3 + \tilde{X}_3 \otimes \exp\left(\frac{z}{2}\tilde{X}_1\right),
\end{aligned} \tag{38}$$

where $z \in \mathbb{R}$ is a parameter. Under the deformed expressions (38) the elements $\tilde{X}_j \in \mathcal{U}_z(\pi(1,1))$, $j = \overline{1,3}$, still satisfy undeformed commutator relationships, that is $\vartheta_{z,ij}(\tilde{X}) = \vartheta_{ij}(X)\big|_{X\Rightarrow\tilde{X}}$ for any $z \in \mathbb{R}$, $i,j = \overline{1,3}$, being given by (37). As a result, we can state that $\tilde{e}_i := D_z(\tilde{X}_i) = \tilde{e}_i(x) = x_i$, where for $x_i \in \mathcal{P}(\pi(1,1))$, $i = \overline{1,3}$, the following Poisson structure

$$\{x_1, x_2\}_{\mathcal{P}(\pi(1,1))} = x_3, \qquad \{x_1, x_3\}_{\mathcal{P}(\pi(1,1))} = x_2, \qquad \{x_3, x_2\}_{\mathcal{P}(\pi(1,1))} = 0 \tag{39}$$

holds. Moreover, since $C = x_2^2 - x_3^2 \in I(\mathcal{P}(\pi(1,1)))$, that is $\{C, x_i\}_{\mathcal{P}(\pi(1,1))} = 0$ for any $i = \overline{1,3}$, one can construct, making use of (31) and (32), integrable Hamiltonian systems on $\overset{(N)}{\otimes}\mathcal{P}(\pi(1,1))$. The same one can do subject to the discussed above Poisson co-algebra $\mathcal{P}_z(so(2,1))$ realized by means of the Poisson manifold $\mathcal{P}(so(2,1))$, taking into account that the following element $C = x_2^2 - x_1^2 - x_3^2 \in I(\mathcal{P}(so(2,1)))$ is a Casimir one.

Now we will consider a special extended Heisenberg–Weil co-algebra $\mathcal{U}_z(h_4)$, called still the oscillator co-algebra. The undeformed Lie algebra h_4 commutator relationships take the form:

$$[n, a_+] = a_+, \qquad [n, a_-] = -a_-, \qquad [a_-, a_+] = m, \qquad [m, \cdot] = 0, \tag{40}$$

where $\{n, a_\pm, m\} \subset h_4$ compile a basis of h_4, $\dim h_4 = 4$. The Poisson co-algebra $\mathcal{P}(h_4)$ is naturally endowed with the Poisson structure like (40) and admits its realization (19) on the Poisson manifold $\mathcal{P}(\mathbb{R}^2)$. Namely, on $\mathcal{P}(\mathbb{R}^2)$ one has

$$e_\pm = D(a_\pm) = \sqrt{p}\exp(\mp q), \qquad e_1 = D(m) = 1, \qquad e_0 = D(n) = p, \tag{41}$$

where $(q,p) \in \mathbb{R}^2$ and the Poisson structure on $\mathcal{P}(\mathbb{R}^2)$ is canonical, that is the same as (25).

Closely related with the relationships (40) there is a generalized $\mathcal{U}_z(su(2))$ co-algebra, for which

$$[x_3, x_\pm] = \pm x_\pm, \qquad [y_\pm, \cdot] = 0, \qquad [x_+, x_-] = y_+ \sin(2zx_3) + y_- \cos(2zx_3)) \frac{1}{\sin z}, \tag{42}$$

where $z \in \mathbb{C}$ is an arbitrary parameter. The co-algebra structure is given now as follows:

$$\begin{aligned}
&\Delta(x_\pm) = c^\pm_{1(2)} e^{izx_3} \otimes x_\pm + x_\pm \otimes c^\pm_{2(1)} e^{-izx_3}, \qquad \Delta(x_3) = I \otimes x_3 + x_3 \otimes I,\\
&\Delta(c^\pm_i) = c^\pm_i \otimes c^\pm_i, \qquad \nu(x_\mp) = -(c^\pm_{1(2)})^{-1} e^{-izx_3} x_\mp e^{izx_3} (c^\pm_{2(1)})^{-1},\\
&\nu(c^\pm_i) = (c^\pm_i)^{-1}, \qquad \nu(e^{\pm izx_3}) = e^{\mp izx_3}
\end{aligned} \tag{43}$$

with $c^\pm_i \in \mathcal{U}_z(su(2))$, $i = \overline{1,2}$, being fixed elements. One can check that the Poisson structure on $\mathcal{P}_z(su(2))$ corresponding to (42) can be realized by means of the canonical Poisson structure on the phase space $\mathcal{P}(\mathbb{R}^2)$ as follows:

$$\begin{aligned}
&[q,p] = i, \qquad D_z(x_3) = q, \qquad D_z(x_\mp) = e^{\pm ip} g_z(q),\\
&g_z(q) = (k + \sin[z(s-q)])(y_+ \sin[(q+s+1)] + y_- \cos[z(q+s+1)])^{1/2} \frac{1}{\sin z},
\end{aligned} \tag{44}$$

where $k, s \in \mathbb{C}$ are constant parameters. Thereby making use of (32) and (33), one can construct a new class of Liouville integrable Hamiltonian flows.

5 The Heisenberg–Weil co-algebra structure and related integrable flows

Consider the Heisenberg–Weil algebra commutator relationships (40) and the following homogenous quadratic forms related with them

$$\left.\begin{aligned}
&x_1x_2 - x_2x_1 - \alpha x_3^2 = 0,\\
&x_1x_3 - x_3x_1 = 0, \qquad x_2x_3 - x_3x_2 = 0
\end{aligned}\right\} R(x), \tag{45}$$

where $\alpha \in \mathbb{C}$, $x_i \in A$, $i = \overline{1,3}$, are some elements of a free associative algebra A. The quadratic algebra $A/R(x)$ can be deformed via

$$\left.\begin{aligned}
&x_1x_2 - z_1x_2x_1 - \alpha x_3^2 = 0,\\
&x_1x_3 - z_2x_3x_1 = 0, \qquad x_2x_3 - z_2^{-1}x_3x_2 = 0,
\end{aligned}\right\} R_z(x), \tag{46}$$

where $z_1, z_2 \in \mathbb{C}\backslash\{0\}$ are some parameters.

Let V be the vector space of columns $X := (x_1, x_2, x_3)^\intercal$ and define the following action h_T: $V \to (V \otimes V^*) \otimes V$, where, by definition, for any $X \in V$ $h_T(X) = T \otimes X$. It is easy to check that conditions (46) will be satisfied if the following relations [9]

$$\begin{aligned}
&T_{11}T_{33} = T_{33}T_{11}, \qquad T_{12}T_{33} = z_2^{-2}T_{33}T_{12}, \qquad T_{21}T_{33} = z_1^2 T_{33}T_{21},\\
&T_{22}T_{33} = T_{33}T_{22}, \qquad T_{31}T_{33} = z_2 T_{33}T_{31}, \qquad T_{32}T_{33} = z_1^{-1}T_{33}T_{32},\\
&T_{11}T_{12} = z_1 T_{12}T_{11}, \qquad T_{21}T_{22} = z_1 T_{22}T_{21},\\
&z_2T_{11}T_{32} - z_2T_{32}T_{11} = z_1z_2T_{12}T_{31} - T_{31}T_{12},\\
&T_{21}T_{32} - z_1z_2T_{32}T_{21} = z_1T_{22}T_{31} - z_2T_{31}T_{22},\\
&T_{11}T_{22} - T_{22}T_{11} = z_1T_{12}T_{21} - z_1^{-1}T_{21}T_{12},\\
&(T_{11}T_{22} - z_1T_{12}T_{21}) = \alpha T_{33}^2 - T_{31}T_{32} + z_1T_{32}T_{31}
\end{aligned} \tag{47}$$

hold. Put now for further convenience $z_1 = z_2^2 := z^2 \in \mathbb{C}$ and compute the "quantum" determinant $D(T)$ of the matrix $T : (A/R_z(x))^3 \to (A/R_z(x))^3$:

$$D(T) = (T_{11}T_{22} - z^{-2}T_{21}T_{12})T_{33}. \tag{48}$$

Remark here that the determinant (48) is not central, that is

$$\begin{aligned} &D^{-1}T_{11} = T_{11}D^{-1}, \qquad D^{-1}T_{12} = z^{-6}T_{12}D^{-1}, \qquad D^{-1}T_{33} = T_{33}D^{-1}, \\ &z^{-6}D^{-1}T_{21} = T_{12}D^{-1}, \qquad D^{-1}T_{22} = T_{22}D^{-1}, \qquad z^{-3}D^{-1}T_{31} = T_{31}D^{-1}, \\ &D^{-1}T_{32} = z^{-3}T_{32}D^{-1}. \end{aligned} \tag{49}$$

Taking into account properties (47)–(49), one can construct the Heisenberg–Weil related co-algebra $\mathcal{U}_z(h)$ being a Hopf algebra with the following coproduct Δ, counit ε and antipode ν:

$$\begin{aligned} &\Delta(T) := T \otimes T, \qquad \Delta\big(D^{-1}\big) := D^{-1} \otimes D^{-1}, \\ &\varepsilon(T) := I, \qquad \varepsilon\big(D^{-1}\big) := I, \qquad \nu(T) := T^{-1}, \qquad \nu(D) := D^{-1}. \end{aligned} \tag{50}$$

Based now on relationships (47), one can easily construct the Poisson tensor

$$\{\Delta(\tilde{T}), \Delta(\tilde{T})\}_{\mathcal{P}_z(h)\otimes\mathcal{P}_z(h)} = \Delta(\{\tilde{T}, \tilde{T}\}_{\mathcal{P}_z(h)}) := \vartheta_z(\Delta(\tilde{T})),$$

subject to which all of functionals (32) will be commuting to each other, and moreover, will be Casimir ones. Choosing some appropriate Hamiltonian functions $\mathcal{H}^{(N)}(\tilde{T}) := \Delta^{(N-1)}(\mathcal{H}(\tilde{T}))$ for $N \in \mathbb{Z}_+$ one makes it possible to present a priori nontrivial integrable Hamiltonian systems. On the other hand, the co-algebra $\mathcal{U}_z(h)$ built by (49) and (50) possesses the following fundamental $\mathcal{R}$-matrix [4] property:

$$\mathcal{R}(z)(T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I)\mathcal{R}(z)$$

for some complex-valued matrix $\mathcal{R}(z) \in \mathrm{Aut}\,(\mathbb{C}^3 \otimes \mathbb{C}^3)$, $z \in \mathbb{C}$. The latter, as is well known [4], gives rise to a regular procedure of constructing of an infinite hierarchy of Liouville-integrable operator (quantum) Hamiltonian systems on related quantum Poissonian phase spaces. On their special cases interesting for applications we plan to go on in another place.

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On Spectrum of Matrix-Valued Continuous Functions of a Family of Commuting Operators

Yevgenii Ye. SAMOILENKO

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *sam7@imath.kiev.ua*

A spectrum of matrix-valued continuous functions of a family of self-adjoint commuting bounded operators on a Hilbert space is studied.

1 Introduction

Let $\widetilde{A} = \{A_i = A_i^*\}_{i=\overline{1,m}} \subset L(H)$ be a family of self-adjoint commuting operators and $\{E_i\}_{i=\overline{1,m}}$ be a family of their spectral measures. A direct product of spectral measures is $\widetilde{E}(\alpha_1 \times \alpha_2 \times \ldots \times \alpha_m) = \overset{m}{\underset{i=1}{\times}} E_i(\alpha_i) = E_1(\alpha_1)E_2(\alpha_2)\cdots E_m(\alpha_m)$, a measure on a measurable space. A support of decomposition of unit E is called $\operatorname{Supp} E = \{\cap\varphi | \varphi = \overline{\varphi} : E(\varphi) = \mathbb{I}\}$, i.e. intersection of all closed sets of full measure. A common spectrum of a family of self-adjoint commuting bounded operators is called $S(\widetilde{A}) = S(\{A_i \,|\, i = \overline{1,m}\}) := \operatorname{Supp} \widetilde{E}$, i.e. the support of product of spectral measures. By definition, the following conclusion may be drawn

$$S(\widetilde{A}) = \operatorname{Supp} \widetilde{E} \subseteq \overset{m}{\underset{i=1}{\times}} \operatorname{Supp} E_i = \overset{m}{\underset{i=1}{\times}} \sigma(A_i),$$

where $\sigma(A_i)$ is a spectrum of operator A_i, $i = \overline{1,m}$. A continuous function of a family of self-adjoint operators $\widetilde{A}$ is

$$f(\widetilde{A}) = \int_{S(\widetilde{A})} f(\lambda_1, \ldots, \lambda_m) d\widetilde{E}(\Lambda),$$

where $\Lambda = (\lambda_1, \ldots, \lambda_m) \in S(\widetilde{A}) \subset \mathbb{R}^m$ (see e.g. [1–4]).

Let us consider a matrix-valued continuous function $\{f_{ij}(t_1, t_2, \ldots, t_m)\}_{i,j=\overline{1,n}}$, where $f_{ij}(t_1, t_2, \ldots, t_m) \in C(\mathbb{R}^m, \mathbb{C})$. The result of this paper is the formula for spectrum of matrix-valued continuous functions of a family of self-adjoint commuting bounded operators.

2 On spectrum of matrix operators

Let $G = \widetilde{F}(\widetilde{A}) = \{F_{ij}(\widetilde{A}) = F_{ij}(A_1, \ldots, A_m) \,|\, i, j = \overline{1,n}\}$, $G : H^n \to H^n$. Let $\widetilde{F}(\Lambda) = \{F_{ij}(\Lambda) = F_{ij}(\lambda_1, \ldots, \lambda_m) \,|\, i, j = \overline{1,n}\}$ be a continuous matrix function $\widetilde{F}(\Lambda) : U(S(\widetilde{A})) \to \mathbb{C}^{n\times n}$, where U is a neighborhood of S and $\widetilde{F}(\widetilde{A})$ is a function on family of self-adjoint commuting bounded operators. Denote $\Delta(G, \lambda) = \Delta(\widetilde{F}(\widetilde{A}), \lambda) := \det(\{F_{ij}(\widetilde{A}) - \delta_i^j \lambda\}_{i,j=1}^n)$, where δ_i^j is a Kroneker symbol and $\Delta(\Lambda, \lambda) := (F_{ij}(\Lambda) - \delta_i^j \lambda)$, where $\Lambda = (\lambda_1, \ldots, \lambda_m) \in S(\widetilde{A})$.

Theorem 1. *A spectrum of the operator G is equal to $\sigma(G) = \{\lambda \in \mathbb{C} \,|\, \exists\, \Lambda \in S : \Delta(\Lambda, \lambda) = 0\}$.*

Proof. ($\Rightarrow$) Let $\forall\, \Lambda \in S(\widetilde{A}) : \Delta(\Lambda, \lambda) \neq 0$, then existence of operator $(G - \lambda\mathbb{I})^{-1}$ means existence of a solution of the system of operator equations $\sum\limits_{j=1}^{n} F_{ij}(\widetilde{A}) x_j = y_i$, $i = \overline{1,n}$, where $x_j, y_i \in H$. The sufficient condition for it is existence of the operator $(\Delta(\widetilde{F}(\widetilde{A}), \lambda))^{-1}$.

From $\Delta(\Lambda,\lambda)\neq 0$ the following conclusion ($\widetilde{E}=\overset{m}{\underset{i,j=1}{\times}}E_{ij}$) may be drawn

$$\mathbb{I}=\int_{\mathbb{R}^m}d\widetilde{E}=\int_{\mathbb{R}^m}\Delta(\Lambda,\lambda)(\Delta(\Lambda,\lambda))^{-1}d\widetilde{E}(\Lambda)=(G-\lambda\mathbb{I})B.$$

It follows that $\exists(G-\lambda\mathbb{I})^{-1}=B\Rightarrow\lambda\in\rho(G)$, where ρ is a resolvent set.

($\Leftarrow$) Let $\exists\Lambda\in S(\widetilde{A}):\Delta(\widetilde{F}(\Lambda,\lambda))=0$, then $\exists\, f$, where f is the eigenvector of operator $\widetilde{F}(\Lambda)$. Denote $\forall\ z\geq 1$:

$$0\neq f^{(z)}:=\widetilde{E}\left(\left(\lambda_1-\frac{1}{z},\lambda_1+\frac{1}{z}\right)\times\cdots\times\left(\lambda_m-\frac{1}{z},\lambda_m+\frac{1}{z}\right)\right)f$$

(notice that E is a projector) and $y^{(z)}:=\frac{f^{(z)}}{\|f^{(z)}\|}$. It follows that

$$(G-\lambda\mathbb{I})y^{(z)}=(\{F_{ij}(\Lambda)-\lambda_{ij}\}_{i,j=\overline{1,n}})y^{(z)}-\Delta(\Lambda,\lambda)y^{(z)}.$$

The condition $\Delta(\Lambda,\lambda)y^{(z)}=0$ is clearly fulfilled, and

$$\|(F_{ij}(\widetilde{A})-\lambda_{ij})y_j^{(z)}\|^2=\int\limits_{\mathbb{R}}|\alpha_{ij}-\lambda_{ij}|^2d(E(\alpha_{ij})y^{(z)},y^{(z)})\to 0,\qquad z\to 0.$$

And finally $\|(G-\lambda\mathbb{I})y^{(z)}\|\to 0$, $z\to 0$, where $\|y^{(z)}\|=1\Rightarrow\lambda\in\sigma(G)$. ■

The following proposition will be useful in examples.

Proposition 1. *Let $A^*=A\subset L(H)$ be a self-adjoint bounded operator. Let $\widetilde{A}=\{f_1(A),\ldots,f_n(A)\}$, where $\{f_i|i=\overline{1,m}\}\subset C(U(\sigma(A)),\mathbb{R})$, U is a neighborhood of $\sigma(A)$, then*

$$S(\{f_1(A),f_2(A),\ldots,f_m(A)\})=\{(f_1(\lambda),f_2(\lambda),\ldots,f_m(\lambda))|\lambda\in\sigma(A)\}.$$

Proof. There is $E_i(\alpha)=\int_{\sigma(A)}\chi(f_i(\lambda))dE$, where $\chi(A)$ is the characteristic function, $i=\overline{1,m}$. Then denote a decomposition $\overset{m}{\underset{i=1}{\times}}\sigma_i=\operatorname{Supp}\widetilde{E}\cup\Theta$, where the common measure of any opening set with Θ is equal to zero, and the common measure of any opening set is not equal to zero if this set contains a point from $\operatorname{Supp}\widetilde{E}$. Since $\widetilde{E}(\alpha_1\alpha_2\cdots\alpha_m)=\int_{\mathbb{R}}\chi_{\alpha_1}(\lambda)\cdots\chi_{\alpha_m}(\lambda)dE(\lambda)=0$ and $f(\sigma(A))=\sigma(f(A))$, then proposition is proved. ■

3 Examples

Example 1. Let $H=L_2([-\pi,\pi],dt)$ and $G=\begin{pmatrix}A_1 & A_2\\ A_3 & A_4\end{pmatrix}$, and $K_j(t):=je^{it}$, $j=\overline{1,4}$, $i=\sqrt{-1}$, and $\widetilde{A}:=\{A_j=\int_{-\pi}^{\pi}K_j(t-\tau)x(\tau)d\tau\,|\,j=\overline{1,4}\}$ family of commuting self-adjoint bounded operators. The common spectrum of $\widetilde{A}$ is equal to

$$S(\widetilde{A})=\{(0,0,0,0),(2\pi,4\pi,6\pi,8\pi)\}=:\{\Lambda_1,\Lambda_2\}.$$

Let us solve the following equations $\Delta(\Lambda_1,\lambda)=0,\Delta(\Lambda_2,\lambda)=0$:

$$\det\begin{pmatrix}-\lambda & 0\\ 0 & -\lambda\end{pmatrix}=0\qquad\text{and}\qquad\det\begin{pmatrix}2\pi-\lambda & 4\pi\\ 6\pi & 8\pi-\lambda\end{pmatrix}=0\quad\Rightarrow$$

$\Rightarrow$ The spectrum of matrix operator is equal to $\sigma(G)=\{0,\pi(5\pm\sqrt{33})\}$.

Example 2. Let $H = L_2([0,1], dt)$ and $f_1, f_2, f_3, f_4 \in C([0,1])$. Let $\widetilde{A} = \{(A_1x)(t) = f_1(t)x(t), (A_2x)(t) = f_2(t)x(t), (A_3x)(t) = f_3(t)x(t), (A_4x)(t) = f_4(t)x(t)\}$ be a family of self-adjoint commuting operators, then

$$S(A_1, A_2, A_3, A_4) = \{f_1(t), f_2(t), f_3(t), f_4(t) | t \in [0,1]\},$$
$$G = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} F_{11}(\widetilde{A}) & F_{12}(\widetilde{A}) \\ F_{21}(\widetilde{A}) & F_{22}(\widetilde{A}) \end{pmatrix},$$

where

$$F_{11}(t_1,t_2,t_3,t_4) = t_1, F_{12}(t_1,t_2,t_3,t_4) = t_2, F_{21}(t_1,t_2,t_3,t_4) = t_3, F_{22}(t_1,t_2,t_3,t_4) = t_4,$$
$$\sigma(G) = \left\{ \lambda \in \mathbb{C} \,|\, \exists\, t \in [0,1] \,:\, \det \begin{pmatrix} f_1(t) - \lambda & f_2(t) \\ f_3(t) & f_4(t) - \lambda \end{pmatrix} = 0 \right\} \Leftrightarrow$$
$$\Leftrightarrow \sigma(G) = \left\{ \frac{f_1(t) + f_4(t) \pm \sqrt{(f_1(t) + f_4(t))^2 + 4(f_2f_3(t) - f_1(t)f_4(t))}}{2} \,\Big|\, t \in [0,1] \right\}.$$

Example 3. Let $(P_1x)(t) = t_1x(t)$, $(P_2y)(t) = 2\pi t_2x(t)$ be two commuting self-adjoint bounded operators, where $x(t) \in L_2([0,1]^2, dt)$, $t = (t_1, t_2)$. The operator $N = P_1(\cos P_2 + i \sin P_2)$ is a normal operator ($N^*N = NN^*$). Let us consider the subalgebra B_V of $C(D, \mathbb{C}^{2\times 2})$ of the form $B_V(D) = \{a \in C(D, \mathbb{C}^{2\times 2}) : a(z) = V^{-1}(z)a(1)V(z), if |z| = 1\}$, where $D = \{z \in \mathbb{C} : |z| \le 1\}$ is the unit disk and $V(z) = \operatorname{diag}\{z^{-1}, 1\}$. The algebra $B_V(D)$ is an example of a non-trivial 2-homogeneous C^*-algebra (see [5]).

It follows that $B(N) = \{a(N) \,|\, a \in B_V(D)\}$ is an example of a non-trivial operator C^*-algebra, because the spectrum of operator N is equal to $\sigma(N) = D$. The following conclusion may be drawn

$$\forall\, a = a(N) \in B_V(N) : a(N) = \begin{pmatrix} a_{11}(N) & a_{12}(N) \\ a_{21}(N) & a_{22}(N) \end{pmatrix} = \begin{pmatrix} a_{11}(f(P_1,P_2)) & a_{12}(f(P_1,P_2)) \\ a_{21}(f(P_1,P_2)) & a_{22}(f(P_1,P_2)) \end{pmatrix},$$

where

$$f(x,y) = x(\cos y + i \sin y),$$
$$S(\{a_{ij}(N), i,j = 1,2\}) = \{\{a_{ij}(t_1(\cos 2\pi t_2 + i \sin 2\pi t_2))\} \,|\, t \in [0,1]^2\},$$
$$\sigma(a(N)) = \left\{ \frac{a_{11}(\tau) + a_{22}(\tau)}{2} \right.$$
$$\left. \pm \frac{\sqrt{(a_{11}(\tau) + a_{22}(\tau))^2 - 4(a_{11}(\tau)a_{22}(\tau) - a_{12}(\tau)a_{21}(\tau))}}{2} \,|\, \tau = f(t_1, 2\pi t_2),\ t \in [0,1]^2 \right\}.$$

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Yangians of Classical Lie Superalgebras: Basic Constructions, Quantum Double and Universal R-Matrix

Vladimir STUKOPIN

Math. Department, Don State Technical University, 1 Gagarin Sq., Rostov-na-Donu, Russia
E-mail: *stukopin@math.rsu.ru, stukopin@mccme.ru*

Some basic results of the theory of Yangians of Lie superalgebras are described. Yangian of basic Lie superalgebra is described as a result of quantization of Lie superbialgebra of polynomial loops. Also, we consider a quantization of Lie superalgebra of twisted currents. Two systems of generators and defining relations are introduced. Equivalence of this systems of generators and defining relations is proved. The notion of double of Yangian and formula for universal R-matrix for double of Yangian are discussed for Yangian of Lie superalgebra $A(m,n)$ type.

1 Introduction

The main goal of this article is to describe the some basic results of the theory of Yangians of Lie superalgebras of classical type. It is well-known that Lie superalgebras of classical type divided into two classes: basic and strange Lie superalgebras. The Lie superalgebras from the first class have characteristics like those of simple Lie algebras. They have a nondegenerate invariant bilinear form (and nontrivial Casimir operator). This characteristic makes possible to define Yangian of basic Lie superalgebra as Yangian of simple Lie algebra. I remind that the notion of Yangian of a simple Lie algebra g was introduced by V. Drinfeld [1] as a deformation of the universal enveloping algebra $U(g[t])$ of a current algebra $g[t]$ and it was extended to the case of a Lie superalgebras of classical type (see [4, 5]). But strange Lie superalgebras P_n, Q_n (see [12]) do not have natural bisuperalgebra structure. But such structure can be defined onto the twisted current superalgebra (see [7]). But we, as distinct from [7] use diagram automorphism and Drinfeld realization of Yangian.

The main tool of applications of the quantum algebras theory in mathematical physics (and physics) is a quantum R-matrix. I remind the definition of an R-matrix [1–3]. An R-matrix is a function $R(u)$ of a complex parameter u with values in $\mathrm{End}\,(V \otimes V)$, where V is a finite-dimensional vector space (superspace) which satisfies the quantum Yang–Baxter equation (QYBE) (or grading quantum Yang–Baxter equation (gQYBE) in the case of superspace):

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v),$$

(here, $R_{12} = R \otimes id \in \mathrm{End}\,(V \otimes V \otimes V)$, etc.) Drinfeld relates to Yangian $Y(g)$ (g is a simple Lie algebra) an element R in some formal completion of $Y(g) \otimes Y(g)$ which is called universal R-matrix (see [1]). Universal R-matrix is defined by the following condition. R-matrix conjugates comultiplication and opposite comultiplication: $\Delta'(a) = R\Delta(u)R^{-1}$. Quantum R-matrix R_V is image of universal R-matrix R under action of the representation operator $\rho_V \otimes \rho_V : Y(g) \otimes Y(g) \to \mathrm{End}\,(V \otimes V)$. The problem of description (or computation) of universal R-matrix is very important as all formulas of quantum R-matrices are consequences of the formula for universal R-matrix. In many cases it reasonable to work with quantum double $DY(g)$ of the Yangian $Y(g)$ if we keep in mind application in quantum field theory. The definition of double for

Hopf algebras was given by V. Drinfeld, explicit formulas for double of Yangian were obtained by F. Smirnov [10] and S. Khoroshkin, V. Tolstoy [9]. The notion of $DY(G)$ for Lie superalgebra G we discuss below (see also [5]). Universal R-matrix for quantum double is a canonical element $R = \sum a_i \otimes a_i'$, where a_i, a_i' are dual bases in $DY(G)$. Yangians of the Lie algebras have been studied at present time in many papers (see [2]). The theory of Yangians for Lie superalgebras (or super Yangians) is less developed than the one for Lie superalgebras (see [4–7]). In this work we made an attempt to develop the theory of Yangians for Lie superalgebras of classical type and generalize some results from [4, 5]. The structure of this work is following. In Section 1 we remind basic definitions of the Lie superalgebras theory. In Section 2 we define bisuperalgebra structure on superalgebras polynomial loops for basic Lie superalgebras and on superalgebra of twisted currents on Lie superalgebra $A(n, n)$. We introduce the main object $Y(G)$, the Yangian of basic Lie superalgebra $Y(G)$ as a quantization of a $U(G[t])$ and the Yangian of strange Lie superalgebra Q_n as a quantization of twisted current Lie superalgebra with values in $A(n, n)$. We describe $Y(G)$ in terms of generators and defining relations. Starting from a system of zero-order generators for which the relations and the comultiplication law are similar to those of the universal enveloping superalgebra, we introduce first-order generators, for which the comultiplication law is more complicated. The remaining relations follow from the compatibility conditions for the algebra and coalgebra structures. From the finite system of generators and relations thus obtained we derive a new system of generators and relations (Section 3) (in the case of the basic superalgebra of the $A(m, n)$ type). In Section 4 we discuss the structure of double and multiplicative formula of universal R-matrix for double of super Yangian $Y(A(m, n))$. The proofs either not adduced or are given only as sketches.

2 Lie superalgebras and superbialgebras

Let us recall some basic definitions from Lie superalgebra theory [8]. Lie superalgebra $G = G_0 \oplus G_1$ is a Z_2-graded algebra with product $[\cdot, \cdot]$, i.e. if $a \in G_\alpha$, $b \in G_\beta$, $\alpha, \beta \in Z_2 = \{0, 1\}$, then $[a, b] \in G_{\alpha+\beta}$. A Lie superalgebra G is superalgebra satisfying the following axioms:

$$[a, b] = -(-1)^{\alpha\beta}[b, a], \qquad [a, [b, c]] = [[a, b], c] + (-1)^{\alpha\beta}[b, [a, c]], \qquad a \in G_\alpha, \qquad b \in G_\beta.$$

Let $A = (a_{ij})$ be a matrix of order r and τ a subset of the set $I = \{1, \dots, r\}$ (as a rule $\tau = \{k\}$, $1 \le k \le r$). Let us assume that Cartan matrix A is a symmetrizable matrix and $B = DA$ is a symmetrical matrix and $D = \operatorname{diag}(d_1, \dots, d_r)$, $d_k \neq 0$, is a diagonal matrix. Let $G = G(A, \tau)$ be the Lie superalgebra with generators x_i^+, x_i^-, h_i, $i \in I$ and the following defining relations:

$$\begin{aligned} &\left[x_i^+, x_i^-\right] = \delta_{ij} h_i, \qquad [h_i, h_j] = 0, \\ &\left[h_i, x_i^\pm\right] = \pm b_{i,j} x_j^\pm, \qquad \left(\operatorname{ad}^{n_{ij}} x_i^\pm\right)(x_j^\pm) = 0, \quad i \neq j, \end{aligned}$$

where $n_{i,j} = 1$ if $b_{i,i} = b_{i,j} = 0$; $n_{i,j} = 2$ if $b_{i,i} = 0$, $b_{i,j} \neq 0$; $n_{i,j} = 1 - 2b_{i,j}/b_{i,i}$, if $b_{i,i} \neq 0$; $\operatorname{ad} a(b) = [a, b]$. Let $\deg(x)$ be a α if $x \in G_\alpha$, $\alpha \in Z_2$.

Let $\Delta = \Delta_0 \cup \Delta_1$ be a root system of G, Δ_i be a root system of G_i, Δ^+, Δ_i^+ be a sets of positive roots of G, G_i respectively, $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be a simple root system. Recall that a simple root α_i is called white if $i \in I \setminus \tau$ and grey or black if $i \in \tau$ and $a_{ii} = 0$ or $a_{ii} = 2$, respectively. Remind also, that a finite-dimensional Lie superalgebra $G = G_0 \oplus G_1$ is of classical type (or classical), if it is simple, and the representation of G_0 in G_1 is completely reducible. We need the description of the Lie superalgebras $A(m, n)$ and Q_n. For $A(m, n)$ the Cartan matrix $A = (a_{i,j})_{i,j=1}^r$, $r = m + n + 1$ is the following: $a_{i,i} = 2$, $i \neq m + 1$, $a_{m+1,m+1} = 0$, $a_{i+1,i} = -1$ for all i, $a_{i,i+1} = -1$, $i \neq m + 1$, $a_{m+1,m} = 1$, and $a_{i,j} = 0$ elsewhere.

For the definition of strange Lie superalgebra Q_n consider diagram automorphism $\sigma : A(n,n) \to A(n,n)$, $\sigma(f_i) = f_{2n+2-i}$, $(f \in \{h, x^+, x^-\})$. The strange classical Lie superalgebra Q_n is the fixed point subalgebra in $A(n,n)$ with respect to the involutive diagram automorphism σ. The subalgebra Q_n has the property that the Cartan matrix H does not coincide with Cartan subalgebras of the even part A_n but admits also an odd part H_1. More precisely, one has $H = H_0 \oplus H_1$, where H_0 is spanned by the $\{h_i\}$ generators and H_1 by the $\{k_i\}$ generators $(1 \le i \le n)$. However, since the $\{k_i\}$ generators are odd, the root generators $x^{\pm}_{\alpha}$ are not eigenvectors of H_1. It is convenient in this case to give the root decomposition with respect to H_0 instead of H. The root system Δ of Q_n coincides then with the root system of A_n. One has $G = G_0 \oplus G_1 = H_0 \oplus (\oplus_{\alpha \in \Delta} G_\alpha)$ with $\dim G_{\alpha \neq 0} = 2$ and $\dim G_{\alpha = 0} = n$. Moreover, the non-zero roots of Q_n are both even and odd. Note also that in $Q_n \otimes Q_n$ are not nontrivial Q_n-invariants.

Let G be a Lie superalgebra with symmetrizable Cartan matrix and nondegenerate Killing form. A bisuperalgebra G is a vector superspace with superalgebra and cosuperalgebra structures compatible in the following sense: the cocommutator map $\varphi : G \to G \otimes G$ must be a 1-cocycle (we assume that G acts on $G \otimes G$ by means of adjoint representations). Let the cocycle φ be a coboundary of element $r \in \bigwedge^2 g$. Consider the current superalgebra $G[t]$ and introduce the cocommutator φ by formula

$$\varphi : a(u) \to [a(u) \otimes 1 + 1 \otimes a(v), r(u,v)],$$

where $r = \frac{t}{u-v}$ and t is a Casimir operator. Cocommutator φ defines on $G[t]$ the bisuperalgebra structure. Let us note that r satisfies the Yang–Baxter equation (or the triangle equation) (see [1]).

Consider the case of Lie superalgebra Q_n. We are going to define the bisuperalgebra structure on the twisted current superalgebra or on the fixed point subalgebra in $A(n,n)[u]$. Let G be a basic superalgebra $A(n,n)$, $\sigma : G \to G$ earlier defined automorphism of order 2, $\sigma^2 = 1$. Then all eigenvalues of σ are ± 1. The automorphism σ is diagonalized and we have the following decomposition: $G = G^0 \oplus G^1$, where $G^j = \mathrm{Ker}\,(\sigma - \epsilon^j I)$, $\epsilon = e^{\pi i} = -1$, $j \in Z_2$; $G^0 = Q_n$. Let us associate subalgebra $G[u]^\sigma$ of algebra $G[u]$ to automorphism σ:

$$G[u]^\sigma = \Big(\bigoplus_{k \in Z_+} \big(G^0 \otimes t^{2k}\big) \Big) \bigoplus \Big(\bigoplus_{k \in Z_+} \big(G^1 \otimes t^{2k+1}\big) \Big).$$

Let us note that

$$G[u]^\sigma = \{a(u) \in G[u] : \sigma(a(u)) = a(-u)\}.$$

Let us define bisuperalgebra structure on the $G[u]^\sigma$. Let us introduce

$$r_\sigma(u,v) = \frac{t}{u-v} + \frac{id \otimes \sigma(t)}{u+v},$$

where $t \in G^{\otimes 2}$ is a Casimir operator. It is easy to check (see [7]) that the function $r_\sigma(u,v)$ is antisymmetric and obeys classical Yang–Baxter equation. Then we can define the cocommutator $\varphi : G[u]^\sigma \to G[u]^\sigma \otimes G[u]^\sigma$ by the formula

$$\varphi : a(u) \to [a(u) \otimes 1 + 1 \otimes a(v), r_\sigma(u,v)].$$

Cocommutator φ defines on a bisuperalgebra structure $G[u]^\sigma$.

3 Quantization

The definition of quantization for Lie bialgebras from [1] can be extended on Lie bisuperalgebras. A quantization of Lie bisuperalgebra A $(\in \{G[t], G[t]^\sigma\})$ is called a Hopf superalgebra A_h over the ring of formal power series $C[[h]]$ such that the following conditions hold:

1) $A_h/hA_h \cong U(A)$, where $U(A)$ is the universal enveloping superalgebra of a Lie superalgebra A;
2) Lie superalgebra A_h is isomorphic to $U(A)[[h]]$, as a vector space;
3) correspondence principle is fulfilled $h^{-1}(\Delta(x) - \Delta^{\mathrm{op}}(x)) \bmod h = \varphi(x \bmod h)$, where Δ is comultiplication, Δ^{op} is opposite comultiplication (i.e., if $\Delta(x) = \sum x'_i \otimes x''_i$, then $\Delta^{\mathrm{op}}(x) = \sum (-1)^{p(x'_i)p(x''_i)} x''_i \otimes x'_i$), cobracket $\varphi : A \to A \bigwedge A$ be a cocycle in A.

From 1), 2) it follows that we can consider the deformations of generators m_i, $m_i u$, $i \in I$, ($m \in \{h, x^{\pm}\}$ in the case of the basic Lie superalgebra and $m \in \{h, k, x^{\pm}\}$ in the case of Q_n), as generators of A_h. Denote these deformations as by $m_{i,0}$, $m_{i,1}$, $i \in I$, correspondingly. From conditions 1), 3) and formula for cocycle it follows that we must introduce the relations and comultiplication laws for $m_{i,0}$ as for m_i. Define a comultiplication law for $h_{i,1}$ in the case of a basic Lie superalgebra by formula:

$$\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + h[h_{i,0} \otimes 1, \Omega_2], \qquad i \in I,$$

where $\Omega_2 = \sum\limits_{\alpha \in \Delta_+} (-1)^{\deg(x^-_\alpha)} x^-_\alpha \otimes x^+_\alpha$ is a part of Casimir operator. Let us define a comultiplication laws for $h_{i,1}$, $k_{i,1}$ in the case of Q_n by formulas

$$\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + h[h_{i,0} \otimes 1, \Omega + id \otimes \sigma(\Omega)], \qquad i \in I,$$
$$\Delta(k_{i,1}) = k_{i,1} \otimes 1 - 1 \otimes k_{i,1} + h[1 \otimes k_{i,0}, \Omega + id \otimes \sigma(\Omega)], \qquad i \in I,$$

where $\Omega = \Omega_2 + 1/2 \sum k_i \otimes k^i$, k^i are dual to k_i. (We also denote odd generator $x^{\pm}_\alpha$) by $y^{\pm}_\alpha$).) Easy to check that the correspondence principle is fulfilled in both cases.

Comultiplication laws for $x^{\pm}_{i,1}$ is

$$\Delta(x^+_{i,1}) = x^+_{i,1} \otimes 1 + (-1)^{\deg(x^-_i)} 1 \otimes x^+_{i,1} + h[x^+_{i,0} \otimes 1, \tilde{\Omega}], \qquad i \in I,$$
$$\Delta(x^-_{i,1}) = x^-_{i,1} \otimes 1 + (-1)^{\deg(x^-_i)} 1 \otimes x^-_{i,1} + h[1 \otimes x^-_{i,0}, \tilde{\Omega}], \qquad i \in I,$$

where $\tilde{\Omega} = \Omega_2$ in the case of the basic Lie superalgebra and $\tilde{\Omega} = \Omega + id \otimes \sigma(\Omega)$ in the case of Q_n.

From the condition of compatibility of superalgebra and cosuperalgebra structures other defining relations for A_h follow. Fixing $h = 1$ we obtain the Yangian of Lie superalgebra G: $Y(G) = A_1$.

We will use the notation $\{a, b\} := ab + (-1)^{\deg(a)\deg(b)} ba$ for anticommutator of elements a, b. Also $(b_{ij})_{i,j=1}^{n-1}$ denotes Cartan matrix for A_{n-1} Lie algebra, $\tilde{b}_{ij} = a_{n,n+i-j}$, where $(a_{ij})_{i,j=1}^{2n-1}$ be a Cartan matrix for $A(n-1, n-1)$ Lie superalgebra.

Theorem 1. 1) *Let G be a basic Lie superalgebra, $G = A(m, n)$. The Yangian $Y(G)$ is a Hopf superalgebra over C with generators $h_{i,0}$, $x^{\pm}_{i,0}$, $h_{i,1}$, $x^{\pm}_{i,1}$, $i \in I$, and defining relations*

$$[h_{i,0}, h_{j,0}] = [h_{i,0}, h_{j,1}] = [h_{i,1}, h_{j,1}] = 0, \tag{1}$$
$$[h_{i,0}, x^{\pm}_{j,0}] = \pm b_{ij} x^{\pm}_{j,0}, \qquad [h_{i,1}, x^{\pm}_{j,0}] = \pm b_{ij} x^{\pm}_{j,1}, \tag{2}$$
$$[x^+_{i,0}, x^-_{j,0}] = \delta_{ij} h_{i,0}, \qquad [x^+_{i,1}, x^-_{j,0}] = \delta_{ij} \left(h_{i,1} + \frac{1}{2} h^2_{i,0} \right), \tag{3}$$
$$[x^{\pm}_{i,1}, x^{\pm}_{j,0}] = [x^{\pm}_{i,0}, x^{\pm}_{j,1}] \pm (b_{ij}/2)(x^{\pm}_{i,0} x^{\pm}_{j,0} + x^{\pm}_{j,0} x^{\pm}_{i,0}), \tag{4}$$
$$(\operatorname{ad} x^{\pm}_{i,0})^{n_{ij}} (x^{\pm}_{j,0}) = 0, \qquad i \neq j, \tag{5}$$
$$[[h_{i,1}, x^+_{i,1}], x^-_{i,1}] + [x^+_{i,1}, [h_{i,1}, x^-_{i,1}]] = 0, \qquad b_{ii} \neq 0, \tag{6}$$
$$[[h_{i,1}, x^+_{j,1}], x^-_{j,1}] + [x^+_{j,1}, [h_{i,1}, x^-_{j,1}]] = 0, \qquad b_{ii} = 0, \qquad b_{ij} \neq 0, \tag{7}$$
$$[[x^{\pm}_{m+1,0}, x^{\pm}_{m+2,0}], [x^{\pm}_{m+1,0}, x^{\pm}_{m,0}]] = 0. \tag{8}$$

The comultiplication law Δ is defined by formulas:

$$\Delta(h_{i,0}) = h_{i,0}\otimes 1 + 1\otimes h_{i,0}, \qquad \Delta(x^{\pm}_{i,0}) = x^{\pm}_{1,0}\otimes 1 + (-1)^{\deg(x^-_i)}1\otimes x^{\pm}_{i,0},$$
$$\Delta(h_{i,1}) = h_{i,1}\otimes 1 + 1\otimes h_{i,1} + [h_{i,0}\otimes 1, \Omega_2],$$
$$\Delta(x^{+}_{i,1}) = x^{+}_{i,1}\otimes 1 + (-1)^{\deg(x^+_i)}1\otimes x^{+}_{i,1} + [x^{+}_{i,0}\otimes 1, \Omega_2],$$
$$\Delta(x^{-}_{i,1}) = x^{-}_{i,1}\otimes 1 + (-1)^{\deg(x^-_i)}1\otimes x^{-}_{i,1} - [1\otimes x^{-}_{i,0}, \Omega_2].$$

2) *The Yangian $Y(Q_n)$ is a Hopf algebra over C with generators $h_{i,0}$, $x^{\pm}_{i,0}$, $k_{i,0}$, $y^{\pm}_{i,0}$, $h_{i,1}$, $x^{\pm}_{i,1}$, $k_{i,1}$, $y^{\pm}_{i,1}$, $1\le i\le n$, (h, x are even, k, y are odd generators). Defining relations for generators h_i, $x^{\pm}_i$ include the relations* (1), (2), (5)–(8) *and the rest defining relations is following:*

$$[h_{i,0}, k_{j,0}] = [h_{i,0}, k_{j,1}] = [h_{i,1}, k_{j,1}] = 0, \qquad [k_{i,1}, k_{j,0}] = [k_{i,0}, k_{j,1}] = 0,$$
$$[h_{i,0}, y^{\pm}_{j,0}] = \pm b_{ij} y^{\pm}_{j,0}, \qquad [k_{i,0}, x^{\pm}_{j,0}] = \pm b_{ij} y^{\pm}_{j,0}, \qquad [k_{i,0}, y^{\pm}_{j,0}] = \mp \tilde{b}_{ij} x^{\pm}_{j,0},$$
$$[h_{i,1}, y^{\pm}_{j,0}] = \pm b_{ij} y^{\pm}_{j,1}, \qquad [k_{i,1}, x^{\pm}_{j,0}] = \pm b_{ij} y^{\pm}_{j,1}, \qquad [k_{i,1}, y^{\pm}_{j,0}] = \mp \tilde{b}_{ij} x^{\pm}_{j,1},$$
$$[x^{+}_{i,0}, y^{-}_{j,0}] = [y^{+}_{i,0}, x^{-}_{j,0}] = \delta_{ij} k_{i,0},$$
$$[y^{+}_{i,0}, y^{-}_{j,0}] = \delta_{ij}\left(-\frac{4}{n+1}\sum_{k=1}^{n} k h_{k,0} + 2\sum_{k=i}^{n} h_{k,0} + 2\sum_{k=i+1}^{n} h_{k,0}\right),$$
$$[x^{+}_{i,1}, x^{-}_{j,0}] = \delta_{ij}\tilde{h}_{i,1} = \delta_{ij}\left(h_{i,1} + \frac{1}{2}\big(h^2_{i,0} + k^2_{i,0}\big)\right),$$
$$[x^{+}_{i,1}, y^{-}_{j,0}] = \delta_{ij}\tilde{k}_{i,1} = \delta_{ij}\left(k_{i,1} + \frac{1}{2}\{h_{i,0}, k_{i,0}\}\right),$$
$$[x^{\pm}_{i,1}, x^{\pm}_{j,0}] = [x^{\pm}_{i,0}, x^{\pm}_{j,1}] \pm (b_{ij}/2)\big(\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} + [y^{\pm}_{i,0}, y^{\pm}_{j,0}]\big),$$
$$[y^{\pm}_{i,1}, y^{\pm}_{j,0}] = [y^{\pm}_{i,0}, y^{\pm}_{j,1}] \pm (\tilde{b}_{ij}/2)\{x^{\pm}_{i,0}, x^{\pm}_{j,0}\} \pm (b_{ij}/2)\{y^{\pm}_{i,0}, y^{\pm}_{j,0}\},$$
$$[x^{\pm}_{i,1}, y^{\pm}_{j,0}] = [x^{\pm}_{i,0}, y^{\pm}_{j,1}] \pm (b_{ij}/2)\{x^{\pm}_{i,0}, y^{\pm}_{j,0}\} \pm (\tilde{b}_{ij}/2)\{y^{\pm}_{i,0}, x^{\pm}_{i,0}\},$$
$$[[h_{i,1}, y^{+}_{i,1}], y^{-}_{i,1}] + [y^{+}_{i,1}, [h_{i,1}, y^{-}_{i,1}]] = 0, \qquad [[k_{i,1}, x^{+}_{i,1}], x^{-}_{i,1}] + [x^{+}_{i,1}, [k_{i,1}, x^{-}_{i,1}]] = 0,$$
$$[[k_{i,1}, y^{+}_{i,1}], y^{-}_{i,1}] + [k_{i,1}, [y^{+}_{i,1}], y^{-}_{i,1}]] = 0,$$
$$[k_{i,0}, k_{j,0}] = \frac{2}{n+1}(2\delta_{i,j} - \delta_{i,j+1} - \delta_{i,j-1})\sum_{k=1}^{n} h_{k,0}$$
$$+ 2(\delta_{i,j} - \delta_{i,j+1})\sum_{k=i}^{n} h_{k,0} + 2(\delta_{i,j} - \delta_{i,j-1})\sum_{k=i+1}^{n} h_{k,0},$$
$$[y^{+}_{i,1}, y^{-}_{j,0}] = \delta_{ij}\left(h_{i,1} + \frac{1}{2}\big(h^2_{i,0} + k^2_{i,0}\big)\right).$$

4 New system of generators

For simplicity we will assume that $G = A(m,n)$. Let us introduce generators $\tilde{x}^{\pm}_{i,k}$, $\tilde{h}_{i,k}$, ($\in Y(G)$), $i\in I$, $k\in Z_+$ by formulas

$$\tilde{x}^{\pm}_{i,k+1} = \pm(a^{-1}_{i,i})[h_{i,1}, \tilde{x}^{\pm}_{i,k}], \qquad i\in I\setminus\tau(i\ne m+1), \tag{9}$$
$$\tilde{x}^{\pm}_{m+1,k+1} = \mp[h_{m,1}, \tilde{x}^{\pm}_{m+1,k}], \tag{10}$$
$$\tilde{h}_{i,k} = [\tilde{x}^{+}_{i,k}, x^{-}_{i,0}], \tag{11}$$
$$\tilde{x}^{\pm}_{i,0} = x^{\pm}_{i,0}, \qquad \tilde{h}_{i,0} = h_{i,0}. \tag{12}$$

Definition 1. Denote by $\bar{Y}(G)$ a superalgebra over C with generators $x^{\pm}_{i,k}$, $h_{i,k}$, $i \in \Gamma = I$, $k \in Z_+$, and defining relations:

$$[h_{i,k}, h_{j,l}] = 0, \qquad \delta_{i,j} h_{i,k+l} = [x^{+}_{i,k}, x^{-}_{j,l}], \tag{13}$$

$$[h_{i,k+1}, x^{\pm}_{j,l}] = [h_{i,k}, x^{\pm}_{j,l+1}] + (b_{ij}/2)(h_{i,k} x^{\pm}_{j,l} + x^{\pm}_{j,l} h_{i,k}), \tag{14}$$

$$[h_{i,0}, x^{\pm}_{j,l}] = \pm b_{ij} x^{\pm}_{j,l}, \tag{15}$$

$$[x^{\pm}_{i,k+1}, x^{\pm}_{j,l}] = [x^{\pm}_{i,k}, x^{\pm}_{j,l+1}] + (b_{ij}/2)(x^{\pm}_{i,k} x^{\pm}_{j,l} + x^{\pm}_{j,l} x^{\pm}_{i,k}), \tag{16}$$

$$\sum_{\sigma} [x^{\pm}_{i,k_{\sigma(1)}}, \dots [x^{\pm}_{i,k_{\sigma(r)}}, x^{\pm}_{j,l}] \dots] = 0, \qquad i \neq j, \qquad r = n_{ij}. \tag{17}$$

The sum is taken over all permutations σ of $\{1, \dots, r\}$. Let us note that the parity function takes the following values on generators: $p(x^{\pm}_{j,k}) = 0$, for $k \in Z_+$, $j \in I \setminus \tau$ $p(h_{i,k}) = 0$, for $i \in I$, $k \in Z_+$, $p(x^{\pm}_{i,k}) = 1$, $k \in Z_+$, $i \in \tau$.

Theorem 2. *Correspondence*

$$\tilde{x}^{\pm}_{i,k} (\in Y(G) \to x^{\pm}_{i,k} (\in \bar{Y}(G), \qquad \tilde{h}_{i,k} (\in Y(G) \to h_{i,k} (\in \bar{Y}(G)$$

defines the isomorphism

$$Y(G) \to \bar{Y}(G).$$

Proof. First of all I will give the sketch of the proof. We can suggest that $\bar{Y}(G)$ is generated by generators $\tilde{x}^{\pm}_{i,k}$, $\tilde{h}_{i,k}$ satisfying relations (1)–(12). Therefore, it is sufficient to prove that relations (13)–(17) follows from (1)–(12) and, again that the relations (1)–(12) follow from relations (13)–(17). The latter fact is almost evident. Really, relations (1)–(8) are contained among relations (13)–(17). The first relation (9) follows from second relation (14) and the definition of $\tilde{h}_{i,1}$. Relation (12) follows from the second relation (13) The inverse fact can be proved as in [5].

The equivalence of defining relations for even generators is similar to the case of Yangians of simple Lie algebras and may be proved by induction. The proof of equivalence of defining relations in the case of odd generators is based on the one for the even generators and is carried out the same way. First of all we deduce (13)–(17) from (1)–(8) for small values of second indices the preparing base of induction. After that we deduce relations (13)–(17) for $i = j$ (for even and odd generators) and finally we deduce (13)–(17) for $i \neq j$ for even and odd indices. ■

5 Double of Yangian and the universal *R*-matrix

Recall the definition of double of Hopf algebra (superalgebra). Let A^0 be a dual A^* with opposite comultiplication. Then the unique quasitriangular Hopf algebra (superalgebra) $D(A)$ exists with the universal R-matrix R such that: 1) $D(A)$ consists of A and A^0 as Hopf subalgebras (subsuperalgebras); 2) R is an image of the canonical element $A \otimes A^0$ under inclusion in $D(A) \otimes D(A)$; 3) the linear map $A \otimes A \hookrightarrow D(A)$, $a \otimes b \mapsto ab$ is a bijection. Let us define the double of the super Yangian. Let $C(G)$ be a superalgebra generated by the elements $x^{\pm}_{ik}$, h_{ik}, $i \in I$, $k \in Z$ with relations (13)–(17). As above, we refer to the second index of the generators as its degree. We obtain that superalgebra $C(G)$ admits Z-filtration:

$$\dots \subset C_n \subset \dots \subset C_{-1} \subset C_0 \subset C_1 \subset \dots \subset C(G).$$

Let $\bar{C}(G)$ be the corresponding formal completion of $C(G)$.

Theorem 3. *Superalgebra $\bar{C}(G)$ isomorphic $DY(G)$.*

Theorem 4. *Let $G = A(m,n)$. Then the universal R-matrix R of the super Yangian double $DY(G)$ can be presented in the factorizable form*

$$R = R_+ R_0 R_-,$$

where

$$R_\pm = \prod_{\alpha\in\Delta_+,\, k\geq 0} \exp\left(N_\alpha x^\pm_{\alpha,k} \otimes x^\mp_{-\alpha,-k-1}\right),$$

where the product is ordering in correspondence with some normal (convex) order (see, for example [9]) and N_α is a some normalizing constant.

The middle term R_0 has a more complicated structure.

Theorem 5. *Let $\varphi^\pm_i(u) = \ln h^\pm_i(u)$, $A = (a_{ij})^n_{i,j=1}$ be a symmetrizable matrix for $A(m,n)$ $(m \neq n)$, $a_{ij}(q) = [a_{ij}]_q = \frac{q^{a_{ij}} - q^{-a_{ij}}}{q-q^{-1}}$, $A(q) = (a_{ij}(q))^n_{i,j=1}$. Let $C(q) = (c_{ij}(q))^n_{i,j=1}$ be a proportional matrix to $A(q)^{-1}$, and T be a shift operator, $Tf(v) = f(v-1)$. Then*

$$R_0 = \prod_{m\geq 0} \exp \sum_{i,j\in I} \mathrm{Res}_{u=v} (\varphi^+_i(u))' \otimes c_{ji}(T^{-1/2}) \varphi^-_j(v + (n+1/2)h),$$

where h is some constant, and operator $T^{-1/2}$ substitutes $c_{ji}(q)$ instead of q.

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Spinor Representation of Lie Algebra for Complete Linear Group

Constantin V. USENKO † *and Bogdan I. LEV* ‡

† *Physics Department, Taras Shevchenko Kyiv University,*
6 Academician Glushkov Ave., 03127 Kyiv, Ukraine
E-mail: *usenko@phys.univ.kiev.ua*

‡ *Institute of Physics of NAS of Ukraine, 46 Nauky Ave., 03028 Kyiv, Ukraine*
E-mail: *lev@iop.kiev.ua*

Spinor representation of group $GL(4,\mathbb{R})$ on special spinor space is developed. Representation space has a structure of the fiber space with the space of diagonal matrices as the base and standard spinor space as typical fiber. Non-isometric motions of the space-time entail spinor transformations which are represented by translation over fibering base in addition to standard $Spin(4,\mathbb{C})$ representation.

1 Introduction

Spinor representation of the group $GL(4,\mathbb{R})$ is needed for correct description of the Fermi fields on Riemann space, such as the space-time of general relativity. It is used for two purposes: to define the connectivity and covariant derivative of spinor field and to define the Lie derivative. Recent publications [1–3] have reminded of this problem. The important problem for definition of Fermi fields on Riemann space is that transformation properties of Dirac equation correspond [4] to $Spin(3,1)$ representation of Lorentz group $SO(3,1)$ only, not the full linear group $GL(4,\mathbb{R})$.

Covariant derivative definition and field equations based on it can be defined by the field of orthonormal basis – tetrad description of curve geometry. In such way the tetrad connectivity is a member of Lorentz group and generates the spinor connectivity as standard $Spin(3,1)$ representation. Spinor representation of group $GL(4,\mathbb{R})$ is needed for investigation of the spinor field symmetry as realization of the space-time symmetry.

In the case when the space-time symmetry subgroup G is different from $SO(3,1)$, the subgroup spinor representation of that symmetry cannot be realized as $Spin(3,1)$ subgroup and, one needs the spinor representation of G. As example we can take the standard model of Universe and its $G(6)$ group of symmetry. It contains the subgroup $G(3)$ of isotropy – subgroup of Lorentz group $SO(3,1)$, and subgroup $G(3)$ of translations. The latter is not a part of the Lorentz group and we can describe translation properties of spinor field (i.e. electron) through spinor representation of group $GL(4,\mathbb{R})$ only.

Here we give results of investigations in special construction for the spinor field on the space-time of general relativity. This is an extension of our investigation [5] of spinor representation for full linear group $GL(4,\mathbb{R})$.

2 Standard construction on Riemann space

For each point of the space-time one constructs the orthonormal basis $\overset{k}{e}_{\mu}(x)$ such that scalar products are

$$\overset{k}{e}_{\mu}(x)\overset{m}{e}_{\nu}(x)g^{\mu\nu} = \eta^{km} = \mathrm{diag}\,(1,-1,-1,-1). \tag{1}$$

Each basis vector is represented by Dirac matrix:

$$\overset{k}{e} \;\Rightarrow\; \gamma^k, \qquad \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{2}$$

Coordinate transformations deal with the coordinate index only:

$$\overset{k}{e}_{\mu'}(x') = \frac{\partial x^\mu}{\partial x^{\mu'}} \overset{k}{e}_\mu(x)$$

and group of invariance for the basis is $SO(3,1)$. This group can be represented by transformations of the spinor field:

$$\overset{k}{e'}_\mu(x) = T^k_m \overset{m}{e}_\mu(x) \;\Rightarrow\; \psi'(x) = U(T)\psi(x), \tag{3}$$

$$U = \exp\left(\frac{i}{4} T_{mn}\sigma^{mn}\right), \qquad \sigma^{nm} = \frac{i}{2}(\gamma^n\gamma^m - \gamma^m\gamma^n). \tag{4}$$

The main problem in spinor representation of basis transformations is in the existence of special type of conjugacy for Dirac spinor: spinor $\overline{\psi}$ is conjugated to $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ if its components are not conjugated only, but are additionally rearranged by the matrix γ^0:

$$\overline{\psi} = (\,\varphi^*\;\chi^*\,)\,\gamma^0. \tag{5}$$

Each transformation modifying that matrix deforms the norm of the spinor space and invariance loses the physical sense.

Covariant derivative can be defined by means of $SO(3,1)$ representation only. The space-time connectivity for basis $\overset{k}{e}_\mu(x+dx)$ is defined by

$$\overset{k}{e}_\mu(x+dx) = \overset{k}{e}_\mu(x) + dx^\nu \omega^k_{m\nu} \overset{m}{e}_\mu(x) \tag{6}$$

This leads to the spinor connectivity

$$\psi(x+dx) = \psi(x) + \frac{i}{4} dx^\nu \omega^k_{m\nu}\eta_{kn}\sigma^{nm}\psi \tag{7}$$

and the covariant derivative

$$\nabla_\mu\psi(x) = \partial_\mu\psi(x) + \frac{i}{4}\omega^k_{m\mu}\eta_{kn}\sigma^{nm}\psi \tag{8}$$

for spinor field.

This is typical approach to involve Fermi fields in general relativity, but it is inappropriate to define the Lie derivative of spinor. Until one considers Lie derivative along Killing vector only, one can keep to previous representation.

If it is needed to involve Lie derivative along non-Killing vector one has to use a corresponding element of group $GL(4,\mathbb{R})$ being outside the Lorentz group. One, may need such a Lie derivative, for example, in the case of investigation of spinor field time dependence for non-static Universe.

3 Point-to-point transformation

We investigate properties of spinor field with respect to motion of the space-time

$$M : x \to y = m(x). \tag{9}$$

Each map $m(x)$ of this motion generates transformation of coordinate basis

$$y = m(x) \;\Rightarrow\; T^{\mu}_{\nu} = \frac{\partial m^{\mu}(x)}{\partial x^{\nu}}. \tag{10}$$

When the motion belongs to neighborhood of identity, this transformation takes exponential form

$$T = \exp(m \cdot t), \tag{11}$$

where

$$t^{\mu}_{\nu} = \frac{\partial \zeta^{\mu}(x)}{\partial x^{\nu}} \tag{12}$$

and vector $\zeta^{\mu}(x)$ determines the direction of motion.

Derivative of basis along motion is the Lie derivative

$$L_{\zeta} \overset{k}{e}_{\mu}(x) = \zeta^{\nu}(x) \partial_{\nu} \overset{k}{e}_{\mu}(x) + \overset{k}{e}_{\nu}(x) \partial_{\mu} \zeta^{\nu}(x). \tag{13}$$

It generates the transformation of basis

$$\overset{k}{e}(x + \tau\zeta) = \overset{k}{e}(x) + \tau L_{\zeta} \overset{k}{e}(x), \tag{14}$$

which can be rewritten as

$$\overset{k}{e}(x + \tau\zeta) = \overset{k}{e}(x) + \tau \zeta^{k}_{m} \overset{m}{e}(x). \tag{15}$$

After integrating we obtain basis transformation as representation of group $GL(4, \mathbb{R})$

$$\overset{k}{e}(m(x)) = \exp\left(m \zeta^{k}_{m}\right) \overset{m}{e}(x). \tag{16}$$

Only in the case of $\zeta^{\mu}(x)$ being a Killing vector, this representation can be continued to the spinor transformation. In general case this does not work and it is needed to extend the spinor space.

3.1 Space of diagonal matrices

A transformation from neighborhood of identity can be represented as a product of two isometrics V, U and dilatation

$$\Delta = \begin{pmatrix} \delta_0 & 0 & 0 & 0 \\ 0 & \delta_1 & 0 & 0 \\ 0 & 0 & \delta_2 & 0 \\ 0 & 0 & 0 & \delta_3 \end{pmatrix}, \qquad T = V \cdot \Delta \cdot U. \tag{17}$$

Both isometrics have spinor representation, but the dilatation has no, because it deforms spinor conjugation. One has to extend the spinor space to represent the subgroup of dilatation.

The subgroup of dilatation is a noncompact Abelian group and has true representations as translations in $\mathbb{R}^4$.

Thus it turns out to be interesting to involve into consideration the space of diagonal matrices D_m

$$D_m = \left\{ \mathrm{diag}\,(d_0, d_1, d_2, d_3) : d_0 \cdot d_1 \cdot d_2 \cdot d_3 \neq 0 \right\}. \tag{18}$$

This space realizes representation of the dilatation subgroup Δ which is extended to the representation of group $G(4,\mathbb{R})$ in such a way:

A point d of D_m is transformed by the element of group g to a symmetric matrix d_g which has the diagonal form d'. This diagonal matrix determines the reflex of d trough transformation $T_g(d) = d'$ and determines also unique element Δ_g of dilatation subgroup. Left isometrics V_g is exactly the same as d_g transformation to d' and right isometrics U_g can by restored uniquely trough

$$U_g = \Delta_g^{-1} V_g^{-1} T_g. \tag{19}$$

3.2 Spinor fiber space

Now we construct for each point d from space of diagonal matrices D_m the spinor space $Spin(4,\mathbb{C})$ with anticommutator

$$\begin{aligned} &\gamma^n\gamma^m + \gamma^m\gamma^n = 2d^{mn}, \\ &d^{mn} = \mathrm{diag}\,(d_0, d_1, d_2, d_3), \qquad (\gamma^0)^2 = d_0 \end{aligned} \tag{20}$$

and with conjugation

$$\overline{\psi} = (\,\varphi^*\ \chi^*\,)\gamma^0. \tag{21}$$

Each spinor space $Spin(4,C)$ realizes spinor representation of isometric group $SO(3,1)$ for metrics d^{mn}. All spinor spaces are isomorphic and can be attached to fiber space with base D_m.

Now non-isometric motion $M : x \to y = m(x)$, for each point x from space-time, which has exponential form (11) is represented as product of two isometrics

$$V_g = \exp(v_m \cdot v_g), \qquad U_g = \exp(u_m \cdot u_g) \tag{22}$$

and dilatation

$$\Delta_g = \exp(d_m \cdot \delta_g) \tag{23}$$

as matrix exponent

$$T_g = \exp(v_m \cdot v_g) \cdot \exp(d_m \cdot \delta_g) \cdot \exp(u_m \cdot u_g). \tag{24}$$

The motion T_g is represented on fiber spinor space in three steps:

1. Right isometrics U_g in start point d

$$U_g : \psi(x;d) \ \Rightarrow\ \exp(u_m \cdot u_g(x))\psi(x;d); \tag{25}$$

2. Translation from start point d to end point $d + \delta_g$ over the base of fiber spinor space and to end point $m(x)$ over space-time

$$\Delta_g : \psi(x;d) \ \Rightarrow\ \exp(u_m \cdot u_g(x))\psi(x;d+\delta_g); \tag{26}$$

3. Left isometrics V_g in end point $d + \delta_g$

$$V_g : \exp(u_m \cdot u_g(x))\psi(x;d+\delta_g) \ \Rightarrow\ \exp(v_m \cdot v_g(m(x)))\exp(u_m \cdot u_g(x))\psi(x;d+\delta_g) \tag{27}$$

for the translated spinor.

As result we have the representation

$$T : \psi(x;d) \ \Rightarrow\ \psi_g(m(x);d) = \exp(v_m \cdot v_g(m(x)))\exp(u_m \cdot u_g(x))\psi(x;d+\delta_g), \tag{28}$$

which preserves the spinor norm, if the measure function on the fibering base is translationally invariant.

4 Example: Lie transformation along time direction

As an example we consider the Lie transformation along time for standard model of Universe. Metrics of space-time for this model can be written as

$$ds^2 = dt^2 - R^2(t)dl^2, \tag{29}$$

where dl^2 is the metrics of corresponding space. Transformation from t_1 to t_2 is dilatation with matrix

$$T = \frac{Rt(t_2)}{R(t_1)}\,\mathrm{diag}\,(0,1,1,1) \tag{30}$$

and is represented simply as translation through base of the spinor fiber space. Corresponding Killing vector acts on the spinor field as derivative along direction $(0,1,1,1)$ over base D_m

$$L_t\psi(d;t,x) = \frac{\partial}{\partial t}\psi(d;t,x) + \left(\frac{\partial}{\partial d_1} + \frac{\partial}{\partial d_2} + \frac{\partial}{\partial d_3}\right)\psi(d;t,x). \tag{31}$$

5 Conclusion

- We have developed the special spinor space which represents the full linear group $GL(4,\mathbb{R})$. It has a structure of the fiber space with the space of diagonal matrices as the base and standard spinor space as typical fiber.
- Non-isometric motions of the space-time entail spinor transformations which are represented by translation over fibering base in addition to standard $Spin(4,\mathbb{C})$ representation.
- Until we do not use the non-isometric motion, spinor fields without overlapping on the base of spinor fiber space are independent. Moreover, each such field can be represented as simple spinor space $\psi(x) \Rightarrow \int \psi(x;d)d^4d$.
- Only if one uses the non-isometric motion of space-time, it is essential to consider the fibering of the spinor space.

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On the Moments of the Traces of Unitary and Orthogonal Random Matrices

Vladimir VASILCHUK

B. Verkin Institute for Low Temperature Physics and Engineering,
47 Lenin Ave., 61103 Kharkiv, Ukraine
E-mail: *vasilchuk@ilt.kharkov.ua*

The complicated moments of the traces of unitary and orthogonal Haar distributed random matrices are studied. The exact formulas for different values of moment and matrix orders are obtained.

1 Introduction

Consider the probability space, whose objects are $n \times n$ unitary matrices, and whose probability measure is unit Haar measure on the group $U(n)$. Denote $\langle \cdot \rangle$ the expectation with respect to the measure. Given a non-negative integer k, consider the moment

$$\left\langle (\operatorname{Tr} U_n)^{a_1} (\operatorname{Tr} U_n^2)^{a_2} \cdots (\operatorname{Tr} U_n^k)^{a_k} \overline{(\operatorname{Tr} U_n)^{b_1}} \cdots \overline{(\operatorname{Tr} (U_n^k)^{b_k}} \right\rangle, \tag{1}$$

where $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ are k-tuples of non-negative integers. If

$$\kappa(a) = \sum_{j=1}^{k} j a_j, \qquad \kappa(b) := \sum_{j=1}^{k} j b_j, \tag{2}$$

and $\kappa(a) \neq \kappa(b)$, then it is easy to see that (1) is equal to zero. Hence, without loss of generality, we can restrict ourselves to the moments for which $\kappa(a) = \kappa(b)$. In this case we call

$$\kappa := \kappa(a) = \kappa(b) \tag{3}$$

the order of the respective moment, and we write

$$m_\kappa^{(n)}(a; b) = \left\langle (\operatorname{Tr} U_n)^{a_1} (\operatorname{Tr} U_n^2)^{a_2} \cdots (\operatorname{Tr} U_n^k)^{a_k} \overline{(\operatorname{Tr} U_n)^{b_1}} \cdots \overline{(\operatorname{Tr} U_n^k)^{b_k}} \right\rangle, \tag{4}$$

In recent paper [3] Diaconis and Evans proved that if $\kappa \leq n$, then

$$m_\kappa^{(n)}(a; b) = \delta_{a,b} \prod_{j=1}^{k} j^{a_j} a_j!, \qquad \delta_{a,b} = \prod_{j=1}^{k} \delta_{a_j, b_j}, \tag{5}$$

Analogous result was obtained for the orthogonal group also (see formula (17) below). The proofs of (5) in [3] were based on the representation theory of the groups $U(n)$ and $O(n)$. Another proof of (5) was given in [6] (Appendix). The proof is based on certain identities for the Töplitz determinants [1] Hughes and Rudnick [5] proved relation (17) for $SO(n)$ and for $\kappa \leq n-1$ using some combinatorial techniques. Similar questions were considered in [2, 4, 9, 11].

We will give below a proof of these relations for the groups $U(n)$ and $O(n)$, by using elementary means.

2 Unitary group

We begin from (5). Our proof is based on the following simple implication of the left invariance of Haar measure on $U(n)$.

Proposition 1. *Let $F(U_n, U_n^*)$ be a continuously differentiable complex valued function. Then for any $n \times n$ Hermitian matrix H we have*

$$\left\langle F_1'(U_n, U_n^*) \cdot HU_n - F_2'(U_n, U_n^*) \cdot U_n^* H \right\rangle = 0, \tag{6}$$

where $F_1'(U_n, U_n^*)$ and $F_2'(U_n, U_n^*)$ are derivatives of F with respect to U_n and U_n^* correspondingly.

The proposition follows from the fact that $\left\langle F\left(e^{itH}U_n, U_n^* e^{-itH}\right)\right\rangle$ is independent of a real parameter t because of the left invariance of Haar measure on $U(n)$.

Choosing in (6) $H = zX^{(x,y)} + \overline{z}\left(X^{(x,y)}\right)^T$, where z is an arbitrary complex number and $X^{(x,y)} = \{\delta_{px}\delta_{qy}\}_{p,q=1}^n$, we conclude that formula (6) is valid also in the case where H is replaced by $X^{(x,y)}$ (in fact, by any real matrix).

Denote a_j the first from the left non-zero index of the k-tuple $a = (0, \dots, 0, a_j, \dots, a_k)$. Then we can write (4) as

$$m_k^{(n)}(a; b) = \sum_{x=1}^{n} \left\langle (U_n^j)_{x,x} (\operatorname{Tr} U_n^j)^{a_j - 1} \cdots (\operatorname{Tr} U_n^k)^{a_k} (\operatorname{Tr} U_n^*)^{b_1} \cdots (\operatorname{Tr} (U_n^*)^k)^{b_k} \right\rangle. \tag{7}$$

We apply the proposition with $H = X^{(x,y)}$ to the function

$$F(U_n, U_n^*) = (U_n^j)_{x,y} (\operatorname{Tr} U_n^j)^{a_j - 1} \cdots (\operatorname{Tr} U_n^k)^{a_k} (\operatorname{Tr} U_n^*)^{b_1} \cdots (\operatorname{Tr} (U_n^*)^k)^{b_k}.$$

Taking into account the relations

$$\begin{aligned}
((U_n^m)_{x,y})' X^{(x,y)} U_n &= \sum_{i=0}^{m-1} (U_n^i X^{(x,y)} U_n^{m-i})_{x,y} = \sum_{i=0}^{m-1} (U_n^i)_{x,x} (U_n^{m-i})_{y,y} \\
&= \delta_{x,x} (U_n^m)_{y,y} + \sum_{i=1}^{m-1} (U_n^i)_{x,x} (U_n^{m-i})_{y,y}, \\
(\operatorname{Tr} U_n^m)' \cdot X^{(x,y)} U_n &= m \operatorname{Tr} U_n^m X = m (U_n^m)_{y,x}, \qquad (\operatorname{Tr} (U_n^*)^m)' X^{(x,y)} U_n^* = m ((U_n^*)^m)_{y,x},
\end{aligned}$$

and similar ones, we obtain

$$\begin{aligned}
&\delta_{xx} \left\langle (U_n^j)_{y,y} \alpha_- \beta \right\rangle + \sum_{i=1}^{j-1} \left\langle (U_n^i)_{x,x} (U_n^{j-i})_{y,y} \alpha_- \beta \right\rangle \\
&\quad + (a_j - 1) j \left\langle (U_n^j)_{x,y} (U_n^j)_{y,x} (\operatorname{Tr} U_n^j)^{a_j - 2} \alpha_+ \beta \right\rangle \\
&\quad + \sum_{l=j+1}^{k} a_l l \left\langle (U_n^j)_{x,y} (U_n^l)_{y,x} \alpha(l) \beta \right\rangle - \sum_{l=1}^{k} b_l l \left\langle (U_n^j)_{x,y} ((U_n^*)^l)_{y,x} \alpha \beta(l) \right\rangle = 0,
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
&\alpha_- = (\operatorname{Tr} U_n^j)^{a_j - 1} \cdots (\operatorname{Tr} U_n^k)^{a_k}, \qquad \beta = (\operatorname{Tr} U_n^*)^{b_1} \cdots (\operatorname{Tr} (U_n^*)^k)^{b_k}, \\
&\alpha_+ = (\operatorname{Tr} U_n^{j+1})^{a_{j+1}} \cdots (\operatorname{Tr} U_n^k)^{a_k}, \qquad \alpha(l) = (\operatorname{Tr} U_n^j)^{a_j - 1} \cdots (\operatorname{Tr} U_n^l)^{a_l - 1} \cdots (\operatorname{Tr} U_n^k)^{a_k}, \\
&\beta(l) = (\operatorname{Tr} U_n^*)^{b_1} \cdots (\operatorname{Tr} (U_n^*)^l)^{b_l - 1} \cdots (\operatorname{Tr} (U_n^*)^k)^{b_k}.
\end{aligned}$$

Applying to (8) the operation $n^{-1}\sum_{x,y=1}^{n}$ and regrouping terms we obtain in view of (7)

$$\begin{aligned}
&m_\kappa^{(n)}(a;b)+\frac{1}{n}\left(\sum_{l=1}^{j-1}m_\kappa^{(n)}(0,\dots,a_l=1,0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k;b)\right.\\
&\quad+j(a_j-1)m_\kappa^{(n)}(0,\dots,0,a_j-2,a'(2j),a_{2j}+1,a''(2j);b)\\
&\quad\left.+\sum_{l=j+1}^{k}la_lm_\kappa^{(n)}(0,\dots,0,a_j-1,a'(l),a_l-1,\tilde a,a_{l+j}+1,a''(l+j);b)\right)\\
&=jb_jm_{\kappa-j}^{(n)}(0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(j),b_j-1,b''(j))\\
&\quad+\frac{1}{n}\left(\sum_{l=1}^{j-1}lb_lm_{\kappa-l}^{(n)}(0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(l),b_l-1,b''(l))\right.\\
&\quad\left.+\sum_{l=j+1}^{k}lb_lm_{\kappa-j}^{(n)}(0,\dots,a_j-1,a';b'(l-j),b_{l-j}+1,\tilde b,b_l-1,b''(l))\right),
\end{aligned}\tag{9}$$

where

$$\begin{aligned}
&a'(l)=(a_{j+1},\dots,a_{l-1}),\qquad a''(l)=(a_{l+1},\dots,a_k),\qquad \tilde a=(a_{l+1},\dots,a_{l+j-1}),\\
&b'(l)=(b_1,\dots,b_{l-1}),\qquad b''(l)=(b_{l+1},\dots,b_k),\qquad \tilde b=(b_{l-j+1},\dots,b_{l-1}).
\end{aligned}$$

It will be important in what follows that the order of all moments in the l.h.s. of (9) is κ, while the orders of all moments in the l.h.s. are less then κ ($\kappa-j$ and $\kappa-l$).

We present now these relations in a more convenient form. Given a non-negative integer K, denote P_K the set of the k-tuples $a=(a_1,\dots,a_k)$, $a_j\ge 0$ such that

$$\sum_{j=1}^{k}ja_j\le K.$$

Consider the vector space $\mathcal{L}_K$ of collections of complex numbers, indexed by pairs $(a;b)$ of k-tuples such that

$$a,\,b\in P_K,\qquad \sum_{j=1}^{k}ja_j=\sum_{j=1}^{k}jb_j\le K,\tag{10}$$

and call the integer κ of (3) the order of a component $v(a;b)$ of $v\in\mathcal{L}_K$, if the indices a and b of the component satisfy (3). We define in $\mathcal{L}_K$ the $\|\cdot\|_\infty$-norm, i.e. if $v\in\mathcal{L}_K$, then

$$\|v\|=\max_{a,b}|v(a;b)|,$$

where the maximum is taken over the pairs (a,b), satisfying (10). Furthermore, we view the expression in the parentheses of the l.h.s. of (9), the first term of the r.h.s., and the expression in the parentheses of the r.h.s. of (9) as the results of action of the linear operators A, B, and C on the vector $m_K^{(n)}$, whose components are the moments (4) of the orders $\kappa\le K$. In other words, if a_j is the first from the left non-zero component of the k-tuple a, then

$$(Av)_\kappa(a;b)=\frac{1}{n}\left(\sum_{l=1}^{j-1}v_\kappa(0,\dots,a_l=1,0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k;b)\right.\tag{11}$$

$$+ j(a_j-1)v_\kappa(0,\dots,0,a_j-2,a'(2j),a_{2j}+1,a''(2j);b)$$
$$+ \sum_{l=j+1}^{k} la_l v_\kappa(0,\dots,0,a_j-1,a'(l),a_l-1,\tilde{a},a_{l+j}+1,a''(l+j);b)\Bigg),$$

$$(Bv)_\kappa(a;b) = jb_j v_{\kappa-j}(0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(j),b_j-1,b''(j)), \tag{12}$$

$$(Cv)_\kappa(a;b) = \frac{1}{n}\Bigg(\sum_{l=1}^{j-1} lb_l v_{\kappa-l}(0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(l),b_l-1,b''(l))$$
$$+ \sum_{l=j+1}^{k} lb_l v_{\kappa-j}(0,\dots,a_j-1,a_{j+1},\dots,a_k;b'(l-j),b_{l-j}+1,\tilde{b},b_l-1,b''(l))\Bigg). \tag{13}$$

With this notation we can rewrite (9) as

$$(I+A)\, m_K^{(n)} = Bm_K^{(n)} + Cm_K^{(n)}. \tag{14}$$

On other hand, if we denote as $\mu_\kappa(a;b)$ the r.h.s. of (5) for $\kappa(a)=\kappa$, then we can easily obtain that the values $\mu_\kappa(a;b)$ verify the following recursive relation for any $a_j \neq 0$

$$\mu_\kappa(a;b) = jb_j\mu_{\kappa-j}(a_1,\dots,a_{j-1},a_j-1,a''(j);b'(j),b_j-1,b''(j)). \tag{15}$$

By using (11)–(15), it is easy to prove the following

Lemma 1. *Let A, B and C be the linear operators, defined by* (11)–(13). *Then*
(i) $||A|| \leq (K-1)/n$,
(ii) if μ_K is the vector of $\mathcal{L}_K$, whose components are $\mu_\kappa(a;b)$, $\kappa \leq K$, then

$$B\mu_K = \mu_K, \qquad C\mu_K = A\mu_K.$$

Remark 1. Relation (15) (and hence its solution (5)) can be also obtained form (14). Indeed, we obtain (15) just passing in (14) to the limit $n \to \infty$ formally. To justify this, using the first assertion of the lemma and induction in K we can prove that the sequence $\{m_K^{(n)}\}_{n=1}^{\infty}$ is uniformly in n bounded and hence compact.

The first assertion of the lemma implies that if $K \leq n$, then the operator $I+A$ is invertible. Hence, for $K \leq n$ (14) (or (9)) is equivalent to

$$m_K^{(n)} = (I+A)^{-1}\left(Bm_K^{(n)} + Cm_K^{(n)}\right).$$

Since $Bm_K^{(n)}$ and $Cm_K^{(n)}$ include the moments whose orders are strictly smaller than K, this relation allows us to find the moments of the order K, provided that the moments of lower orders are known. This suggests the use of the induction in K to prove formula (5).

Indeed, it is easy to check that for $K = 0, 1$ the formula (5) holds. Assume that $m_\kappa^{(n)}(a,b) = \mu_\kappa(a,b)$, $\forall\, \kappa \leq K-1$. The r.h.s. of (12)–(13) contain the components of v, whose orders do not exceed $K-1$. Hence $Bm_K^{(n)} = B\mu_K$, $Cm_K^{(n)} = C\mu_K$, $(B+C)m_K^{(n)} = (I+A)\mu_K$. These facts and the second assertion of the lemma yield the following relation

$$m_K^{(n)} = (I+A)^{-1}(B\mu_K + C\mu_K) = (I+A)^{-1}(I+A)\mu_K = \mu_K,$$

which complete the proof of (5).

3 Orthogonal group

Consider now the probability space, whose objects are $n \times n$ orthogonal matrices, and whose probability measure is the normalized to unit Haar measure on the group $O(n)$ and denote $\langle \cdot \rangle$ the expectation with respect to the measure. Given a non-negative integer k, consider the moment

$$m_\kappa^{(n)}(a) = \left\langle (\operatorname{Tr} O_n)^{a_1} (\operatorname{Tr} O_n^2)^{a_2} \cdots (\operatorname{Tr} O_n^k)^{a_k} \right\rangle, \tag{16}$$

where $a = (a_1, \ldots, a_k)$ is k-tuple.

In recent paper [3] Diaconis and Evans proved that if $\kappa \le n/2$, then

$$m_\kappa^{(n)}(a) = \mathbb{E}\left[\prod_{j=1}^{k} \left(\sqrt{j}\xi_j + y_j \right)^{a_j} \right], \qquad y_j = \left(1 + (-1)^j\right)/2, \tag{17}$$

and ξ_j are i.i.d. standard normal variables.

Hughes and Rudnick [5] we proved (17) for $SO(n)$ and $\kappa \le n-1$ using some combinatorial technic.

As in the previous section we will use the following proposition

Proposition 2. *Let $F(O_n)$ be a continuously differentiable complex-valued function. Then for any $n \times n$ real antisymmetric matrix X we have (cf.* (6)*)*

$$\left\langle F'(O_n) \cdot XO_n \right\rangle = 0, \tag{18}$$

where $F'(O_n)$ is the derivative of F with respect to O_n.

The proposition follows from the fact that $\left\langle F(e^{tX} O_n) \right\rangle$ is independent of a real parameter t.

Denote a_j the first from the left non-zero index of the k-tuple $a = (0, \ldots, 0, a_j, \ldots, a_k)$. Then we can write (16) as

$$m_k^{(n)}(a) = \sum_{x=1}^{n} \left\langle (O_n^j)_{x,x} (\operatorname{Tr} O_n^j)^{a_j - 1} \cdots (\operatorname{Tr} O_n^k)^{a_k} \right\rangle. \tag{19}$$

We apply the proposition with $X^{(x,y)} = \{\delta_{px}\delta_{qy} - \delta_{py}\delta_{qx}\}_{p,q=1}^n$ to the function

$$F(O_n) = (O_n^j)_{x,y} (\operatorname{Tr} O_n^j)^{a_j - 1} \cdots (\operatorname{Tr} O_n^k)^{a_k}.$$

Taking into account the relations

$$\begin{aligned}
\left((O_n^m)_{x,y}\right)' X^{(x,y)} O_n &= \sum_{i=0}^{m-1} \left(O_n^i X^{(x,y)} O_n^{m-i} \right)_{x,y} \\
&= \sum_{i=0}^{m-1} \left((O_n^i)_{x,x} (O_n^{m-i})_{y,y} - (O_n^i)_{x,y} (O_n^{m-i})_{x,y} \right) \\
&= \delta_{x,x} (O_n^m)_{y,y} + \sum_{i=1}^{m-1} (O_n^i)_{x,x} (O_n^{m-i})_{y,y} - \delta_{x,y} (O_n^m)_{x,y} - \sum_{i=1}^{m-1} (O_n^i)_{x,y} \left((O_n^{m-i})^T \right)_{y,x}, \\
(\operatorname{Tr} O_n^m)' \cdot X^{(x,y)} O_n &= m \operatorname{Tr} O_n^m X^{(x,y)} = m (O_n^m)_{y,x} - m (O_n^m)_{x,y} \\
&= m (O_n^m)_{y,x} - m \left((O_n^m)^T \right)_{y,x},
\end{aligned}$$

and similar ones, we obtain

$$\begin{aligned}
&\delta_{xx}\left\langle (O_n^j)_{y,y}\alpha\right\rangle - \delta_{xy}\left\langle (O_n^j)_{x,y}\alpha\right\rangle + \sum_{i=1}^{j-1}\left\langle (O_n^i)_{x,x}(O_n^{j-i})_{y,y}\alpha_-\right\rangle \\
&\quad - \sum_{i=1}^{j-1}\left\langle (O_n^i)_{x,y}(O_n^{j-i})^T_{y,x}\alpha_-\right\rangle + (a_j-1)j\left(\left\langle (O_n^j)_{x,y}(O_n^j)_{y,x}(\operatorname{Tr} O_n^j)^{a_j-2}\alpha_+\right\rangle\right. \\
&\quad \left. - \left\langle (O_n^j)_{x,y}(O_n^j)^T_{y,x}(\operatorname{Tr} O_n^j)^{a_j-2}\alpha_+\right\rangle\right) \\
&\quad + \sum_{l=j+1}^{k} a_l l\left\langle (O_n^j)_{x,y}(O_n^l)_{y,x}\alpha(l)\right\rangle - a_l l\left\langle (O_n^j)_{x,y}(O_n^l)^T_{y,x}\alpha(l)\right\rangle = 0,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
&\alpha_- = (\operatorname{Tr} O_n^j)^{a_j-1}\cdots(\operatorname{Tr} O_n^k)^{a_k}, \qquad \alpha_+ = (\operatorname{Tr} O_n^{j+1})^{a_{j+1}}\cdots(\operatorname{Tr} O_n^k)^{a_k}, \\
&\alpha(l) = (\operatorname{Tr} O_n^j)^{a_j-1}\cdots(\operatorname{Tr} O_n^l)^{a_l-1}\cdots(\operatorname{Tr} O_n^k)^{a_k},
\end{aligned}$$

Applying to (20) the operation $n^{-1}\sum_{x,y=1}^{n}$ and regrouping terms we obtain in view of (19)

$$\begin{aligned}
&m_\kappa^{(n)}(a) + \frac{1}{n-1}\left(\sum_{l=1}^{j-1} m_\kappa^{(n)}(0,\dots,a_l=1,0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k)\right. \\
&\quad + j(a_j-1)m_\kappa^{(n)}(0,\dots,0,a_j-2,a'(2j),a_{2j}+1,a''(2j)) \\
&\quad \left. + \sum_{l=j+1}^{k} l a_l m_\kappa^{(n)}(0,\dots,0,a_j-1,a'(l),a_l-1,\tilde{a},a_{l+j}+1,a''(l+j))\right) \\
&= \frac{n}{n-1}\left(y_j m_{\kappa-j}^{(n)}(0,\dots,a_j-1,a_{j+1},\dots,a_k) + j(a_j-1)m_{\kappa-2j}^{(n)}(0,\dots,a_j-2,a_{j+1},\dots,a_k)\right) \\
&\quad + \frac{1}{n-1}\left(2\sum_{i<j/2} m_{\kappa-2i}^{(n)}(0,\dots,a_{j-2i}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k)\right. \\
&\quad \left. + \sum_{l=j+1}^{k} l a_l m_{\kappa-2j}^{(n)}(0,\dots,a_j-1,a'(l-j),a_{l-j}+1,\hat{a},a_l-1,a''(l))\right),
\end{aligned} \tag{21}$$

where y_j is defined in (17)

$$\begin{aligned}
&a'(l) = (a_{j+1},\dots,a_{l-1}), \qquad a''(l) = (a_{l+1},\dots,a_k), \\
&\tilde{a} = (a_{l+1},\dots,a_{l+j-1}), \qquad \hat{a} = (a_{l-j+1},\dots,a_{l-1}).
\end{aligned}$$

As in the previous section we consider the vector space $\mathcal{L}_K$ of collections of complex numbers, indexed by k-tuples a such that $\sum_{j=1}^{k} j a_j \le K$. and call the integer κ of (3) the order of a component $v(a)$ of $v \in \mathcal{L}_K$, if the index a of the component satisfy (3). We define in $\mathcal{L}_K$ the $\|\cdot\|_\infty$-norm, i.e. if $v \in \mathcal{L}_K$, then

$$\|v\| = \max_{a\in P_K} |v(a)|.$$

Furthermore, we view the expression in the parentheses of the l.h.s. of (21), the first term of the r.h.s., and the expression in the parentheses of the r.h.s. of (21) as the results of action of

the linear operators A, B, and C on the vector $m_K^{(n)}$, whose components are the moments (16) of the orders $\kappa \le K$. In other words, if a_j is the first from the left non-zero component of the k-tuple a, then

$$\begin{aligned}(Av)_\kappa(a) = \frac{1}{n-1}\Bigg(&\sum_{l=1}^{j-1}\nu_\kappa(0,\dots,a_l=1,0,\dots,a_{j-l}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k)\\ &+ j(a_j-1)v_\kappa(0,\dots,0,a_j-2,a'(2j),a_{2j}+1,a''(2j))\\ &+\sum_{l=j+1}^{k} la_l v_\kappa(0,\dots,0,a_j-1,a'(l),a_l-1,\tilde{a},a_{l+j}+1,a''(l+j))\Bigg), \qquad (22)\\ (Bv)_\kappa(a) = {}& y_j\nu_{\kappa-j}(0,\dots,a_j-1,a_{j+1},\dots,a_k)\\ &+ j(a_j-1)\nu_{\kappa-2j}(0,\dots,a_j-2,a_{j+1},\dots,a_k), \qquad (23)\\ (Cv)_\kappa(a) = \frac{1}{n-1}\Bigg(&2\sum_{i<j/2}\nu_{\kappa-2i}(0,\dots,a_{j-2i}=1,0,\dots,a_j-1,a_{j+1},\dots,a_k)\\ &+ y_j\nu_{\kappa-j}(0,\dots,a_j-1,a_{j+1},\dots,a_k)\\ &+ j(a_j-1)\nu_{\kappa-2j}(0,\dots,a_j-2,a_{j+1},\dots,a_k)\\ &+\sum_{l=j+1}^{k} la_l\nu_{\kappa-2j}(0,\dots,a_j-1,a'(l-j),a_{l-j}+1,\hat{a},a_l-1,a''(l))\Bigg). \qquad (24)\end{aligned}$$

With this notation we can rewrite (21) as

$$(I+A)\,m_K^{(n)} = Bm_K^{(n)} + Cm_K^{(n)}. \qquad (25)$$

On other hand, if we denote as $\mu_\kappa(a)$ the r.h.s. of (17) then, using Novikov–Furutzu formula for the average of ξ_j, we obtain for any $a_j \ne 0$ the following recursive relation

$$\begin{aligned}\mu_\kappa(a) = {}& y_j\mu_{\kappa-j}(a_1,\dots,a_{j-1},a_j-1,a''(j))\\ &+ j(a_j-1)\mu_{\kappa-2j}(a_1,\dots,a_{j-1},a_j-2,a''(j)). \qquad (26)\end{aligned}$$

As in the previous section, using (22)–(26), it is easy to prove the Lemma 1 with the only modification of its first assertion $||A|| \le (K-1)/(n-1)$. Thus, the rest of the proof coincides with unitary case.

It must be noted that we can also obtain relation (26) by passing to the limit $n \to \infty$ in (25).

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On Certain Quotient of Temperley–Lieb Algebra

Mariya O. VLASENKO

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *mariyka@imath.kiev.ua*

We consider a certain quotient of Temperley–Lieb algebra of general Coxeter system. This algebra arises as a quotient of Hecke algebra by some more relations than a Temperley–Lieb one. These additional relations correspond to the pairs of commuting generators of Coxeter system in the same way as Temperley–Lieb relations correspond to noncommuting generators. We prove that such algebras are usually well behaved. Especially for an irreducible system whose Coxeter graph does not contain a cycle this quotient is a sum of a matrix algebra and a field for all except a finite set of values of parameter.

1 Hecke deformation of Coxeter group algebra and its quotients

In this section we associate an algebra to a Coxeter group. This algebra is a certain quotient of the Temperley–Lieb algebra associated to the same Coxeter group. In further sections we show that such an algebra usually has simple enough structure.

Let W be a Coxeter group with set $S \subset W$ of generating reflections, let $l : W \longrightarrow \mathbb{N}_0$ be length correspondent to generators S. Let $q \in \mathbb{C}$.

Definition 1. The Hecke algebra $H_q = H_q(W)$ is an associative $\mathbb{C}$-algebra with $\mathbf{1}$, spanned by elements $\{T_w; w \in W\}$ with multiplication

$$T_s T_w = \begin{cases} T_{sw}, & l(sw) = l(w) + 1; \\ (q-1)T_w + qT_{sw}, & l(sw) = l(w) - 1, \end{cases}$$

for $s \in S$ and $w \in W$. Here $T_{\mathbf{1}} = \mathbf{1}_{H_q}$.

Note that H_q is a deformation of Coxeter group algebra $H_1 = \mathbb{C}W$, and its dimension (growth) is not greater then for $\mathbb{C}W$. For the general theory of Hecke algebras arising from Coxeter systems see [1, 2].

Consider for any pair of different reflections $s_i, s_j \in S$ a subgroup $W_{i,j} \subset W$ generated by s_i and s_j. $W_{i,j}$ is a dihedral group with $2m$ elements, where m is an order of element $s_i s_j$ in W.

Definition 2. The *complete* Temperley–Lieb algebra $TL_q^c = TL_q^c(W)$ is a quotient of $H_q(W)$ by relations

$$\sum_{w \in W_{i,j}} T_w = 0 \tag{1}$$

for each pair of different generators $s_i, s_j \in S$.

Note that quotient of $H_q(W)$ by relations (1) for each pair of noncommuting generators $s_i, s_j \in S$ is called the Temperley–Lieb algebra $TL_q = TL_q(W)$ (see [3] and references there). So, TL_q^c is a quotient of TL_q.

Remark 1. There is a way to consider Hecke and Temperley–Lieb algebras where q is indeterminate. For example, in [4, 5] this is done for Coxeter groups of type A (symmetric groups). That is well known that Temperley–Lieb algebra of symmetric group S_n over $\mathbb{C}(q)$ is semisimple,

and its evaluation at $q \in \mathbb{C}\backslash\{0\}$ is also semisimple if and only if $q+q^{-1}+2 \neq 4\cos^2\frac{\pi}{m}$ for some $2 \leq m \leq n$ (see, for example, [2]). In [3] Hecke and Temperley–Lieb algebras of general Coxeter groups are considered over the ring of Laurent polynomials $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ where $q = v^2$. Then a set $\{T_w; w \in W\}$ would be an $\mathcal{A}$-basis of Hecke algebra. In [6] $\mathcal{A}$-basis of Temperley–Lieb algebra TL_q is given as a certain subset of $\{T_w; w \in W\}$.

In this paper all algebras will be algebras with **1** over field $\mathbb{C}$. In the following section we apply combinatorial argumentation based on Diamond lemma to algebras TL_q^c. A restriction to algebras over a field (not ring) is essential here. For Diamond lemma and Groebner basis see, for example, [7].

2 Presentation of TL_q^c via idempotents. Linear basis

Now (and till the end of the paper) we restrict to the class of Coxeter groups (W, S) where $S = \{s_1, \ldots, s_n\}$ is a finite set and for each $1 \leq i < j \leq n$ an element $s_i s_j$ has an order either 2 or 3. Let also $q \neq -1, 0$.

In this section we propose another presentation of an algebra $TL_q^c(W)$, which is useful to find its linear basis and calculate the dimension (or Gelfand–Kirillov dimension).

Consider $p_i = \frac{T_{s_i}+1}{q+1} \in H_q$, and $\tau = \frac{q}{(1+q)^2}$. Then $p_i^2 = p_i$. Consider two cases.

If s_i and s_j commute in W, then T_{s_i} and T_{s_j} commute, and p_i,p_j also commute:

$$p_i p_j = p_j p_i. \tag{2}$$

In TL_q^c we have relation $1 + T_{s_i} + T_{s_j} + T_{s_i}T_{s_j} + T_{s_j}T_{s_i} = 0$, which is equivalent to

$$p_i p_j = p_j p_i = 0. \tag{3}$$

If $s_i s_j$ has order 3 in W, then $T_{s_i}T_{s_j}T_{s_i} = T_{s_j}T_{s_i}T_{s_j}$ in H_q, which is equivalent to

$$p_i p_j p_i - \tau p_i = p_j p_i p_j - \tau p_j. \tag{4}$$

Then in TL_q and TL_q^c there is a relation $1 + T_{s_i} + T_{s_j} + T_{s_i}T_{s_j} + T_{s_j}T_{s_i} + T_{s_i}T_{s_j}T_{s_i} = 0$, which is equivalent to

$$p_i p_j p_i - \tau p_i = p_j p_i p_j - \tau p_j = 0. \tag{5}$$

As it can be seen from the definitions, H_q, TL_q^c and TL_q can be presented via generators T_{s_i}, $s_i \in S$ with relations mentioned above. So, we get the following presentation

Proposition 1. *$H_q(W)$ is generated by idempotents $p_1, \ldots, p_n$ with relations* (4) *for each pair of noncommuting generators $s_i, s_j \in W$ and* (2) *for each pair of commuting generators $s_i, s_j \in W$.*

$TL_q(W)$ is its quotient by relations (5) *for each pair of noncommuting generators $s_i, s_j \in W$.*

$TL_q^c(W)$ is a quotient of $TL_q(W)$ by relations (3) *for each pair of commuting generators $s_i, s_j \in W$.*

In [8] and [9] they study $*$-representations of algebras, generated by projections, which are exactly TL_q^c of Coxeter groups of type A_n and $\widetilde{A_n}$ correspondingly.

In work [10] of the author some generalizations of TL_q^c are considered, where parameter τ in (5) may depend on i, j. It is noted that relations in presentation of TL_q^c by idempotents from Proposition 1 form a Groebner basis (where ordering on generators p_i can be taken arbitrarily, and we consider corresponding homogenous lexicographical ordering on their noncommutative

monomials). In other words, monomials in letters $p_1, \ldots, p_n$, which does not contain the following submonomials

$$p_i p_j p_i \text{ and } p_j p_i p_j \text{ for each non-commuting } s_i, s_j \in W,$$
$$p_i p_j \text{ and } p_j p_i \text{ for each commuting } s_i, s_j \in W$$

together with $\mathbf{1}$ form a linear basis of TL_q^c. Consider the Coxeter graph Γ of Coxeter group W. As its vertices correspond to $s_i \in S$, they also correspond to $\{p_i, 1 \leq i \leq n\}$. So, all elements of this linear basis but $\mathbf{1}$ can be identified with paths in Γ with one restriction: we consider only such paths in which all neighboring edges differ. So, we get the following

Proposition 2. *An algebra $TL_q^c(W)$ is finite-dimensional iff Coxeter graph Γ of W does not contain cycles. The dimension of TL_q^c is*

$$\dim(TL_q^c) = 1 + \sum_{\Gamma' \in \pi_0(\Gamma)} |\Gamma'|^2,$$

where $\pi_0(\Gamma)$ is a set of connected components and $|\Gamma'|$ is a number of vertices in connected component Γ'.

If Γ contains not more than one cycle in each connected component, then the growth of an algebra TL_q^c is linear, i.e. its Gelfand–Kirillov dimension equals 1.

If some connected component of Γ contains more than one cycle, an algebra TL_q^c contains a free subalgebra on two generators.

3 The structure of finite-dimensional algebras TL_q^c

In this section we are going to prove

Theorem 1. *Let Coxeter graph Γ of group W do not contain cycles. Then for almost all $q \in \mathbb{C}$ (except a finite set)*

$$TL_q^c \cong M_1(\mathbb{C}) \bigoplus_{\Gamma' \in \pi_0(\Gamma)} M_{|\Gamma'|}(\mathbb{C}).$$

Remark 2. There are values of q, for which TL_q^c is not semisimple. For example, $TL^s_{\frac{1+\sqrt{-3}}{2}}(S_3)$ is not semisimple as it coincides with $TL_{\frac{1+\sqrt{-3}}{2}}(S_3)$.

Let Γ be a tree with n vertices, and Γ_0 be the set of its vertices. One can take any vertex $\lambda \in \Gamma$ as a root. Then binary relation of being a child is defined as usually on vertices of a rooted tree: $\beta \in \Gamma_0$ is a child of $\alpha \in \Gamma_0$ iff there is an edge between α and β and β is farther from the root λ then α. Define $c_\lambda : \Gamma_0 \longrightarrow 2^{\Gamma_0}$ so that $c_\lambda(\alpha)$ is a set of children vertices of α.

We call $\lambda \in \Gamma_0$ a *matching* vertex if there exists a function $a : \Gamma_0 \longrightarrow \mathbb{C}\backslash\{0\}$ such that

$$a(\alpha) + \tau \sum_{\beta \in c_\lambda(\alpha)} \frac{1}{a(\beta)} = 1 \text{ for each } \alpha \in \Gamma_0, \tag{6}$$

where $\tau = \frac{q}{(q+1)^2}$.

Lemma 1. *If there is a* matching *vertex in graph Γ, then there is an irreducible representation of TL_q^c in dimension $n = |\Gamma|$, in which all generating idempotents are nonzero.*

Proof. We give an explicit construction of such a representation. Consider a vector space V with basis $\{e_\beta; \beta \in \Gamma_0\}$. Let λ be a *matching* vertex. Let $a_\beta = \sqrt{a(\beta)}$, where $a : \Gamma_0 \longrightarrow \mathbb{C}\backslash\{0\}$ be a function from condition (6) above. Let $g_\beta = a_\beta e_\beta + \sum\limits_{\alpha \in c_\lambda(\beta)} \frac{\sqrt{\tau}}{a_\alpha} e_\alpha$. Consider the following action of generating idempotents:

$$p_\beta e_\mu = \begin{cases} a_\beta g_\beta, & \mu = \beta; \\ \dfrac{\sqrt{\tau}}{a_\mu} g_\beta, & \mu \in c_\lambda(\beta); \\ 0, & \text{otherwise.} \end{cases}$$

It follows from (6) that $p_\beta g_\beta = g_\beta$. If vertices α and β are not connected in Γ, then also $c_\lambda(\alpha) \cap c_\lambda(\beta) = \varnothing$, so $p_\beta g_\alpha = p_\alpha g_\beta = 0$. If vertices α and β are connected, then one has $p_\beta g_\alpha = \sqrt{\tau} g_\beta$. So, given above is a representation.

To prove it is irreducible we consider matrix units acting on V:

$$E_{\alpha,\beta} e_\mu = \begin{cases} e_\alpha, & \mu = \beta; \\ 0, & \text{otherwise.} \end{cases}$$

If β is a leaf then $p_\beta = E_{\beta,\beta}$. Suppose that for some α matrix units $E_{\beta,\beta}$ are defined for all $\beta \in c_\lambda(\alpha)$ via $\{p_\mu; \mu \in \Gamma_0\}$. Then

$$E_{\alpha,\alpha} = \frac{1}{a_\alpha^2} \left(1 - \sum_{\beta \in c_\lambda(\alpha)} E_{\beta,\beta}\right) p_\alpha \left(1 - \sum_{\beta \in c_\lambda(\alpha)} E_{\beta,\beta}\right)$$

also is specified. And for each $\beta \in c_\lambda(\alpha)$

$$E_{\beta,\alpha} = \frac{a_\beta}{\sqrt{\tau} a_\alpha} E_{\beta,\beta} p_\alpha E_{\alpha,\alpha}, \qquad E_{\alpha,\beta} = \frac{a_\beta}{\sqrt{\tau} a_\alpha} E_{\alpha,\alpha} p_\alpha E_{\beta,\beta}.$$

Thus an image of TL_q^c in $\mathrm{End}_{\mathbb{C}}(V)$ contains $E_{\alpha,\alpha}$ for all $\alpha \in \Gamma_0$ and $E_{\alpha,\beta}$ for all pairs of connected vertices α and β. So representation constructed above is irreducible as graph Γ is connected. ■

Note that it is easy to verify if some $\lambda \in \Gamma_0$ is *matching*: the values $a(\alpha)$ for $\alpha \in \Gamma_0$ from (6) are defined uniquely by λ. To find them one should put $a(\alpha) = 1$ for all leafs α, and then calculate all other values by recursive formulae (6).

Remark 3. A tree Γ has a *matching* vertex for almost all $q \in \mathbb{C}$. Indeed, when arbitrary vertex λ is taken to be a root, we can define $a(\alpha)$ from (6) with values in $\mathbb{C}(q)$. Then numerators of irreducible fractions $a(\alpha) \in \mathbb{C}(q)$ is a set of certain polynomials $\{P_\alpha^\lambda(q); \alpha \in \Gamma_0\}$, and λ is a *matching* vertex iff $P_\alpha^\lambda(q) \neq 0$ for all $\alpha \in \Gamma_0$. An intersection of this finite sets for all $\lambda \in \Gamma_0$ is exactly the set of such $q \in \mathbb{C}$, which we exclude in Theorem 1 in case of connected Γ.

Now we are going to prove Theorem 1.

Proof. Due to Remark 3 for almost all $q \in \mathbb{C}$ all connected components $\Gamma' \in \pi_0(\Gamma)$ have *matching* vertices. For such a q we can construct a set of irreducible representations $\{V^{\Gamma'}; \Gamma' \in \pi_0(\Gamma)\}$ one for each connected component. Indeed, for each Γ' we can take a representation of corresponding subalgebra from Lemma 1, and induce representation of TL_q^c putting all generating idempotents from other connected components to be zero. Moreover, in this construction representations corresponding to different connected components are nonisomorfic.

There is one more 1-dimensional representation – the trivial one, in which all generating idempotents are zero. It is nonisomorfic to any one from a list $\{V^{\Gamma'}; \Gamma' \in \pi_0(\Gamma)\}$. So, we have a list of pairwise nonisomorfic irreducible representations, and due to Proposition 2

$$\dim(TL_q^c) = \dim(\mathbb{C})^2 + \sum_{\Gamma' \in \pi_0(\Gamma)} \dim\left(V^{\Gamma'}\right)^2.$$

This implies

$$TL_q^c \cong M_1(\mathbb{C}) \bigoplus_{\Gamma' \in \pi_0(\Gamma)} M_{|\Gamma'|}(\mathbb{C})$$

by Jacobson's density theorem. ■

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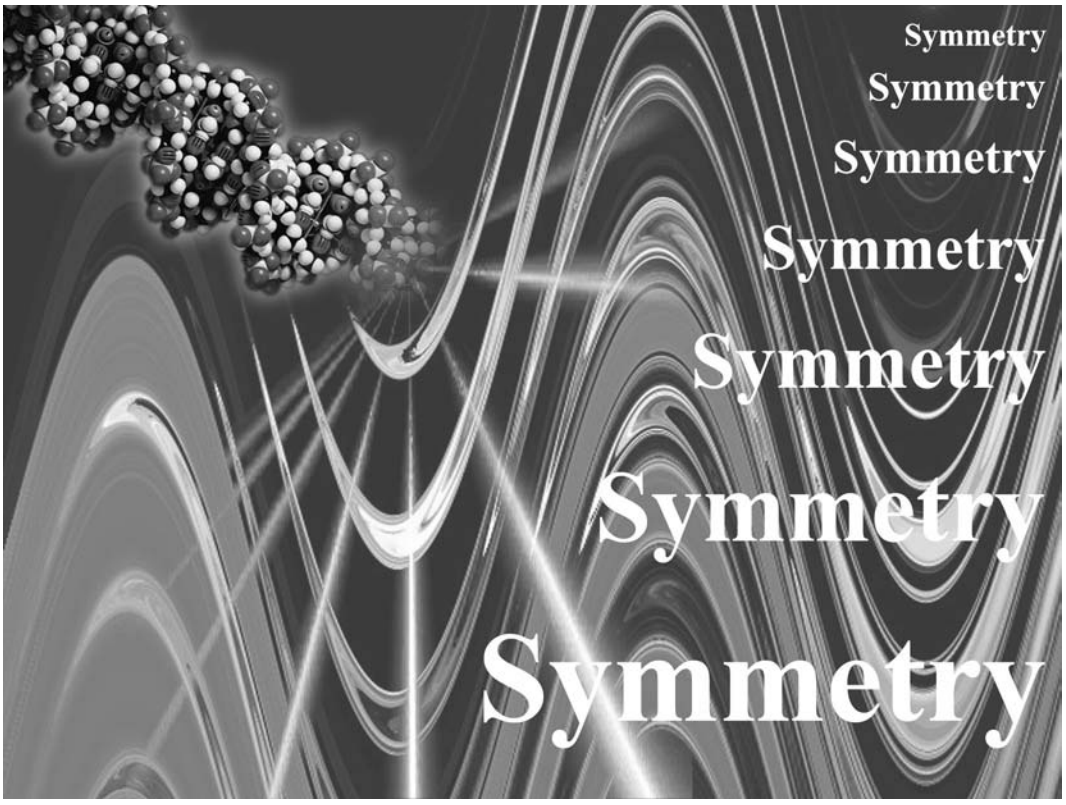

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Torus Doubling Cascade in Problems with Symmetries

Faridon AMDJADI

School of Computing and Mathematical Sciences, Glasgow Caledonian University, Cowcaddens Road, Glasgow G4 OBA, UK
E-mail: *F.Amdjadi@gcal.ac.uk*

Switching the branch of the trivial states, at a Hopf bifurcation, to a branch of the standing and the travelling states is considered. A secondary bifurcation on period solutions, leading to chaotic behaviour, is dealt with the technique of canonical co-ordinates transformation. The techniques that are introduced in this paper are illustrated by a small system of ordinary differential equations.

1 Introduction

In this paper a multiple Hopf bifurcation is considered on the branch of trivial solutions in problems with $O(2)$ symmetry. The theory for this type of bifurcation is well understood and leads us to solutions in the form of standing and travelling waves (van Gils and Mallet-Paret [1], Golubitsky et al [2]). Golubitsky et al [2] distinguish the standing and the travelling (rotating) oscillations in the system of ordinary differential equations (ODEs) by their symmetries, however, these oscillations come from considering the ODE as an amplitude system for some partial differential equation (PDE) model. The triple zero bifurcation equations that we consider in this paper is a normal form for the amplitude equation for a PDE model Arneodo et al [3]. At the multiple Hopf bifurcation point, due to the four-dimensional null space resulting from the Jacobian of the linearized equations, swapping the branches is not straightforward. We show a possible way for swapping the branches, and this involves restricting the solutions in the two-dimensional fixed point subspaces associated with the isotropy subgroups of $O(2) \times S^1$. On the travelling waves we consider a further Hopf bifurcation leading to a tori branch. We deal with this bifurcation using canonical co-ordinates.

2 The travelling waves: multiple Hopf bifurcation

In a previous paper [4] we considered a system of the form

$$\dot{z} = g(z, \lambda), \tag{1}$$

where $z \in \mathbb{C}^3$ and $\lambda \in \mathbb{R}$. The system is symmetric with respect to the group $O(2)$. In problems with $O(2)$ symmetry there are typically many branches of symmetric steady state solutions, and the state space X can be decomposed into a symmetric subspace X^s (reflectional invariant subspace) and an anti-symmetric subspace X^a, i.e. $X = X^s \oplus X^a$, due to the underlying reflection symmetry [5]. On the symmetric solutions $z_s(\lambda)$, the Jacobian matrix can be also decomposed as $g_z(z_s, \lambda) = \mathrm{diag}(g_z^s(z_s, \lambda) : g_z^a(z_s, \lambda))$, where g_z^s and g_z^a are associated with symmetric and anti-symmetric subspaces, respectively [6]. In [4] a Hopf bifurcation from the circle of non-trivial steady states (or relative equilibria) is considered and it is shown that the waves emerging from a Hopf bifurcation change their direction of propagation in a periodic fashion. In that paper [4] we considered a couple of imaginary eigenvalues in the anti-symmetric block of the Jacobian matrix, and due to the symmetry the Hopf bifurcation was not standard. We used the canonical co-ordinates transformation to remove the degeneracy of the system.

The equation we consider in this paper is in the form of (1), which we assume symmetric with respect to the diagonal action of group $O(2)$ defined by

$$r_\alpha z = e^{i\alpha} z, \qquad sz = \bar{z},$$

where $z = (z_1, z_2, z_3, \ldots, z_n) \in \mathbb{C}^n =: X$. The problem has a branch of trivial solutions, with full $O(2)$ symmetry, inherited from the underlying symmetry of the system. We assume that on this branch, at $\lambda = \lambda_0$, the symmetric and the anti-symmetric blocks of the Jacobian matrix satisfy the following conditions

$$g_z^s(0, \lambda_0)\phi_s = \pm i\omega\phi_s, \qquad g_z^a(0, \lambda_0)\phi_a = \pm i\omega\phi_a, \tag{2}$$

where $\phi_s = \phi_{sr} + i\phi_{sj} \in X^s$ and $\phi_a = \phi_{ar} + i\phi_{aj} \in X^a$. Therefore, the Hopf bifurcation on the trivial solution is associated with the two-dimensional irreducible representation of the group $O(2)$. The irreducible representations of $O(2)$ are given by

(i) $r_\alpha = I$, $s = I$, one-dimensional,

(ii) $r_\alpha = I$, $s = -I$, one-dimensional,

(iii) $r_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, $s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, two-dimensional.

Bifurcations associated with the one-dimensional irreducible representations of $O(2)$ are not of interest. The analysis of this type of bifurcation is very similar to that for Hopf bifurcations of vector fields since an additional S^1 symmetry can be defined in this case [7]. Therefore the full symmetry of the system is $O(2) \times S^1$ and the Hopf bifurcation is associated with the two-dimensional irreducible representation of $O(2) \times S^1$. To analyse this bifurcation we linearize the equation (1) at the Hopf bifurcation giving rise to the solution of the form

$$\Phi_k(t) = e^{i\omega t}\phi_k, \qquad k = s, a. \tag{3}$$

We represent the corresponding real eigenspace as $E_i = \mathrm{sp}\{\phi_{sr}, \phi_{sj}, \phi_{ar}, \phi_{aj}\}$ which is a four-dimensional subspace. We can decompose the space X to its isotypic components as $X = X_0 \oplus X_1 \oplus X_2$, where $X_0 = X^s$, X_1 are associated with the one-dimensional representations, and X_2 is associated with the two-dimensional representation. These components are irreducible subspaces. Therefore ϕ_{sr}, ϕ_{sj}, ϕ_{ar}, $\phi_{aj} \in X_2$.

The group actions on the eigenspace can be easily obtained as:

$$\theta \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix} = \begin{bmatrix} \cos\omega\theta & 0 & -\sin\omega\theta & 0 \\ 0 & \cos\omega\theta & 0 & -\sin\omega\theta \\ \sin\omega\theta & 0 & \cos\omega\theta & 0 \\ 0 & \sin\omega\theta & 0 & \cos\omega\theta \end{bmatrix} \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix}, \tag{4}$$

$$r_\alpha \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t & 0 & 0 \\ -\sin\omega t & \cos\omega t & 0 & 0 \\ 0 & 0 & \cos\omega t & \sin\omega t \\ 0 & 0 & -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix},$$

$$s \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_{sr} \\ \phi_{ar} \\ \phi_{sj} \\ \phi_{aj} \end{bmatrix}.$$

Now, to use the results of Golubitsky et at [2] with regards to the isotropy subgroups of $O(2) \times S^1$ on $\mathbb{C}^2$, we identify the eigenspace E_i with $\mathbb{C}^2$ by

$$(x_1, y_1, x_2, y_2) \longleftrightarrow x_1\phi_{sr} + x_2\phi_{sj} + y_1\phi_{ar} + y_2\phi_{aj}, \tag{5}$$

where $x_j + iy_j =: \boldsymbol{z}_j \in \mathbb{C}^2$, $j = 1, 2$. From (4) and (5), we obtain the action of θ on $\mathbb{C}^2$:

$$\theta \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\omega\theta & 0 & \sin\omega\theta & 0 \\ 0 & \cos\omega\theta & 0 & \sin\omega\theta \\ -\sin\omega\theta & 0 & \cos\omega\theta & 0 \\ 0 & -\sin\omega\theta & 0 & \cos\omega\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}.$$

Now, we introduce new co-ordinates $(\tilde{\boldsymbol{z}}_1, \tilde{\boldsymbol{z}}_2) = (\bar{\boldsymbol{z}}_1 - i\bar{\boldsymbol{z}}_2, \boldsymbol{z}_1 - i\boldsymbol{z}_2)$ [2], so that in these new co-ordinates θ acts diagonally on $\mathbb{C}^2$. It can be shown that the representation T of $O(2) \times S^1$ in these new coordinates is given by

$$T(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad T(r_\alpha) = \begin{bmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{bmatrix}, \qquad T(\theta) = \begin{bmatrix} e^{i\omega\theta} & 0 \\ 0 & e^{i\omega\theta} \end{bmatrix}.$$

These results correspond to the results given in [2]. The isotropy subgroups of $O(2) \times S^1$ acting on $\mathbb{C}^2$ are given by Golubitsky, et al [2]. They have shown that there are only two isotropy subgroups, which give two-dimensional fixed point subspaces and these are used here, to reduce the four-dimensional eigenspace E_i to a two-dimensional one. These subgroups and their corresponding fixed point subspaces are given bellow:

	Isotropy subgroup	**Fixed point space**	**dim**
(a)	$Z_2(s) = \{(I, 0), (s, 0)\}$	$\{(\tilde{\boldsymbol{z}}_1, \tilde{\boldsymbol{z}}_1)\}$	2
(b)	$\tilde{SO}(2) = \{(I, 0), (r_{\omega\theta}, \omega\theta)\}$	$\{(\tilde{\boldsymbol{z}}_1, 0)\}$	2

In case (a), from $\tilde{\boldsymbol{z}}_1 = \tilde{\boldsymbol{z}}_2$ and reverting to the original co-ordinates we obtain $y_1 = y_2 = 0$. Thus the identification (5) implies that $(x_1, 0, x_2, 0) \longleftrightarrow x_1\phi_{sr} + x_2\phi_{sj}$. Therefore in this case, subgroup $\sigma_1 = Z_2(s)$ with $\mathrm{Fix}(\sigma_1) \cap E_i = \mathrm{sp}\{\phi_{sr}, \phi_{sj}\}$, reduces the four-dimensional eigenspace E_i to a two-dimensional one, and the Equivariant Hopf Theorem implies that there exists a branch of periodic solutions bifurcating from the trivial solutions at $\lambda = \lambda_0$ having σ_1 as its group of symmetries. This branch corresponds to the standing waves. From $\tilde{\boldsymbol{z}}_2 = 0$ in case (b), we obtain $x_1 = -y_2$ and $y_1 = x_2$. Thus the identification (5) implies that $(x_1, y_1, y_1, -x_1) \longleftrightarrow x_1(\phi_{sr} - \phi_{aj}) + y_1(\phi_{sj} + \phi_{ar})$. Therefore the subgroup $\sigma_2 = \tilde{SO}(2)$, reduces the four-dimensional eigenspace to a two-dimensional one with $\mathrm{Fix}(\sigma_2) \cap E_i = \mathrm{sp}\{\phi_{sj} + \phi_{ar}, \phi_{sr} - \phi_{aj}\}$. Hence the Equivariant Hopf Theorem implies that there exists a branch of periodic solutions bifurcating from the trivial solutions at $\lambda = \lambda_0$ having σ_2 as its group of symmetries. This branch corresponds to the branch of travelling waves. Thus generically there are two branches of periodic solutions bifurcating from the trivial solutions at $\lambda = \lambda_0$.

Once a multiple Hopf bifurcation has been detected at $(0, \lambda_0)$, a starting solution for the two branches of periodic solutions can be obtained in usual way for the variable $z(t)$, using the solution (3). Near the symmetry breaking Hopf bifurcation, the solution of (1) to first order is given by

$$\begin{aligned} &z(t) = \alpha\Phi_j(t) + O\big(\alpha^2\big), \qquad j = s, a, \\ &\lambda = \lambda_0 + O\big(\alpha^2\big), \end{aligned} \tag{6}$$

where $\Phi_j(t)$ is defined by (3). Restricting the solutions to the fixed point subspace with symmetry group σ_1 implies that $\Phi_s(t) = e^{i\omega t}(\phi_{sr} + i\phi_{sj})$ and this solution gives rise to the branch of standing wave solutions. Restricting to the fixed point subspace with the subgroup of symmetry σ_2 leads to a solution of the form $\Phi_a(t) = e^{i\omega t}((\phi_{sj} + \phi_{ar}) + i(\phi_{sr} - \phi_{aj}))$ which gives rise to the branch of travelling wave solutions. Therefore, there exist two branches of solutions bifurcating from trivial solutions at $\lambda = \lambda_0$. Note that ϕ_{kr}, ϕ_{kr}, $k = s, a$ can be obtained by (2).

3 The torus bifurcation: canonical co-ordinates transformation

Assume that a Hopf bifurcation occurs on the branch of travelling wave solutions. The branch switching at this bifurcation is not standard, due to the fact that solutions emanating at this Hopf bifurcation are in the form of torus. This can be achieved by transforming the equations to the canonical co-ordinates [4, 9], so that in this co-ordinates system the drift velocity of the travelling waves will be decoupled, and therefore we observe travelling wave solutions in travelling (rotating) frame as steady state solutions. The torus bifurcation in this co-ordinates system becomes a standard Hopf bifurcation point and the package AUTO [10] deals with it automatically.

In general we make a change of co-ordinates

$$w = W(z) = (w_1(z), w_2(z), \ldots, w_n(z)), \qquad z = (z_1, z_2, \ldots, z_n).$$

Then $(w_1, w_2, \ldots, w_n)$ defines canonical co-ordinates for the one parameter Lie group of transformations $z^* = X(z, \alpha)$, $\alpha \in \mathbb{R}$, if in terms of such co-ordinates the group action is

$$w_i^* = w_i, \quad i = 1, 2, \ldots n-1, \qquad w_n^* = w_n + \alpha.$$

The advantage of these co-ordinates is that we can decouple the velocity of travelling waves. In this paper the canonical co-ordinates is defined as follows:

$$w_1 = \frac{z_2}{z_1}, \quad w_2 = \frac{z_3}{z_1}, \quad \ldots, \quad w_{n-1} = \frac{z_n}{z_1}, \quad r = |z_1|, \quad \phi = \arg(z_1),$$

where $w_j = u_j + iv_j \in \mathbb{C}$, $j = 1, 2, \ldots, n-1$ and $r \in \mathbb{R}$ are all invariant under the rotation and $\phi \to \phi + \alpha$.

4 An example in $\mathbb{C}^3$

We consider the following system

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = z_3, \qquad \dot{z}_3 = -\lambda z_1 - 2.0 z_2 - 1.5 z_3 + |z_1|^2 z_1, \tag{7}$$

and $z = (z_1, z_2, z_3) \in \mathbb{C}^3$, $\lambda \in \mathbb{R}$ [3, 4, 8]. The system is symmetric with respect to the diagonal action of $O(2)$ defined by

$$r_\alpha(z_1, z_2, z_3) = (e^{i\alpha} z_1, e^{i\alpha} z_2, e^{i\alpha} z_3), \qquad s(z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3).$$

The trivial branch of solutions, $z = 0$, $\forall \lambda$, with full $O(2)$ symmetry, is unstable for $\lambda < 0$ and gains stability at $\lambda = 0$ in a subcritical bifurcation to an unstable non-trivial branch of solutions defined by $\lambda = |z_1|^2$. These solutions lie in the symmetric subspace X^s, characterized by $\mathrm{Im}(z_1) = \mathrm{Im}(z_2) = \mathrm{Im}(z_3) = 0$. The Hopf bifurcation from non-trivial states is considered in [4, 8]; here we consider a Hopf bifurcation on trivial states which occurs at $\lambda = 3.0$. The full symmetry of the system (7) is defined as $O(2) \times S^1$ and the Hopf bifurcation is associated with the two-dimensional irreducible representation of $O(2) \times S^1$.

To follow the branches of the standing and the travelling waves, respectively, we start with the initial solutions of the form

$$\begin{aligned}
z_1 &= \alpha(c_1(\cos\omega t\phi_{sr} - \sin\omega t\phi_{sj}) + c_2(\cos\omega t\phi_{sj} + \sin\omega t\phi_{sr})), \ \lambda = 3.0\\
z_2 &= \alpha(c_1(\cos\omega t(\phi_{sj} + \phi_{ar}) - \sin\omega t(\phi_{sr} - \phi_{aj})) + c_2(\cos\omega t(\phi_{sr} - \phi_{aj})\\
&\quad + \sin\omega t(\phi_{sj} + \phi_{ar}))),\\
\lambda &= 3.0,
\end{aligned}$$

where

$$\phi_{sr} = [\ 0,\ 0,\ 0,\ -0.38,\ 0,\ 0.76]^T, \qquad \phi_{sj} = [0,\ 0,\ 0,\ 0,\ 0.54,\ 0]^T$$

and

$$\phi_{ar} = [-0.38,\ 0,\ 0.76,\ 0,\ 0,\ 0]^T, \qquad \phi_{aj} = [0,\ 0.54,\ 0,\ 0,\ 0,\ 0]^T,$$

are the eigenvectors associated with the multiple pure-imaginary eigenvalues, α is a suitable small real value and c_1, c_2 are the arbitrary real constants.

We followed the unstable standing wave solutions and found a symmetry breaking bifurcation at $\lambda = 6.613$ giving rise to a branch of unstable asymmetric standing wave solutions. Note that the standing wave solutions lie in $\mathrm{Fix}(\sigma_1) = X^s$. The branch of asymmetric standing waves terminate at $\lambda = 8.01$ when an asymmetric standing wave collides with an unstable steady state solution in a homoclinic connection. The standing wave solutions terminate at $\lambda = 12.45$ in a heteroclinic connection with the travelling wave and the non-trivial steady states.

Following the travelling waves a torus bifurcation is obtained at $\lambda = 6.176$ giving rise to a branch of stable tori (here after T-branch). Switching the branch of travelling waves to the T-branch is obtained by using the canonical co-ordinate transformations discussed in the previous section. Therefore the system in canonical co-ordinates can be written as:

$$\begin{aligned}
\dot{R} &= 2Ru_1,\\
\dot{u}_1 &= u_2 - u_1^2 + v_1^2,\\
\dot{u}_2 &= -\lambda - 2.0u_1 - 1.5u_2 + R - (u_1u_2 - v_1v_2),\\
\dot{v}_1 &= v_2 - 2.0u_1v_1,\\
\dot{v}_2 &= -2.0v_1 - 1.5v_2 - (u_1v_2 + v_1u_2),\\
\dot{\phi} &= v_1,
\end{aligned} \tag{8}$$

where $R = r^2$.

A bifurcation diagram is given in Fig. 1 (see also Fig. 2 for better understanding of the bifurcations involved). The bifurcation diagram is obtained using both the original (7) and the transformed system (8).

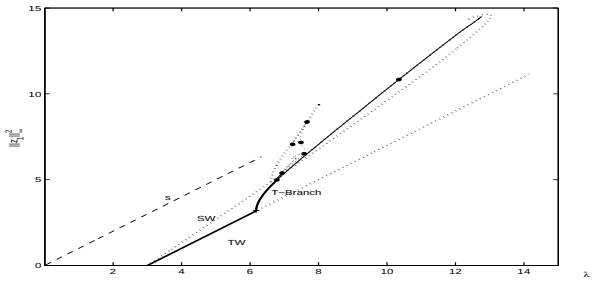

Figure 1. A bifurcation diagram is obtained by solving system (7). The two branches of travelling waves and the standing waves are obtained at $\lambda = 3.0$. Following the travelling waves a torus bifurcation is obtained at $\lambda = 6.176$. Solid circles indicate period doubling bifurcations and unstable branches are shown using dashed lines. A plus sign indicates the torus bifurcation on the travelling wave solutions.

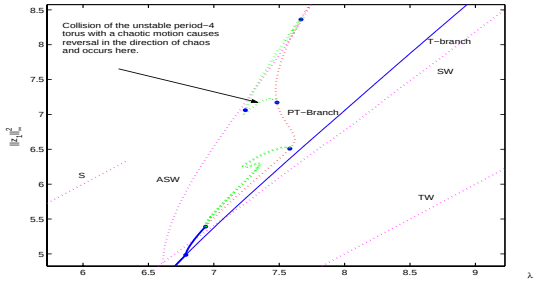

Figure 2. To illustrate a better picture of the bifurcation points and the branches, Fig. 1 is enlarged. The letter T represent a torus; TW (travelling wave); SW (standing Wave); ASW (asymmetric standing wave); PT (period-doubled torus) and s (steady state).

Note that in this new system (8) the rotation ϕ is decoupled from the remaining equations. The drift velocity of the pattern, $\dot{\phi}$, must vanish in the reflection invariant subspace X^s, hence periodic motions in this space are standing waves. The travelling wave solutions are the steady states of the above equations (ignoring the $\dot{\phi}$ equation) and the torus bifurcation appears as

a standard Hopf bifurcation, resulting a standard branch swapping. The torus bifurcation is far from the multiple Hopf bifurcation point which occurred on the trivial states, therefore we do not expect any interaction between these two bifurcations. For better understanding of the motion of the torus we investigate the behaviour of its velocity defined by the variable $\dot{\phi}$ in the transformed equations (8). The torus can be obtained by the numerical integration of the original system (7). A torus and its velocity is shown in Fig. 3.

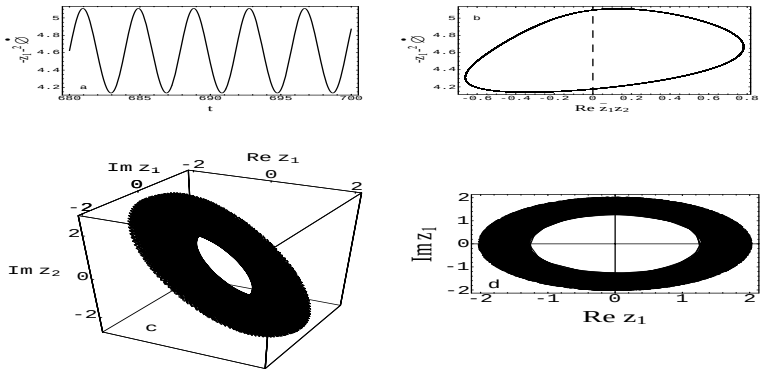

Figure 3. (a): The velocity of the torus is obtained by the numerical integration of the system in the canonical co-ordinates system for $\lambda = 6.39$, it is periodic in time. (b) The phase plot of the velocity in $(\mathrm{Re}\,(\bar{z}_1 z_2), \mathrm{Im}\,\bar{z}_1 z_2))$-plane for $\lambda = 6.39$, it is easy to show that $|z_1|^2\dot{\phi} = \mathrm{Im}\,\bar{z}_1 z_2$. (c): A torus is obtained by the numerical integration of the original equations (7) for $\lambda = 6.39$. (d): A projection of the torus in $(\mathrm{Re}\,z_1, \mathrm{Im}\,z_1)$-plane.

Now, we consider a numerical investigation of chaotic behaviour in the above system (7). In many experiments, changes in the system behavior are studied as some system parameter is varied. We study the qualitative changes in dynamic behavior, which produces the oscillatory waves that reverse their direction of propagation in a chaotic fashion. Therefore we vary the bifurcation parameter λ and observe that the branch of tori loses stability at $\lambda = 6.786$ to a branch of period-doubled tori (PT-branch) and hereafter undergoes torus-doubling cascade bifurcations. PT-branch loses stability at $\lambda = 6.94$ to a branch of period-4 tori and this branch joins with PT-branch at $\lambda = 7.58$. A period-4 tori also bifurcates at $\lambda = 7.48$ from PT-branch. Later we will see that this branch is in the basin of attraction of the chaotic attractors which give rises to the occurrence of reversal in the direction of chaos. This branch collides with PT-branch at $\lambda = 7.60$. As λ increases close to 7.38, the torus-doubled cascade forms a modulated asymmetric strange attractor; the velocity of these modulated waves remain positive for $\lambda < 7.38$ As we vary λ further, a bifurcation occurs at $\lambda = 7.38$. At this point the chaotic attractor gains symmetry, and the velocity of the attractor changes sign irregularly and no longer remains positive (see Fig. 4) indicating chaotic reversal in the direction of propagation. The qualitative changes in chaotic behavior occurs when an asymmetric chaotic attractor describing the chaotic motion collides with an unstable period-4 torus. This increases the size of the attractor, produces a symmetric attractor and hence results in reversal in the direction of propagation (we call this an interior crisis). A crisis is a bifurcation event in which a chaotic attractor and its basin of attraction suddenly disappears or suddenly changes in size as some control parameter is adjusted. In other words, if a parameter is changed in the opposite direction, the chaotic attractor can suddenly appear or the size of the attractor can suddenly be reduced. The interior and the boundary crisis is investigated by Grebogi et al [11] in the context of a one-dimensional quadratic map, when at a tangent bifurcation the stable and unstable orbits are created. The stable fixed point undergoes a period-doubling cascade, forming chaotic behavior and hence colliding with an unstable periodic orbit. The result is the disappearance of the chaotic motion. The mechanism of qualitative changes in this paper involves the symmetry-increasing bifurcation of the strange

attractor in which the asymmetric trajectories collide with an unstable period-4 torus Grebogi et al [12]. In [12] sudden qualitative changes in chaotic dynamical behaviour occur when an attractor collides with an unstable periodic orbit.

In Fig. 4 we show time evolution of the velocity of the travelling waves. The velocity of travelling waves eventually perform complex behaviour. This complex chaotic behaviour reverses its direction of propagation, indicating an irregular sign changes of the wave's velocity.

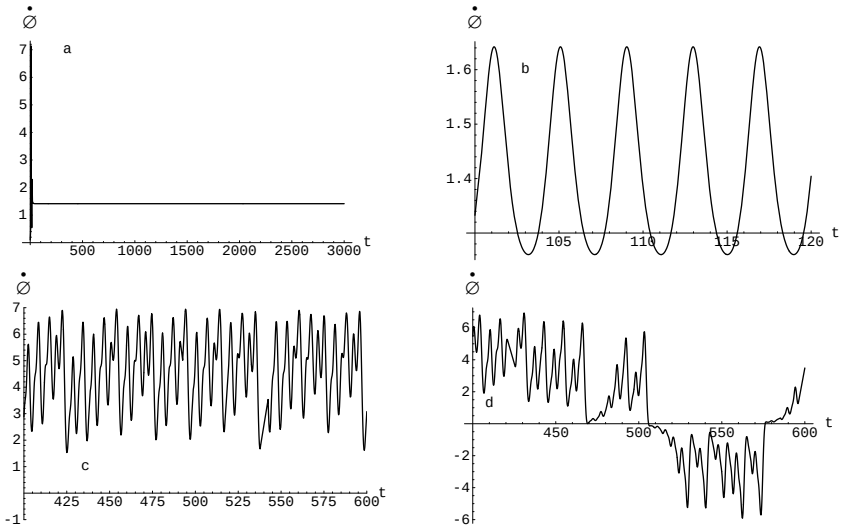

Figure 4.

A detailed study of this problem is submitted to CHAOS, American Institute of Physics [13].

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Nonlinear Symmetries of Variational Calculus and Regularity Properties of Differential Flows on Non-Compact Manifolds

Alexander Val. ANTONIOUK

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *antoniouk@imath.kiev.ua*

This talk is devoted to the discussion of symmetries that arise in the high-order variational equations. Such symmetries lead to a new class of nonlinear estimates on regularity. They permit the work in the case of essentially non-Lipschitz nonlinear differential equations, i.e. when the classical Cauchy–Liouville–Picard regularity scheme fails to work. These estimates are applied to the problem of C^∞ regularity of nonlinear differential flows on manifolds, that could also contain random terms. In particular, we demonstrate that the geometrically correct study of regularity problems for nonlinear flows on manifolds requires introduction of a new type variations with respect to the initial data. These variations are defined via a natural generalization of covariant Riemannian derivatives. We also find how the curvature manifests in the structure of the high-order variational equations.

1 Nonlinear differential equation on manifold with random terms

Origination of different type nonlinearities in differential equations often complicates the study of regular dependence of solutions on the initial data and parameters. The well-known classical regularity scheme, usually attributed to Cauchy, Liouville and Picard, permits to obtain a complete picture of regularity properties for quasi-linear equations (with globally Lipshitz coefficients with all bounded derivatives). For a simple one-dimensional equation

$$y_t(x) = x + \int_0^t F(y_s(x))ds \tag{1}$$

it applies the fixed point techniques for contractive for small t mapping $x \to y_t^x$ to prove the existence and uniqueness of solutions. In a similar way, the application of results on differentiability of implicit function leads to the C^∞ dependence of solution y_t^x on initial data x.

In the case of essentially nonlinear equations with coefficients with unbounded derivatives this scheme does not work. Moreover, the traditional attempt to find good Lipschitz approximations may fail due to the influence of geometry or infinite dimensions. In this talk we demonstrate that the knowledge of *symmetries of high-order differential calculus* permits to work with regularity problems in the essentially nonlinear case. Main attention is devoted to the influence of geometry on regularity properties.

We consider stochastic generalization of equation (1)

$$y_t^x = x + \int_0^t A_0(y_s^x)ds + \sum_\alpha \int_0^t A_\alpha(y_t^x)\delta W_t^\alpha \tag{2}$$

on non-compact smooth oriented connected Riemannian manifold M without boundary. Here coefficients A_0, A_α represent smooth globally defined vector fields, W_t^α denotes a family of one dimensional independent Wiener processes, δW^α means Stratonovich differential, range of index α corresponds to the dimension of manifold.

We intentionally use uncommon for stochastics notation y_t^x for the process, because the results are still valid for the ordinary differential equations ($A_\alpha = 0$). The new geometric effects also arise for this case, i.e. do not follow from the traditional arguments, previously developed in attempts to construct the stochastic differential geometry, e.g. [4–10].

Equation (2) is understood in a sense, that for any C^3 function on manifold equation

$$f(y_t^x) = f(x) + \int_0^t (A_0 f)(y_s^x) ds + \sum_\alpha \int_0^t (A_\alpha f)(y_t^x) \delta W_t^\alpha$$

holds as stochastic equation in $\mathbb{R}^1$. In particular, one can take $f(x) = x^k$ to find its local coordinates representations.

The corresponding semigroup

$$(e^{-tH} f)(x) = \mathbf{E}\, f(y_t^x) \tag{3}$$

provides the solutions to the parabolic Cauchy problem

$$\frac{\partial}{\partial t} u(t,x) + Hu(t,x) = 0, \qquad H = -\frac{1}{2} \sum_\alpha A_\alpha A_\alpha + A_0 \tag{4}$$

and gives therefore a set of actual applications to the problems of infinite dimensional functional and nonlinear analysis, stochastics and mathematical physics, differential geometry and operator theory. We remark that paper can be read in a simpler nonstochastic case for $A_\alpha = 0$.

2 Nonlinear symmetries of differential calculus

The main idea of work in the essentially nonlinear case is to find a set of *a priori estimates on regularity*. They are related with symmetries of high-order derivatives [1].

Let us consider some first order differentiation d on the algebra of states y. Then the high-order differential of nonlinear functional is given by

$$d^n f(y) = f'(y) d^n y + \sum_{j_1 + \cdots + j_s = n,\ s = 2, n-1} f^{(s)}(y) d^{j_1} y \cdots d^{j_s} y + f^{(n)}(y) [dy]^n.$$

The main observation is that above terms $d^n y \sim [dy]^n$ arise simultaneously. This symmetry also manifests in the intermediate terms $d^{j_1} y \cdots d^{j_s} y \sim [dy]^{j_1 + \cdots + j_s} \sim [dy]^n$ and is present in *all* differentials $d^m f$, $m \in \mathbb{N}$.

Returning to equation (1) and writing equations on high-order variations one gets

$$y_t^{(n)} = \partial_x^{(n)} y_t^x = \partial_x^{(n)} x + \int_0^t \partial_x^{(n)} F(y_s^x) ds = \partial_x^{(n)} x + \int_0^t \sum_{\substack{j_1 + \cdots + j_s = n \\ s = 1, \ldots, n}} F^{(s)}(y_s^x) y_t^{(j_1)} \cdots y_t^{(j_s)} ds.$$

So the high-order variation $y_t^{(n)}$ in the l.h.s. becomes *proportional* to the first order variation $y_t^{(1)}$ in n^{th} power in the r.h.s. (for $j_i = 1$). Therefore the behaviours are comparable $\sqrt[n]{y_t^{(n)}} \sim y_t^{(1)}$.

If we introduce a homogeneous with respect to this symmetry expression

$$\rho_n(y,t) = \sum_{j=1,\ldots,n} p_j(y_t^x)\, ||y_t^{(j)}||^{m/j}$$

and impose some reasonable conditions on the behaviour of nonlinearity F and hierarchy of weights p_j, related with the behaviour of nonlinearity F, one comes to the quasi-contractive estimate, e.g. [1,2].

$$\exists\, K \quad \rho_n(y,t) \le e^{Kt} \rho_n(y,0),$$

i.e. **essentially nonlinear** (different powers m/j) **a priori estimate on the regularity** of solution y_t^x with respect to the initial data x (recall that $y_t^{(n)} = \partial_x^{(n)} y_t^x$). Because in the very basis of nonlinear estimates lie monotone methods of nonlinear analysis, such estimates become applicable even in the infinite dimensions.

We also remark, that in the case of globally Lipschitz coefficients with bounded derivatives weights $p_j = 1$. This leads to a qualitively different understanding of the Cauchy–Liouville–Picard regularity scheme, that does not ground on the implicit function techniques.

Turning to our model of stochastic equation on manifold (2), one can also hope to get similar a priori estimates on regularity. Here, however, arises a new fundamental problem: how to define the high-order variations in a geometrically correct way.

3 Generalization of Riemannian covariant derivative or invariance of variations with respect to (y_t^x) coordinate

Today the procedure of the geometrically correct construction of differential flows on manifolds is quite clear. Adding the stochastic terms does not change the picture: in spite of the additional terms, arising in Ito formula, the stochastics could be agreed with the geometry, e.g. via Stratonovich differentials, special second order bundles or orthoframe liftings of diffusions [4–10].

However the further problem of consistency with problematics of **differential geometry**, namely *how the geometrically invariant differentials are constructed from geometrically invariant objects*, in fact, was not considered yet. The attempts to consider variations as defined via derivatives in directions of vector fields, covariant Riemannian derivatives lose an important property of geometric invariance with respect to the process y_t^x.

Let us explain what is meant. Suppose that some process on manifold y_t (of diffusion or any other nature) travels over manifold and enters some vicinity $U \subset M$ with coordinate functions $\varphi = (\varphi^i)_{i=1}^{\dim M}$, $\varphi : U \to \mathbb{R}^{\dim M}$. Then one can speak about the coordinates of process $y_t^i = (\varphi^i) \circ y_t$ when it stays in this vicinity.

Let $\mathcal{D}$ be some first order differentiation operation, correctly defined on diffusion process y_t. It could be of any nature, like ∂_x or stochastic derivative with respect to the random parameter, the principal moment is that the first order differentiation obeys chain rule

$$\mathcal{D}(f \circ y) = (f' \circ y)\, \mathcal{D}y.$$

Because the local coordinate changes $y^{i'} = \varphi^{i'}(y_t) = (\varphi^{i'} \circ \varphi^{\mathrm{inv}})(y^i)_{i=1}^{\dim M}$ represent a particular case of locally defined functions, one gets rule

$$\mathcal{D}y^{m'} = \frac{\partial y^{m'}}{\partial y^m} \mathcal{D}y^m.$$

Therefore *the expression $\mathcal{D}y$ becomes a* **vector field** *with respect to the "coordinate" changes $(y) \to (y')$ of diffusion "variable" y*, though, of course, the diffusion process y_t does not determine some coordinate system, like local coordinate mappings φ, φ' do.

By similar to the classical differential geometry arguments, related with the construction of covariant derivatives, *there is no other way to define correctly the high-order derivatives $(\widetilde{\mathcal{D}})^i y$, but introduce additional terms with connection $\Gamma(y_t)$.*

The correct recurrent definition of the invariant high-order derivative then becomes

$$\widetilde{\mathcal{D}} y^m = \mathcal{D}y^m, \qquad \widetilde{\mathcal{D}}[\,(\widetilde{\mathcal{D}})^i y^m\,] = \mathcal{D}[\,(\widetilde{\mathcal{D}})^i y\,] + \Gamma_{p\ q}^{\ m}(y)\,[\,(\widetilde{\mathcal{D}})^i y^p\,]\mathcal{D}y^q,$$

here must also arise factors $\mathcal{D}y^q$ in the term with $\Gamma(y_t)$.

These additional terms in definition of higher-order derivatives $\widetilde{\mathcal{D}}^n$ guarantee the preservance of vector transformation law with respect to the $(y) \to (y')$ coordinate transformations:

$$(\widetilde{\mathcal{D}})^n y^{m'} = \frac{\partial y^{m'}}{\partial y^m} (\widetilde{\mathcal{D}})^n y^m, \qquad \forall\, n \geq 1.$$

Such $(y) \to (y')$ invariance, or, if one returns to the very beginning, the invariance with respect to changes of local coordinates $(y) \to (y')$ in vicinity, where travels y_t, represents a new purely geometric requirement, which for a long time remained in the shadow.

As a final remark let us also observe that the way to introduce the new type derivatives is independent on particular approach we choose to define the differential flow on manifold, actually **it works for any differential equations (high-order, etc) on manifolds**, because, by consideration above, symbol $y \in M$ must have values in manifold, but nothing more.

Now we can turn to the correct construction of high-order variations of process y_t^x (2) and develop nonlinear estimates on variations to this setting.

4 High order variations on initial data. Tensors on (x) and (y_t^x) coordinates

Consider the first order variation $\frac{\partial (y_t^x)^m}{\partial x^k}$, that represents a vector field on index m for $(y) \to (y')$ "coordinate" transformations and covector field on index k for $(x) \to (x')$ coordinate changes.

From arguments above we can immediately conclude that the definition of *geometrically invariant high-order variations* must include terms with $\Gamma(x)$ and $\Gamma(y)$ to guarantee the preservance of tensorial character on both image $(y) \to (y')$ and domain $(x) \to (x')$ coordinate changes of mapping $x \to y_t^x$.

Definition 1. Variations of differential flow y_t^x with respect to the initial data are defined recurrently by

1st order. $\mathbb{\nabla}^x_k y^m = \dfrac{\partial (y_t^x)^m}{\partial x^k}$ vector on m in (y), covector on k in (x).

High order. For $\gamma = (j_1, \ldots, j_n)$ the high-order variation $\mathbb{\nabla}^x_\gamma y^m = \mathbb{\nabla}^x_{j_n} \cdots \mathbb{\nabla}^x_{j_1} y^m$ is defined by

$$\begin{aligned} \mathbb{\nabla}^x_k (\mathbb{\nabla}^x_\gamma y^m) &= \nabla^x_k (\mathbb{\nabla}^x_\gamma y^m) + \Gamma_{p\ q}^{\ m}(y_t^x) \mathbb{\nabla}^x_\gamma y^p \frac{\partial y^q}{\partial x^k} \\ &= \partial^x_k (\mathbb{\nabla}^x_\gamma y^m) - \underbrace{\sum_{j \in \gamma} \Gamma_{k\ j}^{\ h}(x) \mathbb{\nabla}^x_{\gamma|_{j=h}} y^m}_{\textbf{old covariant derivative}} + \underbrace{\Gamma_{p\ q}^{\ m}(y_t^x) \mathbb{\nabla}^x_\gamma y^p \frac{\partial y^q}{\partial x^k}}_{\textbf{new term}} . \end{aligned} \tag{5}$$

From the point of view of classical Riemannian geometry such definition of the high-order invariant variation of y with terms $\Gamma(x)$, $\Gamma(y)$ and $\frac{\partial y_t^x}{\partial x}$ provides generalization of the classical covariant Riemannian derivative. Unlike all already existing torsion, polynomial connection and other generalizations of variation, defined primarily at point x, it depends not only on initial point of differentiation x, but also on behaviour of process at point y.

The above definition naturally generalizes to the tensors on both (x) and (y_t^x) coordinates: object $u^{(i/\alpha)}_{(j/\beta)}$ represents a tensor on (x) and (y_t^x) coordinates iff it is $T^{p,q}_x M$ tensor on multi-indexes $(i) = (i_1, \ldots, i_p)$, $(j) = (j_1, \ldots, j_q)$ with respect to the local coordinates (x^k) and $T^{r,s} M$ tensor on multi-indexes $(\alpha), (\beta)$ with respect to the local coordinates (y^m). In other words, after the simultaneous change of local coordinate systems $(x^k) \to (x^{k'})$ and $(y^m) \to (y^{m'})$ one has transformation law

$$u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial x^{(i)}}{\partial x^{(i')}} \frac{\partial x^{(j')}}{\partial x^{(j)}} \frac{\partial y^{(\alpha)}}{\partial y^{(\alpha')}} \frac{\partial y^{(\beta')}}{\partial y^{(\beta)}} u^{(i'/\alpha')}_{(j'/\beta')}$$

with Jacobians $\frac{\partial x^{(i)}}{\partial x^{(i')}} = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \cdots \frac{\partial x^{i_p}}{\partial x^{i'_p}}$. Simple examples of such tensors provide already introduced high-order variations or superpositions with usual tensors $u^{(i)}_{(j)}(x) v^{(\alpha)}_{(\beta)}(y^x_t)$.

The invariant (x)-derivative of tensor is defined in a similar to (5) way

$$\mathbb{W}^x_k u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial}{\partial x^k} u^{(i/\alpha)}_{(j/\beta)} + \sum_{s\in(i)} \Gamma_k{}^s{}_h(x) u^{(i/\alpha)|_{s=h}}_{(j/\beta)} - \sum_{s\in(j)} \Gamma_k{}^h{}_s(x) u^{(i/\alpha)}_{(j/\beta)|_{s=h}}$$
$$+ \sum_{\rho\in(\alpha)} \Gamma_\gamma{}^\rho{}_\delta(y^x_t) u^{(i/\alpha)|_{\rho=\delta}}_{(j/\beta)} \frac{\partial y^\delta}{\partial x^k} - \sum_{\rho\in(\beta)} \Gamma_\rho{}^\gamma{}_\delta(y^x_t) u^{(i/\alpha)}_{(j/\beta)|_{\rho=\gamma}} \frac{\partial y^\delta}{\partial x^k}.$$

First line corresponds to the covariant derivative on (x^k) coordinates, second line makes the resulting expression to be tensor with respect to the coordinates (y^m). Next lemma checks the tensorial transformation property for the (x) covariant derivative.

Theorem 1. *The covariant (x) derivative defines a tensor of higher valence, i.e. transformation law holds*

$$\mathbb{W}_k u^{(i/\alpha)}_{(j/\beta)} = \frac{\partial x^{k'}}{\partial x^k} \frac{\partial x^{(i)}}{\partial x^{(i')}} \frac{\partial x^{(j')}}{\partial x^{(j)}} \frac{\partial y^{(\alpha)}}{\partial y^{(\alpha')}} \frac{\partial y^{(\beta')}}{\partial y^{(\beta)}} \mathbb{W}_{k'} u^{(i'/\alpha')}_{(j'/\beta')}.$$

Before turning directly to the proof, let us briefly recall the definition of connection. Connection form is a mapping

$$\mathrm{Hess} : C^\infty(M) \to C^\infty(T^{0,2}M)$$

that fulfills property

$$\mathrm{Hess}\,(h\circ f) = \partial_j h \circ f \,\mathrm{Hess}\, f^j + \partial_{ij} h \circ f \, df^i \otimes df^j.$$

Its components, known as Cristoffel symbols, are defined by taking Hessians of local coordinates

$$\mathrm{Hess}\, x^i = -\Gamma_j{}^i{}_k dx^i \otimes dx^j.$$

In particular, for any function, viewed as a function of local coordinates we have

$$\mathrm{Hess}\,(f\circ x) = (\partial_{ij} f - \Gamma_i{}^k{}_j \partial_k f) dx^i \otimes dx^j.$$

Proof. One should write the expression of (x) derivative and substitute the transformation law of tensor on (x) and (y) coordinates to get

$$\mathbb{W}_k u^{(i/\alpha)}_{(j/\beta)} = [\text{Jacobians}] \frac{\partial x^{k'}}{\partial x^k} \partial_{k'} u^{(i'/\alpha')}_{(j'/\beta')}$$
$$+ \sum_{s\in(i)} [\text{Jacobians with } s\text{-out}] \left\{ \frac{\partial^2 x^s}{\partial x^k \partial x^{s'}} u^{(i'/\alpha')}_{(j'/\beta')} + \Gamma_k{}^s{}_h(x) \frac{\partial x^h}{\partial x^{h'}} u^{(i'/\alpha')|_{s'=h'}}_{(j'/\beta')} \right\} \qquad \text{(A)}$$

$$+ \sum_{\rho\in(\alpha)} [\text{Jacobians with } \rho\text{-out}] \left\{ \frac{\partial^2 y^\rho}{\partial x^k \partial y^{\rho'}} u^{(i'/\alpha')}_{(j'/\beta')} + \Gamma_\gamma{}^\rho{}_\delta(y) \frac{\partial y^\gamma}{\partial y^{\gamma'}} u^{(i'/\alpha')|_{\rho'=\gamma'}}_{(j'/\beta')} \frac{\partial y^\delta(x)}{\partial x^k} \right\} \qquad \text{(B)}$$

$$- \sum_{s\in(j)} (\text{similar to (A)}) - \sum_{\rho\in(\beta)} (\text{similar to (B)}), \qquad \text{(C)}$$

where [Jacobians] means the Jacobians from the transformation law of $u^{(i/\alpha)}_{(j/\beta)}$ and [Jacobians with s-out] means that in product the corresponding Jacobian with index s is omitted. Line (C) is written in analogue to the lines (A)–(B) with corresponding modifications.

Transformation of line (A). By definition of Cristoffel symbols

$$\mathrm{Hess}\,(x^1) = -\Gamma_{2}{}^{1}{}_{3} dx^2 \otimes dx^3 = -\Gamma_{2}{}^{1}{}_{3} \frac{\partial x^2}{\partial x^{2'}} \frac{\partial x^3}{\partial x^{3'}} dx^{2'} \otimes dx^{3'}.$$

From another side from definition of connection

$$\mathrm{Hess}\, x^1(x') = \frac{\partial x^1}{\partial x^{1'}} \mathrm{Hess}\, x^{1'} + \frac{\partial^2 x^1}{\partial x^{2'} \partial x^{3'}} dx^{2'} \otimes dx^{3'} = \left[-\frac{\partial x^1}{\partial x^{1'}} \Gamma_{2'}{}^{1'}{}_{3'} + \frac{\partial^2 x^1}{\partial x^{2'} \partial x^{3'}} \right] dx^{2'} \otimes dx^{3'}$$

that leads to relation

$$\frac{\partial^2 x^1}{\partial x^2 \partial x^{3'}} = \frac{\partial x^{2'}}{\partial x^2} \frac{\partial^2 x^1}{\partial x^{2'} \partial x^{3'}} = \Gamma_{2'}{}^{1'}{}_{3'}(x') \frac{\partial x^1}{\partial x^{1'}} \frac{\partial x^{2'}}{\partial x^2} - \Gamma_{2}{}^{1}{}_{3} \frac{\partial x^3}{\partial x^{3'}} \tag{D}$$

and one can transform each term in (A) to the form

$$\text{(A)} = [\text{Jacobians with } s\text{-out}] \frac{\partial x^s}{\partial x^{s'}} \frac{\partial x^{k'}}{\partial x^k} \Gamma_{k'}{}^{s'}{}_{h'}(x') u^{(i'/\alpha')|_{s'=h'}}_{(j'/\beta')}. \tag{E}$$

Transformation of line (B). Changing the coordinates $(x) \leftrightarrow (y)$ in (D) we get

$$\frac{\partial^2 y^1}{\partial y^2 \partial y^{3'}} = \Gamma_{2'}{}^{1'}{}_{3'}(y') \frac{\partial y^1}{\partial y^{1'}} \frac{\partial y^{2'}}{\partial y^2} - \Gamma_{2}{}^{1}{}_{3}(y) \frac{\partial y^3}{\partial y^{3'}}.$$

This leads to

$$\frac{\partial^2 y^1}{\partial x^k \partial y^{3'}} = \frac{\partial y^2}{\partial x^k} \frac{\partial^2 y^1}{\partial y^2 \partial y^{3'}} = \Gamma_{2'}{}^{1'}{}_{3'}(y') \frac{\partial y^1}{\partial y^{1'}} \frac{\partial y^{2'}}{\partial x^{k'}} \frac{\partial x^{k'}}{\partial x^k} - \Gamma_{2}{}^{1}{}_{3}(y) \frac{\partial y^3}{\partial y^{3'}} \frac{\partial y^2}{\partial x^k}$$

and permits to transform terms in (B) to the form

$$\text{(B)} = [\text{Jacobians with } \rho\text{-out}] \frac{\partial y^\rho}{\partial y^{\rho'}} \frac{\partial x^{k'}}{\partial x^k} \Gamma_{\gamma'}{}^{\rho'}{}_{\delta'} \frac{\partial y^{\delta'}}{\partial x^{k'}} u^{(i'/\alpha')|_{\rho'=\gamma'}}_{(j'/\beta')}. \tag{F}$$

Finally, collecting together terms (E), (F) and transforming the remaining terms in (C) we come to the (x) covariant derivative in new coordinate systems

$$\mathbb{\nabla}_k u^{(i/\alpha)}_{(j/\beta)} = [\text{Jacobians}] \frac{\partial x^{k'}}{\partial x^k} \mathbb{\nabla}_{k'} u^{(i'/\alpha')}_{j'/\beta')}. \qquad \blacksquare$$

An important property of new (x) covariant derivative is a chain rule, which does not hold for traditional covariant derivative.

Corollary 1. *Let $u^{(\alpha)}_{(\beta)}$ be a tensor on manifold M. Then for superposition one has chain rule*

$$\mathbb{\nabla}_k u^{(\alpha)}_{(\beta)}(y(x)) = (\nabla_\ell u^{(\alpha)}_{(\beta)})(y(x)) \frac{\partial y^\ell}{\partial x^k}.$$

Proof. Using definition of (x) derivative we get

$$\mathbb{\nabla}_k u^{(\alpha)}_{(\beta)}(y(x)) = \partial^x_k u^{(0/\alpha)}_{(0/\beta)}(y(x)) + \sum_{\rho \in (\alpha)} \Gamma_{\gamma}{}^{\rho}{}_{\delta}(y) u^{(0/\alpha)|_{s=\gamma}}_{(0/\beta)}(y) \frac{\partial y^\delta}{\partial x^k} - \sum_{\rho \in (\beta)} \Gamma_{\rho}{}^{\gamma}{}_{\delta}(y) u^{(0/\alpha)}_{(0/\beta)|_{\rho=\gamma}} \frac{\partial y^\delta}{\partial x^k}.$$

From chain rule for ∂^x_k and definition of usual covariant derivative follows the statement. $\blacksquare$

5 Role of curvature in the regularity problems

Having in hands the correct procedure of differentiation of tensors of y, such as defined by coefficients $A_\bullet(y_t^x)$ of equation (2), we can find the corresponding variational equations. Taking the partial derivative of (2) we find

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) = \left(\frac{\partial}{\partial x^k}A_0^m(y)\right)dt + \left(\frac{\partial}{\partial x^k}A_\alpha^m(y)\right)\delta W^\alpha.$$

By adding and subtracting the terms with $\Gamma(y)$ to single out the covariant (x)-derivative of vector fields $A_0(y)$, $A_\alpha(y)$ on image coordinates (y) we have

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) = \left(\mathbb{\nabla}_k^x A_0^m(y) - \Gamma_{p\ q}^{\ m}(y)A_0^p\frac{\partial y^q}{\partial x^k}\right)dt + \left(\mathbb{\nabla}_k^x A_\alpha^m(y) - \Gamma_{p\ q}^{\ m}(y)A_\alpha^p\frac{\partial y^q}{\partial x^k}\right)\delta W^\alpha.$$

Noting that the terms near connection contain the differential of process y we finally get invariant form of equation on the first variation

$$\delta\left(\frac{\partial y^m}{\partial x^k}\right) = -\Gamma_{p\ q}^{\ m}(y)\frac{\partial y^p}{\partial x^k}\delta y^q + \mathbb{\nabla}_k^x(A_\alpha^m(y))\delta W^\alpha + \mathbb{\nabla}_k^x(A_0^m(y))dt. \tag{6}$$

Therefore up to the parallel transition term with $\Gamma(y)$ the increments of first order variation are determined by covariant (x)-derivatives of coefficients. We take this observation as the recurrence base in the search for high-order variational equations.

Theorem 2. *Suppose that the equation on variation $\mathbb{\nabla}_\gamma^x y^m, |\gamma| \geq 1$ is written in form*

$$\delta(\mathbb{\nabla}_\gamma^x y^m) = -\Gamma_{p\ q}^{\ m}(\mathbb{\nabla}_\gamma^x y^p)\delta y^q + M_{\gamma\ i}^{\ m}\delta W^i + N_\gamma^m dt. \tag{7}$$

Then the next order variation $\mathbb{\nabla}_k^x\mathbb{\nabla}_\gamma^x y^m = \mathbb{\nabla}_{\gamma\cup\{k\}}^x y^m$ fulfills relation

$$\begin{aligned}\delta(\mathbb{\nabla}_{\gamma\cup\{k\}}^x y^m) = &-\Gamma_{p\ q}^{\ m}(\mathbb{\nabla}_{\gamma\cup\{k\}}^x y^p)\delta y^q + R_{m\ \ell q}^{\ p}(\mathbb{\nabla}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}\delta y^q \\ &+ (\mathbb{\nabla}_k^x M_{\gamma\ i}^{\ m})\delta W^i + (\mathbb{\nabla}_k^x N_\gamma^m)dt.\end{aligned} \tag{8}$$

Therefore the coefficients of variational equations are recurrently related by

$$M_{\gamma\cup\{k\}\ i}^{\ m} = \mathbb{\nabla}_k^x M_{\gamma\ i}^{\ m} + R_{m\ \ell q}^{\ p}(\mathbb{\nabla}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}A_i^q, \qquad N_{\gamma\cup\{k\}}^m = \mathbb{\nabla}_k^x N_\gamma^m + R_{m\ \ell q}^{\ p}(\mathbb{\nabla}_\gamma^x y^p)\frac{\partial y^\ell}{\partial x^k}A_0^q.$$

Above R denotes curvature tensor.

Remark 1. Introduction of new type variations finally let us find a place of curvature (8) in the regularity properties. Having started from variations, defined by derivatives in directions of vector fields or by covariant Riemannian derivatives, we would get a set of (y_t^x) non-invariant equations with exploding number of non-invariant coefficients, e.g. [4, 6]. In other words, additional terms in (5) compactificate the variational equations to a simplest possible form (8), that provides one more argument in favor of introduction of new type variations.

6 Nonlinear estimates on regularity of differential flows on manifolds. Applications to the semigroup properties

From structure of variational equations (8) and chain property of covariant (x)-derivative $\mathbb{\nabla}_k^x u_{(j)}^{(i)}(y_t^x) = [\nabla_\ell u_{(j)}^{(i)}](y_t^x)\frac{\partial y^\ell}{\partial x^k}$ we see that symmetry $\mathbb{\nabla}^x y_t^x \sim \sqrt[n]{(\mathbb{\nabla}^x)^n y_t^x}$ manifests again, leading to the related nonlinear expression

$$\rho_n(y,t) = \sum_{j=1,\dots,n}\mathbf{E}\,p_j(\rho^2(y_t^x,z))\,||(\mathbb{\nabla}^x)^j y_t^x||^{m/j}$$

and nonlinear estimate on regularity. Above z is some fixed point of manifold.

Theorem 3. *Introduce notation* $\widetilde{A_0} = A_0 + \sum\limits_{\alpha} \nabla_{A_\alpha} A_\alpha$. *Suppose conditions hold:*
dissipativity and differential coercitivity*:*

$$\forall\, C\ \exists\, K_C\ \forall\, x \in M \quad \langle \widetilde{A_0}(x), \nabla^x \rho^2(x,z)\rangle + C \sum_{\alpha} \|A_\alpha(x)\|^2 \le K_C(1+\rho^2(x,z)),$$

$$\forall\, C, C'\ \exists\, K_C \quad \langle \nabla \widetilde{A_0}[h], h\rangle + C \sum_{\alpha=1}^{d} |\nabla A_\alpha[h]|^2 + C' \sum_{\alpha} \langle R(A_\alpha, h)A_\alpha, h\rangle \le K_C \|h\|^2.$$

nonlinear behavior of all derivatives of curvature and coefficients*:*

$$\exists\, \mathbf{k}\ \forall\, j \quad \|(\nabla)^j \widetilde{A_\bullet}(x), R(x)\| \le (1+\rho(x,z))^{\mathbf{k}}.$$

Then for polynomial weights $p_j \ge 1$, *which fulfill hierarchy*

$$\forall\, j_1 + \cdots + j_s = i \quad [p_i(z)]^i\, (1+|z|)^{m\mathbf{k}} \le [p_{j_1}(z)]^{j_1} \cdots [p_{j_s}(z)]^{j_s}$$

the **nonlinear estimate on regularity** *holds*

$$\exists\, K \quad \rho_n(y,t) \le e^{Kt} \rho_n(y,0).$$

Proof proceeds similar to [1] with corresponding geometric complications, it will appear in [3].

Turning to the study of regularity properties of semigroup $(P_t f)(x) = \mathbf{E}\, f(y_t^x)$ one can prove representation, that relate the behaviour of covariant derivatives of function and its evolution

$$(\nabla^x)^n P_t f(x) = \sum_{j_1+\cdots+j_s=n,\ s=1,\ldots,n} \mathbf{E}\, \langle (\nabla^y)^s f(y_t^x), (\mathbb{\nabla}^x)^{j_1} y_t^x \otimes \cdots \otimes (\mathbb{\nabla}^x)^{j_s} y_t^x \rangle.$$

In particular, we get next argument in favor of new type variations: they relate classical covariant derivatives of function and its evolution – and only they!

Because in the above representation new type variations $(\mathbb{\nabla}^x)^j y_t^x$ appear as convolutional kernels, one can get the regular properties of semigroup from nonlinear estimates on regularity. This is a subject of [3].

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Regularity Properties of Infinite-Dimensional Evolutions, Related with Anharmonic Lattice Systems

Alexandra Vict. ANTONIOUK

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *antoniouk@imath.kiev.ua*

In this paper we discuss the C^∞ properties of evolution for infinite system of interacting particles. Such evolution is described in terms of semigroup generated by second order differential operator of infinite number of variables. For anharmonic systems this operator has unbounded non-Lipschitz coefficients and thus corresponding semigroup is not strongly continuous in time in the space of continuously differentiable functions. This circumstance makes inapplicable the standard technique of analysis and semigroup theory. Using the connection between the semigroup theory and stochastic differential equations and observation about nonlinear symmetry of high order variational calculus, we state smoothing properties of such infinite-dimensional evolution in the space of continuously differentiable functions. The main attention is devoted to the influence of the nonlinearity on the topology of regularity.

1 Introduction

The appearance of different type of nonlinearities can to a great extent influence the behaviour of a system and usually requires relevant techniques adopted to the description of arising phenomena. Consider the infinite-dimensional system of anharmonic oscillators described by the Gibbs measure with finite range interaction of radius r_0:

$$d\mu = \frac{1}{Z}\exp\left\{-\lambda \sum_{|k-j|\le r_0} b_{k-j}x_k x_j\right\}\prod_{k\in\mathbb{Z}^d}\exp\{-\mathcal{U}(x_k)\}dx_k.$$

It exists and is unique for small $\lambda > 0$, and can be correctly constructed in terms of thermodynamic limit $\Lambda \to \mathbb{Z}^d$ [1, 2]. The associated Hamiltonian, defined on cylinder functions of polynomial behaviour $u, v \in \mathcal{P}_{\mathrm{cyl}}(\mathbb{R}^{\mathbb{Z}^d})$, is given by

$$H_\mu = \sum_{k\in\mathbb{Z}^d}\left\{-\frac{1}{2}\frac{\partial^2}{\partial x_k^2} + (F(x_k) + (Bx)_k)\frac{\partial}{\partial x_k}\right\},$$

where $F(x_k) = \mathcal{U}'(x_k)$ and $(Bx)_k = \sum\limits_{j:|j-k|\le r_0} b_{k-j}x_j$.

We should remark that semigroup of this operator e^{-tH_μ} is well-defined, strongly continuous and even preserves the Sobolev-type spaces of differentiable functions $\mathcal{W}_\Theta$ [3]. This in particular implies that operator H_μ is essentially self-adjoint in $L_2(\mathbb{R}^{\mathbb{Z}^d}, \mu)$ and corresponding dynamics is well-defined. However, spaces $\mathcal{W}_\Theta$ are of very special structure and actually do not give precise information about the *smooth* properties of dynamics $P_t = e^{-tH_\mu}$. Thus, it would be interesting to investigate the properties of semigroup P_t in some spaces with *sup*-topology, like C_b^n (continuously differentiable functions). But in C_b^n semigroup P_t is not strongly continuous on t, and the standard methods of analysis and semigroup theory are not applicable. Nevertheless, such semigroups can be described in the terms of stochastic differential equation in $\mathbb{R}^{\mathbb{Z}^d}$:

$$d\xi_k^0(t,x^0) = dW_k(t) - \{F(\xi_k^0(t,x^0)) + (B\xi^0(t,x^0))_k\}dt, \qquad \xi_k^0(0,x^0) = x_k^0, \quad k\in\mathbb{Z}^d \qquad (1)$$

then

$$(P_t f)(x^0) = \mathbf{E}\, f(\xi^0(t, x^0)), \tag{2}$$

where the cylinder Wiener process $W = \{W_k(t)\}_{k\in\mathbb{Z}^d}$ with values in $\ell_2(a) = \ell_2(a, \mathbb{Z}^d)$, $\sum a_k = 1$ is canonically realized on measurable space $(\Omega = C_0([0,T], \ell_2(a)), \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ with canonical filtration $\mathcal{F}_t = \sigma\{W(s) | 0 \le s \le t\}$ and cylinder Wiener measure $\mathbf{P}$. Processes W_k, $k \in \mathbb{Z}^d$ are independent $\mathbb{R}^1$-valued Wiener processes and $\mathbf{E}$ denotes the expectation with respect to measure $\mathbf{P}$. Further the set of all vectors $a = \{a_k\}_{k\in\mathbb{Z}^d}$ such that $\delta_a = \sup\limits_{|k-j|=1} |a_k/a_j| < \infty$ we denote by $\mathbb{P}$. On the coefficients of H_μ we impose the following conditions: the nonlinear map $F : \mathbb{R}^{\mathbb{Z}^d} \ni x \longrightarrow F(x) = \{F(x_k)\}_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ is generated by the C^∞ monotone function F, $F(0) = 0$, which has polynomial growth on the infinity

$$\exists\, \mathbf{k} \ge -1 \quad \forall\, i \ge 1 \quad |F^{(i)}(x) - F^{(i)}(y)| \le C_i |x - y| (1 + |x| + |y|)^{\mathbf{k}}. \tag{3}$$

Note that for $\mathbf{k} > -1$ the map $F : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ is not locally Lipschitz in any space $\ell_p(c, \mathbb{Z}^d)$.

Due to (2) the investigation of regular properties of semigroup leads to the problem of smooth dependence with respect to the initial data for solutions of stochastic differential equations (1).

In the case of Lipschitz coefficients with bounded derivatives the question of C^∞-smoothness of the stochastic flow and corresponding semigroup has already been solved using the standard fixed point arguments and implicit function technique (see for example [4, 5]). In contrary, equation (1) has coefficients, for which the Lipschitz property breaks even locally on balls in $\ell_p(a)$ and these methods fail to work.

Basing on a simple observation about the symmetry of the system in variations we develop technique, adopted to the investigation of C^∞ smoothness of semigroups and associated stochastic flows in essentially non-Lipschitz case. Our aim is to construct such spaces $C_{(\Theta)^m}$ with sup-topology, that semigroup $P_t : C_{(\Theta)^m} \mapsto C_{(\Theta)^{m+1}}$ and the following estimate is true:

$$\|P_t f\|_{C_{(\Theta)^m}} \le \frac{1}{t^{m/2}} K_{\Theta,m} e M_{\Theta,m} t \|f\|_{C_{(\Theta)^0}}.$$

2 Main definitions, stochastic methods and nonlinear quasi-contractive estimate

To study the smooth properties of semigroup, first of all we need the representation for derivatives of semigroup $\partial_\tau P_t f$, where $\tau = \{k_1, \dots, k_m\}$ is the set of directions, in which differentiation is done and $\partial_\tau = \partial^{|\tau|}/\partial x_{k_1} \cdots \partial x_{k_m}$. Formula (2) leads to

$$\partial_\tau (P_t f)(x^0) = \sum_{s=1}^{m} \sum_{\gamma_1 \cup \cdots \cup \gamma_s = \tau} \mathbf{E}\, \langle \partial^{(s)} f(\xi^0), \xi_{\gamma_1} \otimes \cdots \otimes \xi_{\gamma_s} \rangle. \tag{4}$$

$\partial^{(s)} f = \{\partial_\gamma f\}_{|\gamma|=s}$ denotes above the set of s^{th}-order partial derivatives of function and we used notation

$$\langle \partial^{(s)} f(\xi^0), \xi_{\gamma_1} \otimes \cdots \otimes \xi_{\gamma_s} \rangle = \sum_{j_1,\dots,j_s \in \mathbb{Z}^d} (\partial_{\{j_1,\dots,j_s\}} f)(\xi^0) \xi_{j_1,\gamma_1} \cdots \xi_{j_s,\gamma_s}.$$

Vector $\xi_\tau = \{\xi_{k,\tau}\}_{k\in\mathbb{Z}^d}$ is interpreted as a derivative of ξ^0 with respect to the initial data $x^0 = \{x^0_k\}_{k\in\mathbb{Z}^d}$ and $\xi_{k,\tau} = \frac{\partial^{|\tau|} \xi^0_k(t,x^0)}{\partial x^0_{j_n} \cdots \partial x^0_{j_1}}$ is called below a τ^{th} *variation* of ξ^0. The equation on ξ_τ is derived by the formal successive differentiation of (1) with respect to x^0:

$$\frac{d\xi_{k,\tau}}{dt} = -F'(\xi^0_k)\xi_{k,\tau} - \sum_{j:\, |j-k| \le r_0} b_{k-j} \xi_{j,\tau} - \varphi_{k,\tau}, \qquad \xi_{k,\tau}(0) = x_{k,\tau} \tag{5}$$

where $\varphi_{k,\tau} = \varphi_{k,\tau}(\xi^0, \xi_{\cdot,\gamma}, \gamma \subset \tau, \gamma \neq \tau)$

$$\varphi_{k,\tau} = \sum_{\gamma_1 \cup \cdots \cup \gamma_s = \tau,\ s \geq 2} F^{(s)}(\xi^0) \xi_{k,\gamma_1} \cdots \xi_{k,\gamma_s} \tag{6}$$

In (6) the summation $\sum\limits_{\gamma_1 \cup \cdots \cup \gamma_s = \tau,\, s \geq 2}$ runs on all possible subdivisions of the set $\tau = \{j_1, .., j_n\}$, $j_i \in \mathbb{Z}^d$ on the nonintersecting subsets $\gamma_1, \ldots, \gamma_s \subset \tau$, with $|\gamma_1| + \cdots + |\gamma_s| = |\tau|$, $s \geq 2$, $|\gamma_i| \geq 1$.

It is well-known that for initial data $x^0 \in \ell_{2(\mathbf{k}+1)^2}(a)$ there is a unique *strong solution* to equation (1), i.e. $\ell_2(a)$-continuous $\mathcal{F}_t$-adapted process $\xi^0(t, x^0) \in \mathcal{D}_{\ell_2(a)}(F)$ that fulfills $\mathbf{P}$ a.e. equation (1) in $\ell_2(a)$. For $x^0 \in \ell_2(a)$ the *generalized solution* is obtained as a uniform on $[0, T]$ $\mathbf{P}$ a.e. limit of strong solutions [6–8].

The solution to (5) is understood as the $\mathcal{F}_t$-adapted process $\xi_\gamma(t, x^0)$ that fulfills (5) in $\ell_{m_\gamma}(c_\gamma)$ a.e. on $[0, T]$ and such that for $\mathbf{P}$ a.e. $\omega \in \Omega$ the map $[0, T] \ni t \to \xi_\gamma(t, x^0) \in \ell_{m_\gamma}(c_\gamma)$ is Lipschitz continuous, $\xi_{k,\gamma}(0, x^0) = x_{k,\gamma}$, $k \in \mathbb{Z}^d$, $\xi_\gamma(t, x^0) \in \mathrm{Dom}_{\ell_{m_\gamma}(c_\gamma)}(F'(\xi^0(t, x^0)) + B)$, for a.e. $t \in [0, T]$, and there is a strong $\ell_{m_\gamma}(c_\gamma)$-derivative $d\xi_\gamma(t, x^0)/dt$ a.e. on $[0, T]$.

Now we are going to obtain the representation for semigroup derivatives, which immediately leads to the raise of smoothness for semigroup. To do this we need to recall some definitions of analysis on Wiener space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$ (or Malliavin calculus).

Denote by $\mathcal{J}_{cyl}$ the set of $\mathcal{F}_t$-adapted continuous *integrable cylinder-valued* processes $u_t = \{u_{t,k}\}_{k \in \mathbb{Z}^d}$ for which there exists such set $\Lambda_u \subset \mathbb{Z}^d$, $|\Lambda_u| < \infty$ (support of cylindricity) that for all points $k \notin \Lambda_u$ $u_{t,k} \equiv 0$, $t \in [0, T]$ and

$$\forall\, k \in \Lambda_u \ \ \forall\, p \geq 1 \qquad \mathbf{E} \int_0^T |u_{t,k}|^p dt < \infty.$$

Definition 1. Measurable function G on Ω is *stochastically differentiable in direction* $u \in \mathcal{J}_{\mathrm{cyl}}$ and has directional derivative $D_u G$ if $\exists\, \varepsilon_0 > 0\ \forall\, |\varepsilon| \leq \varepsilon_0$ function $G(\omega_\bullet + \varepsilon \int_0^\bullet u_s ds)$ belongs to $\bigcap\limits_{p \geq 1} L^p(\Omega, \mathbf{P})$ and there is a measurable function $D_u G \in \bigcap\limits_{p \geq 1} L^p(\Omega, \mathbf{P})$ such that

$$\forall\, p \geq 1 \quad \lim_{|\varepsilon| \to 0} \mathbf{E} \left| \frac{G(\omega_\bullet + \varepsilon \int_0^\bullet u_s ds) - G(\omega)}{\varepsilon} - D_u G(\omega) \right|^p = 0.$$

Definition 2. We say that $G \in \mathcal{D}_{\mathrm{loc}}(\Omega)$ (*locally stochastically differentiable*) iff $\forall\, j \in \mathbb{Z}^d$ there is a map $\mathbf{D}_j G \in \bigcap\limits_{p \geq 1} L^p(\Omega, \mathbf{P}, \mathcal{H})$ and $\forall\, u \in \mathcal{J}_{\mathrm{cyl}}\ \exists\, D_u G$ that has representation

$$D_u G = \sum_{j \in \Lambda_u} \langle \mathbf{D}_j G, \int_0^\bullet u_{s,j} ds \rangle_{\mathcal{H}}.$$

Above $\mathcal{H}$ denotes the Cameron–Martin space of absolutely continuous functions $\gamma : [0, T] \to \mathbb{R}^1$, $\gamma(0) = 0$, equipped with the scalar product $\langle \gamma, \gamma \rangle_{\mathcal{H}} = \int_0^T |\dot{\gamma}(s)|^2 ds$.

Derivative D_u has the following properties [9–11].

1. For smooth function $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^1)$ of polynomial with all derivatives behaviour at infinity and $G_1, \ldots, G_n \in \mathcal{D}_{\mathrm{loc}}(\Omega)$ we have $f(G_1, \ldots, G_n) \in \mathcal{D}_{\mathrm{loc}}(\Omega)$ and

$$D_u f(G_1, ..., G_n) = \sum_{i=1}^n [\partial_i f \circ (G_1, \ldots, G_n)]\, D_u G_i, \qquad u \in \mathcal{J}_{\mathrm{cyl}}. \tag{7}$$

2. For real-valued $\mathcal{F}_t$-adapted continuous processes $H_t \in \mathcal{D}_{\mathrm{loc}}(\Omega)$, $t \in [0, T]$ such that $\mathbf{E} \int_0^T |H_s|^p ds < \infty$ and $\mathbf{E} \int_0^T \|\mathbf{D}_j H_s\|_{\mathcal{H}}^p ds < \infty$, for all $p \geq 1$, $j \in \mathbb{Z}^d$ we have

$$\forall\, t \in [0, T]\ \forall\, k \in \mathbb{Z}^d \quad \left\{ \int_0^t H_s ds,\ \int_0^t H_s dW_k(s) \right\} \in \mathcal{D}_{\mathrm{loc}}(\Omega).$$

3. $D_u \int_0^t H_s\,ds = \int_0^t D_u H_s\,ds,$

$D_u \int_0^t H_s\,dW_k(s) = \int_0^t H_s u_{s,k}\,ds + \int_0^t D_u H_s\,dW_k(s), \qquad u \in \mathcal{J}_{\rm cyl}.$

Theorem 1. *Let $\xi^0(t,x^0)$ be a generalized solution to (1), $x^0 \in \ell_2(a)$. For vector $v \in \mathbb{R}^{\mathbb{Z}^d}$ with finite number of nonzero coordinates introduce process*

$$\Gamma_t v = [Id + t(F'(\xi^0(t,x^0)) + B)]v \in \mathcal{J}_{\rm cyl}.$$

Then the derivative in direction $u_t = \Gamma_t v$ gives

$$D_{\Gamma v}\xi^0(t,x^0) = tv. \tag{8}$$

Moreover, the **integration by parts formula** *holds*

$$\mathbf{E}\,\langle \partial f(\xi_t^0), v\rangle_{\ell_2(1)}\Psi = \frac{1}{t}\mathbf{E}\; f(\xi_t^0)\{\Psi \int_0^t \langle \Gamma_s v, dW(s)\rangle_{\ell_2(1)} - D_{\Gamma v}\Psi\} \tag{9}$$

for all $\mathcal{F}_t$-measurable $\Psi \in \mathcal{D}_{\rm loc}(\Omega)$, $f \in \mathcal{P}^\infty_{\rm cyl}(\ell_2(a))$, $t>0$.

The proof is given in [12]. Using this theorem we rewrite the representation of partial derivatives $\partial_\tau P_t f$ (4) in the terms of stochastic derivatives. Introduce notation $\mathbb{D}^k = D_{\Gamma e_k}$ where $e_k = (\dots,0,1_k,0,\dots) \in \mathbb{R}^{\mathbb{Z}^d}$ is k^{th} unit vector. Formula (9) gives for $\mathcal{F}_t$-measurable $\Psi \in \mathcal{D}_{\rm loc}(\Omega)$

$$\forall\, f \in \mathcal{P}^\infty_{\rm cyl}(\ell_2(a)) \quad \mathbf{E}\;\partial_k f(\xi_t^0)\Psi = \frac{1}{t}\,\mathbf{E}\, f(\xi_t^0)\mathbb{D}^*_k\Psi \tag{10}$$

with

$$\mathbb{D}^*_k\Psi = \Psi\int_0^t \langle \Gamma_s e_k, dW(s)\rangle_{\ell_2(1)} - \mathbb{D}^k\Psi.$$

Therefore, the partial derivatives of semigroup permit representation

$$\partial_\tau P_t f(x) = \sum_{\ell=1}^{|\tau|}\sum_{\gamma_1\cup\dots\cup\gamma_\ell=\tau}\sum_{j_1,\dots,j_\ell\in\mathbb{Z}^d} \mathbf{E}\, f(\xi_t^0)\,\frac{\mathbb{D}^*_{j_1}\cdots\mathbb{D}^*_{j_\ell}(\xi_{j_1,\gamma_1}\cdots\xi_{j_\ell,\gamma_\ell})}{t^\ell}. \tag{11}$$

To obtain the raise of smoothness under the action of semigroup P_t we have to investigate the behaviour of derivatives $\mathbb{D}^\beta\xi_\tau = \mathbb{D}^{j_1}\cdots\mathbb{D}^{j_\ell}\xi_\tau$, $\beta = \{j_1,\dots,j_\ell\}$.

Theorem 2. *For initial data $x^0 \in \ell_2(a)$ the coordinates of variations are locally stochastically differentiable $\xi_{k,\tau}(t,x^0) \in \mathcal{D}_{\rm loc}(\Omega)$ and $\mathbb{D}^\beta\xi_{k,\tau} \in \mathcal{D}_{\rm loc}(\Omega)$, $\beta \subset \mathbb{Z}^d$, $|\beta| \ge 1$. The stochastic derivatives $\mathbb{D}^\beta\xi_{k,\tau}$ are represented as a strong solutions in $\ell_{m_\tau}(c_{\tau,\beta})$, $m_\tau = m_1/|\tau|$, to system*

$$\mathbb{D}^\beta\xi_{k,\tau}(t) = \widetilde{x}_{k;\tau,\beta} - \int_0^t [(F'(\xi^0)+B)\mathbb{D}^\beta\xi_\tau]_k ds - \int_0^t \varphi_{k;\tau,\beta}(s)ds, \tag{12}$$

where $\widetilde{x}_{\tau,\beta} = 0$ for $|\beta| \ge 1$ and $\widetilde{x}_{\tau,\varnothing} = \widetilde{x}_\tau$,

$$\varphi_{k;\tau,\beta}(t) = \sum_{\substack{\gamma_1\cup\dots\cup\gamma_\ell=\tau\\ |\gamma_i|\ge 1,\ \ell\ge 1}}\ \sum_{\substack{\sigma_0\cup\dots\cup\sigma_\ell=\beta\\ |\sigma_0|\ge 2-\ell,\ |\sigma_i|\ge 0}} t^{|\sigma_0|}\delta_k^{\sigma_0}F^{(\ell+|\sigma_0|)}(\xi_k^0)\,\mathbb{D}^{\sigma_1}\xi_{k,\gamma_1}\cdots\mathbb{D}^{\sigma_\ell}\xi_{k,\gamma_\ell}$$

and vectors $c_{\tau,\beta} \in \mathbb{P}$ *fulfill hierarchy*

$$\exists\, K_c \quad \delta_k^{\sigma_0}[c_{k;\,\tau,\beta}]^{|\tau|} a_k^{-\frac{\mathbf{k}+1}{2}m_1} \le K_c[c_{k;\,\gamma_1,\sigma_1}]^{|\gamma_1|}\cdots[c_{k;\,\gamma_\ell,\sigma_\ell}]^{|\gamma_\ell|}, \qquad k\in\mathbb{Z}^d. \tag{13}$$

Above $\delta_j^\beta = \prod\limits_{i\in\beta}\delta_j^i$ *is a product of Kronnecker symbols and the subdivisions of sets* $\tau=\gamma_1\cup\cdots\cup\gamma_\ell$, $\beta=\sigma_0\cup\cdots\cup\sigma_\ell$ *are such that* $1\le\ell\le|\tau|$ *and for* $\ell=1$, $|\sigma_0|\ge 1$*; for* $\ell\ge 2$ $|\sigma_0|\ge 0$. *Moreover, at* $t=0$ *there is asymptotic* $\forall\, R>0\ \exists\, K_R\in\bigcap\limits_{p\ge1}L^p(\Omega,\mathbf{P})$ *such that*

$$\forall\,|\beta|\ge 1 \quad \|\mathbb{D}^\beta\xi_\tau(t,x^0)\|_{\ell_{m_\tau}(c_{\tau,\beta})}\le t^{|\beta|+1}K_R(\omega), \qquad t\in[0,T]$$

uniformly on $\max(\|x^0\|_{\ell_2(a)}, \|\widetilde{x}_{\gamma,\varnothing}\|_{\ell_{m_\gamma}(c_{\gamma,\varnothing})},\gamma\subset\tau)\le R$.

The proof of this Theorem is quite complicated and will be given in forthcoming paper [13]. Remark that the equations (12) are obtained by direct action of $\mathbb{D}^\beta$ on equations (5), because (8) and chain rule (7) for F give $\mathbb{D}^\beta F^{(\ell)}(\xi_j^0(t,x^0)) = \delta_j^\beta t^{|\beta|}F^{(\ell+|\beta|)}(\xi_j^0(t,x^0))$.

The property $\mathbb{D}^k\xi_t^0 = te_k$ implies not only the simple integration-by-parts formula, but also a simplified structure of coefficients in (12). In particular, one sees the nonlinear symmetries in this equation, i.e. that the terms $F'(\xi^0)\mathbb{D}^\beta\xi_\tau$, $t^{|\beta|}F^{(1+|\beta|)}(\xi^0)\xi_\tau$, $t^{|\beta|}F^{(|\tau|+|\beta|)}(\xi^0)\xi_{j_1}\cdots\xi_{j_n}$, $\tau=\{j_1,\ldots,j_n\}$ appear in the r.h.s. of (12) simultaneously. Taking into account this reason we introduce a nonlinear expression

$$\rho_{\tau,\beta}(t) = \sum_{\gamma\subset\tau,\ \sigma\subset\beta,\ \gamma\ne\varnothing} \mathbf{E}\, p_{\gamma,\sigma}(\|\xi^0(t,x^0)\|^2_{\ell_2(a)})\|\frac{\mathbb{D}^\sigma\xi_\gamma}{t^{|\sigma|}}\|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})}, \qquad m_\gamma = m_1/|\gamma|.$$

The following Theorem gives a quasi-contractive estimate on $\rho_{\tau,\beta}$, which will be the principal tool for investigation of the raise of smoothness properties of semigroup.

Theorem 3. *Let* F *fulfill* (3), $x^0\in\ell_2(a)$ *and vectors* $c_{\gamma,\sigma}\in\mathbb{P}$ *satisfy hierarchy* (13). *Suppose that monotone functions* $p_{\gamma,\sigma}\in C^2(\mathbb{R}^1_+)$ *are such that* $\exists\,\varepsilon>0\ \exists\, K_p>0\ \forall\, z\in\mathbb{R}_+$

$$\begin{aligned} &p_{\gamma,\sigma}(z)\ge\varepsilon \qquad \textit{and} \qquad (1+z)(|p'_{\gamma,\sigma}(z)|+|p''_{\gamma,\sigma}(z)|)\le K_p p_{\gamma,\sigma}(z),\\ &(p_{\tau,\beta})^{|\tau|}(1+z)^{\frac{\mathbf{k}+1}{2}m_1}\le K_p(p_{\gamma_1,\sigma_1})^{|\gamma_1|}\cdots(p_{\gamma_\ell,\sigma_\ell})z^{|\gamma_\ell|} \end{aligned} \tag{14}$$

for any subdivision $\tau=\gamma_1\cup\cdots\cup\gamma_\ell$, $\beta=\sigma_0\cup\cdots\cup\sigma_\ell$ *such that* $2-\ell\le|\sigma_0|$.

Then there is a constant $M=M_{\tau,\beta}\in\mathbb{R}^1$ *such that the* **quasi-contractive nonlinear estimate** *holds*

$$\rho_{\tau,\beta}(t)\le e^{Mt}\rho_{\tau,\beta}(0). \tag{15}$$

The r.h.s. limit $t=0$ *is substituted by Theorem 2.*

Proof. If we introduce function $h_i(t)=\sum\limits_{\ell=0}^{i}\sum\limits_{\substack{\sigma\subset\beta\\|\sigma|=\ell}}\sum\limits_{\substack{\gamma\subset\tau\\\gamma\ne\varnothing}}\mathbf{E}\,p_{\gamma,\sigma}(\|\xi^0(t,x^0)\|^2_{\ell_2(a)})\left\|\frac{\mathbb{D}^\sigma\xi_\gamma}{t^{|\sigma|}}\right\|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})}$ then it is clear that for estimate (15) it is sufficient to prove that for any $i\in\{0,\ldots,m\}$ $h_i(t)\le e^{M_it}h_i(0)$ or an estimate

$$g_\sigma(t)\le e^{K_1t}g_\sigma(0)+K_2\int_0^t e^{K_1(t-s)}h_{i-1}(s)ds$$

for function $g_\sigma(t)=\sum\limits_{\gamma\subset\tau,\,\gamma\ne\varnothing}\mathbf{E}\,p_{\gamma,\sigma}(\|\xi^0(t,x^0)\|^2_{\ell_2(a)})\left\|\frac{\mathbb{D}^\sigma\xi_\gamma}{t^{|\sigma|}}\right\|^{m_\gamma}_{\ell_{m_\gamma}(c_{\gamma,\sigma})}$.

The proof is based on Ito formula, estimate $H_\mu p_{\gamma,\sigma}(z)\ge K_{\gamma,\sigma}p_{\gamma,\sigma}(z)$, $z\in\mathbb{R}_+$ (see [5, Hint 9]) and Young inequality. ■

The following Proposition is an immediate consequence of Theorem 3.

Proposition 1. *Let F fulfill* (3), $\psi \in \mathbb{P}$ *and function* $Q \in C^2(\mathbb{R}^1)$ *satisfy* $(14)_1$. *Then* $\forall\, n \in \mathbb{N}$ $\exists\, M = M_n(\psi, Q)$ *such that for any* $t \in [0,T]$ *the estimate holds*

$$\mathbf{E}\, Q(\|\xi^0(t,x^0)\|^2_{\ell_2(a)})\, |\mathbb{D}^\beta \xi_{k,\tau}|^{m_\tau} \le \frac{t^{|\beta| m_\tau} e^{Mt} |\tau| \psi_0 Q(\|x^0\|^2_{\ell_2(a)}) (1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2} m_\tau (|\tau|+|\beta|-1)}}{a_k^{\frac{\mathbf{k}+1}{2} m_\tau (|\tau|-1)} \prod\limits_{i\in\beta} a_i^{\frac{\mathbf{k}+1}{2} m_\tau} \prod\limits_{j\in\tau} \psi_{k-j}^{m_1/|\tau|}}$$

for all $1 \le m_1 \le n$, $|\tau| \le m_1$, $|\beta| \le n$ *and* $m_\tau = m_1/|\tau|$.

3 Smoothing properties of semigroups

Proposition 1 shows that for $\mathbb{D}^\beta \xi_\gamma$ there is a *certain ordering of behaviour* with respect to $\beta, \gamma \subset \mathbb{Z}^d$, generated by weights $\left((1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}, \{a_k^{-\frac{\mathbf{k}+1}{2}}\}_{k\in\mathbb{Z}^d} \right)$. Due to the representation (11) it influences corresponding relations between different order derivatives of semigroup P_t and therefore requires a reduction of weights in seminorms on partial derivatives $\partial_\tau P_t f$ in C_Θ, which we are going to introduce.

Let $\Theta = \Theta_0 \cup \cdots \cup \Theta_n$ denote any array of pairs $(p, \mathcal{G}) \in \Theta_i$ with i-tensor $\mathcal{G} = G^1 \otimes \cdots \otimes G^i$, constructed by vectors $G^1, \dots, G^i \in \mathbb{P}$, and monotone functions $p \in C^2(\mathbb{R}^1_+)$ with property $(14)_1$. Array Θ_0 should consist of pairs $(p, \varnothing)$ with empty tensor $\varnothing$ such that $\varnothing \otimes G = G \otimes \varnothing = G$, $G \in \mathbb{P}$. The array $\Theta = \Theta_0 \cup \cdots \cup \Theta_n$, $n \in \mathbb{N}$, is *quasi-contractive with parameter* $\mathbf{k}$ iff for all $(p, \mathcal{G}) \in \Theta_i$, $i = 2, \dots, n$, and all pair of indexes $k, j \in \{1, \dots, i\}$, $k \ne j$, there is $(\widetilde{p}, \widetilde{\mathcal{G}}) \in \Theta_{i-1}$ such that $\exists\, K > 0$ $\forall\, z \in \mathbb{R}_+$

$$(1+z)^{\frac{\mathbf{k}+1}{2}} \widetilde{p}(z) \le K p(z) \qquad \textit{and} \qquad (\widehat{\mathcal{G}}^{\{k,j\}})^\ell \le K \widetilde{\mathcal{G}}^\ell, \qquad \ell = 1, \dots, i-1. \tag{16}$$

Above for tensor $\mathcal{G} = G^1 \otimes \cdots \otimes G^i$, we used notation $\mathcal{G}^i = G^i$ and $(i-1)$-tensor $\widehat{\mathcal{G}}^{\{k,j\}}$ is constructed from i-tensor $\mathcal{G} = G^1 \otimes \cdots \otimes G^i$ by the rule $\widehat{\mathcal{G}}^{\{k,j\}} = G^1 \otimes \cdots \otimes G^{k-1} \otimes G^{k+1} \otimes \cdots \otimes G^{j-1} \otimes A^{-(\mathbf{k}+1)} G^k G^j \otimes G^{j+1} \otimes \cdots \otimes G^i$ with $A^{-(\mathbf{k}+1)} = \{a_k^{-(\mathbf{k}+1)}\}_{k\in\mathbb{Z}^d}$.

Definition 3. We say that $f \in C_\Theta(\ell_2(a))$, $\Theta = \Theta_0 \cup \cdots \cup \Theta_n$ iff $f \in C(\ell_2(a))$ and $\forall\, \tau = \{k_1, \dots, k_i\}$, $|\tau| \le n$ there are partial derivatives $\partial_\tau f = \partial_{k_1} \cdots \partial_{k_i} f \in C(\ell_2(a))$ such that the norm $\|f\|_\Theta = \max\limits_{i=0,\dots,n} |||\partial^{(i)} f|||_{\Theta_i}$ is finite, where

$$|||\partial^{(i)} f|||_{\Theta_i} = \sup_{x\in\ell_2(a)} \max_{(p,\mathcal{G})\in\Theta_i} \frac{|\partial^{(i)} f(x)|_{\mathcal{G}}}{p(\|x\|^2_{\ell_2(a)})}, \qquad |\partial^{(i)} f(x)|^2_{\mathcal{G}} = \sum_{\tau=\{k_1,\dots,k_i\}\in\mathbb{Z}^d} G^1_{k_1} \dots G^i_{k_i} |\partial_\tau f(x)|^2.$$

Above partial derivatives $\partial_\tau f$ are understood in the sense that $\forall\, x^0 \in \ell_2(a)$ $\forall\, h \in \mathbf{X}_\infty([a,b]) = \bigcap\limits_{p\ge1,\, c\in\mathbb{P}} AC_\infty([a,b], \ell_p(c))$ representations hold

$$\forall\, |\tau| = 0, \dots, n-1 \qquad \partial_\tau f(x^0 + h(\bullet))\Big|_a^b = \int_a^b \sum_{k\in\mathbb{Z}^d} \partial_{\tau\cup\{k\}} f(x^0 + h(s)) h'_k(s) ds,$$

where we used notation $AC_\infty([a,b], X) = \{h \in C([a,b], X);\ \exists\, h' \in L^\infty([a,b], X)\}$ for Banach space X.

In [14] it was shown that the semigroup P_t preserves spaces $C_\Theta(\ell_2(a))$ and fulfills estimate $\exists\, M_\Theta, K_\Theta$: $\forall\, f \in C_\Theta(\ell_2(a))$

$$\|P_t f\|_{C_\Theta} \le K_\Theta e^{M_\Theta t} \|f\|_{C_\Theta} \tag{17}$$

if the array Θ is quasi-contractive with parameter $\mathbf{k}$.

Introduce $T_{\mathbf{k}}\Theta = \{(\,(1+z^2)^{\frac{\mathbf{k}+1}{2}}p(z),\, \mathrm{sym}(\mathcal{G}\otimes^{\mathbf{k}+2})\,);\ (p,\mathcal{G})\in\Theta\}$ and denote $(\Theta)^m = \bigcup\limits_{i=0}^{m} T^i_{\mathbf{k}}\Theta$, $(\Theta)^0=\Theta$. Remark that for quasi-contractive with parameter $\mathbf{k}$ array Θ the array $(\Theta)^m$ is also quasi-contractive. This follows from $(\Theta)^i = (\Theta)^{i-1}\cup T_{\mathbf{k}}(\Theta)^{i-1}$ and ordering (16).

Next Theorem gives the raise of smoothness in scale C_Θ under the action of semigroup P_t. Denote by $\mathcal{D}_\Theta$ the closure in C_Θ of $f\in\mathcal{P}^\infty_{cyl}(\ell_2(a))$ such that $\|f\|_{C_\Theta}<\infty$.

Theorem 4. *Let Θ be quasi-contractive array with parameter $\mathbf{k}$ (3). Then $\forall\, m\geq 1\ \exists\, K_{\Theta,m}$, $M_{\Theta,m}$ such that $\forall\, f\in\mathcal{D}_\Theta$ we have $P_tf\in C_{(\Theta)^m}$, $t>0$ and*

$$\|P_tf\|_{C_{(\Theta)^m}}\leq \frac{1}{t^{m/2}}K_{\Theta,m}e^{M_{\Theta,m}t}\|f\|_{C_\Theta},\qquad t>0. \tag{18}$$

Proof. Let $\Theta=\Theta_0\cup\cdots\cup\Theta_n$ be a quasi-contractive array. Consider $f\in\mathcal{P}^\infty_{\mathrm{cyl}}(\ell_2(a))$ such that $\|f\|_{C_\Theta}<\infty$. It is not difficult to see that for (18) it is sufficient to prove: $\forall\, i\in\{0,\dots,n\}\ \forall\, t>0$

$$|||\partial^{(i+1)}P_tf|||_{T_{\mathbf{k}}\Theta_i}\leq\frac{1}{\sqrt{t}}Ke^{Mt}\|f\|_{C_\Theta} \tag{19}$$

or using $P_tf=P_{t/2}P_{t/2}f$ it can be shown inductively that (19) follows from: $\forall\, i=0,\dots,n$ and $\Theta_i\in\Theta$:

$$|||\partial^{(i+1)}P_tf|||_{T_{\mathbf{k}}\Theta_i}\leq Ke^{Mt}\left\{\frac{1}{\sqrt{t}}|||\partial^{(i)}f|||_{\Theta_i}+\max_{\ell=1,\dots,i}(|||\partial^{(\ell)}f|||_{\Theta_\ell},|||\partial^{(\ell)}f|||_{T_{\mathbf{k}}\Theta_{\ell-1}})\right\}.$$

Due to integration-by-parts formula (10) we have

$$\partial_{k_{i+1}}\cdots\partial_{k_1}P_tf=\sum_{\ell=1}^{i}\sum_{\gamma_1\cup\cdots\cup\gamma_\ell=\{k_1,\dots,k_{i+1}\}}\mathbf{E}\,\langle\partial^{(\ell)}f(\xi^0),\xi_{\gamma_1}\otimes\cdots\otimes\xi_{\gamma_\ell}\rangle \tag{20}$$

$$+\frac{1}{t}\sum_{j_1,\dots,j_{i+1}\in\mathbb{Z}^d}\mathbf{E}\,\partial_{j_i}\cdots\partial_{j_1}f(\xi^0)\xi_{j_1,k_1}\cdots\xi_{j_{i+1},k_{i+1}}\int_0^t\langle\Gamma_se_{j_{i+1}},dW(s)\rangle_{\ell_2(1)} \tag{21}$$

$$-\frac{1}{t}\sum_{\ell=1}^{i+1}\sum_{j_1,\dots,j_{i+1}\in\mathbb{Z}^d}\mathbf{E}\,\partial_{j_i}\cdots\partial_{j_1}f(\xi^0)\xi_{j_1,k_1}\cdots\mathbb{D}^{j_{i+1}}\xi_{j_\ell,k_\ell}\cdots\xi_{j_{i+1},k_{i+1}}. \tag{22}$$

The following estimates finish the proof:

$$\begin{aligned}
&|||(20)|||_{T_{\mathbf{k}}\Theta_i}\leq Ke^{Mt}\max_{\ell=1,\dots,i}(|||\partial^{(\ell)}f|||_{\Theta_\ell},|||\partial^{(\ell)}f|||_{T_{\mathbf{k}}\Theta_{\ell-1}})\\
&|||(21)|||_{T_{\mathbf{k}}(p,\mathcal{G})}\leq\frac{1}{\sqrt{t}}Ke^{Mt}|||\partial^{(i)}f|||_{\Theta_i},\\
&|||(22)|||_{T_{\mathbf{k}}(p,\mathcal{G})}\leq Ke^{Mt}|||\partial^{(i)}f|||_{\Theta_i}.
\end{aligned} \tag{23}$$

Estimate (23_1) follows from (17). To obtain (23_2) we apply the Hölder inequality

$$|||(21)|||_{T_{\mathbf{k}}(p,\mathcal{G})}\leq\frac{\left|\sum\limits_{j_1,\dots,j_{i+1}\in\mathbb{Z}^d}\left(\mathbf{E}\,\frac{|\partial_{j_i}\cdots\partial_{j_1}f(\xi^0)|^2}{p^2(z_t)}\right)^{1/2}A_{j_1,\dots,j_{i+1}}\right|_{\mathcal{G}\otimes A^{\mathbf{k}+2}}}{t\,(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}p(\|x^0\|^2_{\ell_2(a)})} \tag{24}$$

with $z_t=\|\xi^0(t,x^0)\|^2_{\ell_2(a)}$ and

$$A^{k_1,\dots,k_{i+1}}_{j_1,\dots,j_{i+1}}=\left(\mathbf{E}\,p^2(z_t)\,|\xi_{j_1,k_1}\cdots\xi_{j_{i+1},k_{i+1}}\int_0^t\langle\Gamma_se_{j_{i+1}},dW(s)\rangle_{\ell_2(1)}|^2\right)^{1/2}.$$

A simple consequence of Ito formula gives

$$\left(\mathbf{E}\,(\int_0^t \langle \Gamma_s e_{j_{i+1}}, dW(s)\rangle_{\ell_2(1)})^{2(i+2)}\right)^{1/2(i+2)} \le Ke^{Mt}\sqrt{t}a_{j_{i+1}}^{-\frac{\mathbf{k}+1}{2}}(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}, \tag{25}$$

the Hölder inequality with $q_\ell = 1/(i+2)$, (25) and Proposition 1 with $|\beta| = 0$, $\tau = \{j\}$ imply

$$\begin{aligned} A^{k_1,\dots,k_{i+1}}_{j_1,\dots,j_{i+1}} &\le \prod_{\ell=1}^{i+1}\left(\mathbf{E}\,p^{2\frac{i+2}{i+1}}|\xi_{j_\ell,k_\ell}|^{2(i+2)}\right)^{1/2(i+2)}\left\{\mathbf{E}\left(\int_0^t\langle\Gamma_s e_{j_{i+1}}, dW\rangle\right)^{2(i+2)}\right\}^{1/2(i+2)} \\ &\le Ke^{Mt}\sqrt{t}(1+\|x^0\|^2_{\ell_2(a)})^{\frac{\mathbf{k}+1}{2}}p(\|x^0\|^2_{\ell_2(a)})\,\frac{a_{j_{i+1}}^{-\frac{\mathbf{k}+1}{2}}}{\prod_{\ell=1}^{i+1}\psi_{j_\ell-k_\ell}}. \end{aligned} \tag{26}$$

Substituting (26) in (24) we obtain

$$\begin{aligned} |||(21)|||_{T_{\mathbf{k}}(p,\mathcal{G})} &\le \frac{Ke^{Mt}}{\sqrt{t}}(\operatorname{tr} a)^{1/2} \\ &\times \sum_{j\in\mathbb{Z}^d}\frac{\delta_a^{\frac{\mathbf{k}+1}{2}|j|}}{\psi_j}\left(\sum_{k_1,\dots,k_i}G^1_{k_1}\cdots G^i_{k_i}\,|\sum_{j_1,\dots,j_i}\left(\mathbf{E}\,\frac{|\partial_{j_i}\cdots\partial_{j_1}f(\xi^0)|^2}{p^2(z_t)}\right)^{1/2}\prod_{\ell=1}^{i}\frac{1}{\psi_{j_\ell-k_\ell}}|^2\right)^{1/2} \\ &\le \frac{K'e^{Mt}}{\sqrt{t}}K_{G,\psi}|||\partial^{(i)}f|||_{(p,\mathcal{G})} \le \frac{K'e^{Mt}}{\sqrt{t}}K_{G,\psi}|||\partial^{(i)}f|||_{\Theta_i}. \end{aligned}$$

Estimate (23_3) can be done in a similar way. ■

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Enhanced Binding in a Model of an Abstract Quantum System Coupled to a Multi-Component Bose Field

Asao ARAI

Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
E-mail: *arai@math.sci.hokudai.ac.jp*

A model of an abstract quantum system coupled to a multi-component Bose field is considered. Under suitable hypotheses, the model has a ground state even if the uncoupled (zero-coupling) system has no ground state.

1 Introduction

A quantum system which has no ground states may be changed, through coupling to a quantum field, to have a ground state. If such a change occurs, then we say that the coupled system has *enhanced binding* with respect to ground state. The phenomenon of enhanced binding, if it occurs, may be regarded as one of the evidences supporting the view point that quantum fields are more fundamental objects underlying the material world.

From this point of view, as well as from a purely mathematical one, it is interesting to clarify whether or not enhanced binding *indeed* occurs in models of a quantum system – typically a system of nonrelativistic quantum particles – coupled to a quantum field.

The problem of enhanced binding was first discussed by Hiroshima and Spohn [11]. They discussed the Pauli–Fierz model in nonrelativistic quantum electrodynamics in the dipole approximation and proved that, under suitable hypotheses, enhanced binding occurs for large coupling constants. Hainzl, Vougalter and Vugalter [10] considered the Pauli–Fierz model without the dipole approximation showing that it has enhanced binding for small coupling constants. The results and the methods in [10] have been extended to the Pauli–Fierz model with spin [7,8] (cf. also [9]).

In a previous paper [6] the enhanced binding problem was considered for a general class of quantum field models, called the generalized spin-boson (GSB) model which describes an abstract quantum system coupled linearly to a Bose field [3–5], and proved, under suitable hypotheses, the existence of enhanced binding for a region of coupling constants. The GSB model was extended to a more general one in [2], whose Hamiltonian is obtained by adding quadratic self-interaction terms of the Bose field to the Hamiltonian of the GSB model, and it was shown that results similar to those in [2] hold also in the extended GSB model.

In this paper we consider a slightly more general model than the extended GSB model in [2] and show that, under suitable hypotheses, enhanced binding occurs in this model too.

The present paper is organized as follows. Section 2 is a preliminary section which recalls basic objects and elementary facts in the theory of the abstract boson Fock space. In Section 3 we define quadratic operators acting in the abstract boson Fock space. In Section 4 we describe the model considered in the present paper. The main theorems are stated in Section 5. The last section is devoted to sketches of proofs of them.

2 Bose fields

We denote the inner product and the norm of a Hilbert space $\mathcal{X}$ by $\langle \cdot, \cdot \rangle_{\mathcal{X}}$ and $\| \cdot \|_{\mathcal{X}}$ respectively, where we use the convention that the inner product is antilinear (resp. linear) in the first (resp.

second) variable. We sometimes omit the subscript $\mathcal{X}$ in $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{X}}$, if there is no danger of confusion.

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. For a subspace $D \subset D(T)$, $T|D$ denotes the restriction of T to D. If T is densely defined, then the adjoint of T is denoted T^*. For linear operators S and T on a Hilbert space, $D(S+T) := D(S) \cap D(T)$ unless otherwise stated.

For each complex Hilbert space $\mathcal{X}$, the boson Fock space $\mathcal{F}_{\rm b}(\mathcal{X})$ over $\mathcal{X}$ is defined by

$$\mathcal{F}_{\rm b}(\mathcal{X}) := \oplus_{n=0}^{\infty} \otimes_{\rm s}^{n} \mathcal{X},$$

where $\otimes_{\rm s}^{n}\mathcal{X}$ denotes the n-fold symmetric tensor product of $\mathcal{X}$ with $\otimes_{\rm s}^{0}\mathcal{X} := \mathbb{C}$ (the set of complex numbers).

The annihilation operator $a(f)$ ($f \in \mathcal{X}$) on $\mathcal{F}_{\rm b}(\mathcal{X})$ is defined to be a densely defined closed linear operator such that, for all $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in D(a(f)^*)$, $(a(f)^*\psi)^{(0)} = 0$ and

$$(a(f)^*\psi)^{(n)} = \sqrt{n} S_n \left(f \otimes \psi^{(n-1)} \right), \qquad n \geq 1,$$

where S_n is the symmetrization operator on $\otimes^n \mathcal{X}$. The adjoint $a(f)^*$, called the creation operator, and the annihilation operator $a(g)$ ($g \in \mathcal{X}$) obey the canonical commutation relations

$$[a(f), a(g)^*] = \langle f, g\rangle_{\mathcal{X}}, \qquad [a(f), a(g)] = 0, \qquad [a(f)^*, a(g)^*] = 0$$

for all $f, g \in \mathcal{X}$ on the dense subspace

$$\mathcal{F}_0(\mathcal{X}) := \{\psi \in \mathcal{F}_{\rm b}(\mathcal{X}) \,|\, \text{there exists a number } n_0 \text{ such that } \psi^{(n)} = 0 \text{ for all } n \geq n_0\},$$

where $[X, Y] := XY - YX$.

We denote by $\Omega := \{1, 0, 0, \ldots\}$ the Fock vacuum in $\mathcal{F}_{\rm b}(\mathcal{X})$. For a subspace $\mathcal{D} \subset \mathcal{X}$ we define a subspace

$$\mathcal{F}_{\rm fin}(\mathcal{D}) := \mathcal{L}\left(\{\Omega,\, a(f_1)^* \cdots a(f_n)^*\Omega \,|\, n \in \mathbb{N}, f_j \in \mathcal{D}, j = 1, \ldots, n\}\right),$$

where $\mathcal{L}(\{\cdot\})$ means the subspace algebraically spanned by the set $\{\cdot\}$. If $\mathcal{D}$ is dense in $\mathcal{X}$, then the subspace $\mathcal{F}_{\rm fin}(\mathcal{D})$ is dense in $\mathcal{F}_{\rm b}(\mathcal{X})$.

Let

$$\phi(f) := \frac{a(f) + a(f)^*}{\sqrt{2}}, \qquad f \in \mathcal{X},$$

which is called the Segal field operator. It is shown that $\phi(f)$ is essentially self-adjoint on $\mathcal{F}_0(\mathcal{X})$ [13, § X.7]. *We denote its closure by the same symbol* $\phi(f)$. The "conjugate momentum" of $\phi(f)$ is defined by

$$\pi(f) := \phi(if), \qquad f \in \mathcal{X}.$$

We have

$$[\phi(f), \pi(g)] = i\mathrm{Im}\left(i\langle f, g\rangle_{\mathcal{X}}\right).$$

For every symmetric operator S on $\mathcal{X}$, one can define a closed symmetric operator $d\Gamma(S)$, called the second quantization of S, by

$$d\Gamma(S) := \oplus_{n=0}^{\infty} S^{(n)},$$

with $S^{(0)} = 0$ and $S^{(n)}$ is the closure of

$$\left(\sum_{j=1}^{n} I \otimes \cdots \otimes \overset{j\text{th}}{\overbrace{S}} \otimes \cdots \otimes I \right) \Bigg| \otimes_{\rm alg}^{n} D(S),$$

where I denotes identity and $\otimes_{\rm alg}^{n}$ algebraic tensor product. If S is self-adjoint, then so is $d\Gamma(S)$.

3 Quadratic operators

In the boson Fock space $\mathcal{F}_{\rm b}(\mathcal{X})$, one can define, in a general manner, linear operators quadratic in the annihilation and the creation operators. Let C be a conjugation on $\mathcal{X}$, i.e., C is an anti-linear, isometric mapping on $\mathcal{X}$ such that $C^2 = I$. Let K be a Hilbert–Schmidt operator on $\mathcal{X}$. Then there exist orthonormal sets $\{f_n\}_{n=1}^M$, $\{g_n\}_{n=1}^M$ in $\mathcal{X}$ and a sequence $\{\lambda_n\}_{n=1}^M$ of positive numbers ($M < \infty$ or M is countably infinite) such that $\sum_{n=1}^{M} \lambda_n^2 < \infty$ and

$$K = \sum_{n=1}^{M} \lambda_n \langle f_n, \cdot \rangle_{\mathcal{X}} g_n,$$

where, in the case $M = \infty$, the sum on the right hand side converges in operator norm [12, Theorems VI.17 and VI.22]. Using this representation of K, we define linear operators $(a^*|K|a^*)_N$ and $(a|K|a)_N$ ($N \in \mathbb{N}$) acting in $\mathcal{F}_{\rm b}(\mathcal{X})$, with domain $\mathcal{F}_{\rm fin}(\mathcal{X})$, by

$$(a^*|K|a^*)_N := \sum_{n=1}^{N\wedge M} \lambda_n a(Cf_n)^* a(g_n)^*, \qquad (a|K|a)_N := \sum_{n=1}^{N\wedge M} \lambda_n a(f_n) a(Cg_n).$$

where $N \wedge M := \min\{N, M\}$ if $M < \infty$ and $N \wedge M := N$ if $M = \infty$. It is easy to see that, for all $\Psi \in \mathcal{F}_{\rm fin}(\mathcal{X})$, the strong limit

$$(a^\#|K|a^\#)\Psi := \text{s-}\lim_{N\to\infty} (a^\#|K|a^\#)_N \Psi$$

exists, where $a^\#$ denotes either a^* or a. Moreover, the operator $(a^\#|K|a^\#)$ with domain $\mathcal{F}_{\rm fin}(\mathcal{X})$ is closable and

$$(a^*|K|a^*)^* \supset (a|K^*|a), \qquad (a|K|a)^* \supset (a^*|K^*|a^*).$$

4 Definition of the model

We consider a model of an abstract quantum system S coupled to an N-component Bose field over $\mathbb{R}^d$ ($d, N \in \mathbb{N}$). We denote the Hilbert space of the system S by $\mathcal{H}$, which is taken to be an arbitrary separable complex Hilbert space. In concrete realizations, S may be a system of nonrelativistic quantum particles or a quantum field system.

The one-particle Hilbert space of the Bose field is taken to be

$$\mathcal{M} := \oplus^N L^2\big(\mathbb{R}^d\big),$$

the N direct sum of $L^2\big(\mathbb{R}^d\big)$. Then the Hilbert space for the Bose field is given by the Fock space $\mathcal{F}_{\rm b}(\mathcal{M})$ over $\mathcal{M}$.

Let μ be a Borel measurable function on $\mathbb{R}^d$ such that $0 < \mu(k) < \infty$ for almost everywhere (a.e.) $k \in \mathbb{R}^d$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Then μ defines a multiplication operator on $L^2\big(\mathbb{R}^d\big)$, which is nonnegative, injective and self-adjoint. We denote it by the same symbol. We define an operator

$$\widehat{\mu} := \oplus^N \mu$$

acting in $\mathcal{M}$.

The Hilbert space of the coupled system of S and the Bose field is given by the tensor product

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\rm b}(\mathcal{M}).$$

Let A be a self-adjoint operator on $\mathcal{H}$, which denotes physically the Hamiltonian of the quantum system S.

We say that a densely defined linear operator on a Hilbert space is Hilbert–Schmidt if it is bounded and its closure is Hilbert–Schmidt. We denote the closure by the same symbol.

In what follows, we denote by C the complex conjugation on $\mathcal{M}$. For a bounded linear operator T on $\mathcal{M}$, we define T_C by

$$T_C := CTC.$$

Let S and T be bounded linear operators on $\mathcal{M}$. We assume the following.

Hypothesis (I).

(I.1) The operators $T\widehat{\mu}S^*$ and $T\widehat{\mu}^{1/2}$ are densely defined and Hilbert–Schmidt.

(I.2) The operator $S\widehat{\mu}S^* + T_C\widehat{\mu}T_C^*$ is densely defined.

Then we can define an operator $H_{\rm b}(\mu, S, T)$ by

$$H_{\rm b}(\mu, S, T) := d\Gamma(S\widehat{\mu}S^* + T_C\widehat{\mu}T_C^*) + (a|T\widehat{\mu}S^*|a) + (a|T\widehat{\mu}S^*|a)^*.$$

It is easy to see that $H_{\rm b}(\mu, S, T)$ is a symmetric operator with

$$D(H_{\rm b}(\mu, S, T)) \supset \mathcal{F}_{\rm fin}\left(D(S\widehat{\mu}S^*) \cap D(T_C\widehat{\mu}T_C^*)\right).$$

We denote the closure of $H_{\rm b}(\mu, S, T)$ by $\bar{H}_{\rm b}(\mu, S, T)$.

The Hamiltonian of the model we consider in the present paper is defined by

$$H := A \otimes I + I \otimes \bar{H}_{\rm b}(\mu, S, T) + \sum_{j=1}^{J} B_j \otimes \phi(g_j) + \sum_{j=1}^{J} K_j \otimes \pi(h_j),$$

where B_j $(j = 1, \ldots, J;\ J \in \mathbb{N})$ is a symmetric operator on $\mathcal{H}$ such that $\cap_{j=1}^{J} D(B_j)$ is dense in $\mathcal{H}$, K_j $(j = 1, \ldots, J)$ is a bounded self-adjoint operator on $\mathcal{H}$ and $g_j, h_j \in \mathcal{M}$, $j = 1, \ldots, J$.

Remark 1. (i) The operator H with $T = 0$, $h_j = 0$ (or $K_j = 0$) and $S\widehat{\mu}S^* = \widehat{\omega} := \oplus^N \omega$ (ω is a nonnegative Borel measurable function on $\mathbb{R}^d$) yields

$$H_{\rm AH} := A \otimes I + I \otimes d\Gamma(\widehat{\omega}) + \sum_{j=1}^{J} B_j \otimes \phi(g_j).$$

This is the Hamiltonian of the GSB model [3]. The existence of ground states of $H_{\rm AH}$ with $N = 1$ was discussed in [3] under the assumption that A has a ground state (cf. also [4] for further extensions). The problem of enhanced binding in the GSB model was considered in [6]. For the absence of ground states of $H_{\rm AH}$, see [5].

(ii) The problem of enhanced binding in the model H with $h_j = 0$ $(j = 1, \ldots, J)$ was discussed in [2]. The results presented below are extensions of those in [2] to the case where $h_j \neq 0$.

Example 1. The Pauli–Fierz Hamiltonian in the dipole approximation [11] is a special case of H with $d = 3$, $N = 2$, $J = 3$. In this sense our model is an abstract generalization of the Pauli–Fierz model in the dipole approximation.

5 Main results

For a self-adjoint operator L on a Hilbert space, we denote its spectrum (resp. essential spectrum) by $\sigma(L)$ (resp. $\sigma_{\rm ess}(L)$).

Definition 1. Let L be a self-adjoint operator on a Hilbert space bounded from below and set

$$E_0(L) := \inf \sigma(L),$$

which is called the ground state energy of L. We say that L has a ground state if $E_0(L)$ is an eigenvalue of L. In that case, each non-zero vector in $\ker(L - E_0(L))$ is called a *ground state* of L.

To state the main results of this paper, we formulate additional hypotheses. For this purpose, we first recall an important notion on commutativity of self-adjoint operators:

Definition 2. We say that two self-adjoint operators S_1 and S_2 on a Hilbert space strongly commute (or S_1 strongly commutes with S_2) if their spectral measures commute.

A family of self-adjoint operators $\{S_j\}_{j=1}^n$ on a Hilbert space is said to be strongly commuting if S_j strongly commutes with S_l for all $j, l = 1, \ldots, n$ with $j \neq l$.

In what follows, we assume that A is of the form

$$A = A_0 + A_1$$

with A_0 a nonnegative self-adjoint operator and A_1 a symmetric operator on $\mathcal{H}$.

We introduce $\tilde{g}_j, \tilde{h}_j \in \mathcal{M}$ $(j = 1, \ldots, J)$ by

$$\tilde{g}_j := S^* g_j - T^* C g_j, \qquad \tilde{h}_j := S^* h_j - T^* C h_j,$$

where S, T and C are operators introduced in the preceding section.

Hypothesis (II). $\tilde{g}_j$, $\tilde{g}_j/\mu^{3/2}, \tilde{h}_j, \tilde{h}_j/\mu \in \mathcal{M}$ $(j = 1, \ldots, J)$ and

$$\langle \tilde{g}_j(k), \tilde{g}_l(k) \rangle_{\mathbb{C}^N}, \langle \tilde{g}_j(k), \tilde{h}_l(k) \rangle_{\mathbb{C}^N} \in \mathbb{R}, \text{ a.e. } k \in \mathbb{R}^d \ (j, l = 1, \ldots, J).$$

Remark 2. Hypothesis (II) implies that (i) $\{\phi(i\widetilde{g}_j/\mu)\}_{j=1}^J$ is a family of strongly commuting self-adjoint operators and each $\phi(i\widetilde{g}_j/\mu)$ strongly commutes with each $\pi(\widetilde{h}_l)$ $(j, l = 1, \ldots, J)$; (ii) $[\phi(\tilde{g}_j), \pi(\tilde{h}_l)] = i\langle \tilde{g}_j, \tilde{h}_l \rangle_{\mathcal{M}}$ on $\mathcal{F}_0(\mathcal{M})$.

Hypothesis (III). The operator A_1 is A_0-bounded, i.e., $D(A_0) \subset D(A_1)$ and there exist constants $a, b \geq 0$ such that, for all $u \in D(A_0)$,

$$\|A_1 u\|_{\mathcal{H}} \leq a \|A_0 u\|_{\mathcal{H}} + b \|u\|_{\mathcal{H}}.$$

Hypothesis (IV). The operator A_0 strongly commutes with each B_j $(j = 1, \ldots, J)$ and

$$D(A_0) \subset \cap_{j,l=1}^J D(B_j B_l).$$

Moreover, there exist constants $c_j, d_j \geq 0$ such that, for all $u \in D(A_0^{1/2})$,

$$\|B_j u\|_{\mathcal{H}} \leq c_j \|A_0^{1/2} u\|_{\mathcal{H}} + d_j \|u\|_{\mathcal{H}} \qquad (j = 1, \ldots, J).$$

Hypothesis (V). The set $\{B_j\}_{j=1}^J$ is a family of strongly commuting self-adjoint operators.

Hypothesis (VI). $D(A_0) \subset \cap_{j=1}^{J} D(B_j A_1) \cap D(A_1 B_j)$ and $[B_j, A_1]|D(A_0)$ is bounded ($j = 1, \ldots, J$). We denote the operator norm of $[B_j, A_1]$ by $\|[B_j, A_1]\|$.

We introduce an operator

$$R_B := \frac{1}{2} \sum_{j,l=1}^{J} \left\langle \frac{\tilde{g}_j}{\sqrt{\mu}}, \frac{\tilde{g}_l}{\sqrt{\mu}} \right\rangle_{\mathcal{M}} B_j B_l.$$

and define

$$A_{\mathrm{ren}} := A - R_B.$$

Under Hypotheses (II)–(IV), we have $D(A_{\mathrm{ren}}) = D(A_0)$.

Hypothesis (VII). The operator A_{ren} is self-adjoint and bounded from below.

Hypothesis (VIII).
(VIII.1) The operator T is Hilbert–Schmidt.
(VIII.2) The operators S and T satisfy the following relations:

$$S^*S - T^*T = I, \qquad SS^* - T_C T_C^* = I, \qquad ST^* = T_C S_C^*, \qquad S^*T_C = T^* S_C.$$

One can prove the following fact:

Theorem 1. *Assume Hypotheses* (I)–(VIII). *Then H is self-adjoint and bounded from below.*

We set

$$\mu_0 := \operatorname{ess.inf}_{k \in \mathbb{R}^d} \mu(k),$$

where ess. inf means essential infimum.

Theorem 2. *Assume Hypotheses* (I)–(VIII). *Suppose that*

$$\{\mu(k) | k \in \mathbb{R}^d\} = [\mu_0, \infty) \tag{1}$$

Then the following (i) *and* (ii) *hold.*
(i) *If $\mu_0 > 0$, then $[E_0(H) + \mu_0, \infty) \subset \sigma_{\mathrm{ess}}(H)$.*
(ii) *If $\mu_0 = 0$, then $\sigma(H) = [E_0(H), \infty)$.*

To establish an existence theorem of a ground state of H *without* the assumption that A has a ground state, we need additional conditions.

Hypothesis (IX). The function μ is continuous on $\mathbb{R}^d$ with

$$\lim_{|k| \to \infty} \mu(k) = \infty$$

and there exist constants $\gamma > 0$ and $c_0 > 0$ such that

$$|\mu(k) - \mu(k')| \leq c_0 |k - k'|^{\gamma} (1 + \mu(k) + \mu(k')), \qquad k, k' \in \mathbb{R}^d.$$

For $s \geq 0$, we introduce constants $C_s(\tilde{g}), D_s(\tilde{h})$ ($\tilde{g} := (\tilde{g}_1, \ldots, \tilde{g}_J), \tilde{h} := (\tilde{h}_1, \ldots, \tilde{h}_J)$) by

$$C_s(\tilde{g}) := \sqrt{2} \sum_{j=1}^{J} \|[B_j, A_1]\| \left\| \frac{\tilde{g}_j}{\mu^s} \right\|_{\mathcal{M}}, \qquad D_s(\tilde{h}) := \sqrt{2} \sum_{j=1}^{J} \|K_j\| \left\| \frac{\tilde{h}_j}{\mu^s} \right\|_{\mathcal{M}}.$$

Provided that $\tilde{g}_j/\mu^s \in \mathcal{M}$ and $\tilde{h}_j/\mu^s \in \mathcal{M}$ $(j = 1, \ldots, J)$ respectively. We define constants F_α $(\alpha = 1, 2, 3)$ by

$$F_1 := C_1(\tilde{g}) + D_0(\tilde{h}), \qquad F_2 := C_2(\tilde{g}) + \frac{1}{2}D_1(\tilde{h}), \qquad F_3 := C_{3/2}(\tilde{g}) + D_{1/2}(\tilde{h}).$$

We set

$$\Sigma(A_{\rm ren}) := \inf \sigma_{\rm ess}(A_{\rm ren}).$$

Generally speaking, the existence of a ground state of H may depend on whether μ_0 is positive or zero [5]. We first state a result on the existence of enhanced binding in the case $\mu_0 > 0$.

Theorem 3. (Enhanced binding in the case $\mu_0 > 0$). *Consider the case $\mu_0 > 0$. Assume Hypotheses* (I)–(IX) *and that*

$$\Sigma(A_{\rm ren}) - E_0(A_{\rm ren}) > \mu_0 + \frac{1}{2}F_3^2 + F_1. \tag{2}$$

Then H has purely discrete spectrum in the interval $[E_0(H), E_0(H) + \mu_0)$. In particular, H has has a ground state.

Remark 3. Condition (2) implies that $E_0(A_{\rm ren})$ is a discrete eigenvalue of $A_{\rm ren}$ and hence $A_{\rm ren}$ has a finite number of ground states. But *A does not necessarily have a ground state.*

Corollary 1. *Under the assumption of Theorem* 3 *and condition* (1)*,*

$$\sigma_{\rm ess}(H) = [E_0(H) + \mu_0, \infty).$$

Theorem 4. (Enhanced binding in the case $\mu_0 = 0$). *Consider the case $\mu_0 = 0$. Assume Hypotheses* (I)–(IX) *with $\tilde{g}_j/\mu^2 \in \mathcal{M}$ $(j = 1, \ldots, J)$ in addition. Suppose that*

$$\Sigma(A_{\rm ren}) - E_0(A_{\rm ren}) > \frac{1}{2}F_3^2 + F_1. \tag{3}$$

and

$$\frac{F_1^2}{\left[\Sigma(A_{\rm ren}) - E_0(H)\right]^2} + \left\{\frac{2F_1^2}{\left[\Sigma(A_{\rm ren}) - E_0(H)\right]^2} + 1\right\}\frac{1}{2}F_2^2 < 1. \tag{4}$$

Then H has a ground state.

Remark 4. In Theorems 3 and 4, *the existence of a ground state of A is not assumed.*

6 Proofs of the main theorems

We give only sketches of proofs of the main theorems stated in the preceding section.

6.1 Proof of Theorem 1

Lemma 1. *There exists a unitary operator W on $\mathcal{F}_{\rm b}(\mathcal{M})$ such that*

$$(I \otimes W)H(I \otimes W)^{-1} = H_1 - \|T\widehat{\mu}^{1/2}\|_{\rm HS}^2,$$

where

$$H_1 := A \otimes I + I \otimes d\Gamma(\widehat{\mu}) + \sum_{j=1}^{J} B_j \otimes \phi(\tilde{g}_j) + \sum_{j=1}^{J} K_j \otimes \pi(\tilde{h}_j)$$

and $\|\cdot\|_{\rm HS}$ denotes Hilbert–Schmidt norm.

Proof. Similar to the proof of [2, Lemma 12]. ■

We introduce a unitary operator

$$U := \prod_{j=1}^{J} e^{-iB_j \otimes \phi(i\tilde{g}_j/\mu)}.$$

Let

$$H_0 := A_{\mathrm{ren}} \otimes I + I \otimes d\Gamma(\widehat{\mu}),$$

$$V_1 := U(A_1 \otimes I)U^{-1} - A_1 \otimes I, \qquad V_2 := \sum_{j=1}^{J} \left(U(K_j \otimes I)U^{-1}\right) I \otimes \pi(\tilde{h}_j).$$

and

$$\widetilde{H}_1 := H_0 + V_1 + V_2.$$

Lemma 2. *Assume Hypotheses* (I)–(VIII). *Then* $UD(H_0) = D(H_0)$ *and, for all* $\Psi \in D(H_0)$,

$$UH_1U^{-1}\Psi = \widetilde{H}_1\Psi.$$

Proof. Similar to the proof of [6, Lemma 3.7]. ■

Using [6, Lemma 3.10] and the well-known estimates

$$\|a(f)\Psi\| \leq \left\| \frac{f}{\sqrt{\mu}} \right\|_{\mathcal{M}} \|d\Gamma(\widehat{\mu})^{1/2}\Psi\|, \qquad \|a(f)^*\Psi\| \leq \left\| \frac{f}{\sqrt{\mu}} \right\|_{\mathcal{M}} \|d\Gamma(\widehat{\mu})^{1/2}\Psi\| + \|f\|_{\mathcal{M}}\|\Psi\|$$

holding for all $\Psi \in D\left(d\Gamma(\widehat{\mu})^{1/2}\right)$ and $f, f/\sqrt{\mu} \in \mathcal{M}$, one can easily see that V_1 and V_2 are infinitesimally small with respect to H_0. Hence, by the Kato–Rellich theorem, $\widetilde{H}_1$ is self-adjoint with $D(\widetilde{H}_1) = D(H_0)$ and bounded from below. By this fact and Lemma 2, H_1 is self-adjoint with $D(H_1) = D(H_0)$ and bounded from below. Theorem 1 now follows from this fact and Lemma 1.

6.2 Proof of Theorem 2

By the self-adjointness of H (Theorem 1), H_1 and Lemma 1, we have

$$\sigma(H) = \sigma(H_1) - \|T\widehat{\mu}^{1/2}\|_{\mathrm{HS}}^2, \qquad \sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_1) - \|T\widehat{\mu}^{1/2}\|_{\mathrm{HS}}^2.$$

In particular

$$E_0(H) = E_0(H_1) - \|T\widehat{\mu}^{1/2}\|_{\mathrm{HS}}^2.$$

On the other hand, one can show in the same way as in [1, Theorem 3.3],

$$[E_0(H_1) + \mu_0, \infty) \subset \sigma_{\mathrm{ess}}(H_1) \qquad (\mu_0 > 0)$$

and

$$\sigma(H_1) = [E_0(H_1), \infty) \qquad (\mu_0 = 0).$$

These facts imply Theorem 2.

6.3 Proofs of Theorems 3 and 4

By Lemma 1 and Theorem 1, it is sufficient to prove that H_1 or $\widetilde{H}_1$ has a ground state. This is done by using Lemma 2. Indeed, one sees that the methods developed in [6] work in the present case too (in [6], $\widetilde{H}_1$ with $V_2 = 0$ is considered). This is due to the fact that the new perturbation term V_2 has properties similar to those of V_1, e.g.,

$$\|V_2\Psi\| \leq D_{1/2}\|I \otimes d\Gamma(\widehat{\mu})^{1/2}\Psi\| + \frac{1}{2}D_0\|\Psi\|, \qquad \Psi \in D\left(I \otimes d\Gamma\left(\widehat{\mu}\right)^{1/2}\right),$$

$$[V_2, I \otimes a(f)]\Phi = -\frac{i}{\sqrt{2}}\sum_{j=1}^{J} U(K_j \otimes I)U^{-1}\langle f, \tilde{h}_j\rangle_{\mathcal{M}}\Phi, \qquad \Phi \in D(I \otimes N_{\rm b}),$$

where $N_{\rm b} := d\Gamma(I)$ is the number operator on $\mathcal{F}_{\rm b}(\mathcal{M})$. It turns out that we need only to shift the constants $c_s(g)$ $(s = 1, 3/2, 2)$ used in Theorems 2.2 and 2.3 in [6], which yields conditions (2)–(4) in the present context.

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First Integrals/Invariants & Symmetries for Autonomous Difference Equations

Leon ARRIOLA

Department of Mathematical & Computer Sciences, University of Wisconsin–Whitewater, 800 West Main Street, Whitewater, WI 53190–1790, USA
E-mail: *arriolal@uww.edu*

For the autonomous kth order difference equation $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$, where $f \in C^1[U]$ with domain $U^{\text{open}} \subset \mathbb{R}^k$, a global first integral/invariant for this difference equation is a nonconstant function $H \in C^1[U]$, with $H : U \to \mathbb{R}$, which remains invariant on the forward orbit $\Gamma^+(x_0, x_1, \ldots, x_{k-1})$. If a first integral exists, then it satisfies a particular functional difference equation and a method of finding solutions is presented. Furthermore, the first integral is constant along the characteristic curves of the associated infinitesimal generator for the Lie group symmetries of the difference equation.

1 Introduction

Example 1. Consider the second order, autonomous difference equation

$$x_{n+2} = (x_{n+1})^2 + x_{n+1} - (x_n)^2 .$$

It will be useful to view this difference equation as the iteration of the mapping

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ f(x,y) \end{pmatrix},$$

where $x \leftrightarrow x_n$, $y \leftrightarrow x_{n+1}$, $z := f(x,y)$, and $f(x,y) = y^2 + y - x^2$. A first integral/invariant is a nonconstant mapping H, which remains invariant on the forward orbit, that is,

$$H(y,z) = H(x,y).$$

For this example, the function $H(x,y) := y - x^2$ is a first integral for this difference equation as is shown here

$$\begin{aligned} H(y,z) &= z - y^2 \\ &= \left(y^2 + y - x^2\right) - y^2 \\ &= H(x,y). \end{aligned}$$

In other words, a first integral is a conservation law for the difference equation.

In this paper, we discuss the existence and construction of first integrals/invariants and the relationship to the one parameter, Lie group symmetries of the difference equation.

2 Notation

Consider the autonomous, kth order ordinary difference equation

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \tag{1}$$

where $u_n := (x_n, x_{n+1}, \ldots, x_{n+k-1}) \in U^{\text{open}} \subset \mathbb{R}^k$ and $f \in C^1[U]$. The forward orbit Γ^+ of the initial condition $u_0 = (x_0, x_1, \ldots, x_{k-1})$ is the sequence $\Gamma^+(u_0) = \{x_0, x_1, x_2, \ldots\}$. The difference equation given in equation (1) can be viewed as the iteration of the mapping $F : \mathbb{R}^k \to \mathbb{R}^k$, where

$$F : \begin{pmatrix} x_n \\ x_{n+1} \\ \vdots \\ x_{n+k-1} \end{pmatrix} \longrightarrow \begin{pmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \end{pmatrix}, \tag{2}$$

that is,

$$u_{n+1} = F(u_n). \tag{3}$$

A first integral [1,2] is a nonconstant function $H \in C^1[U]$ which remains invariant on the forward orbit, that is,

$$H(x_{n+1}, x_{n+2}, \ldots, x_{n+k}) = H(x_n, x_{n+1}, \ldots, x_{n+k-1}), \qquad \forall\, n \in \mathbb{Z}^+,$$

or more concisely,

$$H(u_{n+1}) = H(u_n), \qquad \forall\, n \in \mathbb{Z}^+. \tag{4}$$

Let $D_j f := \partial f / \partial x_{n+j-1}$, for $j = 1, \ldots, k$, in which case the Jacobian of F is given by the companion matrix

$$F'(u_n) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 \\ D_1 f(u_n) & D_2 f(u_n) & D_3 f(u_n) & D_4 f(u_n) & D_5 f(u_n) & \cdots & D_k f(u_n) \end{pmatrix}. \tag{5}$$

Additionally, we will have need for the forward shift operator Ψ, which acts on functions $g \in C^1[U]$, where $g : \mathbb{R}^k \to \mathbb{R}$ and is defined as

$$\Psi\left[g(u_n)\right] := g(u_{n+1}). \tag{6}$$

Inductively, we define

$$\Psi^{n+1} := \Psi\left[\Psi^n\right], \tag{7}$$

where $\Psi^0 := I$ and where I is the identity operator. Using this notation, a first integral is defined by the condition

$$\Psi \circ H = H. \tag{8}$$

Lastly, define X to be the infinitesimal generator of the one parameter Lie group symmetry Φ as the operator [3–5]

$$X := \xi \cdot \nabla = \sum_{j=1}^{k} \xi_j D_j. \tag{9}$$

3 Determining equation of H

Lemma 1. *Let H be a first integral for $u_{n+1} = F(u_n)$, then*

$$\nabla H(u_{n+1}) = F'(u_n)^T \cdot \nabla H(F(u_n)). \tag{10}$$

Proof. Since H is a first integral where $H \in C^1[U]$, take the exterior derivative of equation (4), and use the chain rule to obtain the desired result. ■

Theorem 1. *The function*

$$\alpha := D_k H, \tag{11}$$

is a solution to the functional equation

$$\alpha = \sum_{j=0}^{k-1} \big(\Psi^{k-j-1} D_{j+1} f\big)\big(\Psi^{k-j}\alpha\big). \tag{12}$$

Proof. Expanding the result given in Lemma 1 and examining the coefficient functions of the differentials dx_{n+i-1}, for $i = 1, \ldots, k$, we obtain the alternative form

$$D_1 H = (D_1 f)(\Psi D_k H), \tag{13}$$
$$D_j H = \Psi D_{j-1} H + (D_j f)(\Psi D_k H), \tag{14}$$

for $j = 2, \ldots, k$. These k equations can be reduced to a single equation as follows. For $j = 2$, equation (14) becomes

$$D_2 H = \Psi D_1 H + (D_2 f)(\Psi D_k H).$$

Now take the forward shift of equation (13) to get

$$\Psi D_1 H = (\Psi D_1 f)(\Psi^2 D_k H),$$

and substitute into equation (14) to get

$$D_2 H = (\Psi D_1 f)(\Psi^2 D_k H) + (D_2 f)(\Psi D_k H),$$

in which case

$$\Psi D_2 H = (\Psi^2 D_1 f)(\Psi^3 D_k H) + (\Psi D_2 f)(\Psi^2 D_k H).$$

Similarly,

$$D_3 H = (\Psi^2 D_1 f)(\Psi^3 D_k H) + (\Psi D_2 f)(\Psi^2 D_k H) + (D_3 f)(\Psi D_k H),$$
$$\Psi D_3 H = (\Psi^3 D_1 f)(\Psi^4 D_k H) + (\Psi^2 D_2 f)(\Psi^3 D_k H) + (\Psi D_3 f)(\Psi^2 D_k H).$$

In general,

$$D_j H = \sum_{i=0}^{j-1} \big(\Psi^{j-i-1} D_{i+1} f\big)\big(\Psi^{j-i} D_j H\big), \tag{15}$$

and for $j = k$, we obtain the single functional equation

$$D_k H = \sum_{j=0}^{k-1} \big(\Psi^{k-j-1} D_{j+1} f\big)\big(\Psi^{k-j} D_k H\big). \tag{16}$$

Defining $\alpha = D_k H$ and substituting into equation (16) gives the desired result. ■

Assuming the functional equation given in equation (12) can be explicitly solved, then an autonomous first integral is given by

$$H = \sum_{j=1}^{k} \int D_j H \, dx_{n+j-1}. \tag{17}$$

With this result, we now provide some examples as to how to construct first integrals by finding solutions to equation (12).

Example 2. Recall that $x_{n+2} = (x_{n+1})^2 + x_{n+1} - (x_n)^2$ has a first integral given by $H(x_n, x_{n+1}) = x_{n+1} - (x_n)^2$. The functional equation given in equation (12) reduces to

$$-2y\alpha(z,u) + (2y+1)\,\alpha(y,z) - \alpha(x,y) = 0,$$

where $x = x_n$, $y = x_{n+1}$, $z = x_{n+2}$, and $u = x_{n+3}$. By inspection, a solution is $\alpha(x,y) = c$, where c is an arbitrary constant. Using the fact that $\alpha(x,y) = D_2 H = H_y(x,y)$, then a first integral is

$$H(x,y) = cy + \varphi(x),$$

where φ is an arbitrary function of x. Forcing the condition $H(x,y) = H(y,z)$ gives

$$\varphi(x) = -cx^2,$$

in which case a first integral is

$$H(x,y) = y - x^2,$$

as was found above.

This example is unusual in that a solution can be found by inspection. In general, this is not the case, and therefore we present a technique which can sometimes find solutions to nontrivial problems.

Example 3. Consider the difference equation $x_{n+2} = x_{n+1}\,(x_n + 1)\,/\,(x_{n+1} + 1)$. The function $\alpha = H_y(x,y)$ is a solution of the functional equation

$$\frac{z}{z+1}\alpha(z,u) + \frac{x+1}{(y+1)^2}\alpha(y,z) = \alpha(x,y),$$

where $z = f(x,y) = y(x+1)/(y+1)$ and $u = f(y,z)$. Define the extended solution space as

$$\Omega := \left\{ \alpha \;\middle|\; \frac{z}{z+1}\alpha(z,u) + \frac{x+1}{(y+1)^2}\alpha(y,z) = \alpha(x,y) \right\},$$

and the actual solution space as

$$\tilde{\Omega} := \left\{ \alpha \;\middle|\; \frac{z}{z+1}\alpha(z,u) + \frac{x+1}{(y+1)^2}\alpha(y,z) = \alpha(x,y),\ z = f(x,y),\ u = f(y,z) \right\},$$

in which case $\tilde{\Omega} \subseteq \Omega$. We seek candidates for solutions by finding solutions in Ω and checking whether they satisfy the condition $\Psi \circ H = H$. We begin by taking $\frac{\partial}{\partial x}\big|_{y,z,u \text{ fixed}}$ which gives

$$\alpha(y,z) = (y+1)^2 \alpha_x(x,y).$$

Next, taking $\frac{\partial}{\partial z}\big|_{x,y \text{ fixed}}$ gives $\alpha_z(y,z) = 0$, followed by a backward shift $\Psi^{-1}\alpha_z(y,z) = 0$, gives $\alpha_y(x,y) = 0$, which has solution $\alpha(x,y) = \eta(x)$. Therefore, $H_y(x,y) = \eta(x)$, and after integrating we find that

$$H(x,y) = y\,\eta(x) + \varphi(x),$$

for some arbitrary functions $\eta(x)$ and $\varphi(x)$. Forcing H to be a first integral means that

$$\begin{aligned} &H(y,z) = H(x,y),\\ &z\eta(y) + \varphi(y) = y\eta(x) + \varphi(x). \end{aligned}$$

Replacing $z \leftrightarrow y(x+1)/(y+1)$ gives

$$\frac{y(x+1)}{y+1}\eta(y) + \varphi(y) = y\eta(x) + \varphi(x).$$

Next, take $\frac{\partial}{\partial x}\big|_{y \text{ fixed}}$ followed by $\frac{\partial^2}{\partial y^2}\Big|_{x \text{ fixed}}$, we obtain the second order differential equation in the single variable y

$$y(y+1)^2\eta''(y) + 2(y+1)\eta'(y) - 2\eta(y) = 0,$$

which has solution $\eta(y) = c_1(y+1)$. If we force H to be a first integral, then

$$\begin{aligned} &H(y,z) = H(x,y),\\ &c_1 z(y+1) + \varphi(y) = c_1 y(x+1) + \varphi(x),\\ &c_1\frac{y(x+1)}{y+1}(y+1) + \varphi(y) = c_1 y(x+1) + \varphi(x),\\ &c_1 y(x+1) + \varphi(y) = c_1 y(x+1) + \varphi(x),\\ &\varphi(y) = \varphi(x), \end{aligned}$$

in which case $\varphi(x) = c_2$. It is easily verified that the function H as was found here, namely

$$H(x,y) = y(x+1)$$

is a first integral.

As this example shows, ad hoc methods such as cleverly choosing the sequence of differentiations, followed by checking to see if the condition $\Phi \circ H = H$ can be satisfied, can lead to finding a first integral. A future direction of work is to utilize symbolic packages such as *MATHEMATICA* in order to automate this procedure.

4 Symmetries & first integrals

Let us now examine the relationship between symmetries of a difference equation and first integrals. Recall that a symmetry Φ of a dynamical system F maps solutions into solutions, that is,

$$\begin{array}{ccc} U & \xrightarrow{F} & U \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ U & \xrightarrow[F]{} & U, \end{array}$$

in which case the commutator bracket is zero

$$[\Phi, F] = 0.$$

For autonomous ordinary difference equations, this means that

$$\begin{array}{ccc} u_n = F^n(u_0) & \xrightarrow{F} & u_{n+1} = F \circ F^n(u_0) \\ \Phi \downarrow & & \downarrow \Phi \\ \Phi \circ F^n(u_0) & \xrightarrow[F]{} & F \circ \Phi \circ F^n(u_0) = \Phi \circ F \circ F^n(u_0). \end{array}$$

In the case where we seek continuous one–parameter symmetries, that is, Lie group symmetries, then the associated generator is given by $X := \xi \cdot \nabla = \sum_{j=1}^{k} \xi_j D_j$, in which case

$$\Phi = I + \epsilon X + \mathcal{O}\left(\epsilon^2\right).$$

Consider the second order case

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ z = f(x,y) \end{pmatrix},$$

and let

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y} + \xi_3 \frac{\partial}{\partial z}$$

be the generator of the Lie group symmetry for $z = f(x,y)$, in which case

$$\xi_3 = f_x \xi_1 + f_y \xi_2. \tag{18}$$

Examining the characteristic equation of the generator X gives the relationships

$$\frac{dx}{\xi_1} = \frac{dy}{\xi_2} = \frac{f_x dx + f_y dy}{f_x \xi_1 + f_y \xi_2}, \tag{19}$$

which has solutions

$$\xi_2 dx - \xi_1 dy = 0. \tag{20}$$

We now show that if H exists and Φ is a Lie group symmetry of $z = f(x,y)$, then Φ is also a symmetry of H, in which case H remains constant along the characteristic curve

$$\frac{dy}{dx} = \frac{\xi_2}{\xi_1}. \tag{21}$$

Consider the one–parameter Lie group transformations

$$\begin{aligned} \hat{x} &= x + \epsilon\, \xi_1 + \mathcal{O}\big(\epsilon^2\big), \\ \hat{y} &= y + \epsilon\, \xi_2 + \mathcal{O}\big(\epsilon^2\big), \\ \hat{z} &= z + \epsilon\big(f_x \xi_1 + f_y \xi_2\big) + \mathcal{O}\big(\epsilon^2\big) \end{aligned}$$

and define

$$\Delta\mathcal{H}(x,y,z) := H(y,z) - H(x,y),$$

in which case

$$\begin{aligned}\Delta\mathcal{H}(\hat{x},\hat{y},\hat{z}) &= H(\hat{y},\hat{z}) - H(\hat{x},\hat{y})\\ &= H(y,z) - H(x,y) + \epsilon\Big[\big(f_x(x,y)H_z(y,z) - H_x(x,y)\big)\xi_1\\ &\quad + \big(H_y(y,z) + f_y(x,y)H_z(y,z) - H_y(x,y)\big)\xi_2\Big] + \mathcal{O}\big(\epsilon^2\big).\end{aligned}$$

Since H is a first integral, the determining equations given in equations (13) and (14) are

$$\begin{aligned}H_x(x,y) &= f_x(x,y)H_z(y,z),\\ H_y(x,y) &= H_y(y,z) + f_y(x,y)H_z(y,z),\end{aligned}$$

in which case, up to terms of $\mathcal{O}\big(\epsilon^2\big)$,

$$\Delta\mathcal{H}(\hat{x},\hat{y},\hat{z}) = 0,$$

and therefore Φ is also a symmetry of H. Since H exists, then $H(x,y) = c$, where c is a constant which only depends on the initial condition (x_0, x_1). Taking the exterior derivative gives the equation

$$H_x\,dx + H_y\,dy = 0. \tag{22}$$

Since the 1–forms dx and dy are not identically zero, equations (20) and (22) will have nontrivial solution provided the linear PDE

$$\xi_1 H_x + \xi_2 H_y = 0 \tag{23}$$

is satisfied. In other words, H remains constant along the characteristic curve given in equation (21).

Let us now discuss some implications of equation (23). If specific symmetries of $z = f(x,y)$ are desired, a first integral may be constructed by solving this PDE. In other words, the inverse problem can be studied, that is, given a set of Lie group symmetries, we want to find manifolds $z = f(x,y)$ and the associated first integral.

Example 4. Consider the group of translations defined as the one–parameter transformations

$$\begin{aligned}\hat{x} &= x + \epsilon a + \mathcal{O}\big(\epsilon^2\big),\\ \hat{y} &= y + \epsilon b + \mathcal{O}\big(\epsilon^2\big),\\ \hat{z} &= z + \epsilon(af_x + bf_y) + \mathcal{O}\big(\epsilon^2\big),\end{aligned}$$

where f unspecified. Integrating equation (23) gives

$$H(x,y) = \eta(r),$$

where η is an arbitrary smooth function and

$$r = bx - ay.$$

Consider the identity function $\eta(r) = r$, in which case $\Phi \circ H = H$ gives

$$f(x,y) = (1+\gamma)y - \gamma x,$$

where $\gamma = b/a$. In other words, the class of linear difference equations

$$x_{n+2} = (1+\gamma)x_{n+1} - \gamma x_n$$

has the above Lie group symmetries and has an associated first integral given by

$$H(x,y) = \gamma x - y.$$

Although this example is somewhat simplistic, it does illustrate how the inverse problem may be studied.

5 Summary & future directions

In this paper we have constructed a functional equation, which if solved, will give a first integral. A method of solving this equation is given. Furthermore, a first integral is constant along the characteristic curve associated with the generator of the Lie group symmetries of the difference equation.

Future work is to construct and implement algorithms using symbolic packages such as *MATHEMATICA* in order to search for solutions of the determining functional equation, and hence find first integrals. Furthermore, the solution of the determining functional equation can be thought of as a fixed point of an operator equation. For example, the difference equation $x_{n+2} = f(x_n, x_{n+1})$ has the associated functional equation

$$(\Psi D_1 f)(\Psi^2 \alpha) + (D_2 f)(\Psi \alpha) = \alpha.$$

Define the operator

$$\mathcal{L} := (\Psi D_1 f)\Psi^2 + (D_2 f)\Psi,$$

in which case the solution α to the functional equation is the fixed point of the operator equation

$$\mathcal{L}[\alpha] = \alpha.$$

A sequence of approximations to the solution can be made by using fixed point iteration, that is, defining

$$\alpha_{n+1} := \mathcal{L}[\alpha_n],$$

where α_0 is an initial guess. The obvious question is to determine what are the conditions on the function f that insure that $\mathcal{L}$ is a contraction operator, and hence guarantee that fixed point iteration converges.

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Hidden Symmetry Exposure. The Mechanical Systems with the Hard Structure of Forces

Yuri N. BAZILEVICH

Institute of Continuing Special Education, Pridniprovsk State University of Civil Engineering, 11a Patorzhynskogo Str., 49044 Dnipropetrovsk, Ukraine
E-mail: *bazilevich@ua.fm*

The problem of the optimal decoupling of mechanical systems equations is solved. Gyroscopic and positional non-conservative forces are taken into account. Presence of the latter goes beyond the scope of compact groups theory. The way of the composition of symmetry groups corresponding to the found decoupling is considered. The method for solution of problem of hierarchic decoupling of the equations set is suggested

1 The problem of decoupling

We consider a problem of decoupling of the equations set $B_1\ddot{\boldsymbol{x}} + B_2\dot{\boldsymbol{x}} + B_3\boldsymbol{x} = 0$, where $\boldsymbol{x} \in \mathbb{C}^n$, B_i are complex (generally speaking) n by n matrices. It is necesary to find such transformation

$$\hat{B}_i = HB_iS = \operatorname{diag}(B_{1i}, B_{2i}, \ldots, B_{li}) = \begin{bmatrix} [1] & \ldots & \ldots & 0 \\ \ldots & [2] & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & [l] \end{bmatrix},$$

that all $\{\hat{B}_i\}$ coefficient matrices will have the equal partitioned-diagonal forms, or prove that such transformation does not exist. We want to have l blocks as maximum possible number [1,2].

The second problem is similar to the first, but the matrices will have the partitioned-triangular form:

$$\hat{B}_i = \begin{bmatrix} [1] & //// & //// & //// \\ \cdot & [2] & //// & //// \\ \cdot & \cdot & \cdot & //// \\ 0 & \cdot & \cdot & [l] \end{bmatrix}.$$

We prove that if a B_1 matrix is nonsingular, for the solution of the problems 1 and 2 it is necessary to form auxiliary matrices $C_i = {B_1}^{-1}B_i$, $i = \overline{1,3}$ and for them to solve similar problems, using only similarity transformations:

$$\tilde{C}_i = R^{-1}C_iR. \tag{1}$$

2 Existing method

There exists the method of commutative matrix (see A.K. Lopatin [6], E.D. Yakubovich [9], V.V. Udilov [8] and Bazilevich [2]). Let $\Lambda(C_i)$ be a set of all matrices that are commutative with all matrices $\{C_i\}$. Let a Z matrix be a member of $\Lambda(C_i)$ and have two (or more) different eigenvalues. Vectors of its canonical basis are columns of R transformation matrix of similarity.

This method must be used at first to the parent matrices $\{C_i\}$, then to the blocks obtained consistently. We continue this process until receiving only undecoupling blocks. An application

of methods using a priori information about symmetry of the proper physical system is desirable at the first step. This carries out to get the maximal quantity of blocks. A further increase of quantity of blocks is impossible. The uniqueness theorem confirms that [2].

3 Finding a symmetry group

After implementation of decoupling it is possible to find a symmetry group of the explored physical system. Formative group members will be the matrices $T_\mu = RL_\mu R^{-1}$, where $L_\mu = \mathrm{diag}(E_1, E_2, \ldots, -E_\mu, \ldots, E_l)$, E_i are identity matrices. Their orders are equal to those of corresponding matrices of the obtained subsystems.

The members of obtained set form an Abelian group as $L_j L_\mu = L_\mu L_j$ and $L_\mu^2 = E$. The group members are commutative with the $\{C_i\}$ matrices. Reducing matrices T_μ to the irreducible representations corresponds to decoupling of matrices $\{C_i\}$. In other terms this process corresponds to the similarity transformation (1).

Such group is not unique. Indeed, the given group is Abelian, while symmetry group of the physical system may be not Abelian. It is clear that the matrices of any symmetry group belong to the $\Lambda(C_i)$.

4 Hierarchic decoupling

Reducing matrices to the partitioned-triangular form corresponds to "hierarchic" (vertical) decoupling. Thus, first subsystem does not contain variables of another subsystems. Only variables of first and second subsystems are present in the following subsystem, etc. The number of such subsystems can be greater than at ordinary decoupling.

Subsystems of smaller order made from the diagonal blocks of regenerate matrices are equivalent to the initial equations set from point of dynamic stability. Indeed, a set of eigenvalues of the initial equations set is equal to a union of sets of eigenvalues of subsystems made from the diagonal blocks.

Let us prove that the number of auxiliary subsystems of only the nonsymmetrical coefficient matrices can be more than maximally possible number of independent subsystems to which the given equations set is reduced. Nonsymmetric coefficient matrices are used, for example, in the case of research of dynamic stability of the mechanical systems [5]. This is related to many phenomena: flutter, shimmy wheels of car or front undercarriage of airplane, creep caused by interaction of wheeled pair of railway vehicle with railing and so on. The asymmetry of matrix of position forces makes inapplicable many theorems of qualitative research of motion stability, and also does problems in application of methods of decoupling.

Let $\phi(C_i)$ be the algebra generated by the $\{C_i\}$ matrices. In other terms, this is a set closed with respect to addition of matrices, matrix multiplication and multiplying of matrices on numbers. The first step of the proposed method consists in construction of algebra $\phi(C_i)$. For this reason at first select linearly independent matrices among the parent matrices. We will designate them W_k $(k = \overline{1, r})$. Further we calculate all possible products $Z_{ij} = W_i W_j$. If all Z_{ij} are linear combinations of matrices $\{W_k\}$ then the latter are basic set of $\phi(C_i)$. Otherwise we add the next matrix Z_{ij} to the set $\{W_k\}$ and begin the procedure of verification of all products anew. Criterion of possibility of reducing matrices to the partitioned-triangular form (reducible algebra) is the following: algebra dimension is smaller than n^2, where n is order of matrices.

Further we use a set of the theorems by J.H.M. Wedderburn, E. Artin, E. Noether, and others. Following those theorems a reducible algebra may be semisimple or non-semisimple [7]. In the cases in point, the given matrices are not reducible to the partitioned-diagonal form (if

only were reducible – we would already have done this by the method of commutative matrix). Therefore the algebra is not semisimple.

A non-semisimple algebra has a nontrivial radical ideal. There are computation formulas for its finding [4]. Coordinates $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_r]^T$ of any element of radical ideal in the $\{W_k\}$ basic set satisfy to the equation $D\alpha = 0$, $D = \{d_{ij}\}$, where $d_{ij} = \mathrm{Sp}\,(W_i W_j)$, Sp is the trace of matrix. The intersection of kernels of all matrices of radical ideal is a nontrivial subspace. It is invariant with respect to the parent matrices. We get vectors of this base subspace and orthogonal complement to it and place these vectors as columns of S transformation matrix.

Computational algorithms and computer programs on realization of developed method were made. Computations on hierarchic decoupling of equations of motion of railway vehicles and others systems were performed [3].

5 Example

Let us consider the mechanical system consisting of two bodies (Fig. 1). Let the control device create a force $P = -x_1$, applied to the second body. There is an example of positional non-conservative force. Let $m_1 = m_2 = 1$, $k = 1$.

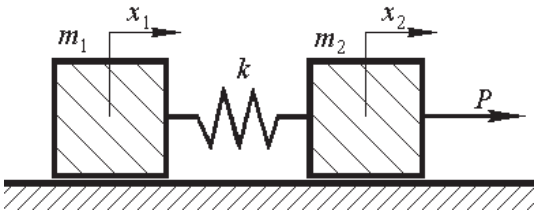

Figure 1.

Let us consider: q_1 is a moving of center of the mass; q_2 is a half of spring pressure size:

$$q_1 = (x_1 + x_2)/2, \qquad q_2 = (x_1 - x_2)/2.$$

The equations of motion have such form:

$$2\ddot{q}_1 + q_1 + q_2 = 0, \qquad 2\ddot{q}_2 - q_1 + 3q_2 = 0.$$

Coefficient matrices are as follows:

$$B_1 = 2E, \qquad B_2 = 0, \qquad B_3 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

Therefore $C_1 = E$, $C_2 = 0$, $C_3 = 0.5 \cdot \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$. We have to resolve the problem of decoupling of this system by similarity transformation.

At first, we will make sure that it is impossible to reduce the given system to the partitioned-diagonal form. For this reason we will find a Z matrix commutative with the given matrices C_i $(i = 1, 2, 3)$. It has a form $Z = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha + 2\beta \end{pmatrix}$, where α, β are arbitrary parameters. So far as the Z matrix has no different eigenvalues, reducing parent matrices $\{C_i\}$ to the partitioned-triangular form is impossible [2].

In order to find if it is possible to reduce matrices $\{C_i\}$ to the partitioned-triangular form or not we will use the algorithm described above.

We consider all possible products of $C_k C_j$ matrices and will check whether the obtained matrices are the linear combination of parent matrices or not. So far as multiplication by

$C_1 = E$ does not change the matrices; it remains to consider the C_3^2 product. We calculate $C_3^2 = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. We check whether this matrix is the linear combination of previous matrices: $C_3^2 = \alpha C_1 + \beta C_3$? This equality corresponds to the equations set:

$$0 = \alpha \cdot 1 + \beta \cdot 0.5,$$
$$1 = \alpha \cdot 0 + \beta \cdot 0.5,$$
$$-1 = \alpha \cdot 0 - \beta \cdot 0.5,$$
$$2 = \alpha \cdot 1 + \beta \cdot 1.5.$$

We obtained that $\alpha = -1$, $\beta = 2$. Consequently, matrix C_2^3 is linear combination of C_1 and C_3 matrices.

So all products belonged to the linear capsule of C_1 and C_3 matrices. Consequently, these matrices form a basic set of $\phi(C_i)$ algebra, generated by matrices $\{C_i\}$. Number r of basis elements equal to 2, that is $r < n^2 \equiv 2^2$. This means that reducing to the triangular form is possible.

We make a $D = \{\mathrm{Sp}\,(C_j C_k)\}$ matrix. All products $C_j C_k$ are already calculated. We get

$$D = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

We make the system of $Dy = 0$ equations:

$$2y_1 + 2y_2 = 0, \qquad 2y_1 + 2y_2 = 0.$$

As a result we get: $\boldsymbol{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We calculate a G matrix:

$$G = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 1 \cdot 1.5 \cdot \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} = 0.5 \cdot \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

$G\xi = 0$ equations have such kind:

$$\xi_1 - \xi_2 = 0, \qquad \xi_1 - \xi_2 = 0.$$

We obtained that the basis in the set of solutions of this system consists of $\boldsymbol{s}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ vector. This vector and $\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ vector are linearly independent. Therefore $R = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Further

$$R^{-1} = \frac{1}{2} \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \tilde{C}_1 = R^{-1} E R = E$$
$$\tilde{C}_3 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot 0.5 \cdot \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot 0.5 \cdot \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

After transformations the initial system is led to the subsystems:

1) $\ddot{y}_2 + y_2 = 0$, 2) $\ddot{y}_1 + y_1 + y_2 = 0$.

Note that G_1 group of matrices $g_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $k \in \mathbb{N}$ is symmetry group of matrices $\{\tilde{C}_i\}$. This group is not compact. Indeed, the subspace $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ (where α is any number) is an invariant subspace in relation to G_1 group and has no direct object invariant with respect to G_1 group.

6 Extensions

A developed method may be applied to equations of evolution of the automatic control systems and to the models of macroeconomics.

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Monopole Equations on 8-Manifolds with Spin(7) Holonomy: An Overview of the Energy Integral

Ayşe Hümeyra BILGE †, *Tekin DERELI* ‡ *and Şahin KOÇAK* §

† *Department of Mathematics, Istanbul Technical University, Istanbul, Turkey*
TUBITAK, Feza Gürsey Institute, Cengelkoy, Istanbul, Turkey
E-mail: *bilge@itu.edu.tr*

‡ *Department of Physics, Koc University, Istanbul, Turkey*
E-mail: *tdereli@ku.edu.tr*

§ *Department of Mathematics, Anadolu University, Eskisehir, Turkey*
E-mail: *skocak@anadolu.edu.tr*

On 8-manifolds with Spin(7) holonomy, we have written an action coupling the curvature of a $U(1)$ bundle to a negative spinor field determined by the Bonan 4-form. The minimizers of this action are the inhomogeneous counterparts of the monopole equations of Corrigan et al. [Corrigan E., Devchand C., Fairlie D. and Nuyts J., *Nucl. Phys. B*, 1983, V.214, 452–464].

1 Introduction

The setup for gauge theories is vector bundles over differentiable manifolds, which are equipped with a fiber metric and a connection compatible with it. The curvature of this connection is a 2-form taking values in the Lie algebra of the structure group of the vector bundle. The connection and the curvature of the vector bundle are usually interpreted as potentials and fields, respectively. Whenever globally meaningful, one can work with additional structures on the base manifold such as other vector bundles, whose sections are also interpreted as physical fields. An "action integral" is a functional of the fields involved, and the crucial point is to write it globally on the manifold. The field equations can be derived either using variational techniques or by determining conditions to attain "topological lower bounds". These topological lower bounds are the integrals of the characteristic classes of the vector bundle and, although related intrinsically to the curvature of the connection, they are determined by topology of the vector bundle [1].

The prototype of these structures is the Yang–Mills theory on four manifolds with an $SU(2)$ vector bundle: The action integral is the L_2 norm of the curvature, which is minimized when the curvature 2-form is self-dual in the Hodge sense, it reaches its topological lower bound and the field equations form a first order elliptic system for the connection. Besides its importance in physics, Yang–Mills theory is extensively used as a tool in low dimensional topology, to compute differentiable invariants of 4-manifolds from the properties of the solution set of Yang–Mills equations [2]. Later Seiberg and Witten [3] proposed to work with a $U(1)$ bundle and an associated spinor bundle, to get a set of inhomogeneous equations. These equations were minimizers of a certain action involving a coupling of the spinor field to the curvature. It turned out that the properties of the solution set of these new equations also yielded the differentiable invariants of the base manifold, but with much ease. A concise review of the Seiberg–Witten theory can be found in [4].

In a series of papers we studied the problem of extending these structures to higher dimensional manifolds [5–7]. The starting point was the definition of "strong self-duality" of 2-forms,

as one cannot talk of self-duality of 2-forms in dimensions other than four. Strong self-duality of a 2-form is defined as the equality of the absolute values of the eigenvalues of the corresponding matrix [5] and we have shown that (i) a 2-form ω in $2n$ dimensions is strongly self-dual if and only if ω^{n-1} is proportional to the Hodge dual of ω, and (ii) a 2-form ω in $4n$ dimensions is strongly self-dual if ω^n is self-dual in the Hodge sense [8]. The first condition was used as a definition of self-duality by Trautman [9] and the second one was used in the work of Grossman et al. [10].

Although this nonlinear criterion encompasses the definitions used in the literature and realizes topological lower bounds in many situations, it is not easily workable, hence a linear notion of self-duality is preferable. The linear subspaces of the strong self-dual forms are closely related to the representations of Clifford algebras and provide such a setting [11]. The dimension of the maximal linear subspaces of strong self-dual 2-forms on R^{2n} is equal to the number of linearly independent vector fields on the sphere S^{2n-1}, known as the Radon–Hurwitz number. These structures however are quite restricted, for example the Radon–Hurwitz number in dimensions $N = 2(2a + 1)$ is just 1. It turns out that $N = 8$ is the most interesting dimension, and the linear subspaces of strong self-dual 2-forms coincide with the linear self-duality definition given first in [12] and [13]. The choice of a maximal linear subspace of strong self-dual 2-forms in 8 dimensions gives rise to a splitting of the 28 dimensional linear space of 2-forms into 7 and 21 dimensional subspaces. The 7 dimensional subspace, consisting of strong self-dual 2-forms, are building blocks of the instanton solution of Grossman et al. [10], while the 21 dimensional subspace leads to the equations given in Corrigan et al. [12]. The first set is overdetermined, but the second one is shown to be an elliptic system for the connection [14]. The question is now whether one can choose these subpaces consistently on the manifold, or whether the equations given in local coordinates are globally meaningful. This question was addressed in Corrigan et al. [12] by requiring that the fourth rank tensor defining self-duality be invariant under a subgroup of the structure group of the tangent bundle of the base manifold. It turns out that when the manifold has Spin(7) holonomy, there is a globally defined 4-form, called the Bonan 4-form [15] or the "calibration form" [16] that gives rise to the self-duality equations. The Bonan 4-forms appears also in a topological lower bound for the Yang–Mills action on 8-manifolds [17].

The structures discussed so far involved a single usually non-Abelian vector bundle over a base manifold. As mentioned above, the Seiberg–Witten theory involves a manifold equipped with a $U(1)$ vector bundle and a spinor bundle. A direct generalization of the Seiberg–Witten theory to 8-manifolds was considered in [18], but leads to trivial results. An interpretation of the Seiberg–Witten equations as projections resulted in a set of monopole equations for a coupling to positive spinors [6]. Finally, in [7] we obtained a set of equations for a coupling to negative spinors as minimizers of a quadratic action integral. In the present paper we outline the construction of the action integral with an emphasis on local expressions.

2 Preliminaries

We start by introducing our notation. We define a Hermitian inner product on $n \times m$ matrices by $(A, B) = \frac{1}{n} \operatorname{tr}(\bar{A}^t B)$. It follows that if X is an m-vector and A is skew-symmetric matrix, then $(AX, X) = \frac{n}{2}(A, X\bar{X}^t - \bar{X}X^t)$. The point-wise norm of a real differential form is defined as $|\omega|^2 = (\omega, \omega) = *(\omega \wedge *\omega)$, where $*$ denotes Hodge dual.

A spin^c structure on a $2n$-dimensional real inner-product space V is a pair (W, Γ), where W is a 2^n dimensional Hermitian vector space, and $\Gamma : V \to \operatorname{End}(W)$ is a linear map satisfying

$$\Gamma(v)^* = -\Gamma(v), \qquad \Gamma(v)^2 = -|v|^2$$

for $v \in V$. Globalization of this construction on each fiber to the tangent bundle of a $2n$-dimensional oriented manifold M will be possible if and only if the second Stiefel–Whitney class $w_2(M)$ has an integral lift, a condition which always holds for manifolds with $Spin(7)$ holonomy.

Given a spin^c structure, the matrix $\Gamma(e_{2n}e_{2n-1}\cdots e_1)$ has eigenvalues $\pm i^n$, and its eigenspaces determine a splitting $W = W^+ \oplus W^-$. The elements of W^+ and W^- are called respectively "positive" and "negative" spinors. It follows that one can write the representation in block form as

$$\Gamma(v) = \begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix} \tag{1}$$

with Γ as given in (1), ρ is block diagonal and one can define the maps

$$\rho^{\pm}(\eta) = \rho(\eta)\mid_{W^{\pm}}, \tag{2}$$

for $2k$-forms η.

The matrices $\gamma(e_i) = \gamma_i$ are characterized by

$$\gamma_1 = I, \qquad \gamma_j^2 + I = 0, \qquad \gamma_j\gamma_k + \gamma_k\gamma_j = 0, \qquad \text{for} \quad j \geq 2, \quad j \neq k. \tag{3}$$

In 8-dimensions, the γ_j's are real, skew-symmetric matrices and the set

$$\{\gamma_2, \gamma_3, \ldots, \gamma_8, \gamma_2\gamma_3, \ldots, \gamma_7\gamma_8\} \tag{4}$$

is orthonormal. The effects of the $\rho^{\pm}$ maps are given by

$$\begin{aligned} &\rho^{\pm}(e_1^* \wedge e_j^*) = \pm\gamma_j, \qquad \rho^{\pm}(e_j^* \wedge e_k^*) = \gamma_j\gamma_k, \\ &\rho^{\pm}(e_1^* \wedge e_j^* \wedge e_k^* \wedge e_l^*) = \pm\gamma_j\gamma_k\gamma_l, \qquad \rho^{\pm}(e_j^* \wedge e_k^* \wedge e_l^* \wedge e_m^*) = \gamma_j\gamma_k\gamma_l\gamma_m. \end{aligned} \tag{5}$$

The γ_j matrices satisfy the relations $\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8 = -I$, which allow the duality identifications

$$\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6 = \gamma_7\gamma_8, \qquad \gamma_2\gamma_3\gamma_4\gamma_5 = -\gamma_6\gamma_7\gamma_8. \tag{6}$$

A Hermitian connection on W, compatible with the Levi-Civita connection of the manifold induces an imaginary valued connection on a certain associated line bundle. The corresponding connection 1-form is denoted by $\mathcal{A}$, its curvature 2-form is F and the Dirac operators $D^{\pm}$ corresponding to $\mathcal{A}$ are differential operators mapping positive spinors to negative spinors and vice versa. The Weitzenböck formula, which is the key in writing of the action integral for the Seiberg–Witten equations, gives

$$\begin{aligned} D^-D^+\phi^+ &= \nabla^*\nabla\phi^+ + \frac{1}{4}s\phi^+ + \rho^+(F)\phi^+, \\ D^+D^-\phi^- &= \nabla^*\nabla\phi^- + \frac{1}{4}s\phi^- + \rho^-(F)\phi^-, \end{aligned} \tag{7}$$

where s is the scalar curvature of M and ∇^* is the L_2-adjoint of ∇. Taking the inner product of (7) with $\phi^{\pm}$ and integrating over M, we obtain

$$\int_M |D^{\pm}\phi^{\pm}|^2 \,\mathrm{dvol} = \int_M \left[|\nabla\phi^{\pm}|^2 + \frac{1}{4}s|\phi^{\pm}|^2 + (\rho^{\pm}(F)\phi^{\pm}, \phi^{\pm})\right] \mathrm{dvol}. \tag{8}$$

Hence if $D^-\phi^- = 0$, from (8) we have

$$\int \left[|\nabla\phi^-|^2 + \frac{1}{4}s|\phi^-|^2\right] \mathrm{dvol} = -\int (\rho^-(F)\phi^-, \phi^-)\,\mathrm{dvol}. \tag{9}$$

We have thus expressed the spinor coupling in terms of the ρ map. Now we will compute the topological term.

The expression of the *Bonan 4-form* Φ in an orthonormal basis $\{e_i\}$, $i = 1, \ldots, 8$, is

$$\begin{aligned}\Phi = [&e_{1234} - e_{1256} - e_{1278} - e_{1357} + e_{1368} - e_{1458} - e_{1467} \\ &+ e_{5678} - e_{3478} - e_{3456} - e_{2468} + e_{2457} - e_{2367} - e_{2358}].\end{aligned} \tag{10}$$

where $e_{ijkl} = e_i^* \wedge e_j^* \wedge e_k^* \wedge e_l^*$. The form Φ is $Spin(7)$ invariant, hence it can be extended to the manifold. Φ is self-dual and it defines a linear map on 2-forms as

$$\omega \to *(\Phi \wedge \omega)$$

with eigenvalues 3 and -1. This map has 7 and 21 dimensional eigenspaces, and a basis consisting of the corresponding eigenvectors are given below. The eigenvectors corresponding to the eigenvalue 3 are

$$\begin{aligned}\omega_{12} &= e_{15} + e_{26} + e_{37} + e_{48}, \\ \omega_{13} &= e_{12} + e_{34} - e_{56} - e_{78}, \\ \omega_{14} &= e_{16} - e_{25} - e_{38} + e_{47}, \\ \omega_{15} &= e_{13} - e_{24} - e_{57} + e_{68}, \\ \omega_{16} &= e_{17} + e_{28} - e_{35} - e_{46}, \\ \omega_{17} &= e_{14} + e_{23} - e_{58} - e_{67}, \\ \omega_{18} &= e_{18} - e_{27} + e_{36} - e_{45}.\end{aligned} \tag{11}$$

The ones corresponding to the eigenvalue -1 can be obtained by changing signs as given in [7]. The eigenvectors $\{\omega_{1j}\}$ and $\{\omega_{jk}\}$ for $j, k = 2, \ldots, 8$ are a basis for local sections of 2-form fields, and the curvature 2-form F has the splitting $F = F^{(7)} + F^{(21)}$, where $F^{(7)} = \sum i a_j \omega_{1j}$, $F^{(21)} = \sum i a_{kl} \omega_{kl}$. It can be seen that

$$F^{(7)} \wedge F^{(21)} \wedge \Phi = 0,$$

hence

$$F \wedge F \wedge \Phi = F^{(7)} \wedge F^{(7)} \wedge \Phi + F^{(21)} \wedge F^{(21)} \wedge \Phi.$$

It follows that one can obtain

$$|F|^2 = \frac{4}{3}|F^{(21)}|^2 - \frac{2}{3} * (F \wedge F \wedge \Phi) \tag{12}$$

that relates the norm of the curvature to the topological term.

3 Seiberg–Witten theory on 4-manifolds

In Seiberg–Witten theory over a 4-manifold, a $U(1)$ connection is coupled to a spinor field. The Seiberg–Witten equations are the minimizers of an action involving the curvature 2-form F and a positive Dirac spinor ϕ^+

$$I(A, \phi^+) = \int_M \left[|F|^2 + |\nabla \phi^+|^2 + \frac{1}{4} s |\phi^+|^2 + \frac{1}{4} |\phi^+|^4 \right] \mathrm{dvol} \geq \int_M \operatorname{tr} F^2, \tag{13}$$

where s is the scalar curvature of the 4-manifold M and A and F are respectively the connection and curvature of a line bundle associated with the spin^c structure on M [4]. The Weitzenböck

formula relates the covariant derivatives $\nabla\phi^{\pm}$ to $D^{\pm}\phi^{\pm}$, compensating for the scalar curvature term and bringing in the coupling $(\rho^{\pm}(F)\phi^{\pm},\phi^{\pm})$.

In 4 dimensions, $\rho^{+}(F)$ is a 2×2 traceless, Hermitian matrix such that $\rho^{+}(F^{-})$ is identically zero. If $D^{+}\phi^{+}=0$, the action (13) can be written as the sum of a topological term and an integrand involving $2|F|^2+(\rho^{+}(F^{+})\phi^{+},\phi^{+})+\frac{1}{4}|\phi^{+}|^4$. It turns out that the integrand vanishes when

$$D^{+}\phi^{+}=0, \qquad \rho^{+}(F^{+})=[\phi^{+}(\bar{\phi}^{+})^{t}]_{o}, \tag{14}$$

where the subscript o denotes the trace-free part of a matrix. From the details of the computation as given in [7] it follows that the term $\frac{1}{4}|\phi^{+}|^4$ is added just to complete the square. Choosing $\rho^{+}(F^{+})$ as in (14) reduces the action to its topological lower bound. In some sense, the spinors $\phi^{\pm}$ are chosen in such a way that the coupling is independent of the part of F, which is expected to be free. This was our idea in looking for Seiberg–Witten type equations on 8-manifolds.

The coupling of a spinor field to the Yang–Mills action in 8-dimensions may follow different paths. A direct generalization of the Seiberg–Witten theory in 4-dimensions has led to trivial results [10]. In our previous paper [11], we defined a generalization by interpreting the right hand sides of the Seiberg–Witten equations as a projection onto a subspace determined by the map ρ^{+} and coupled positive spinors to the curvature. However, it was not possible to express these equations as absolute minimizers of an action.

In [7], we defined monopole equations via a projection using the map ρ^{-}, coupling the curvature to negative spinors. These new set of equations are absolute minimizers of an action, provided that the real and imaginary parts of the negative spinor belongs to certain complemenatary subspaces of negative spinors determined by the image of the Bonan 4-form under the ρ^{-} map, to be described below.

4 The action integral

We will now obtain monopole equations for $F^{(7)}$ and ϕ^{-} derived from an action principle, when ϕ^{-} belongs to a certain subspace. The determination of this subspace in local coordinates is almost trivial, as it imposes itself from the local expression of the ρ. The subtle point is the invariant description of this subspace in terms of the Bonan 4-form, independently of a specific choice of the spin^{c} structure. We refer to [7] for the complete proof of Proposition 1, below.

Proposition 1. *Let M be an 8-manifold with Spin(7) holonomy, with scalar curvature s and Bonan 4-form Φ, (Γ,W) be any spin^{c} structure and W^{-} and ρ^{-} be defined in terms of Γ as in* (2)*. Define a section ϕ of W^{-} by*

$$\phi=(1-P)U+iPU$$

where U is a real section of W^{-} and

$$P=\frac{1}{8}-\frac{1}{16}\rho^{-}(\Phi).$$

Then the monopole equations

$$D^{-}\phi=0, \qquad \rho^{-}(F^{(7)})=\frac{1}{2}(\phi\bar{\phi}^{t}-\bar{\phi}\phi^{t}), \qquad \operatorname{div}A=0$$

are absolute minimizers of the action

$$I(A,U)=\int_{M}\left[|F|^2+|\nabla\phi|^2+\frac{1}{4}s|\phi|^2+|\phi\bar{\phi}^{t}-\bar{\phi}\phi^{t}|^2\right]\operatorname{dvol}$$

and $I(A,U)\geq\frac{2}{3}\int_{M}F\wedge F\wedge\Phi$.

Proof. We give here only a brief outline of the proof, concentrating on local computations which are more illustrative.

Using (9), (12), we can express the action as

$$I(A,U)=\frac{2}{3}\int_M F\wedge F\wedge\Phi+\int_M\left[4|F^{(7)}|^2-(\rho^-(F)\phi,\phi)+\frac{1}{4}s|\phi|^2+|\phi\bar{\phi}^t-\bar{\phi}\phi^t|^2\right]\mathrm{dvol}.$$

At this stage it is useful to have the expression of the $\rho^-(F)$ for a specific choice of spin^c structure that we used in our computations. Other choices lead to similar expressions, and the whole computation can be done in an abstract Clifford algebra setting using the relations (2)–(6). The effect of ρ^+ can be written similarly, but the resulting matrices are nonsingular. This is the main reason for working with negative spinors.

$$\begin{aligned}
\rho^-(\omega_{12})&=-\gamma_5+\gamma_{26}+\gamma_{37}+\gamma_{48}=-4e_{68},\\
\rho^-(\omega_{13})&=-\gamma_2+\gamma_{34}-\gamma_{56}-\gamma_{78}=-4e_{26},\\
\rho^-(\omega_{14})&=-\gamma_6-\gamma_{25}-\gamma_{38}+\gamma_{47}=4e_{46},\\
\rho^-(\omega_{15})&=-\gamma_3-\gamma_{24}-\gamma_{57}+\gamma_{68}=4e_{56},\\
\rho^-(\omega_{16})&=-\gamma_7+\gamma_{28}-\gamma_{35}-\gamma_{46}=4e_{67},\\
\rho^-(\omega_{17})&=-\gamma_4+\gamma_{23}-\gamma_{58}-\gamma_{67}=-4e_{16},\\
\rho^-(\omega_{18})&=-\gamma_8-\gamma_{27}+\gamma_{36}-\gamma_{45}=-4e_{36}.
\end{aligned}$$

It is then possible to check that $|F^{(7)}|^2=|\rho^-(F^{(7)})|^2$. The orthonormality of the set (4) implies that $|\rho^-(\omega_{1j})|^2=4$. On the other hand, from (11) $|\omega_{1j}|^2=4$, hence the ρ^- map is an isometry. It follows that

$$\begin{aligned}
I(A,U)=&\frac{2}{3}\int_M F\wedge F\wedge\Phi\\
&+\int_M\left[4|\rho^-(F^{(7)})|^2-(\rho^-(F)\phi,\phi)+\frac{1}{4}s|\phi|^2+|\phi\bar{\phi}^t-\bar{\phi}\phi^t|^2\right]\mathrm{dvol}.
\end{aligned}$$

The main difficulty is to show that $(\rho^-(F^{(21)})\phi,\phi)=0$ for ϕ given as in the hypothesis of the theorem. If we use the specific spin^c structure corresponding to $\gamma_j=\omega_{1j}$, P is the diagonal matrix with the only nonzero entry, $P_{66}=1$. It can also be seen that the 6th row and the 6th column of $\rho^-(\omega_{jk})$'s are zero, while in $\rho^-(\omega_{1j})$'s non-zero elements are only in the 6th row and the 6th column. Thus, in a local formulation it can be seen that

$$\rho^-(F^{(21)})P=0,\qquad \rho^-(F^{(7)})P+P\rho^-(F^{(7)})=\rho^-(F^{(7)}).$$

As F is pure imaginary and the representation is real, $\rho^-(F)$ is Hermitian but skew-symmetric. Now using $\rho^-(F^{(21)})P=0$, we have

$$(\rho^-(F^{(21)})\phi,\phi)=(\rho^-(F^{(21)})U,(1-P)U+iPU)=(U,\rho^-(F^{(21)})U)=0,$$

where the second equality follows from hermiticity and the third from skew-symmetry. Then

$$(\rho^-(F^{(7)})\phi,\phi)=4(\rho^-(F^{(7)}),\phi\bar{\phi}^t-\bar{\phi}\phi^t),$$

and it can be seen that the action reduces to

$$I(A,U)=\frac{2}{3}\int_M F\wedge F\wedge\Phi+\int_M|2\rho^-(F^{(7)})-(\phi\bar{\phi}^t-\bar{\phi}\phi^t)|^2\mathrm{dvol}$$

which gives the monopole equations. As the left hand side of the matrix equation

$$\rho^-\big(F^{(7)}\big) = \frac{1}{2}(\phi\bar{\phi}^t - \bar{\phi}\phi^t)$$

belongs to a 7-dimensional subspace, we need to check its compatibility. For this we use the orthogonal decomposition of any skew-symmetric matrix A in the form

$$A = (PA + AP) + (I - P)A(I - P).$$

and write

$$\frac{1}{2}(\phi\bar{\phi}^t - \bar{\phi}\phi^t) = \frac{1}{2}[-i(I-P)UU^tP + iPUU^t(I-P)] = i[-UU^tP + PUU^t].$$

Using the properties of the projection operator P, it can be seen that both sides of the equation belong to the same linear subspace. ■

5 The monopole equations

In [6], we have given monopole equations using a projection with $\rho^+(F^{(7)})$, where we denoted $F^{(7)}$ as F^+, and ω_{1j} by f_{j-1}. The same formula (equation (26) in [6]) can be applied as well using a projection with ρ^-. Hence we can write

$$\rho^\pm(F^{(7)}) = \sum_{j=2}^{8} \langle \rho^\pm(\omega_{1j}), \phi^\pm(\phi^\pm)^* \rangle |\rho^\pm(\omega_{1j})|^{-2} \rho^\pm(\omega_{1j}),$$

where ω_{1j}'s are an orthonormal basis for the subspace $F^{(7)}$. We may choose the ω_{1j}'s as above, and with respect to this basis we can write $F^{(7)} = \sum_{j=2}^{8} i a_{1j}\omega_{1j}$, then $\rho^\pm(F^{(7)}) = \sum_{j=2}^{8} i a_{1j}\rho^\pm(\omega_{1j})$, the monopole equations are

$$i a_{1j} = |\rho^\pm(\omega_{1j})|^{-2} \langle \rho^\pm(\omega_{1j}), \phi^\pm(\bar{\phi}^\pm)^t \rangle, \qquad j = 2, \ldots, 8.$$

We give below the monopole equations for the coupling to a negative spinor

$$\begin{aligned}
4a_{12} &= F_{15} + F_{26} + F_{37} + F_{48} = (1/2)(-\phi_6\bar{\phi}_8 + \phi_8\bar{\phi}_6),\\
4a_{13} &= F_{12} + F_{34} - F_{56} - F_{78} = (1/2)(-\phi_2\bar{\phi}_6 + \phi_6\bar{\phi}_2),\\
4a_{14} &= F_{16} - F_{25} - F_{38} + F_{47} = (1/2)(\phi_4\bar{\phi}_6 - \phi_6\bar{\phi}_4),\\
4a_{15} &= F_{13} - F_{24} - F_{57} + F_{68} = (1/2)(\phi_5\bar{\phi}_6 - \phi_6\bar{\phi}_5),\\
4a_{16} &= F_{17} + F_{28} - F_{35} - F_{46} = (1/2)(\phi_6\bar{\phi}_7 - \phi_7\bar{\phi}_6),\\
4a_{17} &= F_{14} + F_{23} - F_{58} - F_{67} = (1/2)(-\phi_1\bar{\phi}_6 + \phi_6\bar{\phi}_1),\\
4a_{18} &= F_{18} - F_{27} + F_{36} - F_{45} = (1/2)(-\phi_3\bar{\phi}_6 + \phi_6\bar{\phi}_3).
\end{aligned}$$

When the spinor belongs to the subspace determined by the Bonan 4-form, we put

$$\bar{\phi}_6 = -\phi_6, \qquad \bar{\phi}_j = \phi_j, \qquad \text{for} \quad j \neq 6.$$

The final step is to show that the resulting equations are elliptic. To check ellipticity of a system of first order partial differential equations in the independent variables $x_1, \ldots, x_k$ and dependent variables $u_1, \ldots, u_n$, we replace $\frac{\partial u_i}{\partial x_j}$ by $\xi_j u_i$, where ξ_j's are indeterminates, and obtain a linear homogeneous system of equations for the u_i's. The characteristic determinant of the

system is the determinant of the coefficient matrix of the corresponding linear system. The system is called *elliptic* if its characteristic determinant has no real roots. As this property is independent of the inmogeneous part, the elipticity of the new set of equations follows from the ellipticity of the equations of Corrigan et al. [12] after checking a nondegeneracy condition arising from the projection of the spinor field to a subspace.

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Separation of Variables in the Null Hamilton–Jacobi Equation

Claudia CHANU

Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy
E-mail: *chanu@dm.unito.it*

This is a summary of the main results contained in a forthcoming joint paper with S. Benenti and G. Rastelli, where we revisit the theory of the separation of variables in the Hamilton–Jacobi equation with a fixed value of the energy. We extend the classical Levi-Civita conditions by using Lagrangian multipliers. By applying the general results to the case of natural Hamiltonians we find conditions generalizing those of Stäckel.

1 Introduction

The study of the Hamilton–Jacobi equation with a fixed value of the energy is a classical topic which is treated in many textbooks. Interesting and deeply examined examples of this kind of equations are for instance

– the Kepler motions with a fixed value of the total energy,

$$\frac{1}{2}p_r^2 + \frac{1}{2r^2}p_\theta^2 - \frac{1}{r} = E;$$

– the so called homogeneous formalism used to deal with time dependent Hamiltonians,

$$\frac{\partial W}{\partial t} + H\left(t, q^i, \frac{\partial W}{\partial q^i}\right) = 0;$$

– the null geodesics in pseudo-Riemannian manifolds,

$$g^{ii}p_i^2 = 0;$$

– the dynamical systems which are Hamiltonian only on a hypersurface. As a simple example, we consider the dynamical system on T^*E_2

$$\dot{x}_i = y_i + f_i(x_i, y_i)\big(y_1^2 + y_2^2 + x_1^2 + x_2^2 - c\big),$$
$$\dot{y}_i = -x_i + g_i(x_i, y_i)\big(y_1^2 + y_2^2 + x_1^2 + x_2^2 - c\big),$$

where $c \in \mathbb{R}$, (x_1, x_2) are Cartesian coordinates in the Euclidean plane E_2, (y_1, y_2) are the associated momenta, f_i and g_i arbitrary functions on T^*E_2. It is a Hamiltonian system only on the hypersurface $\big(y_1^2 + y_2^2 + x_1^2 + x_2^2\big) = 2c$.

All these examples can be considered as special cases of the *null Hamilton–Jacobi equation*

$$\mathcal{H}\left(q^i, \frac{\partial W}{\partial q^i}\right) = 0, \tag{1}$$

where $\mathcal{H}$ is a smooth function on T^*Q. Although separation of variables for such a kind of equations has been studied since the end of 19th century, in our opinion, we still need a better understanding of the theoretical and geometrical framework of this kind of separation of variables.

Our approach that leads to consideration of a new family of equations characterizing the separation of variables of a null Hamiltonian (1), is based upon three elements:

i) a suitable definition of complete solution for a null Hamilton–Jacobi equation;

ii) a geometrical object: the separable connections (as developed by Benenti), which are related to the complete integrability of a special kind of first-order differential systems;

iii) a technical instrument: the Hadamard Lemma.

2 Complete solutions of the null Hamilton–Jacobi equation

First of all, we revisit the concept of complete solution of a null Hamilton–Jacobi equation (1) $\mathcal{H} = 0$ on the cotangent bundle T^*Q of a real n-dimensional manifold Q. We give two definitions, in a sense equivalent, but both essential for the following discussion. The first (and commonly adopted) definition is the "internal complete solution" $W^{(i)}$ which depends on $n-1$ parameters:

Definition 1. An *internal complete solution* of the null Hamilton–Jacobi equation (1) $\mathcal{H} = 0$ is a solution $W^{(i)}(\underline{q}, c_\alpha)$ depending on $n-1$ parameters (c_α) satisfying the *completeness condition*:

$$\operatorname{rank}\left[\frac{\partial^2 W^{(i)}}{\partial q^i \partial c_\alpha}\right] = n-1.$$

The second definition is the "extended complete solution" $W^{(e)}$ depending on n parameters:

Definition 2. An *extended complete solution* of the null Hamilton–Jacobi equation (1) $\mathcal{H} = 0$ is a function $W^{(e)}(\underline{q}, \underline{c})$ depending on n real parameters $\underline{c} = (c_i)$ satisfying the *completeness condition*

$$\det\left[\frac{\partial^2 W^{(e)}}{\partial q^i \partial c_j}\right] \neq 0 \tag{2}$$

and which is a solution of $\mathcal{H} = 0$ for $c_n = 0$.

Remark 1. We recall that in the usual definition a complete solution $W^{(o)}$ of the Hamilton–Jacobi equation $\mathcal{H} = \text{const}$ depends on n parameters and satisfies the completeness condition (2), but is a solution for all values of the parameters.

Remark 2. From a geometrical point of view, $W^{(i)}$ is a Lagrangian foliation of the submanifold $\mathcal{H} = 0$, while $W^{(e)}$ is a Lagrangian foliation of an open neighborhood of $\mathcal{H} = 0$ which is reducible to a foliation of $\mathcal{H} = 0$. An ordinary complete solution $W^{(o)}$ is a Lagrangian foliation compatible with all the submanifolds $\mathcal{H} = h$ $(h \in \mathbb{R})$.

These objects are in a sense equivalent. Indeed, it can be shown that starting from an internal complete solution it is possible to construct an extended one, and vice versa.

Proposition 1. *The null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *admits an extended complete solution* $W^{(e)}$ *if and only if it admits an internal complete solution* $W^{(i)}$.

It is a remarkable fact that the transition from internal to extended complete solutions is compatible with the separation of variables. Indeed, it can be proved that the transition between $W^{(i)}$ and $W^{(e)}$ preserves the separated form of the complete solutions $W^{(i)} = \sum_{j=1}^{n} W_j(q^j, c_\alpha)$, $W^{(e)} = \sum_{j=1}^{n} W_j(q^j, \underline{c})$. Thus,

Proposition 2. *The separation of variables for the null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *is characterized by the existence of one of the two kind of separable complete solutions, internal or extended.*

3 Separable connections and Hadamard Lemma

The second geometrical tool for our analysis are the separable connections, which arise from a general theory, developed by Benenti in [3], related to the connections on cotangent bundles and their links with the theory of separation of variables. It can be proved that the separation of variables in a given coordinate system (q^i) on Q is related to the complete integrability of a special kind of first-order differential systems. The geometrical interpretation of these systems is a special kind of distribution transversal to the fibers Δ which is called *separable connection*. With a given coordinate system (q^i) on Q and with n functions R_i on T^*Q we associate

- the 1-st order differential system
$$\partial_j p_i = \delta_{ij} R_i(\underline{p}, \underline{q}), \tag{3}$$
- n independent vector fields
$$D_i = \partial_i + R_i(\underline{q}, \underline{p})\partial^i,$$
generating a distribution Δ transversal to the fibers and called *separable connection*.

There is a link between the separation for Hamilton–Jacobi equation $\mathcal{H} = 0$ and the complete integrability of (3), which is equivalent to the complete integrability of the Δ. Indeed, the following properties hold:

i) the integral manifolds of Δ are described by $p_i = f_i(q^h)$ where f_i are solutions of the differential system,

ii) the functions f_i are of the form $f_i = \partial_i W(q^h)$,

iii) the function W is additively separable i.e., $W = \sum_i W_i(q^i)$.

By the Frobenius theorem, we find that Δ is completely integrable on an open subset of T^*Q if and only if $[D_i, D_j] = 0$. The commutation of the generators D_i is equivalent to

$$D_i R_j = 0 \qquad (i \neq j).$$

So far no link with the Hamilton–Jacobi equation (1). We add now the requirement that Δ is compatible with the submanifold $\mathcal{H} = 0$ i.e., that the generators D_i are tangent to $\mathcal{H} = 0$,

$$D_i H|_{\mathcal{H}=0} = 0.$$

Then, the conditions for the existence of a separated extended solution $W^{(e)}$ are

$$\begin{aligned} &D_i R_j = 0 \qquad (i \neq j), \\ &D_i \mathcal{H}|_{\mathcal{H}=0} = 0. \end{aligned} \tag{4}$$

The conditions for the existence of a separated internal solution $W^{(i)}$ are

$$\begin{aligned} &D_i \mathcal{H}|_{\mathcal{H}=0} = 0, \\ &D_i R_j|_{\mathcal{H}=0} = 0 \qquad (i \neq j) \end{aligned} \tag{5}$$

since in this case we require that the reduced distribution $\Delta|_{\mathcal{H}=0}$ is completely integrable.

The last tool we use is the following form of the Hadamard Lemma that allows us to deal with differential conditions restricted to a submanifold such as (4) and (5). A proof of the lemma for the one-dimensional case can be found in [1].

Lemma 1. *A smooth function F on T^*Q which vanishes on the submanifold defined by equation $\mathcal{H} = 0$ is of the form*

$$F = \mathcal{H}\lambda,$$

where λ is a suitable smooth function on T^*Q.

4 The general case

By applying the Hadamard Lemma to conditions (4), equivalent to the existence of an extended solution $W^{(e)}$, we get equations

$$L_{ij}(\mathcal{H}) + \mathcal{H}\Big[\lambda_i\big(\partial^j\mathcal{H}\partial^i\partial_j\mathcal{H} - \partial_j\mathcal{H}\partial^i\partial^j\mathcal{H}\big) + \lambda_j(\partial^i\mathcal{H}\partial_i\partial^j\mathcal{H} - \partial^i\partial^j\mathcal{H}\partial_i\mathcal{H}) + \lambda_i\lambda_j\big(\mathcal{H}\partial^i\partial^j\mathcal{H} - \partial^i\mathcal{H}\partial^j\mathcal{H}\big) - \partial_j\lambda_i\partial^i\mathcal{H}\partial^j\mathcal{H} + \partial^j\lambda_i(\partial_j\mathcal{H}\partial^i\mathcal{H} - \lambda_j\mathcal{H}\partial^i\mathcal{H})\Big] = 0, \quad (6)$$

where

$$L_{ij}(\mathcal{H}) = \partial_i\mathcal{H}\partial_j\mathcal{H}\partial^i\partial^j\mathcal{H} + \partial^i\mathcal{H}\partial^j\mathcal{H}\partial_i\partial_j\mathcal{H} - \partial_i\mathcal{H}\partial^j\mathcal{H}\partial^i\partial_j\mathcal{H} - \partial^i\mathcal{H}\partial_j\mathcal{H}\partial_i\partial^j\mathcal{H},$$

are the classical Levi-Civita operators i.e., the left-hand sides of the Levi-Civita separability condition characterizing the ordinary separation of variables

$$L_{ij}(\mathcal{H}) = 0, \qquad (i \neq j).$$

In (6) the additional unknown functions λ_i play the role of Lagrangian multipliers. We call equations (6) *Levi-Civita conditions with Lagrangian multipliers* (LCL). Thus, due to Proposition 2, we have proved

Theorem 1. *The null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *is separable if and only if there exist* n *functions* $\lambda_i(\underline{q}, \underline{p})$ *on* T^*Q *such that Levi-Civita conditions with Lagrangian multipliers* (6) *are satisfied.*

By requiring the existence of an internal separated solution $W^{(i)}$, we get the following criterion equivalent to Theorem 1:

Theorem 2. *The null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *is separable if and only if the Levi-Civita conditions* $L_{ij}(\mathcal{H}) = 0$*, for all* $i \neq j$*, are satisfied on the submanifold* $\mathcal{H} = 0$*.*

It is remarkable that, from Theorem 1, we can derive another characterization of the separation of variables, where the additional functions λ_i are replaced by a single function Λ on T^*Q and the LCL conditions reduce to the usual Levi-Civita separability conditions on the new Hamiltonian $\mathcal{H}/\Lambda$.

Theorem 3. *The null Hamilton–Jacobi equation* (1) $\mathcal{H} = 0$ *is separable in the coordinates* $\underline{q} = (q^i)$ *if and only if there exists a nowhere vanishing function* $\Lambda = \Lambda(\underline{q}, \underline{p})$ *on* T^*Q *such that the conformal Hamiltonian* $\mathcal{H}/\Lambda$ *is separable i.e., for any* $i \neq j$*,*

$$L_{ij}\left(\frac{\mathcal{H}}{\Lambda}\right) = 0.$$

Definition 3. We call the function $\mathcal{J} = \mathcal{H}/\Lambda$ the *conformal Hamiltonian* associated with $\mathcal{H}$ and the function Λ the *conformal factor*.

Remark 3. The transition from a Hamiltonian $\mathcal{H}$ to a conformal Hamiltonian $\mathcal{J} = \mathcal{H}/\Lambda$ is an extension of the so-called *Jacobi transformation* or *Maupertuis transformation* for natural Hamiltonians (see [9] and references cited therein).

The link between the two Hamiltonian systems associated with $\mathcal{H}$ and $\mathcal{J}$ is given by the following properties:

Proposition 3. *On the submanifold $\mathcal{H} = 0$ the Hamiltonian vector fields $X_{\mathcal{H}}$ and $X_{\mathcal{J}}$ are parallel and differ by the factor Λ,*

$$(\Lambda X_{\mathcal{J}})\big|_{\mathcal{H}=0} = X_{\mathcal{H}}\big|_{\mathcal{H}=0},$$

so that the corresponding affine parameters t and $\bar{t}$ are related by equation

$$d\bar{t} = \Lambda dt.$$

Proposition 4. *If we know an ordinary complete solution of the Hamilton–Jacobi equation $\mathcal{J} = h$ for the conformal Hamiltonian $\mathcal{J}$, then, for $h = 0$, we get the orbits on $\mathcal{H} = 0$ of the Hamiltonian vector field $X_{\mathcal{H}}$.*

5 Orthogonal separation for natural Hamiltonians

We apply these general results to the analysis of an important particular case: the natural Hamiltonian in orthogonal coordinates $\frac{1}{2}\, g^{ii} p_i^2 + V(\underline{q})$ on a Riemannian (or pseudo-Riemannian) manifold. The corresponding Hamilton–Jacobi equation with a fixed value E of the energy is associated to the null Hamilton–Jacobi equation

$$\mathcal{H}(\underline{q}, \underline{p}) = \tfrac{1}{2}\, g^{ii} p_i^2 + V(\underline{q}) - E = 0.$$

We examine the following three cases

$$\begin{array}{lll} V = 0, \quad E \neq 0, & \text{non-null geodesics,} & \\ V = 0, \quad E = 0, & \text{null-geodesics,} & \\ V - E \neq 0, & \text{dynamical trajectories with total energy } E. & \end{array} \tag{7}$$

Theorem 4. *A common necessary condition for the orthogonal separability of a natural Hamiltonian in all cases* (7) *is that equations*

$$\frac{1}{g^{hh}} S_{ij}(g^{hh}) = \frac{1}{g^{kk}}\, S_{ij}(g^{kk}), \tag{8}$$

be satisfied for all indices h, k and $i \neq j$, where

$$S_{ij} \colon f \to \partial_i \partial_j f - \partial_j \log g^{ii} \partial_i f - \partial_i \log g^{jj} \partial_j f.$$

Remark 4. The $S_{ij}(\cdot)$ are the second-order differential operators on functions $f(\underline{q})$ on T^*Q that we called *Stäckel operators*. They can be used to characterize the ordinary geodesic orthogonal separation. Indeed, since for the geodesic orthogonal Hamiltonian $H = \frac{1}{2}\, g^{ii} p_i^2$ the Levi-Civita separability conditions are third-degree polynomials in the momenta whose coefficients are (up to a factor) $S_{ij}(g^{kk})$, the Stäckel metrics are characterized by equations

$$S_{ij}(g^{kk}) = 0. \tag{9}$$

Conditions (8) are a non-trivial generalization of the classical Stäckel conditions (9).

Proposition 5. *The necessary conditions* (8) *are equivalent to the existence of a nowhere vanishing function σ on Q such that the conformal metric $\bar{g}^{ii} = g^{ii}/\sigma$ is a Stäckel metric.*

Definition 4. We call *conformal separable coordinates* the orthogonal coordinates satisfying conditions (8).

Necessary and sufficient conditions for the separation of a natural Hamiltonian in the cases (7) are given in the following three theorems:

Theorem 5. *The separation of variables of the Hamilton–Jacobi equation for the non-null geodesic $g^{ii}p_i^2 = 2E$ occurs in given orthogonal coordinates (q^i) if and only if the metric is a Stäckel metric i.e., if and only if* (9) *hold.*

Theorem 6. *The separation of variables of the Hamilton–Jacobi equation for the null geodesic $g^{ii}p_i^2 = 0$ occurs in given orthogonal coordinates (q^i) if and only if* (8) *hold.*

Theorem 7. *The separation of variables of the Hamilton–Jacobi equation with a fixed value E of the energy and for $E - V \neq 0$ occurs in given orthogonal coordinates (q^i) if and only if*

$$\forall\, h,\ \forall\, i \neq j \quad \frac{1}{g^{hh}} S_{ij}(g^{hh}) = \frac{1}{V - E}\, S_{ij}(V).$$

Remark 5. For a non-separable natural Hamiltonian H, there exists at most one value for the energy E such that $H = E$ is separable.

Remark 6. It can be proved [4] that, in a given conformal separable coordinate system (q^i), the most general form for the conformal factor $E - V$ is

$$E - V = \sum_{i=1}^{n} \psi_i(q^i) g^{ii}. \tag{10}$$

Remark 7. By using the conformal Killing two-tensors, we can derive intrinsic versions of the above theorems (see [6, 4]).

6 An illustrative example

In this section we summarize what happens in the two-dimensional case and then we examine a three-dimensional example of conformal separable coordinates in the Euclidean three-space. We recall that (see e.g. [8, 4])

Proposition 6. *A two-dimensional orthogonal coordinate system is conformal separable if and only if, up to a rescaling,*

$$g^{11} = \pm g^{22}.$$

Proposition 7. *Up to a rescaling, every conformal separable system (q^1, q^2) of the Euclidean plane is generated by a non-constant analytic function*

$$F(z) = F(x + iy) = q^1(x, y) + iq^2(x, y).$$

Moreover, (q^1, q^2) are conformal separable coordinates compatible with a natural Hamiltonian $H = G + V$ for a fixed value of the energy E if and only if $E - V$ is of the form

$$E - V = \big(\psi_1(q^1) + \psi_2(q^2)\big)\, g^{11}.$$

Many examples of conformal separable coordinates associated with transformations of the complex plane can be found in [7]. Moreover, in [8] the authors examine different cases of potentials $V \neq 0$ compatible with the coordinates generated by a conformal transformation on $\mathbb{C}$. An explicit solution of a natural Hamilton–Jacobi equation on E_2 that is separable only for a single value of the energy is presented in [4].

Example 1. Let us consider the following natural Hamiltonian on the cotangent bundle of the Euclidean three-space

$$H = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2\right) + \frac{1}{r^2}f_1\left(\frac{\cos\theta}{r}\right) + \frac{1}{r^2}f_2\left(\frac{\sin\theta}{r}\right) + \frac{r^2}{\sin^2\theta}f_3(\phi),$$

where (r,θ,ϕ) are standard spherical coordinates and f_i arbitrary functions. The corresponding Hamilton–Jacobi equation $H = E$ is not solvable by separation of variables (we can check that condition $d(\mathbf{K}dV) = 0$ is never satisfied by a characteristic Killing tensor). However, for $E = 0$ the corresponding null Hamilton–Jacobi equation is separable in the coordinates (μ,ν,ϕ) defined by

$$x = \frac{\mu}{\mu^2+\nu^2}\cos\phi, \qquad y = \frac{\mu}{\mu^2+\nu^2}\sin\phi, \qquad z = \frac{\nu}{\mu^2+\nu^2}, \tag{11}$$

where (x,y,z) are Cartesian coordinates. With respect to these new coordinates we get

$$g^{11} = g^{22} = (\mu^2+\nu^2)^2, \qquad g^{33} = \frac{(\mu^2+\nu^2)^2}{\mu^2}.$$

Since ϕ is ignorable, it is easy to check that the new coordinates fulfill conditions (8). Moreover, according to (10), we have that a natural Hamiltonian is separable in coordinates (μ,ν,ϕ) for the fixed value E of the energy if and only if

$$E - V = (\mu^2+\nu^2)^2\left(\psi_1(\mu) + \psi_2(\nu) + \frac{\psi_3(\phi)}{\mu^2}\right)$$

that for $E = 0$ and $\psi_i = -f_i$ is satisfied by the potential of the considered Hamiltonian. The coordinates (μ,ν,ϕ) defined by (11) are called tangent-sphere coordinates (see [7]). They are constructed by a rotation around a symmetry axis of a planar conformal separable system associated to the transformation of the complex plane $1/\bar{z}$ ($z = x + iy$).

Remark 8. As seen in the previous example, it can be shown that the orthogonal rotational coordinate systems given in [7] (apart from the so-called Hyperbolic coordinates) are conformal separable and the general form of compatible potentials is immediately deducible. All these coordinates systems have $g^{11} = g^{22}$ and $q^3 = \phi$ ignorable. Moreover, all of them allow the R-separation of the Laplace equation. More general conformal separable coordinates of E_3 are the so called confocal cyclides (see the classical book [5]). The corresponding coordinate hypersurfaces are fourth-order surfaces such that the inversion with respect to a sphere of a cyclide is also a cyclide.

7 Final remarks and open problems

For the proofs of theorems and propositions and for more references the reader is referred to [4]. The application of the general theory to other well-known cases leads to many open problems. Work about the case of general non-orthogonal separation and of natural Hamiltonians with vector potential is in progress. However, both problems are very complicated to deal with, since we have to repeat and adapt the study done by Benenti [2] about the reduction of separable coordinates to normal forms.

Another interesting topic currently under investigation is the link between separation for a fixed value of the energy, the multiplicative separation and the R-separation for 2-nd order differential equations (Laplace, Schrödinger, Helmholtz).

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Matrix Impulsive System of Differential Equations with Bilinear Main Part

Vladimir A. CHIRICALOV

1-7 Pechenigivska Str., ap. 112, 04107 Kyiv, Ukraine
E-mail: *cva@skif.kiev.ua*

The main purpose of the paper is to summarize some fundamental results concerning the matrix impulsive differential equations with impulses at fixed points. The theory of impulsive matrix differential equations is interesting in itself and it will assume greater importance in the near future, since the application of the theory to various fields of science is also increasing, especially in the theory of polyphase transmission line and surge phenomena.

1 Introduction

Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbation whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. Thus impulsive differential equations, that is, differential equations involving impulsive effects, appear as a natural description of observed evolution phenomena of several real world problems. In the theory of impulsive systems, there are some problems similar to the ones considered in the theory of ordinary differential equations but there are also problems that are specific to the theory of impulsive systems.

We will be considering the matrix differential equations. It means that a phase space of evolution process be a matrix space $R^{n\times m}$. The point that describes the state of the process at the time t we denote $X(t) \in R^{n\times m}$. The topological product of the phase space $R^{n\times m}$ and real axis R, i.e. $R\times R^{n\times m}$ will be called the extended phase space of the considered evolution process. The process itself goes as follows: the point $P_t = (t, X(t)), X(t) \in R^{n\times m}$, begins its motion from the initial point $P_{t_0} = (t_0, X_0)$ and moves along the curve $\{(t, X(t)); t > t_0\}$ until the time $t_1 > t_0$ at which the point P_t meets the set Γ_1. At $t = t_1$ the operator $\mathcal{M}_1$ transfers the point $X(t_1) \in R^{n\times m}$ into the point $X(t_1 + 0) = X^+(t_1) \in R^{n\times m}$ then the point P_t continues to move further along the curve $\{(t, X(t)); t > t_1\}$ with initial value $(t_1, X^+(t_1))$, $X^+(t_1) = \mathcal{M}_1(X(t_1))$, where $X(t) = X(t, t_1, X_1^+) = X_t(t_1, X_1^+)$ is the solution of equation

$$\frac{dX}{dt} = A(t)X - XB(t) + F(t, X), \tag{1}$$

where $F : R^+ \times R^{n\times m} \to R^{n\times m}$, R^+ is the nonnegative real line. The point P_t moves along the curve until the moment $t_2 > t_1$ at which the point P_t again meet the new set Γ_2. The moment t_i at which the point P_t hits the set Γ_i $(i = 1, 2, \ldots)$ are called moment of impulsive effect. The sets Γ_i in extended phase space $R^+ \times R^{n\times m}$ may be determined by equation $\mathcal{F}_i(t, X) = 0$; $\mathcal{F}_i : R^+ \times R^{n\times m} \to R^+$. When the point P_t hits the set Γ_i at the moment t_i this equation becomes identity: $\mathcal{F}_i(t_i, X(t_i)) \equiv 0$. If this equation is solved with respect to t, we have explicit form of the set $\Gamma_i : t = \tau_i(X)$, $\tau_i : R^{n\times m} \to R^+$. As a special case we are considering the sets Γ_i as a sequence of hyperplane $t = t_i = \text{const}$, where $\{t_i\}$ is a given sequence of times (finite or

infinite). A solution of an impulsive differential system may be a piecewise continuous function having a countable number of discontinuities of the first kind if the integral curve encounters the sets Γ_i at a countable number of points that are not the fixed point of operator $\mathcal{M}_i$. The matrix differential equations without impulses have been considered in the papers [1–6].

2 Matrix differential system with impulses at fixed times

If in a real process described by system (1) the impulses occur at fixed times the mathematical model of this process will be given by the following matrix impulsive system

$$\frac{dX}{dt} = A(t)X - XB(t) + F(t,X), \qquad t \neq t_i, \tag{2}$$

$$\triangle X(t) = D_i X(t)\widetilde{D}_i + \widetilde{F}_i(X(t)), \qquad t = t_i, \tag{3}$$

where $F \in C[R^+ \times \Re, R^{n\times m}]$ is continuous map of the direct product $R^+ \times \Re$ into space of matrices $R^{n\times m}$, $\Re \subset R^{n\times m}$ being an open set, $R^{n\times m}$ is the space of matrices with real entries, $\tilde{F} : R^{n\times m} \to R^{n\times m}$ is nonlinear matrix-valued function, $\triangle X(t_i) = X(t_i+0) - X(t_i)$, $X(t_i+0) = \lim\limits_{h\to 0+} X(t_i+h)$, $D \in R^{n\times n}$, $\widetilde{D} \in R^{m\times m}$. If the matrix-valued functions $\widetilde{F}_i(X)$ and $F(t,X)$ are independent from X we have matrix nonhomogeneous bilinear impulsive system

$$\frac{dX}{dt} = A(t)X - XB(t) + F(t), \qquad t \neq t_i, \tag{4}$$

$$\triangle X(t) = D_i X(t)\widetilde{D}_i + \widetilde{F}_i, \qquad t = t_i, \tag{5}$$

where $A(t)$, $B(t)$, $F(t)$ are Lebesgue summable matrix-valued function of real variable t on the interval $[a,b]$.

The space of matrices $R^{n\times m}$ is topologically isomorphic to the Euclidean space R^{nm} and therefore the space of solutions M of the system (2), (3) is topologically isomorphic to the space of solutions $\widetilde{M}$ of the system of vector differential equations

$$\frac{dx}{dt} = (A(t)\otimes I_m - I_n \otimes B^T(t))x + f(t), \qquad t \neq t_i, \tag{6}$$

$$\triangle x(t) = (D_i \otimes \widetilde{D}_i^T)x(t) + \widetilde{f}_i,\ t = t_i. \tag{7}$$

Here the superscript T denotes the transpose, $\otimes$ is the sign of tensor product of matrices, $x^T = (x_{1,*}, x_{2,*}, \ldots, x_{n,*})$, $f^T = (f_{1,*}, f_{2,*}, \ldots, f_{n,*})$, $\widetilde{f}_i^{\,T} = (\widetilde{f^i}_{1,*}, \widetilde{f^i}_{2,*}, \ldots, \widetilde{f^i}_{n,*})$ $(i = 1,2,\ldots)$ are vectors, $x_{i,*}$ is the i-th row of the matrix X, $f_{i,*}$ is the i-th row of the matrix F. Without lost of generality we take $t_0 = 0$. If $f = 0$, $\widetilde{f} = 0$ we have the homogeneous vector bilinear system of differential equations. It is well known [7] that a solution of the homogeneous vector system of differential equations is $x_t = U_\tau^t x_\tau$, where $x_t = x(t)$, U_τ^t is evolution matrix operator. The structure of the evolution operator for impulsive differential system of equations has been considered in [8, 9]. It should be remarked that U_τ^t for impulsive differential system in the Banach space has been considered in [9] and it was call there *multiplicative Stiltjes integral* $U_\tau^t = \widehat{\int}_\tau^t \exp(dF(s))$ which is the solution of equivalent integral equation [9]

$$U_\tau^t = I + \int_\tau^t (dF(s))U_\tau^s, \tag{8}$$

where I is identity operator, operator-valued function $F(t)$ has bounded variation, and $F(t)$ is not continuously differentiable on $[\tau,t]$, it has finite number of discontinuities of first kind at some moment of time t_i $(i = 1,2,\ldots)$. As was underlined in monograph [7] larger part

of result, concerning the properties of differential system of equations in Banach spaces without impulses, may be generalized into impulsive differential system of equations. We denote $[U_\tau^t] : R^{n\times m} \to R^{n\times m}$ the evolution operator of homogeneous matrix impulsive differential system of equation (2), (3) which corresponding to evolution operator (fundamental matrix) U_τ^t associated homogeneous vector system (6), (7). The vector function $x(t) = U_0^t x_0$ is a solution of homogeneous system (6), (7) for any constant vector x_0 and if x_0 ranges over the whole space R^{nm}, then the family of function $x(t)$ forms a space of solutions $\widetilde{M}$ of the system (6), (7). The matrix function $[U_0^t]X_0$, accordingly, is a solution of the homogeneous system, associated to the system (2), (3), for any constant matrix $X_0 \in R^{n\times m}$ and if X_0 ranges over the whole matrix space $R^{n\times m}$ then the family of matrix functions $X_t = [U_0^t]X_0$ form a space of solutions M of the system (2), (3). The operator $[U_0^t]$ we call *fundamental operator* or *evolution operator* (*transfer operator*) [9]. Some problems of an impulsive matrix differential equations have been considered in [12–16].

3 Homogeneous matrix bilinear impulsive differential system

The solution of the homogeneous system (2), (3) can be written in the form $X_t(X_\tau) = [U_\tau^t]X_\tau$, where $[U_\tau^t]$ is nonsingular solution of homogeneous bilinear matrix impulsive differential system (2), (3), which satisfies the condition $[U_\tau^\tau] = [I]$, $[I]$ is used to denote the unit operator in the space of matrix $R^{n\times m}$. The evolution operator of the equation (2), (3) on the interval of continuity we denote $[\Omega_\tau^t]$: $R^{n\times m} \to R^{n\times m}$, where $[\Omega_\tau^t]Z = \underset{A}{\Omega}{}_\tau^t Z \underset{B}{\Omega}{}_t^\tau$, $Z \in R^{n\times m}$; $\underset{A}{\Omega}{}_\tau^t$, $\underset{B}{\Omega}{}_\tau^t$ are matriciants associated with the matrices $A(t)$ and $B(t)$ respectively. $\underset{A}{\Omega}{}_\tau^\tau = I_n$, $\underset{B}{\Omega}{}_\tau^\tau = I_m$; I_n, I_m are unit matrices in the Euclidean spaces R^n and R^m. Let an interval $[\tau, \tau + h]$ contain a finite number of points of discontinuity t_{j+i}, $(i = \overline{s+1, k})$, $s < k$, if $\tau < t_{j+s+1}$ and $t > t_{j+k}$, then evolution operator of system (2), (3) can be represented as

$$[U_\tau^t] = [\Omega_{t_{j+k}}^t] \prod_\nu \left(([I] + [D_\nu])[\Omega_{t_{\nu-1}}^{t_\nu}] \right), \tag{9}$$

where $\nu = j+k, j+k-1, \ldots, j+s+1$, $(k > s)$, we denote $\tau = t_{j+s}$ and it should be noted that point of time τ is not the point of discontinuity. The evolution operator U_τ^t of the system (6), (7), which corresponding to the operator $[U_\tau^t]$ can be written as

$$U_\tau^t = (\underset{A}{\Omega}{}_{t_{j+k}}^t \otimes \underset{B}{\widetilde{\Omega}}{}_t^{t_{j+k}}) \prod_\nu \left((I_{nm} + D_\nu \otimes \widetilde{D}_\nu^T)(\underset{A}{\Omega}{}_{t_{\nu-1}}^{t_\nu} \otimes \underset{B}{\widetilde{\Omega}}{}_{t_\nu}^{t_{\nu-1}}) \right), \tag{10}$$

where $\nu = j+k, j+k-1, \ldots, j+s+1$, $(k > s)$, $I_{nm} = I_n \otimes I_m$, $\widetilde{\Omega} = \Omega^T$, U_τ^t is a solution of the Cauchy problem for the homogeneous system associated with the system (6), (7), with initial conditions $U_\tau^\tau = I_{nm}$, I_{nm} is unit operator in the Euclidean space R^{nm}. Any matrix solution $Z(t)$ of the vector system (6), (7) can be represented as $Z(t) = U_\tau^t Z(\tau)$. By using the Jacobi formula we get from (10), $\det Z_t = \det U_\tau^t \det Z_\tau$. We use the properties: $\det(A_n \otimes B_m) = (\det A)^n (\det B)^m$, $\det \underset{A}{\Omega}{}_{t_{\nu-1}}^{t_\nu} = \exp \int_{t_{\nu-1}}^{t_\nu} \operatorname{Tr} A(s) ds$, $\det \underset{B}{\Omega}{}_{t_\nu}^{t_{\nu-1}} = \exp(-\int_{t_{\nu-1}}^{t_\nu} \operatorname{Tr} B(s) ds)$. Then we have

$$\det U_\tau^t = \exp\left(\int_\tau^t (n \operatorname{Tr} A(s) - m \operatorname{Tr} B(s)) ds \right) \times \prod_\nu \det(I_{nm} + D_\nu \otimes \widetilde{D}_\nu^T), \tag{11}$$

where $\nu = \overline{j+s+1, j+k}$, $(k > s)$, $\operatorname{Tr} A(s) = \sum\limits_{i=1}^n a_{ii}(s)$, $\operatorname{Tr} B(s) = \sum\limits_{i=1}^m b_{ii}(s)$. Because the matrices $(I_{nm} + D_\nu \otimes \widetilde{D}_\nu^T)$ are nonsingular for all ν, it follows from (11) that the matrix $Z(t)$

is nonsingular if $Z(\tau)$ is such. The operator $U_\tau^t : R^{nm} \to R^{nm}$ is topologically equivalent the operator $[U_\tau^t] : R^{n\times m} \to R^{n\times m}$ therefore the operator $[U_\tau^t]$ is nonsingular in the space $R^{n\times m}$. For a nonsingular operator $[U_\tau^t]$, the inverse operator $[U_\tau^t]^{-1} = [U_t^\tau]$ is given by the following

$$[U_t^\tau] = \left(\prod_\nu [\Omega_{t_\nu}^{t_{\nu-1}}]([I] + [D_\nu])^{-1} \right) [\Omega_t^{t_{j+k}}], \tag{12}$$

where $\nu = \overline{j+s+1, j+k}$, $(k > s)$, $t_{j+s} = \tau$, $\tau < t_{j+s+1}$, $t > t_{j+k}$, $[\Omega_\tau^t]^{-1} = [\Omega_t^\tau]$.

4 Nonhomogeneous bilinear matrix impulsive differential system

We call the system (2), (3) with $F(t) \neq 0$, $\widetilde{F}_i \neq 0$ $(i = 1, 2, \ldots)$ nonhomogeneous bilinear matrix equation

$$\frac{dX}{dt} = A(t)X - XB(t) + F(t,), \qquad t \neq t_i, \tag{13}$$

$$\triangle X(t) = D_i X(t) \widetilde{D}_i + \widetilde{F}_i, \qquad t = t_i, \quad i = 1, 2, \ldots, \tag{14}$$

where $A(t)$, $B(t)$, D_i, $\widetilde{D}_i$ are the same as in system (2), (3), $F : [a, b] \to R^{n\times m}$ is a matrix-valued function, continuous (piecewise continuous) on the interval $[a, b]$, $\widetilde{F}_i \in R^{n\times m}$ are constant matrices, $a < t_i < b$, $a = t_0$. We use change of variables $X(t) = [U_{t_0}^t]Y(t)$, which is called "variation of parameter". Taking into account the properties of evolutionary operator $[U_{t_0}^t]$ we get

$$dY/dt = [U_t^{t_0}]F(t), \qquad t \neq t_i, \qquad \triangle Y(t_i) = [U_{t_i+0}^{t_0}]\widetilde{F}_i, \qquad t = t_i, \quad i = 1, 2, \ldots \tag{15}$$

the jump condition can also be written in the form $\triangle Y(t_i) = [U_{t_i}^{t_0}]\widehat{F}_i$, where $\widehat{F}_i = ([I]+[D_i])^{-1}\widetilde{F}_i$. Because the operators $([I] + [D_i]) : R^{n\times m} \to R^{n\times m}$ are nonsingular for all i, $\widehat{F}_i$ is the unique solution of matrix algebraic equation

$$\widehat{F}_i + D_i \widehat{F}_i \widetilde{D}_i = \widetilde{F}_i. \tag{16}$$

The solution of the equation (16) can be written in the form [7]

$$\widehat{F}_i = \sum_{j_1, j_2, r_1=0, r_2=0}^{n,m,n_{j_1},m_{j_2}} \acute{P}_{j_1} \acute{Q}_{j_1}^{r_1} \widetilde{F}_i \grave{P}_{j_2} \grave{Q}_{j_2}^{r_2} \frac{\partial^r \varphi(\lambda_{j_1}, \mu_{j_2})}{r_{12} \partial \lambda_{j_1}^{r_1} \partial \mu_{j_2}^{r_2}}, \tag{17}$$

where $\varphi(\lambda, \mu) = (1 + \lambda\mu)^{-1}$, $r = r_1 + r_2$, $r_{12} = r_1! r_2!$. The matrices $\acute{Q}_{j_1}$, $\grave{Q}_{j_2}$ are defined by the spectral expansion of the matrices D_j and $\widetilde{D}_j$

$$D_j = \sum_{j_1=1}^{n} (\lambda_{j_1} \acute{P}_{j_1} + \acute{Q}_{j_1}), \qquad \widetilde{D}_j = \sum_{j_2=1}^{m} (\mu_{j_2} \grave{P}_{j_2} + \grave{Q}_{j_2}). \tag{18}$$

Taking into account (16) we get for $t > t_0$

$$Y(t) = Y_0 + \int_{t_0}^{t} [U_\tau^{t_0}] F_1(\tau) d\tau. \tag{19}$$

where $F_1(\tau) = F(\tau) + \sum_i [U_{t_i}^{t_0}]\widehat{F}_i \delta(t - t_i)$.

Corollary 1. *Let* $[U_\tau^t]$ *be a fundamental operator homogeneous matrix system* (2), (3), *with nonsingular operators* $([I]+[D_i])$, $(i=1,2,\ldots)$. *Then every solution of nonhomogeneous matrix bilinear impulsive differential system* (2), (3), *for* $t>t_0$ *is given by the formula*

$$X(t)=[U_{t_0}^t]X_0+\int_{t_0}^t[U_\tau^t]F(\tau)d\tau+\sum_i[U_{t_i}^t]\widehat{F}_i, \tag{20}$$

where i *is defined by the inequalities* $t_0<t_i<t$.

This formula can be written by using Stieltjes integral of operator-valued function $[U_\tau^t]$ by the matrix-valued function $L(t)$, where $L(t)=\int_{t_0}^t(F(\tau)+\sum_i\widehat{F}_i\delta(\tau-t_i))d\tau=\int_{t_0}^tF(\tau)d\tau+\sum_i\widehat{F}_i\eta(\tau-t_i)$ is matrix-valued function of bounded variation on $[t_0,t)$. Here $\delta(t-t_i)$ is the Dirac delta function, $\eta(t-t_i)$ is the Heaviside step-function or jump-function. Both of these functions are generalized functions or distributions

$$X(t)=[U_{t_0}^t]X_0+\int_{t_0}^t[U_\tau^t]dL(t), \tag{21}$$

where $[U_\tau^t]$ is the multiplicative Stiltjes integral [7, 10] or evolution operator of homogeneous bilinear matrix impulsive equation corresponding to (2), (3).

5 Bilinear matrix impulsive system of differential equation with constant coefficients

Let the matrices $A(t)$, $B(t)$, D_i, $\widetilde{D}_i$ in system (2), (3) be constant. Then we have a bilinear matrix impulsive system of differential equations with constant coefficients

$$\frac{dX}{dt}=AX-XB+F(t),\qquad t\neq t_i, \tag{22}$$

$$\triangle X(t)=DX(t)\widetilde{D}+\widetilde{F}_i,\qquad t=t_i,\quad i=1,2,\ldots. \tag{23}$$

At first we consider homogeneous system, i.e. $F(t)\equiv 0$, $\widetilde{F}_i\equiv 0$. Suppose that the times t_i are indexed by the set of natural numbers such that $t_i\to\infty$ for $i\to\infty$. Without loss of generality, we can assume that $t_1>t_0$. It is easy to see that for any (t_0,X_0), there exists a unique solution $X_{t_0}^t$ of (22), (23) for $t\geq t_0$, it can be written as $X_t(t_0,X_0)=[U_{t_0}^t]X_0$, where

$$[U_{t_0}^t]=[\Omega_{t_k}^t]\prod_{\nu=k}^{1}\Big(([I]+[D])[\Omega_{t_{\nu-1}}^{t_\nu}]\Big) \tag{24}$$

and $[\Omega_s^t]Z=e^{A(t-s)}Ze^{-B(t-s)}$. Solutions of the homogeneous matrix system are not invariant with respect to shifts because, due to the times of an impulsive effect $t=t_i$, homogeneous matrix system is not autonomous. However, in some cases expression (24) can be simplified. For example, if the matrices A and D, B and $\widetilde{D}$ commute, then matrix exponent e^{At} commutes with the matrix D and e^{-Bt} commutes with the matrix $\widetilde{D}$ and equality (24) can be written as

$$X_t(t_0,X_0)=([I]+[D])^{p(t,t_0)}e^{A(t-t_0)}X_0e^{-B(t-t_0)}, \tag{25}$$

where $p(t,t_0)$ is the number of points t_i which belong to the segment $[t,t_0]$, i.e. $p(t,t_0)=k$ if $t_k<t<t_{k-1}$, $t_0<t_1$. In particular, if the times t_i are equidistant, $t_i=t_1+(i-1)\theta$ and operator $([I]+[D])$ is nonsingular, then from (25) we get

$$X_t(t_0,X_0)=e^{Ln([I]+[D])p(t,t_0)}e^{A(t-t_0)}X_0e^{-B(t-t_0)},$$

where $p(t,t_0) = (t-t_1)/\theta - \{(t-t_1)/\theta\})$, curly brackets $\{\cdot\}$ means fractional part of the real value $(t-t_1)/\theta$. For the equivalent vector equation (6), (7), we have

$$x_t(x_0) = \Theta(t)e^{S(t-t_1)},$$

where

$$S = \widehat{D} + \frac{1}{\theta}Ln(I_{nm} + D \otimes \widetilde{D}), \widehat{D} = A \otimes I_m - I_n \otimes B^T,$$
$$\Theta(t) = e^{\widehat{D}(t_1-t_0)}e^{-Ln(I_{nm}+D\otimes\widetilde{D})(\{(t-t_1)/\theta\}+1)}.$$

From this, it can be seen that behavior of the solution of an impulsive system is defined by eigenvalues of the matrix S because matrix $\Theta(t)$ is bounded. Consequently, if the real parts of all eigenvalues of the matrix S are negative, then all solutions of homogeneous system corresponding to the system (22), (23) tend to zero for $t \to \infty$.

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Discrete Fourier Inversion of Linear Inhomogeneity

Howard S. COHL

School of Physics, University of Exeter, UK
E-mail: *hcohl@shell42.com*

Solutions to tensor, vector, and scalar linear inhomogeneous partial differential equations can be obtained by discrete Fourier inversion of the linear system. The inverse (Green's function) problem can be cast into sets of single, double, and triple summation/integration expressions by using transcendental eigenfunction expansions in certain three-dimensional triply-orthogonal geometries whose solutions admit simple and R-separation of variables. By reversing and collapsing traditional ordering schemes for the inverse problem, one can derive new special function addition theorems for asymmetric, axisymmetric and cylindrical, orthogonal curvilinear coordinate geometries. In this paper, we introduce such important applications as inhomogeneous Laplace, Helmholtz, wave, Schrödinger, heat, Klein–Gordon, Laplace–Beltrami, biharmonic, triharmonic as well as mixed higher-order harmonic and linear inhomogeneous partial differential equations. Compact expressions are seen to exist in spherical geometries through the utilization of separation angle spherical linear system-function expansions.

1 Introduction

Inhomogeneity is defined as the presence of a source contribution to an otherwise homogeneous linear partial differential equation. Hereafter we refer to linear partial differential equations as linear systems and their solutions as linear system-functions. Inhomogeneity in linear systems is expressed as a non-zero scalar, vector, or tensor valued function ρ. The presence of inhomogeneity usually implies non-zero potential energies through assemblage and confinement. The precise influence due to specific inhomogeneity is often required in many different mathematical and physical contexts. Inhomogeneous variations are described below by adopting general curvilinear coordinate systems, such as spherical and cylindrical, which have been previously studied in the literature [1].

In coordinate systems which allow simple and R-separation of variables for linear systems, rearranging the orderings of multi-summation-integration expressions for infinite-extent Green's functions reveals new special function descriptions. Green's functions in spherical coordinates span an infinite two-dimensional space of quantum numbers (ℓ, m) [2]. Traditional precedent in spherical coordinates $\boldsymbol{x} = (r, \phi, \theta)$ and $\boldsymbol{x}' = (r', \phi', \theta')$ is to sum first over the orbital quantum number ℓ

$$\mathcal{G} = \sum_{\ell=0}^{\infty} \mathcal{G}_\ell, \tag{1}$$

where $\mathcal{G} = \mathcal{G}(\boldsymbol{x}, \boldsymbol{x}', t, t')$, and then to sum second over the azimuthal quantum number m

$$\mathcal{G}_\ell = \sum_{m=-\ell}^{\ell} \mathcal{G}_{\ell,m}. \tag{2}$$

Equation (2) represents an addition theorem for spherical system-functions. By rearranging the traditional ordering we obtain a separate algorithm for summing over the infinite two-

dimensional (ℓ, m) space, first over m

$$\mathcal{G} = \sum_{m=-\infty}^{\infty} \mathcal{G}_m \, e^{im(\phi-\phi')}, \tag{3}$$

and then over ℓ

$$\mathcal{G}_m = \sum_{\ell=|m|}^{\infty} \mathcal{G}_{\ell,m}. \tag{4}$$

The coefficients $\mathcal{G}_m$ are a better basis than $\mathcal{G}_\ell$ since they represent an infinite series over $\mathcal{G}_{\ell,m}$ (equation (4)) rather than a finite series over $\mathcal{G}_{\ell,m}$ (equation (2)). This is also true for linear system-functions in oblate and prolate spheroidal coordinates.

Rotationally invariant coordinate systems possess the azimuthal coordinate ϕ and can therefore be described using Fourier analysis over the azimuthal quantum number m. The coefficients $\mathcal{G}_m$ in cylindrical coordinates are given by the definite integral

$$\mathcal{G}_m = \int_0^{\infty} d\kappa \, \mathcal{G}_m(\kappa). \tag{5}$$

So, in cylindrical coordinates, the problem of obtaining the Fourier coefficients $\mathcal{G}_m$ of a particular linear system is reduced to solving a definite integral which represents the Fourier coefficients [3, 4]. Definite integrals arise for $\mathcal{G}_m$ in parabolic coordinates as well.

Since all linear systems admit solutions through Green's functions, higher-order linear systems represent excellent candidates for discrete azimuthal and separation angle Fourier Green's function analysis. Examples of higher-order linear systems include poly-operator systems such as the biharmonic and triharmonic equations. Poly-operator systems are generated by using integral products of specific linear operators. One can also construct higher-order, hybrid-operator linear systems by introducing different linear operators in a product formalism. There are an infinite number of combinations resulting in linear poly-operator or hybrid-operator inhomogeneous systems. One example which uses the poly-operator formalism has recently been treated by Cohl [5]. There, compact Fourier Green's function expansions are obtained for a biharmonic equation which is equivalent to Poisson's equation. One of the nice features of higher-order poly-operator systems is that the transformed kernel is converted to one which is no longer locally singular. Examples of higher-order inhomogeneous linear systems include the vector and scalar n-Laplace, n-Helmholtz, n-heat, n-wave, n-Schrödinger, n-Klein–Gordon, and n-Laplace–Beltrami equations.

For every linear system, there exist unique multi-summation-integration expressions for the appropriate Green's function in spherical and cylindrical coordinates [6]. The derivation of these addition theorems is critically important to our description and understanding of the properties of linear system-functions [7–10]. Other rearrangements are possible in asymmetric coordinate geometries such as ellipsoidal, paraboloidal, and conical.

2 Separation angle Fourier analysis

The separation angle in an orthogonal curvilinear coordinate system can be obtained by examining the dot product of the observation vector $\boldsymbol{x} = (r, \phi, \theta)$ with the source vector $\boldsymbol{x}' = (r', \phi', \theta')$,

$$\boldsymbol{x} \cdot \boldsymbol{x}' = rr' \cos\gamma, \tag{6}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$ is called the separation angle. When $\gamma = 0$ both vectors have the same orientation, and the relative distances are determined by r and r'. As γ is

increased from zero (radial alignment) to π (radial anti-alignment), the surface of constant γ maps out an infinite cone in space. One may apply a discrete separation angle Fourier analysis of Green's system-functions $\mathcal{G}$, on a sphere, given by

$$\mathcal{G} = \sum_{n=-\infty}^{\infty} \mathcal{G}_n \, e^{in\gamma}, \tag{7}$$

where n is the separation angle quantum number [4, 5] and $\mathcal{G}_n = \mathcal{G}_n(r, r', t, t')$.

Ordinarily, Green's functions in spherical coordinates are expanded in spherical harmonics. One may directly compare the relative strengths of expansions in Legendre polynomials (spherical harmonics, see equation (1)) as compared to discrete Fourier expansions (equation (7)). Discrete separation angle Fourier analysis of functions on spheres is more compact than spherical harmonic analysis (which uses integer degree, integer order, associated Legendre functions). By using separation angle Fourier analysis, one can take advantage of the full spherical component, which is contained entirely within the $n = 0$ term in the expansion. Notice that the Fourier expansion is fully 2π-periodic and necessarily takes full advantage of this particular property of spheres. A second 2π-periodic angle is seen as an azimuthal coordinate about the line of centers which connects the two rays $\gamma = (0, \pi)$. The $n = 0$ term in the expansion represents the offset of a spherical density distribution which is constant at any (ϕ, θ) orientation. It requires an infinite sum over the orbital angular momentum quantum number ℓ to obtain the true spherically symmetric contribution in a modal analysis. For the Laplace equation, the separation angle Fourier coefficients are given by the odd-half-integer degree associated Legendre function of the second kind, which is a toroidal harmonic [11, 12]. We suggest that separation angle Fourier analysis represents a set of spherical linear system-functions which are more fundamental when solving physical problems, and even in pure mathematical contexts on spheres.

3 Newtonian/Coulomb potential self-energy

Here we introduce the potential self-energy W for the inhomogeneous Laplace equation (Poisson's equation)

$$W = \frac{1}{2} \int d^3\boldsymbol{x} \; \Phi(\boldsymbol{x}) \rho(\boldsymbol{x}), \tag{8}$$

where $\boldsymbol{x}$ is the observation vector. $\Phi(\boldsymbol{x})$ is the potential function generated by the source density function $\rho(\boldsymbol{x})$

$$\nabla^2 \Phi(\boldsymbol{x}) = 4\pi \rho(\boldsymbol{x}), \tag{9}$$

where ∇^2 is the Laplacian operator. An integral equivalent representation of equation (9) is given by

$$\Phi(\boldsymbol{x}) = -\frac{1}{2} \int d^3\boldsymbol{x}' \; |\boldsymbol{x} - \boldsymbol{x}'| \; \nabla'^2 \rho(\boldsymbol{x}'), \tag{10}$$

where $\boldsymbol{x}'$ is the source vector. The integral representation of equation (9) is preferable to the standard representation

$$\Phi(\boldsymbol{x}) = -\int d^3\boldsymbol{x}' \; \frac{\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}, \tag{11}$$

because the kernel of integration (Green's function) is better behaved locally. Notice that the kernel of integration $\frac{1}{2}|\boldsymbol{x} - \boldsymbol{x}'|$ is the Green's function for the biharmonic equation [5].

The potential self-energy W is often required in energy minimization techniques in classical and non-classical physics. Due to the global nature of these types of problems, modest errors in the computation of the potential self-energy will often impede the ability of such algorithms, which otherwise yield satisfactory results.

This potential self-energy can be computed as a three-space integral, given that the potential functions are known. The process of obtaining the potential functions is obtained by solving the boundary value problem with the appropriate Dirichlet boundary conditions, which should be obtained by a modal analysis. The choice of a Fourier basis has clear advantages when viewed in terms of a total potential self-energy computation. The choice of expanding physical variables in separation angle or azimuthal angle Fourier bases affords two separate choices in the facility of the computation of the potential self-energy for different types of mass or charge distributions.

The separation distance [5] is given by

$$|\boldsymbol{x}-\boldsymbol{x}'| = \Psi \sum_{m=-\infty}^{\infty} \left(m^2 - \frac{1}{4}\right)^{-1} Q^1_{m-\frac{1}{2}}(\chi)\; e^{im(\phi-\phi')}, \tag{12}$$

where

$$\chi = \frac{R^2 + R'^2 + (z-z')^2}{2RR'}, \tag{13}$$

and

$$\Psi = \frac{\sqrt{\left(R^2 - R'^2\right)^2 + 2\left(R^2 + R'^2\right)(z-z')^2 + (z-z')^4}}{2\pi\sqrt{RR'}}. \tag{14}$$

The corresponding expression in spherical separation angle Fourier modes [5] is given by

$$|\boldsymbol{x}-\boldsymbol{x}'| = \frac{r_>^2 - r_<^2}{2\pi\sqrt{rr'}} \sum_{n=-\infty}^{\infty} \left(n^2 - \frac{1}{4}\right)^{-1} Q^1_{n-\frac{1}{2}}\left(\frac{r^2 + r'^2}{2rr'}\right) e^{in\gamma}. \tag{15}$$

In both cases, equations (12) and (15), one may take advantage of compact Fourier Green's function expansions. When applied to the azimuthal case, one may easily obtain the total potential self-energy for purely axisymmetric ($m = 0$) source distributions. In terms of the separation angle, one may also obtain the total potential self-energy for spherically symmetric ($n = 0$) source distributions. If the spherically symmetric ($n = 0$) contribution is used, the total energy is reduced to a one-dimensional integral over the r' variable. This contribution is easily gained by integrating over the appropriate complete elliptic integrals of the first and second kind (see below). If the axisymmetric ($m = 0$) contribution is needed, the total potential self-energy is reduced to a two-dimensional integral over the R' and z' variables.

3.1 Potential self-energy for axisymmetric source distributions

We now compute the potential self-energy for axisymmetric source distributions. In this case, only the ($m = 0$) term in the integral survives, therefore

$$W = \pi \int_0^{\infty} dR\; R \int_{-\infty}^{\infty} dz\; \rho_0(R,z)\Phi_0(R,z), \tag{16}$$

where (R, z) represent the radial and vertical circular cylindrical positions respectively. The axisymmetric ($m = 0$) Fourier component of a general nonaxisymmetric density distribution is given by

$$\rho_0(R,z) = \frac{1}{2\pi} \int_0^{2\pi} d\phi\; \rho(R,\phi,z), \tag{17}$$

and the azimuthal angle ($m = 0$) Fourier component of the potential is given by

$$\Phi_0(R,z) = -\frac{1}{2}\int d^3\boldsymbol{x}'\ |\boldsymbol{x}-\boldsymbol{x}'|_{m=0}\ \nabla'^2\rho_0(R',z'). \tag{18}$$

The azimuthal angle ($m = 0$) Fourier component of the separation distance is given by [5]

$$|\boldsymbol{x}-\boldsymbol{x}'|_{m=0} = -\frac{2Q^1_{-\frac{1}{2}}(\chi)}{\pi\sqrt{RR'}}\left[\left(R^2-R'^2\right)^2+2\left(R^2+R'^2\right)(z-z')^2+(z-z')^4\right]^{\frac{1}{2}}, \tag{19}$$

where $Q^1_{-\frac{1}{2}}$ is the unit order, negative one-half degree, associated Legendre function of the second kind. The azimuthal angle ($m = 0$) Fourier component can be simplified as

$$|\boldsymbol{x}-\boldsymbol{x}'|_{m=0} = \frac{2}{\pi}\alpha E\left(\sqrt{\frac{4RR'}{(R+R')^2+(z-z')^2}}\right), \tag{20}$$

where E is the complete elliptic integral of the second kind [8] and

$$\alpha = \sqrt{\frac{\left(R^2-R'^2\right)^2+2\left(R^2+R'^2\right)(z-z')^2+(z-z')^4}{(R-R')^2+(z-z')^2}}. \tag{21}$$

Finally, the azimuthal angle ($m = 0$) Fourier component of the potential is given by

$$\Phi_0(R,z) = -2\int_0^\infty dR'\ R'\ \int_{-\infty}^\infty dz'\ \alpha E\left(\sqrt{\frac{4RR'}{(R+R')^2+(z-z')^2}}\right)\nabla'^2\rho_0(R',z'). \tag{22}$$

3.2 Potential self-energy for spherically symmetric source distributions

We now compute the potential self-energy for spherically symmetric source distributions. In this case, only the ($n = 0$) term in the integral survives, therefore

$$W = 2\pi\int_0^\infty dr\ r^2\rho_0(r)\Phi_0(r), \tag{23}$$

where r is the spherical radial coordinate and the spherically symmetric ($n = 0$) Fourier component of a general nonspherical density distribution is given by

$$\rho_0(r) = \frac{1}{4\pi}\int_0^{2\pi} d\phi\ \int_0^\pi d\theta\ \sin\theta\ \rho(r,\phi,\theta). \tag{24}$$

The separation angle ($n = 0$) Fourier component of the potential is given by

$$\Phi_0(r) = -\frac{1}{2}\int d^3\boldsymbol{x}'\ |\boldsymbol{x}-\boldsymbol{x}'|_{n=0}\nabla'^2\rho_0(r'), \tag{25}$$

and the separation angle ($n = 0$) Fourier component of the separation distance is given by

$$|\boldsymbol{x}-\boldsymbol{x}'|_{n=0} = -2\frac{r_>^2-r_<^2}{\pi\sqrt{rr'}}Q^1_{-\frac{1}{2}}\left(\frac{r^2+r'^2}{2rr'}\right), \tag{26}$$

where the subscript notation $r_>$ and $r_<$ represents the greater and lesser of the spherical distances respectively. The separation angle ($n = 0$) Fourier component of the separation distance can be simplified to

$$|\boldsymbol{x}-\boldsymbol{x}'|_{n=0} = \frac{2}{\pi}(r+r')E\left(\frac{2\sqrt{rr'}}{r+r'}\right). \tag{27}$$

Finally, the separation angle ($n = 0$) Fourier component of the potential is given by

$$\Phi_0(r) = -4\int_0^\infty dr'\ r'^2(r+r')E\left(\frac{2\sqrt{rr'}}{r+r'}\right)\nabla'^2\rho_0(r'). \tag{28}$$

4 Summary

In order for one to compute the solution of an arbitrary linear inhomogeneous problem, one has to first obtain the azimuthal Fourier component of the Green's function in question. The Fourier Green's function expansion has been obtained [3, 4, 12] for the Laplace and biharmonic equations. One must also look at the Fourier expansions of other important linear partial differential equations such as the Helmholtz, heat, wave, Schrödinger, Klein–Gordon, Laplace–Beltrami, triharmonic, and higher-order equations. There are an infinite number of product and sum combinations of separate linear operators which can represent even higher-order linear differential operators in space, time, and space-time. By obtaining these results, the integral inverse (Green's function) for that particular problem will have a highly-useful compact Fourier representation. Furthermore, these Green's function expansions will be expressible in all the coordinate systems which allow simple and R-separation of variables (see Moon and Spencer [9, 10] and Miller [7]). Relevant coordinate systems include spherical, cylindrical, parabolic, oblate and prolate spheroidal, and toroidal.

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On Geodesical Extension of the Schwarzshild Space-Time

Valerii S. DRYUMA

Institute of Matematics & Informatics AS Moldova, 5 Academy Str., Kishinev, Moldova
E-mail: *valery@dryuma.com, cainar@mail.md*

The properties of geodesical extension of the Schwarzshild metric for the radial motion of test particle are studied.

1 Introduction

The notion of the Riemann extension of non-Riemannian spaces was introduced first in [1]. Main idea of this theory is to apply the methods of Riemann geometry for studying of the properties of non-Riemannian spaces.

For example, the system of differential equations in form

$$\frac{d^2x^k}{ds^2}+\Pi^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=0 \tag{1}$$

with arbitrary coefficients $\Pi^k_{ij}(x^l)$ can be considered as the system of geodesic equations of affinely connected space with local coordinates x^k.

For the n-dimensional Riemannian spaces with the metrics

$${}^n ds^2=g_{ij}dx^i dx^j$$

the system of geodesic equations looks similar, but the coefficients $\Pi^k_{ij}(x^l)$ now have very special form and depend on the choice of the metric g_{ij}

$$\Pi^i_{kl}=\Gamma^i_{kl}=\frac{1}{2}g^{im}(g_{mk,l}+g_{ml,k}-g_{kl,m}).$$

In order for methods of Riemann geometry to be applied for studying of the properties of the spaces with equations (1) construction of $2n$-dimensional extension of the space with local coordinates x^i was introduced.

The metric of extended space constructs with help of coefficients of equation (1) and looks as follows

$${}^{2n}ds^2=-2\Pi^k_{ij}(x^l)\Psi_k dx^i dx^j+2d\Psi_k dx^k, \tag{2}$$

where Ψ_k are the coordinates of additional space.

The important property of such metric is that the geodesic equations of metric (2) consist of two parts

$$\ddot{x}^k+\Gamma^k_{ij}\dot{x}^i\dot{x}^j=0, \tag{3}$$

and

$$\frac{\delta^2\Psi_k}{ds^2}+R^l_{kji}\dot{x}^j\dot{x}^i\Psi_l=0, \tag{4}$$

where

$$\frac{\delta\Psi_k}{ds} = \frac{d\Psi_k}{ds} - \Gamma^l_{jk}\Psi_l\frac{dx^j}{ds}.$$

The first part (3) of the complete system is the system of equations for geodesics of basic space with local coordinates x^i and it does not contains the coordinates Ψ_k.

The second part (4) of system of geodesic equations has the form of linear 4×4 matrix system of second order ODE's for coordinates Ψ_k

$$\frac{d^2\vec{\Psi}}{ds^2} + A(s)\frac{d\vec{\Psi}}{ds} + B(s)\vec{\Psi} = 0 \tag{5}$$

with the matrix

$$A = A(x^i(s), \dot{x}^i(s)), \qquad B = B(x^i(s), \dot{x}^i(s)).$$

From this point of view we have the case of geodesical extension of the basic space (x^i).

It is important to note that the geometry of extended space is connected with geometry of basic space.

For example the property of such space to be a Ricci-flat one keeps also for the extended space.

This fact give us the possibility to use the linear system of equation (5) for studying of the properties of basic space.

In particular the invariants of the 4×4 matrix-function

$$E = B - \frac{1}{2}\frac{dA}{ds} - \frac{1}{4}A^2$$

under change of the coordinates Ψ_k can be used for that.

The first application of the notion of extended spaces to study of nonlinear second-order differential equations connected with nonlinear dynamical systems was made in paper of the author [2–4].

Here we consider the properties of extended spaces for the Einstein-spaces in General Relativity.

2 The Schwarzschild space-time and geodesic equations

The line element of standard metric of the Schwarzschild space-time in coordinate system x, θ, ϕ, t has the form

$$ds^2 = \frac{1}{(1-2M/x)}dx^2 + x^2(d\theta^2 + \sin^2\theta d\phi^2) - (1-2M/x)dt^2. \tag{6}$$

The geodesic equations of this type of the metric are given by

$$\ddot{x} + \frac{M}{x(2M-x)}\dot{x}^2 + (2M-x)\dot{\theta}^2 + (2M-x)\sin^2\theta\dot{\phi}^2 - \frac{M(2M-x)}{x^3}\dot{t}^2 = 0, \tag{7}$$

$$\ddot{\theta} + \frac{2}{x}\dot{x}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{8}$$

$$\ddot{\phi} + \frac{2}{x}\dot{x}\dot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\phi}\dot{\theta} = 0, \tag{9}$$

$$\ddot{t} - 2\frac{M}{x(2M-x)}\dot{x}\dot{t} = 0. \tag{10}$$

The symbols of Christoffel of the metric (6) look as

$$\Gamma^1_{11}=\frac{M}{x(2M-x)},\qquad \Gamma^1_{22}=(2M-x),\qquad \Gamma^1_{33}=(2M-x)\sin^2\theta,$$
$$\Gamma^1_{44}=-\frac{M(2M-x)}{x^3},\qquad \Gamma^2_{12}=\frac{1}{x},\qquad \Gamma^2_{33}=-\sin\theta\cos\theta,\qquad \Gamma^3_{23}=\frac{\cos\theta}{\sin\theta},$$
$$\Gamma^4_{14}=-\frac{M}{x(2M-x)},\qquad \Gamma^3_{13}=\frac{1}{x}.$$

The equations of geodesic (7)–(10) have the first integrals

$$x^4\dot{x}^2=E^2x^4-(x^2-2Mx)(\mu^2x^2+L^2),\quad x^4\dot{\theta}^2=L^2-\frac{L^2}{\sin^2\theta},\tag{11}$$
$$\dot{\phi}=\frac{L}{x^2\sin^2\theta},\qquad \dot{t}=\frac{E}{(1-2M/x)}.\tag{12}$$

where a dot denotes differentiation with respect to parameter s and (E,μ,L) are the constants of motion.

3 The Riemann extension of the Schwarzschild metric

Now with the help of the formulae (2) we construct the eight-dimensional extension of basic metric (6)

$$\begin{aligned}ds^2=&-\frac{2M}{x(2M-x)}Pdx^2-\frac{2}{x}Qdxd\theta-2(2M-x)Pd\theta^2-\frac{2}{x}Udxd\phi+2\frac{M}{x(2M-x)}Vdxdt\\&-2\frac{\cos\theta}{\sin\theta}Ud\phi d\theta-2((2M-x)\sin^2\theta P-\sin\theta\cos\theta Q)d\phi^2+2\frac{M(2M-x)}{x^3}Pdt^2\\&+2dxdP+2d\theta dQ+2d\phi dU+2dtdV,\end{aligned}\tag{13}$$

where (P,Q,U,V) are the set of additional coordinates.

The eight-dimensional space in local coordinates $(x,\theta,\phi,t,P,Q,U,V)$ with this type of metric is also the Einstein space with the condition on the Ricci tensor

$${}^8R_{ik}=0.$$

The complete system of geodesic equations for the metric (7) is divided into two parts.

The first part coincides with the equations (7)–(10) on the coordinates (x,θ,ϕ,t) and second part forms the linear system of equations for coordinates P, Q, U, V.

They are defined as

$$\begin{aligned}&\ddot{P}+\frac{2M}{x(x-2M)}\dot{x}\dot{P}-\frac{2}{x}\dot{\theta}\dot{Q}-\frac{2}{x}\dot{\phi}\dot{U}-\frac{2M}{x(x-2M)}\dot{t}\dot{V}\\&\quad-\left(\frac{2M}{x^2(x-2M)}\dot{x}^2+\frac{(x-2M)}{x}\dot{\theta}^2+\frac{\sin^2\theta(x-2M)}{x}\dot{\phi}^2+\frac{2M(x-2M)}{x^4}\dot{t}^2\right)P\\&\quad+\left(\frac{4}{x^2}\dot{x}\dot{\theta}-\frac{2\cos\theta}{x}\dot{\phi}^2\right)Q+\left(\frac{4}{x^2}\dot{x}\dot{\phi}+\frac{4\cos\theta}{x\sin\theta}\dot{\theta}\dot{\phi}\right)U+\left(\frac{4M^2}{x^2(x-2m)^2}\dot{x}\dot{t}\right)V=0,\\&\ddot{Q}+2(x-2m)\dot{\theta}\dot{P}-\frac{2}{x}\dot{x}\dot{Q}-\frac{2\cos\theta}{\sin\theta}\dot{\phi}\dot{U}-\frac{2(x-4M)}{x}\dot{x}\dot{\theta}P\\&\quad+\left(\frac{2(x-3M)}{x^2(x-2M)}\dot{x}^2-\frac{2(x-2M)}{x}\dot{\theta}^2-\frac{(x-4M\sin^2\theta)}{x}\dot{\phi}^2+\frac{2M(x-2M)}{x^4}\dot{t}^2\right)Q\end{aligned}$$

$$+\left(\frac{4\cos\theta}{x\sin\theta}\dot{x}\dot{\phi}+\frac{4\cos^2\theta}{\sin^2\theta}\dot{\theta}\dot{\phi}\right)U=0,$$

$$\ddot{U}+2\sin^2\theta(x-2M)\dot{\phi}\dot{P}+2\sin\theta\cos\theta\dot{\phi}\dot{Q}-\left(\frac{2\cos\theta}{\sin\theta}\dot{\theta}+\frac{2}{x}\dot{x}\right)\dot{U}-\frac{2\sin^2\theta(x-4M)}{x}\dot{x}\dot{\phi}P$$
$$-\left(\frac{4\sin\theta\cos\theta}{x}\dot{x}\dot{\phi}+2\dot{\theta}\dot{\phi}\right)Q+\left(\frac{2(x-3M)}{x^2(x-2m)}\dot{x}^2+\frac{4\cos\theta}{x\sin\theta}\dot{x}\dot{\theta}\right.$$
$$\left.+\frac{2(x\cos^2\theta+2M\sin^2\theta)}{x\sin^2\theta}\dot{\theta}^2-\frac{2(x-2M\sin^2\theta)}{x}\dot{\phi}^2\right)U=0,$$

$$\ddot{V}-\frac{2M(x-2m)}{x^3}\dot{t}\dot{P}-\frac{2M}{x(x-2M)}\dot{x}\dot{V}+\frac{4M(x-2M)}{x^4}\dot{x}\dot{t}P$$
$$+\left(\frac{2M(2x-3M)}{x^2(x-2M)^2}\dot{x}^2-\frac{2M}{x}\dot{\theta}^2-\frac{2M\sin^2\theta}{x}\dot{\phi}^2+\frac{2M^2}{x^4}\dot{t}^2\right)V=0.$$

So we get the linear matrix-second order ODE for the coordinates U, V, P, Q

$$\frac{d^2\Psi}{ds^2}+A(x,\theta,\phi,t)\frac{d\Psi}{ds}+B(x,\theta,\phi,t)\Psi=0, \tag{14}$$

where

$$\Psi(s)=\begin{pmatrix}P(s)\\Q(s)\\U(s)\\V(s)\end{pmatrix}$$

and A, B are some 4×4 matrix-functions depending on the coordinates $x(s)$, $\theta(s)$, $\phi(s)$, $t(s)$ and their derivatives.

We shall study this system of equations at the conditions $\theta=\pi/2$, $L=0$, $M=\mu$ and $\phi(s)=\text{const}$. This is simplest case and it corresponds to the radial motion of the test particle in the Schwarzschild space-time.

With these conditions matrix A takes the form

$$\begin{bmatrix}-\frac{2M}{x(-2M+x)}\dot{x} & 0 & 0 & \frac{2M}{x(-2M+x)}\dot{t}\\ 0 & \frac{2}{x}\dot{x} & 0 & 0\\ 0 & 0 & \frac{2}{x}\dot{x} & 0\\ \frac{2M(-2M+x))}{x^3}\dot{t} & 0 & 0 & \frac{2M}{x(-2M+x)}\dot{x}\end{bmatrix},$$

and matrix B has the elements

$$B_{11}=-\frac{\left(2Mx^2+2Mx^2\dot{x}^2-8M^2x\dot{t}^2+8M^3\dot{t}^2\right)}{x^4(x-2M)},\qquad B_{12}=0,\qquad B_{13}=0,$$

$$B_{14}=\frac{4M^2\dot{x}\dot{t}}{x^2(x-2M)^2},\qquad B_{21}=0,$$

$$B_{22}=\frac{\left(2x^3\dot{x}^2-8M^2x\dot{t}^2+8M^3\dot{t}^2+2Mx^2\dot{t}^2-6Mx^2\dot{x}^2\right)}{x^4(x-2M)},\qquad B_{23}=0,\qquad B_{24}=0,$$

$$B_{31}=0,\qquad B_{32}=0,\qquad B_{33}=2\frac{\left(4M^3\dot{t}^2+x^3\dot{x}^2-3Mx^2\dot{x}^2+Mx\dot{t}^2-4M^2x\dot{t}^2\right)}{x^4(x-2M)},$$

$$B_{34}=0,\qquad B_{41}=4\frac{M(x-2M)\dot{x}\dot{t}}{x^4},\qquad B_{42}=0,\qquad B_{43}=0,$$

$$B_{44}=2M\frac{\left(4M^3\dot{t}^2+2x^3\dot{x}^2-3Mx^2\dot{x}^2+Mx^2\dot{t}^2-4M^2x\dot{t}^2\right)}{x^4(x-2M)^2}.$$

The equations for the coordinates of x and t are

$$x^4\dot{x}^2=\mu^2x^4-(x^2-2Mx)(\mu^2x^2)=2M\mu^2x^2,$$
$$\dot{t}=\frac{\mu}{(1-2M/x)}.$$

They have the solutions

$$x(s)=1/4\frac{12^{2/3}\left(\mu\,M^2\,(s-C_1)\,\sqrt{2}\right)^{2/3}}{M} \tag{15}$$

and

$$t(s)=\mu s+2\frac{\mu\sqrt[3]{3}\sqrt[3]{2}M}{\sqrt[3]{M\mu^2}}\sqrt[3]{s}$$
$$+\left(-4\,\mu\,M^{3/2}\mathrm{arctanh}\left(\frac{1}{2}\frac{\sqrt[3]{3}\sqrt[3]{2}\sqrt[6]{M\mu^2}}{\sqrt{M}}\sqrt[3]{s}\right)+C_2\,\sqrt{M\mu^2}\right)\frac{1}{\sqrt{M\mu^2}}, \tag{16}$$

where C_i are parameters.

Now we will solve our matrix system of equations (14) with a given matrix A and B.

The second and third equations of the system are given by

$$\frac{d^2}{ds^2}Q(s)-2\frac{\dot{x}}{x}\frac{d}{ds}Q(s)-\frac{\left(6Mx^2\dot{x}^2+8M^2x\dot{t}-2x^3\dot{x}^2-8M^3\dot{t}^2-2Mx^2\dot{t}^2)\right)}{x^4(x-2M)}Q(s)=0, \tag{17}$$
$$\frac{d^2}{ds^2}U(s)-2\,\frac{\dot{x}}{x}\frac{d}{ds}U(s)-\frac{\left(6Mx^2\dot{x}^2+8M^2x\dot{t}-2x^3\dot{x}^2-8M^3\dot{t}^2-2Mx^2\dot{t}^2)\right)}{x^4(x-2M)}U(s)=0. \tag{18}$$

The first and fourth equations of the given system have the form of 2×2 matrix system of linear second order differential equations with the variable coefficients

$$\frac{d^2}{ds^2}P(s)+\frac{2M\dot{x}}{x(x-2M)}\frac{d}{ds}P(s)-\frac{2M\dot{t}}{x(x-2M)}\frac{d}{ds}V(s)$$
$$-\frac{\left(2\,Mx^2\dot{t}^2+2Mx^2\dot{x}^2-8M^2x\dot{t}^2+8M^3\dot{t}^2\right)}{x^4(x-2M)}P(s)+\frac{4Mx^2\dot{x}\dot{t}}{x^2(x-2M)^2}V(s)=0, \tag{19}$$

and

$$\frac{d^2}{ds^2}V(s)-\frac{2M\left(-2M+x)\right)\dot{t}}{x^3}\frac{d}{ds}P(s)-\frac{2M\dot{x}}{x(x-2M)}\frac{d}{ds}V(s)+\frac{4M(x-2M)\dot{x}\dot{t}}{x^4}P(s)$$
$$+\frac{2M\left(-3Mx^2\dot{x}^2+2x^2\dot{x}^2-4M^2x\dot{t}^2+Mx^2\dot{t}^2+4M^3\dot{t}^2\right)}{x^4(x-2M)^2}V(s)=0. \tag{20}$$

To integrate the equation (17) we transform it with the help of substitution

$$Q(s)=x(s)T(s).$$

As a result we get the equation for the function $T(s)$

$$\frac{d^2T}{ds^2}+\frac{\left(x^4\ddot{x}-2Mx^3\ddot{x}-2Mx^2\dot{x}^2-8M^2x\dot{t}^2+8M^3\dot{t}^2+2Mx^2\dot{t}^2)\right)}{x^4(x-2M)}T(s)=0$$

which takes the form

$$\frac{d^2}{ds^2}T(s) + \frac{M\mu^2}{x^3}T(s) = 0 \tag{21}$$

after substitution here the relation (15) and (16).

In the simplest case

$$x(s) = (9M\mu^2/2)^{1/3}s^{2/3}$$

the solution of equation (21) is

$$T(s) = C_1 s^{2/3} + C_2 s^{1/3},$$

where C_i are the parameters.

As a result we get solutions for the coordinates $Q(s)$ and $U(s)$.

They look as

$$Q(s) = (9M\mu^2/2)^{1/3}(C_1 s + C_2 s^{4/3}), \tag{22}$$

$$U(s) = (9M\mu^2/2)^{1/3}(C_3 s + C_4 s^{4/3}). \tag{23}$$

To integrate the equations for the coordinates $P(s)$ and $V(s)$ we use the relation

$$\dot{x}(s)P(s) + \dot{\theta}(s)Q(s) + \dot{\phi}(s)U(s) + \dot{t}(s)V(s) - \frac{s}{2} = \text{const}, \tag{24}$$

which is consequence of the well-known first integral of geodesic equations of arbitrary Riemann space

$$g_{ik}\dot{x}^i\dot{x}^k = \text{const}.$$

In our case the relation (24) takes the form

$$\dot{x}P(s) + \dot{t}V(s) - \frac{s}{2} = 0 \tag{25}$$

(const = 0) and with the help of this condition the system (19), (20) can be reduced into the two independent equations.

In fact, from the equation (25) we get

$$P(s) = -\frac{2\dot{t}V(s) - s}{\dot{x}} \tag{26}$$

and after substitution of this expression into the equation (20) we get one second-order differential equation for the function $V(s)$.

In fact, with the help of relations (11), (12) we have

$$\frac{d^2}{ds^2}V(s) + \frac{\sqrt{2M\mu^2}}{x(s)^{3/2}}\frac{d}{ds}V(s) - \frac{3M\mu^2}{x(s)^3}V(s) - \frac{\sqrt{2M}}{2x(s)^{3/2}} + \frac{3M\mu}{2x(s)^3}s = 0.$$

Now we put here the expression

$$x(s) = (9M\mu^2/2)^{1/3}s^{2/3}$$

for the function x and as a result we get a simple equation

$$\frac{d^2}{ds^2}V(s) + \frac{2}{3s}\frac{d}{ds}V(s) - \frac{2}{3s^2}V(s) = 0$$

for the coordinate $V(s)$.

Its solution is defined by

$$V(s) = C_5 s + \frac{C_6}{s^{2/3}}. \tag{27}$$

Now from the relation (26) we get the expression for coordinate $P(s)$

$$P(s) = -1/2\,\frac{s^{2/3}\left(6\mu\sqrt[3]{\mu^2 M}s^{5/3}C_5 + 6\mu\sqrt[3]{\mu^2 M}C_6 - 3s^{5/3}\sqrt[3]{\mu^2 M} + 2\sqrt[3]{3}\sqrt[3]{2}sM\right)}{\sqrt[3]{\mu^2 M}\left(s\sqrt[3]{\mu^2 M}3^{2/3}2^{2/3} - 4\sqrt[3]{s}M\right)}. \tag{28}$$

So the formulae (22), (23), (27), (28) present the solutions of geodesic equations for coordinates (Q, U, V, P) of Riemann extension of the Schwarzschild metric for the radial motion of test particle in basic space with coordinates (x, θ, ϕ, t).

The generalization and the interpretation of these solutions will be done later.

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Second Order Differential Operators on the Configuration Spaces

Dmitri L. FINKELSHTEIN

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *fdl@ln.ua*

The procedure of construction of a big class of operators on configuration spaces is presented. Second-order differential operators are studied on configuration space over domain with boundary. Symmetric extensions of the minimal operator are discovered.

1 Introduction

The aim of the present article is construction of a big class of operators on configuration spaces over whole underlying space or only domain of them and discovery of conditions under which these operators will be symmetric. Our underlying space is the usual $\mathbb{R}^d$ with Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ and the fixed measure σ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, such that

$$d\sigma(x) = \rho(x)dx,$$

where $\rho > 0$ a.a., $\rho \in L^1_{\mathrm{loc}}(\mathbb{R}^d, dx)$, $\rho^{1/2} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d, dx)$.

Definition 1. Configuration space $\Gamma = \Gamma_{\mathbb{R}^d}$ is the space of all locally finite subsets (configurations) of $\mathbb{R}^d$:

$$\Gamma = \left\{\gamma \subset \mathbb{R}^d \,\middle|\, |\gamma_\Lambda| < +\infty \ \text{ for any compact } \Lambda \subset \mathbb{R}^d\right\},$$

where $|\cdot|$ means the cardinality of a set and

$$\gamma_\Lambda = \gamma \cap \Lambda.$$

Let us define the σ-algebra $\mathcal{B}(\Gamma)$ as the minimal σ-algebra such that all mappings

$$\Gamma \ni \gamma \longmapsto |\gamma_\Lambda|$$

are $\mathcal{B}(\Gamma)$-measurable for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$, where $\mathcal{B}_c(\mathbb{R}^d)$ is the family of all Borel subsets of $\mathbb{R}^d$ with compact closure.

The space Γ can be naturally embedded into the space $\mathcal{M}(\mathbb{R}^d)$ of all measures on $\mathbb{R}^d$ in the following way

$$\gamma \in \Gamma \leftrightarrow \sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d),$$

where ε_x is a Dirac measure at the point x. Thus Γ can be endowed with topology generated by the weak topology on $\mathcal{M}(\mathbb{R}^d)$. Moreover, it can be shown that the σ-algebra $\mathcal{B}(\Gamma)$ is really the Borel σ-algebra with respect to this topology.

For $f \in C_0(\mathbb{R}^d)$ we can define a pairing between the function and configuration:

$$\langle f, \gamma\rangle := \sum_{x\in\gamma} f(x).$$

This definition is correct since the sum in the r.h.s. is finite due to definition of configurations.

Consider also the class of cylindrical smooth functions: $F \in \mathcal{F}C_b^\infty(\Gamma, \mathcal{D})$ iff

$$F(\gamma) := g_F(\langle\varphi_1, \gamma\rangle, \dots, \langle\varphi_N, \gamma\rangle), \tag{1}$$

where $\varphi_1, \dots, \varphi_N \in \mathcal{D} := C_0^\infty(\mathbb{R}^d)$, $g_F \in C_b^\infty(\mathbb{R}^N)$ (the space of all infinitely differentiable functions on $\mathbb{R}^N$, which are bounded together with all its derivatives).

Note that, if $f \in C_0(\mathbb{R}^d)$ then for any $\gamma \in \Gamma$

$$\langle f, \gamma + \varepsilon.\rangle - \langle f, \gamma\rangle \in C_0(\mathbb{R}^d).$$

Analogously, if $F \in \mathcal{F}C_b^\infty(\Gamma, \mathcal{D})$ then for any $\gamma \in \Gamma$

$$F(\gamma + \varepsilon.) - F(\gamma) \in \mathcal{D}.$$

Consider also the space Γ_Λ of the all configurations of $\Lambda \in \mathcal{B}(\mathbb{R}^d)$:

$$\Gamma_\Lambda = \big\{\gamma \subset \mathbb{R}^d \,\big|\, \gamma \cap \Lambda^c = \varnothing\big\}, \tag{2}$$

where $\Lambda^c := \mathbb{R}^d \setminus \Lambda$.

2 Measures on the configuration spaces

Let us consider a class $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ of the probability measures on $(\Gamma, \mathcal{B}(\Gamma))$, which have all finite local moments; it means that

$$\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \Leftrightarrow \int_\Gamma |\gamma_\Lambda|^n d\mu(\gamma) < +\infty \tag{3}$$

for any $n \in \mathbb{N}$ and for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

We start from the non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable function $r : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$. We suppose that $r(\gamma, x)$ is defined for μ-a.a. $\gamma \in \Gamma$ and a.a. $x \in \mathbb{R}^d$ (note that we always assume that $x \notin \gamma$).

Definition 2. The measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called the Gibbs measure corresponding to r if for any non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable function $h : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$ the following Campbell–Mecke identity holds:

$$\int_\Gamma \sum_{x \in \gamma} h(\gamma, x) d\mu(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} h(\gamma + \varepsilon_x, x) r(\gamma, x) d\sigma(x) d\mu(\gamma). \tag{CM}$$

For examples of the such measures we start from the case when $r \equiv 1$. Mecke [3] proved that there exists only one such measure μ for given Radon measure σ on $\mathbb{R}^d$. This measure is called the Poisson measure with intensity σ and is denoted by π_σ. There exists a direct construction of the Poisson measure. For explore it we start from the space Γ_Λ, where $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Clearly,

$$\Gamma_\Lambda := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_\Lambda^{(n)},$$

where $\Gamma_\Lambda^{(n)}$ is the set of all n-particle configurations (subsets) of Λ, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. There is a bijection $\tilde{\Lambda}^n / S_n \to \Gamma_\Lambda^{(n)}$, where $\tilde{\Lambda}^n := \{(x_1, \dots, x_n) \in \Lambda^n \,|\, x_k \neq x_j, k \neq j\}$, and S_n is the permutation

group over $\{1,\dots,n\}$. Therefore, we can consider on $\Gamma_\Lambda^{(n)}$ the image $\sigma^{(n)}$ of the product measure σ^n under this bijection. Consider also a σ-algebra $\mathcal{B}(\Gamma_\Lambda)$ as the minimal σ-algebra such that all mappings

$$\Gamma_\Lambda \ni \gamma \longmapsto |\gamma_{\Lambda'}| \tag{4}$$

are $\mathcal{B}(\Gamma_\Lambda)$-measurable for any $\Lambda' \in \mathcal{B}_c(\Lambda)$. Then the Poisson measure π_σ^Λ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ is defined as

$$\pi_\sigma^\Lambda := e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}. \tag{5}$$

It can be shown that the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ and that the family of measures $\{\pi_\sigma^\Lambda\}_{\Lambda\in\mathcal{B}_c(X)}$ is consistent. Therefore one has to define the Poisson measure π_σ on $(\Gamma, \mathcal{B}(\Gamma))$ as the projective limit of this family.

Another examples were found by Nguen and Zessin [4]. They showed that a big class of Gibbs measures constructed by the quite general potential satisfy the Campbell–Mecke identity. More precisely, let Φ be a potential, i.e., a measurable function $\Phi : \Gamma_0 \to \mathbb{R} \cup \{+\infty\}$, such that $\Phi(\varnothing) = 0$. Define for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the conditional energy $E_\Lambda^\Phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ such that $E_\Lambda^\Phi(\gamma) = \sum_{\eta\Subset\gamma, |\eta\cap\Lambda|>0} \Phi(\eta)$ if $\sum_{\eta\Subset\gamma, |\eta\cap\Lambda|>0} |\Phi(\eta)| < \infty$ and $E_\Lambda^\Phi(\gamma) = +\infty$ otherwise (the notation $\eta \Subset \gamma$ means that η is a finite subset of γ). Then for fixed $\beta > 0$ we define for $\gamma \in \Gamma$, $\Delta \in \mathcal{B}(\Gamma)$ a specification

$$\Pi_\Lambda^{\sigma,\beta,\Phi}(\gamma, \Delta) = \frac{1_{\{Z_\Lambda^{\sigma,\beta,\Phi}(\gamma)<+\infty\}}}{Z_\Lambda^{\sigma,\beta,\Phi}(\gamma)} \int_\Gamma 1_\Delta(\gamma_{\Lambda^c} \cup \gamma'_\Lambda) e^{-\beta E_\Lambda^\Phi(\gamma_{\Lambda^c}\cup\gamma'_\Lambda)} d\pi_\sigma(\gamma'), \tag{6}$$

where $Z_\Lambda^{\sigma,\beta,\Phi}(\gamma) = \int_\Gamma e^{-\beta E_\Lambda^\Phi(\gamma_{\Lambda^c}\cup\gamma'_\Lambda)} d\pi_\sigma(\gamma')$. A measure $\mu \in \mathcal{M}^1(\Gamma)$ is called the grand canonical Gibbs measure with interaction potential Φ iff for all $\Lambda \in \mathcal{F}_c(\mathbb{R}^d)$ and for all $\Delta \in \mathcal{B}(\Gamma)$ the following Dobrushin–Lanford–Ruelle identity holds

$$\mu(\Delta) = \int_\Gamma \Pi_\Lambda^{\sigma,\beta,\Phi}(\gamma, \Delta) d\mu(\gamma). \tag{7}$$

The set of all such probability measures μ will be denoted by $\mathcal{G}_{gc}(\sigma, \beta\Phi)$.

Therefore, let $\mu \in \mathcal{G}_{gc}(p\,dx, \beta\Phi)$ and let μ has the local first moment (i.e., (3) is true for $n = 1$). Then μ satisfies the Campbell–Mecke identity with

$$r(\gamma, x) = \exp(-\beta E_{\{x\}}^\Phi(\gamma + \varepsilon_x)). \tag{8}$$

Let us recall that

$$E_{\{x\}}^\Phi(\gamma + \varepsilon_x) = \begin{cases} \sum_{\{x\}\subset\eta\Subset\gamma\cup\{x\}} \Phi(\eta), & \text{if } \sum_{\{x\}\subset\eta\Subset\gamma\cup\{x\}} |\Phi(\eta)| < +\infty, \\ +\infty, & \text{otherwise.} \end{cases} \tag{9}$$

Of course, all our considerations will be true in the general situation with function r, but with some additional conditions on it. Thus, if we will want to use our results for examples above, we have to check such conditions for corresponding r.

3 Construction of operators

First of all, for μ-a.a. $\gamma \in \Gamma$ let us consider a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$:

$$d\sigma_\gamma(x) = r(\gamma, x) d\sigma(x).$$

Therefore, we have a family of spaces $\{L^2(\mathbb{R}^d,\sigma_\gamma)\}_{\gamma\in\Gamma}$ (without loss of generality we assume that this family is indexed by all space Γ).

Suppose that we have the family of operators $\{B(\gamma)\}_{\gamma\in\Gamma}$ in the corresponding spaces $\{L^2(\mathbb{R}^d,\sigma_\gamma)\}_{\gamma\in\Gamma}$ such that

$$\mathcal{D}\subset \mathrm{Dom}\,(B(\gamma)),\qquad B(\gamma)1\in L^2(\mathbb{R}^d,\sigma_\gamma)\cap L^1(\mathbb{R}^d,\sigma_\gamma). \tag{10}$$

Suppose also that for $F\in\mathcal{F}C_b^\infty(\Gamma,\mathcal{D})$ there are functions $F_k\in\mathcal{F}C_b^\infty(\Gamma,\mathcal{D})$, $\psi_k:\Gamma\times\mathbb{R}^d\to\mathbb{R}$, $k=1,\dots,n$ such that $\psi_k(\gamma',\cdot)\in L^2(\mathbb{R}^d,\sigma_\gamma)\cap L^1(\mathbb{R}^d,\sigma_\gamma)$ for μ-a.a. $\gamma,\gamma'\in\Gamma$ and for $x\in\gamma$ if we consider $F(\gamma)$ as function of x then

$$B(\gamma')_xF(\gamma)=\sum_{k=1}^n F_k(\gamma)\psi_k(\gamma',x).$$

Then we can consider the bilinear form $\mathcal{E}$ on $\mathcal{F}C_b^\infty(\Gamma,\mathcal{D})$ generated by the family $\{B(\gamma)\}_{\gamma\in\Gamma}$:

$$\mathcal{E}_\mu(F,G)=\int_\Gamma\int_{\mathbb{R}^d} B(\gamma')_x(F(\gamma+\varepsilon_x)-F(\gamma))\big|_{\gamma'=\gamma}\cdot(G(\gamma+\varepsilon_x)-G(\gamma))r(\gamma,x)d\sigma(x)d\mu(\gamma)$$

for $F,G\in\mathcal{F}C_b^\infty(\Gamma,\mathcal{D})$.

Note that for $x\notin\gamma$ one has

$$B(\gamma')_xF(\gamma)=F(\gamma)(B(\gamma')1)(x), \tag{11}$$

since then $F(\gamma)$ is a constant as a function of x.

Proposition 1. *For any $F,G\in\mathcal{F}C_b^\infty(\Gamma,\mathcal{D})$ the following equality holds*

$$\mathcal{E}_\mu(F,G)=\int_\Gamma H_\mu F(\gamma)\cdot G(\gamma)d\mu(\gamma),$$

where

$$\begin{aligned}H_\mu F(\gamma)=&\sum_{x\in\gamma}B(\gamma')_xF(\gamma)\big|_{\gamma'=\gamma-\varepsilon_x}-\sum_{x\in\gamma}F(\gamma-\varepsilon_x)(B(\gamma-\varepsilon_x)1)(x)\\&-\int_{\mathbb{R}^d}B(\gamma)_xF(\gamma+\varepsilon_x)r(\gamma,x)d\sigma(x)+F(\gamma)\left(\int_{\mathbb{R}^d}(B(\gamma)1)(x)r(\gamma,x)d\sigma(x)\right).\end{aligned}$$

Proof.

$$\begin{aligned}&\int_\Gamma\int_{\mathbb{R}^d}B(\gamma)_x(F(\gamma+\varepsilon_x)-F(\gamma))\cdot(G(\gamma+\varepsilon_x)-G(\gamma))r(\gamma,x)d\sigma(x)d\mu(\gamma)\\&\quad=\int_\Gamma\int_{\mathbb{R}^d}B(\gamma)_xF(\gamma+\varepsilon_x)\cdot G(\gamma+\varepsilon_x)r(\gamma,x)d\sigma(x)d\mu(\gamma)\\&\qquad-\int_\Gamma\int_{\mathbb{R}^d}B(\gamma)_xF(\gamma+\varepsilon_x)\cdot G(\gamma)r(\gamma,x)d\sigma(x)d\mu(\gamma)\\&\qquad-\int_\Gamma\int_{\mathbb{R}^d}F(\gamma)\cdot(B(\gamma)1)(x)\cdot G(\gamma+\varepsilon_x)r(\gamma,x)d\sigma(x)d\mu(\gamma)\\&\qquad+\int_\Gamma\int_{\mathbb{R}^d}F(\gamma)\cdot(B(\gamma)1)(x)\cdot G(\gamma)r(\gamma,x)d\sigma(x)d\mu(\gamma)\\&\quad=\int_\Gamma\int_{\mathbb{R}^d}\sum_{k=1}^n F_k(\gamma+\varepsilon_x)\varphi_k(x,\gamma)\cdot G(\gamma+\varepsilon_x)r(\gamma,x)d\sigma(x)d\mu(\gamma)\\&\qquad-\int_\Gamma\int_{\mathbb{R}^d}B(\gamma)_xF(\gamma+\varepsilon_x)\cdot G(\gamma)r(\gamma,x)d\sigma(x)d\mu(\gamma)\end{aligned}$$

$$
\begin{aligned}
&- \int_\Gamma \int_{\mathbb{R}^d} F(\gamma) \cdot (B(\gamma)1)(x) \cdot G(\gamma + \varepsilon_x) r(\gamma, x) d\sigma(x) d\mu(\gamma) \\
&+ \int_\Gamma \int_{\mathbb{R}^d} F(\gamma) \cdot (B(\gamma)1)(x) \cdot G(\gamma) r(\gamma, x) d\sigma(x) d\mu(\gamma) \\
&= \int_\Gamma \sum_{x \in \gamma} \left(\sum_{k=1}^n F_k(\gamma) \varphi_k(x, \gamma - \varepsilon_x) \right) G(\gamma) d\mu(\gamma) \\
&- \int_\Gamma \left(\int_{\mathbb{R}^d} B(\gamma)_x F(\gamma + \varepsilon_x) r(\gamma, x) d\sigma(x) \right) G(\gamma) d\mu(\gamma) \\
&- \int_\Gamma \left(\sum_{x \in \gamma} F(\gamma - \varepsilon_x)(B(\gamma - \varepsilon_x)1)(x) \right) G(\gamma) d\mu(\gamma) \\
&+ \int_\Gamma F(\gamma) G(\gamma) \left(\int_{\mathbb{R}^d} (B(\gamma)1)(x) r(\gamma, x) d\sigma(x) \right) d\mu(\gamma). \qquad \blacksquare
\end{aligned}
$$

Therefore, if $B(\gamma)$ are symmetric operators on $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, \sigma_\gamma)$ for any $\gamma \in \Gamma$, then the corresponding operator H_μ is a symmetric operator on $\mathcal{F}C_b^\infty(\Gamma, \mathcal{D})$ in $L^2(\Gamma, \mu)$.

4 Second order differential operators

In this section we generalized results obtained in [1, 2] for the Poisson measure in our general case. Suppose that $r(\gamma, \cdot) \in C^1(\mathbb{R}^d)$ for μ-a.a. $\gamma \in \Gamma$.

Consider now instead of $B(\gamma)$ the Dirichlet operator $A(\gamma)$ corresponding to the bilinear form

$$
\mathcal{E}_{\sigma_\gamma}(f, g) = \int_{\mathbb{R}^d} \langle \mathcal{A}(\gamma, x) \nabla f(x), \nabla g(x) \rangle d\sigma_\gamma(x),
$$

where $\mathcal{A}(\gamma, x) = (a_{ij}(\gamma, x))_{i,j=1,\dots,d}$ and $a_{ij}(\gamma, \cdot) = a_{ji}(\gamma, \cdot)$ are smooth functions on $\mathbb{R}^d$. Since

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \langle \mathcal{A}(\gamma, x) \nabla f(x), \nabla g(x) \rangle d\sigma_\gamma(x) \\
&\quad = \int_{\mathbb{R}^d} \langle \mathcal{A}(\gamma, x) \nabla f(x), \nabla g(x) \rangle r(\gamma, x) \rho(x) dx \\
&\quad = - \int_{\mathbb{R}^d} \operatorname{div} \left((\mathcal{A}(\gamma, x) \nabla f(x)) r(\gamma, x) \rho(x) \right) g(x) dx \\
&\quad = - \int_{\mathbb{R}^d} (\operatorname{div} (\mathcal{A}(\gamma, x) \nabla f(x))) r(\gamma, x) \rho(x) g(x) dx \\
&\qquad - \int_{\mathbb{R}^d} \langle \mathcal{A}(\gamma, x) \nabla f(x), \nabla (r(\gamma, x) \rho(x)) \rangle g(x) dx \\
&= - \int_{\mathbb{R}^d} \operatorname{div} (\mathcal{A}(\gamma, x) \nabla f(x)) g(x) d\sigma_\gamma(x) \\
&\qquad - \int_{\mathbb{R}^d} \langle \mathcal{A}(\gamma, x) \nabla f(x), \nabla \ln (r(\gamma, x) \rho(x)) \rangle g(x) d\sigma_\gamma(x),
\end{aligned}
$$

then

$$
A(\gamma) f(x) = -\operatorname{div} (\mathcal{A}(\gamma, x) \nabla f(x)) - \langle \mathcal{A}(\gamma, x) \nabla f(x), \beta(\gamma, x) \rangle, \tag{12}
$$

where $\beta(\gamma, x) = \nabla \ln (r(\gamma, x) \rho(x))$.

Since $A(\gamma)1 \equiv 0$ then the operator H_μ corresponding to the family $\{A(\gamma)\}_{\gamma \in \Gamma}$ has the simple form

$$
H_\mu F(\gamma) = \sum_{x \in \gamma} A(\gamma')_x F(\gamma) \big|_{\gamma' = \gamma - \varepsilon_x}.
$$

Let us now study the operator H_μ in the space $L^2(\Gamma_\Lambda, \mu_\Lambda)$, where μ_Λ is the projection of the measure μ onto Γ_Λ and Λ is the regular (bounded or not) domain of $\mathbb{R}^d$ with piecewise C^1 boundary $\partial\Lambda$.

Proposition 2. *The first Green formula for the operator H_μ holds:*

$$\begin{aligned}
&\int_{\Gamma_\Lambda} H_\mu F(\gamma) G(\gamma) d\mu_\Lambda(\gamma) \\
&= \int_{\Gamma_\Lambda} \int_\Lambda \langle \mathcal{A}(\gamma, x) \nabla_x F(\gamma + \varepsilon_x), \nabla_x G(\gamma + \varepsilon_x) \rangle d\sigma_\gamma(x) d\mu_\Lambda(\gamma) \\
&\quad - \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \frac{\partial}{\partial n_s^\gamma} F(\gamma + \varepsilon_s) G(\gamma + \varepsilon_s) d\tilde{\sigma}_\gamma(s) d\mu_\Lambda(\gamma),
\end{aligned}$$

where

$$\frac{\partial}{\partial n_s^\gamma} f(s) = \langle \mathcal{A}(\gamma, s) \nabla f(s), n_s \rangle.$$

Proof. Due to Campbell–Mecke identity one has

$$\int_{\Gamma_\Lambda} H_\mu F(\gamma) G(\gamma) d\mu_\Lambda(\gamma) = \int_{\Gamma_\Lambda} \int_\Lambda (A_\gamma)_x F(\gamma + \varepsilon_x) G(\gamma + \varepsilon_x) d\sigma_\gamma(x) d\mu_\Lambda(\gamma).$$

Then, write for $A(\gamma)$ the first Green formula:

$$\begin{aligned}
&\int_\Lambda A(\gamma) f(x) g(x) d\sigma_\gamma(x) \\
&= -\int_\Lambda \operatorname{div}(\mathcal{A}(\gamma, x) \nabla f(x)) g(x) d\sigma_\gamma(x) - \int_\Lambda \langle \mathcal{A}(\gamma, x) \nabla f(x), \beta(\gamma, x) \rangle g(x) d\sigma_\gamma(x) \\
&= -\int_\Lambda \operatorname{div}(\mathcal{A}(\gamma, x) \nabla f(x)) g(x) r(\gamma, x) \rho(x) d(x) \\
&\quad - \int_\Lambda \langle \mathcal{A}(\gamma, x) \nabla f(x), \beta(\gamma, x) \rangle g(x) d\sigma_\gamma(x) \\
&= \int_\Lambda \langle \mathcal{A}(\gamma, x) \nabla f(x), \beta(\gamma, x) \rangle g(x) d\sigma_\gamma(x) + \int_\Lambda \langle \mathcal{A}(\gamma, x) \nabla f(x), \nabla g(x) \rangle d\sigma_\gamma(x) \\
&\quad - \int_{\partial\Lambda} \langle \mathcal{A}(\gamma, s) \nabla f(s), n_s \rangle g(s) r(\gamma, s) \rho(s) dS(s) \\
&\quad - \int_\Lambda \langle \mathcal{A}(\gamma, x) \nabla f(x), \beta(\gamma, x) \rangle g(x) d\sigma_\gamma(x) \\
&= \int_\Lambda \langle \mathcal{A}(\gamma, x) \nabla f(x), \nabla g(x) \rangle d\sigma_\gamma(x) - \int_{\partial\Lambda} \langle \mathcal{A}(\gamma, s) \nabla f(s), n_s \rangle g(s) d\tilde{\sigma}_\gamma(s),
\end{aligned}$$

where $d\tilde{\sigma}_\gamma(s) = r(\gamma, s)\rho(s)dS(s)$ for $s \in \partial\Lambda$. As a result, we obtain the statement. ■

Corollary 1. *The second Green formula for the operator H_μ has the following form*

$$\begin{aligned}
&\int_{\Gamma_\Lambda} (H_\mu F(\gamma) G(\gamma) - F(\gamma) H_\mu G(\gamma)) d\mu_\Lambda(\gamma) \\
&\quad = \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \left(\frac{\partial}{\partial n_s^\gamma} G(\gamma + \varepsilon_s) \cdot F(\gamma + \varepsilon_s) - \frac{\partial}{\partial n_s^\gamma} F(\gamma + \varepsilon_s) \cdot G(\gamma + \varepsilon_s) \right) d\tilde{\sigma}_\gamma(s) d\mu_\Lambda(\gamma).
\end{aligned}$$

Remark 1. Note that for any function $F \in \mathcal{FC}_b^\infty(\Gamma, \mathcal{D})$ in the form (1)

$$\frac{\partial}{\partial n_s^\gamma} F(\gamma + \varepsilon_s) = \sum_{k=1}^{N} \hat{F}_k(\gamma + \varepsilon_s) \frac{\partial}{\partial n_s^\gamma} \varphi_k(s),$$

where

$$\hat{F}_k(\gamma) := \frac{\partial g_F}{\partial q_k}(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle), \qquad 1 \le k \le N.$$

Let us consider now the minimal operator

$$H_{\min} := (H_\mu, \mathcal{FC}_b^\infty(\Gamma_\Lambda, \mathcal{D}(\Lambda))),$$

which is symmetric in $L^2(\Gamma_\Lambda, \mu_\Lambda)$ (there $\mathcal{D}(\Lambda) = C_0^\infty(\Lambda)$). We define the maximal operator by the standard way:

$$H_{\max} = (H_{\min})^*$$

Proposition 3. *$\mathcal{FC}_b^\infty(\Gamma_\Lambda, \mathcal{D}) \subset \mathrm{Dom}\,(H_{\max})$ and for any $G \in C_b^\infty(\Gamma_\Lambda, \mathcal{D})$*

$$H_{\max} G(\gamma) = H_\mu G(\gamma) + \int_{\partial\Lambda} \frac{\partial}{\partial n_s^\gamma} G(\gamma + \varepsilon_s) d\tilde{\sigma}_\gamma(s)$$

Proof. It follows directly from the second Green formula that for any $G \in \mathcal{FC}_b^\infty(\mathcal{D}, \Gamma_\Lambda)$ and $F \in \mathcal{FC}_b^\infty(\mathcal{D}(\Lambda), \Gamma_\Lambda)$

$$\int_{\Gamma_\Lambda} \left((H_\mu G)(\gamma) F(\gamma) - G(\gamma)(H_\mu F)(\gamma) \right) d\mu_\Lambda(\gamma)$$
$$= -\int_{\Gamma_\Lambda} \int_{\partial\Lambda} F(\gamma) \frac{\partial}{\partial n_s^\gamma} G(\gamma + \varepsilon_s)\, d\tilde{\sigma}_\gamma(s)\, d\mu_\Lambda(\gamma). \qquad \blacksquare$$

In the case when $\mathcal{A}$ is not dependent on γ: $\mathcal{A}(\gamma, x) \equiv \mathcal{A}(x)$ one has that

$$\frac{\partial}{\partial n_s^\gamma} f(s) \equiv \frac{\partial}{\partial \nu_s} f(s) = \langle \mathcal{A}(s) \nabla f(s), n_s \rangle$$

is the usual co-normal derivative.

Thus, if we define the following set of functions that satisfies Neumann-type boundary conditions on $\partial\Lambda$:

$$\mathcal{D}_{\mathcal{N}}(\Lambda) = \left\{ f \in \mathcal{D} \,\middle|\, \frac{\partial}{\partial \nu_s} f(s) = 0,\ s \in \partial\Lambda \right\},$$

then the operator $(H_\mu, \mathcal{FC}_b^\infty(\Gamma_\Lambda, \mathcal{D}_{\mathcal{N}}(\Lambda)))$ will be symmetric in $L^2(\Gamma_\Lambda, \mu_\Lambda)$.

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On the BBGKY Hierarchy Solutions for Many-Particle Systems with Different Symmetry Properties

V.I. GERASIMENKO †, *T.V. RYABUKHA* †, *M.O. STASHENKO* ‡

† *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine*
E-mail: *gerasym@imath.kiev.ua, vyrtum@imath.kiev.ua*

‡ *Volyn State University, 13 Voli Ave., Lutsk, Ukraine*
E-mail: *smo@univer.lutsk.ua*

We construct a new representation to solutions of the initial value problem of the BBGKY hierarchy of equations. Such representation of solutions enable us to describe the cluster nature of evolution of infinite particle systems with various symmetry properties in detail. Convergence of the constructed expansions is investigated in suitable functional spaces.

1 Introduction

Various symmetry properties of many-particle systems are generated by the indistinguishability property of identical particles. Classical identical particles are described by observables and states that are symmetric functions with respect to permutations of their arguments (the phase space coordinates of every particle) [1, 2]. In the quantum case we have additional symmetry properties related to the nature of identical particles (Fermi and Bose particles). Classical many-particle systems can also consist of distinguishable particles. In this case many-particle systems are described by observables and states that are non-symmetric functions of their arguments (non-symmetrical particle systems).

The goal of this paper is to analyze the structure of expansions for the BBGKY hierarchy solutions, which depends on the symmetry properties of many-particle systems. We construct a new representation to solutions of the initial value problem of the BBGKY hierarchy in the form of an expansion over particle clusters whose evolution is governed by the cumulant (semi-invariant) of the evolution operator of the corresponding particle cluster for symmetric and non-symmetric systems.

2 Cluster expansions of evolution operators of symmetrical particle systems

We consider the initial value problem of the BBGKY hierarchy of equations for a classical system of identical particles [1,2]. If $F(0) = (1, F_1(0, x_1), \ldots, F_s(0, x_1, \ldots, x_s), \ldots)$ is a sequence of initial s-particle distribution functions $F_s(0, x_1, \ldots, x_s)$ symmetric in $x_i \equiv (q_i, p_i) \in \mathbb{R}^\nu \times \mathbb{R}^\nu$, $\nu \geq 1$, then a solution $F(t) = (1, F_1(t, x_1), \ldots, F_s(t, x_1, \ldots, x_s), \ldots)$ of the Cauchy problem for the BBGKY hierarchy is represented as the expansion

$$F_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^\nu \times \mathbb{R}^\nu)^n} dx_{s+1} \cdots dx_{s+n}\, \mathfrak{A}_{(n)}(t, x_1, \ldots, x_s; x_{s+1}, \ldots, x_{s+n}) \times F_{s+n}(0, x_1, \ldots, x_{s+n}), \qquad s \geq 1. \tag{1}$$

where the evolution operator $\mathfrak{A}_{(n)}(t)$ is defined as follows. Let $(x_1, \ldots, x_s) \equiv Y$, $(Y, x_{s+1}, \ldots, x_{s+n}) \equiv X$, i.e., $(x_{s+1}, \ldots, x_{s+n}) = X \backslash Y$, and let $|X| = |Y| + |X \backslash Y| = s + n$ denote the number

of elements of the set X. Then we have ($|X\backslash Y| \geq 0$)

$$\mathfrak{A}_{(|X\backslash Y|)}(t, Y, X\backslash Y) = \sum_{P:\{Y, X\backslash Y\}=\bigcup_i X_i} (-1)^{|P|-1}(|P|-1)! \prod_{X_i\subset P} S_{|X_i|}(-t, X_i), \tag{2}$$

where $\sum_P$ is the sum over all possible decompositions of the set $\{Y,\, X\backslash Y\}$ into $|P|$ nonempty mutually disjoint subsets $X_i \subset \{Y,\, X\backslash Y\}$, $X_i \cap X_j = \varnothing$, and the set Y completely belongs to one of the subsets X_i. In formula (2) the evolution operator $S_{|X|}(-t, X)$ of the Liouville equation describes the dynamics of a system of a finite number n of particles, namely,

$$S_n(-t, x_1, \ldots, x_n)\, f_n(x_1, \ldots, x_n) = f_n\big(X_1(-t, x_1, \ldots, x_n), \ldots, X_n(-t, x_1, \ldots, x_n)\big), \tag{3}$$

where $X_i(-t, x_1, \ldots, x_n)$, $i = 1, \ldots, n$, is a solution of the initial value problem for the Hamilton equations of a system of n particles with initial data $X_i(0, x_1, \ldots, x_n) = x_i$, $i = 1, \ldots, n$, ($S_n(0) = I$ is the identity operator). Operator (3) is defined e.g. in the space of integrable functions $f_n \in L^1\left(\mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n}\right) \equiv L^1_n$ [1], in particular, $\|S_n(-t)\|_{L^1_n} = 1$.

The simplest examples of evolution operators $\mathfrak{A}_{(n)}(t)$ (2) have the form

$$\begin{aligned}
&\mathfrak{A}_{(0)}(t, Y) = S_s(-t, Y),\\
&\mathfrak{A}_{(1)}(t, Y, x_{s+1}) = S_{s+1}(-t, Y, x_{s+1}) - S_s(-t, Y) S_1(-t, x_{s+1}),\\
&\mathfrak{A}_{(2)}(t, Y, x_{s+1}, x_{s+2}) = S_{s+2}(-t, Y, x_{s+1}, x_{s+2})\\
&\qquad - S_{s+1}(-t, Y, x_{s+1}) S_1(-t, x_{s+2}) - S_{s+1}(-t, Y, x_{s+2}) S_1(-t, x_{s+1})\\
&\qquad - S_s(-t, Y) S_2(-t, x_{s+1}, x_{s+2}) + 2! S_s(-t, Y) S_1(-t, x_{s+1}) S_1(-t, x_{s+2}).
\end{aligned}$$

Evolution operators (2) are solutions of the following recursion relations ($|X\backslash Y| \geq 0$)

$$S_{|X|}(-t, Y,\, X\backslash Y) = \sum_{P:\{Y,\, X\backslash Y\}=\bigcup_i X_i} \prod_{X_i\subset P} \mathfrak{A}_{(|X_i|-1)}(t, X_i), \tag{4}$$

where $\sum_P$ is the sum as above in formula (2).

We note that recursion relations (4) are typical cluster expansions [3] for the evolution operator $S_{|X|}(-t, Y, X\backslash Y)$ defined by (3). Thus, the operators $\mathfrak{A}_{(|X\backslash Y|)}(t, Y, X\backslash Y)$ (2) have the meaning of the cumulants (semi-invariants) of the operator $S_{|X|}(-t, Y, X\backslash Y)$ describing the evolution of a system of a finite number $|X|$ of particles, i.e., they describe what noninteracting clusters of particles may form a system of the corresponding number of particles in the process of evolution, provided that a cluster of $|Y|$ particles evolves as a single cluster.

The structure of the cluster expansions (4) can be represented in a more explicit and compact form. To this end we consider the set of sequences $\Psi = \big(\Psi_0, \Psi_1(x_1), \ldots, \Psi_n(x_1, \ldots, x_n), \ldots\big)$ of operators Ψ_n of type (3) (Ψ_0 is an operator that multiplies a function by an arbitrary number). In this set we introduce the tensor $*$-product

$$(\Psi_1 * \Psi_2)_{|X|}(X) = \sum_{Y\subset X} (\Psi_1)_{|Y|}(Y)\, (\Psi_2)_{|X\backslash Y|}(X\backslash Y),$$

where $\sum_{Y\subset X}$is the sum over all subsets Y of the set $X \equiv (x_1, \ldots, x_n)$.

As a result, expression (2) can be rewritten in the form

$$\mathfrak{A}(t) = \mathbb{L}\mathrm{n}_*\big(\mathbf{1} + S(-t)\big), \tag{5}$$

where $\mathfrak{A}(t) = (0, (\mathfrak{A}(t))_1(Y), (\mathfrak{A}(t))_2(Y, x_{s+1}), \ldots)$ and $(\mathfrak{A}(t))_{1+n}(Y, x_{s+1}, \ldots, x_{s+n}) = \mathfrak{A}_{(n)}(t, Y,\, X\backslash Y) \equiv \mathfrak{A}_{|Y|+n}(t, Y, X\backslash Y)$. A mapping $\mathbb{L}\mathrm{n}_*$ is defined: $\mathbb{L}\mathrm{n}_*(\mathbf{1} + \Psi) = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \underbrace{\Psi * \cdots * \Psi}_{n}$, ($\Psi = (0, \Psi_1, \ldots, \Psi_n, \ldots)$, and $\mathbf{1} = (1, 0, 0, \ldots)$ is the unit sequence).

We also remark that the cluster expansions (4) have the form

$$\mathbf{1} + S(-t) = \mathbb{E}\mathrm{xp}_*\mathfrak{A}(t),$$

where $\mathbb{E}\mathrm{xp}_*$ is defined as the $*$-exponential mapping: $\mathbb{E}\mathrm{xp}_*\Psi = \mathbf{1} + \sum\limits_{n=1}^{\infty} \frac{1}{n!} \underbrace{\Psi * \cdots * \Psi}_{n}$.

We note that the connections between different representations of the BBGKY hierarchy solutions are considered in [1, 3]. For the first time several first terms of the expansion (1), (2) for the one-particle distribution function were determined in [4, 5].

3 Cumulant representation of BBGKY hierarchy solutions for symmetrical particle systems

We consider the problem of the convergence of the expansion (1), (2) in the space of sequences of integrable functions and prove the solution existence theorem for the initial data from this space.

Let $L^1_\alpha = \bigoplus_{n=0}^{\infty} \alpha^n L^1_n$ be the Banach space of sequences $f = (f_0, f_1(x_1), \ldots, f_n(x_1, \ldots, x_n), \ldots)$ of symmetric integrable functions $f_n(x_1, \ldots, x_n)$ defined on the phase space $\mathbb{R}^{\nu n} \times \mathbb{R}^{\nu n}$ with the norm

$$\|f\| = \sum_{n=0}^{\infty} \alpha^n \|f_n\|_{L^1_n} = \sum_{n=0}^{\infty} \alpha^n \int_{(\mathbb{R}^\nu \times \mathbb{R}^\nu)^n} dx_1 \cdots dx_n \, |f_n(x_1, \ldots, x_n)|,$$

where $\alpha > 1$ is a number; $L^1_{\alpha,0} \subset L^1_\alpha$ is the subspace of finite sequences of continuously differentiable functions with compact supports.

Since on the sequences of integrable functions $f \in L^1_\alpha$ the annihilation operator [1, 2]

$$\big(\mathfrak{a} f\big)_n(x_1, \ldots, x_n) = \int_{\mathbb{R}^\nu \times \mathbb{R}^\nu} dx_{n+1} \, f_{n+1}(x_1, \ldots, x_n, x_{n+1}), \tag{6}$$

is defined, in view of (5) and (6), expression (1), (2) takes the following form in the space L^1_α

$$F(t) = e^{\mathfrak{a}}\mathfrak{A}(t)F(0) = e^{\mathfrak{a}}\mathbb{L}\mathrm{n}_*\big(\mathbf{1} + S(-t)\big)F(0). \tag{7}$$

If $F(0) \in L^1_{s+n}$, then the following estimate is true:

$$\|\mathfrak{A}_{(n)}(t)F_{s+n}(0)\|_{L^1_{s+n}} \leq n! e^{n+2} \|F_{s+n}(0)\|_{L^1_{s+n}}. \tag{8}$$

By virtue of inequality (8), the functions defined by (1) (or (7)) satisfy the following estimate for $\alpha > e$:

$$\|F(t)\|_{L^1_\alpha} \leq c_\alpha \|F(0)\|_{L^1_\alpha}, \tag{9}$$

where $c_\alpha = e^2 \left(1 - \frac{e}{\alpha}\right)^{-1}$ is a constant.

Note that the parameter α can be interpreted as a quantity inversely proportional to the density of the system (the average number of particles in a unit volume).

Thus, according to estimate (9) the following existence theorem is true:

Theorem 1. *If $F(0) \in L^1_\alpha$ is a sequence of nonnegative functions, then for $\alpha > e$, and $t \in \mathbb{R}^1$, there exists a unique solution to the initial value problem for the BBGKY hierarchy, namely the sequence $F(t) \in L^1_\alpha$ of nonnegative functions $F_s(t)$ defined by*

$$F_{|Y|}(t, Y) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d(X \backslash Y) \sum_{P : \{Y, X\backslash Y\} \bigcup_i X_i} (-1)^{|P|-1} \, (|P| - 1)! \times \prod_{X_i \subset P} S_{|X_i|}(-t, X_i) F_{|X|}(0, Y, X \backslash Y), \tag{10}$$

where we have the same notations as for (2). *This solution is a strong solution for* $F(0) \in L^1_{\alpha,0}$ *and a weak one for arbitrary initial data.*

If we consider initial data $F(0)$ from the space $L^\infty_{\xi,\beta}$ of sequences of continuous symmetric functions bounded with respect to the configuration variables [1,2] the problem of the divergence of configuration integrals in each term of series (1), (2) arises. We note that the cumulant structure of expansion (1) allows one to prove the possibility of eliminating this divergence.

4 Cluster expansions of evolution operators of non-symmetrical particle systems

We consider a one-dimensional system of identical particles interacting with their nearest neighbours via the hard-core pair potential Φ. For the configurations ($q_i \in \mathbb{R}^1$ is the position of the i particle center) of such a system the following inequalities must be satisfied: $\sigma + q_i \le q_{i+1}$, where σ is the length of a particle, and the natural way to number the particles is to number by means of the integers from the set $\mathbb{Z}\backslash\{0\}$. The Hamiltonian of the $n = n_1 + n_2$ particle system

$$H_n = \sum_{i\in(-n_2,\ldots,-1,1,\ldots,n_1)} \frac{p_i^2}{2} + \sum_{(i,i+1)\in\big((-n_2,-n_2+1),\ldots,(n_1-1,n_1)\big)} \Phi(q_i - q_{i+1})$$

is a function non-symmetrical [6,7] with respect to permutations of the arguments $x_i \equiv (q_i, p_i) \in \mathbb{R}^1 \times \mathbb{R}^1$.

If $F(0) = \big\{F_s(0, x_{-s_2}, \ldots, x_{s_1})\big\}_{s=s_1+s_2\ge 0}$ is a sequence of initial s-particle distribution functions $F_s(0, x_{-s_2}, \ldots, x_{-1}, x_1, \ldots, x_{s_1})$, $F_0 = 1$, then a solution of the Cauchy problem for the BBGKY hierarchy $F(t) = \big\{F_s(t, x_{-s_2}, \ldots, x_{s_1})\big\}_{s=s_1+s_2\ge 0}$ is represented as the expansion

$$F_s(t, x_{-s_2}, \ldots, x_{s_1}) = \sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1,\ n_2 \ge 0}} \int_{(\mathbb{R}^1\times\mathbb{R}^1)^{n_1+n_2}} dx_{-(n_2+s_2)} \cdots dx_{-(s_2+1)} \times dx_{s_1+1} \cdots dx_{s_1+n_1} \big(\mathfrak{A}_{(n_2,\,n_1)}(t)\, F_{s+n}(0)\big)(x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}), \tag{11}$$

where the evolution operator $\mathfrak{A}_{(n_2,n_1)}(t)$ is defined in the following way. Let $(x_{-s_2}, \ldots, x_{s_1}) \equiv Y$, $(x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}) \equiv X$. The sets X and Y are partially ordered sets, because $\sigma + q_i \le q_{i+1}$. If the subset Y of the set X is treated as one element similar to $(x_{-(n_2+s_2)}, \ldots, x_{-(s_2+1)}, x_{s_1+1}, \ldots, x_{s_1+n_1})$, then for such a partially ordered set we use the notation X_Y. Symbol $|Y| = s = s_1 + s_2$ denotes the number of elements of the set Y and, thus, $|X_Y| = n_1 + n_2 + 1$. Then we have

$$\mathfrak{A}_{(n_2,n_1)}(t, X_Y) = \sum_{P: X_Y = \bigcup_i X_i} (-1)^{|P|-1} \prod_{X_i \subset P} S_{|X_i|}(-t, X_i), \tag{12}$$

where $n_1 + n_2 = n \ge 0$, $\sum_P$ is the sum over all ordered decompositions of the partially ordered set X_Y into $|P|$ nonempty partially ordered subsets $X_i \subset X_Y$, which are mutually disjoint $X_i \bigcap X_j = \varnothing$, and the set Y completely belongs to one of the subsets X_i. As above, in formula (12) the evolution operator $S_{|X|}(-t, X)$ describes the dynamics of a system of a finite number $n = n_1 + n_2$ of particles [2,6].

The evolution operators (12) are solutions of the following recursion relations ($|X_Y| - 1 \ge 0$)

$$S_{|X|}(-t, X) = \sum_{P: X_Y = \bigcup_i X_i} \prod_{X_i \subset P} \mathfrak{A}_{(i_2,\ i_1)}(t, X_i), \tag{13}$$

where $i_1 + i_2 = i = |X_i| - 1 \geq 0$, and $\sum_P$ is the sum given above in formula (12). The recursion relations (13) are cluster expansions for the evolution operator of non-symmetrical particle systems. We note that the structure of cluster expansion (13) is essentially different from the structure of corresponding expansions (4) for the symmetrical systems.

As above, the structure of the cluster expansions (13) can be represented in a more explicit and compact form. In the set of double sequences $\Psi = \{\Psi_{n_1+n_2}(x_{-n_2}, \ldots, x_{n_1})\}_{n_1+n_1=n\geq 0}$ of operators $\Psi_{n_1+n_2}$ we introduce the following tensor $\star$-product

$$(\Psi_1 \star \Psi_2)_{|X|}(X) = \sum_{Y\subset X} (\Psi_1)_{|Y|}(Y)\,(\Psi_2)_{|X\setminus Y|}(X\setminus Y),$$

where $\sum_{Y\subset X}$ is the sum over all partially ordered subsets Y of the partially ordered set $X \equiv (x_{-n_2}, \ldots, x_{n_1})$.

Then expression (12) for the cumulants of non-symmetrical systems can be rewritten in the form

$$\mathfrak{A}(t) = \mathbf{1} - (\mathbf{1} + \mathbf{S}(-\mathbf{t}))^{-\mathbf{1}_\star}, \tag{14}$$

where $\mathfrak{A}(t) = \big(0, (\mathfrak{A}(t))_1(Y), \ldots, (\mathfrak{A}(t))_{1+n_1+n_2}(X_Y), \ldots\big)$ and $(\mathfrak{A}(t))_{1+n_1+n_2}(x_{-(n_2+s_2)}, \ldots, x_{s_1+n_1}, Y, x_{s_1+1}, \ldots, x_{s_1+n_1}) \equiv (\mathfrak{A}(t))_{1+n}(X_Y) = \mathfrak{A}_{(n_2,n_1)}(t, X_Y)$. A mapping $\mathbf{1} - (\mathbf{1} + \cdot)^{-1_\star}$ is defined the by formula: $\mathbf{1} - (\mathbf{1} + \Psi)^{-1_\star} = \sum\limits_{n=1}^{\infty} (-1)^{n-1} \underbrace{\Psi \star \cdots \star \Psi}_{n}$ ($\Psi \equiv (0, \Psi_1, \ldots, \Psi_{1+n_1+n_2}, \ldots)$ and $\mathbf{1} = (1, 0, 0, \ldots)$ is the unit sequence).

The cluster expansions (13) for the evolution operator of non-symmetrical particle systems have the form

$$\mathbf{1} + S(-t) = \big(\mathbf{1} - \mathfrak{A}(t)\big)^{-1_\star},$$

where $(\mathbf{1} - \cdot)^{-1_\star}$ is defined as the $\star$-resolvent: $\big(\mathbf{1} - \Psi\big)^{-1_\star} = \mathbf{1} + \sum\limits_{n=1}^{\infty} \underbrace{\Psi \star \cdots \star \Psi}_{n}$.

5 Cumulant representation of BBGKY hierarchy solutions for non-symmetrical particle systems

We consider the problem of the convergence of expansion (11), (12) in the space of sequences of integrable functions. Let $L^1_\alpha = \sum\limits_{n=0}^{\infty} \bigoplus\limits_{\substack{n\,=\,n_1+n_2\\ n_1,\ n_2\,\geq\,0}} \alpha^{n_1+n_2} L^1_{n_1+n_2}$ be the Banach space of double sequences $f = \{f_n(x_{-n_2}, \ldots, x_{n_1})\}_{n=n_1+n_2\geq 0}$ integrable functions $f_n(x_{-n_2}, \ldots, x_{n_1})$ defined on the phase space $\mathbb{R}^n \times (\mathbb{R}^n \setminus W_n)$ [2,7] with the norm

$$\|f\| = \sum_{n=0}^{\infty} \sum_{\substack{n\,=\,n_1+n_2\\ n_1,\ n_2\,\geq\,0}} \alpha^{n_1+n_2} \int_{(\mathbb{R}^1\times\mathbb{R}^1)^{n_1+n_2}} dx_{-n_2}\cdots dx_{n_1}\,|f_{n_1+n_2}(x_{-n_2}, \ldots, x_{n_1})|,$$

where $\alpha > 1$ is a number; $L^1_{\alpha,0} \subset L^1_\alpha$ is the subspace of finite sequences of continuously differentiable functions with compact supports.

Since on the sequences of integrable functions $f \in L^1_\alpha$ the operators

$$\big(\mathfrak{a}_{(+)} f\big)_n(x_{-n_2}, \ldots, x_{n_1}) = \int_{\mathbb{R}^1\times\mathbb{R}^1} dx_{n_1+1}\, f_{n+1}(x_{-n_2}, \ldots, x_{n_1}, x_{n_1+1}),$$

$$\big(\mathfrak{a}_{(-)} f\big)_n(x_{-n_2}, \ldots, x_{n_1}) = \int_{\mathbb{R}^1\times\mathbb{R}^1} dx_{-(n_2+1)}\, f_{n+1}(x_{-(n_2+1)}, x_{-n_2}, \ldots, x_{n_1})$$

are defined, in view of (14) expansion (11) takes the following form in the space L^1_α

$$F(t) = \big(\mathbf{1} - \mathfrak{a}_{(+)}\big)^{-1}\big(\mathbf{1} - \mathfrak{a}_{(-)}\big)^{-1}\mathfrak{A}(t)F(0)$$
$$= \big(\mathbf{1} - \mathfrak{a}_{(+)}\big)^{-1}\big(\mathbf{1} - \mathfrak{a}_{(-)}\big)^{-1}\big(\mathbf{1} - (\mathbf{1} + S(-t)\big)^{-1^\star}\big)F(0).$$

If $F(0) \in L^1_{s+n}$, then the estimates are true:

$$\|\mathfrak{A}_{(n_2,n_1)}(t)\, F_{s+n}(0)\|_{L^1_{s+n}} \le 2^{\,n_1+n_2}\, \|F_{s+n}(0)\|_{L^1_{s+n}}.$$

By virtue of this inequality, the functions $F(t)$ defined by (11), (12) satisfy the following estimate for $\alpha > 2$,

$$\|F(t)\|_{L^1_\alpha} \le c^2_\alpha \|F(0)\|_{L^1_\alpha}, \tag{15}$$

where $c_\alpha = \big(1 - \frac{2}{\alpha}\big)^{-1}$ is a constant.

Thus, according to (15) the following existence theorem holds:

Theorem 2. *If $F(0) \in L^1_\alpha$ is a sequence of nonnegative functions, then for $\alpha > 2$, and $t \in \mathbb{R}^1$, there exists a unique solution of the initial value problem for the BBGKY hierarchy, namely the sequence $F(t) \in L^1_\alpha$ of nonnegative functions $F_{s_1+s_2}(t)$ defined by*

$$F_{|Y|}(t,Y) = \sum_{n=0}^{\infty} \sum_{\substack{n = n_1 + n_2 \\ n_1,\ n_2 \ge 0}} \int_{(\mathbb{R}^1\times\mathbb{R}^1)^{n_1+n_2}} d(X\backslash Y) \sum_{P: X_Y = \bigcup_i X_i} (-1)^{|P|-1}$$
$$\times \prod_{X_i \subset P} S_{|X_i|}(-t, X_i) F_{|X|}(0, X),$$

where we have the same notations as for relations (12). *This solution is a strong solution for $F(0) \in L^1_{\alpha,0}$ and a weak one for arbitrary initial data in the space L^1_α.*

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CP^{N-1} Harmonic Maps and the Weierstrass System

A.M. GRUNDLAND † *and W.J. ZAKRZEWSKI* ‡

† *Centre de Recherches Mathématiques, Université de Montréal, C. P. 6128, Succ. Centre-ville, Montréal, (QC) H3C 3J7, Canada*
E-mail: *Grundlan@crm.umontreal.ca*

‡ *Department of Mathematical Sciences,University of Durham, Durham DH1 3LE, UK*
E-mail: *W.J.Zakrzewski@durham.ac.uk*

We present a Weierstrass-like system of equations corresponding to the CP^{N-1} harmonic maps. This system consists of four first-order equations for three complex functions which are shown to be equivalent to the CP^{N-1} harmonic maps. When the harmonic maps are holomorphic (or antiholomorphic) one of the functions vanishes and the system of equations reduces to the generalisation of the Weierstrass problem given in [1]. We also discuss a geometrical interpretation of our results and show that the induced metric is proportional to the total energy density of the map.

1 Introduction

This talk is based on our recent paper [2] which is shortly to be published in the Journal of Mathematical Physics. In this paper we discuss two interesting ideas: the CP^1 harmonic maps and their relation to the Weierstrass System. We then show how to generalise this idea to the CP^{N-1} case.

The original idea of looking at this problem was initiated a few years ago by Konopelchenko who, together with his collaborators [3,4], introduced the subject of Weierstrass representations of surfaces immersed in multidimensional spaces. This has generated a lot of interest [5,6] and has led to the connection with the CP^{N-1} harmonic maps [1,2].

These generalisations have also lead to the study of immersed surfaces, whose metric is related to the properties of the corresponding harmonic maps. As is well known (see e.g. [7]), in the CP^1 case all harmonic maps (from S^2) are holomorphic (or antiholomorphic). In the CP^1 case the induced metric is characterised by the holomorphic component of the energy thus showing that, in this case, this characterisation is complete.

Our generalisation [2] covers the case of both holomorphic and non-holomorphic maps for, as is known [7], in the CP^{N-1} case (for $N > 2$) there are harmonic maps which are not holomorphic (or antiholomorphic). We explain in some detail, how when the maps are holomorphic, our generalisation reduces to the one given before [1].

In the next section we briefly review the CP^{N-1} harmonic maps (using the formalism as given in [7]) and in the following sections relate these maps to our version of the Weierstrass problem.

2 CP^{N-1} harmonic maps

CP^{N-1} harmonic maps are maps

$$S^2 \to CP^{N-1} \tag{1}$$

given by the stationary points of the energy

$$\mathcal{L} = \int L\, dx\, dy, \tag{2}$$

where

$$L = \frac{1}{4}(D_\mu z)^\dagger \cdot D_\mu z. \tag{3}$$

The differential operator D_μ in (3) acts on $\psi : S^2 \to CP^{N-1}$ as follows:

$$D_\mu \psi = \partial_\mu \psi - \psi(z^\dagger \cdot \partial_\mu z). \tag{4}$$

Here, $\mu = 1, 2$, and denotes the space coordinates x and y. Moreover, z is a vector field with N components, $z = (z^1, \ldots, z^N)$, which is normalised to 1; i.e. we have

$$z^\dagger \cdot z = 1. \tag{5}$$

Note that for $N = 2$, i.e. for $CP^1 \sim S^2$, we can introduce a complex field W

$$z = \frac{(1, W)}{\sqrt{1 + |W|^2}}. \tag{6}$$

Then, (for CP^1) the Euler Lagrange equations describing harmonic maps are given by

$$\partial\bar{\partial} W - 2\bar{W}\frac{\partial W \bar{\partial} W}{|W|^2 + 1} = 0. \tag{7}$$

where $W = W(\zeta, \bar{\zeta})$ and where

$$\partial = \frac{\partial}{\partial(x + iy)} = \frac{\partial}{\partial\zeta}, \qquad \bar{\partial} = \frac{\partial}{\partial\bar{\zeta}}. \tag{8}$$

Clearly, solutions of (7) are given by

$$W = W(\zeta) \tag{9}$$

or

$$W = W(\bar{\zeta}) \tag{10}$$

i.e. holomorphic and antiholomorphic functions. What is less clear, however, is that all solutions on S^2 are like this; i.e. there are no harmonic functions which are not holomorphic or antiholomorphic.

This, however, is not the case for the CP^{N-1} model [7] for $N > 2$.

3 The Weierstrass system (WS)

To define a WS we consider solutions of a system of equations for 2 complex functions

$$\psi = \psi(\zeta, \bar{\zeta}) \tag{11}$$

and

$$\phi = \phi(\zeta, \bar{\zeta}), \tag{12}$$

which satisfy the following set of first-order equations:

$$\partial\psi = p\phi, \qquad \bar{\partial}\phi = -p\psi, \qquad p = |\phi|^2 + |\psi|^2. \tag{13}$$

Note that $\bar{\partial}\psi$, or $\partial\phi$ have not been specified.

A natural question then arises. Are these two problems related? Obviously the answer is YES. To see this we put

$$W = \frac{\psi}{\bar{\phi}} \tag{14}$$

and then find that

$$\psi = W\frac{(\bar{\partial}\bar{W})^{\frac{1}{2}}}{1+|W|^2}, \qquad \phi = \frac{(\partial W)^{\frac{1}{2}}}{1+|W|^2} \tag{15}$$

satisfy (13).

Moreover, we can also show that the equations of the two systems are fully equivalent.

4 Geometry

To discuss geometrical aspects of the Weierstrass problem [3], and to relate it to the properties of the CP^1 maps, we introduce 3 real functions

$$X_1 = i\int_\gamma [\bar{\psi}^2 + \phi^2]d\zeta - [\psi^2 + \bar{\phi}^2]d\bar{\zeta}, \tag{16}$$

$$X_2 = \int_\gamma [\bar{\psi}^2 - \phi^2]d\zeta + [\psi^2 - \bar{\phi}^2]d\bar{\zeta}, \tag{17}$$

and

$$X_3 = -2\int_\gamma \bar{\psi}\phi d\zeta + \psi\bar{\phi}d\bar{\zeta}, \tag{18}$$

where γ is any curve from a fixed point to ζ.

It is now easy to show that if ψ and ϕ satisfy the equations of the WS then X_i do not depend on the details of the curve γ but only on its endpoints.

Next we treat X_i as components of $\vec{r} = (X_1, X_2, X_3)$ and introduce the metric

$$g_{\zeta\zeta} = (\partial\vec{r}, \partial\vec{r}), \qquad g_{\bar{\zeta}\bar{\zeta}} = (\bar{\partial}\vec{r}, \bar{\partial}\vec{r}), \qquad g_{\zeta\bar{\zeta}} = (\partial\vec{r}, \bar{\partial}\vec{r}). \tag{19}$$

Then, the detailed calculations show that only $g_{\zeta\bar{\zeta}}$ is non-zero and that

$$g_{\zeta\bar{\zeta}} = \frac{|\partial W|^2}{(1+|W|^2)^2} = |Dz|^2. \tag{20}$$

Here D denotes D_μ evaluated with respect to ζ.

Note that, as all harmonic CP^1 maps on S^2 satisfy $W = W(\zeta)$ and $g_{\zeta\bar{\zeta}}$ is the total energy of the map (for antiholomorphic maps we take complex conjugates).

5 The CP^{N-1} case. Some general points

Next we consider the general CP^{N-1} case. Now we have more components, ie if we define $z = \frac{f}{|f|}$ then f has N components.

Note also that harmonic maps are **not** necessarily holomorphic or antiholomorphic. ie the equation

$$\left(1 - \frac{f \otimes f^\dagger}{|f|^2}\right)\left[\partial\bar{\partial} f - \partial f \frac{(f^\dagger \cdot \bar{\partial} f)}{|f|^2} - \bar{\partial} f \frac{(f^\dagger \cdot \partial f)}{|f|^2}\right] = 0, \tag{21}$$

has solutions other than $f = f(\zeta)$ or $f = f(\bar{\zeta})$.

So what should we do? What should we take for functions ψ and ϕ and how many of them should we use? And how to avoid, in the geometrical interpretation, getting only the holomorphic component of the map which was sufficient in the CP^1 case but which is not for CP^{N-1}.

6 Our procedure [2]

To discuss our procedure [2] it is convenient to rewrite the equations of the CP^{N-1} model in terms of the 'projector formalism' (see e.g. [7]). Thus we introduce

$$P = \frac{f \otimes f^\dagger}{|f|^2}, \qquad P^\dagger = P, \qquad P^2 = P. \tag{22}$$

Then, as is easy to check, the equations for the harmonic maps, i.e. (21) are equivalent to

$$[\partial\bar{\partial} P, P] = 0. \tag{23}$$

However, we can rewrite (23) as a conservation law:

$$\partial[\bar{\partial} P, P] + \bar{\partial}[\partial P, P] = 0, \tag{24}$$

i.e.

$$\partial K - \bar{\partial} K^\dagger = 0, \tag{25}$$

where

$$K = [\bar{\partial} P, P] = \frac{\bar{\partial} f \otimes f^\dagger - f \otimes \bar{\partial} f^\dagger}{|f|^2} + \frac{f \otimes f^\dagger}{|f|^4}\left[(\bar{\partial} f^\dagger \cdot f) - (f^\dagger \cdot \bar{\partial} f)\right]. \tag{26}$$

To proceed further it is convenient to put

$$K = M + L, \tag{27}$$

where

$$M = (1 - P)\frac{\bar{\partial} f \otimes f^\dagger}{|f|^2} = (1 - P)\bar{\partial} P \tag{28}$$

and

$$L = -\frac{f \otimes \bar{\partial} f^\dagger}{|f|^2}(1 - P) = -\bar{\partial} P(1 - P). \tag{29}$$

Next we observe that our matrices M and L satisfy the conservation laws separately, i.e. we have

$$\partial M = \bar{\partial} M^{\dagger} \tag{30}$$

and

$$\partial L = \bar{\partial} L^{\dagger}. \tag{31}$$

Note that the two conservation laws are not really independent; they differ from each other by a total divergence (as was drawn to our attention by A. Mikhailov [9]). To see this note that as P is a projector we have

$$(1-P)P = P(1-P) = 0. \tag{32}$$

Then

$$0 = \bar{\partial}\{(1-P)P\} = -\bar{\partial}PP + (1-P)\bar{\partial}P = -\bar{\partial}PP + M \tag{33}$$

and

$$0 = \bar{\partial}\{P(1-P)\} = \bar{\partial}P(1-P) - P\bar{\partial}P = -L - P\bar{\partial}P. \tag{34}$$

Hence

$$M = \bar{\partial}PP, \qquad L = -P\bar{\partial}P \tag{35}$$

and so we see that

$$M - L = \bar{\partial}PP + P\bar{\partial}P = \bar{\partial}P^2 = \bar{\partial}P \tag{36}$$

thus showing that

$$M = L + \bar{\partial}P. \tag{37}$$

Substituting this into (30) we obtain (31).

7 Exploitation of the conservation laws

Let us write out the matrix elements of M and L (using the summation convention). They are

$$M_{ij} = \bar{\Phi}_i^2 \bar{f}_j \tag{38}$$

and

$$L_{ij} = -f_i \bar{\varphi}_j^2, \tag{39}$$

where

$$\varphi_i^2 = \frac{1}{A^2} \bar{f}_k F_{ki}, \qquad A = \bar{f}_l f_l \tag{40}$$

and

$$\Phi_i^2 = \frac{1}{A^2} f_k \overline{G_{ki}}. \tag{41}$$

In these expression we have introduced:

$$F_{ij} = f_i \partial f_j - f_j \partial f_i, \tag{42}$$

and

$$G_{ij} = f_i \bar{\partial} f_j - f_j \bar{\partial} f_i. \tag{43}$$

Note that due to the antisymmetry of F_{ij} and of G_{ij} we have constraints; namely,

$$\bar{f}_k \varphi_k^2 = 0, \qquad f_k \Phi_k^2 = 0. \tag{44}$$

Hence only $(N-1)$ functions φ_i^2 and $(N-1)$ functions Φ_i^2 are linearly independent.

Thus we take as our independent functions, say, $\varphi_2^2, \ldots, \varphi_N^2$ and $\Phi_2^2, \ldots, \Phi_N^2$. In addition, we can set, without any loss of generality, $f_1 = 1$.

Next we note that we can invert the expressions for φ_i^2 and find that

$$\partial f_i = A \left[\varphi_i^2 + f_i \bar{f}_k \varphi_k^2 \right] \tag{45}$$

and

$$\bar{\partial} f_i = A \left[\bar{\Phi}_i^2 + f_i \, \bar{f}_k \bar{\Phi}_k^2 \right]. \tag{46}$$

These two equations will form part of our Weierstrass system as we discuss in the next section.

8 Relation to the Weierstrass system

Next we use the equations of the CP^{N-1} model to calculate $\bar{\partial}\varphi_i^2$ and $\bar{\partial}\Phi_i^2$. We find that

$$\bar{\partial}\varphi_i^2 = -A\varphi_i^2 (\varphi^{\dagger 2} \cdot f) - f_i \left[\left(\varphi^{\dagger 2} \cdot \varphi^2 \right) + (f^\dagger \cdot \varphi^2)(\varphi^{2\dagger} \cdot f \right) \right], \tag{47}$$

and

$$\bar{\partial}\Phi_i^2 = -A\Phi_i^2 (f^\dagger \cdot \bar{\Phi}^2) - \bar{f}_i \left[\left(\Phi^{\dagger 2} \cdot \Phi^2 \right) + (f^\dagger \cdot \bar{\Phi}^2)(\bar{\Phi}^{\dagger 2} \cdot f \right) \right], \tag{48}$$

which, together with (45) and (46) i.e.

$$\partial f_i = A \left[\varphi_i^2 + f_i \bar{f}_k \varphi_k^2 \right],$$
$$\bar{\partial} f_i = A \left[\bar{\Phi}_i^2 + f_i \bar{f}_k \bar{\Phi}_k^2 \right],$$

constitute our **generalisation** of the Weierstrass Problem [2]. Here, as usual, $A = 1 + (f^\dagger \cdot f)$ and all indices, and summations, now go only over $(2, \ldots, N)$.

Note that our procedure has given us four sets of equations for three sets of complex functions, f_i, φ_j and Φ_k. The equations fall into two subclasses (those involving ∂f_i and φ_j and those involving $\partial \bar{f}_i$ and Φ_j). Both sets are equivalent to the same equations for f_i. Note also that instead of taking f_i we could have introduced new functions ψ_i and Ψ_i by, say, $\psi_i = f_i \bar{\varphi}_i$ and $\Psi_i = f_i \bar{\Phi}_i$. Then our sets of functions would have effectively decoupled.

Let us observe that when f_i are holomorphic; i.e. $\bar{\partial} f_i = 0$ then $f^\dagger \cdot \bar{\Phi}^2 = 0$, implying that $|\Phi^2|^2 = 0$. Thus $\Phi_i^2 = 0$ (in analogy with the original Weierstrass system where we had only ϕ and $\psi = f\bar{\phi}$).

9 Geometrical aspects

Now we follow the procedure from Section 4 and introduce real functions X_i which will be treated as components of a vector in a larger dimensional space. To do this we introduce the matrices

$$V = \int_\gamma M d\bar{\zeta} + \int_\gamma M^\dagger d\zeta \tag{49}$$

and

$$W = \int_\gamma L d\bar{\zeta} + \int_\gamma L^\dagger d\zeta \tag{50}$$

and then, for X_i, we take individual entries of each matrix. As $\mathrm{Tr}\, M = \mathrm{Tr}\, L = 0$ V and W have, each, only $N^2 - 1$ independent entries so our construction gives us two vectors in R^{N^2-1}.

Note that W and V do not depend on the contours of integration γ. This, as before, follows from the conservation laws.

Should we consider the entries of W and V separately and so treat our pair of vectors as one vector in a higher dimensional space? This would be a possibility but we do not have to go that far. The reason for this is that, as can be seen by direct calculations, for the solutions (CP^{N-1} harmonic maps) can add our two vectors and obtain a vector which leads to an induced metric with good properties.

Thus we consider

$$X = \int_\gamma (M + L) d\bar{\zeta} + \int_\gamma (M^\dagger + L^\dagger) d\zeta. \tag{51}$$

Then we define the induced metric

$$g_{\alpha\beta} = \sum_{lk} \frac{\partial X_{kl}}{\partial \alpha} \frac{\partial X_{lk}}{\partial \beta}, \tag{52}$$

where α and β stand for ζ or $\bar{\zeta}$.

We find that that for this metric we have

$$g_{\zeta\bar{\zeta}} = |Dz|^2 + |\bar{D}z|^2, \tag{53}$$

which is, of course, the total energy of the harmonic map, while the other components, $g_{\zeta\zeta}$ and $g_{\bar{\zeta}\bar{\zeta}}$, vanish.

The demonstration of this fact is rather complicated as it involves the details of some of the very specific properties of the CP^{N-1} harmonic maps. For more details see [2].

10 Conclusions and some general comments

In this talk we have presented a possible generalization of the Weierstrass problem [2]. Let us stress that this generalization involves more functions than those appearing in the original Weierstrass problem which is very special. However, the original problem has depended on the fact that all harmonic maps from S^2 to CP^1 are either holomorphic or antiholomorphic – which is not the case for CP^{N-1}, when $N > 2$. And even for CP^1 one had to adopt different approaches to holomorphic and antiholomorphic maps (namely, for antiholomorphic maps we had to complex conjugate them – to turn them into holomorphic ones). This is not the case in our construction which holds for all maps.

We can generalise our procedure to other manifolds, such as other Grassmanian models. This involves changing the rank of the projector P in (22). A more interesting generalisation involves establishing the relation to other approaches – say, a lá Bobenko and his collaborators (see e.g. [10]). Such an approach is currently under active study and we hope to report on it in the near future [11].

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Geometrical Interpretation of Constrained Systems

Yurdahan GÜLER

Çankaya University, Department of Mathematics and Computer Science, Ankara, Turkey
E-mail: *yurdahan@cankaya.edu.tr*

Quadratic singular Lagrangians are studied using the Hamilton–Jacobi theory of singular systems. An example is studied.

1 Introduction

The standard approach to classical dynamics is to form a Lagrangian which is a function of n generalized coordinates q_i, n generalized velocities $\dot{q}_i$ and parameter τ. The $2n$ variables q_i, $\dot{q}_i$ form the tangent bundle TQ. The passage from TQ to the cotangent bundle T^*Q is achieved by introducing generalized momenta and a Hamiltonian. However, this procedure requires that the rank of Hessian matrix

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

is n. The investigation of this type of Lagrangians (REGULAR) is a usual procedure. If the rank is less than n, system is called SINGULAR.

The study of singular systems started with DIRAC. He obtained the equations of motion of a singular system using consistency conditions and Poisson brackets. He classified the constraints. To quantize a singular system he introduced the Dirac bracket. Bergmann and his collaborators stressed on the relation between invariance principles and constraints in field theories. In fact, their efforts were to construct a Hamiltonian approach of general relativity to quantize the theory since the Einstein's theory of gravitation is a singular theory to its general covariance.

Singular field theories became the center of interest for physicists after the pioneering work of Faddeev, who introduced the Feynman path integral quantization. Nowadays, singular systems find a very wide range of applications in theoretical physics. An invariance under a global gauge transformation implies a singular theory. Hence, starting from the electromagnetic theory, all gauge theories have singular nature.

2 Hamilton–Jacobi theory of singular systems

The aim is to obtain a valid and consistent Hamilton–Jacobi theory of singular systems. Mathematical method which is used is the Caratheodory's equivalent Lagrangians method. The main point of the method is to define the equivalent Lagrangian (variational principle) and then pass to the phase space. This formulation leads us to a set of Hamilton–Jacobi partial differential equations [1–4].

2.1 Construction of phase space

Let us consider a system which is described by the Lagrangian $L(q_i, \dot{q}_i, \tau)$ such that the rank of the Hessian is $n - p$, $p \leq n$.

Generalized momenta are defined as

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = 1, 2, \ldots, p,$$
$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, \ldots, n-p. \tag{1}$$

Since the rank of the Hessian is $n-p$, one may solve (1) for $\dot{q}_a$ as

$$\dot{q}_a = w_a\left(\tau, q_i, \dot{q}_\mu, p_a\right).$$

So

$$p_\mu = \left.\frac{\partial L}{\partial \dot{q}_\mu}\right|_{\dot{q}_a = w_a} = -H_\mu\left(\tau, q_i, p_a, \dot{q}_\nu\right).$$

Generalized momenta p_μ are NOT independent. Although, it seems that H_μ are functions of $\dot{q}_\nu$, it is a task to show that they do not depend on $\dot{q}_\nu$ explicitly.

Definition 1.

$$H_0 = -L\left(\tau, q_i, \dot{q}_\nu, \dot{q}_a = w_a\right) + p_a w_a + \dot{q}_\mu p_\mu\big|_{p_\mu = -H_\mu}.$$

Like H_μ, H_0 is not an explicit function of $\dot{q}_\mu$.
In fact,

$$\frac{\partial H_0}{\partial \dot{q}_\nu} = -\frac{\partial L}{\partial \dot{q}_\nu} - \frac{\partial L}{\partial \dot{q}_a}\frac{\partial w_a}{\partial \dot{q}_\nu} + p_a \frac{\partial w_a}{\partial \dot{q}_\nu} p_\nu = 0.$$

Therefore, the Hamilton–Jacobi function $S\left(\tau, q_i\right)$ should satisfy the following set of Hamilton–Jacobi Partial Differential Equations (HJPDE) simultaneously for an extremum of the action:

$$p_0 + H_0\left(\tau, q_\nu; q_a, p_a = \frac{\partial S}{\partial q_a}, p_0 = \frac{\partial S}{\partial \tau}\right) = 0,$$
$$p_\mu + H_\mu\left(\tau, q_\nu; q_a, p_a = \frac{\partial S}{\partial q_a}, p_0 = \frac{\partial S}{\partial \tau}\right) = 0$$

or in a compact form

$$H'_\alpha\left(t_p, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}\right) = 0,$$

where

$$H'_\alpha = p_\alpha + H_\alpha, \qquad \alpha = 0, 1, \ldots, p.$$

2.2 The canonical equations

This method leads us to the following equations:

$$dq_r = \frac{\partial H'_\alpha}{\partial p_r} dt_\alpha, \qquad r = 0, 1, \ldots, n, \quad \alpha = 0, 1, \ldots, p,$$
$$dp_r = -\frac{\partial H'_\alpha}{\partial q_r} dt_\alpha,$$
$$dS = \left(-H'_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha.$$

Simultaneous solutions of these equations determine $S\left(t_\alpha, q_a\right)$ uniquely if the initial conditions are given.

Example 1.

$$2L = a_{ij}\left(\tau, q_k\right)\dot{q}_i\dot{q}_j + 2b_i\left(\tau, q_k\right)\dot{q}_i - 2c\left(\tau, q_k\right), \qquad i, j = 1, 2, 3,$$

where

$$a_{ij} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & a_2 \\ 0 & a_2 & -a_2 \end{vmatrix}, \qquad a_1 \neq 0, \quad a_2 \neq 0,$$

$$p_1 = a_1\dot{q}_1, \qquad p_2 = a_2\left(\dot{q}_3 - \dot{q}_2\right) + b, \qquad p_3 = a_2\left(\dot{q}_3 - \dot{q}_2\right),$$

$$w_1 = \frac{p_1}{a_1}, \qquad w_3 = \dot{q}_2 - \frac{p_3}{a_2}, \qquad p_2 = -p_3 + b = -H_2,$$

$$H_0 = -L + p_1w_1 + p_3w_3 + \left(-p_3 + b\right)\dot{q}_2 \qquad \text{or}$$

$$H_0 = \frac{1}{2}\left(\frac{p_1^2}{a_1} - \frac{p_3^2}{a_2}\right) + c.$$

The set of (HJPDE) is

$$\frac{\partial S}{\partial \tau} + H_0\left(q_i, \tau, p_1 = \frac{\partial S}{\partial q_1}, p_3 = \frac{\partial S}{\partial q_3}\right) = 0,$$

$$\frac{\partial S}{\partial q_2} + H_2\left(q_i, \tau, p_1 = \frac{\partial S}{\partial q_1}, p_3 = \frac{\partial S}{\partial q_3}\right) = 0.$$

Differential equations for the characteristics are:

$$dq_1 = \frac{p_1}{a_1}d\tau, \qquad dq_3 = -\frac{p_3}{a_2}d\tau + dq_2,$$

$$dp_i = \left[\frac{1}{2}\left(\frac{p_1^2}{a_1^2}\frac{\partial a_1}{\partial q_i} - \frac{p_3^2}{a_2^2}\frac{\partial a_2}{\partial q_i}\right)\frac{\partial c}{\partial q_i}\right]d\tau + \frac{\partial b}{\partial q_i}dq_2,$$

$$dp_0 = \left[\frac{1}{2}\left(\frac{p_1^2}{a_1^2}\frac{\partial a_1}{\partial q_i} - \frac{p_3^2}{a_2^2}\frac{\partial a_2}{\partial q_i}\right)\frac{\partial c}{\partial \tau}\right]d\tau + \frac{\partial b}{\partial \tau}dq_2.$$

3 Free particle

A particle of mass m is described by generalized coordinates x_i $(i = 1, \ldots, n)$ as

$$2L(x^i, \dot{x}^i) = -mcg_{ij}\dot{x}^i\dot{x}^j, \tag{2}$$

where the metric matrix elements are functions of x_i. One can express the Hamiltonian in the usual way if the rank of the Hessian matrix (metric matrix) is n. In fact, generalized momenta are defined as

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = -mcg_{ij}\dot{x}^j.$$

This definition leads us to the expression

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = -mcg_{ij}\dot{x}^j.$$

Thus,

$$2H = -2L + 2p_i\dot{x}^i = -\frac{1}{mc}g^{km}p_kp_m.$$

Canonical equations

$$\dot{x}^i = \frac{dx^i}{d\tau} = \frac{\partial H}{\partial p_i},$$
$$\dot{p}^i = \frac{dp^i}{d\tau} = -\frac{\partial H}{\partial x_i}$$

lead us to the geodesic equations on the Riemannian manifold identified with the metric g_{ij}. Following calculations give the desired equations

$$\dot{x}_i = -\frac{1}{2mc} g_{ij} p^j,$$
$$\dot{p}_k = -\frac{\partial H}{\partial x^k} = \frac{1}{2mc} \frac{\partial g_{ij}}{\partial x^k} p^i p^j.$$

Thus,

$$\ddot{x}^i = \frac{\partial g^{ij}}{\partial x^k} g_{jm} \dot{x}^k \dot{x}^m - \frac{1}{2} \frac{\partial g_{lk}}{\partial x^j} g^{ji} g^{km} g^{ln} \dot{x}_m \dot{x}_n.$$

After some manupulation we get

$$\ddot{x}^i + \Gamma^i_{km} \dot{x}^k \dot{x}^m = 0, \qquad i = 1, \ldots, n,$$

where

$$\Gamma^i_{km} = -\frac{1}{2} g^{ij} \left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} - \frac{\partial g_{mk}}{\partial x^j} \right).$$

Description of the Lagrangian (2) in terms of the metric g_{ij} gives the geodesic equations.

If the rank of the Hessian matrix is less than n, we have a singular system [5]. Now, the question is whether the geometric approach is applicable or not. To achieve this aim we are going to apply the Canonical formulation to a singular Lagrangian. Let us consider the Lagrangian

$$L = \frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) + \dot{\lambda}_1 \left(y\dot{z} - z\dot{y} \right) + \dot{\lambda}_2 \left(z\dot{x} - x\dot{z} \right) + \dot{\lambda}_3 \left(x\dot{y} - y\dot{x} \right). \tag{3}$$

Physically speaking, this is the Lagrangian which describes a particle such that all three components of angular momentum are conserved. Here $\dot{\lambda}_i$ $(i = 1, 2, 3)$ are Lagrange multipliers.

One can express this Lagrangian as

$$L = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j, \qquad i = 1, \ldots, 6,$$

where

$$\dot{q}_1 \equiv \dot{x}, \qquad \dot{q}_2 \equiv \dot{y}, \qquad \dot{q}_3 \equiv \dot{z}, \qquad \dot{q}_4 \equiv \dot{\lambda}_1, \qquad \dot{q}_5 \equiv \dot{\lambda}_2, \qquad \dot{q}_6 \equiv \dot{\lambda}_3$$

and the 6×6 symmetric matrix g_{ij} is

$$g_{ij} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \\ 0 & -z & y & 0 & 0 & 0 \\ z & 0 & -x & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 & 0 \end{pmatrix} \equiv g_{ji}.$$

One should notice that the rank of g_{ij} is five. Thus, one cannot apply the standard procedure. To find a remedy, we express (3) as

$$L = \frac{1}{2}\left(g_{\nu\mu}\dot{q}_\nu\dot{q}_\mu + 2g_{6\mu}\dot{\lambda}_3\dot{q}_\mu\right), \qquad \mu,\nu = 1,\ldots,5.$$

In order to determine the Hamiltonian, let us define the generalized momenta as

$$\begin{aligned} p_\mu &= \frac{\partial L}{\partial \dot{q}_\mu} = g_{\nu\mu}\dot{q}_\nu + g_{6\mu}\dot{\lambda}_3, \qquad (4)\\ p_{\lambda_3} &= \frac{\partial L}{\partial \dot{\lambda}_3} = g_{6\sigma}\dot{q}_\sigma \\ &= g_{6\sigma}\left[g^{-1}_{\sigma\mu}p_\mu - g^{-1}_{\sigma\mu}a_{6\mu}\dot{\lambda}_3\right] \\ &= g_{66}p_\mu - g_{66} \equiv 0. \end{aligned}$$

Since the rank of g_{ij} is five, the submatrix $g_{\mu\nu}$ is invertible. Thus, using (4) we get

$$\dot{q}_\sigma = g^{-1}_{\sigma\mu}p_\mu - g^{-1}_{\sigma\mu}a_{6\mu}\dot{\lambda}_3, \qquad \sigma = 1,\ldots,5. \qquad (5)$$

Now, we can define the Hamiltonian H_0 as

$$H_0 = -L + p_\mu\dot{q}_\mu + p_{\lambda_3}\dot{\lambda}_3. \qquad (6)$$

Substituting (5) in (6), one obtains

$$H_0 = \frac{1}{2}g^{-1}_{\mu\rho}p_\mu p_\rho, \qquad \mu,\rho = 1,\ldots,5$$

which is independent of $\dot{\lambda}_3$. We expect this result due to the singular nature of the Lagrangian. We have two Hamiltonians to describe the system

$$H_0' = p_0 + \frac{1}{2}g^{-1}_{\mu\rho}p_\mu p_\rho \equiv 0, \qquad (7)$$

$$H' = p_{\lambda_3} \equiv 0. \qquad (8)$$

Canonical equations read as

$$dq_\mu = \frac{\partial H'_\alpha}{\partial p_\mu}dt_\alpha, \qquad dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu}dt_\alpha, \qquad \alpha = 0,6, \qquad dt_0 \equiv dt, \qquad dt_\alpha \equiv d\lambda_3.$$

More explicitly

$$\begin{aligned} dq_\mu &= g^{-1}_{\mu\rho}p_\rho dt, \qquad \mu = 1,\ldots,5,\\ dq_6 &\equiv d\lambda_3,\\ dp_\nu &= -\frac{1}{2}\left(\frac{\partial g^{-1}_{\mu\rho}}{\partial q_\nu}\right)p_\mu p_\rho dt,\\ dp_{\lambda_3} &= 0, \qquad dp_0 = 0. \qquad (9) \end{aligned}$$

The theory forces us to check the consistency conditions i.e. to check whether the variations of the constraints (7) and (8) are zero or not. Equation (9) guarantees that the variation of H' is zero. Besides,

$$dH_0' = dp_0 + \frac{1}{2}\left\{d\left(g^{-1}_{\mu\rho}\right)p_\mu p_\rho + g^{-1}_{\mu\rho}dp_\mu p_\rho + g^{-1}_{\mu\rho}p_\mu dp_\rho\right\}.$$

Using the equations of motion we get

$$dH_0' = \frac{1}{2}\left\{\frac{\partial g^{-1}_{\mu\rho}}{\partial q_\sigma}dq_\sigma p_\mu p_\rho + g^{-1}_{\mu\rho}\left(-\frac{1}{2}\frac{\partial g^{-1}_{\alpha\beta}}{\partial q_\mu}p_\alpha p_\beta dt\right)p_\rho + g^{-1}_{\mu\rho}p_\mu\left(-\frac{1}{2}\frac{\partial g^{-1}_{\alpha\beta}}{\partial q_\rho}p_\alpha p_\beta dt\right)\right\} \equiv 0.$$

Thus, the system is integrable.

Up to this point we have followed the standard canonical formulation of a constrained system. Now, we will propose the geodesic equations

$$\ddot{q}_\mu = \frac{1}{2}\left(g^{-1}\right)_{\mu\nu}\left(\frac{\partial g_{\nu\sigma}}{\partial q_\rho} + \frac{\partial g_{\nu\rho}}{\partial q_\sigma} - \frac{\partial g_{\sigma\rho}}{\partial q_\nu}\right)\dot{q}^\sigma\dot{q}^\rho \tag{10}$$

as equivalent to the equations of motion. In other words, we will show that solutions of (10) satisfy the equations of motion. In fact, equations (10) give

$$\begin{aligned}
\ddot{x} &= \frac{x}{x^2+y^2+z^2}\left[2\dot{\lambda}_2\left(z\dot{x}-x\dot{z}\right) + 2\dot{\lambda}_1\left(y\dot{z}-z\dot{y}\right)\right],\\
\ddot{y} &= \frac{y}{x^2+y^2+z^2}\left[2\dot{\lambda}_1\left(y\dot{z}-z\dot{y}\right) + 2\dot{\lambda}_2\left(z\dot{x}-x\dot{z}\right)\right],\\
\ddot{z} &= \frac{z}{x^2+y^2+z^2}\left[2\dot{\lambda}_2\left(z\dot{x}-x\dot{z}\right) + 2\dot{\lambda}_1\left(y\dot{z}-z\dot{y}\right)\right],\\
\ddot{\lambda}_1 &= \frac{1}{x^2+y^2+z^2}\left[-2\dot{\lambda}_1\left(\frac{x^2\dot{z}+z^2\dot{z}}{z}+y\dot{y}\right) + 2\dot{\lambda}_2 y\left(\frac{z\dot{x}-x\dot{z}}{z}\right)\right],\\
\ddot{\lambda}_2 &= \frac{1}{x^2+y^2+z^2}\left[2x\dot{\lambda}_1\left(\frac{z\dot{y}-y\dot{z}}{z}\right) - 2\dot{\lambda}_2\left(\frac{y^2\dot{z}+z^2\dot{z}}{z}+x\dot{x}\right)\right].
\end{aligned}$$

One should notice that

$$\frac{\ddot{x}}{x} = \frac{\ddot{y}}{y} = \frac{\ddot{z}}{z}.$$

Therefore, equating these ratios to a negative constant k we get the periodic solutions

$$\begin{aligned}
x(t) &= c_1\sin\sqrt{|k|}t + c_2\cos\sqrt{|k|}t,\\
y(t) &= c_3\sin\sqrt{|k|}t + c_4\cos\sqrt{|k|}t,\\
z(t) &= c_5\sin\sqrt{|k|}t + c_6\cos\sqrt{|k|}t.
\end{aligned}$$

These solutions also satisfy the equations of motion.

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$SU(N)$ Flavor Dynamics within a Generalized Heat Kernel Expansion

Brigitte HILLER † *and Alexander A. OSIPOV* †‡

† *Centro de Física Teórica, Departamento de Física da Universidade de Coimbra, 3004-516 Coimbra, Portugal*
E-mail: *brigitte@teor.fis.uc.pt, alexguest@teor.fis.uc.pt*

‡ *Laboratory of Nuclear Problems, JINR, 141980 Dubna, Russia*

The asymptotic expansion of the one-loop effective action $W = \ln\det A$ is derived for the case in which a non-degenerate mass matrix $\mathcal{M} = \mathrm{diag}(m_1, m_2, \ldots)$ is considered, generalizing the standard method of Schwinger–DeWitt. The positively defined elliptic operator $A = U + \mathcal{M}^2$ depends on the external classical fields taking values in the Lie algebra of the internal symmetry group G. The first coefficients of the new asymptotic series are calculated and their relationship with the Seeley–DeWitt coefficients is clarified.

1 Introduction

The existence of fermion families of non-degenerate masses in the Standard Model calls for a field-theoretical framework capable of incorporating in a chiral and gauge covariant way the occurrence of different masses in loops. In QCD, underlying the non-degenerate mass matrix of heavy (constituent) quarks, which results from spontaneous breakdown of chiral symmetry, is a non-degenerate current quark matrix, which dictates the explicit symmetry breaking pattern of the Lagrangian. Both, the covariant and the explicit symmetry breaking terms of the Lagrangian must be preserved in a sensible calculational scheme. Furthermore, in the low-energy regime of QCD, it is more natural to transform to mesonic/baryonic degrees of freedom, which can be obtained by standard bosonization techniques. In this case the bosonization must of course comply with all symmetry requirements of the original Lagrangian.

These issues have been dealt with in various ways, however the symmetries are not automatically fulfilled. For instance [1], through the evaluation of Feynman amplitudes, the symmetry requirements of the original Langrangian have to be checked aposteriori case for case. Perturbative approaches, where the difference in masses is considered to be the small expansion parameter [2], must also be subject to a careful control of symmetries.

In the following we present a technique, which we developed to handle non-degenerate mass matrices, fulfilling by construction all the symmetries of the Lagrangian. It is based on the method of Schwinger–DeWitt [3,4], which is an extremely powerful tool when explicit covariance of calculations of radiative corrections is needed at all intermediate steps. This is the case, for example, for gauge theory [4], quantum gravity [5], chiral field theory [6]; for a recent review see [7].

The main object of study in this formalism is the determinant of the positively defined elliptic operator, which describes quadratic fluctuations of quantum fields in presence of some background fields and contains in compact form the whole information about the one-loop contribution of quantum fields. The proper time formalism of Schwinger allows to evaluate it. The result is an asymptotic expansion of the effective action in powers of proper time with Seeley–DeWitt coefficients a_n [4, 8], which accumulate the whole dependence on background fields. A remarkable property is that every term of the expansion is invariant with respect to

transformations of the internal symmetry group. This is a consequence of the general covariance inherent to the formalism. At the present time the asymptotic coefficients a_n are well known up to and including $n = 5$ for a general operator of Laplace type. Details can be found, for instance, in [6, 9].

In the case of massive quantum fields with a degenerate mass matrix $\mathcal{M} = \mathrm{diag}(m, m, \ldots)$, it is not difficult to derive from the proper time expansion an expansion in inverse powers of m^2, since the mass dependence is easily factorized and a subsequent integration over proper time leads to the desired result. The resulting asymptotic coefficients remain the same. If the mass matrix is not degenerate, i.e. $\mathcal{M} = \mathrm{diag}(m_1, m_2, \ldots)$, than its total factorization is impossible because of the noncommutativity of $\mathcal{M}$ with the rest of the elliptic operator. At the same time a naive factorization by parts breaks the covariant character of the asymptotic series. The natural question arises: is there any simple way to follow which leads to factorization, conserving at the same time the explicit covariance of the expansion? Fortunately, there is such a method [10, 11]. It leads to the generalization of the Seeley–DeWitt coefficients. Here we will not consider the manifest chiral symmetry breaking effect. A careful analysis of this problem would lead us too far away from the subject, leaving the present result for the covariantly transforming terms without changes. We refer to [11] where the symmetry breaking terms have been considered by breaking the $SU(2) \times SU(2)$ chiral symmetry of the Nambu–Jona–Lasinio model [12] by the current quark mass matrix with $\hat{m}_u \neq \hat{m}_d$.

2 Integral representation for the one-loop contribution

The logarithm of a formal determinant describes the low order radiative corrections to the classical theory. Let fermion fields play the role of virtual quanta producing those corrections and the scalar and pseudoscalar mesons be external background fields. In this case the real part of the corresponding effective action can be represented as a proper time integral

$$W[Y] = -\ln|\det D| = \frac{1}{2}\int_0^\infty \frac{dt}{t}\rho(t,\Lambda^2)\mathrm{Tr}\left(e^{-tD^\dagger D}\right). \tag{1}$$

Here the Dirac operator D depends on the background fields, which are collected in the Hermitian second-order differential elliptic operator $D^\dagger D = \mathcal{M}^2 + B = -\partial^2 + Y + \mathcal{M}^2$ in the term Y. We use the Euclidian metric to define the effective action $W[Y]$. To make the integral over t convergent at the lower limit, the regulator $\rho(t, \Lambda^2)$ has been added in (1). All our conclusions are independent of the form of this function, which includes the ultraviolet cut off Λ, in the sense that the generalized Seeley–DeWitt coefficients do not depend on the regulator.

Let us assume that we are dealing with chiral field theory, which possesses the global $U(N_f)_L \times U(N_f)_R$ symmetry if the fermion fields are massless. The typical example is quantum chromodynamics, excluding the $U_A(1)$ anomaly. It is known that the vacuum state of low energy QCD is noninvariant with respect to the action of the chiral group and the whole system makes a phase transition to the state with massive quarks. If one takes into account that the explicit chiral symmetry breaking takes place in QCD through the mass terms of current quarks one can conclude that not equal current quark masses will lead also to not equal constituent quark masses. In order to study this system at large distances we will need the expansion of the effective action in inverse powers of the non degenerate mass matrix

$$\mathcal{M} = \sum_{i=1}^{N_f} \mathcal{M}_i E_i, \qquad (E_i)_{jk} = \delta_{ij}\delta_{ik}, \qquad E_i E_j = \delta_{ij} E_j. \tag{2}$$

The orthonormal basis E_i belongs to the flavor space where the chiral group acts on quarks and background fields in accordance with their transformation properties.

The heat kernel $\mathrm{Tr}\,[\exp(-tD^\dagger D)]$ can be represented [3] as a matrix element of an operator acting in the abstract (unphysical) Hilbert space:

$$\mathrm{Tr}\left(e^{-tD^\dagger D}\right) = \int d^4x\,\mathrm{tr}\,\langle x|e^{-tD^\dagger D}|x\rangle. \tag{3}$$

The plane wave basis $|p\rangle$ significantly simplifies our calculations and leads to the integral representation

$$W[Y] = \frac{1}{2}\int d^4x \int \frac{d^4p}{(2\pi)^4}\int_0^\infty \frac{dt}{t^3}\rho(t,\Lambda^2)e^{-p^2}\mathrm{tr}\left(e^{-t(\mathcal{M}^2+A)}\right)\cdot 1. \tag{4}$$

Here $A = B - 2ip\partial/\sqrt{t}$, and the trace is calculated in flavor space.

3 Asymptotic Schwinger–DeWitt expansion

Before proceeding with our calculations this is the right place to say several words about the standard asymptotic Schwinger–DeWitt expansion. If the mass matrix $\mathcal{M}$ has the degenerate form, than $[\mathcal{M}, A] = 0$ and we find

$$\mathrm{tr}\left(e^{-t(\mathcal{M}^2+A)}\right) = e^{-tm^2}\mathrm{tr}\left(e^{-tA}\right) = e^{-tm^2}\mathrm{tr}\left(\sum_{n=0}^{\infty} t^n a_n\right). \tag{5}$$

Here a_n are the Seeley–DeWitt coefficients which depend on background fields and their derivatives. The integration over momentum and proper time in (4) can be readily done and we obtain the well known result

$$W[Y] = \int \frac{d^4x}{32\pi^2}\sum_{n=0}^{\infty} J_{n-1}\left(m^2\right)\,\mathrm{tr}\,(a_n), \tag{6}$$

where integrals $J_n(m^2)$ are given by

$$J_n(m^2) = \int_0^\infty \frac{dt}{t^{2-n}}e^{-tm^2}\rho\left(t,\Lambda^2\right). \tag{7}$$

Independently on the type of regularization the following property is fulfilled

$$J_n(m^2) = \left(-\frac{\partial}{\partial m^2}\right)^n J_0\left(m^2\right). \tag{8}$$

Choosing $\rho(t,\Lambda^2) = 1 - (1+t\Lambda^2)e^{-t\Lambda^2}$, which corresponds to two subtractions, one finds

$$J_0(m^2) = \Lambda^2 - m^2\ln\left(1+\frac{\Lambda^2}{m^2}\right). \tag{9}$$

We see that functions $J_n(m^2)$, starting from $n > 1$, have the asymptotic behavior $J_n(m^2) \sim m^{-2(n-1)}$ at large values of m^2, i.e. we obtain the inverse mass expansion. To warrant the convergence of the series it is necessary not only that the mass m be different from zero, but also that the background fields change slowly at distances of the order of the Compton wavelength $(1/m)$ of the fermion field. If these criteria are not fulfilled then the creation of real pairs gets essential and this expansion is not useful for calculations.

The remarkable property of the considered expansion is the gauge covariance of the Seeley–DeWitt coefficients.

4 Non-degenerate mass case

Let us return back to the formula (4) and show the way how to develop the above-mentioned tool for the case $[\mathcal{M}, A] \neq 0$. As a first step let us use the formula

$$\mathrm{tr}\left(e^{-t(\mathcal{M}^2+A)}\right) = \mathrm{tr}\left(e^{-t\mathcal{M}^2}\left[1+\sum_{n=1}^{\infty}(-1)^n f_n(t,A)\right]\right), \tag{10}$$

to factorize an exponent with the non commuting mass matrix $\mathcal{M}$. Here

$$f_n(t,A) = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n A(s_1)A(s_2)\cdots A(s_n), \tag{11}$$

where $A(s) = e^{s\mathcal{M}^2}Ae^{-s\mathcal{M}^2}$. It is true, that

$$\mathrm{tr}\left[e^{-t\mathcal{M}^2}f_n(t,A)\right] = \frac{t^n}{n!}\sum_{i_1,i_2,\ldots i_n}^{N_f} c_{i_1 i_2\ldots i_n}(t)\frac{1}{n}\sum_{perm}\mathrm{tr}\,(A_{i_1}A_{i_2}\ldots A_{i_n}), \tag{12}$$

where $A_i = E_i A$ by definition, a second summation means that n possible cyclic permutations of operators inside the trace must be done. The coefficients $c_{i_1 i_2\ldots i_n}(t)$ are totally symmetric with respect to permutations of indices and are calculated easily [11], for instance

$$\begin{aligned}
&c_i(t) = e^{-tm_i^2}, \qquad c_{ij}(t) = \frac{e^{-tm_i^2}-e^{-tm_j^2}}{t\Delta_{ji}},\\
&c_{ijk}(t) = \frac{2}{t^2}\left(\frac{e^{-tm_i^2}}{\Delta_{ji}\Delta_{ki}}+\frac{e^{-tm_j^2}}{\Delta_{kj}\Delta_{ij}}+\frac{e^{-tm_k^2}}{\Delta_{ik}\Delta_{jk}}\right),\\
&c_{ijkl} = \frac{3!}{t^3}\left(\frac{e^{-tm_i^2}}{\Delta_{li}\Delta_{ki}\Delta_{ji}}+\frac{e^{-tm_j^2}}{\Delta_{ij}\Delta_{lj}\Delta_{kj}}+\frac{e^{-tm_k^2}}{\Delta_{jk}\Delta_{ik}\Delta_{lk}}+\frac{e^{-tm_l^2}}{\Delta_{kl}\Delta_{jl}\Delta_{il}}\right).
\end{aligned} \tag{13}$$

Here $\Delta_{ij} \equiv m_i^2 - m_j^2$. In the case of coincidence of indices one can get that $c_i = c_{ii} = c_{ii\ldots i}$.

Therefore the heat kernel is represented as a sum, every term of which contains the coefficient $c_{ij\ldots}(t)$, multiplyed by the corresponding trace from the product of operators A_i. The following integration over t replaces the t-dependent part in these terms by the integrals $J_l(m_i^2)$ and we finally will face the following problem: every term of this expansion will not be invariant with respect to the transformations of the chiral group, although the total effective action $W[Y]$ will possess this property. One needs the algorithm which automatically groups the terms in chiral invariant blocks. This problem has been solved in the papers [11] on the basis of recurrence relations

$$J_l\left(m_j^2\right) - J_l\left(m_i^2\right) = \sum_{n=1}^{\infty}\frac{\Delta_{ij}^n}{2^n n!}\left[J_{l+n}\left(m_i^2\right) - (-1)^n J_{l+n}\left(m_j^2\right)\right]. \tag{14}$$

There it has been shown that the essence of the problem is confined to the correct choosing of the factorized combination, built from the functions $J_i\left(m_j^2\right)$, namely, the effective action $W[Y]$ must be represented in the form

$$W[Y] = \int\frac{d^4x}{32\pi^2}\sum_{i=0}^{\infty} I_{i-1}\,\mathrm{tr}\,(b_i), \qquad I_i \equiv \frac{1}{N_f}\sum_{j=1}^{N_f} J_i\left(m_j^2\right). \tag{15}$$

In this case the background fields are *automatically* combined in the covariant coefficients b_i. For example, for $N_f = 3$ the first four coefficients are

$$
\begin{aligned}
b_0 &= 1, \qquad b_1 = -Y, \qquad b_2 = \frac{Y^2}{2} + \frac{\Delta_{12}}{2}\lambda_3 Y + \frac{1}{2\sqrt{3}}\left(\Delta_{13} + \Delta_{23}\right)\lambda_8 Y, \\
b_3 &= -\frac{Y^3}{3!} - \frac{1}{12}(\partial Y)^2 - \frac{1}{12}\Delta_{12}\left(\Delta_{31} + \Delta_{32}\right)\lambda_3 Y \\
&\quad + \frac{1}{12\sqrt{3}}\left[\Delta_{13}(\Delta_{21} + \Delta_{23}) + \Delta_{23}(\Delta_{12} + \Delta_{13})\right]\lambda_8 Y \\
&\quad + \frac{1}{4\sqrt{3}}\left(\Delta_{31} + \Delta_{32}\right)\lambda_8 Y^2 + \frac{1}{4}\Delta_{21}\lambda_3 Y^2. \qquad (16)
\end{aligned}
$$

Several comments are in order here. First of all it is obvious that in the limiting case of equal masses $m_1 = m_2 = \cdots = m_{N_f}$ our result coincides with the standard Schwinger–DeWitt expansion. If the masses are not equal, the series (15) is a generalization of the well-known result (6). Instead of the asymptotic Seeley–DeWitt coefficients a_n come the coefficients b_n. Indeed one can check that if the operator $D^\dagger D$ transforms in the adjoint representation $\delta(D^\dagger D) = i[\omega, D^\dagger D]$, then b_n are also covariant, i.e. $\delta b_n = i[\omega, b_n]$, where $\omega = \alpha + \gamma_5\beta$ are the parameters of the global infinitesimal chiral transformations. It also should be mentioned that, rigorously speaking, the obtained expansion is not an exact inverse mass expansion. Although the integrals I_l for $l \geq 1$ possess the necessary form for asymptotic behavior $I_{l+1}(m_i^2) \sim m_i^{-2l}$, the coefficients b_l depend on the differences of masses, which may influence the character of the expansion. However general symmetry requirements are a more serious and stringent argument in favor of the obtained series as compared to the result of the thorough study [2] based only on the idea of a $1/m^2$ expansion.

5 Relation between coefficients b_n and a_n

The problem of the calculation of the generalized heat kernel coefficients is a more complicated mathematical problem than the calculation of the standard Seeley–DeWitt coefficients. However one can significantly simplify this problem [13], if from the very beginning one uses the transformation properties of the coefficients b_n. Indeed, let us return back to the formula (5) and extend it to the case $[\mathcal{M}, Y] \neq 0$, omitting for simplicity all terms with derivatives in A. It is remarkable that already at this stage one can take into account the two main conclusions which we have found in the previous section: the form of the factorized part depending on the mass and gauge covariance of coefficients b_n. The first aim can be reached through the definition

$$
\langle A \rangle \equiv \frac{\mathrm{tr}\,(A)}{\mathrm{tr}\,(1)}, \qquad \bar{\mathcal{M}}^2 = \mathcal{M}^2 - \langle \mathcal{M}^2 \rangle, \qquad (17)
$$

the second one through putting

$$
\left(\bar{\mathcal{M}}^2 + Y\right)^n \equiv \sum_{k=0}^{n} (-1)^k C_n^k \langle \left(\bar{\mathcal{M}}^2\right)^{n-k} \rangle b_k. \qquad (18)
$$

Since the left side of equation (18) transforms covariantly, i.e. $\left(\bar{\mathcal{M}}^2 + Y\right)^\Omega = \Omega^{-1}\left(\bar{\mathcal{M}}^2 + Y\right)\Omega$, and on the right side the term $\langle \bar{\mathcal{M}}^2 \rangle$ is invariant with respect to the chiral transformations Ω, it is obvious, that $b_k(Y^\Omega, \bar{\mathcal{M}}^{2\Omega}) = \Omega^{-1} b_k\left(Y, \bar{\mathcal{M}}^2\right)\Omega$. To see how these definitions work let us

consider the exponent in the heat kernel (5)

$$e^{-t(\mathcal{M}^2+Y)} = e^{-t\langle\mathcal{M}^2\rangle}\sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\big(\bar{\mathcal{M}}^2+Y\big)^n$$
$$= e^{-t\langle\mathcal{M}^2\rangle}\langle e^{-t\bar{\mathcal{M}}^2}\rangle\sum_{n=0}^{\infty}\frac{t^n}{n!}b_n = \langle e^{-t\mathcal{M}^2}\rangle\sum_{n=0}^{\infty}\frac{t^n}{n!}b_n. \tag{19}$$

Since

$$\langle e^{-t\mathcal{M}^2}\rangle = \sum_{i=1}^{N_f} e^{-tm_i^2}\langle E_i\rangle = \frac{1}{N_f}\sum_{i=1}^{N_f} e^{-tm_i^2}, \tag{20}$$

the following integration over t leads us to the known result (15), and the coefficients b_n can be found using the definition (18). The same definition allows us to relate b_n with the linear combination of the standard Seeley–DeWitt coefficients $a_i(Y)$, where the replacement $Y \to Y+\bar{\mathcal{M}}^2$ should be done [13]

$$b_n = \sum_{i=0}^{n}\frac{\beta_{n-i}}{(n-i)!}a_i\big(Y\to Y+\bar{\mathcal{M}}^2\big). \tag{21}$$

The parameters β_i depend only on the masses of fermion fields. To establish the form of this dependence it is sufficient to calculate b_n in the simplest case, when in the elliptic operator under consideration all terms with derivatives are omitted. This problem is much simpler to solve than the calculation of the coefficients b_n from the very beginning.

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Variational Multivectors and Brackets in the Geometry of Jet Spaces

Sergei IGONIN [†], *Alexander VERBOVETSKY* [‡] *and Raffaele VITOLO* [§]

[†] *Department of Mathematics, Utrecht University,*
P.O. Box 80010, 3508 TA Utrecht, The Netherlands
E-mail: *igonin@mccme.ru*

[‡] *Independent University of Moscow, 11 B. Vlasevsky Str., 119002 Moscow, Russia*
E-mail: *verbovet@mccme.ru*

[§] *Department of Mathematics, University of Lecce, Via per Arnesano, 73100 Lecce, Italy*
E-mail: *Raffaele.Vitolo@unile.it*

This is a summary of our recent results to appear in [5]. In the framework of jet superspace geometry, we introduce variational multivectors and the variational Poisson and Schouten brackets, which are dual to the variational complex. Their relations with the antibracket in field theory and applications to finding of Hamiltonian structures for evolution equations are outlined. We show also that local variational differential operators of B.L. Voronov, I.V. Tyutin, and Sh.S. Shakhverdiev are determined by variational multivectors and are well-defined on the algebra of multilocal functionals. To achieve the latter result, we show that locally a finite number of smooth action functionals are either linearly dependent or 'almost' functionally independent in a certain sense.

Introduction

In the geometric theory of jet spaces the so-called *variational complex* [12, 13] is of central importance (see, for example, [2, 7] for an introductory treatment). The variational complex is obtained from the de Rham sequence on jet manifolds by a quotient with respect to 'variationally trivial' forms. This complex contains the Euler operator as one of its differentials.

By analogy with ordinary manifolds, where symmetric and skew-symmetric products of vector fields and their Poisson and Schouten brackets are introduced as dual counterparts of forms and the de Rham differential, it is interesting to consider a dual theory to the variational complex and its differential: *variational multivectors* and their brackets, the *variational Poisson and Schouten brackets*.

In this note, we summarize our recent results on this topic; full details can be found in [5].

In Section 1, we set up notation and terminology related to graded vector spaces, graded algebras, and differential operators over such algebras.

Then, we introduce a graded Lie algebra structure on a space of multilinear maps on a vector space. This structure generalizes the Nijenhuis–Richardson and Schouten brackets. By interpreting multilinear maps as differential operators on polynomial functions, we provide an algebraic counterpart for a class of operators recently introduced in [14], namely the *local variational differential operators*.

In Section 2, we specialize our algebraic setting to a geometric situation. Namely, we consider differential operators on *local functionals* (or *actions*). Local functionals are elements of a cohomology group of the *horizontal de Rham complex*. This complex consists of differential forms on the base manifold with coefficients in functions of dependent variables (fields) and a finite number of their derivatives. In other words, local functionals are Lagrangian densities modulo

total divergences. The distinguished expressions of variational multivectors and their bracket are given in Theorems 2, 3.

In Section 3, we consider the above general formalism in different contexts.

First, we observe that the variational Poisson bracket coincides with the Poisson brackets constructed by Kupershmidt [8] on the so-called 'cotangent bundle to a bundle' when restricted to the subspace of polynomial functions.

Second, we consider the skew-symmetric variational multivectors and the *variational Schouten bracket*. The coordinate version of this bracket is well-known under the name *antibracket* (see, e.g., [4] and references therein), so that we obtain here a geometric setting for the antibracket formalism.

This bracket is also of great importance in the Hamiltonian theory of integrable evolution equations, since skew-symmetric variational bivectors whose variational Schouten bracket with itself vanishes are just Hamiltonian operators (see, e.g., [3] and references therein).

As an application of our approach to variational multivectors we discuss the *calculus of local variational differential operators* by B.L. Voronov, I.V. Tyutin, and Sh.S. Shakhverdiev [14]. In the study of the so-called $\delta(0)$-problem in field theory, the authors discovered fundamentally new operators that act on smooth functions $F(S_1,\dots,S_N)$ of local functionals S_i (*multilocal functionals*). We prove that these operators are well-defined, i.e., if $F(S_1,\dots,S_N)$ is an identically zero functional then $\nabla(F(S_1,\dots,S_N))$ is also zero. The crucial part of the proof is to show that all relations in the algebra of multilocal functionals are linear, roughly speaking. This fact looks interesting by itself.

1 The Lie algebra of multilinear mappings: an algebraic model

Graded spaces and graded algebras. Let G be an Abelian group. A vector space V is called *G-graded* if $V=\bigoplus_{g\in G}V_g$ for some vector spaces V_g. *Superspaces* are $\mathbb{Z}_2$-graded vector spaces. The notion of graded space can be carried out in a natural way to subspaces, tensor products, etc. of graded spaces.

A $\Bbbk$-algebra A is called *G-graded* if A is a G-graded vector space and $A_{g_1}A_{g_2}\subset A_{g_1+g_2}$, for all $g_1,\ g_2\in G$.

A *commutation factor* is a pairing $G\times G\to\Bbbk\setminus\{0\}$, $(g_1,g_2)\mapsto\{g_1,g_2\}$, such that

$$\{g_1,g_2\}^{-1}=\{g_2,g_1\},\qquad \{g_1+g_2,g_3\}=\{g_1,g_3\}\{g_2,g_3\}.$$

When $G=\mathbb{Z}$ or $\mathbb{Z}_2$, there is only one nontrivial commutation factor, namely, the *super-commutation factor* $\{g_1,g_2\}=(-1)^{g_1g_2}$. In formulas commutation factors are used according to the following 'generalized rule of signs': whenever an object (i.e., an element of a G-graded vector space) of degree g_1 is interchanged with an object of degree g_2, the multiplier $\{g_1,g_2\}$ is introduced.

For a fixed element $g=(g_1,\dots,g_n)\in G^n=G\oplus\dots\oplus G$ there is a unique function $\epsilon_g\colon S_n\to\Bbbk\setminus\{0\}$ on the permutation group S_n such that

1. $\epsilon_g(\sigma_i)=\{g_i,g_{i+1}\}$ for a transposition $\sigma_i=(i,i+1)$;
2. $\epsilon_g(\sigma'\circ\sigma'')=\epsilon_{\sigma''(g)}(\sigma')\epsilon_g(\sigma'')$, where $\sigma(g)=(g_{\sigma(1)},\dots,g_{\sigma(n)})$.

For the super-commutation factor the function $\epsilon_{(1,\dots,1)}$ coincides with the standard sign of permutations. We shall denote it by ϵ.

For our convenience, if $v\in V_{g_1}$ and $w\in W_{g_2}$ are two homogeneous elements of two G-graded vector spaces V and W, then we write $\{v,w\}$ rather than $\{g_1,g_2\}$.

A G-graded algebra A is called *commutative* if for all $a,\ b\in A$ we have $ab=\{a,b\}ba$.

A left module M over a G-graded commutative algebra A is called *G-graded* if $M = \bigoplus_{g\in G} M_g$ and $A_{g_1} M_{g_2} \subset M_{g_1+g_2}$.

A *G-graded Lie algebra* is a G-graded algebra A such that the multiplication in A, denoted by $[\cdot\,,\cdot]\colon A \otimes_{\Bbbk} A \to A$, satisfies the properties:

$$[a,b] = -\{a,b\}[b,a], \qquad [[a,b],c] = [a,[b,c]] + \{b,c\}[[a,c],b],$$

for all $a, b, c \in A$. For example, if V is a G-graded vector space, then $\mathrm{Hom}_{\Bbbk}(V,V)$, equipped with the commutator, is a G-graded Lie algebra.

A *Lie superalgebra* is $\mathbb{Z}_2$-graded Lie algebra with respect to the super-commutation factor.

Let A be an associative commutative G-graded algebra with unity. A $\Bbbk$-homomorphism $\Delta \in \mathrm{Hom}_{\Bbbk}(A,A)$ is called a *scalar G-graded differential operator* of order k, if for all $a_0, \ldots, a_k \in A$ we have

$$[a_0, [a_1, \ldots, [a_k, \Delta] \ldots]] = 0.$$

In this equality a_i are the operators of left multiplication. Denote by $\mathrm{Diff}_k(A)$ the set of all scalar differential operators of order k. It is clear that $\mathrm{Diff}_0(A) = A$ and $\mathrm{Diff}_k(A) \subset \mathrm{Diff}_l(A)$ for $k \le l$. If $\Delta_1 \in \mathrm{Diff}_k(A)$ and $\Delta_2 \in \mathrm{Diff}_l(A)$, then it is easily seen that $\Delta_1 \circ \Delta_2 \in \mathrm{Diff}_{k+l}(A)$ and $[\Delta_1, \Delta_2] \in \mathrm{Diff}_{k+l-1}(A)$.

The Lie algebra of multilinear mappings. Let V be a G-graded vector space. Let $\{\mathcal{M}_k(V)\}_{k\in\mathbb{N}}$ be a family of graded subspaces of the space of k-linear maps of V with values in V with the following properties:

1. $\mathcal{M}_0(V) = V$;
2. for all $f \in \mathcal{M}_k(V)$, $g \in \mathcal{M}_l(V)$, and $1 \le i \le k$ the $(k+l-1)$-linear map h defined by

$$h(v_1, \ldots, v_{k+l-1}) = f(v_1, \ldots, v_{i-1}, g(v_i, \ldots, v_{i+l-1}), v_{i+l}, \ldots, v_{k+l-1})$$

belongs to $\mathcal{M}_{k+l-1}(V)$.

The space $\mathcal{M}_1(V)$ is a graded Lie algebra with respect to the commutator. Let $\mathfrak{g}$ be a graded Lie subalgebra of $\mathcal{M}_1(V)$. For each $k \ge 1$ define G-graded vector space $\mathfrak{g}^{(k)}$ to be the graded subspace of $\mathcal{M}_k(V)$ such that $f \in \mathfrak{g}^{(k)}$ if

1. f is (graded) symmetric, i.e., $f(v_1, \ldots, v_k) = \epsilon_v(\sigma) f(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$ for all $\sigma \in S_k$;
2. for all $v_1, \ldots, v_{k-1}$ the maps $v \mapsto f(v_1, \ldots, v_{i-1}, v, v_i, \ldots, v_{k-1})$ belong to $\mathfrak{g}$.

Obviously, $\mathfrak{g}^{(1)} = \mathfrak{g}$. By definition, we put $\mathfrak{g}^{(0)} = V$ and $\mathfrak{g}^{(k)} = 0$ for $k < 0$.

Let us introduce the following notation: if f is a k-linear map as above and $v_i \in V$, then $f(v_1, \ldots, v_l)$ for $l \le k$ will stand for the $(k-l)$-linear map obtained by contracting the first l arguments of f with v_i. Then, we can prove the following theorem [5].

Theorem 1. *On the space $\mathfrak{g}^{(*)} = \bigoplus_k \mathfrak{g}^{(k)}$ there exists a unique G-graded Lie algebra structure $[\![\cdot\,,\cdot]\!]$ such that*

1. $[\![\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}]\!] \subset \mathfrak{g}^{(k+l-1)}$;
2. $[\![f, v]\!] = f(v)$ *for* $v \in \mathfrak{g}^{(0)} = V$, $f \in \mathfrak{g}^{(k)}$.

Using the induction, we get the following explicit formula for this bracket:

$$[\![f_1, f_2]\!](v_1, \ldots, v_{k+l-1}) = \sum_{\sigma \in S^l_{k+l-1}} \epsilon_v(\sigma) f_1(f_2(v_{\sigma(1)}, \ldots, v_{\sigma(l)}), v_{\sigma(l+1)}, \ldots, v_{\sigma(k+l-1)})$$

$$- \{f_1, f_2\} \sum_{\sigma \in S^k_{k+l-1}} \epsilon_v(\sigma) f_2(f_1(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l-1)}),$$

where $S^i_n \subset S_n$ is the set of all $(i, n-i)$-*unshuffles*[1], that is, all permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$.

It can be recognized that, by choosing suitable graded vector spaces and algebras, one recovers the *Nijenhuis–Richardson bracket*, the *Schouten bracket* and the *symmetric Schouten concomitant* (see [5] for details and references).

Consider the ideal I in the tensor algebra $T(V)$ generated by $(v \otimes w - \{v, w\} w \otimes v) \in V^{\otimes 2}$ for v, $w \in V$. The algebra $S(V) = T(V)/I$ is a G-graded associative commutative algebra with unity called the *symmetric algebra* of V.

Now, to multilinear maps $f \in \mathfrak{g}^{(k)}$ we assign differential operators on the algebra $S(V)$. Namely, for $k = 0$ we define ∇_f to be the multiplication by $f \in V$. For $k > 0$ it can be proved [5] that there exists a unique differential operator $\nabla_f \in \mathrm{Diff}_k(S(V))$ such that $\nabla_f|_{S^l(V)} = 0$, for $0 \leq l < k$, and $\nabla_f(v_1 \cdots v_k) = f(v_1, \ldots, v_k)$. The operator ∇_f fulfills the expression $\nabla_f(v_1 \cdots v_n) = \sum_{\sigma \in S^k_n} \epsilon_v(\sigma) f(v_{\sigma(1)} \cdots v_{\sigma(k)}) v_{\sigma(k+1)} \cdots v_{\sigma(n)}$ for $n > k$. As one can expect, we have $[\nabla_{f_1}, \nabla_{f_2}] = \nabla_{[\![f_1, f_2]\!]}$ for $f_1 \in \mathfrak{g}^{(k)}$ and $f_2 \in \mathfrak{g}^{(l)}$.

Now we extend the operators ∇_f to a bigger algebra. For this purpose, let us represent the graded space V in the form $V = V_0 \oplus V_+$, where $V_+ = \bigoplus_{g \in G \setminus \{0\}} V_g$. We have $S(V) = S(V_0) \otimes_{\Bbbk} S(V_+)$. The algebra $S(V_0)$ can be identified with an algebra of (polynomial) functions on the dual space V_0^*. Let us extend this algebra to the algebra $\mathcal{A}_0$ of functions on V_0^* of the form $F(v_1, \ldots, v_N)$, where $F \in C^\infty(\mathbb{R}^N)$ is a smooth function in many arguments and $v_i \in V_0$. Assign to all elements of $\mathcal{A}_0$ the degree 0. Denote by $\mathcal{A}(V)$ the graded algebra $\mathcal{A}_0 \otimes_{\Bbbk} S(V_+)$. Then, it is not difficult to prove that, for each $f \in \mathfrak{g}^{(k)}$, the operator ∇_f has the unique extension

$$\nabla_f(F(v_1, \ldots, v_N) \otimes v_+) = \sum_{\substack{0 \leq l \leq k \\ \sigma \in S^l_N}} \frac{\partial^l F}{\partial t_{\sigma(1)} \cdots \partial t_{\sigma(l)}}(v_1, \ldots, v_N) \otimes \nabla_{f(v_{\sigma(1)}, \ldots, v_{\sigma(l)})}(v_+),$$

to the algebra $\mathcal{A}(V)$.

2 Variational Poisson bracket

The jet bundle setting: preliminaries. In this paper we deal with the infinite jet bundles of vector *superbundles*. Jets of purely even bundles have been detailed extensively in the literature (see, e.g., [2,7]), so we will shortly describe the graded setting here. We assume that all manifolds and maps are C^∞.

We say that a vector bundle $\pi\colon E \to M$ is a *superbundle* if it is the direct sum $\pi = \pi^0 \oplus \pi^1$ of two vector bundles $\pi^0\colon E_0 \to M$ and $\pi^1\colon E_1 \to M$.

Recall that each vector bundle α over M determines a supermanifold as follows [1,10,11]. The underlying even manifold is M, and the structure sheaf is the sheaf of sections of the exterior algebra of α^*.

Denote by $\alpha_{k,l}\colon J^k(\alpha) \to J^l(\alpha)$ and $\alpha_k\colon J^k(\alpha) \to M$ the ordinary projections of jet spaces.

Consider the pullback $(\pi^0_k)^*(\pi^1_k)\colon J^k(\pi^0) \times_M J^k(\pi^1) \to J^k(\pi^0)$ of the bundle $\pi^1_k\colon J^k(\pi^1) \to M$ along the projection $\pi^0_k\colon J^k(\pi^0) \to M$. We define the *k-jet space* of π to be the supermanifold that corresponds to the bundle $(\pi^0_k)^*(\pi^1_k)$. It will be denoted by $J^k(\pi)$. In particular, the underlying even manifold of $J^k(\pi)$ is $J^k(\pi^0)$. The natural projections $\pi_k\colon J^k(\pi) \to M$ and $\pi_{k,l}\colon J^k(\pi) \to J^l(\pi)$ for $k > l$ yield a chain whose inverse limit is said to be the *infinite jet space* and is denoted by $J^\infty(\pi)$.

[1]The term *unshuffle* is borrowed from [9] and means separating an ordered set into two subsets, the order within each subset being as in the original set.

Let $x^1,\dots,x^n$ be local coordinates in M, $u^1,\dots,u^{m_0}$ and $u^{m_0+1},\dots,u^m$ be local fiber coordinates in E_0 and E_1 respectively. Then u^j_σ are local derivative coordinates on $J^k(\pi)$. The coordinates x^i and u^j_σ for $j\le m_0$ are even, while u^j_σ for $j>m_0$ are odd.

The superalgebra $\mathcal{F}(\pi)$ of smooth functions on $J^\infty(\pi)$ is defined as the direct limit of the chain of injections $\pi^*_{k,k-1}$. The $\mathcal{F}(\pi)$-module $\Lambda^*(\pi)$ of differential forms on $J^\infty(\pi)$ is defined in the similar way. Below we omit the letter π and write simply $\mathcal{F}$, Λ^k, and so on, when no confusion can arise.

A *horizontal module* is the $\mathcal{F}$-module of sections of $\pi^*_\infty(\alpha)$, where α is a superbundle over M. Denote by $\varkappa$ the horizontal module corresponding to the bundle π itself.

Let P_1 and P_2 be horizontal modules of even vector bundles. A linear differential operator $\Delta\colon P_1\to P_2$ is called *$\mathcal{C}$-differential* if it can be restricted to the graphs of all infinitely prolonged sections of the bundle. The set of all $\mathcal{C}$-differential operators from P_1 to P_2 is denoted by $\mathcal{C}\mathrm{Diff}(P_1,P_2)$. This definition can be generalized to the case of superbundles [5]. In coordinates, $\mathcal{C}$-differential operators have the form of a matrix $(a^\sigma_{ij}D_\sigma)$, where $a^\sigma_{ij}\in\mathcal{F}$, $D_\sigma=D_{i_1}\circ\dots\circ D_{i_r}$ for $\sigma=i_1\dots i_r$ and D_i is the total derivative operator with respect to x^i.

A π_∞-vertical vector field on $J^\infty(\pi)$ is called an *evolutionary field* if it commutes with all D_i (this property does not depend on the choice of coordinates). In coordinates, each evolutionary field is of the form $E_\varphi=\sum_{j,\sigma}D_\sigma(\varphi^j)\partial/\partial u^j_\sigma$, where $\varphi^j\in\mathcal{F}$.

Let P be a horizontal module. For each element $p\in P$ there is a $\mathcal{C}$-differential operator $\ell_p\colon\varkappa\to P$ called the *universal linearization* of p and defined by $\ell_p(\varphi)=(-1)^{p\varphi}E_\varphi(p)$, with $\varphi\in\varkappa$. Here and subsequently, symbols used as the exponents of (-1) stand for the corresponding parities.

A differential form $\omega\in\Lambda^k$ on $J^\infty(\pi)$ is called a *Cartan form* if its pull-back through any prolonged section vanishes. In coordinates, Cartan forms contain factors of the type $\omega^j_\sigma=du^j_\sigma-\sum_i u^j_{\sigma i}\,dx^i$. Denote the module of all Cartan q-forms by $\mathcal{C}\Lambda^q$. Then $d(\mathcal{C}\Lambda^q)\subset\mathcal{C}\Lambda^{q+1}$. Therefore the quotient $\bar d$ of d, acting on $\bar\Lambda^*=\Lambda^*/\mathcal{C}\Lambda^*$, is well defined. Elements of $\bar\Lambda^q$ are called *horizontal forms*. In coordinates, $\bar\Lambda^q$ is generated by $f\,dx^{i_1}\wedge\dots\wedge dx^{i_q}$, where $f\in\mathcal{F}$, and $\bar d(f\,dx^{i_1}\wedge\dots\wedge dx^{i_q})=\sum\limits_{i=1}^n D_i(f)\,dx^i\wedge dx^{i_1}\wedge\dots\wedge dx^{i_q}$. The cohomology $\bar H^i(\pi)$ of the complex $(\bar\Lambda^*,\bar d)$ coincides with the de Rham cohomology of M for all degrees i up to $n-1$ [12, 13].

If P is an $\mathcal{F}$-module, we write $\hat P=\mathrm{Hom}_{\mathcal{F}}(P,\bar\Lambda^n)$ and consider the natural pairing $\langle\cdot\,,\cdot\rangle\colon\hat P\times P\to\bar\Lambda^n$. We recall that for each operator $\Delta\in\mathcal{C}\mathrm{Diff}(P,Q)$ there exists a unique operator $\Delta^*\in\mathcal{C}\mathrm{Diff}(\hat Q,\hat P)$ such that

$$[\langle\hat q,\Delta(p)\rangle]=(-1)^{\Delta\hat q}[\langle\Delta^*(\hat q),p\rangle],\qquad \hat q\in\hat Q,\quad p\in P,$$

where $[\omega]$ denotes the horizontal cohomology class of $\omega\in\bar\Lambda^n$. The operator Δ^* is called *adjoint* to Δ. In coordinates, $(\sum_\sigma a^\sigma_{ij}D_\sigma)^*=(\sum_\sigma(-1)^{|\sigma|}D_\sigma\circ a^\sigma_{ij})^{\mathrm{st}}$, where $a^\sigma_{ij}\in\mathcal{F}$ and the symbol 'st' denotes the supertransposition.

Since evolutionary fields commute with $\bar d$, the cohomology class $[E_\varphi(\omega)]$ for $\omega\in\bar\Lambda^n$ is well defined; denote it by $E_\varphi([\omega])$. We have $E_\varphi([\omega])=[\langle\varphi,\mathcal{E}(\omega)\rangle]$, where $\mathcal{E}(\omega)=\ell^*_\omega(1)$ is the *Euler operator*. In coordinates, $\mathcal{E}(L\,dx^1\wedge\dots\wedge dx^n)=(\delta L/\delta u^j)$, where $\delta L/\delta u^j=\sum_\sigma(-1)^{|\sigma|}D_\sigma(\partial L/\partial u^j_\sigma)$.

Variational Poisson bracket. In this section we use the general construction of Section 1 to introduce a (super)symmetric bracket on multilinear maps of the form $\bar H^n\times\dots\times\bar H^n\to\bar H^n$. Then, we provide a description of maps and the bracket in terms of selfadjoint operators. To this aim, we take $G=\mathbb{Z}_2$ and $V=\bar H^n$. The space $\mathcal{M}_k(\bar H^n)$ is defined to be the set of multilinear maps f of the form

$$f([\omega_1],\dots,[\omega_k])=[\tilde f(\omega_1,\dots,\omega_k)],$$

where $\tilde f\colon\bar\Lambda^n\times\dots\times\bar\Lambda^n\to\bar\Lambda^n$ is a multilinear differential operator. It is easily shown that we can take $\varkappa$ for the algebra $\mathfrak{g}$. The spaces $\mathfrak{g}^{(k)}$, thus constructed, are denoted by $\varkappa^{(k)}$. Elements

of $\varkappa^{(k)}$ will be referred to as *variational (super)symmetric multivectors*. The bracket $[\![\cdot\,,\cdot]\!]$ on $\varkappa^{(k)}$ defined in Section 1 is called the *variational Poisson bracket*.

Let us denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{self}}_{(k)}(P,\hat{P})$ the module of k-linear $\mathcal{C}$-differential operators $\Delta\colon P\times\dots\times P\to\hat{P}$ which are (graded) symmetric and self-adjoint in each argument.

Theorem 2 ([5]). *For each $f\in\varkappa^{(k)}$ there exists a unique $\mathcal{C}$-differential operator $\Delta_f\in\mathcal{C}\mathrm{Diff}^{\mathrm{self}}_{(k-1)}(\hat{\varkappa},\varkappa)$ such that*

$$f(\omega_1,\dots,\omega_k)=[\langle\Delta_f(\mathcal{E}(\omega_1),\dots,\mathcal{E}(\omega_{k-1})),\mathcal{E}(\omega_k)\rangle].$$

From now on, we identify variational multivectors with the corresponding $\mathcal{C}$-differential operators.

Theorem 3 ([5]). *If $\Delta_1\in\varkappa^{(k)}$ and $\Delta_2\in\varkappa^{(l)}$ then for all $\psi_1,\dots,\psi_{k+l-2}\in\hat{\varkappa}$ we have*

$$\begin{aligned}
&[\![\Delta_1,\Delta_2]\!](\psi_1,\dots,\psi_{k+l-2})\\
&\quad=(-1)^{\Delta_1\Delta_2}\Big(\sum_{\sigma\in S^{l-1}_{k+l-2}}\epsilon_\psi(\sigma)(-1)^{\Delta_1\psi_{\sigma(1,l-1)}}\ell_{\Delta_2,\psi_{\sigma(1,l-1)}}(\Delta_1(\psi_{\sigma(l,k+l-2)}))\\
&\quad-\sum_{\sigma\in S^{k}_{k+l-2}}\epsilon_\psi(\sigma)\Delta_2(\ell^*_{\Delta_1,\psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}),\psi_{\sigma(k+1,k+l-2)})\Big)\\
&\quad-\sum_{\sigma\in S^{k-1}_{k+l-2}}\epsilon_\psi(\sigma)(-1)^{\Delta_2\psi_{\sigma(1,k-1)}}\ell_{\Delta_1,\psi_{\sigma(1,k-1)}}(\Delta_2(\psi_{\sigma(k,k+l-2)}))\\
&\quad+\sum_{\sigma\in S^{l}_{k+l-2}}\epsilon_\psi(\sigma)\Delta_1(\ell^*_{\Delta_2,\psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}),\psi_{\sigma(l+1,k+l-2)}),
\end{aligned}\tag{1}$$

where $\ell_{\Delta,\psi_1,\dots,\psi_s}(\varphi)=(-1)^{(\psi_1+\dots+\psi_s)\varphi}\ell_\Delta(\varphi)(\psi_1,\dots,\psi_s)$, for $\varphi\in\varkappa$, and $\psi_{\sigma(k_1,k_2)}$ is the notation for the vector $\psi_{\sigma(k_1)},\dots,\psi_{\sigma(k_2)}$. When $\psi_{\sigma(k_1,k_2)}$ is used as the exponent of (-1) it means the sum of degrees of $\psi_{\sigma(k_1)},\dots,\psi_{\sigma(k_2)}$.

3 Examples and applications

Cotangent bundle to a bundle and the variational Poisson bracket. Here we compare our variational Poisson brackets with the Poisson brackets constructed by Kupershmidt [8]. Let us consider the bundles $\hat{\pi}\colon\hat{E}=E^*\otimes_M\bigwedge^n(T^*M)\to M$ and $\Pi=\pi\oplus\hat{\pi}$. Following Kupershmidt [8] we call the bundle $\Pi_\infty\colon J^\infty(\Pi)\to M$ the *cotangent bundle* to the bundle π.

Let us denote by p^j, $j=1,\dots,m$, the fiber coordinates in $\hat{E}$ dual to u^j with respect to a volume form on M. Then coordinates in $J^\infty(\Pi)$ are x^i, u^j_σ, p^j_σ.

We see that $\varkappa(\Pi)=\varkappa_\Pi\oplus\hat{\varkappa}_\Pi$, where $\varkappa_\Pi=\Gamma(\Pi^*(\pi))$. On the space $\bar{H}^n(\Pi)$ there exists a natural *Poisson bracket* [8] $[\![F,H]\!]=[\langle\mathcal{E}(F),A(\mathcal{E}(H))\rangle]$, for F, $H\in\bar{H}^n(\Pi)$, where $A\colon\hat{\varkappa}(\Pi)\to\varkappa(\Pi)$, $A(\psi,\varphi)=(-\varphi,\psi)$ for $\varphi\in\varkappa_\Pi$ and $\psi\in\hat{\varkappa}_\Pi$. In coordinates we have

$$[\![F,H]\!]=\sum_j\left((-1)^{p^j(F+1)}\frac{\delta F}{\delta p^j}\frac{\delta H}{\delta u^j}-(-1)^{u^jF}\frac{\delta F}{\delta u^j}\frac{\delta H}{\delta p^j}\right).$$

Now, since by definition elements of $\mathcal{F}(\hat{\pi})$ are identified with differential operators from $\Gamma(\hat{\pi})$ to $C^\infty(M)$, we have the natural inclusion $\mathcal{C}\mathrm{Diff}(\hat{\varkappa},\mathcal{F})\to\mathcal{F}(\Pi)$, which uniquely prolongs to the inclusion of algebras $\mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(*)}(\hat{\varkappa},\mathcal{F})\to\mathcal{F}(\Pi)$. This leads to the further inclusions $\mathcal{C}\mathrm{Diff}^{\mathrm{sym}}_{(*)}(\hat{\varkappa},\bar{\Lambda}^n)\to\bar{\Lambda}^n(\Pi)$ and $\varkappa^{(*)}\to\bar{H}^n(\Pi)$, and to the following theorem (see [5] for details and proofs).

Theorem 4 ([5]). *The above Poisson bracket extends the variational Poisson bracket to* $\bar{H}^n(\Pi)$.

Variational Schouten bracket (antibracket) and Hamiltonian formalism. Using an appropriate commutation factor, one easily obtains odd counterparts of the constructions from the previous subsection: super skew-symmetric multivectors and the variational Schouten bracket. This bracket in coordinates coincides with the *antibracket* [4], thus, our results provide a geometrical description of the antibracket.

Let us denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k)}(\varkappa, \hat{\varkappa})$ the module of k-linear $\mathcal{C}$-differential operators $\Delta\colon \varkappa \times \cdots \times \varkappa \to \hat{\varkappa}$ which are skew-symmetric and skew-adjoint in each argument. By analogy with Theorem 2 it can be proved that each super skew-symmetric multivector can be represented through a unique $\mathcal{C}$-differential operator $\Delta \in \mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k-1)}(\hat{\varkappa}, \varkappa)$. Of course, formula (1) holds for the variational Schouten bracket too, *mutatis mutandis*.

The variational Schouten bracket is useful in the Hamiltonian theory of evolution equations. Let us consider a purely even bundle π for the moment. An operator $A \in \mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(1)}(\hat{\varkappa}, \varkappa)$ (variational bivector) is called *Hamiltonian* if $[\![A, A]\!] = 0$. A Hamiltonian operator defines a Lie algebra structure on $\bar{H}^n(\pi)$. An evolution equation $u_t = f$, $f \in \varkappa$ is said to be *Hamiltonian* with respect to A if $A_t - [\![A, f]\!] = 0$.

It can be shown that an evolution equation is Hamiltonian if and only if $\ell \circ A + A \circ \ell^* = 0$, where ℓ is the linearization of the equation at hand. Combining this with other geometric considerations, one obtains an efficient method to find Hamiltonian structures for a given equation [6].

Local variational differential operators. We want to identify the space $\bar{H}^n$ with a subspace of functionals on $\Gamma(\pi)$. Consider first the case of a purely even vector bundle $\pi\colon E \to M$.

A *domain* is an oriented open subset U of M with compact closure $\bar{U}$ and smooth boundary ∂U. Let $U_1, U_2 \subset M$ be two domains such that $B = \partial U_1 = \partial U_2$ and the orientations of U_1, U_2 induce opposite orientations on B. Let $s_i \in \Gamma(\pi, \bar{U}_i)$, $i = 1, 2$, $[s_1]^\infty_x = [s_2]^\infty_x$ for all $x \in B$. Here and below $[s_i]^\infty_x$ is the infinite jet of s_i at x. A horizontal cohomology class $[\omega] \in \bar{H}^n$ gives a well-defined functional on such 4-tuples (U_1, U_2, s_1, s_2) as follows

$$\omega(U_1, U_2, s_1, s_2) = \int_{\bar{U}_1} j_\infty(s_1)^*(\omega) + \int_{\bar{U}_2} j_\infty(s_2)^*(\omega),$$

where $j_\infty(s_i)$ is the section $x \mapsto [s_i]^\infty_x$ of π_∞. It is easily seen that each nonzero class determines a nonzero functional.

Following [14] we call functionals of the form $F(\omega_1, \ldots, \omega_N)$, where F is a smooth function in many arguments, *multilocal functionals*. Denote by $\mathcal{A} = \mathcal{A}(\pi)$ the algebra of multilocal functionals. A natural question arises: what are the relations in $\mathcal{A}$, i.e., for what nonzero smooth functions F and horizontal forms ω_i the expression $F(\omega_1, \ldots, \omega_N)$ induces an identically zero functional? It turns out that at least locally (in a certain sense) all relations in $\mathcal{A}$ are generated by linear relations in $\bar{H}^n$.

Let $U \subset M$ be a domain. For a finite subset $V \subset U$ and $s_0 \in \Gamma(\pi, \bar{U})$ put $\Gamma_{s_0,V} = \{s \in \Gamma(\pi, \bar{U}) \,|\, [s]^\infty_x = [s_0]^\infty_x \ \forall\, x \in V \cup \partial U\}$. Such subsets of $\Gamma(\pi, \bar{U})$ are endowed with the C^∞-topology.

Theorem 5 ([5]). *Let* $s_0 \in \Gamma(\pi, \bar{U})$, V *be a finite subset of* U, *and* $\omega_1, \ldots, \omega_l \in \bar{\Lambda}^n$. *Consider the linear map* $\Psi\colon \Gamma(\pi) \to \mathbb{R}^l$, $\Psi(s) = \bigl(\int_{\bar{U}} j_\infty(s)^*(\omega_1), \ldots, \int_{\bar{U}} j_\infty(s)^*(\omega_l)\bigr)$. *Suppose that there exists a neighborhood* $\Gamma \subset \Gamma_{s_0,V}$ *of* s_0 *such that* $\Psi(\Gamma)$ *does not contain any open subset of* $\mathbb{R}^l$. *Then there is a nontrivial linear combination* $a_1\omega_1 + \cdots + a_l\omega_l$ *that induces a constant functional on a neighborhood* $\Gamma' \subset \Gamma_{s_0,V\cup V'}$ *of* s_0, *where* $V' \subset U$ *consists of* $k < l$ *points.*

Return to a 4-tuple $T = (U_1, U_2, s_1, s_2)$. For finite subsets $V_i \subset U_i$, $i = 1, 2$, set $\Gamma(V_1, V_2) = \{(U_1, U_2, s_1', s_2') \,|\, s_i' \in \Gamma_{s_i,V_i},\ i = 1, 2\}$. Below a *weak neighborhood* of T is a C^∞-topological neighborhood of T in the set $\Gamma(V_1, V_2)$ for some V_1, V_2.

A smooth function $F\colon \mathbb{R}^N \to \mathbb{R}$ is said to be *almost identically zero* around $x \in \mathbb{R}^N$ if for any neighborhood U of x there is a nonempty open subset $U' \subset U$ such that $F|_{U'} = 0$. In particular, in this case all partial derivatives of F vanish at x.

Theorem 6 ([5]). *Suppose that a multilocal functional $F(\omega_1, \dots, \omega_N)$ is zero on some weak neighborhood of T and the function F is not almost identically zero around $\big(\omega_1(T), \dots, \omega_N(T)\big) \in \mathbb{R}^N$. Then there is a nontrivial linear combination $a_1\omega_1 + \dots + a_N\omega_N$ that induces a constant functional on a weak neighborhood of T.*

Corollary 1 ([5]). *For any variational multivector $f \in \varkappa^{(k)}$ the formula*

$$\nabla_f(F(\omega_1, \dots, \omega_N)) = \sum_{1 \le i_1, \dots, i_k \le N} \frac{\partial^k F}{\partial t_{i_1} \dots \partial t_{i_k}}(\omega_1, \dots, \omega_N) \cdot f([\omega_{i_1}], \dots, [\omega_{i_k}]). \tag{2}$$

determines a well-defined differential operator $\nabla_f\colon \mathcal{A} \to \mathcal{A}$ of order k.

Following [14], we call (2) *local variational differential operators*.

If π is a vector superbundle then the horizontal cohomology space is $\mathbb{Z}$-graded $\bar{H}^n = \oplus_{i \ge 0} H_i^n$, the space H_0^n being isomorphic to the n-th horizontal cohomology space of the even component π^0. In this case we define $\mathcal{A}(\pi) = \mathcal{A}(\pi^0) \otimes_{\mathbb{R}} S(\oplus_{i \ge 1} H_i^n)$. Applying Corollary 1 to π^0, one can show that local variational differential operators are well-defined on $\mathcal{A}(\pi)$.

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Dynamics of Sharp Interfaces in One-, Two-Phase Flows in Porous Media: Asymmetry in the Boussinesq and Charny Equations

Anvar KACIMOV † *and Sergio SERRANO* ‡

† *Department of Soil and Water Sciences, Sultan Qaboos University, Al-Khod-123, PO Box 34, Muscat, Oman*
E-mail: *anvar@squ.edu.om*

‡ *Department of Civil & Environmental Engineering, Temple University, 1947 North 12th Str., Philadelphia, PA 19122, USA*
E-mail: *sserrano@temple.edu*

For the Boussinesq and Charny nonlinear diffusion equations we employ the known similarity solutions and derive new ones using the Adomian decomposition method and separation of variables. For a periodic water-drive regime with a heavier fluid sweeping a lighter one from a porous formation we arrive at an explicit analytical expression for the interface and describe the phenomena of "superpropagation" and "counterslumping" that date back to tidal "superelevation" in coastal unconfined aquifers described by J.R. Philip. Similarly, for a two-phase flow with a straight sharp interface separating two fluids of contrasting viscosity the interface in a periodic drive regime propagates deeper than in a constant rate sweep. Applications to groundwater hydrology and petroleum engineering are discussed.

1 Conceptual model

Two-phase flows in porous media are often modeled as two ($i = 1, 2$) continua separated by a sharp interface between two zones in which phase saturations are constant [6]. The interface moves and one phase displaces another either completely or to a residual content, which can be always scaled out. If the fluids and skeleton are incompressible, formation is homogeneous, movement is Darcian, then on each side of the interface the total head h_i satisfies the Laplace equation

$$\Delta h_i(x, y, z, t) = 0, \tag{1}$$

where x, y are the horizontal coordinates, z is the vertical coordinate, $\vec{V}_i = -K_i \nabla h_i$ is the velocity vector, $K_i = k\rho_i g/\mu_i$ are phase conductivities, k is intrinsic permeability, ρ_i and μ_i are fluid densities and viscosities, respectively. If capillary jump across the interface is ignored, then pressure there is continuous as well as the normal component of $\vec{V}$. If the displacement is not full, then k in each zone is not constant (as is postulated in the sharp interface model) but a function of the degree of saturation and (1) does not hold.

Classical groundwater hydrology [6] ignores the second (air) phase and (1) is satisfied for the water phase only. The corresponding free boundary problem is nonlinear owing to the kinematic and dynamic boundary conditions along the water table (phreatic surface) that can result in somewhat counterintuitive shapes, e.g. a wet zone hanging over a dry one [14]. In practice, most unconfined aquifers extend laterally much more than vertically and the free surface slope is gentle. It makes possible a hydraulic approximation or averaging of all flow parameters across the aquifer that eliminates z from (1) and, if y is physically irrelevant (e.g. seepage from irrigation

channels), the free boundary is implicitly incorporated into the governing Boussinesq equation

$$S\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[KH\frac{\partial H}{\partial x}\right] + c(H,x,t), \tag{2}$$

where $H(x,t)$ is the thickness of the saturated zone (water table elevation above a horizontal impermeable bedrock), c is the sink-source (evaporation-infiltration) term and S is specific yield.

2 Similarity solutions and decomposition

Barenblatt in 1952 [5] and Sokolov in 1956 [23] obtained a number of similarity solutions to the nonlinear diffusion equation (2) at $c = 0$. Of particular interest is the instantaneous source solution ABC (Fig. 1a, solid line), which has a finite support $2L(t)$, i.e. two symmetrical tips, which propagate left- and right-ward due to gravitational slumping of a fixed volume of water released as a spike in a dry aquifer.

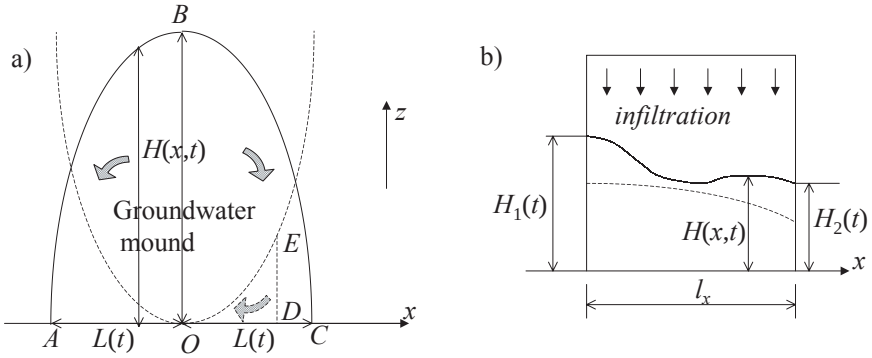

Figure 1.

We demonstrated [13] that the Barenblatt and Sokolov concave parabolas are identical and can be converted into a convex parabola satisfying (2) (Fig. 1a, dashed line). The two parabolas behave differently, in particular, the convex parabola has a frozen tip O, i.e. the two branches can be considered separately and we have localization or a finite size of the saturated zone even at blow-up boundary conditions. The question arises whether the convex parabola can appear in aquifers as a physically real initial condition to the transient equation (2). The answer is yes. We integrated the steady governing equation (LHP in (2) is zero) with a constant reservoir level (ED in Fig. 1a) and evaporation $c = \alpha h$ (α is a constant) and obtained just the convex parabola. Note that this linear evaporation function is an expedient model advocated by some groundwater hydrologists.

We now consider the hydraulic model for a nearly horizontal homogeneous unconfined aquifer of length l_x, bounded by two time-dependent boundary conditions, $H_1(t)$ and $H_2(t)$, respectively, representing the seasonal fluctuations in river stage (Fig. 1b). With location of the origin, $x = 0$, at the left boundary, and the datum at the bottom of the aquifer, (2) can be rewritten as

$$\frac{\partial H}{\partial t} - \frac{\partial}{\partial x}\left[KH\frac{\partial H}{\partial x}\right] = \frac{I}{S} \tag{3}$$

with boundary and initial conditions

$$H(0,t) = H_1(t), \qquad H(l_x,t) = H_2(t), \qquad H(x,0) = H_0(x), \tag{4}$$

where I is the mean daily recharge from rainfall; and $H_0(x)$ is the initial head across the aquifer (Fig. 1b, dashed line). We solve (3), (4) by using one of the Adomian's decomposition

schemes [2–4]. Let us write (3) as

$$\frac{\partial H}{\partial t} = \frac{I}{S} + \frac{KH}{S}\frac{\partial^2 H}{\partial x^2} + \frac{K}{S}\left(\frac{\partial H}{\partial x}\right)^2. \tag{5}$$

Defining the operator $L_t = \partial/\partial t$, and pre-multiplying (5) by the inverse operator L_t^{-1},

$$H = H_0 + \frac{It}{S} + L_t^{-1}N(H), \qquad N(H) = \frac{K}{S}\left[H\frac{\partial^2 H}{\partial x^2} + \left(\frac{\partial H}{\partial x}\right)^2\right]. \tag{6}$$

Now define the series $H = \sum_{n=0}^{\infty} H_n^*$, where the first term satisfies the recharge and the initial condition, that is

$$H_0^* = H_0(x) + \frac{It}{S}. \tag{7}$$

Subsequent terms in the series are given as

$$H_1^* = L_t^{-1}A_0, \qquad H_2^* = L_t^{-1}A_1, \qquad \ldots, \qquad H_{n+1}^* = L_t^{-1}A_n. \tag{8}$$

Taking into account (6), (7) the Adomian polynomials, A_n, in (8) for the nonlinear term, N, are defined as

$$A_0 = N(H_0^*), \qquad A_1 = H_1^*\frac{dN(H_0^*)}{dH_0^*}, \qquad A_2 = H_2^*\frac{dN(H_0^*)}{dH_0^*} + \frac{H_1^{*2}}{2!}\frac{d^2N(H_0^*)}{dH_0^{*2}},$$
$$A_3 = H_3^*\frac{dN(H_0^*)}{dH_0^*} + H_1^*H_2^*\frac{d^2N(H_0^*)}{dH_0^{*2}} + \frac{H_1^{*3}}{3!}\frac{d^3N(H_0^*)}{dH_0^{*3}}, \qquad \ldots. \tag{9}$$

According to (9) the polynomials A_n are generated for each non-linearity so that A_0 depends only on H_0^*, A_1 depends only on H_0^* and H_1^*, A_2 depends only on H_0^*, H_1^*, H_2^*, etc. All of the H_n^* components are calculable. It is now established that the series $\sum_{n=0}^{\infty} A_n$ for $N(H)$ is equal to a generalized Taylor series for $N(H_0)$, that $\sum_{n=0}^{\infty} H_n^*$ is a generalized Taylor series about the function H_0^*, and that the series terms approach zero as $1/(mn)!$, if m is the order of the highest linear differential operator. Since the series converges and does so very rapidly, the n-term partial sum $\Phi_n = \sum_{i=0}^{n-1} H_i^*$ usually serves as an accurate enough and practical solution. Thus, the second term in the series is

$$H_1 = \frac{K}{S}L_t^{-1}\left[H_0^*\frac{\partial^2 H_0^*}{\partial x^2} + \left(\frac{\partial H_0^*}{\partial x}\right)^2\right] = \frac{K}{S}L_t^{-1}\left[\left(H_0(x) + \frac{It}{S}\right)\frac{\partial^2 H_0}{\partial x^2} + \left(\frac{\partial H_0(x)}{\partial x}\right)^2\right]. \tag{10}$$

Higher terms in the series are derived similarly to (10). The third term would require information on the third-order spatial derivative of the initial condition. Clearly $H_0(x)$ must be sufficiently smooth for the calculation of its first and second-order spatial derivative. In practical applications a smooth surface should be fitted through the heads measured at individual wells. The spatial derivatives are then calculated analytically or numerically. In most practical modeling applications [21, 22], the first few terms in the decomposition series constitute an accurate solution. However it has been shown that decomposition series converge to the exact closed-form solution. For example, it has been shown in [18] that decomposition series converge to Sokolov's [23] solution. For more comparisons see [16, 19, 20]. Applications to the hydrodynamic model in porous media, that is the Laplace's equation subject to a dynamic nonlinear free surface are shown in [18]. Applications to the advective-dispersive equation subject to nonlinear reactions are shown in [17].

Several works have been devoted to the convergence problem of decomposition series. See for example the rigorous mathematical framework developed in [1, 7–11].

3 Superpropagation

A fascinating "superelevation" property of (2) has been elucidated in [15]. He considered a BVP with $H(0,t) = H_0 + a\sin\omega t$ and $H_\infty = H_0$ and found that a cyclostationary excited reservoir water level results in H whose time average value is higher than H_0. In other words, the aquifer is prone to accept water rather than to drain. This effect has been later validated experimentally in coastal aquifers subject to tidal variations of the sea level [24]. Clearly, the linear diffusion equation does not exhibit any "superelevation".

We inspected other nonlinear sharp-front equations to find out whether the "superelevation" property is generic. We considered the Charny [6] equation

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[qH + H(H-1)\frac{\partial H}{\partial x}\right] = 0 \tag{11}$$

which describes a sharp interface $H(x,t)$ between two phases in a formation of a constant thickness $b = 1$ with a transient drive of phase 1 displacing phase 2. The nonlinear diffusion equation (11) is based on the same vertical averaging as the Boussinesq equation and invokes additionally the assumption $\mu_1 = \mu_2$. However, $\rho_1 > \rho_2$ that is common in coastal aquifers where denser sea water encroaches into the fresh water zone.

We found [12] a straight line solution, which holds for an arbitrary dimensionless flow rate $q(t)$. In particular, we detected the phenomenon of "counterslumping" when the trailing front B in Fig. 2a slumps for a while against the main flow. For a cyclostationary excitation (11) the interface propagates throughout one period deeper than at a regime with $q = \text{const}$. We called this enhanced sweeping "superpropagation" in congruity with Philip's "superelevation". Note, that we solved PDE (11) in a closed form.

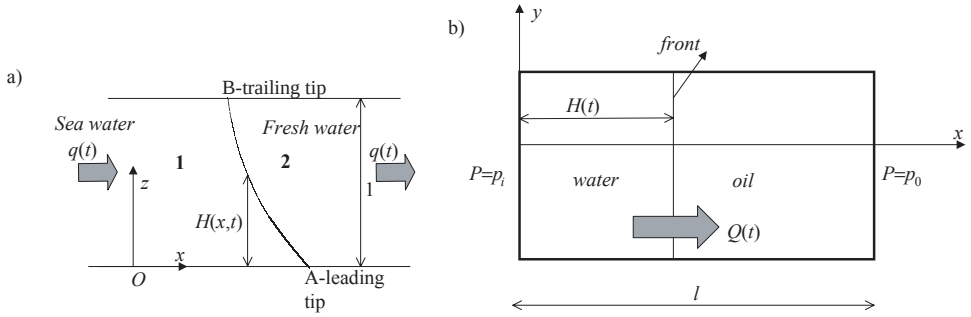

Figure 2.

We tested for the same non-symmetrical effect the Green–Ampt model describing a one-phase transient vertical infiltration. The governing equation is

$$m\frac{dH}{dt} = K\left(1 + \frac{d + h_c}{H}\right), \tag{12}$$

where $H(t)$ is the depth of the infiltration front counted from the soil surface, $d(t)$ is a given ponding water level at the surface and h_c is the constant invoking capillarity of the dry soil. By computer algebra routines we solved equation (12) with a cyclostationary d and arrived at the same conclusion that the wetting front propagates (on the time average) deeper than the front with $d = \text{const}$.

Eventually, we examined a two-phase displacement in x-direction when two phases contrast in viscosity only. The corresponding 1-D ODE according to [6] is

$$m\frac{dH}{dt} = k\frac{p_i - p_0}{\mu_1 H + \mu_2(l - H)}, \tag{13}$$

where $H(t)$ is the front location, l is the finite domain size, $p_i(t)$ and $p_o(t)$ are the injection and abstraction pressures (Fig. 2b). In (13) the densities of the two phases are identical that excludes z. Integrating (13) for a cyclostationary pressure drop $p_i - p_o$ across the layer we discovered the same effect of "superpropagation" as in all previous examples.

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An Elastodynamic State for the Solutions of Axial Symmetric Problems

Necla KADIOGLU and Senol ATAOGLU

Istanbul Technical University, Faculty of Civil Engineering, Maslak 34469 Istanbul, Turkey
E-mail: *kadiog@itu.edu.tr, ataoglu@itu.edu.tr*

The fundamental solution is used for axial symmetric transient problems in BEM formulati on. To check the formulation, a sample problem has been solved in plane strain. The strong singularity of the resulting integral equation has been reduced to weak form. New formulation provides to determine the initial velocity for a transient loading. Some differences have been introduced for the use of generalized functions.

1 Introduction

The aim of this study is to construct an elastodynamic state in an unbounded medium. It is appropriate to use for axial symmetric problems of elastodynamics in reciprocal theorem as fundamental solutions. After finding this elastodynamic state, a problem was solved. For the problem, the presented formulation leads to an integral equation. A numerical approximation is introduced for the solution of this integral equation.

2 A singular elastodynamic state for the solutions of axially symmetric elastodynamic problems

In plane elasticity, field variables are independent of the coordinate x_3. Starting point for construction of the necessary singular solution for plane problems will be a point load of magnitude $\delta(t)$ in the $\underset{\sim}{e}_k$ $(k = 1, 2)$ direction, at the position $\underset{\sim}{y}(y_1, y_2, y_3)$. Corresponding functions $\underset{\sim}{F}^k$ and $\underset{\sim}{G}^k$ can be written as follows [1]:

$$\underset{\sim}{F}^k = \frac{1}{4\pi r}\left\{H\left(t-\frac{r}{c_1}\right)\left(t-\frac{r}{c_1}\right)\right\}\underset{\sim}{e}_k, \tag{1}$$

$$\underset{\sim}{G}^k = \frac{1}{4\pi r}\left\{H\left(t-\frac{r}{c_2}\right)\left(t-\frac{r}{c_2}\right)\right\}\underset{\sim}{e}_k, \tag{2}$$

$$\bar{\rho}^2 = (x_1-y_1)^2 + (x_2-y_2)^2, \qquad r = \sqrt{\bar{\rho}^2 + (x_3-y_3)^2}. \tag{3}$$

It is considered that body force $\underset{\sim}{f}$ is acting along a line on which y_3 varies only between the limits $-\infty$ and $+\infty$. The corresponding functions $\underset{\sim}{\bar{F}}^k$ and $\underset{\sim}{\bar{G}}^k$ to this line load is obtained between the limits [2]

$$\underset{\sim}{\bar{F}}^k = \int_{-\infty}^{+\infty} \underset{\sim}{F}^k\, dy_3, \qquad \underset{\sim}{\bar{G}}^k = \int_{-\infty}^{+\infty} \underset{\sim}{G}^k\, dy_3. \tag{4}$$

But in equations (4), the limits of the integrals are not exactly correct. At a position defined by $\underset{\sim}{x}$, signals propagating with velocities do, however, not arrive until $c_1 t = r$ and $c_2 t = r$, respectively, which implies that the limits of integration of equations (4) will be appropriately

modified. By changing limits and performing integrals, $\underset{\sim}{\bar{F}}^k$ and $\underset{\sim}{\bar{G}}^k$ functions have been found out as follows:

$$\underset{\sim}{\bar{F}}^k = \frac{1}{2\pi}\left\{H\left(t-\frac{\bar{\rho}}{c_1}\right)\left\{t\ln\frac{c_1t+\sqrt{c_1^2t^2-\bar{\rho}^2}}{\bar{\rho}}-\frac{1}{c_1}\sqrt{c_1^2t^2-\bar{\rho}^2}\right\}\right\}\underset{\sim}{e}_k, \tag{5}$$

$$\underset{\sim}{\bar{G}}^k = \frac{1}{2\pi}\left\{H\left(t-\frac{\bar{\rho}}{c_2}\right)\left\{t\ln\frac{c_2t+\sqrt{c_2^2t^2-\bar{\rho}^2}}{\bar{\rho}}-\frac{1}{c_2}\sqrt{c_2^2t^2-\bar{\rho}^2}\right\}\right\}\underset{\sim}{e}_k. \tag{6}$$

And, corresponding displacement vector and stress tensor are

$$\underset{\sim}{\bar{u}}^k = \underset{\sim}{\nabla}\underset{\sim}{\nabla}\cdot(\underset{\sim}{\bar{F}}^k) - \underset{\sim}{\nabla}\times\underset{\sim}{\nabla}\times(\underset{\sim}{\bar{G}}^k), \tag{7}$$

$$\bar{\tau}_{ij}^k = \lambda\,\bar{u}_{l,l}^k\,\delta_{ij} + \mu(\,\bar{u}_{i,j}^k + \bar{u}_{j,i}^k). \tag{8}$$

In an infinite medium, the pair of $\underset{\sim}{\bar{u}}^k$ and $\underset{\sim}{\bar{\tau}}^k$ are the displacement vector and the stress tensor at a point $\underset{\sim}{x}(x_1,x_2)$ due to a body force acting at a specific point $\underset{\sim}{y}(y_1,y_2)$ in the $\underset{\sim}{e}_k$ direction and of magnitude $\delta(t)$. This formulation can also be found in Achenbach's book [2] in terms of $g(t)$ function. It is possible to construct a simpler formulation for axial symmetric problems. Here, cylindrical coordinates will be used. Starting point will be $\underset{\sim}{\bar{F}}^k$ and $\underset{\sim}{\bar{G}}^k$. In cylindrical coordinates, the base vector $\underset{\sim}{e}_{R1}$ on a point $\underset{\sim}{y}_1(R_1,\theta_1)$ can be expressed as in terms of cartesian base vectors, $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ as follow:

$$\underset{\sim}{e}_{R1} = \cos\theta_1\,\underset{\sim}{e}_1 + \sin\theta_1\,\underset{\sim}{e}_2. \tag{9}$$

Now, two singular body force, both having magnitude $g(t)/2R_1$, acting at the points $\underset{\sim}{y}_1$ and $\underset{\sim}{y}_2(R_1,\pi+\theta_1)$ in the directions of $\underset{\sim}{e}_{R_1}$ and $\underset{\sim}{e}_{R_2}$, respectively, are considered. $\underset{\sim}{\bar{F}}^S$ and $\underset{\sim}{\bar{G}}^S$ functions at a point $\underset{\sim}{x}(R,\theta)$ due to these double forces become:

$$\begin{aligned}\underset{\sim}{\bar{F}}^S = {}& \frac{1}{4\pi R_1}\left\{H\left(t-\frac{\bar{\rho}_1}{c_1}\right)\left[\frac{t\ln\left(c_1t+\sqrt{c_1^2t^2-\bar{\rho}_1{}^2}\right)}{\bar{\rho}_1}-\frac{1}{c_1}\sqrt{c_1^2t^2-\bar{\rho}_1{}^2}\right]\right\}\underset{\sim}{e}_{R1}\\ &+\frac{1}{4\pi R_1}\left\{H\left(t-\frac{\bar{\rho}_2}{c_1}\right)\left[\frac{t\ln\left(c_1t+\sqrt{c_1^2t^2-\bar{\rho}_2{}^2}\right)}{\bar{\rho}_2}-\frac{1}{c_1}\sqrt{c_1^2t^2-\bar{\rho}_2{}^2}\right]\right\}\underset{\sim}{e}_{R2},\end{aligned} \tag{10}$$

$$\begin{aligned}\underset{\sim}{\bar{G}}^S = {}& \frac{1}{4\pi R_1}\left\{H\left(t-\frac{\bar{\rho}_1}{c_2}\right)\left[\frac{t\ln\left(c_2t+\sqrt{c_2^2t^2-\bar{\rho}_1{}^2}\right)}{\bar{\rho}_1}-\frac{1}{c_2}\sqrt{c_2^2t^2-\bar{\rho}_1{}^2}\right]\right\}\underset{\sim}{e}_{R1}\\ &+\frac{1}{4\pi R_1}\left\{H\left(t-\frac{\bar{\rho}_2}{c_2}\right)\left[\frac{t\ln\left(c_2t+\sqrt{c_2^2t^2-\bar{\rho}_2{}^2}\right)}{\bar{\rho}_2}-\frac{1}{c_2}\sqrt{c_2^2t^2-\bar{\rho}_2{}^2}\right]\right\}\underset{\sim}{e}_{R2},\end{aligned} \tag{11}$$

$$\bar{\rho_1}^2 = R^2 + R_1^2 - 2RR_1\cos(\theta-\theta_1), \tag{12}$$

$$\bar{\rho_2}^2 = R^2 + R_1^2 + 2RR_1\cos(\theta-\theta_1). \tag{13}$$

After dividing equations (10) and (11)by πR_1 and integrating on the half cylindrical surface whose radius R_1 from $\theta_1=0$ to $\theta_1=\pi$, approaching R_1 to zero under the integral sign before performing the integration, the functions $\underset{\sim}{F}'$ and $\underset{\sim}{G}'$ which arise from a distributed body force in

the direction of $\underset{\sim}{e}_r$ at every point on the cylindrical surface whose radius is $R_1 = 0$, are obtained as follows:

$$\underset{\sim}{F}' = \frac{1}{4\pi}\left\{H\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1}\right)\left[\frac{\sqrt{c_1^2t^2-R^2}}{R}\right.\right.$$
$$\left.\left.+\,\delta\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^2}\right)\left[c_1t\ln\left(\frac{c_1t+\sqrt{c_1^2t^2-R^2}}{R}\right)-\sqrt{c_1^2t^2-R^2}\right]\right]\right\}\underset{\sim}{e}_R, \tag{14}$$

$$\underset{\sim}{G}' = \frac{1}{4\pi}\left\{H\left(t-\frac{R}{c_2}\right)\left(\frac{1}{c_2}\right)\left[\frac{\sqrt{c_2^2t^2-R^2}}{R}\right.\right.$$
$$\left.\left.+\,\delta\left(t-\frac{R}{c_2}\right)\left(\frac{1}{c_2^2}\right)\left[c_2t\ln\left(\frac{c_2t+\sqrt{c_2^2t^2-R^2}}{R}\right)-\sqrt{c_2^2t^2-R^2}\right]\right]\right\}\underset{\sim}{e}_R. \tag{15}$$

And, by substituting equations (14) and (15), the expressions of potentials φ', $\underset{\sim}{\psi}'$, nonzero components of displacement vector, strain and stress tensor, corresponding to this loading, are obtained as follows:

$$\varphi' = \frac{1}{4\pi}\left\{H\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1}\right)\left[-\frac{1}{\sqrt{c_1^2\,t^2-R^2}}\right]\right.$$
$$+\,\delta\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^2}\right)\left[c_1t\ln\frac{c_1t+\sqrt{c_1^2t^2-R^2}}{R}-3\frac{\sqrt{c_1^2t^2-R^2}}{R}\right]$$
$$\left.+\delta^{\cdot}\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^3}\right)\left[-c_1t\ln\frac{c_1t+\sqrt{c_1^2t^2-R^2}}{R}+\sqrt{c_1^2t^2-R^2}\right]\right\}, \tag{16}$$

$$\underset{\sim}{\psi}' = \underset{\sim}{0}, \tag{17}$$

$$u_R' = \frac{1}{4\pi}\left\{H\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1}\right)\left[-\frac{R}{\sqrt{c_1^2t^2-R^2}^3}\right]\right.$$
$$+\,\delta\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^2}\right)\left[-\frac{c_1t}{R^2}\ln\frac{c_1t+\sqrt{c_1^2t^2-R^2}}{R}+2\frac{\sqrt{c_1^2t^2-R^2}}{R^2}+\frac{3}{\sqrt{c_1^2\,t^2-R^2}}\right]$$
$$+\,\delta^{\cdot}\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^3}\right)\left[-\frac{c_1t}{R}\ln\frac{c_1t+\sqrt{c_1^2t^2-R^2}}{R}+4\frac{\sqrt{c_1^2t^2-R^2}}{R}\right]$$
$$\left.+\,\delta^{\cdot\cdot}\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^4}\right)\left[c_1t\ln\frac{c_1t+\sqrt{c_1^2t^2-R^2}}{R}-\sqrt{c_1^2t^2-R^2}\right]\right\}, \tag{18}$$

$$\epsilon_{RR}' = \frac{1}{4\pi}\left\{H\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1}\right)\left[-\frac{1}{\sqrt{c_1^2t^2-R^2}^3}-\frac{3R^2}{\sqrt{c_1^2t^2-R^2}^5}\right]\right.$$
$$+\,\delta\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^2}\right)\left[\frac{2}{R^3}c_1t\ln\frac{c_1t+\sqrt{c_1^2t^2-R^2}}{R}-3\frac{\sqrt{c_1^2t^2-R^2}}{R^3}\right.$$
$$\left.-\,\frac{1}{R\sqrt{c_1^2t^2-R^2}}+4\frac{R}{\sqrt{c_1^2t^2-R^2}^3}\right]$$
$$+\,\delta^{\cdot}\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^3}\right)\left[2\frac{c_1t}{R^2}\ln\frac{c_1t+\sqrt{c_1^2t^2-R^2}}{R}-5\frac{\sqrt{c_1^2t^2-R^2}}{R^2}-\frac{6}{\sqrt{c_1^2t^2-R^2}}\right]$$

$$+\delta^{\cdot\cdot}\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^4}\right)\left[\frac{c_1 t}{R}\ln\frac{c_1 t+\sqrt{c_1^2t^2-R^2}}{R}-5\frac{\sqrt{c_1^2t^2-R^2}}{R}\right]$$
$$+\delta^{\cdot\cdot\cdot}\left(t-\frac{R}{c_1}\right)\left(\frac{1}{c_1^5}\right)\left[-c_1 t\ln\frac{c_1 t+\sqrt{c_1^2t^2-R^2}}{R}+\sqrt{c_1^2t^2-R^2}\right]\Bigg\}, \tag{19}$$

$$\epsilon'_{\theta\theta}=\frac{1}{R}u_R{}', \tag{20}$$
$$\Delta'=\epsilon'_{RR}+\epsilon'_{\theta\theta}, \tag{21}$$
$$\tau'_{RR}=\lambda\Delta'+2\mu\epsilon'_{RR}, \tag{22}$$
$$\tau'_{\theta\theta}=\lambda\Delta'+2\mu\epsilon'_{\theta\theta}, \tag{23}$$
$$\tau'_{zz}=\lambda\Delta'. \tag{24}$$

The pair of $\underset{\sim}{u}'$ and $\underset{\sim}{\tau}'$ forms an elastodynamic state $\mathcal{S}'$ which is appropriate for the solution of the axial symmetric problems of elastodynamics.

3 Sample problem

An infinite medium, without body forces, having a cylindrical cavity is considered. At $t=0$, a constant pressure p_0 is applied on the cavity and maintained. For this problem, cylindrical coordinates (R,θ,z) will be used. And, the outward normal of the boundary is $\underset{\sim}{n}=-\underset{\sim}{e}_R$. The surface tractions on the boundary for $\mathcal{S}$ and for the elastodynamic state $\mathcal{S}'$ are as follows:

$$T_R(a,t)=p_0H^+(t), \tag{25}$$
$$T'_R(a,t)=\underset{\sim}{\tau}'(a,t)\,\underset{\sim}{n}=-\underset{\sim}{\tau}'(a,t)\underset{\sim}{e}_R=-\tau'_{RR}(a,t). \tag{26}$$

And further, for every $R\in[a,\infty)$

$$u_R(R,0)=0. \tag{27}$$

The expression of the dynamic reciprocal identity which is written between $\mathcal{S}$ and $\mathcal{S}'$ elastodynamic states will be reduced to:

$$\int_S \underset{\sim}{T}'*\underset{\sim}{u}\,dS=\int_S \underset{\sim}{T}*\underset{\sim}{u}'\,dS. \tag{28}$$

An integral equation arises substituting equations from (18) to (22) and (25) and (26) in equation (28). After this, changing the loading time of $\underset{\sim}{f}'$ from $t=0$ to $t=a/c_1$, this integral equation converts to:

$$-\int_0^t\Bigg\{(\lambda+2\mu)\left[H(t-\tau)g_0(t-\tau)+\delta(t-\tau)\frac{1}{c_1}g_1(t-\tau)\right.$$
$$\left.+\delta^{\cdot}(t-\tau)\frac{1}{c_1^2}g_2(t-\tau)+\delta^{\cdot\cdot}(t-\tau)\frac{1}{c_1^3}g_3(t-\tau)+\delta^{\cdot\cdot\cdot}(t-\tau)\frac{1}{c_1^4}g_4(t-\tau)\right]$$
$$+\lambda\left[H(t-\tau)f_0(t-\tau)+\delta(t-\tau)\frac{1}{c_1}f_1(t-\tau)+\delta^{\cdot}(t-\tau)\frac{1}{c_1^2}f_2(t-\tau)\right.$$
$$\left.+\delta^{\cdot\cdot}(t-\tau)\frac{1}{c_1^3}f_3(t-\tau)\right]\Bigg\}u_R(a,\tau)d\tau$$
$$=ap_0\int_0^t\Bigg\{H(t-\tau)f_0(t-\tau)+\delta(t-\tau)\frac{1}{c_1}f_1(t-\tau)+\delta^{\cdot}(t-\tau)\frac{1}{c_1^2}f_2(t-\tau)$$
$$+\delta^{\cdot\cdot}(t-\tau)\frac{1}{c_1^3}f_3(t-\tau)\Bigg\}H^+(\tau)d\tau, \tag{29}$$

where

$$
\begin{aligned}
g_0(t) &= -\frac{1}{\sqrt{c_1^2(t+a/c_1)^2-a^2}^3} - \frac{3a^2}{\sqrt{c_1^2(t+a/c_1)^2-a^2}^5},\\
g_1(t) &= \frac{2}{a^3}(c_1t+a)\ln\frac{c_1t+a+\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a} - 3\frac{\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a^3}\\
&\quad - \frac{1}{a\sqrt{c_1^2(t+a/c_1)^2-a^2}} + \frac{4a}{\sqrt{c_1^2(t+a/c_1)^2-a^2}^3},\\
g_2(t) &= \frac{2}{a^2}(c_1t+a)\ln\frac{c_1t+a+\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a}\\
&\quad - 5\frac{\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a^2} - \frac{6}{\sqrt{c_1^2(t+a/c_1)^2-a^2}},\\
g_3(t) &= \frac{1}{a}(c_1t+a)\ln\frac{c_1t+a+\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a} - 5\frac{\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a},\\
g_4(t) &= -(c_1t+a)\ln\frac{c_1t+a+\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a} + \sqrt{c_1^2(t+a/c_1)^2-a^2},\\
f_0(t) &= -\frac{1}{\sqrt{c_1^2(t+a/c_1)^2-a^2}^3},\\
f_1(t) &= -\frac{1}{a^3}(c_1t+a)\ln\frac{c_1t+a+\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a}\\
&\quad + 2\frac{\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a^3} + \frac{3}{a\sqrt{c_1^2(t+a/c_1)^2-a^2}},\\
f_2(t) &= -\frac{1}{a^2}(c_1t+a)\ln\frac{c_1t+a+\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a} + 4\frac{\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a^2},\\
f_3(t) &= \frac{1}{a}(c_1t+a)\ln\frac{c_1t+a+\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a} - \frac{\sqrt{c_1^2(t+a/c_1)^2-a^2}}{a}.
\end{aligned}
\tag{30}
$$

The kernel of equation (29) is strongly singular. The following equalities will be used to reduce the integral equation given in equation (29) to a simpler form

$$\delta(x)f(x) \equiv 0 \qquad \text{for} \quad f(0)=0, \tag{31}$$

$$\delta^{(n)}(x)f(x) \equiv 0 \qquad \text{for} \quad \left.\frac{d^nf(x)}{dx^n}\right|_{x=0} = 0, \tag{32}$$

$$
\begin{aligned}
\int_0^t \delta(t-\tau)f(t-\tau)u(\tau)d\tau &= -H(t-\tau)f(t-\tau)u(\tau)\,|_{\tau=0}^{\tau=t}\\
&\quad + \int_0^t H(t-\tau)[-f^{\cdot}(t-\tau)u(\tau) + f(t-\tau)u^{\cdot}(\tau)]d\tau,
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
\int_0^t \delta^{(n)}(t-\tau)f(t-\tau)u(\tau)d\tau &= -\delta^{(n-1)}(t-\tau)f(t-\tau)u(\tau)\,|_{\tau=0}^{\tau=t}\\
&\quad + \int_0^t \delta^{(n-1)}(t-\tau)[-f^{\cdot}(t-\tau)u(\tau) + f(t-\tau)u^{\cdot}(\tau)]d\tau.
\end{aligned}
\tag{34}
$$

And the new form of the integral equation becomes:

$$
\begin{aligned}
(\lambda+2\mu)\Bigg\{\Bigg|\delta^{\cdot\cdot}(t-\tau)&\frac{1}{c_1^4}u_R(\tau)[-c_1(t-\tau+a/c_1)L(t-\tau)+Q(t-\tau)]\\
&+\delta^{\cdot}(t-\tau)u_R^{\cdot}(\tau)\frac{1}{c_1^4}[-c_1(t-\tau+a/c_1)L(t-\tau)+Q(t-\tau)]
\end{aligned}
$$

$$
\begin{aligned}
&+\delta(t-\tau)\frac{1}{c_1^2}\left[\dot{u_R}(\tau)\frac{1}{ac_1}\{c_1(t-\tau-2a/c_1)L(t-\tau)-5Q(t-\tau)\}\right.\\
&\left.+\frac{1}{c_1^2}\ddot{u_R}(\tau)\{-c_1(t-\tau+a/c_1)L(t-\tau)+Q(t-\tau)\}\right]\\
&+H(t-\tau)\frac{1}{c_1}\left[-\frac{1}{c_1}\dot{u_R}(\tau)\frac{1}{Q(t-\tau)}+u_R(\tau)\left\{-\frac{1}{aQ(t-\tau)}+\frac{c_1(t-\tau+a/c_1)}{Q^3(t-\tau)}\right\}\right]\Bigg|_0^t\\
&+\int_0^t H(t-\tau)\left[\frac{1}{c_1^2}\ddot{u_R}(\tau)\frac{1}{Q(t-\tau)}+\frac{1}{c_1}\dot{u_R}(\tau)\frac{1}{aQ(t-\tau)}+u_R(\tau)\frac{c_1(t-\tau)}{aQ^3(t-\tau)}\right]d\tau\Bigg\}\\
&+\lambda\left\{\left|\dot{\delta}(t-\tau)\frac{1}{ac_1^3}u_R(\tau)[c_1(t-\tau+a/c_1)L(t-\tau)-Q(t-\tau)]\right.\right.\\
&+\delta(t-\tau)\frac{1}{c_1^2}\frac{1}{ac_1}\dot{u_R}(\tau)\{c_1(t-\tau+a/c_1)L(t-\tau)-Q(t-\tau)\}\\
&\left.+H(t-\tau)\frac{1}{c_1}\frac{1}{aQ(t-\tau)}u_R(\tau)\right|_0^t\\
&-\int_0^t H(t-\tau)\left[u_R(\tau)\frac{c_1(t-\tau)}{aQ^3(t-\tau)}+\dot{u_R}(\tau)\frac{1}{c_1}\frac{1}{aQ(t-\tau)}\right]d\tau\Bigg\}\\
&=p_0\left\{\left|-\dot{\delta}(t-\tau)\frac{1}{c_1^3}[c_1(t-\tau+a/c_1)L(t-\tau)-Q(t-\tau)]\right.\right.\\
&-\delta(t-\tau)\frac{1}{ac_1^2}[-c_1(t-\tau)L(t-\tau)+4Q(t-\tau)]-H(t-\tau)\frac{1}{c_1}\frac{1}{Q(t-\tau)}\Big|_0^t\\
&\left.+\int_0^t H(t-\tau)\frac{c_1(t-\tau)}{Q^3(t-\tau)}d\tau\right\},
\end{aligned}
\tag{35}
$$

where

$$
Q(t-\tau)=\sqrt{c_1^2(t-\tau+a/c_1)^2+a^2}, \tag{36}
$$

$$
L(t-\tau)=ln\frac{c_1(t-\tau+a/c_1)+\sqrt{c_1^2(t-\tau+a/c_1)^2+a^2}}{a}. \tag{37}
$$

It must be emphasized that the kernels of the integrals in equation (35) are weakly singular. Using equations (27), (31) to (34) and equating to zero, the multiplier of $\dot{\delta}(t)$ in the remaining term out of the integral in equation (35), the initial velocity $\dot{u_R}(a,0)$ is found as follow:

$$
\dot{u_R}(a,0)=\frac{p_0}{\lambda+2\mu}c_1. \tag{38}
$$

For convenience of notation two new dimensionless quantities are introduced as

$$
t^*=\frac{c_1 t}{a},\qquad U(t^*)=\frac{2\mu}{ap_0}u_R(a,t) \tag{39}
$$

the new form of the integral equation in terms of U and t^* becomes

$$
\begin{aligned}
&\frac{\lambda+2\mu}{2\mu}\int_0^{t^*}H(t^*-\tau^*)\left\{U\left[\frac{(t^*-\tau^*)}{\sqrt{\ }^3}\right]+\dot{U}\left[\frac{1}{\sqrt{\ }}\right]+\ddot{U}\left[\frac{1}{\sqrt{\ }^3}\right]\right\}d\tau^*\\
&-\frac{\lambda}{2\mu}\int_0^{t^*}H(t^*-\tau^*)\left\{U\left[\frac{(t^*-\tau^*)}{\sqrt{\ }^3}\right]+\dot{U}\left[\frac{1}{\sqrt{\ }}\right]\right\}d\tau^*=\int_0^{t^*}H(t^*-\tau^*)\frac{(t^*-\tau^*)}{\sqrt{\ }^3}d\tau^*,
\end{aligned}
\tag{40}
$$

where

$$
\sqrt{\ }=\sqrt{(t^*-\tau^*+1)^2-1}. \tag{41}
$$

And the dimensionless initial velocity is

$$U^{\cdot}(0) = \frac{2\mu}{\lambda + 2\mu}. \tag{42}$$

It is known that U function and first derivative of it being continuous. Using partial integration and substituting equation (42), the last form of the integral equation can be written as follow.

$$\int_0^{t^*} U^{\cdot\cdot} \left[\frac{\lambda + 2\mu}{2\mu} \frac{1}{\sqrt{\ }} + \sqrt{\ } \right] d\tau^* = \frac{t^*}{\sqrt{(t^*+1)^2 - 1}} - \frac{2\mu}{\lambda + 2\mu} \sqrt{(t^*+1)^2 - 1}. \tag{43}$$

Equation (43) is a Volterra integral equation of the first kind [3]. At the same time, it is also an integro-differential equation with a degenerate kernel having $1/\sqrt{t^* - \tau^*}$ singularity. This integral equation can be solved by transform techniques, but because of degenerate kernel, solution yields to an integral for every specific value of t. This integral must also be calculated numerically. Instead of this, another numerical method is introduced. The approximation function for $u_R(a,t)$ has been selected to be the same with the problem given in [1] substituting equation (39) in equation (43) the following equation is obtained between the end values of U of the j-th interval

$$\begin{aligned} &\sum_{k=1}^{j} \left\{ \left\{ [U_{k+1} - U_k] \frac{2}{(t^*_{k+1} - t^*_k)^2} - U^{\cdot}_k \frac{2}{(t^*_{k+1} - t^*_k)} \right\} \right. \\ &\quad \left. \times \left\{ \int_{t^*_k}^{t^*_{k+1}} \left(\sqrt{(t^*_{j+1} - \tau^* + 1)^2 - 1} + \frac{\lambda + 2\mu}{2\mu} \frac{1}{\sqrt{(t^*_{j+1} - \tau^* + 1)^2 - 1}} \right) d\tau^* \right\} \right\} \\ &\quad = \frac{t^*_{j+1}}{\sqrt{(t^*_{j+1} + 1)^2 - 1}} - \frac{2\mu}{\lambda + 2\mu} \sqrt{(t^*_{j+1} + 1)^2 - 1}. \end{aligned} \tag{44}$$

It is noted that:

$$t^*_1 = 0, \qquad U_1 = 0, \qquad U^{\cdot}_1 = U^{\cdot}(0). \tag{45}$$

From first interval, U_2 can be calculated because U_1 and $U^{\cdot}_1$ are known. Then writing $t^* = t^*_{k+1}$ [1], $U^{\cdot}_2$ are calculated. With these values, U_3 becomes the only unknown of equation (44) for $j = 2$. The obtained results (Poisson's ratio = 0.3) are given in Figs. 1 and 2.

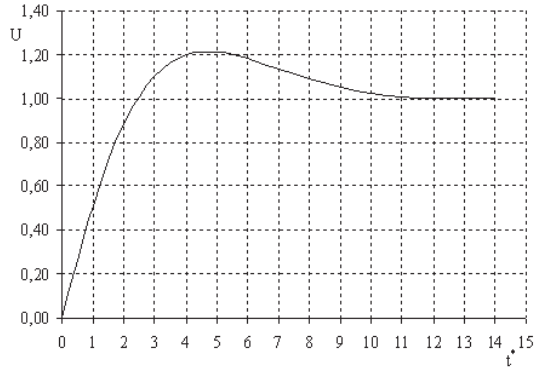

Figure 1. Variation of U on the surface of cylindrical cavity versus t^*.

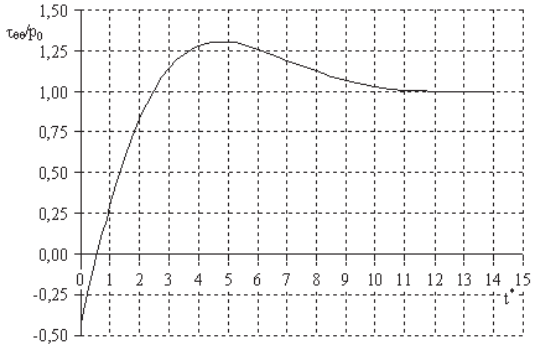

Figure 2. Variation of $\tau_{\theta\theta}/p_0$ on the surface of cylindrical cavity versus t^*.

4 Conclusions

A fundamental elastodynamic state has been derived. An integral equation is obtained by writing the reciprocal identity between one fundamental state and another state which represents the problem. The elastodynamic state can be used for the solutions of the problems having axial symmetry. And, the reciprocal identity which is written for axial symmetric problems is reduced to a Volterra integral equation whose kernel is strongly singular. The resulting integral equation of this problem has been converted to another form whose kernel is weakly singular and has been solved using a simple numerical technique. The most interesting part of the formulation is that it gives the exact value of the initial velocity.

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Spray Algebra

Vyacheslav S. KALNITSKY

StP State University, 28 Universitetsky Ave., 198504 St. Petersburg, Russia
E-mail: *skalnitsky@hotmail.com*

Spray algebra is the algebra of equivariant multilinear tensor fields with respect to the spray of a linear connection. Many natural operations of tensor algebra on tangent bundle have interpretation in term of spray algebra. This give new point of view on well known differential objects and provides us with new means for investigation. In particular, such achievement allows to formulate an analog of so-called Bochner type condition for the case of general linear connection.

1 Introduction

The main purpose of recent report is to construct some tensor algebra on tangent bundle TM of a C^∞-manifold M^n provided with a torsion-free linear connection. This object named by author Spray Algebra is a graded subalgebra of tensor algebra $\mathfrak{T}(TM)$. Some of tensor operations on $\mathfrak{T}(TM)$ save grading and therefore have an interpretation in terms of the spray algebra operations. Many of them appeared earlier in connection with different problems of differential geometry and physics, some objects seem to be described for the first time.

Among operations defined for the leading terms of grading are inner differential, the Jacobi operator, the Killing equation, Ashtekar's pairing and other.

The construction of spray algebra is based on concept of multilinear tensor field on TM. In the case of vector field such ones were investigated by V.V. Kozlov in search of the polynomial integrals of Hamiltonian systems. However, the above concept can be extended over the whole algebra $\mathfrak{T}(TM)$. We describe this construction in details as we cannot give a direct reference.

The great deal of the report is devoted to the explicit description of the structures in particular cases.

The analysis of the exterior differential action on the kernel of Lie differentiation with respect to the spray of some linear connection leads to the classical problems of mobility. This allows to generalize the concept of Bochner type condition for the case of linear connection.

Notations:

$$\bar\imath = (i_1, \dots, i_k), \qquad |\bar\imath| = i_1 + \cdots + i_k, \qquad \|\bar\imath\| = k,$$
$$\bar 1 = (1, \dots, 1), \qquad T_{\bar\imath} dx^{\bar\imath} = T_{i_1 \cdots i_k} dx^{i_1} \cdots dx^{i_k}.$$

2 Multilinear tensor fields

Let us consider a C^∞-manifold M^n and its tangent bundle TM provided with standard maps $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{v})$. For tensor fields of type (k, s) on TM there exists a tensor analog of the (non-tensor) Taylor decomposition with respect to $\boldsymbol{v}$ along the 0-section of TM. We will define it recurrently for the field $\widetilde{T}$. The first term of decomposition is defined by the formula

$$\overset{(0)}{T}(\boldsymbol{x}, \boldsymbol{v}) = \widetilde{T}^{\bar\imath}_{\bar\jmath}(\boldsymbol{x}, 0) \frac{\partial}{\partial v^{\bar\imath}} \otimes dx^{\bar\jmath}.$$

For the tensor field $T = \widetilde{T} - \overset{(0)}{T}$ the following procedure is well defined

$$\overset{(p)}{T} = \lim_{\alpha\to\infty} \alpha^{p+s-k} h_\alpha^* \Big(T - \overset{(1)}{T} - \cdots - \overset{(p-1)}{T} \Big),$$

where h_α^* is the transfer of the field with the fiber-wise homothety of TM. To describe such fields in coordinates let us introduce the **index** of the multi-indices triple $(\bar\imath, \bar\jmath, \bar m)$

$$\mathbf{Ind}(\bar\imath, \bar\jmath, \bar m) = |\bar 1 - [\bar\imath]| + |[\bar\jmath]| + \|\bar m\|,$$

where

$$[\bar\imath] = \left(\left[\frac{i_1 - 1}{n}\right], \ldots, \left[\frac{i_k - 1}{n}\right] \right).$$

The formal Taylor decomposition of T with respect to $\boldsymbol{v}$ has the form

$$T = T^{\bar\imath}_{\bar\jmath|\bar m}(\boldsymbol{x}) v^{\bar m} \frac{\partial}{\partial z^{\bar\imath}} \otimes dz^{\bar\jmath}.$$

The general form of $\overset{(p)}{T}$ follows directly from the tensor coordinate relations

$$\overset{(p)}{T} = \sum_{\mathbf{Ind}(\bar\imath,\bar\jmath,\bar m)=p} T^{\bar\imath}_{\bar\jmath|\bar m}(\boldsymbol{x}) v^{\bar m} \frac{\partial}{\partial z^{\bar\imath}} \otimes dz^{\bar\jmath}.$$

Definition 1. The field $\overset{(p)}{T}$ is called multilinear tensor field of type (k, s, p). The space of such fields is denoted by $\overset{(k,s,p)}{\mathfrak{P}}$.

For instance, the Liouville field $v^i \partial/\partial v^i$ is of type $(1, 0, 1)$, the spray

$$S = v^i \frac{\partial}{\partial x^i} - \Gamma^i_{mn}(\boldsymbol{x}) v^m v^n \frac{\partial}{\partial v^i}$$

is of type $(1, 0, 2)$. Notice that the index is additive with respect to tensor product. It implies that the space

$$\mathfrak{P}(TM) = \oplus_{k,s=0}^{\infty} \oplus_{p=0}^{\infty} \overset{(k,s,p)}{\mathfrak{P}}$$

is graded tensor algebra. Moreover, the straightforward calculation shows that for the vector fields $T \in \overset{(1,0,p)}{\mathfrak{P}}$ and $S \in \overset{(1,0,q)}{\mathfrak{P}}$ holds

$$\mathbf{Ind}([T, S]) = \mathbf{Ind}\, T + \mathbf{Ind}\, S - 1.$$

It means that the space $\oplus_p \overset{(1,0,q)}{\mathfrak{P}}$ is the graded Lie algebra.

3 Spray algebra

Let tensor field T be equivariant with respect to the spray S, i.e.

$$\mathfrak{L}_S T = 0,$$

where $\mathfrak{L}_S$ is the Lie differentiation. Straightforward calculation gives $\overset{(0)}{T} = 0$. For $\mathfrak{L}_S$ and h^*_α commute the following series of equations hold for any $p \geqslant 0$

$$\mathfrak{L}_S \overset{(p)}{T} = 0. \tag{1}$$

The space of solutions of equation (1) will be denoted by $\overset{(k,s,p)}{\mathfrak{Y}}$.

Definition 2. The graded algebra

$$\mathfrak{Y} = \oplus_{k,s=0}^{\infty} \oplus_{p=0}^{\infty} \overset{(k,s,p)}{\mathfrak{Y}}$$

is called *spray algebra* of symmetry linear connection with spray S.

The equation (1) in the standard map on TM is the system of PDE's on tensor field T components. In many interesting cases this system allows the explicit description in terms of tensor fields on manifold M itself and operations over them! This reduction sheds light on many invariant constructions of tensor calculus which turned out to be related. The algebraic structure of spray algebra provides us with new instruments applicable to different problems of differential geometry.

Let us give explicit description of some spaces and operations appeared.

The space $\overset{(0,0,p)}{\mathfrak{Y}}$. For the function f on TM the equation $\mathfrak{L}_S u = f$ is called the kinetic equation. Its solutions are defined up to the equivariant functions u: $\mathfrak{L}_S u = 0$. For the function $\overset{(p)}{u} = u_{|\bar{m}}(\boldsymbol{x})v^{\bar{m}}$ which is the multilinear field in our notations the object $\overset{(p)}{u} = u_{|\bar{m}}(\boldsymbol{x})$ defines the tensor field of type $(0,p)$, $p = \|\bar{m}\|$, on M. The equation (1) can be written in terms of field u

$$\nabla_{(i} \overset{(p)}{u}{}_{|\bar{m})} = 0,$$

where $(,)$ means symmetrization with respect to covariant indices i, $\bar{m}$. Therefore the space $\overset{(0,0,p)}{\mathfrak{Y}}$ is the space of p-Killing fields, i.e. the first integrals of geodesic flow.

The space $\overset{(1,0,p)}{\mathfrak{Y}}$. The elements of this space have the coordinate form

$$(A,B) = A^i_{|\bar{m}}(\boldsymbol{x})v^{\bar{m}}\frac{\partial}{\partial x^i} + B^i_{|\,\bar{n}}(\boldsymbol{x})v^{\bar{n}}\frac{\partial}{\partial v^i},$$

where $\|\bar{m}\| = p-1, \|\bar{n}\| = p$. In terms of components of the objects A and B the equation (1) can be written as generalized Jacobi equation [1]

$$\nabla_{(i}\nabla_j A^l_{|\,\bar{m})} + R^l_{(isj}A^s_{|\,\bar{m})} = 0,$$

where R is curvature tensor. The object B has necessarily the form

$$B_{|\,\bar{n}}v^{\bar{n}} = \overset{h}{\nabla}_i A_{|\,\bar{m}}v^{\bar{m}}v^i,$$

where $\overset{h}{\nabla}$ is horizontal covariant differentiation corresponding to the Berwald connection [2] of semi-basic tensor field formed by A. Therefore the space $\overset{(1,0,p)}{\mathfrak{Y}}$ can be described in terms of generalized Jacobi fields on M. In the case of Hamiltonian systems on two-dimensional manifolds the existence of such fields was investigated by V.V. Kozlov [3], for greater dimensions – by I.A. Taymanov [4].

The space $\overset{(0,1,p)}{\mathfrak{Y}}$. In this case the affine-Killing fields appear as solutions of the equation

$$\nabla_{(i}\nabla_j A_{|l||\,\bar{m})} + A_{(s|\,\bar{m}} R^s_{ij)l} = 0.$$

In the metric case this equation is equivalent to the previous one for dual fields.

In above cases and some others the spaces $\overset{(k,s,p)}{\mathfrak{Y}}$ admit the description in terms of tensor algebra $\mathfrak{T}(M)$. Let us now give the interpretation of some operation on $\mathfrak{T}(TM)$ in $\mathfrak{T}(M)$.

Exterior differential. For $\mathfrak{L}_S$ and d commute the operators

$$d_p : \overset{(0,0,p)}{\mathfrak{Y}} \to \overset{(0,1,p)}{\mathfrak{Y}}$$

are well defined. It is easy to see that d_p are identical injections. For $p = 1$ this fact is equivalent to the Ricci identity and means that every Killing field is an affine-Killing one. The case $p = 2$ was explicitly formulated in [5]. Here we obtained the set of statements for p-Killing fields.

Contraction. Contraction of tensor fields from $\overset{(1,0,p)}{\mathfrak{Y}}$ and $\overset{(0,1,p)}{\mathfrak{Y}}$ get into $\overset{(0,0,p)}{\mathfrak{Y}}$. In language of spray algebra it means, for instance, that for $\vartheta \in \mathfrak{T}^0_1(M)$ and $X \in \mathfrak{T}^1_0(M)$ the field

$$(\nabla\vartheta)(X) - \vartheta(\nabla X) \tag{2}$$

belongs to $\mathfrak{T}^0_0(M)$. This operation is nontrivial pairing of Jacobi fields and affine-Killing ones with value in Killing fields space. The very particular case of the operation (2) was described by Ashtekar [6]. For two Killing fields ϑ and η of metric ω_{ij} he associated the Killing field

$$\eta_j\omega^{ij}(\nabla\vartheta_i) - \vartheta_j\omega^{ij}(\nabla\eta_i),$$

where ∇ is Levi-Civita connection of metric. This operation corresponds to the contraction $\widetilde{\omega}(df, dg)$, where $\widetilde{\omega}$ is equivariant bivector on $\mathfrak{T}(TM)$.

Tensor product. The usual product of equivariant functions on TM in language of spray algebra means the symmetrized tensor product of Killing fields. As a sequence we have the statement that symmetrized product of Killing and affine-Killing fields is the latter again.

Lie bracket. The explicit description of Lie bracket in spray algebra was obtained in [7]. For $A \in \overset{(1,0,p)}{\mathfrak{Y}}$ and $B \in \overset{(1,0,q)}{\mathfrak{Y}}$ the symmetrized expression holds

$$[A, B]_{\mathfrak{Y}} = \mathfrak{S}\left(\nabla_A B - \nabla_B A + (q-1)B(\nabla A) - (p-1)A(\nabla B)\right).$$

4 Bochner complex

Let us begin with the general arguments. For the arbitrary vector field X on a smooth manifold N we introduce the space of equivariant differential p-forms

$$\Omega^X_p = \{\vartheta \in \Omega_p(N) \,|\, \mathfrak{L}_X\vartheta = 0\}.$$

For $\mathfrak{L}_X$ and d commute the exterior differential $d\vartheta$ of equivariant p-form ϑ is also equivariant. Therefore the complex $\Omega^X_\bullet(N)$ with exterior differential is well defined. If $X = 0$ this complex coincides with initial de Rham complex. In the case $N = TM$ and $X = S$ we can apply all above constructions for the complex $\Omega^S_\bullet(TM)$. The decomposition of $\mathfrak{T}(TM)$ induces the

decomposition of the complex and differential on the set of complexes

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & \dots & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow d_{-1} \\
\overset{(0,0,1)}{\Omega_0^S} & \oplus & \overset{(0,0,2)}{\Omega_0^S} & \oplus & \overset{(0,0,3)}{\Omega_0^S} & \dots & \Omega_0^S(TM) \\
\downarrow d_0^1 & & \downarrow d_0^2 & & \downarrow d_0^3 & & \downarrow d_0 \\
\overset{(0,1,1)}{\Omega_1^S} & \oplus & \overset{(0,1,2)}{\Omega_1^S} & \oplus & \overset{(0,1,3)}{\Omega_1^S} & \dots & \Omega_1^S(TM) \\
\downarrow d_1^1 & & \downarrow d_1^2 & & \downarrow d_1^3 & & \downarrow d_1 \\
\overset{(0,2,1)}{\Omega_2^S} & \oplus & \overset{(0,2,2)}{\Omega_2^S} & \oplus & \overset{(0,2,3)}{\Omega_2^S} & \dots & \Omega_2^S(TM) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow
\end{array}
$$

As was shown in Section 3, the operators d_0^p are identical injection of p-Killing fields into the space of affine-Killing fields. In the compact Riemannian case the Jacobi operator is reducible to the Weitzenböck form [8] and the Bochner technique is applicable. The Bochner condition is that guaranteeing every affine mapping be a Killing one, i.e. d_0^1 is a surjection. The strong Bochner condition means Riemannian immovability, i.e. d_{-1}^1 is surjection. All operators in above complexes consist of covariant differentiations and curvature tensor like in the Weitzenböck form. On the basis of these arguments we shall call the complex $\Omega_\bullet^S(TM)$ the Bochner complex and the surjection of the differential will be called the Bochner condition. These notions are defined for the case of non-compact manifolds equipped with a general linear connection.

Acknowledgements

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Quantum Long-Range Interactions in General Relativity

Iosif B. KHRIPLOVICH and Grigory G. KIRILIN

Budker Institute of Nuclear Physics, 630090 Novosibirsk, Russia
E-mail: *khriplovich@inp.nsk.su, kirilin@inp.nsk.su*

We have found one-loop effects in general relativity which can be interpreted as quantum corrections to the Schwarzschild metric. They induce quantum long-range corrections to the Newton law and to gravitational relativistic effects: spin-orbit, spin-spin, and Lense–Thirring interactions.

1 Introduction

It has been recognized long ago that quantum effects can generate long-range corrections in general relativity. Those corrections due to the graviton polarization operator with photons and massless neutrinos in the loop were calculated in [1–4]. The corresponding quantum correction to the Newton potential between two bodies with masses m_1 and m_2 is

$$U_{\gamma\nu} = -\frac{4+N_\nu}{15\pi}\,\frac{k^2\hbar m_1 m_2}{c^3 r^3}, \tag{1}$$

where N_ν is the number of massless two-component neutrinos, k is the Newton gravitational constant (in the text below we put $\hbar = 1$, $c = 1$).

The reason why the problem allows for a closed solution is as follows. The Fourier-transform of $1/r^3$ is

$$\int d\boldsymbol{r}\,\frac{\exp(-i\boldsymbol{q}\boldsymbol{r})}{r^3} = -2\pi\ln\boldsymbol{q}^2. \tag{2}$$

This singularity in the momentum transfer $\boldsymbol{q}$ means that the correction discussed can be generated only by diagrams with two massless particles in the t-channel. The number of such diagrams of second order in k is finite, and their logarithmic part in q^2 can be calculated unambiguously.

The analogous diagrams with gravitons and ghosts in the loop (see Fig. 1; wavy and dashed lines in it refer to gravitons and ghosts, respectively, double wavy lines refer to the background gravitational field) were considered in [1, 5–7].

Clearly, other diagrams with two gravitons in the t-channel contribute as well to the discussed correction $\sim 1/r^3$. Some of these contributions were addressed, along with diagrams 1a,b, in [8–13]. However in these papers the set of considered diagrams was incomplete, and the results for these diagrams were incorrect. For the first time the complete set of relevant diagrams was pointed out in [14], however, only one of them was calculated therein correctly.

The problem of quantum correction to the Newton law, which is certainly interesting from the theoretical point of view, was then addressed in our previous article [15]. In it all relevant diagrams, except one (see Fig. 3b below), were calculated correctly.

In a recent paper [16] our criticisms are acknowledged, though implicitly. The diagram 3b is calculated in [16] correctly, and the results for all other contributions agree with ours.

The content of our present work is as follows. We demonstrate in an elementary way that the discussed corrections are the same both for scalar and (after averaging over spins) for spinor particles. The fact was proven previously in [17] by direct calculation of loop diagrams.

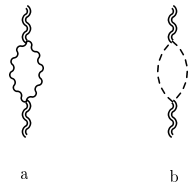

Figure 1. Graviton loops.

Then, using the background field technique [7], we construct effective amplitudes which describe quantum power corrections in general relativity. Since the derived corrections are universal (i.e. the same for scalar and spinor particles), in the limit when one of the interacting particles is heavy, these corrections can be interpreted as corrections to the Schwarzschild metric. (Our results for the latter differ from the results of [17].) In this way we not only simplify the calculation of the corrections to the Newton law, but obtain rather easily quantum corrections to gravitational relativistic effects.

We have also cross-checked and confirmed the result for the corrections to the Newton law by calculation in another technique (used previously in [14]) that with the gravitational variables $\psi^{\mu\nu} = \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}$ in the harmonic gauge $\partial_\mu \psi^{\mu\nu} = 0$.

2 Effective amplitudes

Let us sketch first of all our proof of the fact that the corrections $\sim \ln|q^2|$ for spinor particles coincide, after averaging over spins, with those for scalar ones. To this end, it is sufficient in fact to consider tree diagrams, Fig. 2, which are building blocks of the logarithmic loops. Here and below wavy lines denote gravitons again, and solid lines refer to scalar or spinor particles. We average the spinor diagrams over the spins, and single out from the numerators of diagrams 2a,b, both for scalar and spinor particles, the structures that cancel the denominators therein. Thus obtained contact contributions combine with the initial contact diagram 2c into an effective seagull which is the same both for spins 0 and 1/2. As to the true s- and u-pole contributions in diagrams 2a,b, which are left after this procedure, they also coincide for scalar and spinor particles with the adopted accuracy.

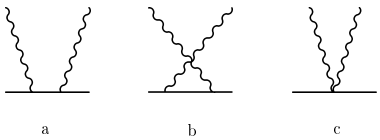

Figure 2. Tree diagrams.

We start the discussion of the loops with the vacuum polarization diagrams, Fig. 1. The effective Lagrangian corresponding to the sum of these loops, derived in [7], can be rewritten for our purpose as [8]

$$L_{rr} = -\frac{1}{1920\pi^2}\ln|q^2|\left(42R_{\mu\nu}R^{\mu\nu} + R^2\right); \tag{3}$$

here $R_{\mu\nu}$ is the Ricci tensor, $R = R^\mu_\mu$ is the scalar curvature. We are interested at the moment in the situation when at least one of the particles is considered in the static limit. In this case $|q^2| \to \boldsymbol{q}^2$, and in the coordinate representation we obtain

$$L_{rr} = \frac{1}{3840\pi^3 r^3}\left(42R_{\mu\nu}R^{\mu\nu} + R^2\right). \tag{4}$$

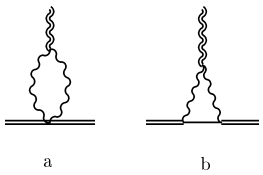

Figure 3. Vertex diagrams.

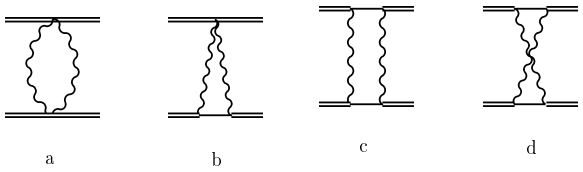

Figure 4.

The next set of diagrams, Fig. 3, refers to the vertex part. The corresponding effective operator is

$$L_{rt} = -\frac{k}{8\pi^2 r^3}\left(3R_{\mu\nu}T^{\mu\nu} - 2RT\right), \qquad T = T^{\mu}_{\mu}. \tag{5}$$

Here $T^{\mu\nu}$ is the energy-momentum tensor (EMT) averaged over spin.

At last diagrams of Fig. 4. Their sum is

$$L_{tt} = \frac{k^2}{\pi r^3}\,T^2. \tag{6}$$

In fact, the box diagrams in Fig. 4 not only contribute to amplitude (6). They also generate a more complicated amplitude which cannot be reduced to a product of energy-momentum tensors. We will come back to this irreducible amplitude later.

In virtue of the Einstein equations

$$R_{\mu\nu} = 8\pi k\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{7}$$

the three Lagrangians (4), (5), (6) can be conveniently combined into

$$L_{\rm tot} = -\frac{k^2}{60\pi r^3}\left(138\,T_{\mu\nu}T^{\mu\nu} - 31T^2\right). \tag{8}$$

3 Quantum corrections to metrics

Now quantum corrections to some gravitational effects can be most easily derived as follows. Let us split the total EMT $T_{\mu\nu}$ into those of a static central body and of a light probe one, $T^o_{\mu\nu}$ and $t_{\mu\nu}$, respectively. Then, by variation the resulting expression in $t^{\mu\nu}$ we obtain a tensor which can be interpreted as a quantum correction $h^{(q)}_{\mu\nu}$ to the metric created by the central body:

$$h^{(q)}_{\mu\nu} = \frac{k^2}{15\pi r^3}\left(138T^o_{\mu\nu} - 31g_{\mu\nu}T^o\right). \tag{9}$$

It follows immediately from this expression that

$$h^{(q)}_{00} = \frac{107}{15}\,\frac{k^2 M}{\pi r^3}, \tag{10}$$

where M is the mass of the central body.

The calculation of the space components $h^{(q)}_{mn}$ demands in fact some modification of formula (9). The point is that we work with the gauge condition $h^{\mu}_{\nu;\,\mu} - (1/2)h^{\mu}_{\mu;\,\nu} = 0$ for the graviton field. It is only natural to demand that the resulting effective field $h^{(q)}_{mn}$ should satisfy the same condition which simplifies now to $h^{(q)\mu}{}_{\nu,\,\mu} - (1/2)h^{(q)\mu}{}_{\mu,\,\nu} = 0$. Thus obtained space metric is

$$h^{(q)}_{mn} = \frac{k^2 M}{\pi r^3}\left\{\frac{31}{15}\delta_{mn} - \frac{76}{15}\left[\frac{r_m r_n}{r^2} + \ln\left(\frac{r}{r_0}\right)\left(\delta_{mn} - 3\frac{r_m r_n}{r^2}\right)\right]\right\}. \tag{11}$$

Technically, the expression in square brackets in (10) originates from the terms containing structures of the type $\partial_\mu T^{\mu\nu}$. Generally speaking, they arise when calculating Lagrangians (5), (6), and (8), but are omitted therein since they vanish on-mass-shell. Thus these terms are absent in (9). But they can be restored by rewriting the net result (8) with the Einstein equations (7) as

$$L_{tot} = -\frac{1}{3840\pi^3 r^3}\left(138 R_{\mu\nu}R^{\mu\nu} - 31R^2\right), \tag{12}$$

and then attaching energy-momentum tensors to the double wavy lines. The presence of $\ln(r/r_0)$, where r_0 is some normalization point, is quite natural here if one recalls $\ln \boldsymbol{q}^2$ in the momentum representation. Fortunately, this term in square brackets does not influence physical effects.

Our results (10), (11) differ from the corresponding ones of [17]. The main reason is that the contributions of diagrams 4 to metric are omitted in [17]. This omission does not look logical to us: on-mass-shell one cannot tell these diagrams from others (see (8), (12)). Besides, the Fourier-transformation of $(q_m q_n/\boldsymbol{q}^2 \ln \boldsymbol{q}^2)$ is performed in [17] incorrectly, with a wrong result ($r_m r_n/r^2$ only) for the term in the square brackets in (11).

4 Quantum corrections to gravitational effects

We start with the correction to the Newton law. In line with (10), we should take into account here the above mentioned contribution of the box diagrams 4, which cannot be reduced to metric. In the static limit for both particles this irreducible contribution is [15, 16, 19]

$$-\frac{23}{3}\,\frac{k^2 Mm}{\pi r^3}. \tag{13}$$

The net correction to the Newton law is

$$U^q(r) = -\frac{41}{10}\,\frac{k^2 Mm}{\pi r^3}. \tag{14}$$

Let us go over now to the quantum correction to the interaction of the orbital angular momentum $\boldsymbol{l}$ of a light particle with its own spin $\boldsymbol{s}$, i.e. to the common gravitational spin-orbit interaction. It is most easily obtained with the general expression for the frequency $\boldsymbol{\omega}$ of the spin precession in a gravitational field derived in [18]. For a nonrelativistic particle in a weak static centrally-symmetric field this expression simplifies to

$$\omega_i = \frac{1}{2}\,\epsilon_{imn}(\gamma_{mnk} v_k + \gamma_{0n0} v_m). \tag{15}$$

Here

$$\gamma_{mnk} = \frac{1}{2}\,(\partial_m h_{nk} - \partial_n h_{mk}), \qquad \gamma_{0n0} = -\frac{1}{2}\,\partial_n h_{00}$$

are the Ricci rotation coefficients, $\boldsymbol{v}$ is the particle velocity (the present sign convention for $\boldsymbol{\omega}$ is opposite to that of [18]). A simple calculation results in

$$U^q_{ls}(r) = -\frac{169}{20}\,\frac{k^2}{\pi r^5}\,\frac{M}{m}\,(\boldsymbol{ls}). \tag{16}$$

Now let us derive the quantum correction to the classical velocity-dependent gravitational interaction. We start with the amplitude (8) written in the momentum representation:

$$L_{\rm tot} = \frac{k^2}{30}\,\ln|q^2|\left(138 T_{\mu\nu}T^{\mu\nu} - 31T^2\right). \tag{17}$$

As distinct from the previous corrections, here we go beyond the static approximation, and expand $\ln|q^2| = \ln(\boldsymbol{q}^2 - \omega^2)$ to first order in ω^2. Then (in spirit of [20] where this trick was applied to the calculation of the classical velocity-dependent c^{-2} correction to the Newton law) we rewrite ωT_{00} as $q_n T_{0n}$, neglect terms of 4th and higher orders in c^{-2}, and come back to the coordinate representation. The resulting velocity-dependent quantum correction is

$$U^q_{vv}(\boldsymbol{r}) = -\frac{k^2 m_1 m_2}{60\pi r^3}\,[445(\boldsymbol{v}_1\boldsymbol{v}_2) + 321(\boldsymbol{n}\boldsymbol{v}_1)(\boldsymbol{n}\boldsymbol{v}_2)], \qquad \boldsymbol{n} = \frac{\boldsymbol{r}}{r}. \tag{18}$$

With formula (18) we derive (in spirit of [21], § 106, Problem 4) the quantum correction to the interaction of the orbital momentum $\boldsymbol{l}$ of a light particle with the internal angular momentum (spin) $\boldsymbol{s}$ of a compound central body, i.e. to the Lense–Thirring effect. The result is

$$U^q_{\rm LT}(r) = -\frac{69}{5}\,\frac{k^2}{\pi r^5}\,(\boldsymbol{l}\boldsymbol{s}). \tag{19}$$

In the same way one obtains with (18) the correction to the spin-spin interaction of two compound bodies:

$$U^q_{ss}(\boldsymbol{r}) = \frac{69}{10}\,\frac{k^2}{\pi r^5}\,[3(\boldsymbol{s}_1\boldsymbol{s}_2) - 5(\boldsymbol{n}\boldsymbol{s}_1)(\boldsymbol{n}\boldsymbol{s}_2)]. \tag{20}$$

The quantum correction to the classical velocity-dependent gravitational interaction is generated also by the irreducible parts of the box diagrams (a sort of relativistic analogue of (13)). To derive it, we expand the corresponding expressions for those diagrams to first order in the small parameter $(p_1p_2 - m_1m_2)/m_1m_2$ (but not to zeroth order, as it was the case with (13)). Thus obtained net result can be written in the momentum representation as

$$-k^2 m_1 m_2 \ln\left(\boldsymbol{q}^2 - \omega^2\right)\frac{2}{3}\left(23 + \frac{524}{5}\,\frac{p_1p_2 - m_1m_2}{m_1m_2}\right). \tag{21}$$

Now we expand again $\ln(\boldsymbol{q}^2 - \omega^2)$ to first order in ω^2, and use the obvious identity

$$\omega p_0 \approx \omega m \approx m\boldsymbol{v}\boldsymbol{q}.$$

At last, going over into the coordinate representation (and omitting the zeroth-order contribution (13)), we arrive at the following result for the irreducible contribution to the quantum velocity-dependent correction:

$$U^{q,\,irr}_{vv}(\boldsymbol{r}) = \frac{k^2 m_1 m_2}{10\pi r^3}\,[311(\boldsymbol{v}_1\boldsymbol{v}_2) + 115(\boldsymbol{n}\boldsymbol{v}_1)(\boldsymbol{n}\boldsymbol{v}_2)]. \tag{22}$$

From (22) we obtain now the irreducible contributions to the quantum corrections to the Lense–Thirring and spin-spin interactions, respectively:

$$U^{q,\,irr}_{LT}(r) = \frac{262}{5}\,\frac{k^2}{\pi r^5}\,(\boldsymbol{l}\boldsymbol{s}), \tag{23}$$

$$U^{q,\,irr}_{ss}(\boldsymbol{r}) = -\frac{131}{5}\,\frac{k^2}{\pi r^5}\,[3(\boldsymbol{s}_1\boldsymbol{s}_2) - 5(\boldsymbol{n}\boldsymbol{s}_1)(\boldsymbol{n}\boldsymbol{s}_2)]. \tag{24}$$

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On Symmetries in (Anti)Causal (Non)Abelian Quantum Theories

Frieder KLEEFELD

CFIF, Instituto Superior Técnico, Av. Rovisco Pais, P-1049-001 Lisboa, Portugal
E-mail: *kleefeld@cfif.ist.utl.pt*

(Anti)causal boundary conditions being imposed on the (seemingly) Hermitian Quantum Theory (HQT) as described in standard textbooks lead to an (Anti)Causal Quantum Theory ((A)CQT) with an indefinite metric (see e.g. [1–6]). Therefore, an (anti)causal neutral scalar field is not Hermitian, as one (anti)causal neutral scalar field consists of a non-Hermitian linear combination of two (Hermitian) acausal fields. Fundamental symmetries in (A)CQT are addressed. The quantum theoretical (transition) probability and antiparticle concepts are revised. Imaginary parts of cross sections and refraction indices are related.

1 Introduction to the idea of (anti)causal fields

Textbooks (e.g. [7]) on Quantum Field Theory (QFT) declare the neutral scalar Klein–Gordon (KG) field to be a *Hermitian* (*shadow* [8, 10]) *field* $\varphi(x) = \varphi^+(x)$ with real mass $m = m^*$ and Lagrangian $\mathcal{L}^0_\varphi(x) = \frac{1}{2}\left((\partial\varphi(x))^2 - m^2\varphi(x)^2\right)$. Note that $\varphi(x)$ is representing *one* (real) field-theoretical degree of freedom! Its equation of motion is $(\partial^2 + m^2)\varphi(x) = 0$ yielding – strictly speaking – a principal value propagator [10] $\mathrm{P}\,\frac{i}{p^2-m^2}$. Causality is typically enforced *afterwards* by the use of *causal* or *anticausal* Feynman propagators ($i/(p^2-m^2+i\varepsilon)$ (causal), $i/(p^2-m^2-i\varepsilon)$ (anticausal)) corresponding – strictly speaking – to the *causal KG equation* $(\partial^2+m^2-i\varepsilon)\phi(x) = 0$ or *anticausal KG equation* $(\partial^2 + m^2 + i\varepsilon)\phi^+(x) = 0$, respectively [8, 9, 1–3]. The *causal KG field* $\phi(x) = (\varphi_1(x) + i\varphi_2(x))/\sqrt{2}$ and the *anticausal KG field* $\phi^+(x) = (\varphi_1(x) - i\varphi_2(x))/\sqrt{2}$ are *non-Hermitian* and represented by *two Hermitian shadow fields* $\varphi_j(x) = \varphi_j^+(x)$ $(j = 1, 2)$ yielding *two* (real) field-theoretical degrees of freedom. I.e. imposing causal boundary conditions on Quantum Theory (QT) leads (already at zero temperature) to a *doubling* of degrees of freedom like in Thermal Field Theory [11] or Open Quantum Systems [12]. The non-Hermitian nature of QT should not surprise[1], but be taken into account!

[1] In-fields and out-fields fulfil same *causal* KG equations: $(\partial^2 + m^2 - i\varepsilon)\phi_{\mathrm{in}}(x) = 0$, $(\partial^2 + m^2 - i\varepsilon)\phi_{\mathrm{out}}(x) = 0$. Hence, the Hilbert space of out-states is not obtained from the Hilbert space of in-states by Hermitian conjugation. Therefore we claim that in Quantum Mechanics (QM) "bra's" (in our notation: $\langle\!\langle \cdots |$) are not obtained by Hermitian conjugation from "ket's" (in our notation: $|\cdots\rangle = \langle\cdots|^+$)! An unexpected result in (A)CQT is obtained when considering the causal/anticausal KG, Dirac and Schrödinger equations for a complex mass $M = m - \frac{i}{2}\Gamma \simeq -i\varepsilon$. In deriving the standard continuity equations as described in text books, i.e.

$$\begin{aligned}
&\text{KG:} && \partial_\mu[\phi^+(x)\partial^\mu\phi(x) - \phi(x)\partial^\mu\phi^+(x)] = 2i\varepsilon\phi^+(x)\phi(x),\\
&\text{Dirac:} && \partial_\mu[i\bar\psi(x)\gamma^\mu\psi(x)] = -2i\varepsilon\bar\psi(x)\psi(x),\\
&\text{Schrödinger:} && i\partial_t[\psi^+(x)\psi(x)] + \frac{1}{2m}\,\vec\nabla\cdot[\psi^+(x)\,\vec\nabla\,\psi(x) - \psi(x)\,\vec\nabla\,\psi^+(x)]\\
& && = \psi^+(x)[V(x) - V^+(x)]\psi(x) - \frac{i\varepsilon}{2m}[\psi^+(x)\,\vec\nabla^{\,2}\psi(x) + \psi(x)\,\vec\nabla^{\,2}\psi^+(x)],
\end{aligned}$$

we have to observe that all these currents are not conserved due to the finite imaginary part of the mass (here $-\varepsilon$) or non-Hermitian *causal potentials* $V(x)$ being Laplace-transforms of *causal* propagators! The non-conservation of the Schrödinger current indicates a breakdown of the traditional probability interpretation of Schrödinger theory by Max Born in (A)CQT. Note that the "massless" causal Dirac equation $(i\,\partial\!\!\!/ + i\varepsilon)\psi(x) = 0$ is not chiral invariant!

2 Introduction to the "Nakanishi model"

In the year 1972 N. Nakanishi [8, 9] investigated for curiosity the so-called "Complex-Ghost Relativistic Field Theory", i.e. a theory for a KG field $\phi(x)$ with complex mass $M := m - \frac{i}{2}\Gamma$ (and the Hermitian conjugate field $\phi^+(x)$)[2]. In the following we want to introduce immediately isospin and to consider for convenience a set of N equal complex mass KG fields $\phi_r(x)$ $(r = 1, \ldots, N)$ (i.e. a charged "Nakanishi field" with isospin $\frac{N-1}{2}$) described by the "Nakanishi Lagrangian"

$$\mathcal{L}_0(x) = \sum_r \left\{ \frac{1}{2}\left((\partial\phi_r(x))^2 - M^2\phi_r(x)^2\right) + \frac{1}{2}\left((\partial\phi_r^+(x))^2 - M^{*2}\phi_r^+(x)^2\right)\right\}.$$

The Lagrange equations of motion for the causal and anticausal "Nakanishi field" $\phi_r(x)$ and $\phi_r^+(x)$, i.e. $(\partial^2 + M^2)\phi_r(x) = 0$ and $(\partial^2 + M^{*2})\phi_r^+(x) = 0$, are solved by a Laplace-transform[3]. The "Nakanishi model" is quantized by claiming *Canonical equal-real-time commutation relations*[4]. The non-vanishing commutation relations in configuration space are $(r, s = 1, \ldots, N)$:

$$[\phi_r(\vec{x}, t), \Pi_s(\vec{y}, t)] = i\,\delta^3(\vec{x} - \vec{y})\delta_{rs}, \qquad [\phi_r^+(\vec{x}, t), \Pi_s^+(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y})\delta_{rs}.$$

The resulting non-vanishing momentum-space commutation relations are $(r, s = 1, \ldots, N)$:

$$[a(\vec{p}, r), c^+(\vec{p}\,', s)] = (2\pi)^3 2\omega(\vec{p})\delta^3(\vec{p} - \vec{p}\,')\delta_{rs},$$
$$[c(\vec{p}, r), a^+(\vec{p}\,', s)] = (2\pi)^3 2\omega^*(\vec{p})\delta^3(\vec{p} - \vec{p}\,')\delta_{rs}.$$

The Hamilton operator is derived by a standard Legendre transform (see e.g. [3, 2])[5]:

$$H_0 = \sum_r \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{1}{2}\omega(\vec{p})\big(c^+(\vec{p}, r)a(\vec{p}, r) + a\,(\vec{p}, r)c^+(\vec{p}, r)\big)$$
$$+ \sum_r \int \frac{d^3p}{(2\pi)^3 2\omega^*(\vec{p})} \frac{1}{2}\omega^*(\vec{p})\big(a^+(\vec{p}, r)c(\vec{p}, r) + c(\vec{p}, r)a^+(\vec{p}, r)\big).$$

The "Nakanishi-KG propagator" is obtained by *real-time ordering* of causal KG fields [8, 3, 15][6]:

$$\Delta_N(x - y)\delta_{rs} := -i\langle\langle 0|T[\phi_r(x)\phi_s(y)]|0\rangle \stackrel{!}{=} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2 - M^2}\delta_{rs}. \tag{1}$$

The anticausal "Nakanishi-KG propagator" is obtained by Hermitian conjugation or by a vacuum expectation value of an *anti-real-time ordered* product of two anticausal fields.

[2] The formalism was later (1999-2000) independently rederived by the author (see e.g. [1–3]). N. Nakanishi called the non-Hermitian fields $\phi(x)$ and $\phi^+(x)$ "Complex Ghosts".

[3] The result decomposing in "positive" & "negative" complex frequencies (e.g. $\phi_r(x) = \phi_r^{(+)}(x) + \phi_r^{(-)}(x)$) is:

$$\phi_r(x) = \int \frac{d^4p}{(2\pi)^3} \text{“}\delta\big(p^2 - M^2\big)\text{”} e^{-ip\cdot x} a(p, r) = \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \big[a(\vec{p}, r)e^{-ipx} + c^+(\vec{p}, r)e^{ipx}\big]\big|_{p^0=\omega(\vec{p})},$$
$$\phi_r^+(x) = \int \frac{d^4p}{(2\pi)^3} \text{“}\delta\big(p^2 - M^{*2}\big)\text{”} e^{ip^*\cdot x} a^+(p, r) = \int \frac{d^3p}{(2\pi)^3 2\omega^*(\vec{p})} \big[c(\vec{p}, r)e^{-ip^*x} + a^+(\vec{p}, r)e^{ip^*x}\big]\big|_{p^0=\omega(\vec{p})},$$

where we defined $a(\vec{p}) := a(p)|_{p^0=\omega(\vec{p})}$ and $c^+(\vec{p}) := a(-p)|_{p^0=\omega(\vec{p})}$ on the basis of the complex "frequency" $\omega(\vec{p}) := \sqrt{\vec{p}^2 + M^2}$ $(\omega(\vec{0}) := M)$. The meaning of the symbolic delta-distribution "$\delta\big(p^2 - M^2\big)$" for complex arguments has been illuminated by N. Nakanishi [13, 9]. Nowadays it may be embedded in the framework of (tempered) Ultradistributions [14].

[4] The standard Canonical conjugate momenta to the (anti)causal fields $\phi_r(x)$ and $\phi_r^+(x)$ are given by $\Pi_r(x) := \delta\mathcal{L}_0(x)/\delta(\partial_0\phi_r(x)) \stackrel{!}{=} \partial_0\phi_r(x)$ and $\Pi_r^+(x) := \delta\mathcal{L}_0(x)/\delta(\partial_0\phi_r^+(x)) \stackrel{!}{=} \partial_0\phi_r^+(x)$.

[5] c^+ are *creation operators* of Bosonic *particles*, while a^+ are *creation operators* of Bosonic *holes*. c are *annihilation operators* of Bosonic *holes*, while a are *annihilation operators* of Bosonic *particles*. The antiparticle/antihole concept will be sketched in Section 11. (Anti)particles propagate towards the future, (anti)holes towards the past.

[6] For intermediate states with complex mass these propagators lead to *Poincaré covariant* results. At each interaction vertex coupling to *intermediate* complex mass fields there holds *exact* 4-momentum conservation. Only if complex mass fields with finite Γ appeared as asymptotic states, then Poincaré covariance would be violated!

3 (Anti)causal quantum mechanics

The representation independent, time-dependent Schrödinger equation is $i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle$. Its adjoint is given by $-i\partial_t\langle\langle\psi(t)| = \langle\langle\psi(t)|H$. In 1-dim. QM we consider the Hamilton operator of the (anti)causal Harmonic Oscillator [9,2–6,16] (see equation (2)) $H = H_C + H_A = \frac{1}{2}\omega[c^+, a]_\pm + \frac{1}{2}\omega^*[a^+, c]_\pm = \omega(c^+a \pm \frac{1}{2}) + \omega^*(a^+c \pm \frac{1}{2})$ ($\pm$ for Bosons/Fermions[7]) with[8]

$$\begin{pmatrix} [c,c^+]_\mp & [c,a^+]_\mp \\ [a,c^+]_\mp & [a,a^+]_\mp \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{“indefinite metric”},$$

yielding $[H_C, H_A] = 0$[9]. The right/left eigensystem of the Hamilton operator is found by diagonalizing the stationary Schrödinger equation $(H-E)|\psi\rangle = 0$ & its adjoint $\langle\langle\psi|(H-E) = 0$. The resulting (normalized) right eigenstates $|n,m\rangle$ and left eigenstates $\langle\langle n,m|$ for the eigenvalues $E_{n,m} = \omega(n \pm \frac{1}{2}) + \omega^*(m \pm \frac{1}{2})$ are given by $|n,m\rangle := \frac{1}{\sqrt{n!m!}}(c^+)^n(a^+)^m|0\rangle$ and $\langle\langle n,m| := \frac{1}{\sqrt{m!n!}}\langle\langle 0|c^m a^n$ (Bosons: $n,m \in \{0,1,2,\ldots\}$; Fermions: $n,m \in \{0,1\}$). The (bi)orthogonal eigenstates are complete: $\langle\langle n',m'|n,m\rangle = \delta_{n'n}\delta_{m'm}$, $\sum_{n,m}|n,m\rangle\langle\langle n,m| = \mathbf{1}$. In holomorphic representation (see e.g. [18]) the time-dependent Schrödinger equation and its adjoint are:

$$i\partial_t\langle\langle z,z^*|\psi(t)\rangle = \int dz'dz'^*\langle\langle z,z^*|H|z',z'^*\rangle\langle\langle z',z'^*|\psi(t)\rangle,$$
$$-i\partial_t\langle\langle\psi(t)|z,z^*\rangle = \int dz'dz'^*\langle\langle\psi(t)|z',z'^*\rangle\langle\langle z',z'^*|H|z,z^*\rangle.$$

The holomorphic representation of the Hamilton operator ($H(z,z^*) = H_C(z) + H_A(z^*)$) is:

$$\begin{aligned} H(z',z'^*;z,z^*) &= \langle\langle z',z'^*|H|z,z^*\rangle = H(z,z^*)\langle\langle z',z'^*|z,z^*\rangle \\ &= \left(-\frac{1}{2M}\frac{d^2}{dz^2} + \frac{1}{2}M\omega^2z^2 - \frac{1}{2M^*}\frac{d^2}{dz^{*2}} + \frac{1}{2}M^*\omega^{*2}z^{*2}\right)\langle\langle z',z'^*|z,z^*\rangle. \end{aligned}$$

The resulting stationary Schrödinger equations in holomorphic representation are given by $H(z,z^*)\langle\langle z,z^*|\psi\rangle = E\langle\langle z,z^*|\psi\rangle$, $\langle\langle\psi|z,z^*\rangle H(z,z^*) = E\langle\langle\psi|z,z^*\rangle$[10]. Note the non-trivial identities $\int dzdz^*|z,z^*\rangle\langle\langle z,z^*| = \mathbf{1}$, $\langle\langle z,z^*|z',z'^*\rangle = \delta(z-z')\delta(z^*-z'^*)$.

4 Lorentz transformations/covariance of (anti)causal systems

A Lorentz transformation $\Lambda^\mu{}_\nu$ for a given metric $g_{\mu\nu}$ is defined by $\Lambda^\mu{}_\rho g_{\mu\nu}\Lambda^\nu{}_\sigma = g_{\rho\sigma}$. Let n^μ be a timelike unit 4-vector ($n^2 = 1$) and ξ^μ an arbitrary complex 4-vector with $\xi^2 \neq 0$. We want to construct [1–3] a Lorentz transformation $\Lambda^\mu{}_\nu(\xi)$ relating the 4-vector ξ^μ with its “restframe”, i.e. $\xi^\mu = \Lambda^\mu{}_\nu(\xi)n^\nu\sqrt{\xi^2}$ and $n_\nu\sqrt{\xi^2} = \xi_\mu\Lambda^\mu{}_\nu(\xi)$. After defining the *inversion matrix* $P^\mu_\nu :=$

[7] The Fermionic oscillator we tend to denote by $H = \frac{1}{2}\omega\{d^+, b\} + \frac{1}{2}\omega^*\{b^+, d\}$ with $\{b, d^+\} = 1$ etc.

[8] All other possible commutators (Bosons) or anticommutators (Fermions) of a, c, a^+, and c^+ vanish!

[9] A “Hermitian” Hamilton operator quantized with a non-trivial (e.g. indefinite) metric is called *pseudo-Hermitian*! Pseudo-Hermiticity (& pseudo-unitarity) presently promoted [17] by M. Znojil and A. Mostafazadeh. Ideas go back to names like W. Pauli, P.A.M. Dirac, S.N. Gupta, K. Bleuler, and E.C.G. Sudarshan.

[10] The eigensolutions for the eigenvalues $E_{n,m}$ are given by ($(M\omega)^{-1/2}$ = complex oscillator length):

$$\langle\langle z,z^*|n,m\rangle = i^{n+m}\sqrt{\frac{|M\omega|}{2^{n+m}n!m!\pi}}\exp\left(-\frac{1}{2}\left(M\omega z^2 + M^*\omega^*z^{*2}\right)\right)H_n\left(z\sqrt{M\omega}\right)H_m\left(z^*\sqrt{M^*\omega^*}\right),$$
$$\langle\langle n,m|z,z^*\rangle = (-i)^{m+n}\sqrt{\frac{|M\omega|}{2^{m+n}m!n!\pi}}\exp\left(-\frac{1}{2}\left(M^*\omega^*z^{*2} + M\omega z^2\right)\right)H_m\left(z^*\sqrt{M^*\omega^*}\right)H_n\left(z\sqrt{M\omega}\right).$$

$2n^\mu n_\nu - g^\mu{}_\nu$ we find *two "symmetric"* and *two "asymmetric"* matrices $\Lambda^\mu{}_\nu(\xi)$ solving the defining equation $\Lambda^\mu{}_\rho\, g_{\mu\nu}\, \Lambda^\nu{}_\sigma = g_{\rho\sigma}$, i.e. $(\xi\cdot n := \xi^\mu n_\mu)$:

$$\Lambda^\mu{}_\nu(\xi) = \pm\left\{ g^\mu{}_\nu - \frac{\sqrt{\xi^2}}{\sqrt{\xi^2}\mp\xi\cdot n}\left[n^\mu \mp \frac{\xi^\mu}{\sqrt{\xi^2}}\right]\left[n_\nu \mp \frac{\xi_\nu}{\sqrt{\xi^2}}\right]\right\},$$
$$\Lambda^\mu{}_\nu(\xi) = \pm\left\{ g^\mu{}_\rho - \frac{\sqrt{\xi^2}}{\sqrt{\xi^2}\mp\xi\cdot n}\left[n^\mu \mp \frac{\xi^\mu}{\sqrt{\xi^2}}\right]\left[n_\rho \mp \frac{\xi_\rho}{\sqrt{\xi^2}}\right]\right\} P^\rho{}_\nu.$$

For the real Lorentz group the 4 solutions are related to the well known ortho-chronous/non-ortho-chronous proper/improper Lorentz transformations! Convince yourself that for the metric $+,-,-,-$ one of the "asymmetric" Lorentz boosts is given by:

$$\Lambda^\mu{}_\nu(\xi) = \begin{pmatrix} \frac{\xi^0}{\sqrt{\xi^2}} & \frac{\vec{\xi}^T}{\sqrt{\xi^2}} \\ \frac{\vec{\xi}}{\sqrt{\xi^2}} & 1_3 + \frac{\vec{\xi}\vec{\xi}^T}{\sqrt{\xi^2}(\sqrt{\xi^2}+\xi^0)} \end{pmatrix} \Rightarrow \Lambda^\mu{}_\nu(p)|_{p^0=\omega(\vec{p})} = \begin{pmatrix} \frac{\omega(\vec{p})}{M} & \frac{\vec{p}^T}{M} \\ \frac{\vec{p}}{M} & 1_3 + \frac{\vec{p}\vec{p}^T}{M(M+\omega(\vec{p}))} \end{pmatrix}.$$

In the right expression we chose $\xi^\mu = p^\mu$ with $p^2 = M^2$ and $M = m - i\frac{\Gamma}{2}$. Some properties of $\Lambda^\mu{}_\nu(p)$ have already discussed in Refs. [1–3][11].

5 The (anti)causal Dirac theory

The causal Dirac equation and its relatives obtained by Hermitian conjugation and/or transposition are [1, 2, 4–6] $(i\overrightarrow{\partial\!\!\!/} - M)\psi_r(x) = 0$, $(i\overrightarrow{\partial\!\!\!/} - \bar{M})\psi^c_r(x) = 0$, $\overline{\psi^c_r}(x)(-i\overleftarrow{\partial\!\!\!/} - M) = 0$, and $\bar{\psi}_r(x)(-i\overleftarrow{\partial\!\!\!/} - \bar{M}) = 0$ ($M := m - \frac{i}{2}\Gamma$, $\bar{M} := \gamma_0 M^+ \gamma_0$). Note that $r = 1, \ldots, N$ is an isospin index and $\psi_r(x)$, $\bar{\psi}_r(x)$, $\psi^c_r(x) = C\gamma_0\psi^*_r(x)$, $\overline{\psi^c_r}(x) = \psi^T_r(x)C$ are Grassmann fields. The underlying Lagrange density is given by [1, 2, 4–6] (see also [21]) ($N = 1$ yields neutrinos!)

$$\mathcal{L}^0_\psi(x) = \sum_r \frac{1}{2}\left(\overline{\psi^c_r}(x)\left(\frac{1}{2}i\overleftrightarrow{\partial\!\!\!/} - M\right)\psi_r(x) + \bar{\psi}_r(x)\left(\frac{1}{2}i\overleftrightarrow{\partial\!\!\!/} - \bar{M}\right)\psi^c_r(x)\right).$$

4-spinors $u(p,s) \equiv v(-p,s)$ in complex 4-momentum space ($s = \pm\frac{1}{2}$) are introduced by the defining equation $(\not{p} - \sqrt{p^2})u(p,s) = 0 \Leftrightarrow \overline{u^c}(p,s)(-\not{p} - \sqrt{p^2}) = 0$. The spinors are normalized according to $\mathrm{sgn}\,[\mathrm{Re}\,(p^0)]\sum_s u(p,s)\overline{v^c}(p,s) = \not{p} + \sqrt{p^2}$ for $\mathrm{Re}\,[p^0] \neq 0$. Equations of motions are solved by a Laplace-transformation[12]. Note that the spinors obey the analyticity proper-

[11] G.P. Pron'ko [19] could of course argue that such a Lorentz boost between $\vec{p}$ and $\vec{p}'$ *"... understood literarily leads to nonsense because the transformed space components of the momentum become complex. ..."* Certainly this argument is *only* true for complex mass fields with finite Γ being treated as *asymptotic states*. Yet – as argued in the context of equation (1) – for complex mass fields in *intermediate states* Poincaré invariance is completely restored! The crucial difference between traditional HQT and (A)CQT is that in HQT fields are claimed to be representations of the covering group of the real Lorentz group $L^\uparrow_+$, while in (A)CQT even asymptotic (anti)causal states are representations of the covering group of the complex Lorentz group $L_+(C)$ (or, more generally, the covering group of the respective Poincaré group) [20].

[12] The result decomposing in "positive" & "negative" complex frequencies (e.g. $\psi_r(x) = \psi^{(+)}_r(x) + \psi^{(-)}_r(x)$) is:

$$\psi_r(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})}\left[e^{-ip\cdot x} b_r(p,s)u(p,s) + e^{ip\cdot x} b_r(-p,s)v(p,s)\right]\big|_{p^0=\omega(\vec{p})},$$
$$\psi^c_r(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2\omega^*(\vec{p})}\left[e^{ip^*\cdot x} b^+_r(p,s)u^c(p,s) + e^{-ip^*\cdot x} b^+_r(-p,s)v^c(p,s)\right]\big|_{p^0=\omega(\vec{p})},$$
$$\overline{\psi_r}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2\omega^*(\vec{p})}\left[e^{ip^*\cdot x} b^+_r(p,s)\bar{u}(p,s) + e^{-ip^*\cdot x} b^+_r(-p,s)\bar{v}(p,s)\right]\big|_{p^0=\omega(\vec{p})},$$
$$\overline{\psi^c_r}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})}\left[e^{-ip\cdot x} b_r(p,s)\overline{u^c}(p,s) + e^{ip\cdot x} b_r(-p,s)\overline{v^c}(p,s)\right]\big|_{p^0=\omega(\vec{p})}.$$

Note that $d^+_r(p,s) := b_r(-p,s)$, $\{b_r(\vec{p},s), d^+_{r'}(\vec{p}',s')\} = (2\pi)^3\, 2\omega(\vec{p})\,\delta^3(\vec{p}-\vec{p}')\delta_{ss'}\delta_{rr'}, \ldots$.

ty $u^c(p,s) = u(-p^*,s) = v(p^*,s)$. The (anti)causal Dirac equation is Lorentz covariant due to standard transformation properties of spinors and γ-matrices, i.e. $u(p) = S(\Lambda(p))u(\sqrt{p^2}n)$ and $S^{-1}(\Lambda(p))\gamma^\mu S(\Lambda(p)) = \Lambda^\mu{}_\nu(p)\gamma^\nu$. The causal "Nakanishi–Dirac propagator" of a spin 1/2 Fermion is obtained by *standard Fermionic real-time ordering* of causal Dirac fields:

$$S_N(x-y)_{\alpha\beta}\delta_{rs} := -i\langle\!\langle 0|T[(\psi_r(x))_\alpha(\overline{\psi^c_s}(y))_\beta]|0\rangle \stackrel{!}{=} \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2-M^2}(\not{p}+M)_{\alpha\beta}\delta_{rs}.$$

The anticausal "Nakanishi-Dirac propagator" is obtained by Hermitian conjugation or by a vacuum expectation value of a *anti-real-time ordered* product of two anticausal Dirac fields.

6 (Anti)causal massive and "massless" vector fields

With[13] $\vec{e}^{(i)}(i = x,y,z)$ and $\vec{e}^{(i)}\cdot\vec{e}^{(j)} = \delta^{ij}$ we define the polarization vectors $\varepsilon^{\mu(i)}(p) := \Lambda^\mu{}_\nu(p)\varepsilon^{\nu(i)}(\sqrt{p^2},\vec{0}) = \Lambda^\mu{}_\nu(p)(0,\vec{e}^{(i)})^\nu$. In the chosen unitary gauge they obey $p^\mu\varepsilon^{(i)}_\mu(p) = 0$, $\varepsilon^{\mu(i)}(p)\varepsilon^{(j)}_\mu(p) = -\delta^{ij}$, and $\sum_i \varepsilon^{\mu(i)}(p)\varepsilon^{\nu(i)}(p) = -g^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}$. Based on these polarization vectors the Bosonic field operators for (anti)causal vector fields (respecting (anti)causal commutation relations) are easily introduced according to ($r = 1,\ldots,N =$ isospin index)

$$V^\mu_r(x) = \sum_j \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})}\varepsilon^{\mu(j)}(p)\big[e^{-ip\cdot x}a_r(p,j) + e^{ip\cdot x}a_r(-p,j)\big]\big|_{p^0=\omega(\vec{p})},$$

$$(V^\mu_r(x))^+ = \sum_j \int \frac{d^3p}{(2\pi)^3 2\omega^*(\vec{p})}\varepsilon^{\mu(j)}(p^*)\big[e^{ip^*\cdot x}a^+_r(p,j) + e^{-ip^*\cdot x}a^+_r(-p,j)\big]\big|_{p^0=\omega(\vec{p})}.$$

7 Norm conservation in QM, Klein–Gordon- and Dirac-theory

(Anti)causal KG/Dirac fields are decomposed into positive/negative complex frequency parts (e.g. $\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x)$ and $\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x)$). Subtraction of equations of motion for $\phi^{(\pm)}(x)$, $\psi^{(\pm)}$, $\psi(x)$ and respective adjoints[14] yields the following continuity equations: KG: $\partial_\mu[\phi^{(\mp)}(x)\partial^\mu\phi^{(\pm)}(x) - (\partial^\mu\phi^{(\mp)}(x))\phi^{(\pm)}(x)] = 0$, Dirac: $\partial_\mu[i\overline{\psi^{(\mp)c}}(x)\gamma^\mu\psi^{(\pm)}(x)] = 0$, Schrödinger: $i\partial_t[\tilde{\psi}(x)\psi(x)] + \frac{1}{2M}\overrightarrow{\nabla}\cdot[\tilde{\psi}(x)\overrightarrow{\nabla}\psi(x) - (\overrightarrow{\nabla}\tilde{\psi}(x))\psi(x)] = 0$. Note that all currents are conserved and in general non-zero, even for the neutral KG field! The observed norm conservation for (anti)causal KG & Dirac fields and Schrödinger wavefunctions is related to the (complex) energy conservation and hence related to the probability conservation! We conclude that the Schrödinger norm is $\int d^3x\,\tilde{\psi}(x)\psi(x)$, and *not* $\int d^3x|\psi(x)|^2$!

8 Charge conservation in QM, Klein–Gordon- and Dirac-theory

We will introduce simply charged (anti)causal systems according to the isospin concept. For the KG theory we define $\phi_\pm(x) := \big(\phi_1(x) \pm i\phi_2(x)\big)/\sqrt{2}$, for the Dirac theory we define $\psi_\pm(x) := \big(\psi_1(x) \pm i\psi_2(x)\big)/\sqrt{2}$, while for the Schrödinger theory we define $\psi_\pm(x) := \big(\psi_1(x) \pm i\psi_2(x)\big)/\sqrt{2}$ and $\tilde{\psi}_\pm(x) := \big(\tilde{\psi}_1(x) \pm i\tilde{\psi}_2(x)\big)/\sqrt{2}$[15]. Subtraction of causal equations of motion and the

[13]The problem of constructing (anti)causal vector fields is twofold. First one has to be aware that even a "massless" (anti)causal field has to be treated as if it were massive due to the at least infinitesimal imaginary part of its complex mass. That even non-Abelian massive vector fields can be treated consistently within QFT without relying on a Higgs mechanism has been shown by Jun-Chen Su in a renormalizable and unitary formalism [22]. Secondly, one has to be able to construct polarization vectors based on a boost of complex mass fields.

[14]KG: $(\partial^2 + M^2)\phi^{(\pm)}(x) = 0$, $\phi^{(\pm)}(x)(\overleftarrow{\partial}^{\,2} + M^2) = 0$; Dirac: $(i\not{\partial} - M)\psi^{(\pm)}(x) = 0$, $\overline{\psi^{(\pm)c}}(x)(-i\overleftarrow{\not{\partial}} - M) = 0$; Schrödinger: $i\partial_t\psi(x) = \Big(-\frac{1}{2M}\overrightarrow{\nabla}^{\,2} + V(x)\Big)\psi(x)$, $-i\tilde{\psi}(x)\overleftarrow{\partial}_t = \tilde{\psi}(x)\Big(-\frac{1}{2M}\overleftarrow{\nabla}^{\,2} + V(x)\Big)$.

[15]The neutral theory would follow by setting either $\phi_1(x)$ ($\psi_1(x)$, $\tilde{\psi}_1(x)$) or $\phi_2(x)$ ($\psi_2(x)$, $\tilde{\psi}_2(x)$) equal to zero.

respective adjoints[16] leads to the following continuity equations reflecting charge conservation: KG: $\partial_\mu[\phi_\mp(x)\partial^\mu\phi_\pm(x)-(\partial^\mu\phi_\mp(x))\phi_\pm(x)]=0$, Dirac: $\partial_\mu[i\overline{\psi^c_\mp}(x)\gamma^\mu\psi_\pm(x)]=0$, Schrödinger: $i\partial_t[\tilde{\psi}_\mp(x)\psi_\pm(x)]+\frac{1}{2M}\overrightarrow{\nabla}\cdot[\tilde{\psi}_\mp(x)\overrightarrow{\nabla}\psi_\pm(x)-(\overrightarrow{\nabla}\tilde{\psi}_\mp(x))\psi_\pm(x)]=0$. Note that currents and charges vanish for neutral KG and Dirac fields! The reason is a cancellation of underlying norm currents! The neutral Dirac field does *not* admit any *Abelian* gauge couplings due to $[\overline{\psi^c}(x)\not{A}(x)\psi(x)]^T=-\overline{\psi^c}(x)\not{A}(x)\psi(x)$, $[\overline{\psi^c}(x)\sigma^{\mu\nu}F_{\mu\nu}(x)\psi(x)]^T=-\overline{\psi^c}(x)\sigma^{\mu\nu}F_{\mu\nu}(x)\psi(x)$. The concept of local (non)Abelian gauge invariance in (A)CQT is sketched in the footnote[17].

9 Shadow fields – the Hermiticity content of the (anti)causal Klein–Gordon- and Dirac-theory

It is instructive to decompose an (A)CQT into its Hermitian components. Hermitian fields underlying non-Hermitian (anti)causal fields are here called "shadow fields" [8, 10][18]. Consider e.g. (anti)causal Lagrangians of neutral (anti)causal spin 0 Bosons and and spin 1/2 Fermions:

$$\begin{aligned}\mathcal{L}^0_\phi(x)&=\frac{1}{2}\big((\partial\phi(x))^2-M^2(\phi(x))^2\big)+\frac{1}{2}\big((\partial\phi^+(x))^2-M^{*2}(\phi^+(x))^2\big),\\ \mathcal{L}^0_\psi(x)&=\frac{1}{2}\left(\overline{\psi^c}(x)\left(\frac{1}{2}i\overleftrightarrow{\not\partial}-M\right)\psi(x)+\bar{\psi}(x)\left(\frac{1}{2}i\overleftrightarrow{\not\partial}-\bar{M}\right)\psi^c(x)\right).\end{aligned}\tag{2}$$

$\phi(x)$, $\phi^+(x)$, $\psi(x)$, $\psi^c(x)$ are decomposed in Hermitian shadow fields $\phi_{(1)}(x)$, $\phi_{(2)}(x)$, $\psi_{(1)}(x)$, $\psi_{(2)}(x)$ by $\phi(x)=:(\phi_{(1)}(x)+i\phi_{(2)}(x))/\sqrt{2}$, $\phi^+(x)=:(\phi_{(1)}(x)-i\phi_{(2)}(x))/\sqrt{2}$, and $\psi(x)=:(\psi_{(1)}(x)+i\psi_{(2)}(x))/\sqrt{2}$, $\psi^c(x)=:(\psi_{(1)}(x)-i\psi_{(2)}(x))/\sqrt{2}$, yielding the decomposed Lagrangians

$$\begin{aligned}\mathcal{L}^0_\phi(x)&=\frac{1}{2}\big((\partial\phi_{(1)}(x))^2-\mathrm{Re}[M^2](\phi_{(1)}(x))^2\big)-\frac{1}{2}\big((\partial\phi_{(2)}(x))^2-\mathrm{Re}[M^2](\phi_{(2)}(x))^2\big)\\&\quad+\mathrm{Im}[M^2]\phi_{(1)}(x)\phi_{(2)}(x),\\ \mathcal{L}^0_\psi(x)&=\frac{1}{2}\overline{\psi}_{(1)}(x)\left(\frac{1}{2}i\overleftrightarrow{\not\partial}-\mathrm{Re}[M]\right)\psi_{(1)}(x)-\frac{1}{2}\overline{\psi}_{(2)}(x)\left(\frac{1}{2}i\overleftrightarrow{\not\partial}-\mathrm{Re}[M]\right)\psi_{(2)}(x)\\&\quad+\frac{1}{2}\mathrm{Im}[M]\big(\overline{\psi}_{(2)}(x)\psi_{(1)}(x)+\overline{\psi}_{(1)}(x)\psi_{(2)}(x)\big).\end{aligned}\tag{3}$$

Note that Bosonic & Fermionic shadow fields are described by *principal value propagators* and *interact* with each other. One shadow field has *positive* norm, one has *negative* norm. If one would remove the interaction term, one would introduce interactions between causal and anticausal fields (e.g. $\phi(x)\phi^+(x)$) leading to a *violation of causality* and the *loss of analyticity* in QT.

10 Chiral symmetries in (anti)causal Dirac theory

Chiral rotations of (anti)causal fields. Consider the (anti)causal Lagrangian given by equation (2). Perform the following chiral rotation of (anti)causal fields: $\psi(x)\rightarrow\exp(i\gamma_5\alpha)\psi(x)$,

[16] KG: $(\partial^2+M^2)\phi_\pm(x)=0$, $\phi_\pm(x)(\overleftarrow{\partial}^{\,2}+M^2)=0$; Dirac: $(i\not\partial-M)\psi_\pm(x)=0$, $\overline{\psi^c_\pm}(x)(-i\overleftarrow{\not\partial}-M)=0$, Schrödinger: $i\partial_t\psi_\pm(x)=\left(-\frac{1}{2M}\overrightarrow{\nabla}^{\,2}+V(x)\right)\psi_\pm(x)$, $-i\tilde{\psi}(x)\overleftarrow{\partial}_t=\tilde{\psi}(x)\left(-\frac{1}{2M}\overleftarrow{\nabla}^{\,2}+V(x)\right)$.

[17] The Dirac Lagrangian with minimally coupled (non)Abelian gauge fields is given by the expression $\mathcal{L}(x)=\overline{\psi^c_+}(x)(\frac{1}{2}i\overleftrightarrow{\not\partial}+g\not{A}(x)-M)\psi_-(x)+\overline{\psi}_-(x)(\frac{1}{2}i\overleftrightarrow{\not\partial}+g^*\gamma^\mu A^+_\mu(x)-\bar{M})\psi^c_+(x)$ (with $\psi_\pm(x):=(\psi_1(x)\pm i\psi_2(x))/\sqrt{2}$). It is invariant under the local gauge transformations $g\not{A}'=g\not{A}+[\not\partial,\Lambda(x)]$, $\psi'_-(x)=\exp(i\Lambda(x))\psi_-(x)$, $\psi'_+(x)=\exp(-i(\Lambda(x))^T)\psi_+(x)$. Non-Abelian case: $A_\mu(x)=A^a_\mu(x)\lambda^a/2$ and $\Lambda(x)=\Lambda^a(x)\lambda^a/2$. *Non-Abelian* gauge fields admit minimal coupling even to neutral Fermions, *if* $[A^\mu(x)]^T=-A^\mu(x)$, as $[\overline{\psi^c}(x)\not{A}(x)\psi(x)]^T=\overline{\psi^c}(x)\not{A}(x)\psi(x)$ and $[\overline{\psi^c}(x)\sigma^{\mu\nu}F_{\mu\nu}(x)\psi(x)]^T=\overline{\psi^c}(x)\sigma^{\mu\nu}F_{\mu\nu}(x)\psi(x)$.

[18] Note that E.C.G. Sudarshan (1972) used the term "shadow state" with a different meaning!

$\psi^c(x) \to \exp(i\gamma_5\alpha)\psi^c(x)$. The resulting continuity-like equation is $\partial_\mu[\overline{\psi^c}(x)\gamma^\mu\gamma_5\psi(x)] \propto M$. Of course there exist the respective Hermitian conjugate/transposed continuity-like equations!

Chiral rotations of shadow fields. Consider the (anti)causal Lagrangian equation (3). Perform the following chiral rotation of the shadow fields: $\psi_{(1)}(x) \to \exp(i\gamma_5\alpha)\psi_{(1)}(x)$, $\psi_{(2)}(x) \to \exp(-i\gamma_5\alpha)\psi_{(2)}(x)$. The resulting continuity-like equation is $\partial_\mu[\overline{\psi}(x)\gamma^\mu\gamma_5\psi(x)] \propto \mathrm{Re}[M]$. Also here there exist the respective Hermitian conjugate/transposed continuity-like equations!

Chiral symmetry for massive Fermions. Define $\psi_R(x) := P_R\psi(x)$, $\psi_L(x) := P_L\psi(x)$, $P_R := (1+\gamma_5)/2$, $P_L := (1-\gamma_5)/2$. Define also $\chi_\pm(x) := (\psi_R(x) \pm i\psi_L(x))/\sqrt{2}$. Consider the (anti)causal Lagrangian equation (2) in the new fields, i.e.

$$\begin{aligned}\mathcal{L}^0_\psi(x) &= \frac{1}{2}\left\{\overline{\psi^c_L}(x)\frac{i}{2}\overset{\leftrightarrow}{\partial}\!\!\!\!\!/\,\psi_R(x) + \overline{\psi^c_R}(x)\frac{i}{2}\overset{\leftrightarrow}{\partial}\!\!\!\!\!/\,\psi_L(x) - M\big(\overline{\psi^c_R}(x)\psi_R(x) + \overline{\psi^c_L}(x)\psi_L(x)\big)\right\} + \text{h.c.}\\ &= \frac{1}{2}\left\{\overline{\chi^c_+}(x)\frac{1}{2}\overset{\leftrightarrow}{\partial}\!\!\!\!\!/\,\chi_+(x) - \overline{\chi^c_-}(x)\frac{1}{2}\overset{\leftrightarrow}{\partial}\!\!\!\!\!/\,\chi_-(x) - M\big(\overline{\chi^c_+}(x)\chi_-(x) + \overline{\chi^c_-}(x)\chi_+(x)\big)\right\} + \text{h.c.}\end{aligned}$$

Note that the Lagrangian is invariant under the chiral rotations $\chi_\pm(x) \to \exp(\pm i\gamma_5\alpha)\chi_\pm(x)$ even for *arbitrary complex Fermion mass* M!

11 New antiparticle concept and the intrinsic parities of anti-Bosons and anti-Fermions

In (A)CQT antiparticles are isospin partners of particles[19]. This holds for Bosons and Fermions. It leads – like in the Bosonic case – to the fact that the anti-Fermions have the same intrinsic parity as the Fermions[20]. The components of Bosonic or Fermionic field operators characterized by phasefactors with "negative" (complex) frequency, i.e. $\exp(-i\omega(\vec{k})t)$ and $\exp(-i\omega^*(\vec{k})t)$, are responsible for the annihilation of (anti)particles and respective (anti)holes, and not for their creation. The traditional identification of annihilation operators of negative frequency states with antiparticles gets lost for fields described by a complex mass with a finite imaginary part. In this situation traditional HQT – in contrast to (A)CQT – ceases to be applicable.

12 Conjugate T-matrix $\overline{T}_{fi}$, dual vacuum, transition probabilities and (anti)causal cross sections

As $|\psi(x)|^2$ is not a probability density in (anti)causal Schrödinger theory, $|T_{fi}|^2$ is *not* to be interpreted as a *transition probability* in (anti)causal scattering theory! In (anti)causal scattering theory we have instead to consider a quantity $\overline{T}_{fi}T_{fi}$, where $\overline{T}_{fi}$ ($\neq T^+_{fi}$) is called the *conjugate T-matrix*. The construction of the explicit analytical expression for the conjutate T-matrix $\overline{T}_{fi}$ showed up to be a non-trivial task. For brevity we want to give here the final result without proof. We assume the causal T-matrix T_{fi} to be determined by the standard expression $(2\pi)^4\delta^4(P_f - P_i)iT_{fi} = \langle\langle 0|\mathcal{A}(\vec{p}\,'_{N_f})\cdots\mathcal{A}(\vec{p}\,'_1)T[\exp(iS_{\rm int})-1](\mathcal{C}(\vec{p}_1))^+\cdots(\mathcal{C}(\vec{p}_{N_i}))^+|0\rangle_c$ with $\mathcal{A}(\vec{p}\,'_j) \in \{a(\vec{p}\,'_j), b(\vec{p}\,'_j)\}$ and $\mathcal{C}(\vec{p}_j) \in \{c(\vec{p}_j), d(\vec{p}_j)\}$. Call N_F the overall number of Fermionic operators in the initial and final state. Then the conjugate causal T-matrix $\overline{T}_{if}$ is given by:

$$\begin{aligned}&(2\pi)^4\delta^4(P_f - P_i)(-i)\overline{T}_{if}\\ &\quad = \langle\langle\bar{0}|\big(\mathcal{A}(\vec{p}_{N_i})\cdots\mathcal{A}(\vec{p}_1)T[\exp(-iS_{int})-1](\mathcal{C}(\vec{p}\,'_1))^+\cdots(\mathcal{C}(\vec{p}\,'_{N_f}))^+\big)^T|\bar{0}\rangle_c\end{aligned}$$

[19] Charged pions (π^+, π^-) are e.g. represented by the isospin combination $\pi_\pm(x) = (\pi_1(x) \pm i\pi_2(x))/\sqrt{2}$, while the positron ($e^+$) and electron ($e^-$) are represented by $e_\pm(x) = (e_1(x) \pm ie_2(x))/\sqrt{2}$. $\pi_1(x)$, $\pi_2(x)$ or $e_1(x)$, $e_2(x)$ are non-Hermitian fields describing, respectively, pairs of causal neutral particles with equal complex mass.

[20] In spite of this feature (A)CQT reproduces exactly the high precision results of Quantum Electrodynamics.

$$\stackrel{!}{=} (-1)^{N_F(N_F-1)/2}\langle\!\langle\bar{0}|(\mathcal{C}(\vec{p}\,'_{N_f}))^+\cdots(\mathcal{C}(\vec{p}\,'_1))^+\big(T[\exp(-iS_{\rm int})-1]\big)^T\mathcal{A}(\vec{p}_1)\cdots\mathcal{A}(\vec{p}_{N_i})|\bar{0}\rangle_c$$
$$= (-1)^{N_F(N_F-1)/2}\langle\!\langle\bar{0}|(\mathcal{C}(\vec{p}\,'_{N_f}))^+\cdots(\mathcal{C}(\vec{p}\,'_1))^+\overline{T}[\exp(-iS_{\rm int}^T)-1]\mathcal{A}(\vec{p}_1)\cdots\mathcal{A}(\vec{p}_{N_i})|\bar{0}\rangle_c$$
$$\stackrel{!}{=} (-1)^{N_F(N_F-1)/2}\langle\!\langle\bar{0}|(\mathcal{C}(\vec{p}\,'_{N_f}))^+\cdots(\mathcal{C}(\vec{p}\,'_1))^+\overline{T}[\exp(-iS_{\rm int})-1]\mathcal{A}(\vec{p}_1)\cdots\mathcal{A}(\vec{p}_{N_i})|\bar{0}\rangle_c$$

with $\mathcal{A}(\vec{p}_j)\in\{a(\vec{p}_j),b(\vec{p}_j)\}$ and $\mathcal{C}(\vec{p}\,'_j)\in\{c(\vec{p}\,'_j),d(\vec{p}\,'_j)\}$ and $|\bar{0}\rangle$ (and $\langle\!\langle\bar{0}|$) being the *dual vacuum* annihilating creation operators and creating annihilation operators. We used the useful transposition identity $\big(T\big[\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\big]\big)^T=\overline{T}\big[(\mathcal{O}(x_1))^T\cdots(\mathcal{O}(x_n))^T\big]$ holding for Bosonic and Fermionic operators. $\overline{T}_{fi}T_{fi}$ for a causal process (to be identified with the transition probability) and therefore also the respective causal cross section are not necessarily real numbers[21].

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[21] Only if the underlying theory represented by a causal Lagrangian or causal Hamiltonian is probability conserving, i.e. non-absorptive, $\overline{T}_{fi}T_{fi}$ and therefore also the causal cross-section will be quasi-real, i.e. infinitesimally close to a real number. Hence, for selective (so called "inelastic") causal processes in probability non-conserving theories (e.g. open quantum systems) the respective causal cross-sections will develop a finite imaginary part. Within particle physics this new feature complements in a beautiful manner, what is well understood for a long time in theoretical optics, i.e. that the imaginary part of the refractive index of a material is related to its absorption coefficient. Similar arguments hold for anticausal processes and respective anticausal cross-sections.

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Coarse-Grained Structure of a Physical (Strange) Attractor. Analytical Solution

Maria K. KOLEVA

Institute of Catalysis, Bulgarian Academy of Science, 1113 Sofia, Bulgaria
E-mail: *mkoleva@bas.bg*

The structure of the physical and strange attractors is inherently associated with the boundedness of fluctuations. The idea behind the boundedness is that a stable long-term evolution of any natural and engineered system is possible if and only if the fluctuations that the system exerts are bounded so that the system permanently stays within its thresholds of stability. It has been established that the asymptotic structure of the physical and strange attractors is identical. Now it is found out that though the non-asymptotic behavior is universal it can be very different, namely: on coarse-graining the physical attractors can exhibit a variety of behavior while the strange attractors always have hyperuniversal properties. Yet, under certain levels of coarse-graining both physical and strange attractors match non-asymptotically a variety of noise type behavior.

1 Introduction

The notion of a physical attractor is inherently consistent with the notion of the boundedness of fluctuations. The idea behind the boundedness is that a stable long-term evolution of any natural and engineered system is possible if and only if the fluctuations that the system exerts are bounded so that the system permanently stays within its thresholds of stability. Thus, the thresholds of stability match the boundaries of a finite volume in the phase space so that every phase trajectory is confined in that volume called physical attractor [1]. Further, we have proved [1] that any bounded irregular sequence (BIS) has certain asymptotic properties insensitive to the increment statistics. These properties are called $1/f^{\alpha}$-type noise. They are: (a) the power spectrum comprises a continuous band that uniformly fits the shape $1/f^{\alpha(f)}$ where $\alpha(f)$ monotonically increases starting from 1 at $f = 1/T$ (T is the length of the sequence) up to p as f approaches infinity (p is arbitrary but $p > 2$); (b) the physical (strange) attractor is non-homogeneous; (c) the Kolmogorov entropy is finite. It has been proven also that the strange attractors that are related to the simulated dynamical systems share the above properties.

The present task is to reveal the influence of the increment statistics onto the properties of the bounded irregular sequences after coarse-graining. The coarse-graining is an inevitable part of recording data at every experiment. Therefore, it is important to elucidate how it impacts the experimentally recorded time series. Taking into account that a long-term boundedness is possible if and only if the size of correlations among increments does not exceed a certain finite value, it is to be anticipated that the coarse-grained structure of any BIS is universal, i.e. insensitive to the increment statistics. Indeed, an arbitrary increase of the increment correlation size certainly makes the amplitude of fluctuations to exceed the thresholds of stability. To the most surprise, it turns out that on the coarse-graining there are 3 types of non-asymptotic behavior of every BIS: (i) the low level whose major property is that the structure of any BIS matches the white noise behavior; (ii) the meso-level where the sequence of large-scaled fluctuations matches telegraphic noise behavior. At the same time, the large-scaled fluctuations retain a certain specific property related to the increment statistics; (iii) macro-level where the properties of any BIS are hyperuniversal, namely insensitive to any detail of the increment

statistics. It should be stressed that the hyperuniversal properties on that level of coarse-graining are due to the fact that under coarse-graining the symmetric random walk with a constant step appears as the global attractor for any fractal Brownian motion. This is the subject of § 3. On the other hand, it should be stressed that the $1/f^{\alpha}$-type noise properties listed above remain unaffected by the coarse-graining. Thus, both white noise and telegraphic noise behavior are rather alias effects: they are typical for sequences whose length does not exceed a certain one. The asymptotic behavior of every BIS, regardless to the level of coarse-graining, is always covered by the $1/f^{\alpha}$-type noise properties established in [1].

The above separation is made on the grounds of certain Poissonian properties of the large-scaled fluctuations appearance. The particularity of that behavior is imposed by the boundedness and the lack of long-range correlations among increments. More precisely, the lack of long-range correlations among increments ensures a uniform convergence of the average to the mean of any BIS. In turn, it provides the stationarity of the large-scaled fluctuations appearance. The existence of mean and average for any BIS is guaranteed by the Lindeberg theorem that states: each BIS has a finite mean and a finite variance regardless to the increment statistics [2]. The interplay between all these properties renders that each large-scaled fluctuation can be approximated by an excursion: a trajectory of a walk originating at the mean value of a given sequence at time t and returning to it for the first time at time $t + \Delta$. This interval is called the next duration of an excursion. The distinctive property of each excursion is that the random walk of the increment renders certain relation between the amplitude and the duration of the excursion. In § 2 it is worked out that the boundedness provides each excursion to be loaded in a specific "embedding interval". The length of that interval is exerted as a random choice from a certain range of almost equiprobable values. The presence of embedding time intervals is the warrant of the "pulse"-like behavior of any BIS, namely: the BIS is a succession of excursions separated by quiescent intervals whose lengths vary with the "time". It should be stressed that this behavior is robust to the coarse-graining and it is natural on the quantum level [3]. The importance of the "pulse"-like behavior is twofold: (i) it provides that the fluctuations remain bounded at any scale of averaging, respectively at any level of coarse-graining; (ii) each BIS remains a BIS on coarse-graining and thus ensures the robustness to coarse-graining of the $1/f^{\alpha}$-type noise properties.

2 "Embedding" intervals

The target of the present and the next section are the properties of the pulse-like behavior of the coarse-grained BIS's. We start with the excursion embedding. Further it is proven that this property is imposed by the boundedness and is insensitive to the increment statistics. The presence of the embedding intervals renders that any successive excursions are separated by non-zero quiescent intervals. This is the key property for the pulse-like performing of the excursion sequences. The present task is to establish the relation between the duration of the embedding intervals and the duration of the corresponding excursions. That relation is based on the notion of an excursion: a trajectory of a walk originating at the mean value of a given sequence at time t and returning to it for the first time at time $t + \Delta$. Therefore, the probability for an excursion of duration Δ is determined by the degree of correlation between any two points of a sequence. On the other hand, the probability that any two points of a sequence, separated by a distance η, have the same value is given by the autocorrelation function $G(\eta)$. A generic property of the BIS's [1] is that the autocorrelation function of any of them can be defined for sequences of arbitrary but finite length T. Yet, its shape is universal, namely:

$$G(\eta, T) \propto 1 - \left(\frac{\eta}{T}\right)^{\nu(\eta/T)}.$$

where $\nu\left(\frac{\eta}{T}\right)$ is an everywhere continuous function monotonically decreasing between the following limits:

$$\nu\left(\frac{\eta}{T}\right) \to p-1 \qquad \text{as} \qquad \frac{\eta}{T} \to 0,$$
$$\nu\left(\frac{\eta}{T}\right) \to 0 \qquad \text{as} \qquad \frac{\eta}{T} \to 1.$$

Then, the probability that an excursion of duration Δ happens in an interval T reads:

$$P(\Delta,T) = \frac{1}{T}\int_0^{\Delta} G(\eta,T)\,d\eta = \int_0^{\Delta/T}\left(1-\varepsilon^{\nu(\varepsilon)}\right)\,d\varepsilon.$$

By using the standard calculus [3], the integration of any power function of non-constant exponent $\alpha(x)$ ($\alpha(x)$ is an everywhere continuous function) reads:

$$J = \int_a^b x^{\pm\alpha(x)}\,dx = \frac{b^{1\pm\alpha(b)}}{1\pm\alpha(b)} - \frac{a^{1\pm\alpha(a)}}{1\pm\alpha(a)}.$$

Thus,

$$P(\Delta,T) = \frac{\Delta}{T}\left(1-\left(\frac{\Delta}{T}\right)^{\nu(\Delta/T)}\right).$$

The $P(\Delta,T)$ dependence only on the ratio Δ/T verifies that each excursion is "embedded" in a certain interval so that no other excursion happens in that interval.

The next step is to work out the shape of $P(\Delta,T)$. Its role is crucial for the behavior of the excursion sequence. To elucidate this point let us consider two extreme cases:

- $P(\Delta,T)$ is a sharp single-peaked function. Then, it ensures a single value of the most probable ratio $\frac{\Delta}{T}$. In other words, the duration of the most probable embedding interval associated with an excursion of duration Δ_0 has a single value T_0 such that $\frac{\Delta_0}{T_0}$ is the peak value of $P(\Delta,T)$.
- $P(\Delta,T)$ has a gently sloping maximum. Then, the relation between Δ and T behaves as a multi-valued function: a range of nearly equiprobable values of T corresponds to each most probable Δ. In the course of the time this produces a variability of the duration of the "embedding" intervals even when the sequence comprises identical excursions. Then, the excursion sequence always exhibits a non-periodic ("pulse"-like) behavior.

The establishing of the particular shape of $P(\Delta,T)$ requires the knowledge about the explicit shape of $\nu\left(\frac{\Delta}{T}\right)$. Next it is worked out on the grounds of the proof that neither BIS sustained to an arbitrary length comprises any long-ranged increment correlations. The general restriction on the increment correlation size requires a uniform contribution to the power spectrum of all scales, i.e. there are no "special" frequencies at the power spectrum.

The only factor that can modify the power spectrum is the non-constant exponent $\alpha(f)$ of its shape $1/f^{\alpha(f)}$. The boundedness requires the monotonic decay of $\alpha(f)$ in the limits $(1,p]$ not specifying its shape [1]. Our task now is to establish the shape(s) of $\alpha(f)$ that fits the lack of long-range increment correlations. By virtue of the strict monotony of the power spectrum the required criterion is that neither any of its components nor any of its derivatives of an arbitrary order has a specific contribution. Simple calculations yield that it is achieved if and only if the shape of $\alpha(f)$ is the linear dependence, namely:

$$\alpha(f) = 1+\gamma f. \tag{1}$$

Equation (1) provides that the $1/f^{\alpha(f)}$ derivative of an arbitrary order has the same sign throughout the entire frequency interval of the power spectrum. On the contrary, any non-linear $\alpha(f)$ makes each order derivative of $1/f^{\alpha(f)}$ change its sign at certain frequencies. Thus, only the linear dependence does not introduce any additional scale to those inherent for the increment statistics.

Because of the diffeomorfism between $\alpha(f)$ and $\nu\left(\frac{\Delta}{T}\right)$ the shape of the latter reads:

$$\nu\left(\frac{\Delta}{T}\right) = (p-1)\left(1-\frac{\Delta}{T}\right).$$

The plot of $P(\Delta, T)$ with the above shape of $\nu\left(\frac{\Delta}{T}\right)$ shows that it has a gently sloping maximum: indeed, the values of $P(\Delta, T)$ in the range $\frac{\Delta}{T} \in [0.25, 0.4]$ vary by less than 7 %. Outside this range $P(\Delta, T)$ decays sharply. Thus, though $P(\Delta, T)$ is a single-valued function, it provides a multi-valued relation between the most probable values of Δ and T, namely: a certain range of nearly equiprobable values of T is associated with each Δ. In the course of the time the multi-valued relation is exerted as a random choice of the duration of the "embedding" intervals even when the sequence comprises identical excursions. Thus, the multi-valued relation between Δ and T provides that any coarse-grained BIS is again a BIS. In turn, this proves that the $1/f^{\alpha}$-type noise properties are robust to coarse-graining.

3 Symmetric random walk as the global attractor for the fractal Brownian motion

Every BIS comprises fine-structure fluctuations superimposed on the large-scaled ones. The former are created by the short-range increment walk and the latter appear as successive excursions from the mean value of the BIS. It is to be expected that the fine structure fluctuations strongly depend on the particularities of the increment statistics while the large scaled ones have certain generic properties robust to that statistics. One such property is that there is a certain relation between the amplitude A of each fluctuation and its duration Δ. Next it is worked out for the case when the increment size is much smaller than the thresholds of stability. Then the fluctuations can be approximated by excursions created by continuous fractal Brownian motion of the increment walk in the course of the time. The latter renders certain relation between the amplitude and the duration of the excursion, namely: $\sqrt{\langle A^2 \rangle} \propto \Delta^{\beta(\Delta)}$, β is set by the particularity of the increment statistics; the averaging is over the sample realizations. The dependence of β on Δ comes from the interplay between the finite radius of the increment correlations and the amplitude of the excursion itself that is limited only by the thresholds of stability. Next it is proven that each time when $a \ll A$ the fractal Brownian motion uniformly approaches the symmetric random walk with a constant step. Indeed, since the increment size is permanently bounded, the mean square deviations (m.s.d.) of all the trajectories of the same number of steps N is confined in a finite range. Therefore, these m.s.d. also form a BIS. Further, according to the Lindeberg theorem [2] the latter has a finite mean and a finite variance and thus it is a subject to the Central Limit Theorem. As a result, the m.s.d. are Gaussianly distributed. In turn, the sizes of the successive trajectories of a given are predominantly equal to the corresponding mean. So, the increment correlations are to be associated with a single trajectory whose number of steps m is related to the increment correlation size and the exponent $\tilde{\beta}$ is specific to the increment statistics. In other words, the increment correlations create "blobs" whose size is $\sqrt{\langle a^2 \rangle} \propto m^{\tilde{\beta}}$. Then, the large excursions can be approximated by a symmetric random walk with constant step equal to the size of the blobs. Thus the dependence of any large scale excursions on its duration reads:

$$\sqrt{\langle A^2 \rangle} \propto N^{0.5} m^{\tilde{\beta}}.$$

It is obvious that when $N \gg m$ the dependence tends to:

$$\sqrt{\langle A^2 \rangle} \propto N^{0.5} a,$$

where a is considered a constant independent of N. So, the symmetric random walk with constant step appears as the global attractor for any fractal Brownian motion.

It is worth noting that the finite size of the "blobs" ensures the uniform convergence of the average to the mean of the original BIS. Indeed, the distinctive property of any fractal Brownian is that any power $\beta \neq 0.5$ comes from certain correlation between the current increment μ_i and the corresponding step τ_i. Then the average is:

$$A = \sum_{i=1}^{N} \mu_i(\tau_i)\tau_i = \sum_{i=1}^{N} (-1)^{\gamma_i} \tau_i^{\beta_i} \tag{2}$$

and correspondingly the m.s.d.:

$$\langle A^2 \rangle \propto \left\langle \sum_{i=1}^{N} (\mu_i(\tau_i)\tau_i)^2 \right\rangle = \left\langle \sum_{i=1}^{N} \tau_i^{2\beta_i} \right\rangle,$$

where the averaging is over the different samples of the trajectory. The property of the above relations is that whenever the probabilities for γ_i odd and even are not permanently equal there is a correlation between the increment and the corresponding step. So A is certainly non-zero which immediately makes the deviation from the mean non-zero. Moreover, equation (2) yields that A can become arbitrarily large on increasing N. On the contrary, a permanent equal probability for γ_i odd and even implies independence from one another of the increments and the steps. It yields $A = 0$ which guarantees the uniform convergence of the average to the mean.

4 Statistics of the coarse-grained BIS's

Our first task is to elucidate that the excursion appearance meets the three generic features of a Poissonian process, namely: (i) the excursion appearance is a stationary process; (ii) the successive excursions are independent of one another events; (iii) no more than one event can be developed at any instant. The stationarity of the excursion sequences is guaranteed by the following interplay between the boundedness and the lack of any long-ranged increment statistics. Indeed, the Lindeberg theorem [2] states that any BIS has a finite mean and a finite variance insensitively to the increment statistics. Obviously, the coarse-graining affects neither the presence nor the value of the mean. Further, the lack of long-ranged increment correlations guarantees the uniform convergence of the average to the mean. This, in turn, verifies the stationarity of the excursion appearance. This along with the embedding of each excursion in a larger interval is a warrant that the successive excursions are independent of one another events.

On the other hand, both the boundedness and the lack of long-ranged increment correlations render that every BIS is subject to the central limit theorem. So, the departure from the mean obeys the Gaussian distribution. Our task now is to elucidate the interplay between the Poissonian properties and the Gaussian ones under the coarse-graining. It is obvious that the coarse-graining makes all the excursions of the amplitude less than the size of coarse-graining A_{cgr} contribute to the quiescent intervals. Then, the excursion of a given amplitude appears in the time course with the probability that reads:

$$P(A) = \frac{A^{1/\beta(A)} \exp\left(-A^2/A_\sigma^2\right)}{(1+\eta) A^{1/\beta(A)} \exp\left(-A^2/A_\sigma^2\right) + \int_0^{A_{\mathrm{cgr}}} A^{1/\beta(A)} \exp\left(-A^2/A_\sigma^2\right)\, dA}. \tag{3}$$

The required probability $P(A)$ is given by the ratio between the weighted duration $A^{1/\beta(A)} \times \exp\left(-A^2/A_\sigma^2\right)$ of an excursion of amplitude A and the weighted length of the corresponding embedding interval. The latter comprises the weighted duration of the excursion itself, the weighted duration of the natural embedding interval expressed through η and the weighted duration of the quiescent interval due to the coarse-graining $\int_0^{A_{\rm cgr}} A^{1/\beta(A)} \exp\left(-A^2/A_\sigma^2\right)\, dA$; A_σ is the variance of the BIS and in the present consideration it is a parameter. The Poissonian properties of the excursion appearance ensure that $P(A)$ holds only for $A \geq A_{\rm cgr}$. Otherwise $P(A) \equiv 0$ since all the excursions of amplitude less than the size of coarse-graining $A_{\rm cgr}$ are "smoothed out" and contribute to the quiescent intervals.

$P(A)$ exhibits 3 different types of behavior with respect to the ratio A/A_σ. In the case $A \ll A_\sigma$ the following approximation holds:

$$J(A_{\rm cgr}) = \int_0^{A_{\rm cgr}} A^{1/\beta(A)} \exp(-A^2/A_\sigma^2)\, dA \propto A_{\rm cgr}^{1+1/\beta(A_{\rm cgr})}.$$

In the case when $A_{\rm cgr} \leq A \ll A_\sigma$ it yields:

$$P(A) \approx \frac{1}{1 + A/A_\sigma}.$$

So, in this case the fluctuations of different amplitudes are equiprobable which is a distinctive property of the white noise. Therefore, the BIS behaves as a white noise sequence. Yet, a warning is necessary: this behavior is available up to a certain finite length of the sequence. That length is proportional to the inverse of the probability for appearance of an excursion of amplitude A. So, it is of the order of $\exp\left(-\frac{A^2}{A_\sigma^2}\right)$. The asymptotic behavior of the sequence is covered by the $1/f^\alpha$-type noise properties listed in the Introduction.

The other extreme case is $A_\sigma \gg A_{\rm cgr}$. The corresponding approximation of $J(A)$ reads:

$$J(A) \approx \text{const}.$$

Then for $A_{\rm cgr} \leq A \gg A_\sigma$ $P(A)$ becomes:

$$P(A) \propto A^{1/\beta(A)} \exp\left(-\frac{A^2}{A_\sigma^2}\right). \tag{4}$$

In this case the excursion sequence behavior matches telegraphic noise, namely the excursions are almost identical separated by large quiescent intervals. An immediate result of (4) is that the amplitude of these excursions is set by the most probable one, namely $A = A_{\rm cgr}$. Thus, the length of the sequence that matches the telegraphic noise behavior is proportional to $\exp\left(-\frac{A_{\rm cgr}^2}{A_\sigma^2}\right)$. The asymptotic behavior of the sequence is again covered by the $1/f^\alpha$-type noise properties listed in the Introduction. It is worth noting that this type of telegraphic noise differs from the dichotomic one at the power spectrum shape. The dichotomic noise has an exponential power spectrum while the telegraphic noise that comes from a BIS has the power spectrum of the shape $1/f^{\alpha(f)}$. The exponential dependence in the former case comes from the second kind discontinuities while only first kind discontinuities are available for the bounded sequences.

The behavior of any coarse-grained BIS is inherently related to the incremental statistics trough the explicit dependence of $P(A)$ on $\beta(A)$ in (3). However, when the size of coarse-graining is much larger than the size of the symmetric random walk $\beta(A)$ turns constant equal to 0.5. Then, $P(A)$ gets insensitive to the details of the increment statistics and all the characteristics of the coarse-grained BIS become hyperuniversal, i.e. insensitive to any particularity of the increment statistics. The hyperuniversal behavior of any BIS appears whenever $a \ll A_{\rm cgr}$. Since a is finite, there is always a large enough size of coarse-graining such that $a \ll A_{\rm cgr}$.

So, at large enough sizes of coarse-graining the behavior of the BIS is to be hyperuniversal. It is worth noting, however, that since a and A_σ are independent parameters, it is possible that $a \ll A_\sigma$. Then the behavior of the corresponding BIS is hyperuniversal at almost every level of coarse-graining.

These considerations yield an apparent distinction between the physical attractors and the strange attractors that originate from the simulated dynamical systems. The strange attractors arise at unstable solutions where the inevitable round-off at simulation is amplified due to a positive Lyapunov coefficient [1]. Since the round-off to the higher and the lower value is permanently equiprobable, the power β at the corresponding fractal Brownian motion is 0.5 for the excursions of all sizes. Hence, the strange attractors have a hyperuniversal structure at every level of coarse-graining. On the contrary, the physical attractors exhibit a variety of behavior under the coarse-graining.

5 Conclusion

The major physical result of the present work is that the boundedness is incompatible with long-range increment correlations. This result gives rise to several very important consequences:

(i) $1/f^\alpha$-type noise properties of each BIS are scale-free and robust to coarse-graining;

(ii) pulse-like behavior is also typical for every BIS and is robust to coarse-graining. Being natural on the quantum level [3], it is self-reproduced on every level of coarse-graining through the existence of embedding intervals introduced by the boundedness and the coarse-graining. The major feature of the embedding introduced by the boundedness is that its length is a random choice from a certain range at any point of the sequence. In turn, the variability of that length ensures the irregularity of every BIS, even of that comprising identical excursions. This lays a base to the insensitivity of the $1/f^\alpha$-type noise properties to the coarse-graining;

(iii) the symmetric random walk with a constant step appears as the global attractor for any fractal Brownian motion. In turn, the size of that step provides the scale beyond which both the structure and the succession of the excursions are insensitive to the incremental statistics. Below that scale these properties are affected by the incremental statistics through the power exponent β of the fractal Brownian motion that approximates the excursions.

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Computer Algebra Study of Spectral Invariants of Differential Operators on Curved Manifolds

Vladimir V. KORNYAK

Joint Institute for Nuclear Research, 141980 Dubna, Russia
E-mail: *kornyak@jinr.ru*

We consider asymptotic heat kernel expansion for elliptic differential operators acting on compact closed curved manifolds. The coefficients in this expansion are quantities of fundamental importance in quantum field theory, quantum gravity, spectral geometry and topology of manifolds. Obtaining explicit expressions for these quantities is very laborious task, especially in the problems of modern physics studying complicated operators (high order and *nonminimal*) in complicated geometric environment (in the presence of torsion and gauge fields in addition to the Riemannian curvature tensor). In fact, the task cannot be accomplished without computer algebra tools. In this paper we describe a covariant algorithm for computing the heat kernel coefficients. With the help of a ***C*** implementation of the algorithm some new results were obtained. The most significant of them concern nonminimal operators and manifolds with torsion.

1 Introduction

Determination of the internal structure of an object via the spectra of different radiations and waves around the object is one of the archetypal problems in physics. Particular instances of this problem arise practically in all fields of physics ranging from experimental physics of elementary particles to problems in seismology and medical tomography. More restricted but more precise mathematical version of this problem may be formulated as follows. A manifold (bundle) equipped with such structures as metric, curvature, torsion, gauge fields etc. and an elliptic (pseudo)differential operator acting on this manifold are given. What information about the manifold can one obtain by studying the spectral properties of the operator? M. Kac phrases the problem in an evocative title of his paper "Can one hear the shape of a drum?". Historically first studies were devoted mainly to the Laplace operator acting on a Riemannian manifold and for a while there was even impression that geometrical properties of the manifold can be completely restored by the spectrum. Only in 1964, J. Milnor [1] found the first counter-example, namely, a pair of 16-dimensional tori with different (flat) Riemannian metrics but with identical spectra of the Laplace operator acting on the tori. Later on many similar examples of *multiply connected* isospectral manifolds have been constructed, and recently, in 1999, D. Schüth [2] constructed continuous isospectral families of metrics on the product of spheres $S^4 \times S^3 \times S^3$, the first example of closed *simply connected* isospectral but non-isometric Riemannian manifolds.

Nevertheless, many important properties of manifold can be restored by the spectrum of elliptic operator acting on it. In particular, such global geometric invariants as dimension, volume, and total scalar curvature, are known to be spectrally determined. Moreover, various manifolds such as round spheres of dimension ≤ 6 and 2-dimensional flat tori are uniquely determined by the spectra of the Laplacian.

Whereas the classical spectral geometry deals mainly with Laplace operator acting on Riemannian manifold, the needs of modern physics force to consider similar problems for operators more general than Laplacian and acting on manifolds more general than Riemannian.

Approaches, based on studying *asymptotical* properties of spectrum, make the problem available for computational methods. The main and most constructive approach consists in

investigation of the *heat kernel expansion.* This approach can be described briefly as follows. Starting with an elliptic operator A of the order $2r$, acting on a bundle whose base is a compact close n-dimensional manifold M, and introducing an additional "temporal" variable t one can construct the *heat* operator $A-\frac{\partial}{\partial t}$. Then one can compute the short-time asymptotic expansion of the diagonal elements of the kernel of this heat operator:

$$\langle x|e^{-tA}|x\rangle \sim \sum_{m\geq 0} E_m(x|A)t^{\frac{m-n}{2r}}, \qquad t\to +0. \tag{1}$$

The coefficients $E_m(x|A)$ in this expansion are spectral invariants of the operator A, and encode information about the asymptotic properties of the spectrum. The E_m's are of fundamental importance in spectral geometry and topology of manifolds, mathematical physics, quantum field theory and quantum gravity. These coefficients are called the *heat invariants* or *heat kernel coefficients*. They are also widely known under the (different combinations of) names Hadamard, DeWitt, Seeley, Gilkey, according to papers of these authors [3–6]. Hereinafter we shall use the term *DeWitt–Seeley–Gilkey (DWSG) coefficients*, since these authors pay special attention to the important for us differential geometric (covariant) aspect of the problem.

Computation of the DWSG coefficients in the case of curved space-time includes complicated manipulations with tensor expressions, integrations, leading in many cases to expressions in terms of hypergeometric and gamma functions, and simplification of these expressions. To cope with these difficulties we elaborated (and implemented in $\boldsymbol{C}$) an algorithm [7].

With the help of our program we succeed in obtaining some new results. Most substantial and difficult among them is computation of complete expression [8] of E_4 for so-called *nonminimal* operator important in the quantum theory of gauge fields and quantum gravity. Note that just this coefficient is most important for the physical 4-dimensional space-time in view of the *Atiyah–Bott formula* [9,10] expressing the index of elliptic operator A on n-dimensional manifold in terms of the coefficient E_n. Moreover, we obtained some new results for higher order operators and for manifolds with torsion. In particular, the spectral invariant E_2 for nonminimal operator on manifold with torsion has been computed in [11].

The problems with torsion are especially difficult from the computational standpoint and we focused special attention on them. The torsion is defined as antisymmetric part of affine (or linear) connection $T^{\lambda}{}_{\mu\nu}=\Gamma^{\lambda}{}_{\nu\mu}-\Gamma^{\lambda}{}_{\mu\nu}$, where $\Gamma^{\lambda}{}_{\mu\nu}$ are the coefficients of connection. The Einstein's General Relativity is based on a special connection called *Levi-Civita* connection, i.e., *symmetric* and compatible with metric affine connection. This torsionless connection can be expressed completely in terms of metric. The General Relativity describes well the interaction of the matter with the gravity as far as macroscopic bulk matter is considered. However, on the microscopic level, it seems reasonable to take into account the influence of spin on the geometry of space-time. To describe the interaction of spinning particles with the gravitation, a gravitation theory should include the non-vanishing torsion. In 1922 Elie Cartan first pointed out that there is no *a priori* reason to assume an affine connection to be symmetric in the context of General Relativity. He proposed also a theory of gravitation with torsion which is known now as the Einstein–Cartan theory. The torsion arises naturally in the different (based on Poincaré and affine groups) gauge theories of gravity developed in the recent years. Moreover, all kinds of modern superstring theories also predict the existence of torsion.

2 Operators on manifolds and applications of heat kernel

Let us consider a compact closed n-dimensional manifold M equipped with Riemann metric g and linear (or affine) connection defined in the bundle of linear (affine) frames over M. We assume that the connection (locally expressed in terms of the *Christoffel symbols* $\Gamma^{\alpha}{}_{\mu\nu}$) is compatible

with the metric g, but not necessarily symmetric. We shall denote the curvature and torsion of this connection by the symbols $R^{\alpha}{}_{\beta\mu\nu}$ and $T^{\alpha}{}_{\mu\nu}$, respectively.

In more general setup, we consider also a vector bundle over M with the gauge connection in this bundle. The curvature of the gauge connection we denote by the symbol $W_{\mu\nu}$.

Using all these connections we can construct the covariant differentiation D_μ acting in the smooth sections of the bundle E. Using the operators D_μ one can construct covariant differential operators of different kinds. For example, the operator $\Box = g^{\mu\nu}D_\mu D_\nu$ is a covariant generalization of the Laplace operator.

Typical examples of so-called *minimal* operators studied in physics and mathematics are $-\Box + X$, and $\Box^2 + V^{\mu\nu}D_\mu D_\nu + N^\mu D_\mu + X$. Here $V^{\mu\nu}$, N^μ, X are matrix valued tensor fields (sections of bundle of endomorphisms $\operatorname{End} E$ of the bundle E). We write explicitly only "Lorentz" (i.e., corresponding to the coordinates in the manifold M) indices. The "gauge" indices are assumed implicitly. The term *minimal* means the operator symbol is Lorentzian scalar and its leading term is power of the Laplace operator.

Operators of more complicated type with non-Laplacian leading terms are called *nonminimal*. The simplest example of nonminimal operator is the Navier–Lamé operator of classical elasticity, $\mu\Delta\vec{V} + (\lambda+\mu)\nabla(\nabla\vec{V})$ (the Lamé constants, λ and μ, characterize the material). H. Weyl was apparently the first who investigated the asymptotics of spectrum of operators of the like type. We shall consider nonminimal operator of the form

$$-g^{\mu\nu}\Box + aD^\mu D^\nu + X^{\mu\nu}, \tag{2}$$

where $X^{\mu\nu}$ is a tensor field, a is a scalar parameter.

In recent years nonminimal operators of a similar sort have been encountered by physicists studying the quantization of gauge fields. For example, the quantization of Yang–Mills field in an arbitrary covariant background gauge leads to the operator $A^{ab}_{\mu\nu} = -\delta_{\mu\nu}\Box^{ab} - \left(\frac{1}{\alpha} - 1\right) D^{ac}_\mu D^{cb}_\nu - 2f^{acb}G^c_{\mu\nu}$, where D_μ is a covariant derivative containing the external field potential A_μ, $G_{\mu\nu}$ is a corresponding field strength and f^{abc} are the structure constants of a corresponding Lie algebra. An analogous operator arises also in quantum gravity.

We present below some examples of applications of the DWSG coefficients in different fields of mathematics and physics:

- **Spectral geometry and topology.** The index of elliptic operator A on a n-dimensional manifold M can be expressed via the Atiyah–Bott formula:

  $$\operatorname{ind}(A) \equiv \dim\operatorname{Ker} A - \dim\operatorname{Coker} A = \operatorname{Tr}\left[e^{-tA^+A} - e^{-tAA^+}\right].$$

  For the trace of the operator exponent the following asymptotic formula holds $\operatorname{Tr} e^{-tA} = \sum\limits_i e^{-t\lambda_i} \sim \sum\limits_{m\geq 0} B_m(A)t^{\frac{m-n}{2r}}$, $t \to 0_+$, where $B_m(A)$ are invariants of the manifold expressed in terms of the DWSG coefficients: $B_m(A) = \int_M d^nx\sqrt{g}\operatorname{tr} E_m(x|A)$. Here, Tr means the trace of the operator in the functional space, tr means the trace with respect to the Lorentz indices, g is the determinant of metric. In particular cases the invariants $B_m(A)$ can be interpreted as: B_0 is the volume of the manifold; if $n = 2$ and $A = -\Box$, then $E_2 = (4\pi)^{-1}R/6$ and $B_2 \sim \int d^2x\sqrt{g}R$ is the *Gauss–Bonnet invariant* of two-dimensional manifold; if $m = n$, then B_m is the index of A.

- **Korteweg–de Vries hierarchy.** The DWSG coefficients, computed for the Sturm–Liouville (1-dimensional Schrödinger) operator $A = -\frac{d^2}{dx^2} - U$, form the hierarchy of the higher Korteweg–de Vries equations. The mth KdV equation takes the form $\frac{\partial U}{\partial\tau} = \frac{\partial}{\partial x}G_m(U)$, where $G_m = \frac{m!}{2(m/2)!}E_m$. This approach extends somewhat the standard construction of the KdV hierarchy. For example, one can consider the matrix valued operator and construct the matrix KdV equation: $\frac{\partial U}{\partial\tau} = \frac{\partial}{\partial x}G_4(U) = \partial^3_x U + 3(\partial_x U)U + 3U(\partial_x U)$.

- **Spectral ζ-function and functional determinants in quantum field theory** (one-loop divergences of the effective action, the Green functions and the axial and trace anomalies etc.). Many problems in the theoretical physics are reduced to the computation of the functional integrals of the form $Z = \int D\Phi\, e^{-S(\Phi)}$ (or $Z = \int D\Phi\, e^{-iS(\Phi)}$). In accordance with the Laplace method (or saddle-point method) the main contribution into the integral gives the vicinity of the stationary point $\overline{\Phi}$ of the action $S(\Phi)$: $\frac{\delta S}{\delta \Phi}\big|_{\Phi=\overline{\Phi}} = 0$. The integral can be written approximately as $Z = \int D\phi e^{-S(\overline{\Phi}+\phi)} = e^{-S(\overline{\Phi})} \int D\phi e^{-\phi S''(\overline{\Phi})\phi + O(\phi^3)} \sim e^{-S(\overline{\Phi})} \det{}^{-1/2} S''(\overline{\Phi})$. In this formula $S''(\overline{\Phi}) \equiv A$ and $\det A = \prod_i \lambda_i$, that is, we have to compute the determinant of a matrix of infinite size. To cope with this difficulty we can use the regularization with the help of spectral (also called *generalized*) ζ-function which is an analog of the Riemann ζ-function with eigenvalues of the operator A in place of integer numbers in the classical Riemann function $\zeta(s) = \sum_i \frac{1}{\lambda_i^s} \equiv \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{Tr}\, e^{-tA}$. As is seen from the definition of the spectral ζ-function $\ln\det A = -\zeta'(s)\big|_{s=0}$. On the other hand, $\mathrm{Tr}\, e^{-tA}$ can be expressed via the DWSG coefficients.

Besides there are many other applications like *Huygens' principle for hyperbolic operators*, *high temperature expansion of the partition function* in statistical physics etc.

3 Algorithm for computing the DWSG coefficients

Following the approach based on the covariant generalization of the pseudodifferential calculus [12], let us consider a positive elliptic operator which spectrum lies inside a contour C. The Dunford formula $e^{-tA} = \int_C \frac{id\lambda}{2\pi} e^{-t\lambda}(A-\lambda)^{-1}$, allows to express the heat operator $\exp(-tA)$ in terms of the resolvent $(A-\lambda)^{-1}$ of the operator A. In the pseudodifferential calculus the following asymptotic representation for the matrix elements of the resolvent is used

$$G(x,x',\lambda) \equiv \langle x|\frac{1}{A-\lambda}|x'\rangle = \int \frac{d^n k}{(2\pi)^n \sqrt{g(x')}}\, e^{il(x,x',k)} \sigma(x,x',k;\lambda), \tag{3}$$

where $\sigma(x,x',k;\lambda)$ is an amplitude, $l(x,x',k)$ is a (real) phase function which is a biscalar with respect to general coordinate transformations, k is a wave vector.

The resolvent of operator A satisfies the equation $(A-\lambda)G = \mathbb{1}$ (unit operator) which leads to an equation for the amplitude:

$$(A(x, D_a + iD_a l) - \lambda)\sigma(x,x',k;\lambda) = I(x,x'), \tag{4}$$

where $I(x,x')$ is a transport function having both bundle and Lorentz indices.

The covariant generalization of the pseudodifferential calculus is reduced to the proper definitions of the phase and transport functions occurring in formulas (3) and (4). Suitable definitions, suggested by H. Widom, can be described as follows. In the flat space the phase function is a linear function of the coordinates x, $l(x,x',k) = k_a(x-x')^a$, and the transport function is (in the functional space) the unit matrix $I(x,x') = \mathbb{1}$, i.e., constant. One can provide these properties by postulating vanishing of all derivatives for the transport function I and derivatives of the order > 1 for the phase function l. The covariant analog of these properties of the phase and transport functions is that the correspondent *symmetrized* covariant derivatives should be equal to zero at the point $x = x'$, i.e.:

$$[\{D_{a_1} \dots D_{a_m}\} l] = 0, \quad m > 1; \qquad [\{D_{a_1} \dots D_{a_m}\} I] = 0, \quad m \geq 1, \tag{5}$$

where $\{\dots\}$ means symmetrizing in all Lorentz indices, and $[\dots]$ means transition to coincidence limit $(x = x')$.

Equations (5) together with the "initial conditions" $[l]=0$, $[D_a l]=k_a$ and $[I]=\mathbb{1}$ allow one to compute the coincidence limits (these are just only values we need for the *diagonal* elements of the heat kernel) for nonsymmetrized covariant derivatives $[D_{a_1}\dots D_{a_m}l]$ and $[D_{a_1}\dots D_{a_m}I]$. The coincidence limits of nonsymmetrized derivatives are obtained directly from (5) by reducing all terms to a unified index ordering with the help of the Ricci identity for the commutator of covariant derivatives. The resulting expressions are polynomials in the curvature tensors R^a_{bcd} and W_{ab}, torsion T^a_{bc}, and covariant derivatives of all these tensors. The functions $l(x,x',k)$ and $I(x,x')$, introduced with the help of relations (5), play an important role in so-called *intrinsic symbolic calculus* developed by Widom. In fact, just these universal functions manifest the geometric properties of a base manifold and a bundle.

Expanding the amplitude σ in the degrees of homogeneity of k: $\sigma=\sum\limits_{m=1}^{\infty}\sigma_m(x,x',k;\lambda)$, we obtain the recursion equations for σ_m from equation (4). Solving the recursion equations we obtain σ_m. The heat invariants are expressed in terms of integrals of the coincidence limits $[\sigma_m]$ by the formula

$$E_m(x|A)=\int\frac{d^nk}{(2\pi)^n\sqrt{g}}\int_C\frac{id\lambda}{2\pi}e^{-\lambda}[\sigma_m](x,k,\lambda)\equiv J([\sigma_m]). \tag{6}$$

The integrals in (6) can be expressed in terms of gamma and Gauss hypergeometric functions for a wide class of operators A. The typical integral of terms in (6) takes the form

$$J\left(\frac{k^{2p}k_{a_1}\dots k_{a_{2s}}}{(k^{2r}-\lambda)^l[(1-a)k^{2r}-\lambda]^m}\right)$$
$$=g_{\{a_1\dots a_{2s}\}}\frac{\Gamma\left((p+s+n/2)/r\right)}{(4\pi)^{n/2}2^sr\Gamma(n/2+s)\Gamma(l+m)}F\left(m,(p+s+n/2)/r;l+m;a\right), \tag{7}$$

where $g_{\{\mu_1\dots\mu_{2s}\}}$ is a symmetrized sum of products of metric tensors.

Since m and l are integer numbers, the hypergeometric function in (7) can be expressed in terms of elementary functions with the help of one of the *Gauss relations* which is a second order recurrence with respect to the parameter a:

$$a(1-z)F(a+1,b;c;z)=(2a-c-az+bz)F(a,b;c;z)+(c-a)F(a-1,b;c;z). \tag{8}$$

To finish transition to the elementary functions we should solve this recurrence with the following initial conditions: $F(0,b;m;z)=1$, $F(1,b;m;z)=(m-1)!\frac{(-z)^{1-m}}{(1-b)_{m-1}}\Big[(1-z)^{m-b-1}-\sum\limits_{k=0}^{m-2}\frac{(b-m+1)_k}{k!}z^k\Big]$, $m=1,2,\dots$, $m-b\neq 1,2,\dots$.

Here, $(a)_k=a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol (shifted factorial).

Besides, some additional work on structuring large output expressions is needed to make them as short and readable as possible. For this purpose we should first of all to find a suitable basis of independent elements in accordance with some criteria. The problem of constructing such bases is very difficult and up to now have no satisfactory algorithmic solution. As one can see, any coefficient E_m in expansion (1) is a linear combination of tensor monomials of a given mass dimensionality. These monomials are constructed from tensors by multiplication and index contraction. Factors in the monomials could be, in general case, non-commutative and posses symmetries leading to non-obvious linear dependencies among the monomials. Simplifying the tensor expressions one should take into account the symmetry properties of the tensors $R^a{}_{bcd}$, $T^a{}_{bc}$, W_{ab}, the Ricci identity:

$$[D_a,D_b]\varphi^{b_1\dots b_l}_{a_1\dots a_k}=\sum_{i=1}^{l}R^{b_i}{}_{jab}\varphi^{b_1\dots b_{i-1}jb_{i+1}\dots b_l}_{a_1\dots a_k}-\sum_{i=1}^{k}R^{j}{}_{a_iab}\varphi^{b_1\dots b_l}_{a_1\dots a_{i-1}ja_{i+1}\dots a_k}$$

$$+\ T^{j}{}_{ab}D_j\varphi^{b_1\dots b_l}_{a_1\dots a_k} + W_{ab}\varphi^{b_1\dots b_l}_{a_1\dots a_k}, \tag{9}$$

the Bianchi identities for both affine $R^a{}_{bcd}$ and gauge W_{ab} curvatures:

$$D_aR^b{}_{cde} + D_dR^b{}_{cea} + D_eR^b{}_{cad} + T^i{}_{ad}R^b{}_{cei} + T^i{}_{de}R^b{}_{cai} + T^i{}_{ea}R^b{}_{cdi} = 0, \tag{10}$$

$$D_aW_{bc} + D_bW_{ca} + D_cW_{ab} + W_{ai}T^i{}_{bc} + W_{bi}T^i{}_{ca} + W_{ci}T^i{}_{ab} = 0, \tag{11}$$

the cyclic identity:

$$\begin{aligned} &R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} + D_bT^a{}_{cd} + D_cT^a{}_{db} + D_dT^a{}_{bc} + T^a{}_{bi}T^i{}_{cd} \\ &\quad + T^a{}_{ci}T^i{}_{db} + T^a{}_{di}T^i{}_{bc} = 0, \end{aligned} \tag{12}$$

and also different consequences of these identities obtained by covariant differentiations and index contractions. In addition, tensor expressions are invariant with respect to renaming of dummy indices (i.e., with respect to the group S_n of all permutations of these indices, where n is the number of pairs of dummy indices in a given tensor monomial). All these identities and symmetries make the problem of choice of canonical basis rather nontrivial. Certainly, the problem have no unique solution and different bases are useful for different purposes. The problems with torsion are especially difficult, as the Bianchi and cyclic identities are non-linear in the presence of torsion. It may happen, that, in general, the problem of canonizing tensor expressions has no algorithmic solution at all, the usual situation for non-commutative and non-associative structures. Taking all the above into account, we elaborated a heuristic approach based on choosing a certain ordering of tensors and tensor indices and reducing tensor monomials to minimal ones with respect to this ordering. Of course, such approach does not ensure the full independence of the tensor invariants in the final expressions (especially in the case of high degree tensor polynomials arising in the DWSG coefficients of high order). Nevertheless, the approach works quite satisfactory and allows, in most cases, to decrease the expressions essentially.

In the case of non-minimal and high-order operators, the scalar coefficients at tensor invariants depend on the dimension of the manifold n and parameters of operator (like a in (2)). As a rule, there are many linear dependencies among these scalar coefficients. The dependences should be eliminated by the linear algebra methods to make the resulting formulas as compact as possible.

The algorithm has been implemented in the ***C*** language. The ***C*** code of total length about 11000 lines contains about 250 functions for different manipulations with tensors and scalars. These functions are gathered into two programs ***CoincidenceLimits*** and ***DWSGCoefficient***.

The ***CoincidenceLimits*** program computes the coincidence limits of non-symmetrized derivatives of the functions $l(x,x',k)$ and $I(x,x')$ and writes them to the disk. Once computed and stored they are used in many calculations for different operators A.

4 New results

With the help of the programs ***CoincidenceLimits*** and ***DWSGCoefficient*** we have obtained several new results. Some of them were not known entirely, others were known only partly or at different simplifying assumptions. Among the new results we can mention the following:

- E_4 with gauge and Riemann curvatures and torsion for operator $-\Box + X$.
- E_2, E_4 (gauge, Riemann curvatures; torsion) and E_6 (without torsion) for operator $-\Box + B_iD^i + X$. For example, $E_2 = (4\pi)^{-\frac{n}{2}}\left\{-X - \frac{1}{4}B^2 + \frac{1}{6}R + D_i\left(\frac{1}{2}B^i + \frac{1}{6}T^i\right) + \frac{1}{12}T^2 - \frac{1}{24}T_{ijk}T^{ijk} - \frac{1}{12}T_{ijk}T^{jik}\right\}$, where $T_i = T^j{}_{ji}$, $T^2 = T_iT^i$, $B^2 = B_iB^i$.
- E_2, E_4 (gauge, Riemann curvatures; torsion) for operator $\Box^2 + V^{\mu\nu}D_\mu D_\nu + N^\mu D_\mu + X$.

As to the nonminimal operator $-g^{\mu\nu}\Box + aD^\mu D^\nu + X^{\mu\nu}$, we succeeded in computing the coefficient E_2 in the presence of gauge and Riemannian curvatures and torsion and the coefficient E_4 without torsion. In view of importance of the last result, let us consider it in more detail.

Recall from Atiyah–Bott formula that the coefficient E_4 is connected intimately just with the dimension of physical space-time $n = 4$ due to the fact that the Atiyah–Singer index of an elliptic operator on a manifold of dimension n can be expressed in terms of E_n with the help of integration. The full expression for E_4 can not be present here (one can find it in [8]), since it consists in 73 tensor terms with 43 different scalar coefficients (with 28 linear dependencies among them). Here we reproduce only the trace (with respect to Lorentz indices) $\mathrm{tr}_L E_4$. The expression for the trace still contains some novelties in comparison with the result of Branson, Gilkey and Pierzchalski [13], since they computed $\mathrm{tr}_L E_4$ without gauge field and neglecting the terms with total derivative (essential if the topology of manifold is non-trivial).

$$\begin{aligned}\mathrm{tr}_L E_4 = (4\pi)^{-\frac{n}{2}}\Big\{ &- C_1\Box X_i{}^i - C_2 D_i D_j X^{ij} + C_3\left(X_i{}^i X_j{}^j + X_{ij}X^{ij}\right)\\ &+ C_4 X_{ij}X^{ji} + C_5 X_{ij}W^{ij} - C_6 W_{ij}X^{ij} + C_7 W_{ij}W^{ij} + C_8 R_{ijkl}R^{ijkl}\\ &- C_9 R_{ij}X^{ij} - C_{10}R_{ij}R^{ij} + C_{11}\Box R + C_{12}R^2 - C_{13}R X_i{}^i\Big\}, \end{aligned} \tag{13}$$

where

$$\begin{aligned}C_1 = {}& \frac{1}{96a^2\left(\frac{n}{2}-2\right)_4}\Big\{(1-a)^{1-\frac{n}{2}}\left(96 - 96a - 48an + 8a^2n + 6a^2n^2 + a^2n^3\right)\\ &- 96(1-a)^2 + 32a^2n + 2a^2n^2 - 5a^2n^3 + a^2n^4\Big\},\\ C_2 = {}& \frac{1}{48a^2\left(\frac{n}{2}-2\right)_4}\Big\{(1-a)^{1-\frac{n}{2}}\left(-48n + 48a + 24an - 4a^2n + a^2n^3\right)\\ &+ 48n - 48a - 72an + 24an^2 + 48a^2 + 4a^2n - 24a^2n^2 + 5a^2n^3\Big\},\\ C_3 = {}& \frac{1}{16a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}(-4 + 2a + an) + 4 - 2a + an\Big\},\\ C_4 = {}& \frac{1}{16a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}(4 + 4n - 6a - 3an) - 4 - 4n + 6a - 3an - 2an^2 + an^3\Big\},\\ C_5 = {}& \frac{1}{192a^2\left(\frac{n}{2}-2\right)_5}\Big\{(1-a)^{-1-\frac{n}{2}}\left(192n + 48n^2 + 576a + 192an - 216an^2 - 24an^3\right.\\ &- 1152a^2 - 944a^2n + 136a^2n^2 + 62a^2n^3 + 2a^2n^4 + 576a^3 + 664a^3n + 86a^3n^2\\ &\left.- 31a^3n^3 - 5a^3n^4 - 104a^4n\right) - 62a^4n^2 - a^4n^3 + 2a^4n^4 - 192n - 48n^2 - 576a\\ &- 384an + 72an^2 + 576a^2 + 272a^2n - 208a^2n^2 + 10a^2n^3 + 4a^2n^4 - 24a^3n + 26a^3n^2\\ &- 9a^3n^3 + a^3n^4\Big\},\\ C_6 = {}& \frac{1}{192a^2\left(\frac{n}{2}-2\right)_5}\Big\{(1-a)^{-1-\frac{n}{2}}\left(-192n - 48n^2 + 384a + 816an + 96an^2 + 12an^3\right.\\ &- 768a^2 - 1120a^2n - 148a^2n^2 - 8a^2n^3 - 2a^2n^4 + 384a^3 + 584a^3n + 150a^3n^2 + 7a^3n^3\\ &\left.- 88a^4n - 58a^4n^2 - 5a^4n^3 + a^4n^4\right) + 192n + 48n^2 - 384a - 624an + 48an^2 + 12an^3\\ &+ 384a^2 + 304a^2n - 164a^2n^2 + 8a^2n^3 + 2a^2n^4 - 24a^3n + 26a^3n^2 - 9a^3n^3 + a^3n^4\Big\},\\ C_7 = {}& \frac{1}{48a\left(\frac{n}{2}-1\right)_2}\Big\{(1-a)^{1-\frac{n}{2}}\left(-96 + 22an + an^2 + 2a^2n - a^2n^2\right) + 96 - 96a + 26an\\ &- 3an^2 + an^3\Big\},\\ C_8 = {}& \frac{(1-a)^{2-\frac{n}{2}} - 16 + n}{180}, \end{aligned}$$

$$C_9 = \frac{1}{24a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}\big(12n-12a-8an-an^2+2a^2n+a^2n^2\big)-12n+12a$$
$$+8an-5an^2\Big\},$$
$$C_{10} = \frac{1}{1440a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}\big(-360n+360a+296an+60an^2+an^3-112a^2n$$
$$-60a^2n^2-2a^2n^3-4a^3n+a^3n^3\big)+360n-360a-296an+116an^2-an^3+an^4\Big\},$$
$$C_{11} = \frac{1}{480a^2\left(\frac{n}{2}-2\right)_4}\Big\{(1-a)^{1-\frac{n}{2}}\big(480-240n-240a-120an+16a^2n+26a^2n^2$$
$$+11a^2n^3+a^2n^4+4a^3n+4a^3n^2-a^3n^3-a^3n^4\big)-480+240n+720a-360an$$
$$+120an^2-240a^2+104a^2n-70a^2n^2+15a^2n^3-5a^2n^4+a^2n^5\Big\},$$
$$C_{12} = \frac{1}{576a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}\big(-144+72a+56an+12an^2+an^3-16a^2n-12a^2n^2$$
$$-2a^2n^3-4a^3n+a^3n^3\big)+144-72a+16an-16an^2-an^3+an^4\Big\},$$
$$C_{13} = \frac{1}{48a\left(\frac{n}{2}-1\right)_3}\Big\{(1-a)^{-\frac{n}{2}}\big(-24+12a+8an+an^2-2a^2n-a^2n^2\big)+24-12a$$
$$-an^2+an^3\Big\}.$$

As one can see, the scalar coefficients in (13) depend in a rather non-trivial way not only on the gauge parameter a, but also on the space dimension n. This is a typical property of nonminimal and high order operators [12].

Acknowledgements

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Numerical Simulation of Replacing Oil by Water in a Scale-Invariant Porous Medium

Ekaterina P. KUROCHKINA † *and Olga N. SOBOLEVA* ‡

† *Institute of Thermophysics SB RAS, 630090 Novosibirsk, Russia*
E-mail: *kurochkina@itp.nsc.ru*

‡ *Institute of Computational Mathematics and Mathematical Geophysics SB RAS, 630090 Novosibirsk, Russia*
E-mail: *olga@nmsf.sscc.ru*

The objective of the paper is to study the interface between the fluids having different viscosities in their joint movement through the nonhomogeneous random scale-invariant porous media. The surface tension and the force of gravity are not taken into account. Porosity and permeability have pulsations from an extreme wide range of scales and logarithmic-normal statistics. Statistical parameters of the interface are found. The problem is numerically solved in a unit size cube with 256^3 grid in spatial variables.

1 Introduction

The interface of oil displaced by water is unstable. This instability has the origin of different viscosities and explains the viscous fingering between fluids while the displacement takes place [1]. Another origin of the complex interface is the fluctuations of permeability of the porous media. Geological porous media are characterized by multiple-scale heterogeneities that have a significant influence on the transport of water and oil. The permeability heterogeneity is often identified as the main factor determining transport and dispersion of impurities in the subsurface [2]. In this paper, both factors are studied and compared in a direct computer experiment. We concentrate on them and do not discuss other important factors such as the surface tension and the gravitational force. The physical properties of such media are simulated by means of random fields. The probability distribution is simplified by the refined scaling hypothesis by Kolmogorov [3]. This hypothesis is associated with the fractal properties of the porous media that have a certain experimental support [4]. In particular, correlation functions of the parameters are asymptotically described by power laws. This fact enables us to use the scaling theory for modeling permeability and porosity.

2 Statement of problem

A displacement of a more viscous fluid (oil) by a less viscous one (water) in the 3D-scale-invariant porous medium is considered. At a low Reynolds number, Darcy's law gives the velocity $\boldsymbol{v}(\boldsymbol{x}) = -\varepsilon(\boldsymbol{x})\nabla p$, where p is pressure, $\varepsilon(\boldsymbol{x})$ is permeability. The incompressibility condition brings about the equation for p:

$$\nabla\left[\varepsilon\left(\boldsymbol{x}\right)\nabla p\left(\boldsymbol{x}\right)\right] = 0, \tag{1}$$

where the permeability $\varepsilon(\boldsymbol{x})$ is equal to $\varepsilon_1(\boldsymbol{x})$ if the first phase (water), is located at point $\boldsymbol{x}$ and equal to $\varepsilon_2(\boldsymbol{x})$, if the second phase (oil) is located at this point $\boldsymbol{x}$. The displacement of one fluid by another one is displayed with passive particles which label the interface between fluids. The

trajectories of particles that label the interface between fluids are calculated from the equations:

$$m\left(\boldsymbol{x}\right)\frac{d\boldsymbol{x}_i\left(t\right)}{dt}=\boldsymbol{v}(\boldsymbol{x}),\qquad \boldsymbol{x}_i(0)=\boldsymbol{x}_{i0}, \tag{2}$$

where i stands for the number of particle $i=1,\ldots,N$, $m\left(\boldsymbol{x}\right)$ is porosity. We solve equation (2) using the pressure from equation (1) and find the new coordinates of our phases. For the next time step we solve equation (1) with the new permeability coefficient $\varepsilon(\boldsymbol{x})$. A more viscous fluid (oil, solute) is displaced by a less viscous one (water). The ratio of permeabilities is determined by the ratio between viscosities of phases. The medium does not change while the flow moves. How does heterogeneity of a porous medium affect the average front coordinate, the variance of width of the front, etc.?

For describing functions of permeability and porosity we use the scaling theory proposed by Kolmogorov [3]. The full details of the scaling approach for the porous medium are considered in [5]. Here, we only briefly outline such an approach. The fluctuations of $\varepsilon(\boldsymbol{x})$, $m\left(\boldsymbol{x}\right)$ of various sizes may be identified by a spatial smoothing at $l\to 0$, $\varepsilon(\boldsymbol{x},l)\to\varepsilon(\boldsymbol{x})$, $m\left(\boldsymbol{x},l\right)\to m\left(\boldsymbol{x}\right)$. We assume that there exists such a minimum scale l_0, that $\varepsilon(\boldsymbol{x},l_0)\approx\varepsilon(\boldsymbol{x}), m\left(\boldsymbol{x},l_0\right)\approx m\left(\boldsymbol{x}\right)$.

The dimensionless field $\psi(\boldsymbol{x},l,l')=\varepsilon(\boldsymbol{x},l)/\varepsilon(\boldsymbol{x},l')$ is similar to the dimensionless ratios of fields according to Kolmogorov (1962). Namely, the field $\psi(\boldsymbol{x},l,l')$ is assumed to be translatory homogeneous, isotropic and of the scaling symmetry. The latter means that $\psi(\boldsymbol{x},l,l')$ has the same probability distribution as the field $\psi(K\boldsymbol{x},Kl,Kl')$ where K is a positive number. The field $\psi(\boldsymbol{x},l,l')$ has too many arguments. We define a simpler field having the same information. To introduce such a field we use the following identity that holds by definition of $\psi(\boldsymbol{x},l,l')$:

$$\psi(\boldsymbol{x},l,l'')=\psi(\boldsymbol{x},l,l')\psi(\boldsymbol{x},l',l'').$$

Making l' to be infinitesimal as opposed to l, we obtain a simpler field $\varphi(\boldsymbol{x},l)=\frac{d\psi(\boldsymbol{x},l,ly))}{ldy}|_{y=1}$ that has the same information as $\psi(\boldsymbol{x},l,l')$. The permeability $\varepsilon(\boldsymbol{x})$ is expressed via $\varphi(\boldsymbol{x},l)$ as

$$\varepsilon\left(\boldsymbol{x}\right)=\varepsilon_0\exp\left[-\int_{l_0}^{L}\varphi(\boldsymbol{x},l)\frac{dl}{l}\right]. \tag{3}$$

Doing it in the same way for porosity, we obtain

$$m\left(\boldsymbol{x}\right)=m_0\exp\left[-\int_{l_0}^{L}\chi\left(\boldsymbol{x},l\right)\frac{dl}{l}\right], \tag{4}$$

where L is the scale of the largest fluctuations.

We assume the scale invariance, spatial homogeneity and isotropy to be present for the second centered correlation functions. Here we use a simple model, in which the statistical distributions for $\varphi(\boldsymbol{x},l)$, $\chi\left(\boldsymbol{x},l\right)$ are supposed to be Gaussian-distributed and delta-correlated in the scale logarithm for $\boldsymbol{x}=\boldsymbol{y}$:

$$\Phi^{\varphi\varphi}\left(\boldsymbol{x},\boldsymbol{x},l,l'\right)=\left\langle\varphi(\boldsymbol{x},l)\varphi(\boldsymbol{x},l')\right\rangle_c=\Phi_0^{\varphi\varphi}\delta\left(\ln l-\ln l'\right), \tag{5}$$

where $\left\langle\right\rangle_c$ denotes the central moment,

$$\Phi^{\chi\chi}\left(\boldsymbol{x},\boldsymbol{x},l,l'\right)=\left\langle\chi(\boldsymbol{x},l)\chi(\boldsymbol{x},l')\right\rangle_c=\Phi_0^{\chi\chi}\delta\left(\ln l-\ln l'\right). \tag{6}$$

The correlation between porosity and permeability is determined by the correlation between $\varphi(\boldsymbol{x},l)$ and $\chi(\boldsymbol{x},l)$:

$$\Phi^{\varphi\chi}\left(\boldsymbol{x},\boldsymbol{x},l,l'\right)=\left\langle\varphi(\boldsymbol{x},l)\chi(\boldsymbol{x},l')\right\rangle_c=\Phi_0^{\varphi\chi}\delta\left(\ln l-\ln l'\right). \tag{7}$$

Thus, we do not violate the scale invariance of porosity and permeability. We consider a simple model, when statistical distributions for $\varphi(\boldsymbol{x},l)$, $\chi(\boldsymbol{x},l)$ are assumed to be Gaussian, and hence distributions of porosity and of permeability are logarithmic-normal.

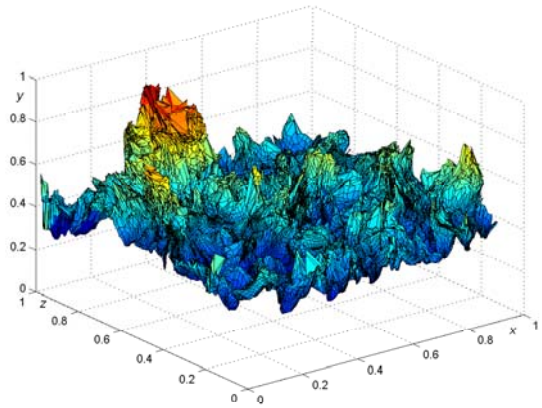

Figure 1. The interface between phases. The ratio of viscosities $\nu = 1$.

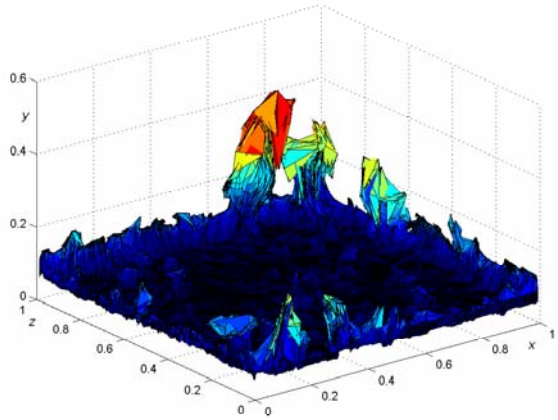

Figure 2. The interface between phases for the ratio of viscosities $\nu = 0.01$.

3 Results of numerical modeling

Equations (1), (2) are numerically solved in the cube with edges L_0. A constant pressure is set at the faces $y = 0$, $y = L_0$: $p\,(x, y, z)\,|_{y=0} = p_1$, $p\,(x, y, z)\,|_{y=L_0} = p_2$, $p_1 > p_2$. The pressure of the other faces of the cube is set by a linear dependence in terms of y: $p = p_1 + (p_2 - p_1)\, y/L_0$. The main filtration flow is directed along the axis y. For the numerical calculation we use dimensionless variables. All lengths are measured in units of L_0, the permeabilities being measured in units of the less viscous phase ε_{01}. Thus, it is sufficient to solve the problem in a unit size cube with a unit pressure difference and $\varepsilon_{01} = 1$. First we calculate the field of porosity and the field of permeability replacing integrals (3), (4) by the finite difference formula, in which it is convenient to use the logarithm with base 2:

$$\varepsilon\,(\boldsymbol{x}) = \varepsilon_{0j} \exp\left[-\ln 2 \sum_{k=-8}^{0} \varphi(\boldsymbol{x}, \tau_k)\delta\tau\right],$$
$$m\,(\boldsymbol{x}) = m_0 \exp\left[-\ln 2 \sum_{k=-8}^{0} \chi(\boldsymbol{x}, \tau_k)\delta\tau\right], \tag{8}$$

where $j = 1, 2$, $\varepsilon_{01} = 1$, $\varepsilon_{02} = \mu_1/\mu_2$, μ_1, μ_2 are viscosities of phases, $l_k = 2^{\tau_k}$; $\tau_k = k\delta\tau$, $\delta\tau = 1$ is the discretization step of the scale logarithm, $\tau_k = 0, -1, \ldots, \log_2 \frac{1}{256} = -8$. We use a 256^3 grid in spatial variables. The fields $\varphi(\boldsymbol{x}, l_k)$ and $\chi(\boldsymbol{x}, l_k)$ are generated independently for every scale l_k because they are delta-correlated in terms of the scale logarithm. The common indices of degree are summarized over the statistically independent layers. For every τ_k, we obtain the two fields:

$$\varphi(\boldsymbol{x}, \tau_k) = \sqrt{\frac{\Phi_0^{\varphi\varphi}}{\ln 2}}\zeta(\boldsymbol{x}, \tau_k) + \langle\varphi\rangle \ln 2,$$
$$\chi(\boldsymbol{x}, \tau_k) = \sqrt{\frac{\Phi_0^{\chi\chi}}{\ln 2}}\left(r\zeta\,(\boldsymbol{x}, \tau_k) + \sqrt{1 - r^2}\varsigma\,(\boldsymbol{x}, \tau_k)\right) + \langle\chi\rangle \ln 2,$$

where $-1 \le r \le 1$. The fields $\zeta(\boldsymbol{x}, \tau_k)$, $\varsigma\,(\boldsymbol{x}, \tau_k)$ are independent, Gaussian, having a unit variance, zero average and the same correlation functions:

$$\langle\zeta(\boldsymbol{x}, \tau_i)\zeta(\boldsymbol{y}, \tau_j)\rangle_c = \langle\varsigma(\boldsymbol{x}, \tau_i)\varsigma(\boldsymbol{y}, \tau_j)\rangle_c = \exp\left[-\frac{(\boldsymbol{x} - \boldsymbol{y})^2}{2^{2\tau_i}}\delta_{ij}\right]. \tag{9}$$

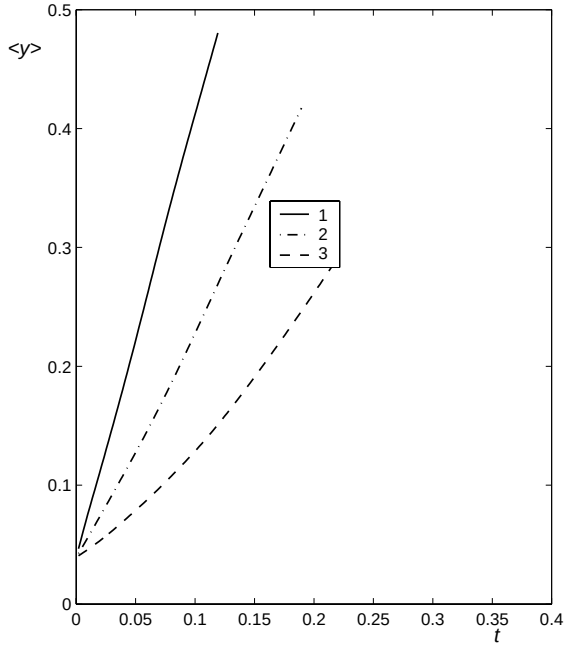

Figure 3. The average position of the front for the different ratios of viscosities: 1 – $\nu = 1$; 2 – $\nu = 0.5$; 3 – $\nu = 0.2$.

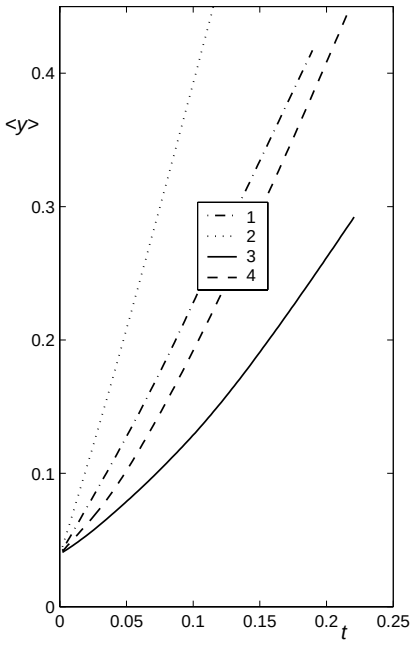

Figure 4. 1 – heterogeneous medium $\nu = 0.5$; 2 – the constant permeability and porosity are equal to mean values, $\nu = 0.5$; 3 – heterogeneous medium, $\nu = 0.2$; 4 – constant porosity and permeability $\nu = 0.2$.

Thus, we consider a simple correlation between the fields. The natural condition for the porosity $0 \le m(\boldsymbol{x}) \le 1$ is satisfied by the choice of parameters $\Phi_0^{\chi\chi}$, $\langle\chi\rangle$.

A particular structure of correlation functions enables us to present a common matrix as product of four correlation matrices. These matrices have a relatively low dimension. This fact allows us to apply the algorithm described in [6]. Equation (1) is solved by the iterative method. In the present calculations, the two upper and the three lower layers are left empty, i.e. the functions φ, χ in these layers are equal to zero. Two empty upper layers indicate to the fact that the largest scale $L = 1/8$. This allows us to replace the probable mean quantities by the space-averaged values. The three lower layers are also left empty to provide a good difference approximation for equation (1).

The parameters of calculation are $\Phi_0^{\varphi\varphi} = 0.3$, $\langle\varphi\rangle = 0.15$, $m_0 = 1$, $\Phi_0^{\chi\chi} = 0.05$, $\langle\chi\rangle = 0.7$, $\Phi^{\phi\chi} = \sqrt{0.3 * 0.05}$, the coefficient correlation between porosity and permeability $\gamma = 0.92$, the variance of permeability of the first phase is equal to 1.297, the mean porosity being approximately equal to 0.15. Evolution of the front depending on the ratio of viscosities ν is displayed in Figs. 1, 2[1].

Fig. 1 shows the interface between the same phases $\nu = 1$. In Fig. 2 we have the interface between the phases for the ratio of viscosities $\nu = 0.01$ ($\nu = \mu_1/\mu_2$). The geometry of both media and the moment of time are the same. The interface between fluids has an unstable behavior and "viscous fingering" for the $\nu = 0.01$. The numerical simulation shows that within the domain of ratio of viscosities for oil $0.01 \le \nu \le 1$, the variances of the fronts strongly depend on heterogeneity of the porous media. In the Hele–Shaw cell permeability heterogeneities are, as a rule, not considered [1]. Fig. 3 shows average coordinates of the front depending on time for $\nu = 0.5$, 0.2. Such ratios of viscosities are typical of Siberian reservoirs. The comparison of the average coordinates of the front in a heterogeneous medium with the coordinates of the

[1]Figures in colour will be available only in electronic version.

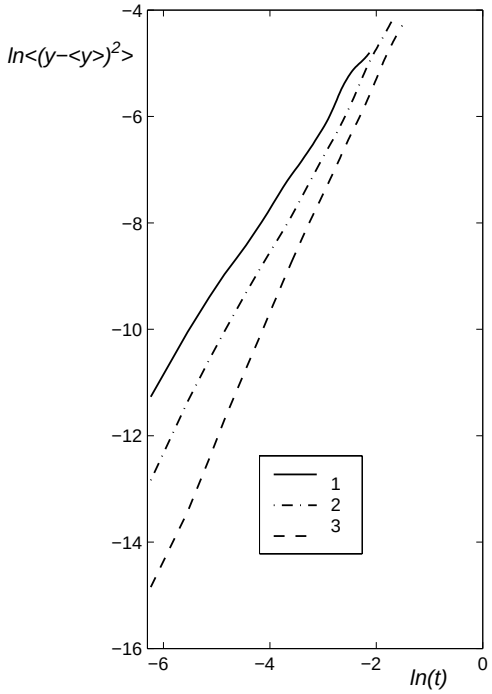

Figure 5. The logarithm of variance of the front along axis y for the different viscosities ratios: 1 – $\nu = 1$; 2 – $\nu = 0.5$; 3 – $\nu = 0.2$.

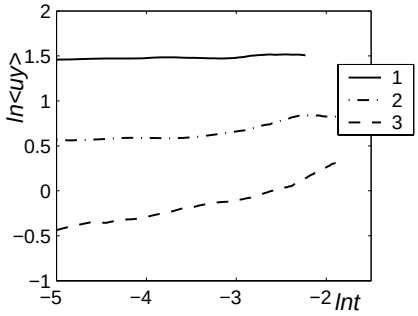

Figure 6. 1 – The logarithm of average velocity of the front for $\nu = 1.0$; 2 – $\nu = 0.5$; 3 – $\nu = 0.2$.

front for a medium with constant permeability and porosity is displayed in Fig. 4.

The variances of width of the front for different ν are displayed in the logarithm coordinates in Fig. 5. The slope angles α increase if the viscosities ratio decreases:

$$\nu = 1, \quad \alpha = 1.56; \qquad \nu = 0.5, \quad \alpha = 1.84; \qquad \nu = 0.2, \quad \alpha = 2.24.$$

The time dependence of variance of the front is a power function of time with the index of power α. This fact is a consequence of scale-invariance of medium.

The ratio of the longitudinal component and the transverse one of the diffusion tensor is constant and approximately equal to 3.5 for $\nu = 1$. (This value can also be evaluated by the perturbation theory). Non-diagonal components of the tensor are small for our problem. For $\nu = 0.5$, the ratio of the longitudinal component and the transverse one calculated for water at the moment t_{end} is approximately equal to 5.0. The average velocities along the axis y are displayed in Fig. 6.

4 Summary

We have considered 3-D model of displacement of immiscible fluids in fractal porous media. The "viscous fingering" occurs, similar to the flow in a horizontal square Hele–Show cell, although we do not consider the surface tension force. Different spatial moments of the front are calculated, and their statistical characteristics are presented. Dispersion of the water front is described by the second order diffusion tensor D_{ij}. All components of D_{ij} are computed. The value of the transverse diffusion is investigated and compared to the longitudinal one, while statistical parameters of the medium and viscosities of the fluids vary.

Acknowledgements

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Flow Through a Porous Medium with Multiscale Log-Stable Permeability

Gennady A. KUZ'MIN † *and Olga N. SOBOLEVA* ‡

† *Institute of Thermophysics SB RAS, 630090 Novosibirsk, Russia*
E-mail: *kuzmin@itp.nsc.ru*

‡ *Institute of Computational Mathematics and Mathematical Geophysics SB RAS, 630090 Novosibirsk, Russia*
E-mail: *olga@nmsf.sscc.ru*

We consider single-phase flow of an incompressible fluid through a random scaling porous medium. The joint multi-point probability distribution for the porosity and permeability is supposed to be log-stable and satisfy the conditions of the Kolmogorov's refined scaling hypothesis. A subgrid model is derived which is similar to Landau–Lifschitz formula. The similarity of the present method to the Wilson's renormalization group is noticed.

1 Introduction

We study the multiscale heterogeneity in the subsurface hydrodynamics using the subgrid method similar to that of Wilson's renormalization group. The renormalization group (RG) method was originally developed in the quantum field theory to remove singularities in the perturbation theory. It was used in the variety of problems including the statistical physics, the turbulence theory, certain geophysical problems, and so on (see for example [1]). We use Wilson's formulation of the RG that has some application to subgrid modelling in computational hydrodynamics. Our use of the RG method is motivated by the refined scaling properties [2] that were experimentally observed in the subsurface hydrodynamics [3]. Our previous papers [4–6] describe the general method of analysis and the RG model based on the Kolmogorov's refined scaling to study the subgrid flows. The results conform with those obtained with other formulations of the RG methods (see, for example, [7]). We have considered the logarithmic-normal theory as a simple problem, and the log-stable medium was proposed as a more general task. The present paper is devoted to the flows in the porous self-similar media, described by the selected log-stable probability distributions.

2 The refined scaling of the porous media

Let us consider the scaling hypothesis for the permeability field $\varepsilon(\boldsymbol{x})$. The fluctuations of $\varepsilon(\boldsymbol{x})$ of various sizes may be identified by the spatial smoothing. We use the Fourier filter that omits all the Fourier harmonics whose wavelength is shorter than some threshold value l, when obtaining $\varepsilon_l(\boldsymbol{x})$. At $l \to 0$, $\epsilon_l(\boldsymbol{x}) \to \epsilon(\boldsymbol{x})$. The dimensionless field $\psi(\boldsymbol{x}, l, l') = \epsilon_{l'}(\boldsymbol{x})/\epsilon_l(\boldsymbol{x})$ is similar to the dimensionless fields by Kolmogorov [2]. We assume the similarity hypothesis. Namely, the field $\psi(\boldsymbol{x}, l, l')$ is assumed to be translatory homogeneous, isotropic and of the scaling symmetry. The latter means that $\psi(\boldsymbol{x}, l, l')$ has the same probability distributions as $\psi(K\boldsymbol{x}, Kl, Kl')$, where K a positive number. When introducing the scaling hypothesis, one usually uses the notion of the scale dimension Δ_ψ, that means that the fields $\psi(\boldsymbol{x}, l, l')$ and $K^{\Delta_\psi}\psi(K\boldsymbol{x}, Kl, Kl')$ have the same probability distribution. We suppose that for our nondimensional function ψ $\Delta_\psi = 0$.

The field $\psi(\boldsymbol{x}, l, l')$ has too many arguments. We define a simpler field that contains the same information. To introduce such a field we use the following identity that holds by definition of $\psi(\boldsymbol{x}, l, l')$,

$$\psi(\boldsymbol{x}, l, l'') = \psi(\boldsymbol{x}, l, l')\psi(\boldsymbol{x}, l', l''). \tag{1}$$

Identity (1) results in the exponential representation of ψ and ε (see (3)). The derivation of the representation that is not very rigorous, is the following one.

Let us consider the limit $l'' \to l'$. In the first order in $l''-l'$, we obtain the differential equation:

$$\frac{\partial\psi(\boldsymbol{x}, l, l')}{\partial l'} = \frac{1}{l'}\psi(\boldsymbol{x}, l, l')\varphi(\boldsymbol{x}, l'), \tag{2}$$

where $\varphi(\boldsymbol{x}, l') = \left.\frac{\partial\psi(\boldsymbol{x}, l', l'y)}{l'\partial y}\right|_{y=1}$ is a dimensionless field. From the definition of $\psi(\boldsymbol{x}, l, l')$, we obtain the equation that expresses the field $\psi(\boldsymbol{x}, l, l')$ or $\varepsilon(\boldsymbol{x}, l)$ in terms of $\varphi(\boldsymbol{x}, l)$

$$\frac{\partial\varepsilon_l(\boldsymbol{x})}{\partial \ln l} = \varphi(\boldsymbol{x}, l)\varepsilon_l(\boldsymbol{x}). \tag{3}$$

The self-similar fluctuations may be observed within a finite range of scales $l_0 < l < L$. Equation (3) should be supplemented by the boundary condition on any end of the range (l_η, L). The solution to (3) is the following

$$\varepsilon_l(\boldsymbol{x}) = \varepsilon_0 \exp\left[-\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]. \tag{4}$$

The permeability $\epsilon(\boldsymbol{x})$ at the smallest scale l_0 is determined by formula (4) with $l \to l_0$

$$\varepsilon(\boldsymbol{x}) = \varepsilon_0 \exp\left[-\int_{l_0}^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]. \tag{5}$$

Formula (5) will be used independent of its derivation. It determines the statistical model by defining the statistical distribution for $\varphi(\boldsymbol{x}, l_1)$. When deriving (4) we assumed that the derivative of ψ on l exists. Let us note, however, that a similar but more general exponential solution to (1) is expressed via integral over measure. For simplicity, we use expressions (4), (5).

The scaling models are defined by the field $\varphi(\boldsymbol{x}, l)$ which is assumed to be homogeneous, isotropic and statistically invariant to the scale transform $\varphi(\boldsymbol{x}, l) \to \varphi(K\boldsymbol{x}, Kl)$. The discrete approximation of the fields $\varphi(\boldsymbol{x}, l_1)$ will be considered to render the probabilistic models. In these approximations, the fields $\varphi(\boldsymbol{x}, l)$, $\varphi(\boldsymbol{y}, l')$ with different scales l, l', at any $\boldsymbol{x}$, $\boldsymbol{y}$ are considered to be statistically independent. This supposition is usually assumed in the scaling models and reflects the decay of statistical dependence when the scales of fluctuations become different in the order of magnitude. The latter was proposed in [2].

To describe the probability distribution of $\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}$ for large L/l we use the theorem about sums of independent variables [8, 9]. If the dispersion of $\varphi(\boldsymbol{x}, l)$ at a given point exists, then the theorem says that the factor $\int_{l_0}^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}$ for very large L/l_0 tends to a normal field. In the opposite case (the second correlation function does not exist) the integral over $\frac{dl}{l}$ tends to a field described by a stable distribution. For simplicity, the same distribution is assumed for $\varphi(\boldsymbol{x}, l)$, $l_0 < l < L$.

3 Subgrid modeling

We present a subgrid model for a single-phase flow of incompressible fluid through the scale-invariant porous rock with log-stable distribution, keeping in mind that similar methods might be useful for the wave field evolution and for other related characteristics.

The problem is formulated as follows. Let an incompressible fluid steadily flow through a porous medium with the fluctuating permeability coefficient $\epsilon(\boldsymbol{x})$. The simplest model is Darcy's law which expresses the velocity $\boldsymbol{v} = -\epsilon(\boldsymbol{x})\nabla p$ via the pressure p. The incompressibility condition $\operatorname{div} \boldsymbol{v} = 0$ leads to the equation for p:

$$\frac{\partial}{\partial x_j}\left[\varepsilon(\boldsymbol{x})\frac{\partial}{\partial x_j}p(\boldsymbol{x})\right] = 0. \tag{6}$$

An extremely wide range of scales is supposed to exist $L/l_0 \gg 1$. The direct computation of the pressure field $p(\boldsymbol{x})$ from the equation is impossible or requires enormous computational costs. We derive from (6) an equation that describes only the fluctuating pressure field of the largest-scale fluctuations outside the self-similar range.

The spatial smoothing is defined in the previous section for the permeability field. Here we use instead the statistical smoothing of $\varepsilon(\boldsymbol{x})$. The statistical distributions for the fields $\varphi(\boldsymbol{x}, l_1)$ with $l_0 < l_1 < L$ determine a statistical model for the medium. The function $\varepsilon(\boldsymbol{x})$ is divided into two components with respect to the scale l. The large-scale component $\varepsilon(\boldsymbol{x}, l)$ is obtained from $\varepsilon(\boldsymbol{x})$ via statistical average over all $\varphi(\boldsymbol{x}, l_1)$ with $l_1 < l$, $\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}) - \varepsilon(\boldsymbol{x}, l)$ being a complementary small-scale component

$$\varepsilon(\boldsymbol{x}, l) = \varepsilon_0 \exp\left[-\int_l^L \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]\left\langle \exp\left[-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]\right\rangle_< \tag{7}$$

$$\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}, l)\left[\frac{\exp\left[-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]}{\left\langle \exp\left[-\int_{l_0}^l \varphi(\boldsymbol{x}, l_1)\frac{dl_1}{l_1}\right]\right\rangle_<} - 1\right], \tag{8}$$

where $\langle\cdot\rangle_<$ denotes averaging over $\varphi(\boldsymbol{x}, l_1)$ of small scales l_1, $D = 1, 2, 3\ldots$ is the spatial dimension. The probability distribution for p is determined by ε from equation (6). We define the large-scale (ongrid) pressure field $p(\boldsymbol{x}, l)$ as averaged solution to (6), where the large-scale component $\varepsilon(\boldsymbol{x}, l)$ is fixed, ε' may be random, $p(\boldsymbol{x}, l) = \langle p(\boldsymbol{x})\rangle_<$. The complementary subgrid component $p' = p(\boldsymbol{x}) - p(\boldsymbol{x}, l)$ cannot generally be rejected from the filtered equation

$$\nabla\left[\varepsilon(\boldsymbol{x},l)\nabla p_l(\boldsymbol{x}) + \langle\varepsilon'(\boldsymbol{x})\nabla p'(\boldsymbol{x})\rangle_<\right] = 0, \tag{9}$$

because the second term may be essential. The choice of the form of the second term defines the subgrid model.

We use the perturbation theory employing the amplitude of the subgrid fluctuations as a small parameter to evaluate the suitable subgrid model. The actual effective parameter of expansion will be estimated later. In the Wilson's RG, initial l is only slightly greater than the smallest scale l_0. This makes it possible to derive the differential equation in l.

Subtracting (9) from (6), we obtain the equation for the subgrid pressure p'

$$\nabla_i\left[\varepsilon(\boldsymbol{x})\nabla p(\boldsymbol{x})\right] - \nabla_i\left[\varepsilon(\boldsymbol{x},l)\nabla_i p(\boldsymbol{x},L) + \langle\varepsilon'(\boldsymbol{x})\nabla_i p'(\boldsymbol{x})\rangle_<\right] = 0. \tag{10}$$

Equation (10) is used to find the subgrid pressure $p'(\boldsymbol{x})$ in terms of the ongrid components. The equation cannot be solved exactly, therefore the perturbation theory is used. The left-hand side of the equation contains the terms of first and second order in the subgrid fluctuations. In this case, we neglect the terms of the second order. To simplify the problem further and to derive the differential equation, let us consider equation (9) in the large-scale limit. This is typical of the Wilson's RG calculations. In the Fourier representation, one considers the zero wavenumber limit in the ongrid expression. The powers of wavenumbers correspond to the gradients of various orders in the spatial representations. The second term in (9) is the average

of product of the two subgrid terms. Their aggregate wavenumber may be small if only both factors have almost equal (but oppositely directed) large wavevectors. We conclude that in the above equation, those terms are largest which have a maximum derivative of the subrid terms in the physical space. Thus we arrive at the reduced equation for the subgrid pressure $p'(\boldsymbol{x})$:

$$\Delta p'(\boldsymbol{x}) = -\frac{1}{\varepsilon(\boldsymbol{x}, l)} \nabla \varepsilon'(\boldsymbol{x}) \cdot \nabla p(\boldsymbol{x}, l)$$

that reproduces the Landau–Lifschits model [10]. They estimated the contribution of the random fine-dispersed dielectric permeability to the effective one, that is

$$p'(\boldsymbol{x}) = \int G(\boldsymbol{x} - \boldsymbol{x}') \frac{1}{\varepsilon(\boldsymbol{x}', l)} \nabla_j \varepsilon'(\boldsymbol{x}') \cdot d\boldsymbol{x}' \nabla_j p(\boldsymbol{x}, l),$$

where $G(\boldsymbol{x} - \boldsymbol{x}')$ is the Green's function for the Laplace equation. Again, we have to retain the terms with lower derivatives of the large-scale field $p(\boldsymbol{x}, l)$ treating them as constants in the integrand. Substituting this solution into the filtered equation, we come to the following expression for the subgrid term in (9)

$$\langle \varepsilon'(\boldsymbol{x}) \nabla_i p'(\boldsymbol{x}) \rangle_< = -\frac{1}{\varepsilon(\boldsymbol{x}, l)} \int \nabla_i G(\boldsymbol{x} - \boldsymbol{x}') \langle \varepsilon'(\boldsymbol{x}) \nabla'_j \varepsilon'(\boldsymbol{x}') \rangle_< \cdot d\boldsymbol{x}' \nabla_j p(\boldsymbol{x}, l).$$

Integrating by parts and taking into account the isotropy of $\varepsilon'(\boldsymbol{x})$ we obtain

$$\langle \varepsilon'(\boldsymbol{x}) \nabla_i p'(\boldsymbol{x}) \rangle_< = -\frac{1}{D\varepsilon(\boldsymbol{x}, l)} \langle \varepsilon'(\boldsymbol{x}) \varepsilon'(\boldsymbol{x}) \rangle_< \nabla_i p(\boldsymbol{x}, l). \tag{11}$$

This term is of the Landau–Lifschitz type. The subgrid formula is supported by the following direct numerical experiments for the logarithmic-normal scaling distribution (see also [5, 6]).

We consider the displacement of a front between two similar fluids while they flow through the nonhomogeneous porous medium. The interface between the fluids are labelled by the particles that passively flow with the fluid. Viscosities and another physical parameters of the fluids are the same. The passive labels are described by the equation

$$m(\boldsymbol{x}) \frac{d\boldsymbol{x}^{(i)}}{dt} = -\varepsilon(\boldsymbol{x}) \nabla p(\boldsymbol{x}).$$

The porosity $m(\boldsymbol{x})$ is modelled similarly to$\varepsilon(\boldsymbol{x})$. We consider various self-similar correlations. An example of probability density is shown in Fig. 1.

Let us now consider the correlation functions of the log-stable permeability. The stable distribution is described by the four indices α, β, γ, λ. The index $0 < \alpha \leq 2$ is the most important one. The Gaussian case corresponds to $\alpha = 2$. The other values of α describe various stable distributions. They have the probability distributions but do not have moments of order $n \geq \alpha$. The indices β, γ, λ describe the skewness of the distribution, its center, and scale. Boufadel et al [11] argues that the experimental data about certain sandstones may have the log-stable distributions with $\alpha \approx 1.86$, $\beta = 1$.

Expression (11) contains the second correlation function of the subgrid permeability. Contrary to $\phi(\boldsymbol{x}, l)$, the moments of the function $\varepsilon'(\boldsymbol{x})$ are determined by the exponential function (8) and exist at $\beta = 1$. This assertion uses theorem 2.5.2 [9]. The case $\beta = 1$ at $\alpha < 2$ is the only one when the necessary moments exist. To define the log-stable distributed $\varepsilon(\boldsymbol{x})$ we use the discrete approximation. In formula (4), the integral is replaced by the finite difference formula

$$\varepsilon(\boldsymbol{x}) = \varepsilon_0 2^{\left[-\delta\tau \sum\limits_{i=0}^{N} \varphi(\boldsymbol{x}, \tau_i)\right]}, \tag{12}$$

where $l = 2^\tau$, $\delta\tau = \delta \log_2 l$ is a step in logarithm of scale.

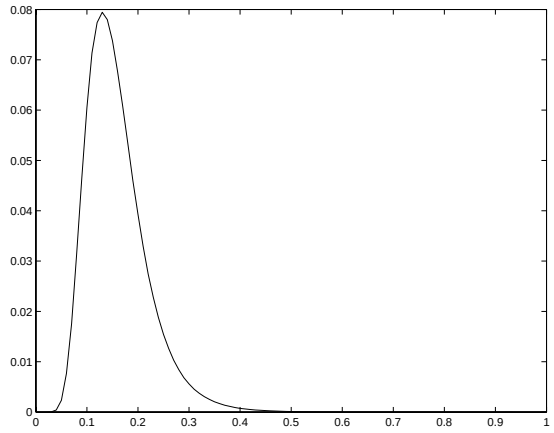

Figure 1. The probability density of porosity.

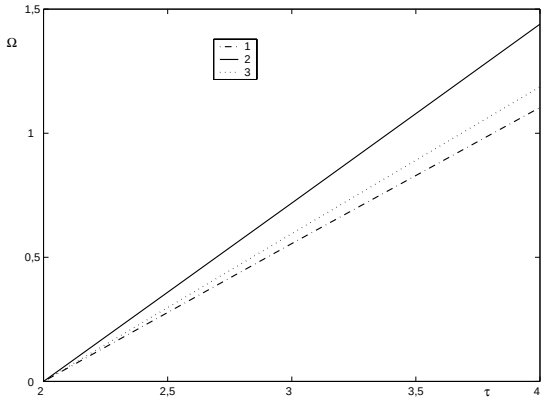

Figure 2. Direct numerical test of the subgrid formula. The flow is along the axis y. $\tau = \log_2 l$, where l is the minimum scale in (4), $\Omega = \mathrm{Log}_2\langle (dy/dt)/u\rangle$, u is some suitable scale of velocity. 1 – negative correlation of m, ε; 2 – the intermediate positive correlation of m, ε; 3 – the positive correlation of m, ε. The theoretical formula fits the computed values.

We define the stable field $\varphi(\boldsymbol{x}, \tau_i)$ by a simple formula, via a discrete sum of the same independent stable variables ζ_k [12], expression 2.3.6 $\varphi(\boldsymbol{x}_j, \tau) = a_{jk} Y_k$. The sum over repeated indices is assumed. The similarity to the Wiener's formula, which expresses the Gaussian field in terms of the white Gaussian noise is obvious. Here the stable process is expressed in terms of the stable white noise. Average of (5) over subgrid fluctuations of $\varphi(\boldsymbol{x}, \tau)$ leads to products of expressions which are similar to characteristic functions of a stable variable of the form

$$\left\langle \varepsilon_0 2^{\left[-\delta\tau \sum\limits_{l=1}^{N} \varphi(\boldsymbol{x},\tau_l)\right]} \right\rangle_{<} = \prod_{\boldsymbol{j}}^{l} \left\langle \exp\left(-a^l_{\boldsymbol{i},\boldsymbol{j}} \zeta^l_{\boldsymbol{j}} \delta\tau\right) \right\rangle_{<} .$$

All the layers are averaged over the statistically independent layers which correspond to different scales, labelled by the index $l = 1, 2, \ldots, N$; $\delta\tau$ is a step in the logarithm of scale, $\boldsymbol{i}$, $\boldsymbol{j}$ correspond to the points of the spatial grid. The kernel $a^l_{\boldsymbol{ij}}$ denotes the spherical and scale-symmetric function $a^l_{\boldsymbol{ij}} \equiv a^l\left(\frac{|\boldsymbol{i}-\boldsymbol{j}|}{l}\right)$. The averages are calculated using the stable distributions contained in [9]. Integrals converge for both $\beta = 1$, $\alpha > 1$ and $\beta = 1$, $\alpha < 1$. The result equals

$$\left\langle \varepsilon_0 2^{\left[-\delta\tau \sum\limits_{l'<l} \varphi(\boldsymbol{x},\ln l')\right]} \right\rangle_{<} = \exp\left\{ \delta\tau \sum^{l'<l} \lambda \left[-a^{l'}_{\boldsymbol{ij}} \gamma + \left(a^{l'}_{\boldsymbol{ij}}\right)^{\alpha} \right] \right\}.$$

Taking into account the layers $l' > l$ that are not averaged, we obtain

$$\varepsilon(\boldsymbol{x}, l) = \varepsilon_0 \exp\left[-\sum_{\boldsymbol{j}}^{l'>l} a^{l'}_{\boldsymbol{ij}} \zeta^{l'}_{\boldsymbol{j}} + \left\{ \sum_{\boldsymbol{j}}^{l'<l} \lambda \left[-a^{l}_{\boldsymbol{ij}} \gamma + \left(a^{l}_{\boldsymbol{ij}}\right)^{\alpha} \right] \right\} \right],$$

$$\varepsilon'(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}, l) \left\{ \exp\left[-\sum_{\boldsymbol{j}}^{l'<l} a^{l'}_{\boldsymbol{ij}} \zeta^{l'}_{\boldsymbol{j}} + \lambda \left\{ \sum_{\boldsymbol{j}}^{l'<l} \lambda \left[a^{l'}_{\boldsymbol{ij}} \gamma - \left(a^{l'}_{\boldsymbol{ij}}\right)^{\alpha} \right] \right\} \right] - 1 \right\}.$$

According to (11), the second moment of the subgrid permeability is needed. Similar direct evaluation yields:

$$\left\langle \varepsilon'(\boldsymbol{x})\,\varepsilon'(\boldsymbol{x})\right\rangle_{<} = \varepsilon(\boldsymbol{x},l)\,\varepsilon(\boldsymbol{x},l) \times\left\{\left\langle \exp\left[-2\sum_{\boldsymbol{j}}^{l'<l} a_{\boldsymbol{ij}}^{l'}\zeta_{\boldsymbol{j}}^{l'}\right]\right\rangle \exp\left[2\lambda\left\{\widehat{\sum_{\boldsymbol{j}}^{l}}\left[a_{\boldsymbol{ij}}^{l}\gamma-\left(a_{\boldsymbol{ij}}^{l}\right)^{\alpha}\right]\right\}\right]-1\right\}.$$

In the first order of the perturbation theory,

$$\left\langle \varepsilon'(\boldsymbol{x})\,\nabla p'(\boldsymbol{x})\right\rangle_{<} \approx -\frac{\lambda\sum_{\boldsymbol{j}}^{l'<l}\left[\left(2^{\alpha}-2\right)\left(a_{\boldsymbol{ij}}^{l'}\right)^{\alpha}\right]}{D}\,\varepsilon(\boldsymbol{x},l)\,\nabla p(\boldsymbol{x},l).$$

Substituting this into (9), we can conclude that the ongrid pressure is described by formula (5) in which ε_0 is replaced by the effective constant ε_{0l}. The latter satisfies the equation

$$\frac{d\ln\varepsilon_{0l}}{d\ln l}=\lambda\sum^{l'<l}\left[-a_{ij}^{l'}\gamma+\left(1-\frac{2^{\alpha}-2}{D}\right)\left(a_{ij}^{l'}\right)^{\alpha}\right].$$

Let us consider a particular simple case in which the statistical smoothing is most similar to the spatial one

$$\sum_j\left(a_{ij}^{l}\right)^{\alpha}=\gamma,\qquad \sum_j a_{ij}^{l}=1.$$

In this case, the statistical smoothing does not change the average permeability that is equal to ε_0. However the effective permeability is described by the equation

$$\frac{d\ln\varepsilon_{0l}}{d\ln l}=-\frac{2^{\alpha}-2}{D}\lambda\gamma.$$

The effective permeability by the main scale L equals to ε_{00}

$$\varepsilon_0=\varepsilon_{00}\left(\frac{l_0}{L}\right)^{-\frac{2^{\alpha}-2}{D}\lambda\gamma}. \tag{13}$$

Equality (13) is one of the main results of the present paper. It is similar to the Landau–Lifschitz formula but describes the log-stable media. It contains the two constants ε_0 and ε_{00}. The latter describes the mean flow through the media $\overline{\boldsymbol{v}}=\varepsilon_{00}\nabla\overline{p}$. The former is the constant of the self-similar permeability (4). The fact that those constants are different, is the evidence that the subgrid fluctuations of the pressure are essential. It is evident that the effect of fluctuations changes its sign when α crosses 1. On the other hand, it is seen that the power in (13) is small at large D and/or $\alpha\to 1$. This gives the hope that $\frac{2^{\alpha}-2}{D}\lambda\gamma$ may be used as a parameter of expansion in the perturbation theory. To reveal this possibility we consider the terms of higher orders.

Let us again consider equation (10) and substitute $\varepsilon(\boldsymbol{x})=\varepsilon(\boldsymbol{x},l)+\varepsilon'(\boldsymbol{x})$. The fluctuations of pressure are the sum of terms of different orders

$$p'(\boldsymbol{x})=p_1'(\boldsymbol{x})+p_2'(\boldsymbol{x})+p_3'(\boldsymbol{x})+\cdots.$$

We have considered the term of the first order. The terms of the second order yield the equation for $p_2'(\boldsymbol{x})$

$$\nabla_m\left[\varepsilon(\boldsymbol{x},l)\nabla_m p_2'(\boldsymbol{x})\right]+\nabla_m\left[\varepsilon'(\boldsymbol{x})\,\nabla_m p_1'(\boldsymbol{x})-\langle\varepsilon'(\boldsymbol{x})\nabla_m p_1'(\boldsymbol{x})\rangle_{<}\right]=0. \tag{14}$$

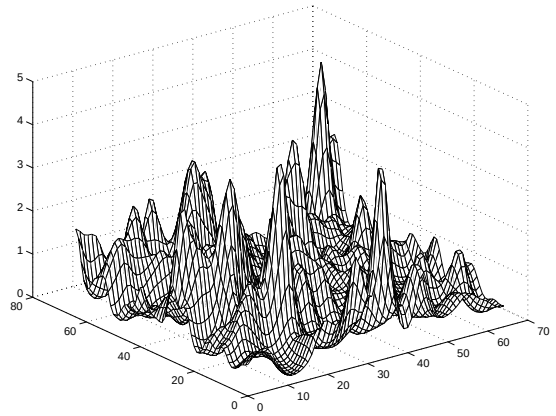

Figure 3. Typical realizations of the logarithmic-normal process.

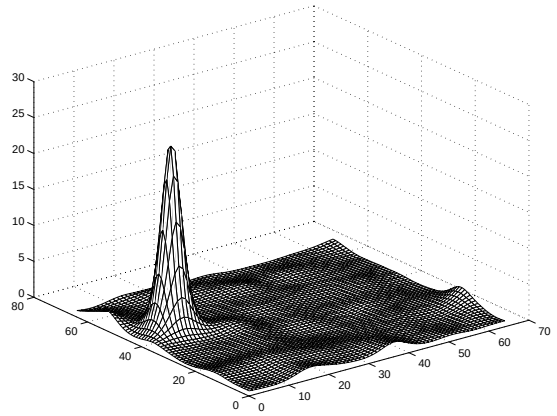

Figure 4. A realization of the log-stable process, $\alpha = 1.86$.

The solution for $p_2'(\boldsymbol{x})$ is obtained as for $p_1'(\boldsymbol{x})$ in the large-scale limit

$$p_2'(\boldsymbol{x}) = -\frac{1}{\varepsilon(\boldsymbol{x},l)^2}\int G(\boldsymbol{x}-\boldsymbol{x}_1) \times \nabla_{1m}\left[\begin{array}{c} \varepsilon'(\boldsymbol{x}_1)\nabla_{1m}\left\{\int d\boldsymbol{x}' G(\boldsymbol{x}_1-\boldsymbol{x}')\nabla_j\varepsilon'(\boldsymbol{x}')\nabla_j p(\boldsymbol{x}_1,l)\right\} \\ -\langle\varepsilon'(\boldsymbol{x}_1)\nabla_{1m}\left\{\int d\boldsymbol{x}' G(\boldsymbol{x}_1-\boldsymbol{x}')\nabla_j\varepsilon'(\boldsymbol{x}')\nabla_j p(\boldsymbol{x}_1,l)\right\}\rangle_< \end{array}\right],$$

When evaluating this integral the third-order correlation function appear which can be calculated in the same manner as above.

$$\begin{aligned}
&\left\langle\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\varepsilon'(\boldsymbol{x})\right\rangle_< \\
&\quad= \varepsilon^3(\boldsymbol{x},l)\left\langle\left\{\exp\left[-\sum_{\boldsymbol{j}}^{l'<l} a_{\boldsymbol{ij}}^{l'}\zeta_{\boldsymbol{j}}^{l'} + \lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[a_{\boldsymbol{ij}}^{l'}\gamma - \left(a_{\boldsymbol{ij}}^{l'}\right)^{\alpha}\right]\right\}\right] - 1\right\}^3\right\rangle_< \\
&\quad\approx \varepsilon^3(\boldsymbol{x},l)\left\{\begin{array}{c} \exp\left[(3^\alpha-3)\lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[\left(a_{\boldsymbol{ij}}^{l'}\right)^{\alpha}\right]\right\}\right] \\ -3\exp\left[(2^\alpha-2)\lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[a_{\boldsymbol{ij}}^{l'}\gamma - \left(a_{\boldsymbol{ij}}^{l'}\right)^{\alpha}\right]\right\}\right] + 2 \end{array}\right\} \\
&\quad\approx \varepsilon^3(\boldsymbol{x},l)\left\{\begin{array}{c} \exp\left[(3^\alpha-3)\lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[\left(a_{\boldsymbol{ij}}^{l'}\right)^{\alpha}\right]\right\}\right] \\ -3\exp\left[(2^\alpha-2)\lambda\left\{\sum_{\boldsymbol{j}}^{l'<l}\lambda\left[a_{\boldsymbol{ij}}^{l'}\gamma - \left(a_{\boldsymbol{ij}}^{l'}\right)^{\alpha}\right]\right\}\right] + 2 \end{array}\right\}.
\end{aligned}$$

Obviously, this term is of the next order in the parameter of expansion. The general formula is as follows

$$\left[\varepsilon(\boldsymbol{x},l)\Delta p_{n+1}'(\boldsymbol{x})\right] + \nabla_i\left[\varepsilon'(\boldsymbol{x})\nabla_i p_n'(\boldsymbol{x}) - \langle\varepsilon'(\boldsymbol{x})\nabla_i p_n'(\boldsymbol{x})\rangle_<\right] = 0.$$

Iterating the formula and inserting the result into (9), it is possible to conclude that the high-order terms are smaller in the parameter of expansion. Iterating the formula and inserting the result into (9), one sees that the high-order terms are smaller in the parameter of expansion.

The comparison of the direct numerical simulation with the derived formula is complicated by the following fact. Fig. 3 shows the comparison of the logarithmic-normal ($\alpha = 2$) and Fig. 4 – log-stable (at $\alpha = 1.86$) realizations of permeability.

As is known and confirmed by Fig. 4, the stable processes (and the log-stable) are statistically represented by rare but large events. In order that the statistical characteristics of such processes be found, one needs greater spatial regions. For this reason, the direct numerical test of formula (13) for various values of α will be published in a separate paper.

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Asymptotic Approach to the Quantum Mechanical Problem of the Three Coulomb Centers

Volodymyr LAZUR, Svitlana MYHALYNA and Mykhaylo KHOMA

Uzhgorod National University, 32 Voloshyna Str., 88000 Uzhgorod, Ukraine
E-mail: *lazur@univ.uzhgorod.ua, khoma@uzhgorod.ukrsat.com*

The asymptotic properties of the solution of quantum-mechanical three Coulomb centers problem eZ_1ZZ are studied. Within the framework of the perturbation theory the asymptotic formulas for energies of eZ_1ZZ system are obtained at large separation L between interacting fragments. Asymptotic for one-electron three-center wave function of the hydrogen-like atomic ion in the vicinity of two Coulomb centers $Z + Z$ are constructed. As the applications of obtained results the leading term of the asymptotic of exchange interactions between hydrogen-like molecular ion eZZ with nuclei of different elements are calculated.

1 Introduction

The solutions of a Schrödinger equation with potentials of two and three Coulomb centers represents considerable interest from the point of view of different problems relating few-body systems, considered in adiabatic approximation. In molecular physics these systems play the same fundamental role as atom of hydrogen in atomic physics [1]. The results obtained while solving many-center Coulomb problems have also numerous and relevant applications in physics of slow atomic-molecular collisions, scattering theory, nuclear physics, spectroscopy of complex chemical compounds etc. However till now there is a deep asymmetry in progressing the theory and methods of a two-center eZ_1Z_2 and three-center $eZ_1Z_2Z_3$ Coulomb problems solving. Comparatively with effective asymptotic and numerical methods of eZ_1Z_2 problem solving only a few attempts to solve the three-center problem $eZ_1Z_2Z_3$ using the different approximate methods (see, for example, [2, 3]) are represented now. The reason of such situation is probably that the Schrödinger equations with potential of two Coulomb centers enables variable separation in prolate spheroidal coordinates, so that the problem of finding integrals of motion for system eZ_1Z_2 is reduced to a one-dimensional spectral problem. The possibility of variable separation shows that the system eZ_1Z_2 has higher dynamical symmetry in addition to the geometrical one. This additional dynamical symmetry results in that together with the operator L_z of the electron angular momentum projections to an internuclear axis, the so-called operator Λ of the separation constant [1] commuted with a Hamiltonian of a eZ_1Z_2 system. Naturally the question arises, whether it is possible to find the operator Λ, commuting with a Hamiltonian in many-center Coulomb problems, so that to achieve separation of variables in them also? Coulson and Joseph [4] on an example of a four-center problem showed, that the answer is negative. More general result was obtained by Roy [5], who investigated a dynamical symmetry of multi-center problems. From his work follows, in particular, that the Schrödinger equations of a problem $eZ_1Z_2Z_3$ are not separable in any orthogonal coordinate system and it is necessary to deal with partial differential equations that essentially complicate the problem of calculation electronic wave functions and energy terms. The above mentioned features essentially handicap in-depth analysis of a problem $eZ_1Z_2Z_3$ and require to use some approximate methods. The development of the majority of them is expedient for different limit cases and, in particular, for study of asymptotic properties of the solutions of $eZ_1Z_2Z_3$ problem, both for large and on small intercentre separation.

In the present work we study the asymptotic behavior of discrete spectrum of eZ_1ZZ system (one electron and three fixed nuclear charges: Z_1 and $Z_2 = Z_3 = Z$). This system can serve as a model for collision systems consisting of three ions with closed electronic shells (two of them being identical) and one "active" bound electron. We consider the Schrödinger equation for the problem of electron motion in the field of three Coulomb centers:

$$\hat{H}\Psi \equiv \left(-\frac{\triangle}{2} - \frac{Z_1}{|\vec{r} - \vec{R}_1|} - \frac{Z}{|\vec{r} - \vec{R}_2|} - \frac{Z}{|\vec{r} - \vec{R}_3|} \right) \Psi(\vec{r}; \vec{Q}) = E(\vec{Q})\Psi(\vec{r}; \vec{Q}), \tag{1}$$

where $\vec{r}$ is the radius-vector of the electron, $\vec{R}_i$ is the radius-vector of the i-th nucleus ($i = 1, 2, 3$), $r_i = |\vec{r} - \vec{R}_i|$ is the distance from the electron to i-th nucleus, $E(\vec{Q})$ and $\Psi(\vec{r}, \vec{Q})$ are the electron energy and wave function, respectively, that depend on three coordinates $Q_1 = L$, $Q_2 = R$, $Q_3 = \beta$, $\vec{Q} = (L, R, \beta)$, the meaning of which is shown in Fig. 1.

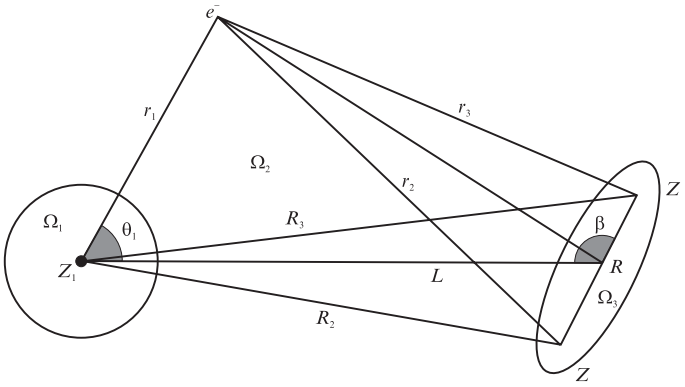

Figure 1. Geometry of quasi-molecule eZ_1ZZ and used notation.

2 Asymptotic expressions for the potential energy surfaces of eZ_1ZZ system

In the limit $L \to \infty$ the solutions of equation (1) are localized either near the nucleus Z_1 or near the two identical charges (ions), Z+Z. Thus, Ψ_I is the wave function that corresponds to the case when system eZ_1ZZ is separated as a hydrogen-like atomic ion eZ_1 and two identical charges, and Ψ_{II} corresponds to the case of infinitely separated a hydrogen-like molecular ion eZZ and a charge Z_1. The energies $E(\vec{Q})$ of eZ_1ZZ system in the limit $L \to \infty$ can be classified in an analogous manner: $E_I(E_{II})$ energies go over into the energy levels of isolated atomic (molecular) ion $eZ_1(eZZ)$ for asymptotically large L. We characterize $E_I(\vec{Q})$ and Ψ_I by the set of (parabolic) quantum numbers $I = [n_1 n_2 m]$ which describe the states of isolated hydrogen-like ion eZ_1; for $E_{II}(\vec{Q})$ and Ψ_{II} will be characterized by the set of (spheroidal) quantum numbers $II = [k, q, m']$ which describe the states of molecular ion eZZ. The function Ψ_I we expand over the Coulomb parabolic functions [6] $\varphi_{n_1 n_2 m}(\mu, \nu, \varphi_1)$:

$$\Psi_I = \sum_{n_1' n_2' m'} a_{n_1' n_2' m'}(R_2, R_3, \theta_2, \theta_3)\varphi_{n_1' n_2' m'}(\mu, \nu, \varphi_1), \tag{2}$$

$$\varphi_{n_1 n_2 m}(\mu, \nu, \varphi_1) = \frac{\sqrt{2}Z_1^{3/2}}{n^2} f_{n_1 m}\left(\frac{Z_1\mu}{n}\right) f_{n_2 m}\left(\frac{Z_1\nu}{n}\right) \frac{e^{im\varphi_1}}{\sqrt{2\pi}}, \tag{3}$$

where

$$f_{pm}(\rho)=\frac{1}{|m|!}\sqrt{\frac{(p+|m|)!}{p!}}\Phi(-p,|m|+1,\rho)\exp(-\rho/2)\rho^{|m|/2}, \tag{4}$$

Φ is the confluent hypergeometric function of the first kind [7], n is the principal quantum number, and $\mu=r_1(1+\cos(\theta_1))$ and $\nu=r_1(1-\cos(\theta_1))$ are the parabolic coordinates. We represent the wave function $\Psi_{II}(\vec{r};\vec{Q})$ as expansion

$$\Psi_{II}(\vec{r};\vec{Q})=\Psi_i(\xi,\eta,\varphi;\widetilde{\xi},\widetilde{\eta},R)=\sum_j\sum_{m_j}a_{ijm_j}(\widetilde{\xi},\widetilde{\eta},R)\varphi_{jm_j}(\xi,\eta,\varphi;R) \tag{5}$$

over the two-center wave functions φ_{jm_j} of eZZ discrete spectrum

$$\varphi_{jm_j}(\xi,\eta,\varphi;R)=N_{jm_j}\Pi_{jm_j}(\xi;R)S_{jm_j}(\eta;R)\frac{\exp(im_j\varphi)}{\sqrt{2\pi}}, \tag{6}$$

where N_{jm_j} is the normalization factor, $\Pi_{jm_j}(\xi;R)$ and $S_{jm_j}(\eta;R)$ are the quasiradial and quasiangular spheroidal wavefunctions [1], m_j is the projection of angular momentum on the axis $\vec{R}$, and j designate all other quantum numbers [1,8]. The coordinates $\widetilde{\xi}$, $\widetilde{\eta}$ used in equation (5), are defined by (see Fig. 1)

$$\begin{aligned}\widetilde{\xi}&=\left(\sqrt{L^2-RL\cos(\beta)+R^2/4}+\sqrt{L^2+RL\cos(\beta)+R^2/4}\right)R^{-1},\\ \widetilde{\eta}&=\left(\sqrt{L^2-RL\cos(\beta)+R^2/4}-\sqrt{L^2+RL\cos(\beta)+R^2/4}\right)R^{-1}.\end{aligned} \tag{7}$$

The energies E_I of the system eZ_1+Z+Z (electron is predominantly localized in the Ω_I-region; see Fig. 1) in the first order of perturbation theory are given by

$$E_I(\vec{Q})\equiv E_{n_1n_2m}(R_2,R_3,\widetilde{\gamma})=-\frac{Z_1^2}{2n^2}-\left(\frac{Z}{R_2}+\frac{Z}{R_3}\right)+\frac{3Zn\triangle}{2Z_1}\left(\frac{1}{R_2^4}+\frac{1}{R_3^4}+\frac{2\cos\widetilde{\gamma}}{R_2^2R_3^2}\right)^{1/2}, \tag{8}$$

where $\triangle=n_1-n_2$, and $\widetilde{\gamma}$ is the angle between $\vec{R}_2$ and $\vec{R}_3$. At sufficiently large distances L between Z_1 and eZZ, the potential energy surfaces of $eZZ+Z_1$ quasi-molecular system in the second order approximation of perturbation theory are given by the expression

$$\begin{aligned}E_I(\vec{Q})\equiv E_i(\widetilde{\xi},\widetilde{\eta},R)&=\varepsilon_i(R)-\frac{Z_1R^2}{4\widetilde{\xi}}\left[1+\left(\frac{1}{3}+\frac{3\widetilde{\eta}^2-1}{2}A_{ii}^{(2)}(R)\right)\frac{1}{\widetilde{\xi}^2}\right]+\\ &+\frac{Z_1^2R^4}{16\widetilde{\xi}^4}\left[(1-\widetilde{\eta}^2)\alpha_i^{(+)}(R)+\widetilde{\eta}^2\alpha_i^{(-)}(R)\right],\qquad \alpha_i^{(\pm)}(R)=\sum_j{}'\frac{\left[A_{ii}^{(2)}(R)\right]^2}{\varepsilon_i(R)-\varepsilon_j(R)},\end{aligned} \tag{9}$$

where coefficients $A_{ij}^{(n)}(R)$ are given as

$$\begin{aligned}A_{ij}^{(n)}(R)&=\frac{2^n n!(n-\kappa)!^2}{(2n)!(n+\kappa)!}N_{ij}(R)\\ &\times\left(\int_{-1}^{1}S_i(\eta;R)P_n^\kappa(\eta)S_j(\eta;R)d\eta\int_1^\infty\Pi_i(\xi;R)P_n^\kappa(\xi)Pi_j(\xi;R)\xi^2d\xi\right.\\ &\left.-\int_{-1}^{1}S_i(\eta;R)P_n^\kappa(\eta)S_j(\eta;R)\eta^2d\eta\int_1^\infty\Pi_i(\xi;R)P_n^\kappa(\xi)Pi_j(\xi;R)d\xi\right),\end{aligned} \tag{10}$$

$\kappa=|m_i-m_j|$, $N_{ij}(R)=N_i(R)N_j(R)$, and $\varepsilon_i(R)$ are the energies of hydrogen-like molecular ion eZZ (the subscript m_j in the notations for $\Pi_{jm_j}(\xi;R)$, $S_{jm_j}(\eta;R)$ and N_{jm_j} has been omitted). Formulae similar to (8) and (9) for the specific case of $Z_1+H_2^+$ system have been derived in [2,3].

3 Asymptotic expression for the wave function of $A^{(Z_1-1)+}$ ion in the vicinity of two Coulomb centers $Z+Z$

The asymptotic form of three-center wave function $\Psi_I(\vec{r};\vec{Q})$ of ion $A^{(Z_1-1)+}$ in vicinity of two Coulomb centers $Z+Z$ is necessary to calculate the asymptotic of two-electron exchange interaction (the matrix element for two-electron transition) between highly charged ion and diatomic molecule. The wave function $\Psi_I(\vec{r};\vec{Q})$ satisfies the following integral relation

$$\Psi_I(\vec{r};\vec{Q}) = -\frac{1}{2}\int_S d\vec{S}\left[\Psi_I(\vec{r}\,';\vec{Q})\vec{\nabla}G_{E_I}(\vec{r},\vec{r}\,';\vec{Q}) - G_{E_I}(\vec{r},\vec{r}\,';\vec{Q})\vec{\nabla}\Psi_I(\vec{r}\,';\vec{Q})\right], \tag{11}$$

where S is the surface that surrounds the volume containing the nuclei $Z+Z$. The one-electron three-center Green's function $G_{E_I}(\vec{r},\vec{r}\,';\vec{Q})$, appearing in equation (11), satisfies the equation:

$$\left(-\frac{\triangle}{2} + V_1\left(|\vec{r}+\vec{L}|\right) - \frac{Z}{|\vec{r}+\vec{R}/2|} - \frac{Z}{|\vec{r}-\vec{R}/2|} - E_I\right) G_{E_I}(\vec{r},\vec{r}\,';\vec{Q}) = \delta(\vec{r}-\vec{r}\,'). \tag{12}$$

Here E_I is the energy of the system eZ_1+Z+Z when the electron is predominantly localized in the Ω_I-region; see Fig. 1. For $L\to\infty$, E_I goes into the energy levels of isolated atomic ion eZ_1. The potential $V_1(r_1)$ in equation (12) is a spherically symmetric effective potential which characterizes the field of A^{Z_1+} atomic ion and asymptotically behaves as

$$V_1(r_1) \underset{r_1\to\infty}{\to} -\frac{Z_1}{r_1}. \tag{13}$$

At large distances between Z_1 and $Z+Z$, we represent G_{E_I} as product $G_{E_I} = G^{(0)}_{E_I^{(0)}}\chi(L)$, where $\chi(L)$ is a correction function obtained in [9] and $G^{(0)}_{E_I^{(0)}}$ is the Green's function for two identical Coulomb centers $Z+Z$:

$$\left(-\frac{\triangle}{2} - \frac{Z}{|\vec{r}+\vec{R}/2|} - \frac{Z}{|\vec{r}-\vec{R}/2|} - E_I^{(0)}\right) G^{(0)}_{E_I^{(0)}}(\vec{r},\vec{r}\,';\vec{R}) = \delta(\vec{r}-\vec{r}\,'). \tag{14}$$

Here $E_I^{(0)}$ is the electron energy of isolated ion eZ_1. Our work is dedicated to investigation of the two-center Coulomb Green's function $G^{(0)}_{E_I^{(0)}}$ has been studied in [10] in detail. Using the results of [10], one obtains the following expression for $\Psi_I\left(\vec{r};\vec{Q}\right)$:

$$\Psi_I\left(\vec{r};\vec{Q}\right) = D_I(\alpha_1,L)\sum_{\ell=0}^{\infty}\sum_{m\geq m_1}^{+\ell}\widetilde{B}^{\ell}_{mm_1}(p_I;\alpha,\beta)\,\Pi^{(1)}_{m\ell}(p_I,\xi)\,\bar{S}_{m\ell}(p_I,\eta)\,\frac{e^{im\varphi}}{\sqrt{2\pi}}, \tag{15}$$

$$D_I(\alpha_1,L) = 2\widetilde{A}_{\alpha_1\ell_1}\left(\frac{2}{\alpha_1}\right)^{m_1}(2\alpha_1)^{2Z/\alpha_1}\,m_1!B_{\ell_1m_1}e^{-2Z/\alpha_1}L^{\frac{Z_1+2Z}{\alpha_1}-m_1-1}e^{-\alpha_1 L}, \tag{16}$$

$$\widetilde{B}^{\ell}_{mm_1}(p_I;\alpha,\beta) = (-1)^{m_1+m}N^{-1}_{m\ell}(p_I)\sum_{s=0,1}^{\infty}(-1)^s C_{m_1,m+s}d^{m\ell}_s(p_I)\,D^{m+s\,*}_{m_1m}(\alpha,\beta,0), \tag{17}$$

$$B_{\ell_1m_1} = \frac{1}{2^{m_1}m_1!}\left(\frac{2\ell_1+1}{2}\frac{(\ell_1+m_1)!}{(\ell_1-m_1)!}\right)^{1/2}, \qquad C_{m_1,m+s} = \frac{(m+m_1+s)!}{2^{m_1}m_1!(m+s-m_1)!}. \tag{18}$$

The coefficient $\widetilde{A}_{\alpha_1\ell_1}$ is related to the normalization constant of asymptotic wave function of eZ_I system, $D^{m+s\,*}_{m_1m}(\alpha,\beta,0)$ is the Wigner rotation matrix [6], $\alpha_1=\sqrt{-2E_I}$ and $p_I = R\sqrt{-2E_I}/2$.

The regular solution $\Pi^{(1)}_{m\ell}(p_I,\xi)$ is given as expansion over the Coulomb radial wave function

$$\Pi^{(1)}_{m\ell}(p_I,\xi)=\left(\frac{\xi-1}{\xi+1}\right)^{m/2}\sum_{s=-\infty}^{\infty}h_s\left(\alpha_1,\lambda_{m\ell},\nu\,|p_I\right)R^{(1)}_{\nu+s}(x),\qquad x=p_I\left(\xi+1\right), \tag{19}$$

$$R^{(1)}_{\nu+s}(x)=x^{\nu+s}e^{-x}\Phi\left(-\alpha_1+\nu+s+1,2\nu+2s+2;2x\right), \tag{20}$$

where $\Phi(a,b,x)$ is the confluent hypergeometric function and $\lambda_{m\ell}$ is the value of separation constant [1] for the two-center system $Z+Z$ with given electron energy E_I and internuclear distance R. Asymptotic behavior at large ξ of the $\Pi^{(1)}_{m\ell}(p_I,\xi)$ solutions leads to the normalization condition for coefficients h_s:

$$\sum_{s=-\infty}^{\infty}h_s\frac{\Gamma\left(2s+2\nu+2\right)}{2^{s+\nu}\Gamma\left(s+\nu+1-\tilde{\alpha}\right)}=1. \tag{21}$$

The procedure for calculation of the acceptable values $\nu=\nu_{m\ell}(p_I)$ (which ensure convergence of the expansion (19)) is described in [11, 12].

4 Exchange interaction of hydrogen-like molecular ion with a nucleus

Results obtained above can be used for study of the many problems of atomic theory and, in particular, of the problems of slow ion-molecules collision. Consider the following charge-exchange reaction:

$$eZZ+Z_1\to eZ_1+Z+Z \tag{22}$$

at low collision velocities. The exchange interaction $\triangle(\vec{Q})$ between adiabatic electronic states of quasi-molecules $eZZ+Z_1$ and eZ_1+Z+Z is given by the expression

$$\triangle(\vec{Q})=\int_S d\vec{S}\left[\Psi^*_I\vec{\nabla}\Psi_{II}-\Psi^*_{II}\vec{\nabla}\Psi_I\right]. \tag{23}$$

Using for Ψ_I and Ψ_{II} the expansions (2) and (5), we obtain (to the leading order of $1/L$)

$$\begin{aligned}\triangle(L,\beta)=&\frac{Z_1^{3/2}2^{Z_1/\alpha_2+1/2}}{n^2(n-1)!}\bar{A}(\beta)\\&\times\left(\frac{Z_1}{n}\right)^{n-1}L^{n+\frac{2Z}{\alpha_2}-1}\exp\left[-\frac{L}{2}\left(\alpha_2+\frac{Z_1}{n}\right)-\frac{1}{2}\left(\frac{Z_1}{\alpha_2}+\frac{2nZ}{Z_1}\right)\right].\end{aligned} \tag{24}$$

The coefficient $\bar{A}(\beta)$ is related to the normalization constant of asymptotic wave function of eZZ system; $\alpha_2=\sqrt{-2E^{(0)}_{II}(R)}$, $E^{(0)}_{II}(R)$ is the energy of molecular ion eZZ, and n is a principal quantum number of the state described by Ψ_I wave function. Obtained result for matrix element of exchange interaction (24) usually used for calculation of the total cross-sections of the one-electron transition at low energy ion-molecule collision [9].

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Symmetry Analysis in Spikes (Bursts) Recognition and Classifications

Alexander MAKARENKO †, *Anatoliy POLIARUSH* † *and Igor TETKO* ‡

† *Institute of Applied System Analysis, National Technical University of Ukraine (KPI), 37 Peremogy Ave., Kyiv, Ukraine*
E-mail: *makalex@mmsa.ntu-kpi.kiev.ua, apolixus@yahoo.com*

‡ *Institute of Bioorganic and Petroleum Chemistry of NAS of Ukraine, Kyiv, Ukraine; Institute for Bioinformatics, Ingolstraedter Lanstrasse 1, D-85764, Neuherberg, Germany*
E-mail: *itetko@vvclab.org*

New approach to spikes (burst) recognition is described. The symmetry properties of recognized objects are the foundation of the methodology. The objects are described by ordinary differential equations in special phase space. Then the symmetry is connected with phase flow of ODE systems. Neuronal spikes recognition is the example of approach application.

1 Introduction

In recent research of data processing, pattern recognition, modelling the problem of recognition of abrupt changes and intrinsic structures is very important. Especially interesting are also applications of recognition of the structural elements in time series data. The examples are the cycles in economics, spikes in neurophysiology, textures in pattern recognition. Analysis follows to the conclusion that structural properties are geometrical properties of some objects. This requires the application of geometrical approach to geometrical properties recognition. Proposed study covers the problem of recognition of objects that have the form of mappings of a segment onto a manifold. The method presented is based on symmetry analysis.

2 Abstract statement of the problem

Let us consider the following function of class k:

$$\varphi : [a; b] \to M, \tag{1}$$

where $[a; b] \subset \mathbb{R}$, and M is some manifold of class k. We shall designate the set of such functions as Φ. So, functions (1) describe geometrical objects in M.

As is known, manifolds are metrizable. Let us designate one of possible distances

$$d_M : M \times M \to [0; +\infty)\,. \tag{2}$$

Similarly we may introduce a distance function between geometrical objects in M

$$d : \Phi \times \Phi \to [0; +\infty)\,. \tag{3}$$

Consider an arbitrary transformation T that acts on Φ:

$$T : \Phi \to \Phi. \tag{4}$$

Then we can define classes $K_T(\varepsilon)$ of geometrical objects in M:

$$K_T : (0; +\infty) \to 2^{\Phi}, \tag{5}$$

where 2^{Φ} is a set of subsets in Φ, and

$$K_T(\varepsilon) = \{\varphi : d(T\varphi, \varphi) < \varepsilon\}. \tag{6}$$

Suppose that $A \subset 2^{\Phi}$ consists of several classes $K_{T_i}(\varepsilon_i)$, $i = 1, \ldots, n$. Then the problem of classification consists in finding T_i if A is given.

3 Geometrical objects described by an ordinary differential equation

Suppose we have n ordinary differential equations of order m:

$$\begin{aligned}
&\frac{dy^{(0)}}{dt} = y^{(1)}, \qquad \frac{dy^{(2)}}{dt} = y^{(2)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{dy^{(m-1)}}{dt} = f_i\left(y^{(0)}, y^{(1)}, \ldots, y^{(m-1)}\right),
\end{aligned} \tag{7}$$

where $i = 1, \ldots, n$, and f is bounded function defined on some area $D \subset \mathbb{R}^m$. Then all functions $y(t)$ that are solutions of (7) induce a set of geometrical objects on $\mathbb{R}^m$. In this particular case (1) takes the form

$$\varphi : A_{\varphi} \subset \mathbb{R} \to \mathbb{R}^m, \tag{8}$$

where

$$\varphi : t \mapsto \left(y^{(0)}(t), y^{(1)}(t), \ldots, y^{(m-1)}(t)\right)^T \tag{9}$$

and φ is a function of class 1. Then for an arbitrary point $p \in D$ there exists the only function $y_p(t)$, that $\left(y_p^{(0)}(0), y_p^{(1)}(0), \ldots, y_p^{(m-1)}(0)\right)^T = p$. Let us define such mapping $X$of D into $\mathbb{R}^m$, that

$$X : p \mapsto \left(y_p^{(1)}(0), y_p^{(2)}(0), \ldots, y_p^{(m)}(0)\right)^T. \tag{10}$$

It is a vector field on D, induced by equation (7). Similarly, we can define a local one-parametric group G, that acts on D:

$$g_{\tau} : p \mapsto \left(y_p^{(0)}(\tau), y_p^{(1)}(\tau), \ldots, y_p^{(m-1)}(\tau)\right)^T. \tag{11}$$

Consider the action of G on set Φ of geometrical objects represented by functions

$$\phi : A_{\phi} \subset \mathbb{R} \to D. \tag{12}$$

by formula

$$g_{\tau} : \Phi \to \Phi, \tag{13}$$

$$g_{\tau}(\varphi)(t) = g_{\tau}(\varphi(t - \tau)). \tag{14}$$

Let us introduce a set of functions

$$Q_{\tau} : \Phi \to [0; +\infty), \tag{15}$$

that act as

$$Q_{\tau} : \varphi \mapsto d(g_{\tau}(\varphi), \varphi). \tag{16}$$

It is obvious that $Q_{\tau}(\varphi) = 0$ for any admissible τ if and only if φ is a solution of (7). Then we have a simple criterion of evaluation of adjustment of the geometrical object to given differential equations (7).

4 Neuronal spikes recognition as example of symmetry analysis application

Analysis of populations of neurons represents an important step for better understanding of brain functioning. The present theories of brain functioning put accent on the neuronal interactions in terms of syncronized firing across many neurons, or spatio-temporal interactions and presence of specific patterns. Many such neuronal interactions cannot be observed with recordings from single neuron. Thus, analysis of neuronal populations does not simply provide an additive scheme for increase experimental data but makes possible to detect some features of brain functioning that could never be observed with recording of only single neuron. We represent at Fig. 1 typical example of spike in neurophysiological data.

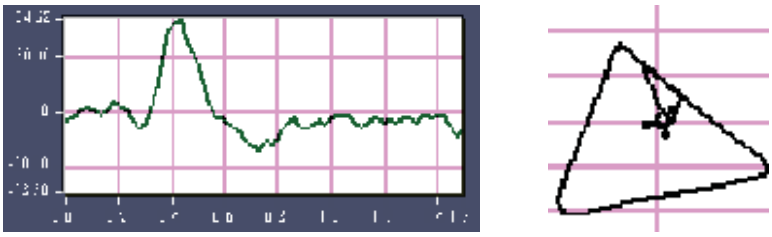

Figure 1. Spike example and its portrait in phase space. The centers of the spikes are at 0.41 ms. The axis x corresponds to time and axis y corresponds to voltage in electrode.

The basic hypothesis used to detect and separate action potential of neurons assumes that spikes generated by the same neurons have similar shapes and these shapes are unique and conservative for each recorded neuron. The shape of neuron spikes detected by the electrode depends on the distance, relative position, properties of the media, e.g., presence and distribution of glia cells, between electrode and neurons. The shape also depends on resistance of electrode and neuron. Thus, even if two neurons are identical, their action potential detected at the microelectrode can be different and, thus, spikes from such neurons could be separated.

The number of different methods used in the neurophysiology for spike sorting dramatically increases. One of the most popular methods is template matching. These techniques use templates that represent some typical waveform shapes of neurons in time domain. A classification of a candidate spike is done by its comparison to all templates and selecting the best matching one.

There are many basic problems that should be solved for successful spike sorting. The first basic question concerns the number of different types of neurons that should be separated in the experimental data. The usual practice is to use a "supervisor", i.e. the experimentator, who can provide a preliminary classification of data, e.g. selection of template spikes, based on his experience and knowledge. However, even if the number of classes is identified, the separation of spikes remains very difficult problem due to extracellular and intracellular noise that can disturb the form of the action potential.

The extracelular noise is usually taken into account by the most of models as an additive noise. The intracellular noise that can produce variation in the spike waveform is more difficult to account for. Recently we have proposed a new method for spike sorting that consider the problem of spike sorting in phase space and describe the spike waveform as an ordinary differential equation with perturbation. This approach made possible to account for both extracellular and intracellular noise. The differential equation describing the activity of a neuron was supposed to have a limit trajectory in phase space, and noise was treated as deviation of the signal from that trajectory. Current study provides further development of this idea. In contrast to that we proposed a numerical method that takes into account that the variety of spike waveforms

generated by the same neuron cannot be explained only by influence of both kinds of noise on some "typical" for a given neuron impulse, but it should be considered as a whole.

Let us suppose that every observed spike $x_i(t)$ of the i-th neuron is a solution of an ordinary differential equation with perturbation

$$\frac{d^3x}{dt^3} = f_i\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) + F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right), \tag{17}$$

where $F(\cdot)$ is a perturbation function. The perturbation function $F(x, \ldots, t)$, bounded by a small value, is a random process with zero mean and small correlation time $\tau^* \ll T$. The solution of the equation

$$\frac{d^3x}{dt^3} = f_i\left(x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right) \tag{18}$$

describes a self-oscillating system. The duration of spikes is limited, so we can suppose that $x(t)$ is defined on some interval $(a; b)$. In practice parameters a and b are chosen manually by an expert. Set of differential equations

$$\frac{dx^{(0)}}{dt} = x^{(1)}, \qquad \frac{dx^{(1)}}{dt} = x^{(2)}, \qquad \frac{dx^{(2)}}{dt} = f_i\left(x^{(0)}, x^{(1)}, x^{(2)}\right), \tag{19}$$

where $x^{(k)}$ is the k-th derivative of $x(t)$ is equivalent to (18).

If function f_i is defined in open domain $D \subset \mathbb{R}^3$ then any solution $x(t)$ of (19) induces phase trajectory $\vec{X}(t) = \left(x^{(0)}(t), x^{(1)}(t), x^{(2)}(t)\right)^T$. In this case the trajectory is also an integral curve for (19). The theorem about existence of a solution for an ordinary differential equation gives that for an arbitrary point $p \in D$ there exists such integral curve $\vec{X}_p(t)$ that $\vec{X}_p(t_0) = p$ for some t_0 and $\vec{X}_p(t)$ is a solution of (19).

The equation (18) induces a local one-parameter group G on D. Let us define some action $g_\tau \in G$ on D. Consider integral curve $\vec{X}_p(t)$ that passes p when $t = t_0$. Then

$$g_\tau : p \mapsto \vec{X}_p(\tau + t_0), \tag{20}$$

i.e., $g_\tau(p)$ describes the state of system (19) at time moment $\tau + t_0$ under initial conditions p.

If the action of G on D is known we can introduce a criterion that allows to determine whether an arbitrary function $x(t)$, defined on interval $(a; b)$, is a solution of (19). That is, $\vec{X}(t) = \left(x^{(0)}(t), x^{(1)}(t), x^{(2)}(t)\right)^T$ is an integral curve for (19) if and only if

$$g_\tau\left(\vec{X}(t - \tau)\right) = \vec{X}(t) \tag{21}$$

for any $t \in (a; b)$ and any permissible $\tau \in (-\varepsilon; +\varepsilon)$.

Therefore, the new theoretical background uses a wider class of differential equations, not necessarily describing self-oscillating systems. Another essential difference is that the activity of a neuron is classified with a symmetry transformation in phase space. Criterion of classification is the steadiness of the portrait in phase space of a spike against the transformation that corresponds to the given neuron (under the action of local one-parameter group). A computational method for modelling transformations is introduced and tested.

The algorithm to separate neuronal spikes several intermediate steps:

1. Spike detection from the noisy signal;
2. Calculation of distances between the phase trajectories of the detected spikes;
3. Detection of spikes that hypothetically belong to the same neuron;

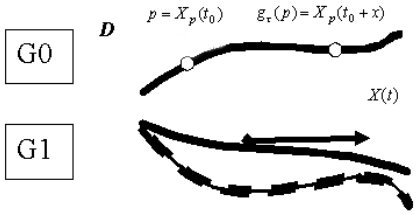

Figure 2. A vector field and its integral curves that correspond to solutions of equation (3). The action of g_t on point p is shown. The curve G0 corresponds to trajectory of equations (19) invariant under the action of one-parameter group. Any other curve G1 (not the solution of the system (19)) is not invariant under the action of group which corresponds to system (19). The difference between G1 and transformed curve G1 (dashed line) is the measure of dissymmetry and is the tool for classification.

4. Building a numerical model of transformation group that corresponds to every neuron observed;
5. Classification of spikes.

The three first steps were performed similarly to our previous analysis and are described in details in elsewhere [1]. In this article we only briefly summarize the main implementation details and parameters of calculations of these steps that are important to reproduce our work. The vector field approach (steps 4–5) is described and main differences between old and new algorithm are emphasized.

The first 4 steps corresponded to the training phase on which the number of spike classes was estimated and the vector field for each class was constructed. In order to perform this analysis a few dozens of spike occurrences, usually corresponding to several minutes of recording time were required. Like in our previous approach a human expert could participate at step 3 were the number of spike classes was detected.

The method described in the article is development of the method based on template matching in phase space. The stages of spike detection and forming spike classes are generally similar to the latter. The essential difference of the method presented from template matching is implementation of spike classification after spike classes are formed. The template matching method characterizes the whole class of spikes generated by the same neuron by the phase portrait of its typical spike.

In contrast to other methods, the method based on symmetry analysis uses computational modelling of vector field that conforms the differential equation describing the activity of the chosen neuron and thus it is more stable against alterations of spike form. It showed better results than the former and can be useful in experiments, where spikes tend to change their form. More details are adduced in [1–4].

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Visualisation and Animation in Mathematics and Physics

Eberhard MALKOWSKY

Department of Mathematics, Faculty of Science and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia and Montenegro
E-mail: *ema@BankerInter.net*

We give some applications of our own software for geometry and differential geometry and its extensions [5,1,2,4,6–9] to the visualisation and animation of certain topics in mathematics and physics such as the graphical representation of potential surfaces, the growth of crystals, neighbourhoods in weak topologies and the study of isometric maps.

1 Introduction

Visualisation and animation are of vital importance for the modern methods in mathematical education. They strongly support the understanding of concepts in mathematics and physics. We think that the application of a commercial graphics software package is neither a satisfactory approach for the illustration of the theoretical concepts, nor can it be used as their substitute. It should not be the aim of education to teach students to use some software package by instructing them which keys to press, and how to move the mouse, regardless of how convenient this may seem. The emphasis should be put on teaching the fundamental theoretical facts.

In view of this, we developed an *open software* in PASCAL on programme level which provides the basic tools for computer graphics, in order to offer an alternative to existing graphics software packages.

The main purpose of our software is to visualise the classical results in differential geometry on PC screens, plotters, printers or any other postscript device, but it also has extensions to physics, chemistry, crystallography and the engineering sciences. To the best of our knowledge, no other comparable, comprehensive software of this kind is available.

The software is *open* which means that its source files are accessible to the users, thus enabling them to apply it in the solutions of their own problems. This makes it extendable and flexible, and applicable to both teaching and research in many fields. In contrast to this, almost all other available graphics packages are *closed*; in general, the area below the user interface is inaccessible and consequently the software cannot be extended beyond the scope of the solutions it offers.

The software uses *OOP*, *object oriented programming*, and its programming language is *PASCAL*. The software is self-contained in the sense that no graphics package is needed other than PASCAL.

2 Wulff's crystals and potential surfaces

Here we deal with the graphical representation of crystals and their *potential surfaces*[1]. According to *Wulff's principle* [10], the shape of a crystal is uniquely defined by its surface energy function. A surface energy function is a real-valued function depending on a direction in space.

Let ∂B^n denote the unit sphere in $\mathbb{R}^{n+1}$ and $F : \partial B^n \to \mathbb{R}$ be a surface energy function then

$$PM = \{\vec{x} = F(\vec{e})\vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in \partial B^n\}$$

is an n-dimensional manifold which represents F.

[1]Figures in colour will be available only in electronic version.

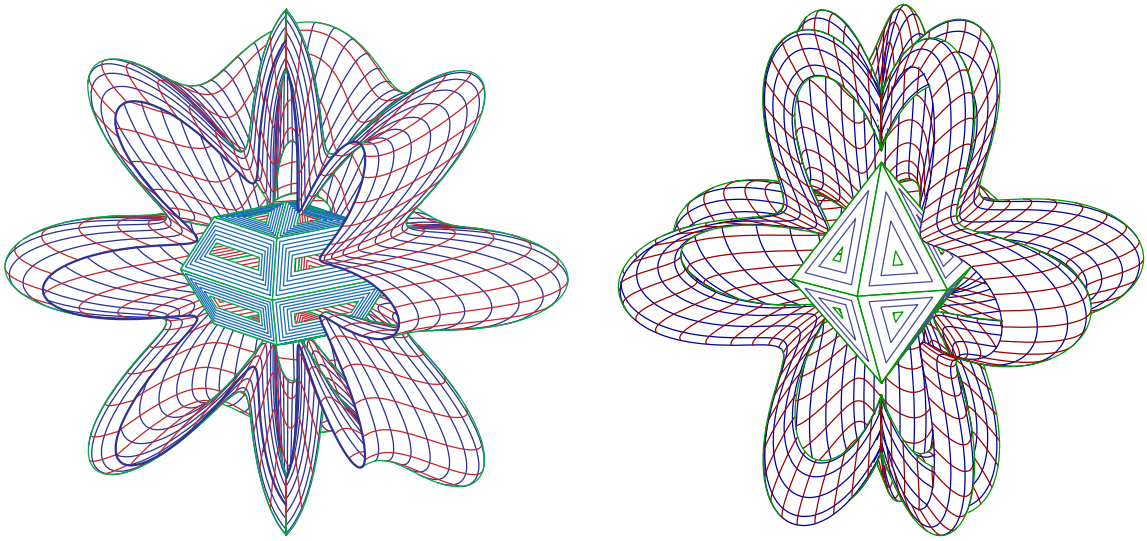

Figure 1. Potential surface and corresponding Wulff's crystals.

If $n = 1$, then $\vec{e} = \vec{e}(u) = \{\cos u, \sin u\}$ for $u \in (0, 2\pi)$ and we obtain a *potential curve* with a parametric representation

$$PC = \{\vec{x} = f(u)(\cos u, \sin u) : u \in (0, 2\pi)\}, \quad \text{where} \quad f(u) = F(\vec{e}(u)).$$

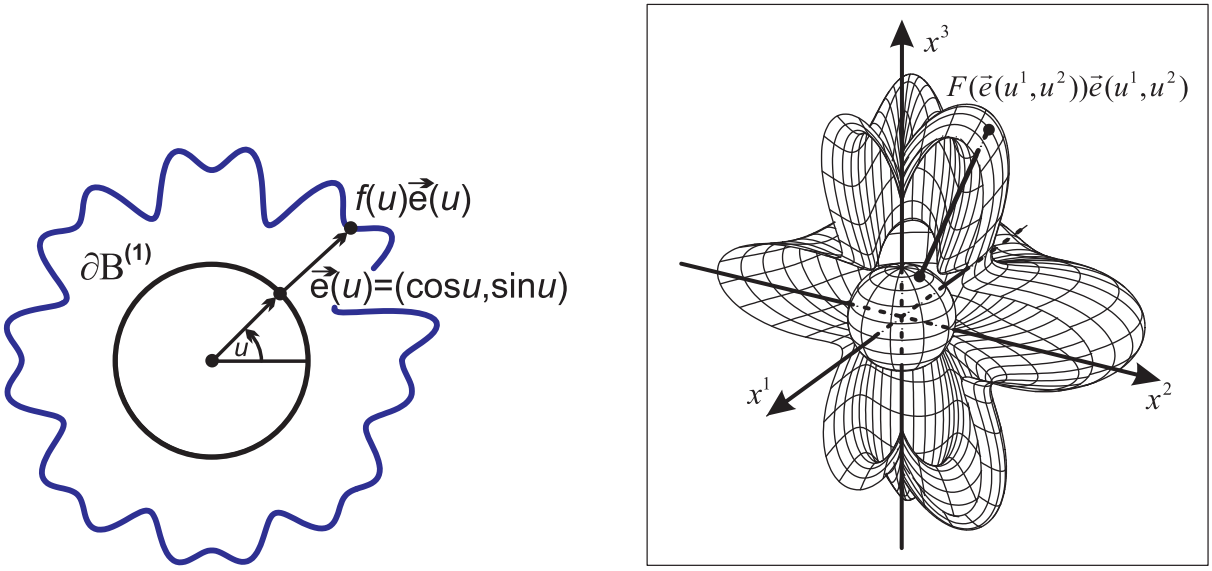

Figure 2. A potential curve and a potential surface.

If $n = 2$, then $\vec{e} = \vec{e}(u^1, u^2) = \{\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1\}$ for $(u^1, u^2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi)$ and we obtain a *potential surface* with a parametric representation

$$PS = \left\{\vec{x} = f\left(u^1, u^2\right)\left(\cos u^1 \cos u^2, \cos u^1 \sin u^2, \sin u^1\right) : \left(u^1, u^2\right) \in R\right\}$$

where $f(u^1, u^2) = F(\vec{e}(u^1, u^2))$.

The intersection of a potential surface and a sphere with radius r and centre in the origin is a curve which represents the constant values r of the function F; it is a so-called *equipotential line*.

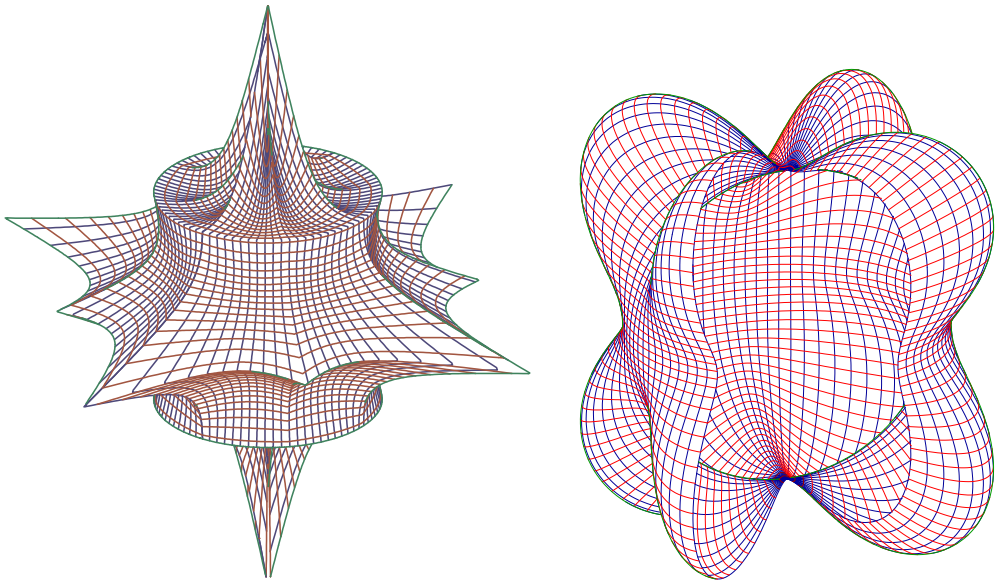

Figure 3. Potential surfaces.

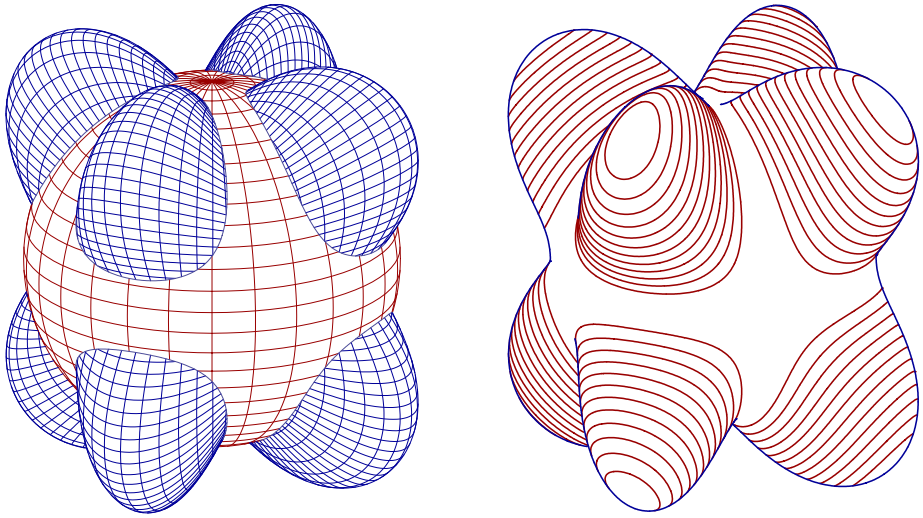

Figure 4. A potential surface and its intersection with a sphere and equipotential lines.

Wulff gave the following geometric principle of construction for crystals [10].

Theorem 1. Wulff's principle *For every $\vec{e} \in \partial B^n$, let $E_{\vec{e}}$ denote the hyperplane orthogonal to $\vec{e}$ and through the point P with position vector $\vec{p} = F(\vec{e})\vec{e}$, and $H_{\vec{e}}$ be the half space which contains the origin 0 and has the boundary $E_{\vec{e}} = \partial H_{\vec{e}}$. Then the crystal C_F which has F as its surface energy function is uniquely determined and given by*

$$C_F = \bigcap_{\vec{e} \in \partial B^n} H_{\vec{e}} = \bigcap_{\vec{e} \in \partial B^n} \{\vec{x} : \vec{x} \bullet \vec{e} \leq F(\vec{e})\}.$$

Since Wulff's construction in Theorem 1 is far from applicable for the graphical representation of crystals, we give two more useful results.

Theorem 2 ([1, Satz 6.1]). *Let $F : \partial B^n \to \mathbb{R}^+$ be a continuous function. Then a point X is on the boundary ∂C_F of Wulff's crystal C_F corresponding to F if and only if*

$$F(\vec{e}) \geq \vec{x} \bullet \vec{e} \text{ for all } \vec{e} \in \partial B^n \quad \text{and} \quad F(\vec{e}_0) = \vec{x} \bullet \vec{e}_0 \text{ for some } \vec{e}_0 \in \partial B^n.$$

Theorem 3 ([1, Satz 6.2]). *Let $F : \partial B^n \to \mathbb{R}^+$ be a continuous function and $CF : \partial B^n \to \mathbb{R}^+$ be defined by*

$$CF(\vec{e}) = \inf \left\{ F(\vec{u})(\vec{e} \bullet \vec{u})^{-1} : \vec{u} \in \partial B^n \text{ and } \vec{e} \bullet \vec{u} > 0 \right\}.$$

Then a parametric representation for ∂C_F is

$$\vec{x}(u^1, u^2) = CF(\vec{e}(u^1, u^2))\vec{e}(u^1, u^2) \text{ for } (u^1, u^2) \in (-\pi/2, \pi/2) \times (0, 2\pi).$$

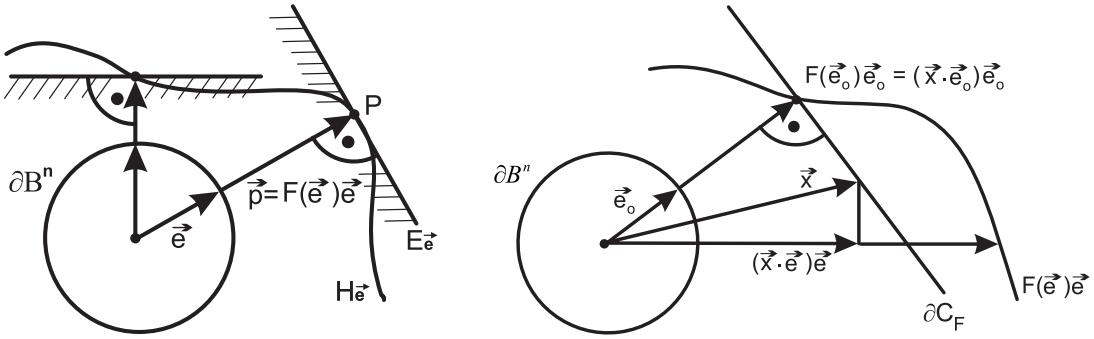

Figure 5. Wulff's constructions according to Theorems 1 and 2.

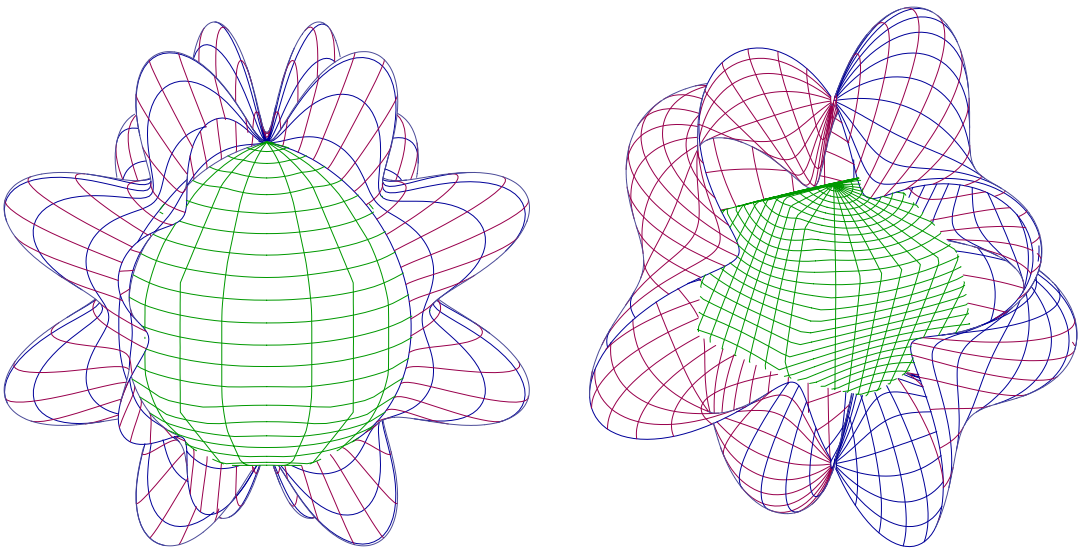

Figure 6. Wulff's crystals constructed by Theorems 2 and 3.

Although we have used both Theorems 2 and 3 to develop algorithms and programmes for the graphic representation of Wulff's crystals, in some cases a parametric representation can explicitly be given for the boundary of a Wulff's crystal, that is for the function CF.

One such case is when the function F is equal to a norm in three-dimensional space. Then if $F = \|\cdot\|$, the boundary of Wulff's crystal corresponding to F is a sphere with respect to the dual norm of $\|\cdot\|$. Here we represent the potential surfaces and corresponding Wulff's crystals for the ℓ_1 and ℓ_∞ norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ given by

$$\|\vec{x}\|_1 = |x_1| + |x_2| + |x_3| \quad \text{and} \quad \|\vec{x}\|_\infty = \max_{1 \le k \le 3} |x_k| \quad \text{for} \quad \vec{x} = \{x_1, x_2, x_3\}.$$

The two pictures are dual to one another.

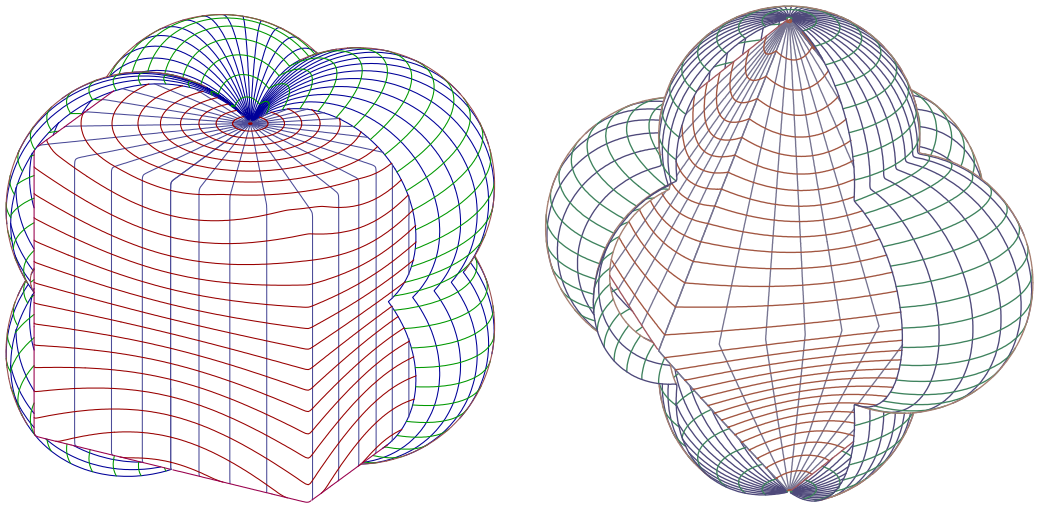

Figure 7. Wulff's crystals corresponding to the ℓ_1 and ℓ_∞ norms.

3 Weak topologies

Let $(X, \mathcal{T})$ be a topological space. A subbase for the topology $\mathcal{T}$ is a collection $\Sigma \subset \mathcal{T}$ such that given any point $x \in X$ and a neighbourhood N of x there exists a finite collection $\{N_1, \ldots, N_n\}$ of members of Σ with $x \in \bigcap_{k=1}^{n} N_k \subset N$. It is well known that if $X \neq \varnothing$ and Σ is a collection of sets with $\bigcup \Sigma = X$ then there is a unique topology $\mathcal{T}$ such that Σ is a subbase for $\mathcal{T}$; $\mathcal{T}$ is the weakest topology with $\Sigma \subset \mathcal{T}$, it is called the topology generated by Σ and consists of $\varnothing$, X and all unions of finite intersections of members of Σ.

Let X be a set, $(Y, \mathcal{T})$ be a topological space and $f : X \to Y$ be a map. Then the topology on X generated by the collection $\{f^{-1}(O) : O \in \mathcal{T}\}$ is called the weak topology by f, written $w(X, f)$; then f is continuous as a map from $(X, w(X, f))$ to $(Y, \mathcal{T})$ and $w(X, f)$ is the weakest topology on X such that this is true. If $\Sigma(Y)$ is a subbase for $\mathcal{T}$ then $\Sigma = \{f^{-1}(G) : G \in \Sigma(Y)\}$ is a subbase for $w(X, f)$.

If the topology of Y is metrizable and given by the metric d, we may use the concept of the weak topology by f to define a metric ρ on X by $\rho = d \circ f$.

Example 1. Here we consider $Y = \mathbb{R}^n$ with

$$d_p(y, \tilde{y}) = \begin{cases} \sum_{k=1}^{n} |y_k - \tilde{y}_k|^p & (0 < p < 1), \\ \left(\sum_{k=1}^{n} |y_k - \tilde{y}_k|^p \right)^{1/p} & (1 \leq p < \infty), \end{cases}$$

and $d_\infty(y, \tilde{y}) = \max\limits_{1 \leq k \leq n} |y_k - \tilde{y}_k|$, that is the limit case $d_\infty(y, \tilde{y}) = \lim\limits_{p \to \infty} d_p(y, \tilde{y})$. If we put $\|y\|_p = d_p(y, 0)$, then $\| \cdot \|_p$ is a norm for $1 \leq p \leq \infty$; the linear topology given by d_p for $0 < p < 1$ is not locally convex. We use the concept of the weak topology to introduce metrics on certain subsets of $S \subset \mathbb{R}^2$ and $\mathbb{R}^3$ and represent neighbourhoods of zero with respect to these metrics. The functions $f : S \to \mathbb{R}^2$ or $f : S \to \mathbb{R}^3$ have the special form

$$f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \quad \text{or} \quad f(x_1, x_2, x_3) = (f_1(x_1), f_2(x_2), f_3(x_3)).$$

Fig. 9 shows the images under the function $f = (\tan \pi/2, \tan \pi/2, \tan \pi/2)$ of the spheres $S_{p,r}(0) = \{x \in \mathbb{R}^3 : d_p(x) = r\}$ for $p = 1.7$ and $r = .55$ and $r = .85$. An animation for this is available.

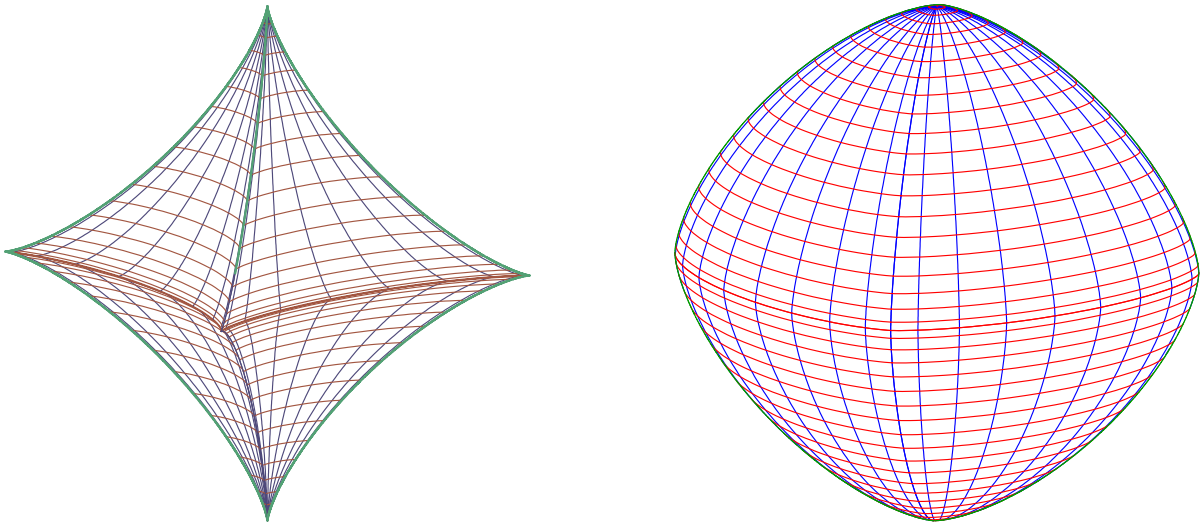

Figure 8. Spheres with respect to the metrics $d_{3/4}$ and $d_{3/2}$.

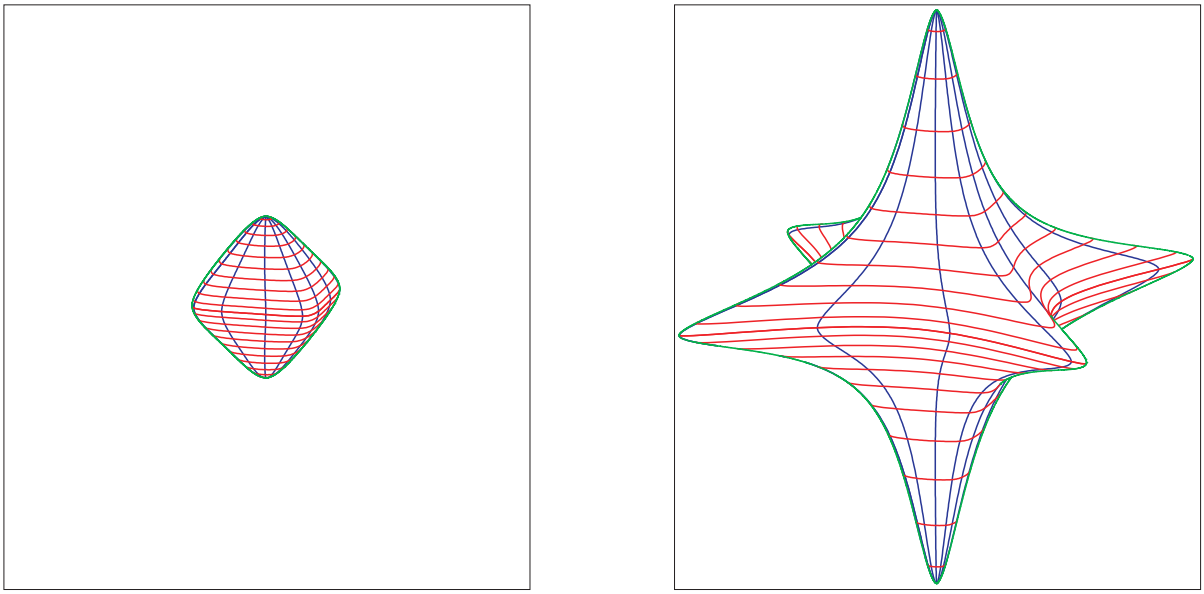

Figure 9. Images of spheres under the function f.

We use the functions $f : S_f = \mathbb{R} \times (0, \infty) \times \mathbb{R} \to \mathbb{R}^3$ and $g : S_g = \mathbb{R} \times \mathbb{R} \times (-1, 1) \to \mathbb{R}^3$ with $f(x_1, x_2, x_3) = (x_1, \log x_2, x_3)$ and $g(x_1, x_2, x_3) = (x_1, x_2, \tan(x_3\pi/2))$ to introduce metrics on S_f and S_g, and write $\rho_f = d_2 \circ f$ and $\rho_g = d_1 \circ g$.

4 Isometric maps of surfaces

Let $D \subset \mathbb{R}^2$ be a domain and S be a surface given by a parametric representation

$$\vec{x}(u^i) = \left(x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)\right) \quad \text{for } (u^i) = (u^1, u^2) \in D$$

with coordinate functions $x^k \in C^r(D)$ $(r \geq 1)$ and $\vec{x}_1 \times \vec{x}_2 \neq \vec{0}$ on D, where $\vec{x}_k = \partial\vec{x}/\partial u^k$ for $k = 1, 2$. The functions $g_{ik} : D \to \mathbb{R}$ with $g_{ik} = \vec{x}_i \bullet \vec{x}_k$ for $i, k = 1, 2$ are called the first fundamental coefficients of S.

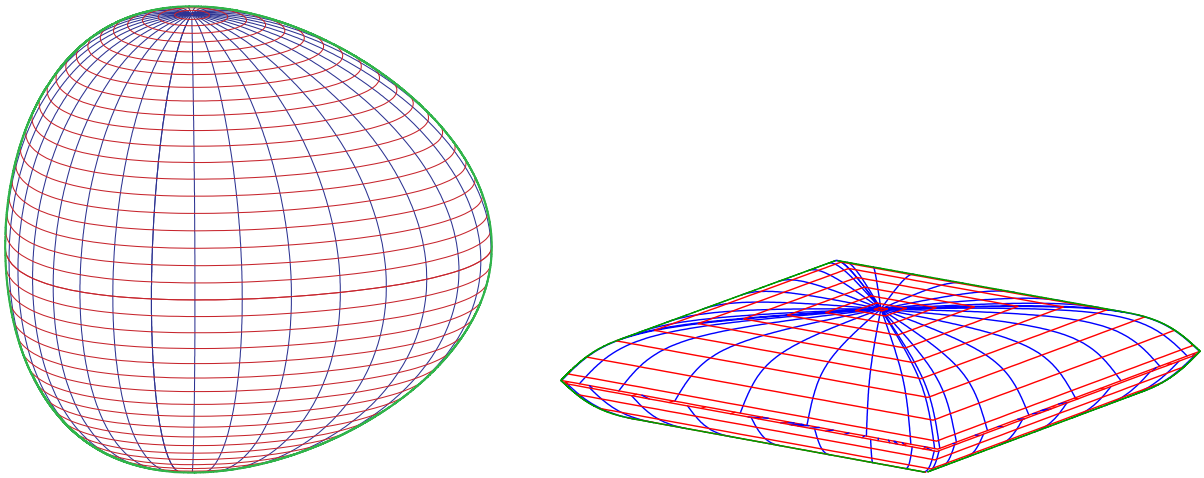

Figure 10. Spheres centred in the origin of S_f and S_g with respect to ρ_f and ρ_g.

Let S and S^* be surfaces with parametric representations $\vec{x}(u^i)$ and $\vec{x}^*(\tilde{u}^i)$ and $F : S \to S^*$ a map onto S^* given by the functions $h^i \in C^r(D)$ with $\tilde{u}^i = h^i(u^1, u^2)$ $(i = 1, 2)$ and non-vanishing Jacobian. We may introduce new parameters u^{*i} for S^* by putting $\tilde{u}^i = h^i(u^{*1}, u^{*2})$ $(i = 1, 2)$. Then the map F is given by $u^{*i} = u^i$ $(i = 1, 2)$, and S and S^* are said to have the same parameters.

A map $F : S \to S^*$ is called *isometric* if the length of every arc on S is the same as that of its corresponding image. It is well known that a map $F : S \to S^*$ is isometric if and only if the first fundamental coefficients g_{ik} and g^*_{ik} of S and $S^*$with respect to the same parameters (u^j) and (u^{*j}) satisfy $g_{ik}(u^j) = g^*_{ik}(u^{*j})$ for $i, k = 1, 2$ ([3, Satz 57.1, p. 213]).

We consider surfaces of revolution and screw surfaces given by parametric representations

$$\vec{x}(u^i) = (u^1 \cos u^2, u^1 \sin u^2, h(u^1)) \quad \text{and} \quad \tilde{\vec{x}}(\tilde{u}^i) = (\tilde{u}^1 \cos \tilde{u}^2, \tilde{u}^1 \sin \tilde{u}^2, c\tilde{u}^2 + f(\tilde{u}^1)),$$

where c is a constant.

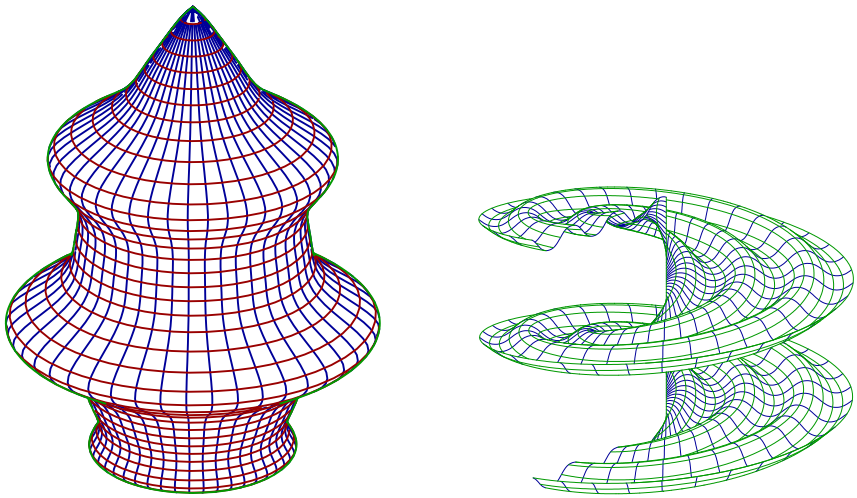

Figure 11. A surface of revolution and a screw surface.

It is well known that every screw surface S can be mapped isometrically onto a surface of revolution (cf. [3, Satz 57.4 (Bour), p. 217]).

Example 2. We consider the catenoid with a parametric representation

$$\vec{x}(u^i) = \left(u^1 \cos u^2, u^1 \sin u^2, a \operatorname{arcosh}(u^1/a)\right), \quad \text{where } a > 0 \text{ is a constant.}$$

We put $k^2 = a^2 - c^2$ for $0 \leq c \leq a$. Then the catenoid is isometric with any of the screw surfaces given by the function f with

$$f(u^{*1}) = k \log\left(\sqrt{(u^{*1})^2 + c^2} + \sqrt{(u^{*1})^2 - k^2}\right) - c \arctan\left(\frac{k}{c}\sqrt{\frac{(u^{*1})^2 + c^2}{(u^{*1})^2 - k^2}}\right).$$

The special cases $c = 0$ and $c = a$ yield the catenoid and the helikoid.

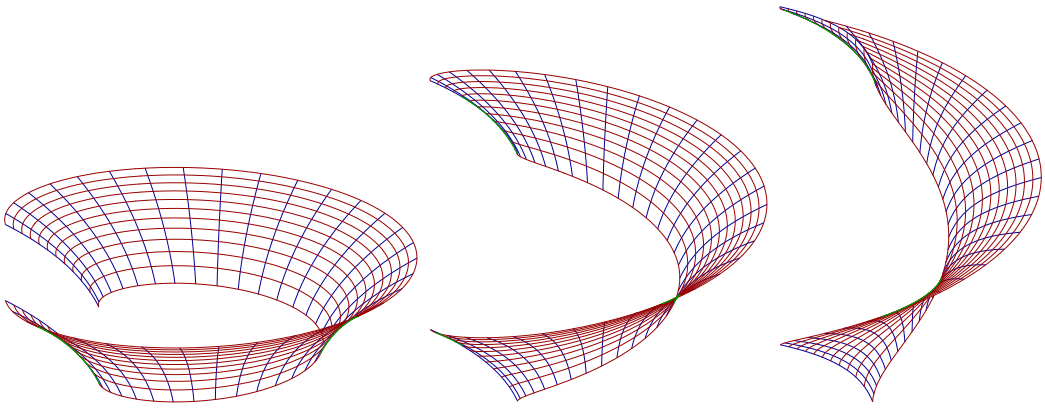

Figure 12. Surfaces of revolution with $c = a/8, a/2, 7a/8$, isometric with a catenoid and a helikoid.

Animations are available for Figs. 8, 9 and Example 2.

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One Frequency Asymptotic Solutions to Differential Equation with Deviated Argument and Slowly Varying Coefficients

Giovanni MATARAZZO [†], *Anna POMPEI* [‡] *and Valeriy SAMOYLENKO* [§]

[†] *Department of Computer Engineering and Applied Mathematics, Salerno University, via Ponte Don Melillo, 84084 Fisciano (SA), Italy*
E-mail: *matarazz@diima.unisa.it*

[‡] *Department of Mathematics, University of Catania, viale A. Doria 6, 95125 Catania, Italy*

[§] *Department of Mechanics and Mathematics, Kyiv National Taras Shevchenko University, 64 Volodymyrs'ka Str., 01033 Kyiv-33, Ukraine*
E-mail: *vsam@univ.kiev.ua*

The object of this paper is to study a problem of constructing of an approximate one-frequency solutions for weakly nonlinear ordinary differential equations with deviated argument and slowly varying coefficients. On the basis of asymptotic techniques of nonlinear mechanics an algorithm for asymptotic integration of differential equation under consideration is given.

1 Introduction and formulation of the problem

Among numerous modern analytic methods of studying nonlinear oscillatory processes asymptotic methods are often applied [1]. The effectiveness of these methods in various fields of applied mathematics and mechanics is now recognized all over the world because those methods enable not only to construct approximate solutions of corresponding differential equations, but also to examine qualitative properties of nonlinear processes described by given equations.

In present paper we consider the problem of construction of one-frequency asymptotic approximations for weakly nonlinear multi-dimensional oscillatory systems with delay and slowly varying parameters. Such a problem is of a certain practical importance, through its connection with studying of non-stationary processes in oscillatory systems. Similar problems may occur, for example, in the systems with changing mass and inflexibility, in particular, in numerous problems of electronics and radiotechnics, connected with the problems of modulation, charged particles acceleration phenomena, control theory, etc.

The purpose of the present paper is to develop techniques for constructing of one-frequency asymptotic solution to the system of weakly nonlinear ordinary differential equations with deviated argument and slowly varying coefficients of the following form

$$\sum_{s=1}^{N}\left(\alpha_{rs}(\tau)\frac{dx_s(t)}{dt}+\beta_{rs}(\tau)\frac{dx_s(t-\sigma(\tau))}{dt}+\gamma_{rs}(\tau)x_s(t)+\delta_{rs}(\tau)x_s(t-\sigma(\tau))\right)$$
$$=\varepsilon f_r\left(\tau,\theta,x_1(t),\dots,x_N(t),x_1(t-\sigma(\tau)),\dots,x_N(t-\sigma(\tau))\right), \tag{1}$$

where ε is a small parameter; $\tau=\varepsilon t$ is *the slow time*; $d\theta/dt=\nu(\tau)$ is an instant frequency of an external periodical force; $\sigma(\tau)\geq\sigma_0>0$ is a delay; functions $f_r(\tau,\theta,x_1,\dots,x_N,y_1,\dots,y_N)$ are supposed to be 2π-periodic with respect to θ of the following representation

$$f_r(\tau,\theta,x_1,\dots,x_N,y_1,\dots,y_N)=\sum_{|m|\leq M_r}f_{rm}(\tau,x_1,\dots,x_N,y_1,\dots,y_N)e^{im\theta}.$$

We also supposed the coefficients $f_{rm}(\tau, x_1, \ldots, x_N, y_1, \ldots, y_N)$, $|m| \leq M_r$, $r = \overline{1,N}$ to be polynomials in $x_1, \ldots, x_N$; $y_1, \ldots, y_N$ infinite differentiable with respect to τ. The values $\alpha_{rs} = \alpha_{rs}(\tau)$, $\beta_{rs} = \beta_{rs}(\tau)$, $\gamma_{rs} = \gamma_{rs}(\tau)$, $\delta_{rs} = \delta_{rs}(\tau)$, $r, s = \overline{1,N}$, $\nu = \nu(\tau)$, $\sigma = \sigma(\tau)$ are supposed to be sufficiently smooth in variable τ.

We assume that the characteristic equation of system (1), i.e. equation $\det \|\kappa_{rs}(\lambda)\| = 0$, where

$$\kappa_{rs}(\lambda) = \lambda\alpha_{rs}(\tau) + \gamma_{rs}(\tau) + (\lambda\beta_{rs}(\tau) + \delta_{rs}(\tau))e^{-\lambda\sigma(\tau)},$$

has $2N$ solutions $\lambda_j = \pm i\omega_j(\tau)$, $j = \overline{1,N}$, at every moment of a slow time τ. At the same time the other roots of characteristic equation are supposed to have negative real parts.

Thus, fundamental frequencies $\omega_j = \omega_j(\tau)$, $j = \overline{1,N}$, of the unperturbed system (1) (when $\varepsilon = 0$) at every τ satisfy the characteristic equation $\det \|\kappa_{rs}(i\omega_j(\tau))\| = 0$, and are smooth enough in τ. We also assume that the unperturbed system (1) has only the trivial equilibrium point $x_1 = x_2 = \cdots = x_N = 0$, and the conditions of non-existence of interior resonance hold.

Let $\varphi_k^{(j)} = \varphi_k^{(j)}(\tau)$, $j, k = \overline{1,N}$ be so-called normal functions being nontrivial solutions to a system of linear algebraic homogeneous equations

$$\sum_{s=1}^{N} \kappa_{rs}(i\omega_j(\tau))\varphi_s^{(j)}(\tau) = 0, \qquad j, r = \overline{1,N},$$

and $\chi_k^{(j)} = \chi_k^{(j)}(\tau)$, $j, k = \overline{1,N}$ be solutions of the conjugate system

$$\sum_{s=1}^{s=N} \bar{\kappa}_{rs}(i\omega_j(\tau))\chi_s^{(j)}(\tau) = 0, \qquad j, r = \overline{1,N},$$

that satisfy normalization conditions.

For every value of parameter τ the unperturbed system (1) has a set of periodic solutions

$$x_s(t) = a\varphi_s^{(j)}(\tau)\cos(\omega_j(\tau)t + \varphi), \qquad j, s = \overline{1,N},$$

where $a = a(\tau)$ and $\varphi = \varphi(\tau)$ are arbitrary values called normal unperturbed oscillations of the system (1).

Basing on ideas of asymptotic techniques of nonlinear mechanics [1–4] we describe an algorithm allowing us to find an approximate solution to the problem (1) asymptotically close to normal unperturbed oscillations of system (1)

$$x_s(t) = a\varphi_s^{(1)}(\tau)\cos(\omega_1(\tau)t + \varphi), \qquad s = \overline{1,N}.$$

2 Asymptotic expansion

In the context of solving the problem of construction of approximate (asymptotic) solutions, non-resonance and resonance cases are usually studied, so solutions to the problem are found separately for each of these cases. Dependence of both of the fundamental frequency $\omega(\tau)$ and the frequency of external force $\nu(\tau)$ on slow time τ does not allow to apply such an approach to the problem under our consideration. As it is known [1,5], the problem is connected with possible transfers of the system (1) from one, for example, non-resonance state, to another resonance state, and vice versa, provoking varying of the frequencies $\omega(\tau)$, $\nu(\tau)$ with slow time τ.

Thus, taking into consideration dependence of frequencies of the system (1) on slow time τ we seek an asymptotic solution of the problem (1) in the following form:

$$x_s(t) = a\varphi_s^{(1)}(\tau)\cos\varphi + \sum_{k=1}^{\infty} \varepsilon^k U_{ks}(\tau, a, \theta, \varphi), \qquad s = \overline{1,N}, \tag{2}$$

where $\varphi = \frac{p}{q}\theta + \psi$; numbers p, q are distinct natural members and depend on correlation between frequencies $\omega_1(\tau)$, $\nu(\tau)$; functions $a(t)$ and $\psi(t)$ satisfy the following differential equations

$$\frac{da}{dt} = \sum_{k=1}^{\infty} \varepsilon^k A_k(\tau, a, \psi), \qquad \frac{d\psi}{dt} = \omega_1(\tau) - \frac{p}{q}\nu(\tau) + \sum_{k=1}^{\infty} \varepsilon^k B_k(\tau, a, \psi), \tag{3}$$

where functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$ for any $k \in \mathbb{N}$ are 2π-periodic in variable ψ.

Functions $U_{ks}(\tau, a, \theta, \varphi)$, $k \in \mathbb{N}$, $s = \overline{1, N}$, are supposed to be 2π-periodic in variables θ, ψ without first harmonics in their Fourier series expansion on φ, i.e.,

$$\int_0^{2\pi} U_{ks}(\tau, a, \theta, \varphi) e^{\pm i\varphi}\, d\varphi = 0, \qquad k \in \mathbb{N}, \quad s = \overline{1, N}. \tag{4}$$

Conditions (5) allow us to construct asymptotic solutions without secular terms.

In order to find functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$, $U_k(\tau, a, \theta, \varphi)$, $k \in \mathbb{N}$, it becomes necessary to substitute expansion (2) into equation (1), taking into account differential equations (3), expand an obtained relation into series in small parameter ε and finally equate terms at the same degree of ε. Let us introduce the following notations [6]:

$$\begin{aligned}
&Q_0 = \left(\omega_1(\tau) - \frac{p}{q}\nu(\tau)\right)\frac{\partial}{\partial\psi}, \qquad P_0 = \omega_1(\tau)\frac{\partial}{\partial\varphi} + \nu(\tau)\frac{\partial}{\partial\theta}, \qquad \mathcal{N}_0 = \sum_{k=0}^{\infty} \frac{(-\sigma(\tau))^{k+1}}{(k+1)!} Q_0,\\
&M_{rs} = \alpha_{rs}(\tau)P_0 + \gamma_{rs}(\tau), \qquad N_{rs} = \beta_{rs}(\tau)P_0 + \delta_{rs}(\tau),\\
&\mathcal{L}_r = \sum_{s=1}^{N} \varphi_s^{(1)}(\tau)\,\{\alpha_{rs}(\tau) + \delta_{rs}(\tau)\mathcal{N}_0 \cos(\omega_1(\tau)\sigma(\tau))\\
&\qquad + \beta_{rs}(\tau)[(\mathcal{N}_0 Q_0 + 1)\cos(\omega_1(\tau)\sigma(\tau)) + \omega_1(\tau)\mathcal{N}_0 \sin(\omega_1(\tau)\sigma(\tau))]\},\\
&\mathcal{E}_r = \sum_{s=1}^{N} \varphi_s^{(1)}(\tau)\,\{\delta_{rs}(\tau)\mathcal{N}_0 \sin(\omega_1(\tau)\sigma(\tau))\\
&\qquad + \beta_{rs}(\tau)[(\mathcal{N}_0 Q_0 + 1)\sin(\omega_1(\tau)\sigma(\tau)) - \omega_1(\tau)\mathcal{N}_0 \cos(\omega_1(\tau)\sigma(\tau))]\}.
\end{aligned} \tag{5}$$

By substitution of formulas (5) into the right part of equation (1), by standard way we obtain the following relations

$$\begin{aligned}
&\sum_{s=1}^{N} [M_{rs}U_{ks}(\tau, a, \theta, \varphi) + N_{rs}U_{ks}(\tau, a, \theta - \nu\sigma, \varphi - \omega_1\sigma)]\\
&\qquad = F_{rk}(\tau, a, \theta, \varphi) - (\mathcal{L}_r A_k + a\mathcal{E}_r B_k)\cos\varphi + (\mathcal{E}_r A_k - a\mathcal{L}_r B_k)\sin\varphi,
\end{aligned} \tag{6}$$

where an explicit form of functions $F_{rk}(\tau, a, \theta, \varphi)$, $k \in \mathbb{N}$, $r = \overline{1, N}$, is found after the sequel determination of functions $A_m(\tau, a, \psi)$, $B_m(\tau, a, \psi)$, $U_m(\tau, a, \theta, \varphi)$, $m = \overline{1, k-1}$, $r = \overline{1, N}$. To realize it we have to use the 2π-periodicity property of functions $A_k(\tau, a, \psi)$, $B_k(\tau, a, \psi)$, $U_k(\tau, a, \theta, \varphi)$, $F_{rk}(\tau, a, \theta, \varphi)$, $r = \overline{1, N}$, $k \in \mathbb{N}$, with respect to variables ψ, φ, θ correspondingly, and represent these functions by means of their Fourier series as follows

$$\begin{aligned}
U_{rk}(\tau, a, \theta, \varphi) &= \sum_{m,n=-\infty}^{+\infty} U_{rkmn}(\tau, a) e^{i(m\theta + n\varphi)},\\
F_{rk}(\tau, a, \theta, \varphi) &= \sum_{m,n=-\infty}^{+\infty} F_{rkmn}(\tau, a) e^{i(m\theta + n\varphi)},
\end{aligned}$$

$$A_k(\tau,a,\psi)=\sum_{n=-\infty}^{+\infty}A_{kn}(\tau,a)e^{in\psi},\qquad B_k(\tau,a,\psi)=\sum_{n=-\infty}^{+\infty}B_{kn}(\tau,a)e^{in\psi},\tag{7}$$

where

$$U_{rkmn}(\tau,a)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}U_{rk}(\tau,a,\theta,\varphi)e^{-i(m\theta+n\varphi)}\,d\theta d\varphi,$$
$$F_{rkmn}(\tau,a)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}F_{rk}(\tau,a,\theta,\varphi)e^{-i(m\theta+n\varphi)}\,d\theta d\varphi,$$
$$A_{kn}(\tau,a)=\frac{1}{2\pi}\int_0^{2\pi}A_k(\tau,a,\psi)e^{-in\psi}\,d\psi,\qquad B_{kn}(\tau,a)=\frac{1}{2\pi}\int_0^{2\pi}B_k(\tau,a,\psi)e^{-in\psi}\,d\psi.$$

Taking into consideration the relation $\varphi-\frac{p}{q}\theta=\psi$ we separate resonance and non-resonance terms in Fourier expansion (7) of functions $F_{rk}(\tau,a,\theta,\varphi)$, $k\in\mathbb{N}$, in the following way

$$F_{rk}(\tau,a,\theta,\varphi)=\sum_{mq+(n\pm1)p\neq0}F_{rkmn}(\tau,a)e^{i(m\theta+n\varphi)}+\sum_{mq+(n\pm1)p=0}F_{rkmn}(\tau,a)e^{i(m\theta+n\varphi)}.$$

In order for the equations (6) to have 2π-periodic in φ, θ solutions it is necessary that the following condition holds:

$$\sum_{r=1}^{N}\chi_r^{(1)}\left[\mathcal{L}_rA_k+a\mathcal{E}_rB_k)\cos\varphi+(\mathcal{E}_rA_k-a\mathcal{L}_rB_k)\sin\varphi\right]$$
$$=\sum_{mq+(n\pm1)p=0}F_{rkmn}(\tau,a)e^{i(m\theta+n\varphi)},\qquad m,n\in\mathbb{Z},\quad k\in\mathbb{N},\tag{8}$$

where $F_{rkmn}(\tau,a)$ are Fourier coefficients of functions $F_{rk}(\tau,a,\theta,\varphi)$.

The latter term in (8) can be given as

$$\sum_{mq+(n\pm1)p=0}F_{rkmn}(\tau,a)e^{i(m\theta+n\varphi)}$$
$$=\sum_{n=-\infty}^{+\infty}\left[F_{rk,-pn,qn-1}(\tau,a)e^{i\varphi}+F_{rk,-pn,qn+1}(\tau,a)e^{-i\varphi}\right]e^{iqn\psi}.$$

Using both condition (4) and relation (6)–(8) we find

$$\sum_{s=1}^{N}\kappa_{rs}(m\nu+n\omega_1)U_{ksmn}(\tau,a)=F_{rkmn}(\tau,a),\qquad r=\overline{1,N},\tag{9}$$

where $mq+(n\pm1)p\neq0$, $m,n\in\mathbb{Z}$.

Evidently, the system of algebraic equations (9) has a unique solution

$$U_{ksmn}(\tau,a)=\sum_{r=1}D_{ks}(m\nu+n\omega_1)F_{rkmn}(\tau,a)\left(\det\|\kappa_{rs}(m\nu+n\omega_1)\|\right)^{-1},$$

where $D_{ks}(\lambda)$ are corresponding minors of the determinant $\det\|\kappa_{rs}(\lambda)\|$, if $mq+(n\pm1)p\neq0$, $m,n\in\mathbb{Z}$. In case $mq+(n\pm1)p=0$, $m,n\in\mathbb{Z}$, we put $U_{ksmn}(\tau,a)=0$.

Functions $A_k(\tau,a,\psi)$, $B_k(\tau,a,\psi)$, $k\in\mathbb{N}$, are defined from the system of differential equations

$$\mathcal{L}A_k+a\mathcal{E}B_k=G_k(\tau,a,\psi),\qquad \mathcal{E}A_k-a\mathcal{L}B_k=H_k(\tau,a,\psi),\tag{10}$$

where

$$\mathcal{L} = \sum_{r=1}^{N} \chi_r^{(1)}(\tau)\mathcal{L}_r, \qquad \mathcal{E} = \sum_{r=1}^{N} \chi_r^{(1)}(\tau)\mathcal{E}_r,$$

$$G_k(\tau,a,\psi) = \sum_{r=1}^{N} \frac{\chi_r^{(1)}(\tau)}{4\pi^2} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} F_{rk}(\tau,a,\theta,\varphi) e^{im(p\theta-q\varphi)} \left(e^{i\varphi}+e^{-i\varphi}\right) e^{imq\psi}\, d\theta\, d\varphi\,,$$

$$H_k(\tau,a,\psi) = \sum_{r=1}^{N} \frac{i\chi_r^{(1)}(\tau)}{4\pi^2} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} F_{rk}(\tau,a,\theta,\varphi) e^{im(p\theta-q\varphi)} \left(e^{i\varphi}-e^{-i\varphi}\right) e^{imq\psi}\, d\theta\, d\varphi,$$

To solve equations (10), we use the Fourier representations (7) and easily find

$$A_k(\tau,a,\psi) = \sum_{n=-\infty}^{+\infty} \frac{1}{4\pi^3} e^{in\psi} \left(\mathcal{L}_{0n}^2+\mathcal{E}_{0n}^2\right)^{-1} \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} \sum_{r=1}^{N} \chi_r^{(1)}(\tau)$$
$$\times F_{rk}(\tau,a,\theta,\varphi) e^{im(p\theta-q\varphi)} (\mathcal{L}_{0n}\cos\varphi - \mathcal{E}_{0n}\sin\varphi) e^{i(mq-n)\psi}\, d\theta d\varphi d\psi,$$

$$B_k(\tau,a,\psi) = \sum_{n=-\infty}^{+\infty} \frac{1}{4a\pi^3} e^{in\psi} \left(\mathcal{L}_{0n}^2+\mathcal{E}_{0n}^2\right)^{-1} \int_0^{2\pi} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi}\int_0^{2\pi} \sum_{r=1}^{N} \chi_r^{(1)}(\tau)$$
$$\times F_{rk}(\tau,a,\theta,\varphi) e^{im(p\theta-q\varphi)} (\mathcal{L}_{0n}\sin\varphi + \mathcal{E}_{0n}\cos\varphi) e^{i(mq-n)\psi}\, d\theta d\varphi d\psi,$$

where

$$\mathcal{L}_{0n} = \sum_{s,r=1}^{N} \chi_r^{(1)}(\tau)\varphi_s^{(1)}(\tau) \Big[\alpha_{rs}(\tau) + \delta_{rs}(\tau)\rho(nl(\tau))\cos\left(\omega_1(\tau)\sigma(\tau)\right) +$$
$$+ \beta_{rs}(\tau) e^{-n\sigma(\tau)l(\tau)} \cos\left(\omega_1(\tau)\sigma(\tau)\right) + \beta_{rs}(\tau)\omega_1(\tau)\rho(nl(\tau))\sin\left(\omega_1(\tau)\sigma(\tau)\right)\Big],$$

$$\mathcal{E}_{0n} = \sum_{s,r=1}^{N} \chi_r^{(1)}(\tau)\varphi_s^{(1)}(\tau) \Big[\delta_{rs}(\tau)\rho(nl(\tau))\sin\left(\omega_1(\tau)\sigma(\tau)\right)$$
$$+ \beta_{rs}(\tau) e^{-n\sigma(\tau)l(\tau)} \sin\left(\omega_1(\tau)\sigma(\tau)\right) - \beta_{rs}(\tau)\omega_1(\tau)\rho(nl(\tau))\cos\left(\omega_1(\tau)\sigma(\tau)\right)\Big],$$

$$\rho(nl) = \begin{cases} (e^{-n\sigma l(\tau)}-1)/(inl) & \text{if} \quad nl \neq 0, \\ 0 & \text{if} \quad nl = 0. \end{cases}$$

Thus, the problem of construction of one-frequency solutions to equations (1) asymptotically close to normal unperturbed oscillations of system (1) is solved.

3 Conclusion

In the paper a problem of construction of an approximate one-frequency solutions to weakly nonlinear ordinary differential equations with deviated argument and slowly varying coefficients is studied. An algorithm for asymptotic integration of differential equation under consideration is given.

The approach proposed above allows to build asymptotical solution to weakly nonlinear ordinary the first order differential equations with deviated argument and slowly varying coefficients. Having defining approximate solution (functions $A_k(\tau,a,\psi)$, $B_k(\tau,a,\psi)$, $U_{rk}(\tau,a,\theta,\varphi)$, $k \in \mathbb{N}$, $r = \overline{1,N}$) it is possible to study stationary regimes and their stability as well as processes connected with passing through resonance zones [2].

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The WKB Method for the Dirac Equation with Vector-Scalar Potentials in 2+1 and 3+1 Dimensions

Olexander K. REITY, Vasyl' V. RUBISH and Svitlana I. MYHALYNA

Uzhgorod National University, 32 Voloshyna Str., Uzhgorod, Ukraine
E-mail: *reiti@univ.uzhgorod.ua, vrubish@univ.uzhgorod.ua*

In the framework of the Dirac approach we have developed the relativistic version of the WKB method for centrally symmetrical potential with the mixed Lorentz structure. We have obtained relativistic wavefunctions of light quark and the new quantization rule containing the spin-orbit interaction. These gives us the possibility of finding the energy levels and decay width of hydrogen-like quark-systems.

1 Introduction

Dirac equations with potentials of a concrete type, for which it is possible to write the exact solution, are encountered quite seldom. Most often either numerical or asymptotic methods are used for searching the solutions. In many theoretical and applied problems the possibility of obtaining the asymptotic solution allows to carry out the most complete analysis of a problem. Therefore, hardly there is a necessity to explain in detail importance of creating and investigating asymptotic methods for the solving the Dirac equation. The Wentzel–Kramers–Brillouin quasi-classical approximation (or WKB method) is one of basic and most universal asymptotic methods of solving problems of theoretical and mathematical physics (see, for example, [1–4]), for which the exact solutions are either unknown or rather difficult to find. As it is known [1–4], in the case of a Coulomb field this method has high accuracy even for small values of quantum numbers. In contrast to the perturbation theory, the given approach is not connected with a smallness of interaction and consequently has wider applicability area allowing to study qualitative legitimacies in behaviour and of properties of quantum mechanical systems. The new application of quasi-classical approach can be low-energy sector of QCD (the energy spectrum of hadrons, confinement study, decay widths of hadrons), where standard approaches based on perturbative theories are inapplicable because of the fact that interaction between quarks in this area is not small.

2 The WKB method for the Dirac equation in a centrally symmetrical field

We obtain the formulae of the quasi-classical approximation for solutions of the Dirac equation with potential of centrally symmetry having mixed Lorenz structure. For this purpose we search wavefunctions of the stationary states (in standard representation) in the form of

$$\Phi = r^{-1} \begin{pmatrix} F(r)\Omega_{jlM}\left(\boldsymbol{n}\right) \\ iG(r)\Omega_{jl'M}\left(\boldsymbol{n}\right) \end{pmatrix}, \tag{1}$$

where Ω is the spherical spinor, j and M are the total angular moment and projection of j ($j = l \pm 1/2$) respectively, l is the orbital moment ($l + l' = 2j$), $\boldsymbol{n} = \boldsymbol{r}/r$.

After the separation of the angular variables the Dirac equation with centrally symmetrical potential containing both vector $V(r)$ and scalar $S(r)$ parts takes the form ($\hbar = c = 1$)

$$\frac{dF}{dr} + \frac{k}{r}F - \frac{1}{\hbar}\left[(E - V(r)) + (m + S(r))\right]G = 0,$$
$$\frac{dG}{dr} - \frac{k}{r}G + \frac{1}{\hbar}\left[(E - V(r)) - (m + S(r))\right]F = 0, \tag{2}$$

where F and G are the radial functions, $k = \mp(j + 1/2)$ for states with $j = l \pm 1/2$, E is the energy of level.

For finding quasi-classical solutions of the system (2) it is convenient to write equations (2) in the matrix form [5]:

$$\Psi' = \frac{1}{\hbar}D\Psi, \qquad \Psi = \left\{ \begin{array}{c} F \\ G \end{array} \right\},$$
$$D = \left(\begin{array}{cc} -\hbar k/r & E - V(r) + m + S(r) \\ -E + V(r) + m + S(r) & \hbar k/r \end{array} \right). \tag{3}$$

Here a Planck constant $\hbar$ remains in the explicit form, the prime denotes the derivative with respect to r. We will regard the solution of the matrix equation (3) as the formal expansion in powers of $\hbar$:

$$\Psi = \varphi \exp\left(\int^r y dr\right), \qquad y(r) = \frac{1}{\hbar}y_{-1}(r) + y_0(r) + \hbar y_1(r) + \cdots,$$
$$\varphi(r) = \sum_{n=0}^{\infty} \hbar^n \varphi^{(n)}(r), \tag{4}$$

where the upper (lower) component $\varphi_F^{(n)}$ ($\varphi_G^{(n)}$) of the vector $\varphi^{(n)}$ corresponds to the radial wavefunction $F(G)$. Having substituted (4) into (3) and equated to zero the coefficients of each power of $\hbar$, we arrive at the recurrent system

$$(D - y_{-1})\,\varphi^{(0)} = 0, \qquad (D - y_{-1})\,\varphi^{(n+1)} = \varphi^{(n)'} + \sum_{k=0}^{n} y_{n-k}\varphi^{(k)}, \quad n = 0, 1, \ldots. \tag{5}$$

Using the first two equation of system (5) by means of the left and right vectors technique we find the terms y_{-1}, y_0 and $\varphi^{(0)}$. So we have obtained (up to a normalization constant)

$$\Psi = \left\{ \begin{array}{c} F \\ G \end{array} \right\} = \left[2q\left(q \pm \frac{k}{r}\right)\right]^{-1/2}$$
$$\times \exp\left\{\pm\int^r q dr + \frac{1}{2}\int^r \frac{(m+S)\,V' + (E-V)\,S'}{q\,(q \pm kr^{-1})}\,dr\right\}\left(\begin{array}{c} m + S + E - V \\ kr^{-1} \pm q \end{array} \right), \tag{6}$$
$$q = \sqrt{(m+S)^2 - (E-V)^2 + (k/r)^2}, \tag{7}$$

(here and hereinafter $\hbar = 1$).

The effective energy and effective potential

$$E_{\text{ef}} = \frac{E^2 - m^2}{2m}, \qquad U_{\text{ef}}(r, E) = \frac{E}{m}V + S + \frac{S^2 - V^2}{2m} + \frac{k^2}{2mr^2} \tag{8}$$

correspond to the Dirac system (2). So $U(r, E)$ looks like a potential with a barrier (Fig. 1). We consider the most general case, when the effective potential is barrier type (Fig. 1). Then wavefunction has the different form in the three regions: 1) potential well $r_0 < r < r_-$ ($q^2 < 0$); 2) the below-barrier region $r_- < r < r_+$ ($q^2 > 0$); 3) the classically allowed region with continuum spectrum $r > r_+$ ($q^2 < 0$), where r_0, r_- and r_+ are turning points.

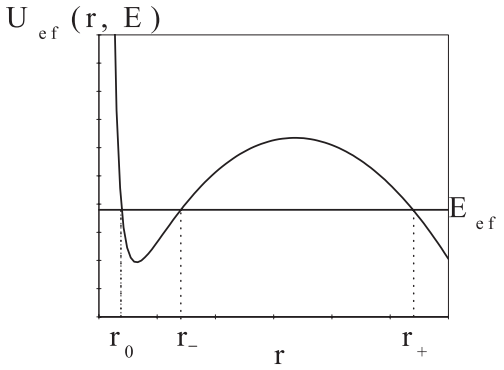

Figure 1. The type of the effective potential $U_{\rm ef}(r,E)$.

3 The wavefunction of the Dirac particle in classically allowed and forbidden regions

The wavefunction of state has the different form in the various regions.

I. The region $r_0 < r < r_-$ is classically allowed; there the wavefunctions (6) oscillate

$$F = C_1^{\pm}\left[\frac{E-V+m+S}{p}\right]^{1/2}\cos\Theta_1, \qquad G = C_1^{\pm}\mathrm{sgn}\,k\left[\frac{E-V-m-S}{p}\right]^{1/2}\cos\Theta_2, \quad (9)$$

where

$$p(r) = \sqrt{(E-V)^2-(m+S)^2-(k/r)^2} \quad (10)$$

is the quasi-classical momentum for the radial motion of a particle, $C_1^{\pm}$ is a normalization constant,

$$\Theta_1 = \int_{r_-}^{r}\left(p+\frac{kw}{pr}\right)dr+\frac{\pi}{4}, \qquad \Theta_2 = \int_{r_-}^{r}\left(p+\frac{k\widetilde{w}}{pr}\right)dr+\frac{\pi}{4}, \quad (11)$$

$$w = \frac{1}{2}\left(\frac{V'-S'}{m+S+E-V}-\frac{1}{r}\right), \qquad \widetilde{w} = \frac{1}{2}\left(\frac{V'+S'}{m+S-E+V}+\frac{1}{r}\right).$$

Signs $\pm$ correspond to values $k>0$ and $k<0$ respectively. If the level width Γ is small (it will be shown later) the wavefunction of quasi-stationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r<r_0$ and $r>r_-$ [1]

$$\int_{r_0}^{r_-}\left(F^2+G^2\right)dr = 1. \quad (12)$$

Here $\cos^2\Theta_i(r)$ can be replaced with average value 1/2:

$$\left|C_1^{\pm}\right| = \left[\int_{r_0}^{r_-}\frac{E-V(r)}{p(r)}dr\right]^{-1/2} = \left(\frac{2}{T}\right)^{1/2}, \quad (13)$$

where T is the oscillation period of a relativistic particle inside a potential well.

II. The below-barrier region $r_- < r < r_+$ is classically forbidden. Here $p=iq$, and quantities q, y_{-1} and y_0 are real. As it is known [1], the wavefunction should exponentially decrease inside this region. So the solutions of the Dirac system (2) in the below-barrier region for $k<0$ are

$$\Psi = \frac{C_2^-}{\sqrt{qQ}}\exp\left[-\int_{r_+}^{r}\left(q-\frac{(m+S)V'+(E-V)S'}{2qQ}\right)dr\right]\begin{pmatrix} m+S+E-V \\ -Q\end{pmatrix}, \quad (14)$$

where $Q = q+|k|/r$.

III. The result for region of continuum ($r > r_+$) is the most interesting. Here the wavefunction corresponds to the divergent wave (taking off particle): (for $k < 0$)

$$\Psi = \frac{C_3^-}{\sqrt{pP}} \exp\left[\int_{r_+}^{r} \left(ip - \frac{(m+S)\,V' + (E-V)\,S'}{2pP}\right) dr\right] \begin{pmatrix} m+S+E-V \\ iP \end{pmatrix}, \tag{15}$$

where $P = p + i\,|k|\,/r$.

For bypass of these points and matching the solutions we employ the usual method [2]. Closely to the r_- and r_+ the system (2) is reduced to the Schrödinger equation with the effective potential linearly depending on $r - r_\pm$, the solution of which is expressed by the Airy function; one can use a the more elegant Zwaan method [1, 3]. So the relation between the constants in various regions is of the form

$$C_2^\pm = -iC_3^\pm = \mp\frac{C_1^\pm}{2}\left[\frac{E - V(r_-) + m + S(r_-)}{|k|\,r_-^{-1}}\right]^{\pm\frac{1}{2}} \times \exp\left[-\int_{r_-}^{r_+}\left(q \pm \frac{(m+S)\,V' + (E-V)\,S'}{2qQ}\right) dr\right]. \tag{16}$$

Let us find the equation determining the real part of the level energy E and width Γ of quasi-stationary levels $E = E_r - i\Gamma/2$. Neglecting the barrier penetrability from (9) we obtain the quantization condition:

$$\int_{r_0}^{r_-}\left(p + \frac{kw}{pr}\right) dr = \left(n_r + \frac{1}{2}\right)\pi, \qquad n_r = 0, 1, 2, \ldots, \tag{17}$$

where n_r is the radial quantum number.

Let us proceed to calculation of the level width Γ. Having used the explicit expressions (15) for functions F and G, relation between normalization constants (16) and expression for $C_1^\pm$ (13), we get the tunneling probability:

$$\Gamma = -2\mathrm{Im}\left[G^*F\right]_{r\to\infty} = \frac{1}{T}\exp\left[-2\int_{r_-}^{r_+}\left(q - \frac{kw}{qr}\right) dr\right]. \tag{18}$$

However, the formulae (9)–(18) essentially differ from the formulae of nonrelativistic quasi-classics (in particular, by the relativistic expression for the quasimomentum p, taking into account spin-orbit interaction and additional pre-exponent multiplier) and are more complicated. Their application to concrete problems does not encounter difficulties, since all quantities in functions F and G are expressed in quadratures.

In (2+1) dimensions the Dirac system differs from (2) by replacements $G \to -G$ and $k \to -(l + 1/2)$, where $l + 1/2$ is the eigenvalue of the total angular moment J_z [6]. So all the expressions for wavefunctions, quantization condition and decay width in (2+1)-dimensional case can be obtained from formulae (9)–(18) by these replacements.

4 Approbation of the results obtained

4.1. For testing our version of the WKB method, let us obtain the Sommerfeld–Dirac formula known in atomic physics. We choose $V(r) = -\frac{\alpha Z}{r}$ and $S(r) = 0$ ($\alpha \approx 1/137$ is the fine structure constant). Having calculated the integral from r_0 to r_- in the quantization condition (17), we

arrive at the Sommerfeld–Dirac formula

$$E = m\left[1+\left(\frac{\alpha Z}{n+\sqrt{k^2-(\alpha Z)^2}}\right)^2\right]^{-1/2},$$
$$n = n_r + \frac{(1+\operatorname{sgn} k)}{2} = \begin{cases} 0,1,2\ldots, & \text{for } k<0,\\ 1,2\ldots, & \text{for } k>0.\end{cases}$$

Here $n_r = 0,1,2,\ldots$ is the radial quantum number from the quantization condition (17).

4.2. There is one more interesting case, in which it is possible to get an exact solution of the Dirac equation with potential of the type $V(r) = -\frac{\alpha}{r}$, $S(r) = -\frac{\alpha'}{r}$. In the same way as that by means of which the Coulomb potential $V(r)$ is derived from the exchange of massless photon between the nucleus and the leptons orbiting around it, the scalar potential $S(r)$ is created by the exchange of massless scalar mesons. The quantization condition (17) gives result coinciding with the expression obtained in [7]:

$$E = m\left\{\frac{-\alpha\alpha'}{\alpha^2+(n+\gamma)^2} \mp \left[\left(\frac{\alpha\alpha'}{\alpha^2+(n+\gamma)^2}\right)^2 - \frac{\alpha'^2-(n+\gamma)^2}{\alpha^2+(n+\gamma)^2}\right]^{1/2}\right\},$$
$$\gamma = \sqrt{k^2-\alpha^2+\alpha'^2}.$$

4.3. We consider the potential $V(r) = S(r) = ar^2/4$, a is the force of an oscillator. Integration in quantization condition (17) from r_0 to r_- gives the following result

$$\frac{(E-m)}{2}\sqrt{\frac{E+m}{2a}} - \frac{|k|}{2} - \frac{1}{4}\operatorname{sgn} k = n_r + \frac{1}{2},$$

input $\mathcal{K} = |k| + (1+\operatorname{sgn} k)/2$, get

$$(E-m)\sqrt{2(E+m)} = (4n_r + 2\mathcal{K} + 1)\sqrt{a}.$$

The latter equation for energy is cubic, its real solution is

$$E_{n_r,\mathcal{K}} = \left(2m + 8\cdot 2^{2/3}m^2A^{-1/3} + 2^{1/3}A^{1/3}\right)/6,$$

where $A = -B + \sqrt{B^2 - 1024m^6}$, $B = 32m^3 - 27a(1+2\mathcal{K}+4n_r)^2$. This expression completely coincides with result obtained in [8].

4.4. Let us consider a motion of a massless fermion in the scalar field $S(r) = -\frac{\alpha'}{r} + \sigma r$, $\sigma > 0$; $V(r) = 0$. Integration in quantization condition (17) from r_0 to r_- gives following result

$$\frac{E^2+2\sigma(\alpha'-\gamma)}{4\sigma} - \frac{k}{\sigma(r_0+r_-)\pi}\left[2r_0\left(\frac{\Pi(\nu_+^2,\chi)}{r_0^2-P_+^2} + \frac{\Pi(\nu_-^2,\chi)}{r_0^2-P_-^2}\right)\right.$$
$$\left. - \left(\frac{1}{r_0+P_+} + \frac{1}{r_0+P_-}\right)F(\chi)\right] = \left(n_r+\frac{1}{2}\right), \tag{19}$$

where

$$r_{0,-} = \frac{1}{2^{1/2}\sigma}\sqrt{E^2+2\alpha'\sigma \mp \sqrt{(E^2+2\alpha'\sigma)^2-(2\sigma\gamma)^2}}, \qquad \gamma = \sqrt{k^2+\alpha'^2},$$
$$P_\pm = \frac{1}{2\sigma}(-E \pm \sqrt{E^2+4\alpha'\sigma}), \quad \chi = \sqrt{\frac{E^2+2\sigma(\alpha'-\gamma)}{E^2+2\sigma(\alpha'+\gamma)}}, \qquad \nu_\pm^2 = \chi\,\frac{P_\pm + r_0}{P_\pm - r_0}.$$

Calculating the integrals approximately in (17), one obtains the following spectrum instead of (19)

$$\frac{E^2_{n_r,k}}{4\sigma} = n_r + \frac{1}{2} + \frac{\gamma-\alpha'}{2} + \frac{k}{4\gamma} + \frac{\sigma k}{2\pi E^2_{n_r,k}}\left(0.38 + \ln\frac{E^2_{n_r,k}}{\sigma\gamma}\right) + O\left(\left(\frac{\sigma\gamma}{E^2_{n_r,k}}\right)^2\right). \qquad (20)$$

The form (20) is exact for small $(\gamma\sigma/E^2_{n_r,k})^2$, and the accuracy can be tested by comparison with the exact formula (19); two sets of lowest levels with $k = \pm 1$ coincide within 2 % and for higher levels the precision is smaller than 1 %. At $\alpha' = 0$ expression (20) is reduced to Simonov's result [9].

In case of $E^2_{n_r,k} \gg \sigma\gamma$ in formula (20) expression $\left(0.38 + \ln E^2_{n_r,k}/\sigma\gamma\right)/\pi$ tends to 1 and one can obtain following expression for energies

$$\varepsilon_{n_r,k} = \frac{E_{n_r,k}}{\sqrt{\sigma}} = \pm\sqrt{N - \alpha' + \left[(N-\alpha')^2 + 2k\right]^{1/2}}, \qquad N = 2n_r + 1 + \gamma + \frac{k}{2\gamma}. \qquad (21)$$

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Asymptotical Expansions for One-Phase Soliton-Type Solution to Perturbed Korteweg–de Vries Equation

Yuliya SAMOYLENKO

Kyiv National Taras Shevchenko University, 64 Volodymyrska Str., 01033 Kyiv-33, Ukraine
E-mail: *yusam@univ.kiev.ua*

Korteweg–de Vries equation with coefficients depending on a small parameter is studied. The asymptotical expansions for one-phase soliton-type solutions are constructed.

1 Introduction

Korteweg–de Vries equation is one of the most known nonlinear differential equation and is a fundamental one of modern physics. It was deduced by D.J. Korteweg and G. de Vries [1] in order to describe so-called solitary waves firstly observed by J. Scott-Russel [2] in 1834.

The secondary discovery of this outstanding equation is connected with attempt to solve famous Fermi–Pasta–Ulam problem [3] in 1965, when M.D. Kruskal and N.J. Zabusky [4] found relation of the system of nonlinear oscillators of Toda type to the Korteweg–de Vries equation. Later it appeared as object for studying in different fields of physics, hydrodynamics, solid body theory, plasma theory, quantum mechanics and others [5, 6].

During the last 35 years a lot of papers were devoted to consideration of different properties of solutions to the Korteweg–de Vries equation, in particular, to finding its solutions in exact form via the inverse scattering transform initiated by C.S. Gardner, J.M. Green, M.D. Kruskal, R.M. Miura and P.D. Lax [7, 8] and developed in [9–12].

On the other hand, while considerating processes with small dispersion a problem of studying Korteweg–de Vries equation with varying coefficients and a small parameter arises. Similar problems can be researched through one of approaches developed on the base of a small parameter technique being well-known to be effective in studying different nonlinear problems [13, 14].

2 Statement of a problem and preliminary notes

We consider Korteweg–de Vries equation with varying coefficients and a small parameter of the following form

$$u_{xxx} = a(x,\varepsilon)u_t + b(x,\varepsilon)uu_x. \tag{1}$$

Functions $a(x,\varepsilon)$, $b(x,\varepsilon)$ are assumed to be represented as

$$a(x,\varepsilon) = \frac{1}{\varepsilon}\sum_{k=0}^{\infty} a_k(x)\varepsilon^k, \qquad b(x,\varepsilon) = \frac{1}{\varepsilon}\sum_{k=0}^{\infty} b_k(x)\varepsilon^k,$$

where $x \in \mathbb{R}^1$, $t \in (0;T)$, $a_k(x), b_k(x) \in C^{(\infty)}(\mathbb{R}^1)$, $k \geq 0$.

By using the approach proposed in [15–18] we develop an algorithm for constructing asymptotical solutions to problem (1).

Let us consider infinite differentiable functions $f = f(x,t,\tau)$, $(x,t,\tau) \in (\mathbb{R}^1_x \times (0;T) \times \mathbb{R}^1_t)$, such that for any compact $K \subset \mathbb{R}^1_x \times (0;T)$ and any non-negative integer numbers n, m, q, α uniformly with respect to variables (x,t) the following conditions are fulfilled:

1) the relation

$$\lim_{\tau\to\infty}\tau^n\frac{\partial^{m+q+\alpha}}{\partial x^m\partial t^q\partial\tau^\alpha}f(x,t,\tau)=0 \tag{2}$$

takes place;

2) there exists an infinite differentiable function $f^-(x,t)$ such that

$$\lim_{\tau\to-\infty}\tau^n\frac{\partial^{m+q+\alpha}}{\partial x^m\partial t^q\partial\tau^\alpha}\left[f(x,t,\tau)-f^-(x,t)\right]=0,\qquad (x,t)\in K. \tag{3}$$

Let the linear space of such functions $f=f(x,t,\tau)$, $(x,t,\tau)\in(\mathbb{R}^1_x\times(0;T)\times\mathbb{R}^1_t)$ be denoted $G=G(\mathbb{R}^1_x\times(0;T)\times\mathbb{R}^1_\tau)$.

Let $G_0=G_0(\mathbb{R}^1_x\times(0;T)\times\mathbb{R}^1_\tau)$ be a linear subspace of the space $G=G(\mathbb{R}^1_x\times(0;T)\times\mathbb{R}^1_\tau)$ consisting of functions $f(x,t,\tau)$ satisfying on every compact $K\subset\mathbb{R}^1_x\times(0;T)$ additionally to the conditions (2), (3) the equality

$$\lim_{\tau\to-\infty}f(x,t,\tau)=0,\qquad (x,t)\in K,$$

is satisfied uniformly with respect to (x,t).

Definition 1. A function $u=u(x,t,\varepsilon)$ is called one-phase soliton-type one if for any integer $N\geq 0$ function $u(x,t,\varepsilon)$ can be represented as

$$u(x,t,\varepsilon)=\sum_{k=0}^{2N}\varepsilon^{\frac{k}{2}}\left[u_k(x,t)+v_k(x,t,\tau)\right]+O\big(\varepsilon^{N+1}\big), \tag{4}$$

where $\tau=\frac{x-\varphi(t)}{\sqrt{\varepsilon}}$, $\varphi(t)\in C^{(\infty)}(0;T)$ is a scalar real function; functions $u_k(x,t)$ are infinite differentiable; $v_0(x,t,\tau)\in G_0$, $v_k(x,t,\tau)\in G$, $k=1,2,\ldots,2N$.

Function $S=x-\varphi(t)$ is called a phase of one-phase soliton-type solution (4).

Below we propose algorithm of constructing asymptotical solution to the equation (1). Firstly we establish a form of asymptotical solution to problem (1). In particular, it is not complicated to prove that asymptotical solution to the problem (1) can be represented as

$$u(x,t,\varepsilon)=\bar{U}(x,t,\varepsilon)+\sqrt{\varepsilon}G_\varepsilon(S,x,t)+\sqrt{\varepsilon}F_\varepsilon(S,x,t),$$

where $\bar{U}(x,t,\varepsilon)$ is infinite differentiable function; $G_\varepsilon(S,x,t)$, $F_\varepsilon(S,x,t)$ are infinite differentiable functions sufficiently quickly tending to zero as $|x|\to\infty$ and satisfying conditions

$$\|\sqrt{\varepsilon}G_\varepsilon(S,x,t)\|_C\leq C_1,\qquad \|F_\varepsilon(S,x,t)\|_C\leq C_2$$

with constants C_1, C_2 not depending on ε and norm $\|\cdot\|_C$ of continuous in $(x,t)\in\mathbb{R}^1_x\times[0;T]$ function space.

3 Scheme of constructing asymptotical solution

Solution to the equation (1) is searched in the form of following asymptotical series:

$$u(x,t,\varepsilon)=Y_N(x,t,\tau,\varepsilon)+O\big(\varepsilon^{N+1}\big), \tag{5}$$

where

$$Y_N(x,t,\tau,\varepsilon)=\sum_{k=0}^{2N}\varepsilon^{\frac{k}{2}}\left[u_k(x,t)+v_k(x,t,\tau)\right],\qquad \tau=\frac{x-\varphi(t)}{\sqrt{\varepsilon}}.$$

Function $U_N(x,t,\varepsilon) = \sum\limits_{k=0}^{2N} \varepsilon^{\frac{k}{2}} u_k(x,t)$ is a regular part of asymptotic (5), and function $V_N(x,t,\varepsilon)$ $= \sum\limits_{k=0}^{2N} \varepsilon^{\frac{k}{2}} v_k(x,t,\tau)$ is a singular part of asymptotic (5), i.e. $Y_N = U_N + V_N$.

At the first step the asymptotical solution to the problem (1) is constructed in some μ-neighborhood of curve $\Gamma = \{(x,t) \in \mathbb{R}^1 \times (0;T) : x = \varphi(t)\}$, i.e. in the domain:

$$\Omega_\mu(\Gamma) = \{(x,t) \in \mathbb{R}^1 \times (0,T) : |x - \varphi(t)| < 2\mu\},$$

where μ is a small positive parameter, as well as a function $\varphi(t)$ is defined later.

By direct calculations we find derivatives $u_t(x,t,\varepsilon)$, $u_x(x,t,\varepsilon)$, $u_{xxx}(x,t,\varepsilon)$, substitute them into equation (1) and obtain the relation:

$$\begin{aligned}
&\frac{\partial^3 Y_N}{\partial x^3} + \frac{3}{\sqrt{\varepsilon}} \frac{\partial^3 Y_N}{\partial x^2 \partial \tau} + \frac{3}{\varepsilon} \frac{\partial^3 Y_N}{\partial x \partial \tau^2} + \frac{1}{\varepsilon^{\frac{3}{2}}} \frac{\partial^3 Y_N}{\partial \tau^3} \\
&\qquad = a(x,\varepsilon) \left(\frac{\partial Y_N}{\partial t} - \frac{1}{\sqrt{\varepsilon}} \frac{\partial Y_N}{\partial \tau} \varphi'(t) \right) + b(x,\varepsilon) \left(\frac{\partial Y_N}{\partial x} + \frac{1}{\sqrt{\varepsilon}} \frac{\partial Y_N}{\partial \tau} \right) Y_N + g_N(x,t,\tau,\varepsilon). \quad (6)
\end{aligned}$$

Here $g_N(x,t,\tau,\varepsilon) = O\big(\varepsilon^{N+1}\big)$ is an infinite differentiable function of its arguments that are defined recursively in N.

To find equations for regular part of asymptotics from relation (6) we calculate limit as $\tau \to +\infty$ and equate coefficients at the same powers ε. As result, we have system of partial differential equations for functions u_k, $k = 0, \ldots, 2N$:

$$\begin{aligned}
&a_0(x) \frac{\partial u_0}{\partial t} + b_0(x) \frac{\partial u_0}{\partial x} u_0 = 0, \\
&a_0(x) \frac{\partial u_k}{\partial t} + b_0(x) u_0(x,t) \frac{\partial u_k}{\partial x} + b_0(x) u_k(x,t) \frac{\partial u_0}{\partial x} = f_k(x,t,u_0,u_1,\ldots,u_{k-1}), \quad (7)
\end{aligned}$$

where functions $f_k(t,x,u_0,u_1,\ldots,u_{k-1})$, $k = 1,\ldots,2N$, are defined recurrently. Additionally the equations (7) for any $k = 0,\ldots,2N$ are assumed to have infinite differentiable solutions.

3.1 Defining singular part of asymptotics

Differential equations for singular part of asymptotics (5) are deduced from relation (6) after reducing (7). Later functions $v_k(x,t,\tau)$, $k = 0,1,\ldots,2N$, are defined on the curve $x = \varphi(t)$ and continued into domain $\Omega_\mu(\Gamma)$. Let values of functions $v_k(x,t,\tau)$, $k = 0,1,\ldots,2N$, on the curve $x = \varphi(t)$ be denoted $v_k(t,\tau)$. As result, we have the following system of partial differential equations:

$$\begin{aligned}
&\frac{\partial^3 v_0}{\partial \tau^3} + a_0(\varphi) \frac{\partial v_0}{\partial \tau} \varphi'(t) - b_0(\varphi) \left[u_0(\varphi,t) \frac{\partial v_0}{\partial \tau} + v_0 \frac{\partial v_0}{\partial \tau} \right] = 0, \\
&\frac{\partial^3 v_k}{\partial \tau^3} + a_0(\varphi) \frac{\partial v_k}{\partial \tau} \varphi'(t) - b_0(\varphi) \left[u_0(\varphi,t) \frac{\partial v_k}{\partial \tau} + v_k \frac{\partial v_0}{\partial \tau} + v_0 \frac{\partial v_k}{\partial \tau} \right] = \mathcal{F}_k(t,\tau), \quad (8)
\end{aligned}$$

where $k = 1,\ldots,2N$, functions

$$\mathcal{F}_k(t,\tau) = F_k(t, v_0(x,t,\tau), \ldots, v_{k-1}(x,t,\tau), u_0(x,t), \ldots, u_k(x,t))\Big|_{x=\varphi(t)}$$

are defined recurrently. Namely, first we find function $v_0(t,\tau)$ from (8), later extend it into domain $\Omega_\mu(\Gamma)$. At the next step we calculate the function

$$\mathcal{F}_1(t,\tau) = F_1(t, v_0(x,t,\tau), u_0(x,t), u_1(x,t))\Big|_{x=\varphi(t)},$$

define function $v_1(t,\tau)$, extend it into the domain $\Omega_\mu(\Gamma)$, and so on.

3.2 Solvability of equations for singular part of asymptotics

Consider the first equation in (8). It can be written as follows (after integration in τ):

$$\frac{d^2v_0}{d\tau^2} = -a_0(\varphi)v_0(\varphi,\tau)\varphi'(t) + b_0(\varphi)u_0(\varphi,t)v_0(t,\tau) + \frac{1}{2}b_0(\varphi)v_0^2(\varphi) + c_1(t). \tag{9}$$

Due to condition $v_0(t,\tau) \in G_0$, we can take $c_1(t) \equiv 0$.

Multiplying equation (9) by $dv_0/d\tau$ and integrating it in τ one receives:

$$\left(\frac{dv_0}{d\tau}\right)^2 = -a_0(\varphi)v_0^2(t,\tau)\varphi'(t) + \frac{1}{2}b_0(\varphi)v_0^3(t,\tau) + c_2(t).$$

Due to condition $v_0(t,\tau) \in G_0$ we can take $c_2(t) \equiv 0$.

Thus, a solution to the equation (9) in the space G_0 is represented as

$$v_0(t,\tau) = A[\varphi]\,\mathrm{ch}^{-2}((\tau + C_0)H[\varphi]), \tag{10}$$

where

$$A[\varphi] = -2\frac{a_0(\varphi(t))\varphi'(t) - b_0(\varphi(t))u_0(\varphi(t),t)}{b_0(\varphi(t))}, \qquad H[\varphi] = \frac{2\sqrt{A}}{b_0(\varphi(t))}$$

provided that $A[\varphi] > 0$.

So, the following lemma is proved.

Lemma 1. *Let $A[\varphi] > 0$. Then a solution of the first equation of system* (8) *in the space G_0 exists and is given by formula* (10)*.*

Differential equations of system (8) for $k = 1, 2, \ldots, 2N$ can be written in the following operator form

$$\hat{L}v_k = F_k, \tag{11}$$

where operator $\hat{L}$ is

$$\hat{L} = \frac{\partial^3}{\partial\tau^3} + \left[a_0(\varphi(t))\varphi'(t) - b_0(\varphi(t)) - b_0(\varphi(t))u_0(\varphi(t),t)\right]\frac{\partial}{\partial\tau} - \frac{\partial v_0}{\partial\tau}b_0(\varphi(t)).$$

The following lemma is true.

Lemma 2. *Suppose functions $F_k(t,\tau) \in G_0$, $k \geq 1$. Then operator equations* (11) *are solvable in the space G if and only if the conditions*

$$\int_{-\infty}^{+\infty} F_k(t,\tau)v_0(t,\tau)d\tau = 0, \qquad k \geq 1, \tag{12}$$

are satisfied.

Relation (12) is called an orthogonality condition. In the case, when the orthogonality condition (12) is fulfilled, the general solution to the equation (8) in the space G can be written as

$$\nu_k(t,\tau) = z_k(t,\tau) + c_k v_{0\tau}, \qquad k \geq 1,$$

where c_k is constant of integrability, $z_k(t,\tau)$ is a particular solution to the non-homogeneous equation (8):

$$z_k(t,\tau) = v_{0\tau}\int_{-\infty}^{\tau} v_{0\tau}^{-2}(t,\tau_1)\int_{-\infty}^{\tau_1}\Phi_k(t,\tau_2)v_{0\tau}(t,\tau_2)d\tau_2 d\tau_1, \qquad k = 1, 2, \ldots, 2N.$$

3.3 Definition of the phase function $\varphi(t)$

The function of phase $\varphi(t)$ is defined from ordinary differential equation deduced from orthogonality condition (12). In particular, if one takes the function

$$\mathcal{F}_1(t,\tau) = a_0(\varphi)\frac{\partial v_0}{\partial t} + b_0(\varphi)\frac{\partial u_0(x,t)}{\partial x}\bigg|_{x=\varphi(t)} \cdot v_0 + b_0(\varphi)u_1(\varphi)\frac{\partial v_0}{\partial \tau}, \tag{13}$$

then after substituting function $v_0(t,\tau)$ into (13) and using (12), finds

$$\frac{d\varphi}{dt} = -\frac{\frac{\partial u_0(x,t)}{\partial x}\Big|_{x=\varphi} b_0(\varphi)A[\varphi]H^2[\varphi]}{2a_0(\varphi)A'[\varphi]H[\varphi] - A[\varphi]H'[\varphi]a_0(\varphi)}. \tag{14}$$

Since under rather general conditions on $a_0(x)$, $b_0(x)$ functions $A[\varphi]$, $H[\varphi]$ may be considered to be infinite differentiable ones and consequently, the differential equation (14) may be assumed to have unique infinite differentiable solution.

3.4 Continuation of singular part of asymptotics into domain $\Omega_\mu(\Gamma)$

We define functions $v_k(x,t,\tau)$, $k \geq 0$, in the closure of domain $\Omega_\mu(\Gamma)$. For $k = 0$ solution of equation (8) is deduced above (see formula (10)). For $k = 1, 2, \ldots, 2N$ a solution to the equation (8) has a form

$$v_k(t,\tau) = \nu_k(t)\eta_k(t,\tau) + \psi_k(t,\tau), \qquad k = 1, \ldots, 2N,$$

where

$$\lim_{\tau\to-\infty}\eta = 1, \quad \eta \in G,$$
$$\nu_k(t,\tau) = -\left[a_0(\varphi(t))\varphi'(t) - b_0(\varphi(t))\right]^{-1}\lim_{\tau\to-\infty}\Phi_k(t,\tau),$$
$$\Phi_k(t,\tau) = -\int\limits_{-\infty}^{\tau}\mathcal{F}_k(t,\tau)d\tau + E_k(t), \qquad \lim_{\tau\to+\infty}\Phi_k(t,\tau) = 0,$$
$$\psi_k(t,\tau) = \psi_{k,1}(t,\tau) + c_k(t)v_{0\tau}(t,\tau),$$

a function $\psi_{k,1}$ belongs to the space G_0, $c_k(t)$ is a constant of integration.

Let us consider the Cauchy problem:

$$\Lambda u_k^-(x,t) = f_k(x,t),$$
$$u_k^-(x,t)\Big|_\Gamma = \nu_k(t), \qquad k = 1, \ldots, 2N, \tag{15}$$

where the differential operator Λ is

$$\Lambda = a_0(x)\frac{\partial}{\partial t} + b_0(x)u_0(x,t)\frac{\partial}{\partial x} + b_0(x)\frac{\partial u_0(x,t)}{\partial x}.$$

Since the curve Γ is transversal to a characteristics of differential operator Λ for any $t \in [0;T]$, the problem (15) is correctly posed, and because of it there exists a solution $u_k^-(x,t) \in C^{(\infty)}(\Omega_\mu(\Gamma))$ for small enough μ.

Continuation of functions $v_k(t,\tau)$, $k = 0, 1, \ldots, 2N$ into domain $\Omega_\mu(\Gamma)$ is defined as

$$v_0(x,t,\tau) = v_0(t,\tau), \qquad v_k(x,t,\tau) = u_k^-(x,t)\eta(t,\tau) + \psi_k(t,\tau).$$

4 Constructing global asymptotical solution

Let us note domains

$$D^- = \{(x,t) \in \mathbb{R}^1 \times (0;T) : \varphi(t) - x \geq \mu\},$$
$$D^+ = \{(x,t) \in \mathbb{R}^1 \times (0;T) : x - \varphi(t) \geq \mu\}.$$

For all $x < \varphi(t)$, $t \in (0;T)$, functions $u_k^-(x,t)$, $k = 1, \ldots, 2N$ are defined as infinite differentiable solutions to the problem (15).

Asymptotical solution to the equation (1) is found through gluing functions defined before. Let $\chi(\xi) \in C^\infty(\mathbb{R}^1) : \chi(\xi) = 1$ if $\xi \geq 2$ and $\chi(\xi) = 0$ if $\xi \leq 1$.

Thus, the following result is established.

Theorem 1. *Let us suppose:*
1) *functions* $a_k(x), b_k(x) \in C^{(\infty)}(\mathbb{R}^1)$, $k \geq 0$*;*
2) *the inequality* $A[\varphi] > 0$*, where function* $\varphi(t)$ *is a solution to the equation* (14)*, takes place;*
3) *functions* $F_k(t,\tau)$, $k = 1, \ldots, N$*, belong to the space* G_0*;*
4) *the orthogonality condition* (12) *is fulfilled.*
Then the function

$$Y_N(x,t,\varepsilon) = \begin{cases} Y_N^-(x,t,\varepsilon), & (x,t) \in D^- \backslash \Omega_\mu(\Gamma), \\ Y_N^+(x,t,\varepsilon), & (x,t) \in D^+ \backslash \Omega_\mu(\Gamma), \\ Y_N(x,t,\tau,\varepsilon), & (x,t) \in \Omega_\mu(\Gamma), \end{cases}$$

where

$$Y_N^-(x,t,\varepsilon) = u_0(x,t) + \sum_{k=1}^{2N} \varepsilon^{\frac{k}{2}} \left[u_k(x,t) + u_k^-(x,t)\right], \qquad (x,t) \in D^-,$$

$$Y_N^+(x,t,\varepsilon) = \sum_{k=1}^{2N} \varepsilon^{\frac{k}{2}} u_k(x,t), \qquad (x,t) \in D^+,$$

$$Y_N(x,t,\tau,\varepsilon) = \sum_{k=0}^{2N} \varepsilon^{\frac{k}{2}} \left[u_k(x,t) + v_k(x,t,\tau)\right], \qquad (x,t) \in \Gamma, \qquad \tau = \frac{x - \varphi(t)}{\sqrt{\varepsilon}},$$

is an asymptotical expansion for solution to the Korteweg–de Vries equation (1) *on the interval* $(0;T)$*, i.e. for any compact* $K \subset \mathbb{R}^1_x \times (0;T)$

$$\max_{(x,t)\in K} |u(x,t,\varepsilon) - Y_N(x,t,\varepsilon)| = O\big(\varepsilon^{N+1}\big), \qquad N \in \mathbb{N}.$$

5 Conclusions

In the paper the problem of constructing asymptotical solutions to the Korteweg–de Vries equation with coefficients depending on a small parameter is studied. The algorithm of constructing asymptotical solutions is developed.

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Axially Symmetric Black Hole Skyrmions

Nobuyuki SAWADO and Noriko SHIIKI

Department of Physics, Tokyo University of Science, Noda, Chiba 278-8510, Japan
E-mail: *sawado@ph.noda.tus.ac.jp, norikoshiiki@mail.goo.ne.jp*

It has been known that a $B = 2$ skyrmion is axially symmetric. We consider the Skyrme model coupled to gravity and obtain static axially symmetric black hole solutions numerically. The black hole skyrmion no longer has integer baryonic charge but has fractional charge outside the horizon as in the spherically symmetric case. Therefore, the solution represents a black hole partially swallowing a deuteron.

1 Introduction

It has been shown that the no-hair conjecture for black holes [1] is violated when some nonlinear matter fields are considered. The first counter example was provided by Luckock and Moss [2] who found the Schwarzschild black hole with Skyrme hair. The presence of the horizon in the core of skyrmion unwinds the skyrmion, leaving fractional baryon charge outside the horizon. The full Einstein–Skyrme system was solved later by Droz *et al.* to obtain spherically symmetric black holes with Skyrme hair [3]. Other counterexamples include static spherically symmetric black holes in the Einstein–Yang–Mills (EYM) [4], the Einstein–Yang–Mills–Dilaton (EYMD) [5, 6] and the Einstein–Yang–Mills–Higgs (EYMH) theory [7]. More interestingly, it has been also shown that these Einstein–Yang–Mills theories have static axially symmetric black hole solutions [8, 9].

Motivated by the axially symmetric hairy black holes in Refs. [8, 9], we shall study the Einstein–Skyrme model with axial symmetry. It has been shown that a $B = 2$ skyrmion is axially symmetric and represents a deuteron [10]. Our model, therefore, provides a convenient framework to study the interactions between a deuteron and a black hole. By examining the baryon number of the solution, the absorption of the deuteron by the black hole is observed as in the spherically symmetric case. We expect our solutions to be stable as skyrmions are topologically stable objects.

2 The model

The Skyrme model is an effective theory of QCD based on pion fields alone [11]. At low energy, the symmetry of the strong interaction is broken spontaneously and hence the Skyrme Lagrangian retains the chiral symmetry. The Skyrme model coupled to gravity is defined by

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_G, \tag{1}$$

where

$$\mathcal{L}_S = \frac{f_\pi^2}{16} g^{\mu\nu} \mathrm{tr}\big(U^{-1}\partial_\mu U U^{-1}\partial_\nu U\big) + \frac{1}{32a^2} g^{\mu\rho} g^{\nu\sigma} \mathrm{tr}\left([U^{-1}\partial_\mu U, U^{-1}\partial_\nu U][U^{-1}\partial_\rho U, U^{-1}\partial_\sigma U]\right),$$
$$\mathcal{L}_G = \frac{1}{16\pi G} R.$$

Let us introduce an ansatz for the metric given in Ref. [8]

$$ds^2 = -f dt^2 + \frac{m}{f}(dr^2 + r^2 d\theta^2) + \frac{l}{f} r^2 \sin^2\theta d\varphi^2, \tag{2}$$

where $f = f(r, \theta)$, $m = m(r, \theta)$, and $l = l(r, \theta)$.

The axially symmetric Skyrme field can be parameterized by

$$U = \cos F(r,\theta) + i\vec{\tau}\cdot\vec{n}_R \sin F(r,\theta), \tag{3}$$

with $\vec{n}_R = (\sin\Theta\cos n\varphi, \sin\Theta\sin n\varphi, \cos\Theta)$. In terms of F and Θ, the Lagrangian (1) has the form

$$\mathcal{L}_S = \mathcal{L}_S^{(1)} + \mathcal{L}_S^{(2)}, \tag{4}$$

where

$$\mathcal{L}_S^{(1)} = -\frac{f_\pi^2 f}{8m}\left\{(\partial_r F)^2 + \frac{1}{r^2}(\partial_\theta F)^2 + \left[(\partial_r\Theta)^2 + \frac{1}{r^2}(\partial_\theta\Theta)^2\right]\sin^2 F + \frac{n^2}{r^2\sin^2\theta}\frac{m}{l}\sin^2\Theta\sin^2 F\right\},$$

$$\mathcal{L}_S^{(2)} = -\frac{1}{2a^2r^2}\left(\frac{f}{m}\right)^2\left\{(\partial_r F\partial_\theta\Theta - \partial_\theta F\partial_r\Theta)^2 + \frac{n^2}{\sin^2\theta}\frac{m}{l}\left[(\partial_r F)^2 + \frac{1}{r^2}(\partial_\theta F)^2\right]\sin^2\Theta\right\}\sin^2 F$$
$$- \frac{1}{2a^2r^2}\left(\frac{f}{m}\right)^2\left\{\frac{n^2}{\sin^2\theta}\frac{m}{l}\left[(\partial_r\Theta)^2 + \frac{1}{r^2}(\partial_\theta\Theta)^2\right]\sin^2 F\sin^2\Theta\right\}\sin^2 F.$$

Since we are interested in $B = 2$, we shall take the winding number $n = 2$.

The baryon current in curved space-time is obtained by taking the space-time covariant derivative ∇_μ,

$$b^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{tr}(U^{-1}\nabla_\nu U U^{-1}\nabla_\rho U U^{-1}\nabla_\sigma U). \tag{5}$$

The baryon number then is given by integrating b^0 over the hypersurface $t = 0$,

$$B = \int dr d\theta d\varphi\sqrt{g^{(3)}}\, b^0 = -\frac{1}{\pi}\int dr d\theta\,(\partial_r F\partial_\theta\Theta - \partial_\theta F\partial_r\Theta)\sin\Theta(1-\cos 2F)$$
$$= -\frac{1}{\pi}\int dF\wedge d\Theta\sin\Theta(1-\cos 2F) = \left[\frac{1}{2\pi}(2F - \sin 2F)\cos\Theta\right]_{F_0,\Theta_0}^{F_1,\Theta_1},$$

where (F_0,Θ_0) and (F_1,Θ_1) are the values at the inner and outer boundary, respectively. In flat space-time [10], we have

$$(F_0,\Theta_0) = (\pi,0) \ \text{ and } \ (F_1,\Theta_1) = (0,\pi), \tag{6}$$

which gives $B = 2$. In the presence of a black hole, the integration should be performed from the horizon to infinity, which changes the values of F_0 and allows the B to be fractional.

The energy density of the skyrmion outside the horizon can be obtained by the zero-zero component of the stress-energy tensor $-T_0^0$,

$$\epsilon = -T_0^0$$
$$= \frac{f_\pi^2}{8}\frac{f}{m}\left[(\partial_r F)^2 + \frac{1}{r^2}(\partial_\theta F)^2 + \left\{(\partial_r\Theta)^2 + \frac{1}{r^2}(\partial_\theta\Theta)^2\right\}\sin^2 F + \frac{n^2}{r^2\sin^2\theta}\frac{m}{l}\sin^2 F\sin^2\Theta\right]$$
$$+ \frac{1}{2a^2r^2}\frac{f^3}{m^2}\left[(\partial_{[r}F\partial_{\theta]}\Theta)^2 + \frac{n^2}{\sin^2\theta}\frac{m}{l}\left\{(\partial_r F)^2 + \frac{1}{r^2}(\partial_\theta F)^2\right\}\sin^2\Theta\right.$$
$$\left. + \frac{n^2}{\sin^2\theta}\frac{m}{l}\left\{(\partial_r\Theta)^2 + \frac{1}{r^2}(\partial_\theta\Theta)^2\right\}\sin^2 F\sin^2\Theta\right]\sin^2 F \tag{7}$$

3 Boundary conditions

Let us consider the boundary conditions for the chiral fields and metric functions with help of Ref. [8]. At the horizon $r = r_h$, the time-time component of the metric satisfies

$$g_{tt} = -f(r_h, \theta) = 0. \tag{8}$$

Regularity of the metric at the horizon requires

$$m(r_h, \theta) = l(r_h, \theta) = 0. \tag{9}$$

The boundary conditions for $F(r, \theta)$ and $\Theta(r, \theta)$ at the horizon are obtained by expanding them at the horizon and inserting into the field equations derived from $\delta\mathcal{L}_S/\delta F = 0$ and $\delta\mathcal{L}_S/\delta\Theta = 0$ respectively,

$$\partial_r F(r_h, \theta) = \partial_r \Theta(r_h, \theta) = 0. \tag{10}$$

The condition that the space-time is asymptotically flat requires

$$f(\infty, \theta) = m(\infty, \theta) = l(\infty, \theta) = 1. \tag{11}$$

The boundary conditions for F and Θ at infinity remain the same as in flat space-time

$$F(\infty, \theta) = 0, \qquad \partial_r \Theta(\infty, \theta) = 0. \tag{12}$$

For the solution to be axially symmetric, we have

$$\partial_\theta f(r, 0) = \partial_\theta m(r, 0) = \partial_\theta l(r, 0) = 0, \tag{13}$$

$$\partial_\theta f\left(r, \frac{\pi}{2}\right) = \partial_\theta m\left(r, \frac{\pi}{2}\right) = \partial_\theta l\left(r, \frac{\pi}{2}\right) = 0. \tag{14}$$

Likewise for F,

$$\partial_\theta F(r, 0) = \partial_\theta F\left(r, \frac{\pi}{2}\right) = 0. \tag{15}$$

Regularity on the axis and axisymmetry impose the boundary conditions on Θ as

$$\Theta(r, 0) = 0, \qquad \Theta\left(r, \frac{\pi}{2}\right) = \frac{\pi}{2}. \tag{16}$$

4 Numerical results and discussions

For the purpose of numerical computation, we shall introduce a dimensionless radial coordinate x and coupling constant α,

$$x = a f_\pi r, \qquad \alpha = \pi G f_\pi^2. \tag{17}$$

Then, in this system, free parameters are only x_h and α. Fig. 1 shows dependence of the metric function f on θ with $\alpha = 1.0$. For smaller values of α, the results are slightly lower than that of $\alpha = 1.0$. Other metric functions l and m exhibit similar behavior as f. The Skyrme functions F and Θ are shown in Figs. 2, 3. α dependence as well as θ dependence of F is rather small. Θ is less distorted from $\Theta = \theta$ for smaller α. We show the energy density of the Skyrme fields ($\epsilon = -T_0^0$) in Figs. 4, 5 with $\alpha = 0.01$, 1.0. As can be seen, the density becomes dumbbell in shape with the highest along z-axis while in flat space-time it is toroidal. As α becomes small, it approaches to a more spherical shape. It is interesting to see how the presence of the black hole

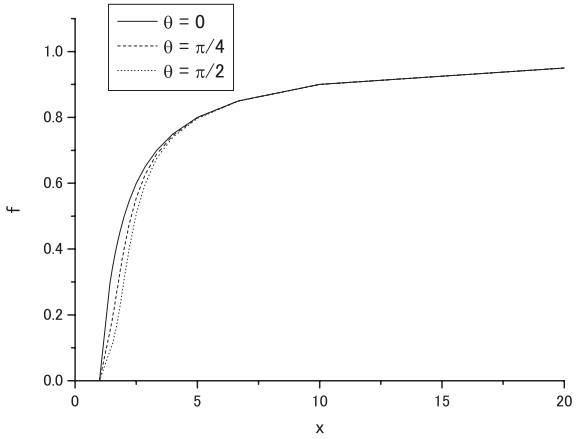

Figure 1. The metric function f as a function of x with $\alpha = 1.0$ and $x_h = 1.0$.

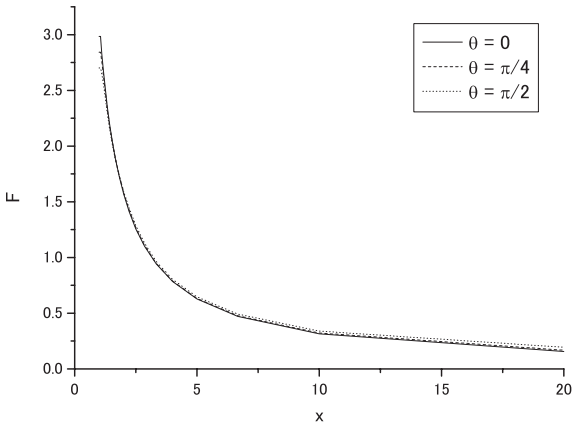

Figure 2. The Skyrme function F as a function of x with $\alpha = 1.0$ and $x_h = 1.0$.

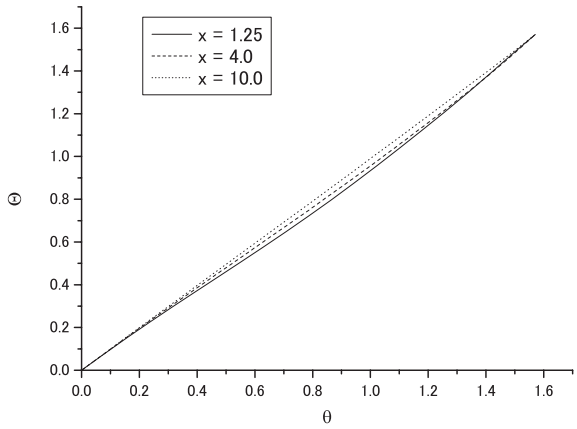

Figure 3. The Skyrme function Θ as a function of θ with $\alpha = 1.0$ and $x_h = 1.0$.

affects the shape of the skyrmion. The domain of existence of the solutions in the parameter space is shown in Fig. 6. For $\alpha \gtrsim 2.0$, there exists no solution since the chiral fields become too massive for the black hole to support outside the horizon. It is also observed that the black hole has a finite minimum size unlike the spherically symmetric case. Hence one cannot recover regular solutions as the limit of zero horizon size. Fig. 7 shows the dependence of the baryon number on α and x_h. It is observed that the baryon number gets more absorbed by the black hole in increase of the size of the black hole and the coupling constant.

We suspect that our solutions should be stable since the skyrmions are highly stable objects. This statement may be verified by applying the catastrophe theory of hairy black holes proposed in Ref. [12].

Finally, recent studies of theories with large extra dimensions indicate that a true Planck scale is of order a TeV and the production rate of black holes massive than the Planck scale become quite large [13–15]. It will be interesting to extend our model to higher dimensions since it bring us an interesting possibility that the deuteron black holes could be produced in the LHC by collision of two protons.

Inclusion of gauge fields will be also interesting to study electrically charged deuteron black holes [16].

Acknowledgements

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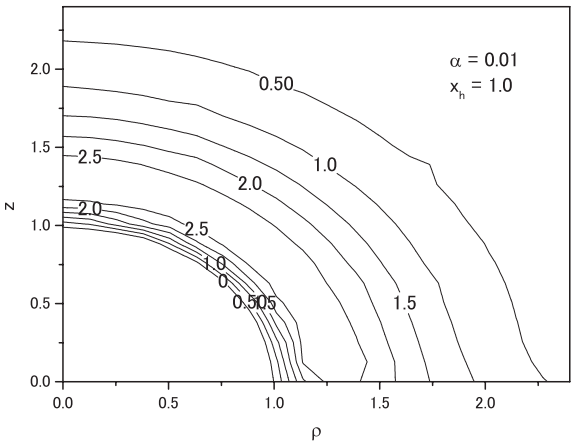

Figure 4. A contour plot of the energy density ϵ in cylindrical coordinates ρ and z with $x_h = 1.0$.

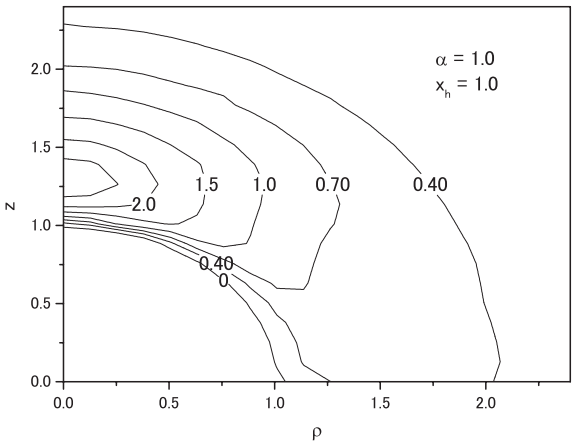

Figure 5. A contour plot of the energy density ϵ in cylindrical coordinates ρ and z with $x_h = 1.0$.

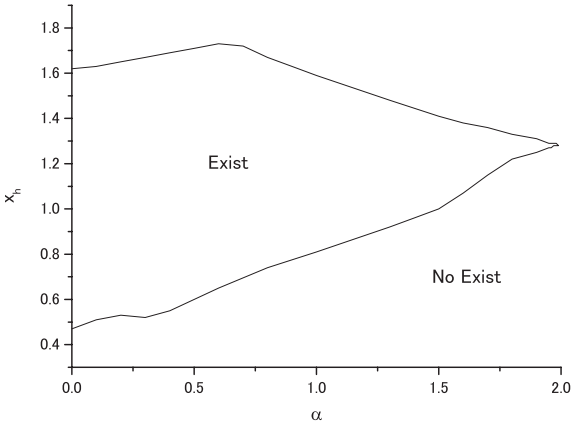

Figure 6. The domain of existence of the solution. For $\alpha \gtrsim 2.0$, there exists no non-trivial solution.

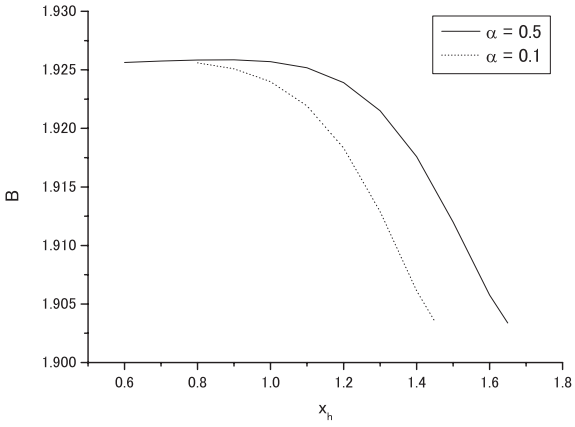

Figure 7. The dependence of the baryon number on the size of the horizon.

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Quantum Phase Space in Relativistic Theory: the Case of Charge-Invariant Observables

A.A. SEMENOV †, *B.I. LEV* † *and C.V. USENKO* ‡

† *Institute of Physics of NAS of Ukraine, 46 Nauky Ave., 03028 Kyiv, Ukraine*
E-mail: *sem@iop.kiev.ua, lev@iop.kiev.ua*

‡ *Physics Department, Taras Shevchenko Kyiv University, 6 Academician Glushkov Ave., 03127 Kyiv, Ukraine*
E-mail: *usenko@phys.univ.kiev.ua*

Mathematical method of quantum phase space is very useful in physical applications like quantum optics and non-relativistic quantum mechanics. However, attempts to generalize it for the relativistic case lead to some difficulties. One of problems is band structure of energy spectrum for a relativistic particle. This corresponds to an internal degree of freedom, so-called charge variable. In physical problems we often deal with such dynamical variables that do not depend on this degree of freedom. These are position, momentum, and any combination of them. Restricting our consideration to this kind of observables we propose the relativistic Weyl–Wigner–Moyal formalism that contains some surprising differences from its non-relativistic counterpart.

This paper is devoted to the phase space formalism that is specific representation of quantum mechanics. This representation is very close to classical mechanics and its basic idea is a description of quantum observables by means of functions in phase space (symbols) instead of operators in the Hilbert space of states.

The first idea about this representation has been proposed in the early days of quantum mechanics in the well-known Weyl work [1]. Let us put the operator $\hat{A}$, which acts in the Hilbert space of states of a quantum system, into correspondence to a function on the phase space (symbol) $A(p,q)$ according with the following rule

$$\hat{A} = \int_{-\infty}^{+\infty} A(p,q)\,\hat{W}(p,q)\,dpdq, \tag{1}$$

where $\hat{W}(p,q)$ is an operator generalization of δ-function that is called operator of quasiprobability density. This operator is defined as Fourier image of operator exponent (operator of representation of the Heisenberg–Weyl group)

$$\hat{W}(p,q) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} \exp\left\{\frac{i}{\hbar}\left[Q(p-\hat{p}) - P(q-\hat{q})\right]\right\} dQdP. \tag{2}$$

Nowadays, this transformation is known as Weyl transform.

In Table 1 one can find correspondences among some constructions in classical mechanics and in the Hilbert space and phase space representations of quantum mechanics. First of all, it is related to two binary operations, namely, usual product and bracket. In classical mechanics there exist conventional commutative multiplication of two functions and Poisson bracket. It is well-known that corresponding operations in the Hilbert space representation of quantum mechanics are non-commutative products of two observables and commutator. In the phase space representation of quantum mechanics we deal with so-called star product and Moyal bracket.

Table 1. Some constructions in classical mechanics and in quantum mechanics (Hilbert space and phase space representations) are shown. First line corresponds to the usual product of two observables. Brackets (Poisson, commutator, Moyal) are shown in the second line. Evolution equations in Heisenberg and Schrödinger representations are presented in third and fourth lines respectively. In the fifth line one can see expressions for mean value of an observable. Expressions for a pure state are presented in the sixth line.

Classical mechanics	Quantum mechanics (Hilbert space)	Quantum mechanics (phase space)
$A(p,q)B(p,q)$	$\hat{A}\hat{B}$	$A(p,q) \star B(p,q)$, $\star = \exp\left\{\frac{i\hbar}{2}\left(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q}\right)\right\}$
$\{A(p,q), B(p,q)\}_P$ $= A(p,q)\left(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q}\right)B(p,q)$	$\left[\hat{A},\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$	$\{A(p,q), B(p,q)\}_M$ $= \frac{1}{i\hbar}\left(A(p,q)\star B(p,q) - B(p,q)\star A(p,q)\right)$ $= \frac{2}{\hbar}A(p,q)\sin\left\{\frac{\hbar}{2}\left(\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \overleftarrow{\partial_p}\overrightarrow{\partial_q}\right)\right\}B(p,q)$
$\partial_t A = \{A,H\}_P$	$\partial_t \hat{A} = \frac{1}{i\hbar}\left[\hat{A},\hat{H}\right]$	$\partial_t A = \{A,H\}_M$
$\partial_t \rho = \{H,\rho\}_P$	$\partial_t \hat{\rho} = \frac{1}{i\hbar}\left[\hat{H},\hat{\rho}\right]$	$\partial_t W = \{H,W\}_M$
$\bar{A} = \int_{-\infty}^{+\infty} A(p,q)\,\rho(p,q)\,dpdq$	$\bar{A} = \mathrm{Tr}\left(\hat{A}\hat{\rho}\right)$	$\bar{A} = \int_{-\infty}^{+\infty} A(p,q)\,W(p,q)\,dpdq$
$\rho(p,q) = \delta(p - p(t))\,\delta(q - q(t))$	$\hat{\rho} = \lvert\psi\rangle\langle\psi\rvert$	$W(p,q)$ $= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\psi^*\left(p + \frac{P}{2}\right)\psi\left(p - \frac{P}{2}\right)e^{-\frac{i}{\hbar}Pq}dP$

One can see a very important property: when $\hbar \to 0$, quantum mechanics in the phase space transforms to classical mechanics.

Now, it is possible to write down quantum counterparts for the evolution equations (Hamilton or Liouville), expressions for expected values of observables, and distribution function for a pure state. In quantum mechanics this function has been presented firstly by Wigner in 1932 [2], and nowadays is well-known as Wigner function.

This formalism is very useful in a lot of branches of modern physics, e.g. in quantum optics (see for an example [3].) The fact that Wigner function does not satisfy Kolmogorov axiomatic for probability has helped us to better understand the nature of quantum non-locality [4, 5].

However, generalization of this formalism to the relativistic case meets some problems. A very interesting feature is the fact that Weyl transform is not Lorentz invariant in an explicit form. We do not consider this question in the paper. Some consideration of it and references one can find in [6, 7]. Here, we concentrate our attention on another problem, that is absence of a well-defined position operator in relativistic case.

Consider the simplest example, a scalar charged particle in a constant magnetic field. This system is described by the Klein–Gordon equation

$$-\hbar^2\partial_t^2\Psi = \left(c^2\left(\hat{p} - eA(q)\right)^2 + m^2c^4\right)\Psi. \tag{3}$$

This is well-known that energy spectrum of this equations is subdivided in two bands: lower and upper (see Fig. 1)

$$E(n,\pm) = \pm mc^2\sqrt{1 + \frac{2}{mc^2}e(n)}, \tag{4}$$

where $e(n)$ is the spectrum of a non-relativistic particle. Moreover, according to modern notions about the field theory, one supposes that lower band is occupied by particles (so-called Dirac Sea).

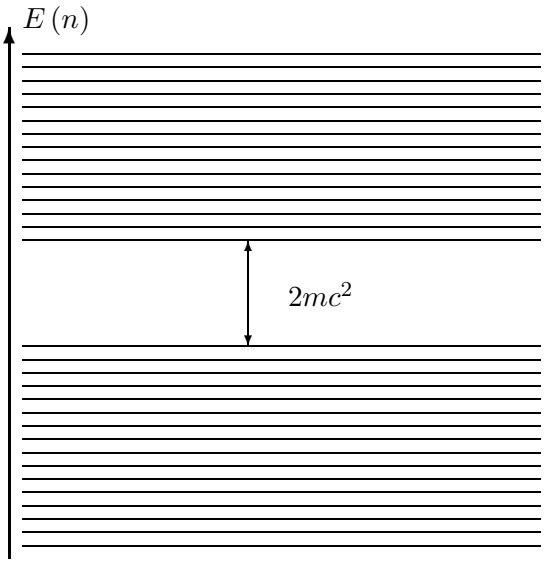

Figure 1. Energy spectrum of a relativistic particle.

The Klein–Gordon equation is a second-order equation in time. One can rewrite it in accordance with the Feshbach–Villars formalism [8] as a system of two first-order equations. To achieve this one should provide changing variables

$$\Psi = \frac{1}{\sqrt{2}}\left(\varphi + \chi\right), \qquad i\hbar\partial_t \Psi = \frac{mc^2}{\sqrt{2}}\left(\varphi - \chi\right). \tag{5}$$

As a result we can write down the Klein–Gordon equation in the form of the Schrödinger equation

$$i\hbar\partial_t \psi = \hat{H}\psi, \tag{6}$$

with two-component wave function $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ and Hamiltonian $\hat{H} = (\tau_3 + i\tau_2)\, e(\hat{n}) + \tau_3 mc^2$. In the last equation $e(\hat{n})$ is non-relativistic Hamiltonian, and τ_i are the Pauli matrices. In fact, they present a specific internal degree of freedom, so-called charge variable.

The point is that eigenfunctions of position and momentum operators belong to both bands. Hence, the question about measurement of position and momentum in the relativistic case is not so trivial.

If one has an aim to construct the phase space formalism for a relativistic particle, the charge variable should be taken into account as an independent degree of freedom, and τ_i have to be considered as a specific part of the phase space. Such a kind of formalism would be very useful. However, we will consider a simpler case.

Let us restrict our attention only to such observables that commute with all matrices τ_i

$$\left[\hat{A}, \tau_i\right] = 0, \qquad \text{for} \quad i = 1, 2, 3. \tag{7}$$

In fact, this means that these observables are operator functions of position and momentum and do not depend on τ_i

$$\hat{A} = F(\hat{p}, \hat{q}). \tag{8}$$

We call elements of corresponding subalgebra of the dynamical algebra as *charge-invariant observables*. Indeed, all these operators are invariant relatively to unitary transformations (or, to be more precise, generalized unitary transformations [8]) in charge subspace of the Hilbert space.

A lot of important observables belong to the subalgebra of charge-invariant observables. These are position, momentum, second moments, *etc.* However, it is worth noting that Hamiltonian and current do not belong to this subalgebra. Hence, one cannot use formalism presented here for those ones.

Using the phase space formalism, one can provide analysis for such subalgebra of dynamical algebra. An interesting feature of these observables is the fact that even $[A]$ and odd $\{A\}$ parts of corresponding operators (diagonal and not diagonal components of matrices 2×2 in the energy representation, see [8]) are not independent. In [6, 7], one can find a proof of the following constraint for them (in the energy representation)

$$\{A\}_{nm} = \frac{E(m) - E(n)}{E(m) + E(n)} \tau_1 [A]_{nm}. \tag{9}$$

Another very important property of charge-invariant observables is the fact that it is possible for them to present usual (scalar, not matrix-valued) Wigner function with conventional rule for the calculation of mean values

$$\overline{A} = \int_{-\infty}^{+\infty} A(p,q) W(p,q) dp dq. \tag{10}$$

Nevertheless, there are very important differences from the non-relativistic case. First of all, this object is a sum of four components

$$W(p,q) = \sum_{\alpha=\pm 1} \left(W_{[\alpha]}(p,q) + W_{\{\alpha\}}(p,q) \right). \tag{11}$$

Namely, these are two even components, which correspond to particle and antiparticle

$$W_{[\pm]}(p,q) = \sum_{n,m=0}^{\infty} \varepsilon(m,n) W_{nm}(p,q) C^*_{m;\pm} C_{n;}{}^{\pm}, \tag{12}$$

and two odd (non-physical) components, which correspond to the interference between particle and antiparticle

$$W_{\{\pm\}}(p,q) = \sum_{n,m=0}^{\infty} \chi(m,n) W_{nm}(p,q) C^*_{m;\pm} C_{n;}{}^{\mp}. \tag{13}$$

In equations (12) and (13) $C_{n;}{}^{\pm}$ is a wave function in the energy representation, $W_{nm}(p,q)$ is a Hermitian generalization of the Wigner function (see [7, 9]), functions

$$\varepsilon(n,m) = \frac{E(n) + E(m)}{2\sqrt{E(n) E(m)}} \tag{14}$$

and

$$\chi(n,m) = \frac{E(n) - E(m)}{2\sqrt{E(n) E(m)}} \tag{15}$$

are so-called ε- and χ-factors.

One can find evolution equations (quantum Liouville equation) for each component separately

$$\partial_t W_{[\pm]}(p,q,t) = \pm \left\{ E(p,q), W_{[\pm]}(p,q,t) \right\}_M, \tag{16}$$

$$\partial_t W_{\{\pm\}}(p,q,t) = \mp \left[E(p,q), W_{\{\pm\}}(p,q,t) \right]_M, \tag{17}$$

where

$$E(p,q) = \sqrt[\star]{m^2 c^4 + c^2 \left(p - eA(q)\right)^2} \tag{18}$$

is the effective Hamiltonian. Here one has two surprising features as well.

First of all, the symbol of the Hamiltonian is defined by means of square root in a sense of star product. This means that Weyl symbol of the Hamiltonian differs from the Hamilton function in classical mechanics. In non-relativistic mechanics they coincide.

Second, the evolution equation for the odd components is formulated using "anti-Moyal" bracket, which is counterpart of the anti-commutator. This means that evolution of odd part is a non-unitary operation.

Moreover, there are once more, the most important property. Definition of even component include a very important multiplier, ε-factor (14). It is absent in the non-relativistic theory. Consider its properties more closely:

1. $\varepsilon(n,m) = \varepsilon(m,n)$;
2. $\varepsilon(n,n) = 1$;
3. $\varepsilon(n,m) > 1$, if $n \neq m$.

This is very similar to the case, when one considers an arbitrary system in an environment. For the Wigner function in this case one can write down the following expression

$$W(p,q) = \sum_{n=0}^{\infty} W_{nn}(p,q)\,|C_n|^2 + \sum_{n\neq m} a(m,n) W_{nm}(p,q) C_m^* C_n. \tag{19}$$

However, non-diagonal components of $a(m,n)$ (counterpart of ε-factor) are always smaller than 1. Therefore, in the relativistic case, influence of vacuum is inverted to the decoherence process. This means that relativistic system has more quantum non-local properties than its non-relativistic counterpart.

It is possible to consider this question from the more formal point of view. It is well-known that not every normalized function can be regarded as Wigner function presenting a real quantum physical state [10]. In the non-relativistic case there exists so-called quantization condition that selects Wigner functions from set of functions on the phase space. In our case this condition can be presented in the following way (for a free particle).

Theorem 1. *For the functions $W_{[\pm]}(p,q)$ and $W_{\{\pm\}}(p,q)$ to be even and odd components of the Wigner function for charge-invariant observables, it is necessary and sufficient that all constraints hold true (see [6]) and the following conditions are satisfied:*

$$\frac{\partial^2}{\partial p_1 \partial p_2} \ln \int_{-\infty}^{+\infty} W_{[\pm]}\left(\frac{1}{2}(p_1+p_2), q\right) \exp\left(\frac{i}{\hbar}(p_1-p_2)q\right) dq = -\frac{c^4 p_1 p_2}{E(p_1)E(p_2)(E(p_1)+E(p_2))^2}, \tag{20}$$

$$\frac{\partial^2}{\partial p_1 \partial p_2} \ln \int_{-\infty}^{+\infty} W_{\{\pm\}}\left(\frac{1}{2}(p_1+p_2), q\right) \exp\left(\frac{i}{\hbar}(p_1-p_2)q\right) dq = -\frac{c^4 p_1 p_2}{E(p_1)E(p_2)(E(p_1)-E(p_2))^2}. \tag{21}$$

A proof of this criterion can be found in [6], and formulation for a more general case in [7].

The main difference from the non-relativistic case is the fact that in the right-hand-side of equation (20) one has an expression that differs from 0. In the non-relativistic case this is exactly 0.

Consider a simple example. Fig. 2 shows the Wigner function for the coherent state of a free particle [11]. In fact, this is a Gauss distribution for non-strong space localization. However, when localization along position is very strong, additional "vacuum fluctuations" appear. They

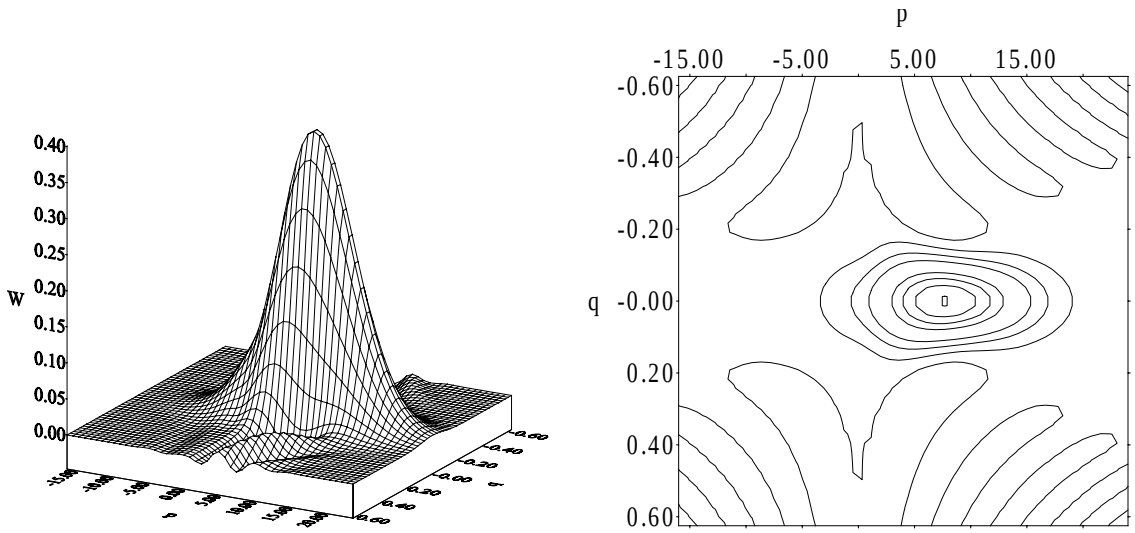

Figure 2. Wigner function and its contours for the coherent state of a free relativistic particle. Position is given in the units of Compton wavelength, momentum is given in mc units.

counteract to the strong localization. This means that coherent state (which is classical in terms of work [5]) manifests a quantum feature.

For a conclusion, we note that quantum quasidistributions of relativistic systems are more different from classical distributions than their non-relativistic counterparts. This means that relativistic systems are more interesting objects for investigation of quantum non-locality and non-classicality.

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Semiclassical Approach to the Geometric Phase Theory for the Hartree Type Equation

Alexander SHAPOVALOV †, *Andrey TRIFONOV* ‡ *and Alexander LISOK* ‡

† *Tomsk State University, 36 Lenin Ave., 634050 Tomsk, Russia*
E-mail: *shpv@phys.tsu.ru*

‡ *Tomsk Polytechnic University, 30 Lenin Ave., 634034 Tomsk, Russia*
E-mail: *trifonov@phtd.tpu.edu.ru*

Quasi-energy states and a spectrum of quasi-energies asymptotic in small parameter $\hbar$ ($\hbar \to 0$) are constructed for a multidimensional Hartree type equation with non-local nonlinearity and with an external field cyclic in time. The quasi-energy states are a special case of trajectory coherent solutions of the Hartree type equation, which belong to the class of semiclassically concentrated functions. A function of this class describes a solitary wave localized in a neighborhood of a phase trajectory in the space of moments of the solution. The phase trajectory is closed due to the configuration of the external field. The Aharonov–Anandan geometric phases, which characterize a system "as a whole", are found for the quasi-energy states in a semiclassical approximation accurate to $O(\hbar^{3/2})$, $\hbar \to 0$.

1 Introduction

Quantum systems exposed to external periodic fields have nontrivial topological properties and are characterized by the geometric phase (GP) of a wave function [1, 2]. Systems of this type possess a set of quasi-energy states (QES's) and a spectrum of quasi-energies, which were originally introduced in [3, 4]. Mathematically, QES's and GP's are relevant to the properties of a quantum system as a whole. In particular, GP is related to a special kind of gauge symmetry. Details of the GP theory and its applications can be found in the reviews [5, 6].

In constructing quasi-energy states and geometric phases for similar nonlinear systems described by partial differential equations one faces the problem of integrability of nonlinear equations with external fields (variable coefficients) responsible for the nontrivial geometry and topology of the system. The usual symmetry analysis, when directly applied to these systems, fails since ordinary symmetry structures are effectively calculated for nonlinear equations with constant coefficients (see, for example, [7–9]). It should be noted that the above gauge symmetries can be of interest in the symmetry analysis of nonlinear equations of mathematical physics. Gauge symmetries result from the global properties of the geometry of a system and are constructed differently than standard symmetry analysis structures.

In this work, based on the Maslov complex germ method [10] we construct quasi-energy states asymptotic in small parameter $\hbar$, $\hbar \to 0$, in the class of semiclassical concentrated solutions of a multidimensional Hartree type equation (HTE) with a non-local nonlinearity and an external field periodic in time. The solution construction technique developed by the authors [11, 12] relies upon the periodic solutions of the dynamic system of Hamilton–Ehrenfest equations. The existence of periodic phase trajectories is provided by the external field.

The geometric phases for the QES under consideration are obtained in explicit form.

2 Problem statement and notations

Consider the Hartree type equation

$$\{-i\hbar\partial_t + \hat{\mathcal{H}}(t) + \varkappa\hat{V}(t, \Psi(t))\}\Psi(\vec{x}, t) = 0, \qquad \Psi \in L_2\big(\mathbb{R}^3_x\big). \tag{1}$$

Here, $\partial_t = \partial/\partial t$; $\hbar$ is a "small parameter", $\hbar \in [0,1)$, $\vec{x} \in \mathbb{R}^3$; $|\Psi|^2 = \Psi\Psi^*$, and the function Ψ^* is complex conjugate to Ψ. The scalar product of the functions Ψ and Φ in the space $L_2\big(\mathbb{R}^3_x\big)$ is denoted as

$$\langle \Psi|\Phi\rangle = \int_{\mathbb{R}^3} \Psi^*(\vec{x})\Phi(\vec{x})d\vec{x}. \tag{2}$$

Following quantum mechanical terms, we shall refer to the solutions of equation (1) as states.

The linear operator $\hat{\mathcal{H}}(t) = \mathcal{H}(\hat{z},t)$ is a Weyl-ordered function [13] of time t and operators

$$\hat{z} = (\hat{\vec{p}}, \vec{x}), \qquad \hat{\vec{p}} = -i\hbar\nabla = -i\hbar\frac{\partial}{\partial\vec{x}}, \tag{3}$$

for which the following commutation relations are valid:

$$[\hat{z}_k, \hat{z}_j]_- = i\hbar J_{kj}, \qquad k,j = \overline{1,6}, \tag{4}$$

where $J = \|J_{kj}\|_{6\times 6}$ is the unit symplectic matrix

$$J = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}_{6\times 6}, \qquad \mathbb{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

In quantum mechanical terms the linear operator $\hat{\mathcal{H}}(t)$ is written as

$$\hat{\mathcal{H}}(t) = \frac{1}{2m}\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2 - e\langle\vec{E}(t),\vec{x}\rangle + \frac{k}{2}\vec{x}^{\,2}. \tag{6}$$

The external field in the operator (6) is the superposition of a constant magnetic field $\vec{H} = (0,0,H)$ with vector potential $\vec{A} = \frac{1}{2}\vec{H}\times\vec{x}$, an electric field $\vec{E}(t) = (E\cos\omega t, E\sin\omega t, 0)$ periodic in time with frequency ω, and an oscillator field with potential $\frac{k}{2}\vec{x}^2$. The nonlocal operator $\hat{V}(t,\Psi(t))$ in equation (1) has the form

$$\hat{V}(t,\Psi(t))\Psi(\vec{x},t) = \int_{\mathbb{R}^3} V(\vec{x},\vec{y})|\Psi(\vec{y},t)|^2 d\vec{y}\Psi(\vec{x},t), \tag{7}$$

$$V(\vec{x},\vec{y}) = V_0 \exp\left[-\frac{(\vec{x}-\vec{y})^2}{2\gamma^2}\right]. \tag{8}$$

The quantities E, H, V_0, k, ω, γ, $\varkappa$, e, and c are real parameters; $\langle\vec{x},\vec{y}\rangle = \sum\limits_{j=1}^{3} x_j y_j$ is the Euclidean scalar product of the vectors $\vec{x}$, $\vec{y}$, $\vec{x}^{\,2} = \langle\vec{x},\vec{x}\rangle$.

Below we shall use the following notation:

$$\omega_H = \frac{eH}{mc} \quad \text{is the cyclotron frequency,} \tag{9}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{is the oscillator frequency,} \tag{10}$$

$$\omega_a = \omega_0\sqrt{1+\left(\frac{\omega_H}{2\omega_0}\right)^2}, \tag{11}$$

$$\omega_{\rm nl} = \sqrt{\frac{|\tilde{\varkappa}V_0|}{m\gamma^2}} \quad \text{is the "nonlinear frequency",} \qquad \tilde{\varkappa} = \varkappa\|\Psi\|^2, \quad \eta = \operatorname{sign}(\tilde{\varkappa}V_0). \tag{12}$$

Equation (1) has many physical applications. In particular, in the Bose–Einstein condensate theory, the solution Ψ has the meaning of wave function, and the nonlocal potential $V(\vec{x},\vec{y})$ describes the coupling of condensate particles which are in an external field (see, e.g., the reviews [14, 15].

Let us define the geometric phase for the Hartree type equation (1) based on the general concept of geometric phases in quantum mechanics, proposed by Aharonov and Anandan for cyclic states [2] (see also [5, 16]).

Quasi-energy states $\Psi_{\mathcal{E}}(\vec{x},t,\hbar)$ in quantum mechanics were introduced by Zel'dovich [3] and Ritus [4] by the following condition:

$$\Psi_{\mathcal{E}}(\vec{x},t,\hbar)=\mathrm{e}^{-i\mathcal{E}t/\hbar}\varphi_{\mathcal{E}}(\vec{x},t,\hbar),\qquad \text{where}\quad \varphi_{\mathcal{E}}(\vec{x},t+T,\hbar)=\varphi_{\mathcal{E}}(\vec{x},t,\hbar). \tag{13}$$

The quantity $\mathcal{E}$ entering into equation (13) is called quasi-energy and defined modulo $\hbar\omega$, $(\omega=2\pi/T)$, i.e. $\mathcal{E}'=\mathcal{E}+m\hbar\omega$, $m\in\mathbb{Z}$.

States of this type play a key role in studying quantum mechanical systems exposed to strong periodic external fields when standard methods of the non-stationary perturbation theory are inapplicable.

Quasi-energy states are special cases of cyclic states introduced by Aharonov and Anandan [2] and have nontrivial GP's.

For the Hartree type equation (1), by a cyclic state on a time interval $[0,T]$, we mean the solution $\Psi(t)$ possessing the property

$$\Psi(t)=\mathrm{e}^{if(t)}\varphi(t),\qquad t\in[0,T],\quad f(T)-f(0)=\phi(\mathrm{mod}\ 2\pi),\quad \varphi(T)=\varphi(0). \tag{14}$$

The complete phase ϕ of the function (14) is subdivided into two summands: the dynamic phase

$$\delta=-\frac{1}{\hbar}\int_0^T dt\frac{\langle\Psi(t)|[\hat{\mathcal{H}}(t)+\varkappa\hat{V}(t,\Psi(t))]|\Psi(t)\rangle}{\langle\Psi(t)|\Psi(t)\rangle} \tag{15}$$

and the Aharonov–Anandan geometric phase

$$\gamma=i\int_0^T dt\frac{\langle\varphi(t)|\dot{\varphi}(t)\rangle}{\langle\varphi(t)|\varphi(t)\rangle}. \tag{16}$$

Let us call the quasi-energy state for the Hartree type equation a solution $\Psi_{\mathcal{E}}(\vec{x},t,\hbar)$ of (1) for which we have

$$\Psi_{\mathcal{E}}(\vec{x},t+T,\hbar)=\mathrm{e}^{-i\mathcal{E}T/\hbar}\Psi_{\mathcal{E}}(\vec{x},t,\hbar). \tag{17}$$

It can be easily verified that equations (13) and (17) are equivalent. It is also obvious that quasi-energy states are a special case of the cyclic states (14). The set of the $\mathcal{E}$ values constitutes a spectrum of quasi-energies.

Comparing (14) and (17), we obtain that the function $f(t)$ entering into a quasi-energy state is

$$f(t)=-\mathcal{E}t/\hbar, \tag{18}$$

and for the complete phase ϕ, in accordance with (14), we have

$$\phi=-\mathcal{E}T/\hbar\pmod{2\pi}. \tag{19}$$

By virtue of (15)–(19), the Aharonov–Anandan phase $\gamma_{\mathcal{E}}$ corresponding to the quasi-energy state $\Psi_{\mathcal{E}}(\vec{x},t,\hbar)$ can be determined by the formula

$$\gamma_{\mathcal{E}}=-\frac{\mathcal{E}T}{\hbar}+\frac{1}{\hbar}\int_0^T dt\frac{\langle\Psi_{\mathcal{E}}|[\hat{\mathcal{H}}(t)+\varkappa\hat{V}(t,\Psi(t))]|\Psi_{\mathcal{E}}\rangle}{\langle\Psi_{\mathcal{E}}|\Psi_{\mathcal{E}}\rangle}\pmod{2\pi}. \tag{20}$$

We now state for equation (1) the problem of finding, in a semiclassical approximation, of the quasi-energy states and the quasi-energy spectrum in the class of trajectory concentrated functions [11, 12]

$$\mathcal{P}^t_\hbar = \mathcal{P}^t_\hbar\big(Z(t,\hbar), S(t,\hbar)\big) = \left\{\Phi : \Phi(\vec{x},t,\hbar) = \varphi\left(\frac{\Delta\vec{x}}{\sqrt{\hbar}}, t, \hbar\right)\exp\left[\frac{i}{\hbar}(S(t,\hbar) + \langle\vec{P}(t,\hbar), \Delta\vec{x}\rangle)\right]\right\}, \tag{21}$$

where the function $\varphi(\vec{\xi}, t, \hbar)$ belongs to the Schwartz space $\mathbb{S}$ with respect to the variable $\vec{\xi} \in \mathbb{R}^3$, smoothly depends on t, and is regular in $\sqrt{\hbar}$, $\hbar \to 0$. Here, $\Delta\vec{x} = \vec{x} - \vec{X}(t,\hbar)$; the real function $S(t,\hbar)$ and the 6-component vector function $Z(t,\hbar) = (\vec{P}(t,\hbar), \vec{X}(t,\hbar))$, which specify the class $\mathcal{P}^t_\hbar(Z(t,\hbar), S(t,\hbar))$, regularly depend on $\sqrt{\hbar}$ in the neighborhood of $\hbar = 0$ and *are to be determined*. In the cases where this does not lead to ambiguity, we use the shorthand symbol $\mathcal{P}^t_\hbar$ for $\mathcal{P}^t_\hbar(Z(t,\hbar), S(t,\hbar))$.

The functions of the class $\mathcal{P}^t_\hbar$ are normalizable $\|\Phi(t)\|^2 = \langle\Phi(t)|\Phi(t)\rangle$ in the space $L_2(\mathbb{R}^3_x)$ with respect to the scalar product (2). If $\Phi(t)$ is a solution of equation (1), then $\|\Phi(t)\|^2 = \|\Phi(0)\|^2$; therefore, in what follows the argument t in $\|\Phi(t)\|^2$ will be omitted.

Let us define for a linear operator $\hat{A} : \mathcal{P}^t_\hbar \to \mathcal{P}^t_\hbar$ and $\Psi \in \mathcal{P}^t_\hbar$ its mean value as

$$\langle\hat{A}\rangle = \frac{1}{\|\Psi\|^2}\langle\Psi|\hat{A}|\Psi\rangle. \tag{22}$$

For a solution Ψ of equation (1), we have

$$\frac{d\langle\hat{A}(t)\rangle}{dt} = \left\langle\frac{\partial\hat{A}(t)}{\partial t}\right\rangle + \frac{i}{\hbar}\langle[\hat{\mathcal{H}}(t) + \varkappa\hat{V}(t,\Psi(t)), \hat{A}(t)]\rangle, \tag{23}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of the linear operators $\hat{A}$ and $\hat{B}$.

The vector function $Z(t,\hbar) = (\vec{P}(t,\hbar), \vec{X}(t,\hbar))$ is a parameter of the class $\mathcal{P}^t_\hbar$ of the form (21), which is to be determined.

When constructing a solution $\Psi \in \mathcal{P}^t_\hbar$ of equation (1), the vector function $Z(t,\hbar)$ will be chosen to satisfy the condition

$$\vec{X}(t,\hbar) = \langle\vec{x}\rangle, \qquad \vec{P}(t,\hbar) = \langle\hat{\vec{p}}\rangle. \tag{24}$$

Let us construct asymptotic solutions for equation (1) with an accuracy of $O(\hbar^{3/2})$, $\hbar \to 0$, which correspond to the leading term of the asymptotic expansion. To do this, we expand the function $V(\vec{x}, \vec{y})$ of the form (8) in the operator (7) in a Taylor power series of $\Delta\vec{x} = \vec{x} - \vec{X}(t,\hbar)$, $\Delta\vec{y} = \vec{y} - \vec{X}(t,\hbar)$ and restrict ourselves to the terms of the order two inclusive in $\Delta\vec{x}$ and $\Delta\vec{y}$. Then equation (1) can be written

$$\left\{-i\hbar\partial_t + \tilde{\mathcal{H}}(t) + \langle\mathcal{H}_z(t), \Delta\hat{z}\rangle + \frac{1}{2}\langle\Delta\hat{z}, \tilde{\mathcal{H}}_{zz}(t)\Delta\hat{z}\rangle\right\}\Psi(\vec{x},t) = O\big(\hbar^{3/2}\big). \tag{25}$$

Here, we take into account (24) and use the following notation: $\Delta\hat{z} = (\Delta\hat{\vec{p}}, \Delta\vec{x}) = (\hat{\vec{p}} - \vec{P}(t,\hbar), \vec{x} - \vec{X}(t,\hbar))$. The central moments of the function $\Psi(\vec{x},t)$ are written as a (6×6)-matrix $\Delta_2(t)$ of the form

$$\Delta_2(t) = \begin{pmatrix} \sigma_{pp}(t) & \sigma_{px}(t) \\ \sigma_{xp}(t) & \sigma_{xx}(t) \end{pmatrix}, \tag{26}$$

where $\sigma_{pp}(t) = (\sigma_{p_k p_l}(t)) = (\langle \Delta\hat{p}_k \Delta\hat{p}_l \rangle)$, $\sigma_{xp}(t) = (\sigma_{x_k p_l}(t)) = \left(\frac{1}{2}\langle \Delta x_k \Delta\hat{p}_l + \Delta\hat{p}_l \Delta x_k \rangle\right)$, $\sigma_{xx}(t) = (\sigma_{x_k x_l}) = (\langle \Delta x_k \Delta x_l \rangle)$;

$$\tilde{\mathcal{H}}(t) = \mathcal{H}(t) + \tilde{\varkappa} V_0 - \eta \frac{m\omega_{\rm nl}^2}{2} \sum_{j=1}^{3} \sigma_{x_j x_j}, \tag{27}$$

$$\mathcal{H}(t) = \frac{1}{2m}\vec{P}^2 + \frac{m\omega_a^2}{2}\left(X_1^2 + X_2^2\right) + \frac{m\omega_0^2}{2} X_3^2 + \frac{\omega_H}{2}(P_1 X_2 - P_2 X_1) - eE(X_1 \cos\omega t + X_2 \sin\omega t), \tag{28}$$

$$\mathcal{H}_z(t) = \begin{pmatrix} \mathcal{H}_p(t) \\ \mathcal{H}_x(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{m}P_1 + \frac{\omega_H}{2} X_2 \\ \frac{1}{m}P_2 - \frac{\omega_H}{2} X_1 \\ \frac{1}{m}P_3 \\ -\frac{\omega_H}{2}P_2 + m\omega_a^2 X_1 - eE\cos\omega t \\ \frac{\omega_H}{2}P_1 + m\omega_a^2 X_2 - eE\sin\omega t \\ m\omega_0^2 X_3 \end{pmatrix}, \tag{29}$$

where $\vec{P} = \vec{P}(\hbar, t)$, $\vec{X} = \vec{X}(\hbar, t)$ and the notation (9)–(12) is used;

$$\tilde{\mathcal{H}}_{zz}(t) = \begin{pmatrix} \tilde{\mathcal{H}}_{pp}(t) & \tilde{\mathcal{H}}_{px}(t) \\ \tilde{\mathcal{H}}_{xp}(t) & \tilde{\mathcal{H}}_{xx}(t) \end{pmatrix}, \tag{30}$$

$$\tilde{\mathcal{H}}_{pp}(t) = (\tilde{\mathcal{H}}_{p_k p_l}(t)) = \operatorname{diag}\left(\frac{1}{m}, \frac{1}{m}, \frac{1}{m}\right), \tag{31}$$

$$\tilde{\mathcal{H}}_{xx}(t) = (\tilde{\mathcal{H}}_{x_k x_l}(t)) = \operatorname{diag}\left(m(\omega_a^2 - \eta\omega_{\rm nl}^2), m(\omega_a^2 - \eta\omega_{\rm nl}^2), m(\omega_0^2 - \eta\omega_{\rm nl}^2)\right), \tag{32}$$

$$\tilde{\mathcal{H}}_{px}(t) = (\tilde{\mathcal{H}}_{p_k x_l}(t)) = \begin{pmatrix} 0 & \frac{\omega_H}{2} & 0 \\ -\frac{\omega_H}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

In [11, 12], equation (25) is called an associated linear Schrödinger equation. Its Hamiltonian is square in $\Delta\hat{z}$:

$$\hat{\tilde{\mathcal{H}}}(t) = \tilde{\mathcal{H}}(t) + \langle \mathcal{H}_z(t), \Delta\hat{z} \rangle + \frac{1}{2}\langle \Delta\hat{z}, \tilde{\mathcal{H}}_{zz}(t)\Delta\hat{z} \rangle. \tag{34}$$

The solutions of the linear equation (25) enable us to find approximate solutions of the nonlinear Hartree type equation (1).

3 The Hamilton–Ehrenfest system

The solution Ψ of equation (1) is constructed in the class of trajectory concentrated functions (21) with the help of the solutions of the dynamic Hamilton–Ehrenfest equations (HEE's) for the centered moments of the field Ψ [11,12]. To obtain a solution Ψ exactly up to $O(\hbar^{3/2})$, it suffices to take into account the moments of the second order. It should also be noted that to construct the quasi-energy states (17) for equation (1), one must take the periodic solutions of the HEE's.

Let us substitute the operators $\hat{\vec{p}}$, $\vec{x}$, and $\Delta_2(t)$ into (23) instead of the operator $\hat{A}$. In view of (24) and the notation (26)–(33), we obtain the following HEE's:

$$\dot{Z}(t,\hbar) = J\mathcal{H}_z(t), \tag{35}$$

$$\dot{\Delta}_2(t) = J\tilde{\mathcal{H}}_{zz}(t)\Delta_2 - \Delta_2\tilde{\mathcal{H}}_{zz}(t)J. \tag{36}$$

Obviously, equations (35), which describe the motion of the centroid of the field Ψ, are integrable independently of equations (36) for the second moments $\Delta_2(t)$. Therefore, we first find the periodic solution of the system (35) and then integrate equations (36). The elementary particular periodic solution of the system (35) can be taken in the form

$$Z_0(t,\hbar) = (\vec{P}_0(t,\hbar), \vec{X}_0(t,\hbar)),$$

$$\vec{P}_0(t,\hbar) = \left(-m\left(\frac{\omega_H}{2}+\omega\right)\xi\sin\omega t, m\left(\frac{\omega_H}{2}+\omega\right)\xi\cos\omega t, 0\right), \tag{37}$$

$$\vec{X}_0(t,\hbar) = (\xi\cos\omega t, \xi\sin\omega t, 0), \tag{38}$$

$$\xi = \frac{eE}{m(\omega_0^2 - \omega_H\omega - \omega^2)}. \tag{39}$$

For the solution (37), the Hamiltonian (34) of equation (25) is periodic with the period $T = 2\pi/\omega$. In what follows we assume that $\xi^{-1} \neq 0$ which means the absence of a resonance.

In [11,12], the system (36) is shown to be equivalent to the system of equations in variations

$$\dot{A}(t) = J\tilde{\mathcal{H}}_{zz}(t)A(t), \tag{40}$$

where $A(t)$ is a 6×6 nonsingular matrix and $\tilde{\mathcal{H}}_{zz}(t)$ has the form of (30).

Solving the system (40), one can find the trajectory-coherent states (TCS's) for the associated linear Schrödinger equation (25) in explicit form using well-known methods (see, e.g., [17]). These TCS's, in turn, lead to approximate TCS's for the nonlinear Hartree type equation (1) in the class of trajectory concentrated functions (21) [11,12].

Taking into account the periodic behavior of the Hamiltonian (34), (37), we now state the problem of constructing quasi-energy states in the class of TCS's (quasi-energy TCS's).

To do this, we use the solutions of the system (40) that satisfy the Floquet condition

$$a(t+T) = \exp(i\Omega T)a(t), \tag{41}$$

where $a(t)$ is a column of the matrix $A(t)$ and Ω is a real constant. The fundamental matrix of the system (40), (41) is written as

$$A(t) = (a_1(t), a_2(t), a_3(t), a_1^*(t), a_2^*(t), a_3^*(t)),$$

where the columns of the matrix $A(t)$ are denoted by $a_j(t)$, $j = \overline{1,3}$ and $a_j^*(t)$ is complex conjugate to $a_j(t)$ normalized by the condition $\langle a_j, J^T a_l^*\rangle = 2i\delta_{jl}$, $\langle a_j, J^T a_l\rangle = 0$. The columns are given by the following expressions:

$$a_1(t) = \frac{e^{i\omega_+ t}}{\sqrt{2}}\left(g_0, ig_0, 0, -\frac{i}{g_0}, \frac{1}{g_0}, 0\right)^T, \qquad a_2(t) = \frac{e^{i\omega_- t}}{\sqrt{2}}\left(g_0, -ig_0, 0, -\frac{i}{g_0}, -\frac{1}{g_0}, 0\right)^T,$$

$$a_3(t) = e^{i\omega_s t}\left(0, 0, g_s, 0, 0, \frac{-i}{g_s}\right)^T. \tag{42}$$

Here

$$\omega_+ = \sqrt{\omega_a^2 - \eta\omega_{\rm nl}^2} + \frac{\omega_H}{2}, \tag{43}$$

$$\omega_- = \sqrt{\omega_a^2 - \eta\omega_{\rm nl}^2} - \frac{\omega_H}{2} \tag{44}$$

are combined frequencies, $\omega_{\rm s}=\sqrt{\omega_0^2-\eta\omega_{\rm nl}^2}$, $g_0=\sqrt{\frac{m}{2}(\omega_++\omega_-)}$, $g_{\rm s}=\sqrt{m\omega_{\rm s}}$,

$$A(t)=\begin{pmatrix} B(t) & B^*(t)\\ C(t) & C^*(t)\end{pmatrix}, \tag{45}$$

where

$$B(t)=\begin{pmatrix} \frac{g_0e^{i\omega_+t}}{\sqrt{2}} & \frac{g_0e^{i\omega_-t}}{\sqrt{2}} & 0\\ \frac{ig_0e^{i\omega_+t}}{\sqrt{2}} & \frac{-ig_0e^{i\omega_-t}}{\sqrt{2}} & 0\\ 0 & 0 & g_{\rm s}e^{i\omega_st}\end{pmatrix},\qquad C(t)=\begin{pmatrix} \frac{-ie^{i\omega_+t}}{\sqrt{2}g_0} & \frac{-ie^{i\omega_-t}}{\sqrt{2}g_0} & 0\\ \frac{e^{i\omega_+t}}{\sqrt{2}g_0} & \frac{-e^{i\omega_-t}}{\sqrt{2}g_0} & 0\\ 0 & 0 & \frac{-ie^{i\omega_st}}{g_{\rm s}}\end{pmatrix}.$$

The periodic motion (37), (38) is stable in the linear approximation provided $\omega_0^2>\eta\omega_{\rm nl}^2$. This is assumed to be fulfilled.

The solution of the system (26), (36) is presented via the blocks $B(t)$ and $C(t)$ of the matrix $A(t)$ as

$$\begin{aligned}\sigma_{xx}(t)&=\frac{\hbar}{4}(C(t)D_\nu C^+(t)+C^*(t)D_\nu C^T(t)),\\ \sigma_{pp}(t)&=\frac{\hbar}{4}(B(t)D_\nu B^+(t)+B^*(t)D_\nu B^T(t)),\\ \sigma_{px}(t)&=\frac{\hbar}{4}(B(t)D_\nu C^+(t)+B^*(t)D_\nu C^T(t)),\end{aligned}$$

where the diagonal matrix $D_\nu=\mathrm{diag}\,(2\nu_1+1,2\nu_2+1,2\nu_3+1)$, $\nu_1,\nu_2,\nu_3=0,1,2,\dots$ and the symbol $(^+)$ implies Hermitian conjugation. The matrices σ_{xx} and σ_{pp} are diagonal as well and are equal to

$$\begin{aligned}\sigma_{xx}(t)&=\frac{\hbar}{m}\,\mathrm{diag}\left(\frac{\nu_1+\nu_2+1}{\omega_++\omega_-},\frac{\nu_1+\nu_2+1}{\omega_++\omega_-},\frac{2\nu_3+1}{2\omega_s}\right),\\ \sigma_{pp}(t)&=\frac{\hbar m}{4}\,\mathrm{diag}\,((\omega_++\omega_-)(\nu_1+\nu_2+1),(\omega_++\omega_-)(\nu_1+\nu_2+1),2\omega_s(2\nu_3+1)),\end{aligned}\tag{46}$$

The nonzero elements of the matrix $\sigma_{xp}(t)$ are $\sigma_{p_1x_2}(t)=-\sigma_{p_2x_1}=\frac{\hbar}{2}(\nu_1-\nu_2)$. Thus, the solution (37) and the relevant matrix of the second moments Δ_2 constitute a periodic solution of the HEE's (35), (36).

4 The quasi-energies and geometric phases

Let us construct the quasi-energy TCS's of the associated linear Schrödinger equation (25) for the periodic solutions of the HEE's (35), (36).

Based on these solutions, we can find, in a semiclassical approximation, the corresponding solutions of the Hartree type equation (1), which coincide with the quasi-energy TCS's of equation (25) for the time zero. We shall refer to these solutions of equation (1) as semiclassical quasi-energy trajectory coherent states (SQETCS's).

Let us construct the SQETCS's following [11,12]. The linear Schrödinger equation (25) with the squared Hamiltonian (34) admits a solution in the form of a Gaussian wave packet:

$$\Phi(\vec{x},t)=N_\hbar\exp\left\{\frac{i}{\hbar}\left[S(t,\hbar)+\langle\vec{P}_0(t,\hbar),\Delta\vec{x}\rangle+\frac{1}{2}\langle\Delta\vec{x},Q(t,\hbar)\Delta\vec{x}\rangle\right]\right\},\tag{47}$$

where the normalization factor $N_\hbar$, the real function $S(t,\hbar)$, and the complex matrix $Q(t,\hbar)$ are to be determined. The Hamiltonian (34) of the linear Schrödinger equation (25), whose coefficients (26)–(33) are specified by solving the HEE's (37), (46), is periodic in time with period $T = 2\pi/\omega$. In this case, there exists a solution of the form (47) satisfying the quasi-energy condition (17). Following quantum mechanical terms, we treat this solution as a vacuum state and denote it as $|0,t\rangle$. One can directly verify that

$$|0,t\rangle = N_\hbar \exp\left\{\frac{i}{\hbar}\left[\frac{1}{2}eE\xi - \tilde{\varkappa}V_0 - \frac{\hbar}{2}(\omega_+ + \omega_- + \omega_\mathrm{s}) + \eta\hbar\omega_\mathrm{nl}^2\left(\frac{\nu_1+\nu_2+1}{\omega_+ + \omega_-} + \frac{2\nu_3+1}{4\omega_s}\right)\right]t \right.$$
$$- \frac{i}{\hbar}\left(\frac{\omega_H}{2} + \omega\right) m\xi(\Delta x_1 \sin\omega t - \Delta x_2 \cos\omega t)$$
$$\left. - \frac{m}{4\hbar}[(\omega_+ + \omega_-)(\Delta x_1^2 + \Delta x_2^2) + 2\omega_\mathrm{s}\Delta x_3^2]\right\}, \tag{48}$$

where ξ is of the form (38).

It can easily be seen that the operators

$$\hat{a}_j(t) = N_j\langle a_j(t), J^T\Delta\hat{z}\rangle, \qquad j = \overline{1,3}, \tag{49}$$

where N_j is a normalization factor and the vector function $a_j(t)$ is the jth column of (42), are the symmetry operators for the linear Schrödinger equation (25) commuting with the equation operator,

$$-i\hbar\partial_t\hat{a}_j(t) + [\hat{\tilde{\mathcal{H}}}(t), \hat{a}_j(t)] = 0.$$

The trajectory coherent states $|\nu,t\rangle$ for equation (25) are determined as a result of the action of the creation operators $\hat{a}_j^+(t)$ on the vacuum state (48)

$$|\nu,t\rangle = \prod_{j=1}^{3}\frac{1}{\sqrt{\nu_j!}}(\hat{a}_j^+(t))^{\nu_j}|0,t\rangle. \tag{50}$$

Here, the operators $\hat{a}_j^+(t)$ are Hermite conjugate to $\hat{a}_j(t)$ with respect to the scalar product (2). Substituting equations (48) and (49) into (50), we obtain explicit expressions of the TCS's for equation (25):

$$|\nu,t\rangle = \exp\left[-i(\omega_+\nu_1 + \omega_-\nu_1 + \omega_\mathrm{s}\nu_3)t\right]\prod_{j=1}^{3}N_{a_j}^{\nu_j}\frac{\hbar^{\frac{1}{2}(\nu_1+\nu_2+\nu_3)}}{2^{\frac{\nu_1+\nu_2}{2}}}\sqrt{\frac{\nu_1!\nu_2!}{\nu_3!}}$$
$$\times\sum_{k_1=0}^{\nu_1}\sum_{k_2=0}^{\nu_2}\frac{(-1)^{\nu_2+\nu_3-k_1}\ i^{\nu_1+\nu_2-k_1-k_2}}{k_1!k_2!(\nu_1-k_1)!(\nu_2-k_2)!}H_{k_1+k_2}\left(\frac{\Delta x_1 g_0}{\sqrt{\hbar}}\right)$$
$$\times H_{\nu_1+\nu_2-k_1-k_2}\left(\frac{\Delta x_2 g_0}{\sqrt{\hbar}}\right)H_{\nu_3}\left(\frac{\Delta x_3 g_\mathrm{s}}{\sqrt{\hbar}}\right)|0,t\rangle. \tag{51}$$

The states (51) satisfy the condition (17) and they are the quasi-energy TCS's for equation (25). When $|\nu,0\rangle$ coincides with the initial conditions for equation (1), expression (51) is a semiclassical approximation for the quasi-energy TCS of the Hartree type equation (1) with an accuracy of $O(\hbar^{3/2})$, $\hbar \to 0$. For example, expression (48) is a solution of the linear Schrödinger equation (25) for any $\nu = (\nu_1,\nu_2,\nu_3)$. However, for $t = 0$ the matrix Δ_2 of the form (26), (46) with $\nu = (\nu_1,\nu_2,\nu_3) = (0,0,0)$ corresponds to the state $|0,0\rangle$. Thus, the function (48) will be an asymptotic solution, up to $O(\hbar^{3/2})$, of the Hartree type equation (1) only if $\nu = (\nu_1,\nu_2,\nu_3) = (0,0,0)$.

Substituting the function (51) into condition (17)

$$|\nu, t+T\rangle = \exp\left[-\frac{i}{\hbar}\mathcal{E}_\nu T\right]|\nu, t\rangle,$$

we obtain the spectrum of quasi-energies $\mathcal{E}_\nu$ for equation (1)

$$\mathcal{E}_\nu = -\frac{eE}{2}\xi + \tilde{\varkappa}V_0 + \hbar\left[\left(\omega_+ - \frac{\eta\omega_{\rm nl}^2}{\omega_+ + \omega_-}\right)\left(\nu_1 + \frac{1}{2}\right)\right.$$
$$\left. + \left(\omega_- - \frac{\eta\omega_{\rm nl}^2}{\omega_+ + \omega_-}\right)\left(\nu_2 + \frac{1}{2}\right) + \left(\omega_s - \frac{\eta\omega_{\rm nl}^2}{2\omega_{\rm s}}\right)\left(\nu_3 + \frac{1}{2}\right)\right] + O\big(\hbar^{3/2}\big). \tag{52}$$

The Aharonov–Anandan geometric phases (20) related to the SQETCS's (51) are written as

$$\gamma_{\mathcal{E}_\nu} = -\frac{T}{\hbar}\mathcal{E}_\nu + \frac{1}{\hbar}\int_0^T \langle\hat{\tilde{\mathcal{H}}}(t)\rangle dt, \tag{53}$$

where $\mathcal{E}_\nu$ and the operator $\hat{\tilde{\mathcal{H}}}(t)$ are of the form (52) and (34), respectively.

Calculation of the mean value for the operator $\hat{\tilde{\mathcal{H}}}(t)$ according to the formula (22) for the functions of the form (51) yields

$$\langle\hat{\tilde{\mathcal{H}}}(t)\rangle = \tilde{\mathcal{H}}(t) + \frac{\hbar}{2}(\omega_+ + \omega_-)(\nu_1 + \nu_2 + 1) + \hbar\omega_{\rm s}\left(\nu_3 + \frac{1}{2}\right) + \frac{\hbar\omega_H}{4}(\nu_1 - \nu_2),$$
$$\tilde{\mathcal{H}}(t) = \frac{m\xi^2}{2}(3\omega^2 + 2\omega\omega_H - \omega_0^2) + \tilde{\varkappa}V_0 - \frac{\eta\hbar\omega_{\rm nl}^2}{\omega_+ + \omega_-}(\nu_1 + \nu_2 + 1) - \frac{\eta\hbar\omega_{\rm nl}^2}{2\omega_{\rm s}}\left(\nu_3 + \frac{1}{2}\right). \tag{54}$$

Let us now substitute the quasi-energies (52) and (54) into the formula (53). Then we obtain the following expression for the Aharonov–Anandan geometric phase corresponding to the quasi-energy TCS's of equation (1) in the semiclassical approximation:

$$\gamma_{\mathcal{E}_\nu} = \frac{T}{2\hbar}m(2\omega^2 + \omega\omega_H)\xi^2 + O\big(\hbar^{3/2}\big), \tag{55}$$

where ξ has the form (39).

For more detailed analysis of quasi-energies and geometric phases it is useful to compare the expressions (52) and (55) with similar ones for the one-dimensional case.

5 One-dimensional case

Consider equation (1) where Ψ belongs to the space $L_2(\mathbb{R}^1_x)$, the linear operator $\hat{\mathcal{H}}(t)$ is taken as

$$\hat{\mathcal{H}}(t) = \frac{\hat{p}^2}{2m} - eEx\cos\omega t + \frac{k}{2}x^2, \tag{56}$$

and the nonlocal operator $\hat{V}(t, \Psi(t))$ is

$$\hat{V}(t, \Psi(t))\Psi(x, t) = \int_{-\infty}^{+\infty} V(x, y)|\Psi(y, t)|^2 dy\Psi(x, t),$$
$$V(x, y) = V_0 \exp\left[-\frac{(x-y)^2}{2\gamma^2}\right]. \tag{57}$$

As the magnetic field is missing ($H = 0$) in the case under consideration, the cyclotron frequency (9) is zero ($\omega_H = 0$) and from (11) we have $\omega_a = \omega_0$. Putting $\omega_H = 0$ into (43), (44) one has

$$\omega_+ = \omega_- = \omega_{\rm s} = \sqrt{\omega_0^2 - \eta\omega_{\rm nl}^2}.$$

The Hamilton–Ehrenfest system (35), (36) corresponding to (56), (57) accurate to $O(\hbar^{3/2})$ is

$$\dot p = -kx + eE\cos\omega t, \qquad \dot x = \frac{p}{m}, \tag{58}$$

$$\dot\sigma_{xx} = \frac{2}{m}\sigma_{xp}, \qquad \dot\sigma_{xp} = \frac{1}{m}\sigma_{pp} - m\omega_{\rm s}^2\sigma_{xx}, \qquad \dot\sigma_{pp} = -2m\omega_{\rm s}^2\sigma_{xp}. \tag{59}$$

The periodic solution (37), (38) with the period $T = 2\pi/\omega$ for the Hamilton system (58) takes the form

$$P_0(t,\hbar) = -m\omega\xi\sin\omega t, \qquad X_0(t,\hbar) = \xi\cos\omega t.$$

Here and below

$$\xi = \frac{eE}{m(\omega_0^2 - \omega^2)}.$$

The matrix $A(t)$ in the system of equations in variations (40) has now the size 2×2 with one-dimensional blocks $B(t)$ and $C(t)$ in (45). Accordingly, the system (40) becomes

$$\dot B = -m\omega_{\rm s}^2 C, \qquad \dot C = \frac{B}{m}. \tag{60}$$

The Floquet solution (41)

$$a(t) = \begin{pmatrix} B(t) \\ C(t) \end{pmatrix}$$

of the system (60) normalized by the condition $\langle a, J^T a^*\rangle = 2i$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, is found as

$$a(t) = \frac{\exp(i\omega_{\rm s}t)}{\sqrt{m\omega_{\rm s}}}\begin{pmatrix} im\omega_{\rm s} \\ 1 \end{pmatrix}.$$

Then the quasi-energy TCS's similar to (51) for the one-dimensional HTE (1) with operators (56), (57) have the form

$$|n,t\rangle = \frac{i^n}{\sqrt{2^n n!}}N_\hbar \exp\left\{\frac{i}{\hbar}\left[\frac{1}{4}eE\xi t - \frac{3\omega^2 - \omega_0^2}{8\omega}m\xi^2\sin 2\omega t - \tilde\varkappa V_0 t\right.\right. \tag{61}$$
$$\left.\left. -\hbar\left(\omega_{\rm s} - \frac{\eta\omega_{\rm nl}^2}{2\omega_{\rm s}}\right)\left(n+\frac{1}{2}\right)t\right] - \frac{i}{\hbar}m\omega\xi\sin\omega t\Delta x - \frac{m\omega_{\rm s}}{2\hbar}\Delta x^2\right\} H_n\left(\sqrt{\frac{m\omega_{\rm s}}{\hbar}}\Delta x\right).$$

Here $\Delta x = x - \xi\cos\omega t$, $N_\hbar$ is a normalization factor, $n \in \mathbb{Z}_+$. The condition (17) yields the spectrum of quasi-energies $\mathcal{E}_n$ for the functions (61) as

$$\mathcal{E}_n = -\frac{1}{4}eE\xi + \tilde\varkappa V_0 + \hbar\left(\omega_{\rm s} - \frac{\eta\omega_{\rm nl}^2}{2\omega_{\rm s}}\right)\left(n+\frac{1}{2}\right) + O\big(\hbar^{3/2}\big), \tag{62}$$

and the Aharonov–Anandan geometric phase is

$$\gamma_{\mathcal{E}_n} = \frac{T}{2\hbar}m\omega^2\xi^2 + O\big(\hbar^{3/2}\big). \tag{63}$$

6 Concluding remarks

The spectrum of quasi-energies (52) includes the sum of energies of three oscillators with shifted combined frequencies $\omega_\pm$ and ω_s the shift of which depends on nonlinearity. The geometric phase (55) in the approximation $O(\hbar^{3/2})$ under consideration does not include quantum corrections and nonlinearity. Similar is true for the one-dimensional case (62), (63). The following explanation to this can be proposed. The nonlinearity and the quantity $\hbar$ in this approximation do not enter to the solution (37), (38) of the Hamiltonian system (35) that specifies the geometry of the physical system. As a consequence, the geometric phase also does not include these quantities. In addition, the geometric phase for the three-dimensional case (55) (if $\omega_H = 0$) is twice as large as that for the one-dimensional case (63) since the centroid motion of the field Ψ is plane in the three-dimensional case (see (37), (38)).

In the limit $T \to \infty$ ($\omega \to 0$) the operator $\hat{\mathcal{H}}(t)$ of the form (6) (in the three-dimensional case) and (56) (in the one-dimensional case) does not depend on t any more and we denote $\lim\limits_{\omega \to 0} \hat{\mathcal{H}}(t) = \hat{\mathcal{H}}$. Then the expressions (51), (52) and (61), (62) determine the discrete spectral series of the nonlinear spectral problem

$$(\hat{\mathcal{H}} + \varkappa \hat{V}(\Psi)\}\Psi = \mathcal{E}\Psi.$$

This corresponds to a stationary solution of the Hamilton–Ehrenfest system (35), (36) and (58), (59), respectively (see also [10]).

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New Frobenius Structures on Hurwitz Spaces in Terms of Schiffer and Bergmann Kernels

Vasilisa SHRAMCHENKO

Department of Mathematics and Statistics, Concordia University,
7141 Sherbrooke West, Montreal H4B 1R6, Quebec, Canada
E-mail: *vasilisa@mathstat.concordia.ca*

New family of flat potential (Darboux–Egoroff) metrics on the Hurwitz spaces and corresponding Frobenius structures are found. We consider a Hurwitz space as a real manifold. Therefore the number of coordinates is twice as big as the number of coordinates used in the construction of Frobenius structure on Hurwitz spaces found by B. Dubrovin more than 10 years ago. The branch points of a ramified covering and their complex conjugates play the role of canonical coordinates on the constructed Frobenius manifolds. We introduce a new family of Darboux–Egoroff metrics in terms of the Schiffer and Bergmann kernels, find corresponding flat coordinates and a prepotential of associated Frobenius manifolds.

1 Introduction

The aim of the present work is to construct a new class of Frobenius manifolds associated to the Hurwitz spaces. First, we describe Dubrovin's structure of Frobenius manifolds on Hurwitz spaces. Those manifolds have complex dimension L, where L is complex dimension of the Hurwitz space. In our approach we consider the Hurwitz space as a real manifold, therefore the constructed here Frobenius manifolds have $2L$ coordinates.

We start with construction of a new family of flat diagonal metrics on the Hurwitz space written in terms of the Schiffer and Bergmann reproducing kernels on a ramified covering of Riemann sphere with simple branch points. Each of these metrics gives a Frobenius algebra on the tangent space. Then we find flat coordinates of the metrics and prepotentials F of the Frobenius structures. The prepotential F satisfies the WDVV system with respect to the flat coordinates for the corresponding metric according to the one-to-one correspondence between Frobenius structures and solutions to the WDVV equations [1].

2 Frobenius manifolds and WDVV equations

The theory of Frobenius manifolds is related to various branches of mathematics such as singularity theory, reflection groups, algebraic geometry, isomonodromic deformation theory, boundary-value problems, and Painlevé equations. It was developed by B. Dubrovin as a geometric interpretation of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of the following type:

$$F_i F_1^{-1} F_j = F_j F_1^{-1} F_i, \qquad i,j = 1,\dots,n,$$

where F_i is the matrix

$$(F_i)_{mn} = \frac{\partial^3 F}{\partial t^i \partial t^m \partial t^n}$$

for the function F of n variables $t^1,\dots,t^n$ with the normalization: F_1 is a constant nondegenerate matrix.

Any solution $F(t)$, $t = (t^1, \dots, t^n) \in M$ of these WDVV equations which satisfies an additional assumption of quasihomogenuity,

$$F(c^{d_1}t^1, \dots, c^{d_n}t^n) = c^{d_F}F(t^1, \dots, t^n) \qquad \text{for some } c \neq 0 \text{ and } d_1, \dots, d_n, d_F \in \mathbb{R},$$

determines in the domain M the structure of Frobenius manifold and vice versa (see [1]).

Definition 1. An algebra A over $\mathbb{C}$ is called (commutative) Frobenius algebra if:

- it is a commutative associative $\mathbb{C}$-algebra with a unity e.
- it is supplied with a $\mathbb{C}$-bilinear symmetric nondegenerate inner product having the property $\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle$

Definition 2. M is Frobenius manifold if a structure of Frobenius algebra is specified on any tangent plane T_tM at any point $t \in M$ smoothly depending on the point such that

1) the inner product $\langle , \rangle$ is a flat metric on M,
2) the unity vector field e is covariantly constant w.r.t. the Levi-Civita connection ∇ for the metric $\langle , \rangle$: $\nabla e = 0$,
3) the tensor $(\nabla_z \boldsymbol{c})(u, v, w)$ is symmetric in four vector fields u, v, w, z, where $\boldsymbol{c}$ is the following symmetric 3-tensor: $\boldsymbol{c}(u, v, w) = \langle u \cdot v, w\rangle$,
4) a vector field E is determined on M such that $\nabla(\nabla E) = 0$,

$$[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y,$$
$$E\langle x, y\rangle - \langle [E, x], y\rangle - \langle x, [E, y]\rangle = (2 - \nu)\langle x, y\rangle. \tag{1}$$

for some constant ν, the charge of Frobenius manifold.

We take the local coordinates $\lambda_1, \dots, \lambda_L$ on the Hurwitz space to be canonical coordinates for multiplication, i.e. $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$, where $\partial_i = \partial_{\lambda_i}$. Then, the unit vector field is $e = \sum_{i=1}^{L} \partial_i$, and the Euler vector field is given by $E = \sum_{i=1}^{L} \lambda_i \partial_i$. Then in these coordinates the metric is diagonal: $d\boldsymbol{s}^2 = \sum_{i=1}^{L} \eta_{ii} d\lambda_i^2$, where η_{ii} are functions of $\{\lambda_i\}$. The rotation coefficients of the metric are defined as

$$\beta_{ij} = \frac{\partial_j \sqrt{\eta_{ii}}}{\sqrt{\eta_{jj}}}, \qquad i \neq j.$$

The flatness of a potential metric is provided by the system

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj} \qquad \text{for different } i, j, k, \tag{2}$$

$$\sum_k \partial_k \beta_{ij} = 0. \tag{3}$$

(The metric is potential if and only if all rotation coefficients are symmetric with respect to indices: $\beta_{ij} = \beta_{ji}$.) The existence of the Euler vector field on the manifold (see the definition of Frobenius manifolds) implies another relation on rotation coefficients:

$$\sum_k \lambda_k \partial_k \beta_{ij} = -\beta_{ij}. \tag{4}$$

The definition of Frobenius manifold implies the existence of the function F such that the 3-tensor c is given by its third derivatives

$$\boldsymbol{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C}) = \frac{\partial^3 F}{\partial t^A \partial t^B \partial t^C},$$

where $\{t^A\}$ is the set of flat coordinates for the metric. This function F is called the prepotential of the Frobenius manifold.

3 Hurwitz spaces

Hurwitz space is the moduli space of the pairs $(\mathcal{L},\lambda)$, where $\mathcal{L}$ is a compact Riemann surface of genus g and λ is a meromorphic function on $\mathcal{L}$ of degree N. This function λ realizes the surface $\mathcal{L}$ as an N-fold ramified covering of $\mathbb{C}P^1$. We consider the space $M = M_{g;n_0,\dots,n_m}$, which is the moduli space of the genus g coverings of $\mathbb{C}P^1$ with fixed type of ramification over the point at infinity: $n_0,\dots,n_m \in \mathbb{N}$, $\sum\limits_{k=0}^{m}(n_k+1) = N$.

Denote by $\lambda_1,\dots,\lambda_L$ the finite critical values of the function λ (the branch points of the covering): $\lambda_j = \lambda(P_j)$, $d\lambda|_{P_j} = 0$, $j = 1,\dots,L$, and consider them as local coordinates on M. We assume that all the branch points λ_j are simple.

4 Kernels on Riemann surfaces

1. The kernel $W(P,Q)$ defined by $W(P,Q) = \mathrm{d}_P\mathrm{d}_Q \log E(P,Q)$ is the unique symmetric differential on $\mathcal{L}\times\mathcal{L}$ with the second order pole at $P = Q$ and properties:

$$\oint_{a_j} W(P,Q) = 0, \qquad \oint_{b_j} W(P,Q) = 2\pi i\omega_j(P), \qquad j = 1,\dots,g,$$

where $\{a_j,b_j\}_{j=1}^g$ is the canonical basis of cycles on $\mathcal{L}$; $\{\omega_j(P)\}_{j=1}^g$ is the corresponding set of holomorphic differentials normalized by $\oint_{a_i}\omega_j = \delta_{ij}$; and $E(P,Q)$ is the prime form on the surface $\mathcal{L}$.

Variational formulas for $W(P,Q)$ [2,3]:

$$\frac{dW(P,Q)}{d\lambda_k} = \frac{1}{2}\frac{W(P,P_k)}{dx_k}\frac{W(Q,P_k)}{dx_k},$$

$x_k = \sqrt{\lambda-\lambda_k}$ is the local parameter near ramification point P_k and the notation $W(P,P_k)/dx_k$ should be understood as $W(P,P_k)/dx_k = (W(P,Q)/dx(Q))\,|_{Q=P_k}$.

2. The Schiffer kernel $\Omega(P,Q)$ is the symmetric differential on $\mathcal{L}\times\mathcal{L}$ [4] defined as

$$\Omega(P,Q) = W(P,Q) - \pi\sum_{i,j=1}^{g}(\operatorname{Im}\mathbb{B})^{-1}_{ij}\omega_i(P)\omega_j(Q).$$

3. The Bergmann kernel $B(P,\bar{Q})$ is the following differential on $\mathcal{L}\times\mathcal{L}$ [4]:

$$B(P,\bar{Q}) = \pi\sum_{i,j=1}^{g}(\operatorname{Im}\mathbb{B})^{-1}_{ij}\omega_i(P)\overline{\omega_j(Q)},$$

where $\mathbb{B}$ is the symmetric matrix of b-periods of ω_j: $\mathbb{B}_{ij} = \oint_{b_i}\omega_j$ ($\operatorname{Im}\mathbb{B}$ is positive definite). The Bergmann and Schiffer kernels have the following properties:

$$\oint_{a_i}\Omega(P,Q) = -\oint_{a_i}B(\bar{P},Q), \qquad \oint_{b_i}\Omega(P,Q) = -\oint_{b_i}B(\bar{P},Q),$$

where integration is taken with respect to the first argument.

Variational formulas for the Schiffer and Bergmann kernels:

$$\frac{d\Omega(P,Q)}{d\lambda_k} = \frac{1}{2}\frac{\Omega(P,P_k)}{dx_k}\frac{\Omega(Q,P_k)}{dx_k}, \qquad \frac{d\Omega(P,Q)}{d\bar{\lambda}_k} = \frac{1}{2}\frac{B(P,\bar{P}_k)}{\overline{dx_k}}\frac{B(Q,\bar{P}_k)}{\overline{dx_k}},$$
$$\frac{dB(P,\bar{Q})}{d\lambda_k} = \frac{1}{2}\frac{\Omega(P,P_k)}{dx_k}\frac{B(\bar{Q},P_k)}{dx_k}, \qquad \frac{dB(P,\bar{Q})}{d\bar{\lambda}_k} = \frac{1}{2}\frac{B(P,\bar{P}_k)}{\overline{dx_k}}\frac{\overline{\Omega(Q,P_k)}}{\overline{dx_k}}.$$

Here as before $\Omega(P,P_k)/dx_k$ stands for $(\Omega(P,Q)/dx_k(Q))\,|_{Q=P_k}$.

To prove these formulas one uses the Rauch variational formulas for the holomorphic normalized differentials and for the matrix of b-periods [2]:

$$\frac{d\omega_j(P)}{d\lambda_k} = \frac{1}{2}\omega_j(P_k)W(P,P_k), \qquad \frac{d\mathbb{B}_{ij}}{d\lambda_k} = \pi i\,\omega_i(P_k)\omega_j(P_k),$$

and the variational formula for the kernel $W(P,Q)$ given above.

5 Dubrovin's Frobenius structure on Hurwitz spaces

We start with a reformulation of Dubrovin's structure of the Frobenius manifold on moduli space $M = M_{g;n_0,\dots,n_m}$ of coverings $(\mathcal{L},\lambda)$ of genus $g \geq 1$ using the kernel $W(P,Q)$. The branch points $\lambda_1,\dots,\lambda_L$ are used as local coordinates on the space M.

The flat metric on M. For any two tangent vectors ∂', ∂'' consider the inner product $\langle \partial', \partial'' \rangle_\phi = \Omega_{\phi^2}(\partial' \cdot \partial'')$, where

- ϕ is one of the primary differentials listed below;
- Ω_{ϕ^2} is a one form on M: $\Omega_{\phi^2} = \sum\limits_{i=1}^{L} \left(\mathrm{res}_{P_i} \frac{\phi^2}{d\lambda}\right) d\lambda_i$;
- multiplication of the tangent vector fields is defined as follows: $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$ for $\partial_i = \partial/\partial\lambda_i$.

Primary differentials. (z_i is a local parameter near ∞^i ; $z_i^{-n_i-1} = \lambda$.)

Type1. $\phi_{t^{i;\alpha}}(P) = \dfrac{1}{\alpha}\mathrm{res}_{Q=\infty^i} z_i^{-\alpha}(Q)W(P,Q) \sim z_i^{-\alpha-1}dz_i, \quad P \sim \infty^i,$
$i = 0,\dots,m; \quad \alpha = 1,\dots,n_i.$

Type2. $\phi_{v^i}(P) = \mathrm{res}_{Q=\infty^i}\lambda(Q)W(P,Q) \sim -d\lambda, \quad P \sim \infty^i; \quad i = 1,\dots,m.$

Type3. $\phi_{w^i}(P) = \text{v.p.}\displaystyle\int_{\infty^0}^{\infty^i} W(P,Q), \quad \mathrm{res}_{\infty^i}\phi_{w^i} = 1; \quad \mathrm{res}_{\infty^0}\phi_{w^i} = -1; \quad i = 1,\dots,m.$

Type4. $\phi_{r^i}(P) = -\displaystyle\oint_{a_i} \lambda(Q)W(P,Q), \quad \phi_{r^i}(P+b_i) = \phi_{r^i}(P) + 2\pi i d\lambda; \quad i = 1,\dots,g.$

Type5. $\phi_{s^i}(P) = \dfrac{1}{2\pi i}\displaystyle\oint_{b_i} W(P,Q), \quad \oint_{a_i} \phi_{s^i} = 1; \quad i = 1,\dots,g.$

Note that all above differentials have zero a-periods except for $\oint_{a_j} \phi_{s^i} = \delta_{ij}$.

Theorem 1. *With $\phi = \phi_{t^{i;\alpha}}$; $\phi = \sum\limits_{i=1}^{m} \delta_i\phi_{v^i}$; $\phi = \sum\limits_{i=1}^{m} \delta_i\phi_{w^i}$; $\phi = \sum\limits_{i=1}^{g} \delta_i\phi_{r^i}$; $\phi = \sum\limits_{i=1}^{g} \delta_i\phi_{s^i}$ the metric $\langle,\rangle_\phi$ is flat and its rotation coefficients are given by the kernel $W(P,Q)$ at the branch points* [3]*:*

$$\beta_{ij} = \frac{1}{2}\frac{W(P_i,P_j)}{dx_i\,dx_j}.$$

The rotation coefficients satisfy equations (2)–(4)*. Requirement* (1) *holds for the metric.*

Flat coordinates for the metric $\langle,\rangle_\phi$. (I.e. the coordinates with respect to which the metric is constant.)

$$t^{i;\alpha} = \mathrm{res}_{P=\infty^i} z_i^\alpha(\text{v.p.}\int_{\infty^0}^{P} \phi)d\lambda, \qquad i = 0,\dots,m; \quad \alpha = 1,\dots,n_i,$$

$$v^i = -\text{v.p.}\int_{\infty^0}^{\infty^i} \phi, \qquad w^i = -\mathrm{res}_{\infty^i}\lambda\phi, \qquad i = 1,\dots,m,$$

$$r^k = \frac{1}{2\pi i}\oint_{b_k}\phi, \qquad s^k = -\oint_{a_k}\lambda\phi, \qquad k=1,\dots,g.$$

Denote the flat coordinates by t^A, i.e., let $t^A \in \{t^{i;\alpha}; v^i, w^i; r^k, s^k \,|\, i=0,\dots,m,\ \alpha=1,\dots,n_i; k=1,\dots,g\}$.

Prepotential of the Frobenius structure is a function $F(t^A)$ the third derivatives of which are given by the 3-tensor $\mathbf{c}$ from the definition of Frobenius manifold:

$$\frac{\partial^3 F(t)}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} = \mathbf{c}(\partial_{t^A},\partial_{t^B},\partial_{t^C}) = \langle \partial_{t^A}\cdot\partial_{t^B},\partial_{t^C}\rangle_\phi.$$

Let $\omega^{(1)}$ and $\omega^{(2)}$ be two differentials on $\mathcal{L}$ holomorphic outside of infinity with the following behaviour at infinities:

$$\omega^{(i)} = \sum_{k=-k_1}^{\infty} c^{(i)}_{k,a} z_a^k dz_a + \frac{1}{n_a+1} d\left(\sum_{k>0} r^{(i)}_{k,a}\lambda^k\log\lambda\right), \qquad P\sim\infty^a,$$

where $k_1\in\mathbb{Z}$ and $c^{(i)}_{k,a}$, $r^{(i)}_{k,a}$ are some coefficients. Denote also for $k=1,\dots,g$:

$$\oint_{a_k}\omega^{(i)} = A^{(i)}_k,$$
$$\omega^{(i)}(P^{a_k}) - \omega^{(i)}(P) = dp^{(i)}_k(\lambda), \qquad p^{(i)}_k(\lambda) = \sum_{s>0} p^{(i)}_{sk}\lambda^s,$$
$$\omega^{(i)}(P^{b_k}) - \omega^{(i)}(P) = dq^{(i)}_k(\lambda), \qquad q^{(i)}_k(\lambda) = \sum_{s>0} q^{(i)}_{sk}\lambda^s.$$

Here $\omega(P^{a_k})-\omega(P)$ denotes an additive transformation of differential under analytic continuation along the cycle a_k on the Riemann surface.

Definition 3. For two such differentials define a pairing:

$$\langle \omega^{(i)}\ \omega^{(j)}\rangle = \sum_{a=0}^{m}\left[\sum_{n\geq 0}\frac{c^{(i)}_{-n-2,a}}{n+1}c^{(j)}_{n,a} + c^{(i)}_{-1,a}\,\text{v.p.}\int_{P_0}^{\infty^a}\omega^{(j)} - \text{v.p.}\int_{P_0}^{\infty^a} r^{(i)}_{n,a}\lambda^n\omega^{(j)}\right]$$
$$+\frac{1}{2\pi i}\sum_{k=1}^{g}\left[-\oint_{a_k} q^{(i)}_k(\lambda)\omega^{(j)} + \oint_{b_k} p^{(i)}_k(\lambda)\omega^{(j)} + A^{(i)}_k\oint_{b_k}\omega^{(j)}\right].$$

P_0 is the point on the curve $\mathcal{L}$ such that $\lambda(P_0)=0$; all basic cycles $\{a_k\}$ and $\{b_k\}$ start at P_0.

Consider the multivalued function on $\mathcal{L}$: $p(P) = \text{v.p.}\int_{\infty^0}^{P}\phi$. The differential $pd\lambda$ is of the above type.

Theorem 2. *The prepotential of the Frobenius structure is* $F = \frac{1}{2}\langle pd\lambda\ pd\lambda\rangle$.

Theorem 3. *For the second derivatives of the prepotential we have* $\partial_{t^A}\partial_{t^B}F = \langle \phi_{t^A}\ \phi_{t^B}\rangle$.

6 New Frobenius structure on Hurwitz spaces

We consider the space $M = M_{g;n_0,\dots,n_m}$ as a real manifold with the coordinates $\{\lambda_1,\dots,\lambda_L; \bar\lambda_1,\dots,\bar\lambda_L\}$. On this space we introduce a Frobenius structure analogous to the one described in Section 5. The new Darboux–Egoroff metric, corresponding flat coordinates and primary differentials are written in terms of the Schiffer and Bergmann kernels which depend on both $\{\lambda_k\}$ and $\{\bar\lambda_k\}$.

The flat metric on M. For two tangent vectors ∂', ∂'' consider the inner product $\langle\partial',\partial''\rangle_{\phi+\psi} = \Omega_{\phi^2+\psi^2}(\partial'\cdot\partial'')$, where

- ϕ is "holomorphic" and ψ is "antiholomorphic" parts of one of the primary differentials listed below;
- $\Omega_{\phi^2+\psi^2}$ is a one form on M: $\Omega_{\phi^2+\psi^2} = \sum\limits_{i=1}^{L} \left(\mathrm{res}_{P_i} \frac{\phi^2}{d\lambda}(d\lambda_i) + \tilde{\mathrm{res}}_{P_i} \frac{\psi^2}{d\bar{\lambda}}(d\bar{\lambda}_i) \right)$, where we denote $\tilde{\mathrm{res}} f = \overline{\mathrm{res} \bar{f}}$;
- multiplication of the tangent vector fields is defined as follows: $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$, where i and j run through the index sets $\{k\}_{k=1}^{L}$ and $\{\bar{k}\}_{k=1}^{L}$, and $\partial_i = \partial/\partial\lambda_i$, $\partial_{\bar{i}} = \partial/\partial\bar{\lambda}_i$.

Primary differentials. (z_i is a local parameter near ∞^i; $z_i^{-n_i-1} = \lambda$.)

Type 1. $i = 0, \dots, m$; $\alpha = 1, \dots, n_i$.

a) $\Phi_{t^{i;\alpha}}(P) = \phi_{t^{i;\alpha}}(P) + \psi_{t^{i;\alpha}}(P) = \dfrac{1}{\alpha}\mathrm{res}_{Q=\infty^i} z_i^{-\alpha}(Q)\Omega(P,Q) + \dfrac{1}{\alpha}\mathrm{res}_{Q=\infty^i} z_i^{-\alpha}(Q)B(\bar{P},Q)$
$\sim (z_i^{-\alpha-1} + O(1))dz_i + O(1)d\bar{z}_i, \quad P \sim \infty^i.$

b) $\Phi_{t^{\overline{i;\alpha}}}(P) = \phi_{t^{\overline{i;\alpha}}}(P) + \psi_{t^{\overline{i;\alpha}}}(P) = \overline{\psi_{t^{i;\alpha}}(P)} + \overline{\phi_{t^{i;\alpha}}(P)} = \overline{\Phi_{t^{i;\alpha}}(P)}.$

Type 2. $i = 1, \dots, m$.

a) $\Phi_{v^i}(P) = \phi_{v^i}(P) + \psi_{v^i}(P) = \mathrm{res}_{Q=\infty^i}\lambda(Q)\Omega(P,Q) + \mathrm{res}_{Q=\infty^i}\lambda(Q)B(\bar{P},Q)$
$\sim -d\lambda + O(1)(d\bar{z}_i + dz_i), \quad P \sim \infty^i.$

b) $\Phi_{v^{\bar{i}}}(P) = \phi_{v^{\bar{i}}}(P) + \psi_{v^{\bar{i}}}(P) = \overline{\psi_{v^i}(P)} + \overline{\phi_{v^i}(P)} = \overline{\Phi_{v^i}(P)}.$

Type 3. $i = 1, \dots, m$.

a) $\Phi_{w^i}(P) = \phi_{w^i}(P) + \psi_{w^i}(P) = \mathrm{v.p.} \displaystyle\int_{\infty^0}^{\infty^i} \Omega(P,Q) + \int_{\infty^0}^{\infty^i} B(\bar{P},Q),$

$\mathrm{res}_{\infty^i}\Phi_{w^i} = 1; \quad \mathrm{res}_{\infty^0}\Phi_{w^i} = -1.$

b) $\Phi_{w^{\bar{i}}}(P) = \phi_{w^{\bar{i}}}(P) + \psi_{w^{\bar{i}}}(P) = \overline{\psi_{w^i}(P)} + \overline{\phi_{w^i}(P)} = \overline{\Phi_{w^i}(P)}.$

Type 4. $k = 1, \dots, g$.

$$\phi_{r^k}(P) + \psi_{r^k}(P) = -\oint_{a_k} \lambda(Q)\Omega(P,Q) - \oint_{a_k} \lambda(Q)B(\bar{P},Q),$$
$$\Phi_{r^k}(P) = \phi_{r^k}(P) + \overline{\psi_{r^k}(P)} + \psi_{r^k}(P) + \overline{\phi_{r^k}(P)}.$$

Type 5. $k = 1, \dots, g$.

$$\phi_{u^k}(P) + \psi_{u^k}(P) = \oint_{b_k} \lambda(Q)\Omega(P,Q) + \oint_{b_k} \lambda(Q)B(\bar{P},Q),$$
$$\Phi_{u^k}(P) = \phi_{u^k}(P) + \overline{\psi_{u^k}(P)} + \psi_{u^k}(P) + \overline{\phi_{u^k}(P)}.$$

Type 6. $k = 1, \dots, g$.

$$\Phi_{s^\alpha}(P) = \phi_{s^\alpha}(P) + \psi_{s^\alpha}(P) = \frac{1}{2\pi i}\oint_{b_\alpha} \Omega(P,Q) + \frac{1}{2\pi i}\oint_{b_\alpha} B(\bar{P},Q).$$

Type 7. $k = 1, \dots, g$.

$$\Phi_{t^\alpha}(P) = \phi_{t^\alpha}(P) + \psi_{t^\alpha}(P) = -\frac{1}{2\pi i}\oint_{a_\alpha} \Omega(P,Q) - \frac{1}{2\pi i}\oint_{a_\alpha} B(\bar{P},Q).$$

For all these differentials ϕ-part is a differential with respect to a local parameter z and ψ-part is a differential with respect to $\bar{z}$. For brevity we denote by Φ_{ξ^A} an arbitrary differential from the above list. a- and b-periods of Φ_{ξ^A} are non-zero only for two types of primary differentials: $\oint_{a_\alpha} \Phi_{\xi^A} = \delta_{\xi^A,s^\alpha}$; $\oint_{b_\alpha} \Phi_{\xi^A} = \delta_{\xi^A,t^\alpha}$.

Flat coordinates for the metric $\langle , \rangle_{\phi+\psi}$.

$$t^{i;\alpha} = \mathrm{res}_{P=\infty^i} z_i^\alpha \left(\mathrm{v.p.} \int_{\infty^0}^{P} \phi \right) d\lambda, \qquad t^{\overline{i;\alpha}} = \tilde{\mathrm{res}}_{P=\infty^i} \bar{z}_i^\alpha \left(\mathrm{v.p.} \int_{\infty^0}^{P} \psi \right) d\bar{\lambda},$$

$$
\begin{aligned}
&i=0,\dots,m, \qquad \alpha=1,\dots,n_i,\\
&v^i=-\text{v.p.}\int_{\infty^0}^{\infty^i}\phi, \qquad v^{\bar i}=-\text{v.p.}\int_{\infty^0}^{\infty^i}\psi, \qquad i=1,\dots,m,\\
&w^i=-\text{res}_{\infty^i}\lambda\phi, \qquad w^{\bar i}=-\tilde{\text{res}}_{\infty^i}\bar\lambda\psi, \qquad i=1,\dots,m,\\
&r^k=\frac{1}{2\pi i}\oint_{b_k}\phi, \quad u^k=\frac{1}{2\pi i}\oint_{a_k}\phi, \qquad k=1,\dots,g,\\
&s^k=-\oint_{a_k}(\lambda\phi+\bar\lambda\psi), \qquad t^k=-\oint_{b_k}(\lambda\phi+\bar\lambda\psi), \qquad k=1,\dots,g.
\end{aligned}
$$

We denote $\tilde{\text{res}}f=\overline{\text{res}\bar f}$ and the flat coordinates by ξ^A, i.e., $\xi^A\in\{t^{i;\alpha},t^{\overline{i;\alpha}};v^i,v^{\bar i},w^i,w^{\bar i};r^k,u^k,s^k,t^k\,|\,i=0,\dots,m,\ \alpha=1,\dots,n_i;\ k=1,\dots,g\}$. The number of flat coordinates equals $2L$ by virtue of the Riemann–Hurwitz formula.

Theorem 4. *The rotation coefficients of the metric $\langle,\rangle_{\phi+\psi}$ are given by:*

$$\beta_{ij}=\frac{1}{2}\frac{\Omega(P_i,P_j)}{dx_i\,dx_j}, \qquad \beta_{i\bar j}=\frac{1}{2}\frac{B(P_i,\bar P_j)}{dx_i\,d\bar x_j}, \qquad \beta_{\bar i\bar j}=\overline{\beta_{ij}}.$$

They satisfy equations (2)–(4). *The metrics satisfy* (1) *with the Euler field* $\sum\limits_{k=1}^{L}(\lambda_k\partial_{\lambda_k}+\bar\lambda_k\partial_{\bar\lambda_k})$, *where in* (3) *and* (4) *the sum over the index set* $\{1,\dots,L;\bar 1,\dots,\bar L\}$ *is understood.*

Prepotential of the Frobenius structure is a function $F(\xi^A)$ the third derivatives of which are given by the 3-tensor $\boldsymbol{c}$ from the definition of Frobenius manifold:

$$\frac{\partial^3F(\xi)}{\partial\xi^A\partial\xi^B\partial\xi^C}=\boldsymbol{c}(\partial_{\xi^A},\partial_{\xi^B},\partial_{\xi^C})=\langle\partial_{\xi^A}\cdot\partial_{\xi^B},\partial_{\xi^C}\rangle_{\phi+\psi}.$$

To write an expression for prepotential we define a new pairing of multivalued differentials on $\mathcal{L}$ as follows. Let $\omega^{(i)}(P)$ be a differential on $\mathcal{L}$, such that it can be decomposed in holomorphic ($\omega^{(i)}_{1,0}$) and antiholomorphic ($\omega^{(i)}_{0,1}$) parts: $\omega^{(i)}=\omega^{(i)}_{1,0}+\omega^{(i)}_{0,1}$, which are regular outside infinities and have the following behaviour near $P\sim\infty^a$:

$$
\begin{aligned}
\omega^{(i)}_{1,0}(P)&=\sum_{n=-n_1}^{\infty}c^{(i)}_{n,a}z_a^n dz_a+\frac{1}{n_a+1}\,d\left(\sum_{n>0}r^{(i)}_{n,a}\lambda^n\log\lambda\right),\\
\omega^{(i)}_{0,1}(P)&=\sum_{n=-n_2}^{\infty}c^{(i)}_{\bar n,a}\bar z_a^n d\bar z_a+\frac{1}{n_a+1}\,d\left(\sum_{n>0}r^{(i)}_{\bar n,a}\bar\lambda^n\log\bar\lambda\right),
\end{aligned}
$$

where $n_1,n_2\in\mathbb{Z}$; $c^{(i)}_{n,a}$, $r^{(i)}_{n,a}$, $c^{(i)}_{\bar n,a}$, $r^{(i)}_{\bar n,a}$ are some coefficients. Denote also for $k=1,\dots,g$:

$$
\begin{aligned}
&\oint_{a_k}\omega^{(i)}=A^{(i)}_k, \qquad \oint_{b_k}\omega^{(i)}=B^{(i)}_k,\\
&\omega^{(i)}_{1,0}(P^{a_k})-\omega^{(i)}_{1,0}(P)=dp^{(i)}_k(\lambda), \qquad p^{(i)}_k(\lambda)=\sum_{s>0}p^{(i)}_{sk}\lambda^s,\\
&\omega^{(i)}_{1,0}(P^{b_k})-\omega^{(i)}_{1,0}(P)=dq^{(i)}_k(\lambda), \qquad q^{(i)}_k(\lambda)=\sum_{s>0}q^{(i)}_{sk}\lambda^s.\\
&\omega^{(i)}_{0,1}(P^{a_k})-\omega^{(i)}_{0,1}(P)=dp^{(i)}_{\bar k}(\bar\lambda), \qquad p^{(i)}_{\bar k}(\bar\lambda)=\sum_{s>0}p^{(i)}_{\bar s\bar k}\bar\lambda^s,\\
&\omega^{(i)}_{0,1}(P^{b_k})-\omega^{(i)}_{0,1}(P)=dq^{(i)}_{\bar k}(\bar\lambda), \qquad q^{(i)}_{\bar k}(\bar\lambda)=\sum_{s>0}q^{(i)}_{\bar s\bar k}\bar\lambda^s.
\end{aligned}
$$

Again, $\omega(P^{a_k}) - \omega(P)$ denotes an additive transformation of differential under analytic continuation along the cycle a_k on the Riemann surface.

Definition 4. For two such differentials define the pairing:

$$\begin{aligned}\langle \omega^{(i)}\ \omega^{(j)} \rangle = \sum_{a=0}^{m} \Bigg[& \sum_{n\geq 0} \frac{c^{(i)}_{-n-2,a}}{n+1} c^{(j)}_{n,a} + c^{(i)}_{-1,a} \text{v.p.} \int_{P_0}^{\infty^a} \omega_{1,0}(j) - \text{v.p.} \int_{P_0}^{\infty^a} r^{(i)}_{n,a} \lambda^n \omega^{(j)}_{1,0} \\ & + \sum_{n\geq 0} \frac{c^{(i)}_{-\overline{n-2},a}}{n+1} c^{(j)}_{\bar{n},a} + c^{(i)}_{-\bar{1},a} \text{v.p.} \int_{P_0}^{\infty^a} \omega^{(j)}_{0,1} - \text{v.p.} \int_{P_0}^{\infty^a} r^{(i)}_{\bar{n},a} \bar{\lambda}^n \omega^{(j)}_{0,1} \Bigg] \\ & + \frac{1}{2\pi i} \sum_{k=1}^{g} \Bigg[- \oint_{a_k} q^{(i)}_k(\lambda) \omega^{(j)}_{1,0} + \oint_{a_k} q^{(i)}_{\bar{k}}(\bar{\lambda}) \omega^{(j)}_{0,1} + \oint_{b_k} p^{(i)}_k(\lambda) \omega^{(j)}_{1,0} \\ & - \oint_{b_k} p^{(i)}_{\bar{k}}(\bar{\lambda}) \omega^{(j)}_{0,1} + A^{(i)}_k \oint_{b_k} \omega^{(j)}_{1,0} - B^{(i)}_k \oint_{a_k} \omega^{(j)}_{1,0} \Bigg].\end{aligned}$$

P_0 is the point on the curve $\mathcal{L}$ such that $\lambda(P_0) = 0$; all basic cycles $\{a_k\}$ and $\{b_k\}$ start at P_0.

Consider the multivalued differential Φ on $\mathcal{L}$: $\Phi(P) = \left(\text{v.p.} \int_{\infty^0}^{P} \phi\right) d\lambda + \left(\text{v.p.} \int_{\infty^0}^{P} \psi\right) d\bar{\lambda}$, where ϕ and ψ are the corresponding parts of some primary differential of one of the listed types.

The pairing is defined for this differential and the following theorem holds.

Theorem 5. *The prepotential of the Frobenius structure is* $F = \frac{1}{2}\langle \Phi\ \Phi \rangle$.

Theorem 6. *For the second derivatives of the prepotential we have* $\partial_{\xi^A} \partial_{\xi^B} F = \langle \Phi_{\xi^A}\ \Phi_{\xi^B} \rangle$.

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Long-Range Order in Classical Lattice Linear Oscillator Systems and Peierls Argument

Wolodymyr SKRYPNIK

Institute of Mathematics of NAS Ukraine, 3 Tereshchenkivs'ka Str., 01601 Kyiv-4, Ukraine
E-mail: *skrypnik@imath.kiev.ua*

Existence of the ferromagnetic long-range order is proven for Gibbs classical lattice systems of linear oscillators with a strong polynomial pair nearest neighbor ferromagnetic potential.

1 Introduction

Mathematical theory of phase transitions began its quick development half a century ago by describing critical properties of the Ising-type finite valued spin systems in terms of a contour technique [1]. This technique gave rise to Pirogov–Sinai (PS) theory, based on a knowledge of ground states, which allows to calculate a phase diagram at low temperatures, and can be successfully applied not only to classical spin systems with finite number of ground states but also to quantum ones. Generalization of the PS theory to Gibbs lattice systems with discrete or continuous set of ground states, such as systems of rotators and oscillators, characterized by continuous variables associated to lattice sites of the hypercubic lattice $\mathbb{Z}^d$, demands a profound combinatorial and probabilistic technique. Such a technique was proposed in [2] for linear lattice oscillator systems with nearest neighbor interaction. In this important paper the intuitive idea that the number of phases is determined by the number of global minima of the polynomial external field was justified. The technique of this paper was used also in proving of existence of an order parameter (the ground state expectation value of the squared absolute value of the Higgs field) in the $SU(n)$ lattice Higgs gauge field model [3]. In the infinite gauge interaction limit the Euclidean action, i.e. the potential energy of the Gibbs system, is reduced to the potential energy of oscillator system from the previous paper.

If one deals with classical systems of oscillators interacting with spins or fermions having simple self-interaction then, integrating out the variables corresponding to the spins and fermions in the partition function, one obtains an effective lattice oscillator system. Such an effective oscillator system is derived in the stationary quantum Holstein model.

To describe a phase diagram of a Gibbs system with a continuous set of ground states one has at first to look at decrease of correlations and then find long-range order (lro) and the associated order parameter. For different nearest neighbor lattice anisotropic systems with nearest neighbor (n-n) interaction, including linear oscillator systems, occurrence of lro was established with the help of the generalized Peierls argument in [4]. Lro means that there is no decrease of correlations. For high temperatures or weak coupling there is always a decrease of correlations for usual interactions. The order parameter always has a jump at critical values of the temperature or a coupling constant, i.e. it is a discontinuous function of these parameters. This fact can often be attributed to a breakdown of a discrete symmetry, for instance, Z_2 symmetry associated to a change of signs of all (infinite) spin variables when potential energy is an even function.

Since general oscillator systems are quite important there emerged a necessity to generalize the Peierls argument to non-nearest neighbor interactions. This argument has been already generalized for the linear oscillator system appeared in the stationary Holstein model in [5] but the technique of this paper gave no hint how to do this in a general case.

2 Peierls argument and lro

The Peierls argument is based on the contour bound for the Gibbs average on a compact connected set Λ (hypercube)

$$\langle \prod_{\langle x,y\rangle\in\Gamma} \chi_x^+\chi_y^-\rangle_\Lambda \le e^{-|\Gamma|E},$$

where the contour energy $E > 0$ is independent of Λ and Γ, Γ is the set of nearest neighbors with left and right sides in Γ belonging to different sides of a contour γ that is a completely connected (without holes) set of faces of completely connected union of unit cubes, centered at lattice sites, i.e. Γ is a set of nearest neighbors associated to γ, $|\Gamma| = 2|\gamma|$ is the number of lattice sites in Γ or twice the hyper-area of the contour γ, γ is closed if the faces adjacent to the boundary of Λ are added,

$$\chi_x^+(q_\Lambda) = \chi_{(0,\infty)}(q_x), \qquad \chi_x^-(q_\Lambda) = \chi_{(-\infty,0)}(q_x), \qquad q_x \in \mathbb{R},$$

$\chi_{(a,b)}$ is the characteristic function of the open interval (a, b). q_x may be one of the component of a many-component oscillator or rotator.

For classical linear oscillator systems with the potential energy U and the inverse temperature β the Gibbs average is given by

$$\langle F_X\rangle_\Lambda = Z_\Lambda^{-1}\int F_X(q_X)e^{-\beta U(q_\Lambda)}dq_\Lambda = \int F_X(q_X)\rho^\Lambda(q_X)dq_X, \qquad q_X = (q_x, x \in X),$$
$$\rho^\Lambda(q_X) = Z_\Lambda^{-1}\int e^{-\beta U(q_\Lambda)}dq_{\Lambda\setminus X}, \qquad Z_\Lambda = \int e^{-\beta U(q_\Lambda)}dq_\Lambda.$$

Here the integrations are performed over $R^{|\Lambda|}$, $R^{|X|}$, $R^{|\Lambda\setminus X|}$ and ρ^Λ are the correlation functions, where $|X|$ is the cardinality (number of sites) of X.

When E diverges at zero temperature or infinite strength of nearest neighbor ferromagnetic interaction (coupling constant) then the Peierls argument yields inequality [4,6] for sufficiently low temperature or large coupling constant

$$\langle \chi_x^+\chi_y^-\rangle_\Lambda \le ae^{-2dE}. \tag{1}$$

where a is a positive constant independent of Λ. This leads immediately to ferromagnetic lro for unit spins s_x, where a' is independent of Λ, inequality $\langle m_\Lambda^2\rangle_\Lambda \ge a'$ and non-triviality of the order parameter

$$m_\Lambda = |\Lambda|^{-1}\sum_{x\in\Lambda} s_x.$$

in the thermodynamic limit, i.e. $\Lambda \to \mathbb{Z}^d$. This order parameter is an analog of the Ising magnetization.

Indeed, from $\chi_x^{+(-)} = \frac{1}{2}[1 \pm s_x]$ one obtains

$$4\langle \chi_x^+\chi_y^-\rangle_\Lambda = 1 + \langle s_x\rangle_\Lambda - \langle s_y\rangle_\Lambda - \langle s_x s_y\rangle_\Lambda.$$

For systems invariant under the transformation of changing signs of the oscillator variables the third and the second terms in the right-hand side of latter equality are equal to zero and

$$\langle s_x s_y\rangle_\Lambda = 1 - 4\langle \chi_x^+\chi_y^-\rangle_\Lambda.$$

The average in the right-hand side of this equality is arbitrarily small due to (1) and the unit spin lro occurs.

To derive (1) one has to insert

$$1=\prod_{l\in\Lambda}(\chi_l^+ + \chi_l^-) = \sum_{s_\Lambda, s_x=\pm 1}\prod_{l\in\Lambda}\chi_l^{s_l}$$

into the left-hand side of the contour bound. The sum over spin configurations is equivalent to a sum over contours $\gamma_{x,y}$ separating x, y which may end on the boundary of Λ and be non-closed (the resummation over spin configurations outside of the set of nearest neighbors Γ built on the contour γ should be performed). That is, the left-hand side of (1) is bounded by

$$\sum_{\gamma_{x,y}\in\Lambda}\langle\chi_x^+\chi_y^-\prod_{x',x''\in\Gamma(\gamma_{x,y})}\chi_x^+\chi_{x'}^-\rangle_\Lambda \le \sum_{\gamma_{x,y}\in\Lambda}\langle\prod_{x',x''\in\Gamma(\gamma_{x,y})}\chi_{x'}^+\chi_{x''}^-\rangle_\Lambda \le \sum_{\gamma}\langle\prod_{x',x''\in\Gamma(\gamma)}\chi_{x'}^+\chi_{x''}^-\rangle_\Lambda,$$

where γ is a closed contour. The sum over the closed contours is bounded by the right-hand side of (1) [7, Lemmas 5.3.5–6].

From the Frohlich–Lieb argument [4] one derives the following inequality for oscillator systems an arbitrary positive r

$$\langle\sigma_x\sigma_y\rangle_\Lambda \ge r^2\langle s_x s_y\rangle_\Lambda - \langle\sigma_x^2\sigma_y^2\rangle_\Lambda^{\frac{1}{2}} - 4, \qquad \sigma_x(q_\Lambda)=q_x\in\mathbb{R}.$$

For systems with potential energy satisfying super-stability and regularity conditions [7, 12] the second term in the right-hand side of the inequality is uniformly bounded in Λ and unit spin lro implies occurrence of the ferromagnetic oscillator lro

$$\langle\sigma_x\sigma_y\rangle_\Lambda \ge a'' > 0,$$

where a'' does not depend on Λ. This, of course, means non-triviality of the oscillator order parameter

$$M_\Lambda = |\Lambda|^{-1}\sum_{x\in\Lambda}\sigma_x.$$

The subtle point is that the contour bound has to hold for systems with non-zero nearest-neighbor ferromagnetic interaction.

3 Results

Let us consider the linear oscillator system with the potential energy (an even function)

$$U(q_\Lambda)=\sum_{x\in\Lambda}u(q_x)+\sum_{\langle x,y\rangle\in\Lambda}\phi(q_x,q_y)+U'(q_\Lambda), \qquad q_x\in\mathbb{R},$$

where u is the external field which is a bounded below even polynomial of $2n$-th degree ($2\le n\in\mathbb{Z}$)

$$\phi(q_x,q_y)=-g(q_x^k q_y^l + q_x^l q_y^k), \qquad ptk+l=2n_0,$$

u does not depend on g, U' is expressed as a finite sum of products $q_{x_j}^{l_j}$ with negative coefficients and U satisfies the super-stability and regularity conditions which almost guaranteed existence of the thermodynamic limit. g is the strength of the ferromagnetic nearest-neighbor interaction. There is an remarkably simple derivation of the contour bound, proposed in [8] for the case $n_0=1$, which is based on the fact that the Gibbs average satisfies GKS inequality (generalized Griffiths inequality) [9] and the bound

$$\chi_x^+\chi_y^- \le e^{-\frac{g\beta}{2}(\sigma_x^k\sigma_y^l+\sigma_x^l\sigma_y^k)}. \tag{2}$$

This bound follows from

$$\chi_x^+\chi_y^- = e^{-\frac{g\beta}{2}(\sigma_x^k\sigma_y^l+\sigma_x^l\sigma_y^k)}e^{\frac{g\beta}{2}(\sigma_x^k\sigma_y^l+\sigma_x^l\sigma_y^k)}\chi_x^+\chi_y^- \le e^{-\frac{g\beta}{2}(\sigma_x^k\sigma_y^l+\sigma_x^l\sigma_y^k)}\chi_x^+\chi_y^- \le e^{-\frac{g\beta}{2}(\sigma_x^k\sigma_y^l+\sigma_x^l\sigma_y^k)}.$$

Here one takes into account that $\sigma_x \ge 0$, $\sigma_y \le 0$. The latter inequality is employed for the proof of the contour bound as follows

$$\langle \prod_{\langle x,y\rangle\in\Gamma} \chi_x^+\chi_y^- \rangle_\Lambda \le \langle e^{-\frac{g\beta}{2}\sum\limits_{\langle x,y\rangle\in\Gamma}(\sigma_x^k\sigma_y^l+\sigma_x^l\sigma_y^k)} \rangle_\Lambda$$
$$= \left(\langle e^{\frac{g\beta}{2}\sum\limits_{\langle x,y\rangle\in\Gamma}(\sigma_x^k\sigma_y^l+\sigma_x^l\sigma_y^k)} \rangle_{\Lambda[\Gamma]} \right)^{-1} \le e^{-\frac{g\beta}{2}\sum\limits_{\langle x,y\rangle\in\Gamma}\langle(\sigma_x^k\sigma_y^l+\sigma_y^k\sigma_y^l)\rangle_{\Lambda[\Gamma]}} = e^{E_\Gamma}.$$

where $\langle\cdot,\cdot\rangle_{\Lambda[\Gamma]}$ is the average corresponding to the potential energy

$$U_\Gamma(q_\Lambda) = U(q_\Lambda) + \frac{g}{2}\sum_{x,y\in\Gamma}(q_x^k q_y^l + q_x^l q_y^k).$$

In latter line we applied the Jensen inequality. Taking into account that the Gibbs average is a monotone increasing function in interaction (a consequence of the GKS inequality) we obtain

$$E_\Gamma \ge E|\Gamma|, \qquad E = \beta g\langle\sigma^k\sigma'^l + \sigma^l\sigma'^k\rangle,$$

where

$$\langle\sigma^k\sigma'^l + \sigma^l\sigma'^k\rangle = Z^{-1}(2)\int (q_1^k q_2^l + q_1^l q_2^k)e^{-\beta u(q_1,q_2)}dq_1dq_2,$$
$$Z(2) = \int e^{-\beta u(q_1,q_2)}dq_1dq_2, \qquad u(q_1,q_2) = u_0(q_1) + u_0(q_2) - g\big(q_1^k q_2^l + q_1^l q_2^k\big),$$

k, l are positive integers, $k = 2s-1$, $l = 2(n_0-s)-1$, i.e. $k+l = 2n_0$ and β is the inverse temperature. This expression for E is the generalized BF contour bound.

We show [10] that E diverges at infinity in g and that means that the unit spin and oscillator lro occur in the system at sufficiently large g. We establish that

$$q_x^k q_y^l + q_x^l q_y^k = q_x^{2n_0} + q_y^{2n_0} - (q_x - q_y)^2 Q(q_x, q_y),$$

where Q is a homogeneous positive polynomial. This means that the effective external potential $u(q) - gq^{2n_0}$ has minima whose depth depends on g. This representation implies that ground states are ferromagnetic.

More interesting systems are described by finite-range U' which has non-ferromagnetic terms. In this case the above arguments are not useful. Let us consider the case

$$\phi(q_x,q_y) = g_0(q_x-q_y)^{2n_1}Q'(q_x,q_y), \qquad g_0 = g^{\frac{\xi}{2(n-n_0)}} = z^{-\xi}, \qquad u(q) = u_0(q) - gq^{2n_0},$$

where u_0 is a bounded below even polynomial with $2n$ degree, Q' is the positive symmetric even homogeneous polynomial with the degree $2n_2$, $n_2 < n - n_1$ and $Q(1,1) = 1$. The case of non-positive translation invariant U' small for large g and $Q' = 1$, $n_1 = 1$, $n_0 = 1$, $g_0 = g$, $u_0 = \eta q^{2n}$ was considered in [6, 10–11].

A derivation of the contour bound was based on the superstability bound for the rescaled correlation function (the variables are rescaled by $g^{-\frac{1}{2}}$) translated by the minimum e_0 of the rescaled external field u_g and the analog of (2)

$$\chi^+(q_x)\chi^-(q_y) \le \exp\{\beta[Q_g(q_x,q_y) - e_0]\}, \tag{3}$$

where

$$Q_g(q_x,q_y)=e_0^{-1}\left[(q_x-q_y)^2+\frac{4}{3}\left(\left|q_x^2-e_0^2\right|+\left|q_y^2-e_0^2\right|\right)\right].$$

It is derived easily from two inequalities ($R=e_0$, $c=\beta e_0^{-1}$)

$$\chi^+(q_x)\chi^-(q_y)\le e^{-c[R^2-(q_x-q_y)^2]},\qquad |q_x|,|q_y|\ge 2^{-1}R,$$
$$\chi^+(q_x)\chi^-(q_y)\le e^{-c\left[R^2-\frac{4}{3}(|q_x^2-R^2|+|q_y^2-R^2|)\right]},\qquad |q_x|,|q_y|\le 2^{-1}R.$$

For $|q_x|\le\frac{R}{2}$, $|q_y|\ge\frac{R}{2}$ the second term in the expression for Q is not less than cR^2.

More precisely, we demand that the following super-stability and regularity conditions hold

$$U(q_\Lambda)\ge\sum_{x\in\Lambda}u^-(q_x),\qquad u^-(q)=u(q)-\zeta v^0(zq)-\zeta_0,\qquad z=g^{-\frac{1}{2(n-1)}},$$

$$|W'(q_X;q_Y)|\le\sum_{x\in X,\,y\in Y}\Psi'(|x-y|)\left(v^0(z^{1-n}q_x)+v^0(z^{1-n}q_y)\right),\qquad v^0(q)=\sum_{j=1}^{n-1}q^{2j},$$

where ζ, ζ_0 are positive constants, $\Psi'\in L^1(\mathbb{Z}^d)$ and

$$W'(q_X;q_Y)=U'(q_\Lambda)-U'(q_Y)-U'(q_X),\qquad \Lambda=X\cup Y.$$

The second term in the expression for u^- guarantees that the contribution of the non-positive part of U' is small for large g.

Let $\rho_*^\Lambda(q_X)$, U_*, u_*, u_*^- denote, respectively, the rescaled and translated by e_0, correlation functions, potential energy, potentials u and u^- and put. From the translation invariant character of the Lebesque measure it follows that ρ_*^Λ is expressed in terms of U_*. In our method we have to rely on the following important theorem [6].

Theorem 1. *For the correlation functions ρ_*^Λ the following superstability bound is valid*

$$\rho_*^\Lambda(q_X)\le\exp\left\{|X|e_*(g)-\beta\left[U_*^+(q_X)+\sum_{x\in X}u_*^+(q_x)\right]\right\},$$

where

$$u_*^+=u_*^--3\epsilon v^0,\qquad \epsilon=z^{2n(n-1)},$$

$e_*(g)$ depends neither on oscillator variables nor Λ, U_*^+ is the positive part of U_* generated by a pair potential and for arbitrary $\delta>0$ and sufficiently large g

$$\delta e_0\ge e_*(g).$$

From (3) and the superstability bound it follows that for $e_0>1$ the contour bound holds with

$$E=\beta e_0-e_*(g)+2\ln I_*\big(g,Q^0\big),$$

where

$$I_*(g,Q^0)=\int e^{-\beta[u_*^+(q)-e_0^{-1}Q^0(q;e_0)]}dq,\qquad Q^0(q;e_0)=\frac{4}{3}|q(q+2e_0)|.$$

The term with U_*^+ in the super-stability bound plays an important role: it cancels the term generated by the first term in the expression for Q_g. It is not difficult to see that $\ln I_*(g, Q^0)$ diverges at infinity in g as δe_0, where $\delta > 0$ is arbitrary small. That is, E diverges in g at infinity as e_0, which is proportional to $g^{\frac{n}{2(n-1)}}$, and this proves occurrence of lro. If one uses the usual Ruelle superstability [12, 13] bound without the term U_*^+ in its right-hand side then the first term in Q_g yields an additional term $\beta e_0^{-1} q^2$ under the sign of the exponent in the expression for I_* and the method fails.

By the similar arguments we prove occurrence of lro for the above pair potential ϕ and non-translation invariant U' satisfying the superstability and regularity conditions guaranteeing that non-ferromagnetic part of U' is small at large g [14]. We also establish how g_0 has to depend on g in order to preserve lro. g_0 should not be very small to permit U_*^+ to cancel the contribution of the first term in the expression for Q_g in and it cannot be very large because it has to permit the rescaled and translated by e_0 ferromagnetic n-n interaction to satisfy the uniform in g regularity condition.

Resume. Our results show that the ferromagnetic lro in lattice linear oscillator systems occurs only if a depth of the effective potential and the strength of the polynomial ferromagnetic n-n interaction are large and correlated and the non-ferromagnetic part of the potential energy is sufficiently small.

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Generalizations of Schouten–Nijenhuis Bracket

Dmitrij V. SOROKA and Vyacheslav A. SOROKA

Kharkiv Institute of Physics and Technology, 61108 Kharkiv, Ukraine
E-mail: *dsoroka@kipt.kharkov.ua, vsoroka@kipt.kharkov.ua*

The Schouten–Nijenhuis bracket is generalized for the superspace case and for the Poisson brackets of opposite Grassmann parities.

1 Introduction

Recently a recipe for the construction of new Poisson brackets from the bracket with a definite Grassmann parity was proposed [1]. This recipe is based on the use of exterior differentials of diverse Grassmann parities. It was indicated in [1] that this recipe leads to the generalizations of the Schouten–Nijenhuis bracket [2–8] on both the superspace case and the case of the brackets with diverse Grassmann parities. In the present report we give the details of these generalizations[1].

2 Poisson brackets related with the exterior differentials

Let us recall the recipe for the construction from a given Poisson bracket of a Grassmann parity $\epsilon \equiv 0, 1 \pmod 2$ of another one.

A Poisson bracket, having a Grassmann parity ϵ, written in arbitrary non-canonical phase variables z^a

$$\{A, B\}_\epsilon = A \overleftarrow{\partial}_{z^a} \omega_\epsilon^{ab}(z) \overrightarrow{\partial}_{z^b} B, \tag{1}$$

where $\overleftarrow{\partial}$ and $\overrightarrow{\partial}$ are right and left derivatives respectively, has the following main properties:

$$g(\{A, B\}_\epsilon) \equiv g_A + g_B + \epsilon \pmod 2,$$
$$\{A, B\}_\epsilon = -(-1)^{(g_A+\epsilon)(g_B+\epsilon)}\{B, A\}_\epsilon,$$
$$\sum_{(ABC)} (-1)^{(g_A+\epsilon)(g_C+\epsilon)}\{A, \{B, C\}_\epsilon\}_\epsilon = 0,$$

which lead to the corresponding relations for the matrix ω_ϵ^{ab}

$$g\left(\omega_\epsilon^{ab}\right) \equiv g_a + g_b + \epsilon \pmod 2, \tag{2}$$
$$\omega_\epsilon^{ab} = -(-1)^{(g_a+\epsilon)(g_b+\epsilon)}\omega_\epsilon^{ba}, \tag{3}$$
$$\sum_{(abc)} (-1)^{(g_a+\epsilon)(g_c+\epsilon)}\omega_\epsilon^{ad}\partial_{z^d}\omega_\epsilon^{bc} = 0, \tag{4}$$

where $\partial_{z^a} \equiv \partial/\partial z^a$ and $g_a \equiv g(z^a)$, $g_A \equiv g(A)$ are the corresponding Grassmann parities of phase coordinates z^a and a quantity A and a sum with a symbol (abc) under it designates a summation

[1] Concerning the generalizations of the Schouten–Nijenhuis bracket see also [9, 10].

over cyclic permutations of a, b and c. We shall consider the non-degenerate matrix ω_ϵ^{ab} which has an inverse matrix $\omega_{ab}^\epsilon(-1)^{g_b\epsilon}$ (a grading factor is chosen for the convenience)

$$\omega_\epsilon^{ab}\omega_{bc}^\epsilon(-1)^{g_c\epsilon} = \delta_c^a$$

(there is no summation over ϵ in the previous relation) with the properties

$$\begin{aligned}
&g(\omega_{ab}^\epsilon) \equiv g_a + g_b + \epsilon \pmod 2,\\
&\omega_{ab}^\epsilon = (-1)^{(g_a+1)(g_b+1)}\omega_{ba}^\epsilon,\\
&\sum_{(abc)}(-1)^{(g_a+1)g_c}\partial_{z^a}\omega_{bc}^\epsilon = 0.
\end{aligned}$$

The Hamilton equations for the phase variables z^a, which correspond to a Hamiltonian H_ϵ ($g(H_\epsilon) = \epsilon$),

$$\frac{dz^a}{dt} = \{z^a, H_\epsilon\}_\epsilon = \omega_\epsilon^{ab}\partial_{z^b}H_\epsilon \tag{5}$$

can be represented in the form

$$\frac{dz^a}{dt} = \omega_\epsilon^{ab}\partial_{z^b}H_\epsilon \equiv \omega_\epsilon^{ab}\frac{\partial(d_\zeta H_\epsilon)}{\partial(d_\zeta z^b)} \stackrel{\text{def}}{=} (z^a, d_\zeta H_\epsilon)_{\epsilon+\zeta}, \tag{6}$$

where d_ζ ($\zeta = 0, 1$) is one of the exterior differentials d_0 or d_1, which have opposite Grassmann parities 0 and 1 respectively and following symmetry properties with respect to the ordinary multiplication

$$\begin{aligned}
&d_0z^a d_0z^b = (-1)^{g_ag_b}d_0z^b d_0z^a,\\
&d_1z^a d_1z^b = (-1)^{(g_a+1)(g_b+1)}d_1z^b d_1z^a
\end{aligned} \tag{7}$$

and exterior products

$$\begin{aligned}
&d_0z^a \wedge d_0z^b = (-1)^{g_ag_b+1}d_0z^b \wedge d_0z^a,\\
&d_1z^a \tilde\wedge d_1z^b = (-1)^{(g_a+1)(g_b+1)}d_1z^b \tilde\wedge d_1z^a.
\end{aligned} \tag{8}$$

We use different notations $\wedge$ and $\tilde\wedge$ for the exterior products of d_0z^a and d_1z^a respectively.

By taking the exterior differential d_ζ from the Hamilton equations (5), we obtain

$$\frac{d(d_\zeta z^a)}{dt} = (d_\zeta\omega_\epsilon^{ab})\frac{\partial(d_\zeta H_\epsilon)}{\partial(d_\zeta z^b)} + (-1)^{\zeta(g_a+\epsilon)}\omega_\epsilon^{ab}\partial_{z^b}(d_\zeta H_\epsilon) \stackrel{\text{def}}{=} (d_\zeta z^a, d_\zeta H_\epsilon)_{\epsilon+\zeta}. \tag{9}$$

As a result of equations (6) and (9) we have by definition the following binary composition for functions F and H of the variables z^a and their differentials $d_\zeta z^a \equiv y_\zeta^a$

$$(F,H)_{\epsilon+\zeta} = F\Big[\overleftarrow{\partial}_{z^a}\,\omega_\epsilon^{ab}\,\overrightarrow{\partial}_{y_\zeta^b} + (-1)^{\zeta(g_a+\epsilon)}\,\overleftarrow{\partial}_{y_\zeta^a}\,\omega_\epsilon^{ab}\,\overrightarrow{\partial}_{z^b} + \overleftarrow{\partial}_{y_\zeta^a}\,y_\zeta^c\left(\partial_{z^c}\omega_\epsilon^{ab}\right)\overrightarrow{\partial}_{y_\zeta^b}\Big]H. \tag{10}$$

By using relations (2)–(4) for the matrix ω_ϵ^{ab}, we can establish the following properties for the binary composition (10)

$$\begin{aligned}
&g[(F,H)_{\epsilon+\zeta}] \equiv g_F + g_H + \epsilon + \zeta \pmod 2,\\
&(F,H)_{\epsilon+\zeta} = -(-1)^{(g_F+\epsilon+\zeta)(g_H+\epsilon+\zeta)}(H,F)_{\epsilon+\zeta},\\
&\sum_{(EFH)}(-1)^{(g_E+\epsilon+\zeta)(g_H+\epsilon+\zeta)}(E,(F,H)_{\epsilon+\zeta})_{\epsilon+\zeta} = 0,
\end{aligned}$$

which mean that the composition (10) satisfies all the main properties for the Poisson bracket with the Grassmann parity equal to $\epsilon+\zeta$. Thus, the application of the exterior differentials of opposite Grassmann parities to the given Poisson bracket results in the brackets of the different Grassmann parities.

By transition to the co-differential variables $y_a^{\epsilon+\zeta}$, related with differentials y_ζ^a by means of the matrix ω_ϵ^{ab}

$$y_\zeta^a = y_b^{\epsilon+\zeta}\omega_\epsilon^{ba}, \tag{11}$$

the Poisson bracket (10) takes a canonical form[2]

$$(F,H)_{\epsilon+\zeta} = F\left[\overleftarrow{\partial}_{z^a}\overrightarrow{\partial}_{y_a^{\epsilon+\zeta}} - (-1)^{g_a(g_a+\epsilon+\zeta)}\overleftarrow{\partial}_{y_a^{\epsilon+\zeta}}\overrightarrow{\partial}_{z^a}\right]H, \tag{12}$$

that can be proved with the use of the Jacobi identity (4).

The bracket (10) is given on the functions of the variables z^a, y_ζ^a

$$F = \sum_p \frac{1}{p!} y_\zeta^{a_p}\cdots y_\zeta^{a_1} f_{a_1\ldots a_p}(z), \qquad g(f_{a_1\ldots a_p}) = g_f + g_{a_1} + \cdots + g_{a_p},$$

whereas this bracket, rewritten in the form (12), is given on the functions of variables z^a and $y_a^{\epsilon+\zeta}$

$$F = \sum_p \frac{1}{p!} y_{a_p}^{\epsilon+\zeta}\cdots y_{a_1}^{\epsilon+\zeta} f^{a_1\ldots a_p}(z), \qquad g(f^{a_1\ldots a_p}) = g_f + \epsilon p + g_{a_1} + \cdots + g_{a_p}.$$

We do not exclude a possibility of the own Grassmann parity $g_f \equiv g(f)$ for a quantity f. By taking into account relation (11), we have the following rule for the rising of indices:

$$f^{b_1\ldots b_p} = (-1)^{\sum\limits_{k=1}^{p-1}[g_{b_1}+\cdots+g_{b_k}+k(\epsilon+\zeta)](g_{b_{k+1}}+g_{a_{k+1}}+\epsilon)}\omega_\epsilon^{b_p a_p}\cdots\omega_\epsilon^{b_1 a_1} f_{a_1\ldots a_p}.$$

Note that the quantities $f_{a_1\ldots a_p}$ and $f^{a_1\ldots a_p}$ have in general the different symmetry and parity properties.

In the case $\zeta = 1$, due to relations (7), (8), the terms in the decomposition of a function $F(z^a, y_1^a)$ into degrees p of the variables y_1^a

$$F = \sum_p \frac{1}{p!} y_1^{a_p}\cdots y_1^{a_1} f_{a_1\ldots a_p}(z)$$

can be treated as p-forms and the bracket (10) can be considered as a Poisson bracket on p-forms so that being taken between a p-form and a q-form it results in a $(p+q-1)$-form[3]. Thus, the bracket (10) is a generalization of the bracket introduced in [12] on the superspace case and on the case of the brackets (1) with arbitrary Grassmann parities ϵ ($\epsilon = 0, 1$).

3 Generalizations of the Schouten–Nijenhuis bracket

If we take the bracket in the canonical form (12), then we obtain the generalizations of the Schouten–Nijenhuis bracket [2,3] (see also [4–8,12]) onto the cases of superspace and the brackets

[2]There is no summation over ϵ in relation (11).

[3]Concerning a Poisson bracket between 1-forms and its relation with the Lie bracket of vector fields see in the book [11].

of diverse Grassmann parities. Indeed, let us consider the bracket (12) between monomials F and H having respectively degrees p and q

$$F = \frac{1}{p!} y_{a_p}^{\epsilon+\zeta} \cdots y_{a_1}^{\epsilon+\zeta} f^{a_1 \ldots a_p}(z), \qquad g(f^{a_1 \ldots a_p}) = g_f + p\epsilon + g_{a_1} + \cdots + g_{a_p},$$

$$H = \frac{1}{q!} y_{a_q}^{\epsilon+\zeta} \cdots y_{a_1}^{\epsilon+\zeta} h^{a_1 \ldots a_q}(z), \qquad g(h^{a_1 \ldots a_q}) = g_h + q\epsilon + g_{a_1} + \cdots + g_{a_q}.$$

Then as a result we obtain

$$\begin{aligned}(F,H)_{\epsilon+\zeta} = {} & \frac{(-1)^{[g_{b_1}+\cdots+g_{b_{q-1}}+(q-1)(\epsilon+\zeta)](g_f+g_l+p\zeta)}}{p!(q-1)!} \\ & \times y_{b_{q-1}}^{\epsilon+\zeta} \cdots y_{b_1}^{\epsilon+\zeta} y_{a_p}^{\epsilon+\zeta} \cdots y_{a_1}^{\epsilon+\zeta} \left(f^{a_1 \ldots a_p} \overleftarrow{\partial}_{z^l} \right) h^{b_1 \ldots b_{q-1} l} \\ & - \frac{(-1)^{(g_l+\epsilon+\zeta)(g_f+p\epsilon+g_{a_2}+\cdots+g_{a_p})+[g_{b_1}+\cdots+g_{b_q}+q(\epsilon+\zeta)][g_f+\epsilon+(p-1)\zeta]}}{(p-1)!q!} \\ & \times y_{b_q}^{\epsilon+\zeta} \cdots y_{b_1}^{\epsilon+\zeta} y_{a_p}^{\epsilon+\zeta} \cdots y_{a_2}^{\epsilon+\zeta} f^{l a_2 \ldots a_p} \partial_{z^l} h^{b_1 \ldots b_q}. \end{aligned} \tag{13}$$

3.1 Particular cases

Let us consider the formula (13) for the particular values of ϵ and ζ.

1. We start from the case which leads to the usual Schouten–Nijenhuis bracket for the skew-symmetric contravariant tensors. In this case, when $\epsilon = 0$, $\zeta = 1$ and the matrix $\omega_0^{ab}(x) = -\omega_0^{ba}(x)$ corresponds to the usual Poisson bracket for the commuting coordinates $z^a = x^a$, we have

$$\begin{aligned}(F,H)_1 = {} & \frac{(-1)^{(q-1)(g_f+p)}}{p!(q-1)!} \Theta_{b_{q-1}} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_1} \left(f^{a_1 \ldots a_p} \overleftarrow{\partial}_{x^l} \right) h^{b_1 \ldots b_{q-1} l} \\ & - \frac{(-1)^{g_f(q+1)+q(p-1)}}{(p-1)!q!} \Theta_{b_q} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_2} f^{l a_2 \ldots a_p} \partial_{x^l} h^{b_1 \ldots b_q}, \end{aligned} \tag{14}$$

where $\Theta_a \equiv y_a^1$ are Grassmann co-differential variables related owing to (11) with the Grassmann differential variables $\Theta^a \equiv d_1 x^a$

$$\Theta^a = \Theta_b \omega_0^{ba}.$$

When Grassmann parities of the quantities f and h are equal to zero $g_f = g_h = 0$, we obtain from (14)

$$(F,H)_1 \stackrel{\text{def}}{=} (-1)^{(p+1)q+1} \Theta_{a_{p+q}} \cdots \Theta_{a_2} [F,H]^{a_2 \ldots a_{p+q}},$$

where $[F,H]^{a_2 \ldots a_{p+q}}$ are components of the usual Schouten–Nijenhuis bracket (see, for example, [7]) for the contravariant antisymmetric tensors[4]. This bracket has the following symmetry property

$$[F,H] = (-1)^{pq} [H,F]$$

and satisfies the Jacobi identity

$$\sum_{(FHE)} (-1)^{ps} [[F,H],E] = 0,$$

where s is a degree of a monomial E.

[4]Here and below we use the same notation $[F,H]$ for the different brackets. We hope that this will not lead to the confusion.

2. In the case $\epsilon = \zeta = 0$ and $\omega_0^{ab}(x) = -\omega_0^{ba}(x)$ we obtain the bracket for symmetric contravariant tensors (see, for example, [6])

$$(F,H)_0 = \frac{1}{p!(q-1)!} y^0_{b_{q-1}} \cdots y^0_{b_1} y^0_{a_p} \cdots y^0_{a_1} \left(\partial_{x^l} f^{a_1 \ldots a_p}\right) h^{b_1 \ldots b_{q-1} l}$$
$$- \frac{1}{(p-1)!q!} y^0_{b_q} \cdots y^0_{b_1} y^0_{a_p} \cdots y^0_{a_2} f^{l a_2 \ldots a_p} \partial_{x^l} h^{b_1 \ldots b_q} \stackrel{\text{def}}{=} y^0_{a_{p+q}} \cdots y^0_{a_2} [F,H]^{a_2 \ldots a_{p+q}},$$

where commuting co-differentials y^0_a connected with commuting differentials $y^0_0 \equiv d_0 x^a$ in accordance with (11)

$$y^a_0 = y^0_b \omega^{ba}_0$$

and the bracket $[F,H]^{a_2 \ldots a_{p+q}}$ has the following symmetry property

$$[F,H] = -(-1)^{g_f g_h} [H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{g_e g_h} [E,[F,H]] = 0.$$

3. By taking the Martin bracket [13] $\omega_0^{ab}(\theta) = \omega_0^{ba}(\theta)$ with Grassmann coordinates $z^a = \theta^a$ ($g_a = 1$) as an initial bracket (1), we have in the case $\zeta = 0$ for antisymmetric contravariant tensors on the Grassmann algebra

$$(F,H)_0 = \frac{(-1)^{(q-1)(g_f+1)}}{p!(q-1)!} \Theta_{b_{q-1}} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_1} (f^{a_1 \ldots a_p} \overleftarrow{\partial}_{\theta^l}) h^{b_1 \ldots b_{q-1} l}$$
$$+ \frac{(-1)^{(q-1)g_f + p}}{(p-1)!q!} \Theta_{b_q} \cdots \Theta_{b_1} \Theta_{a_p} \cdots \Theta_{a_2} f^{l a_2 \ldots a_p} \partial_{\theta^l} h^{b_1 \ldots b_q}$$
$$\stackrel{\text{def}}{=} \Theta_{a_{p+q}} \cdots \Theta_{a_2} [F,H]^{a_2 \ldots a_{p+q}},$$

where the Grassmann co-differentials Θ_a related with the Grassmann differentials Θ^a as

$$d_0 \theta^a \equiv \Theta^a = \Theta_b \omega^{ba}_0.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{g_f g_h} [H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{g_e g_h} [E,[F,H]] = 0.$$

4. By taking the Martin bracket again, in the case $\zeta = 1$

$$d_1 \theta^a \equiv x^a = x_b \omega^{ba}_0$$

we obtain for the symmetric tensors on Grassmann algebra

$$(F,H)_1 = \frac{1}{p!(q-1)!} x_{b_{q-1}} \cdots x_{b_1} x_{a_p} \cdots x_{a_1} (f^{a_1 \ldots a_p} \overleftarrow{\partial}_{\theta^l}) h^{b_1 \ldots b_{q-1} l}$$
$$- \frac{1}{(p-1)!q!} x_{b_q} \cdots x_{b_1} x_{a_p} \cdots x_{a_2} f^{l a_2 \ldots a_p} \partial_{\theta^l} h^{b_1 \ldots b_q}$$
$$\stackrel{\text{def}}{=} x_{a_{p+q}} \cdots x_{a_2} [F,H]^{a_2 \ldots a_{p+q}}.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H]=-(-1)^{(g_f+p+1)(g_h+q+1)}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)}(-1)^{(g_e+s+1)(g_h+q+1)}[E,[F,H]]=0.$$

5. In general, if we take the even bracket in superspace with coordinates $z^a=(x,\theta)$, where x and θ are respectively commuting and anticommuting (Grassmann) variables, then in the case $\zeta=1$ we have

$$\begin{aligned}(F,H)_1&=\frac{(-1)^{(g_{b_1}+\cdots+g_{b_{q-1}}+q-1)(g_f+g_l+p)}}{p!(q-1)!}y^1_{b_{q-1}}\cdots y^1_{b_1}y^1_{a_p}\cdots y^1_{a_1}(f^{a_1\ldots a_p}\overleftarrow{\partial}_{z^l})h^{b_1\ldots b_{q-1}l}\\&-\frac{(-1)^{(g_l+1)(g_f+g_{a_2}+\cdots+g_{a_p})+(g_{b_1}+\cdots+g_{b_q}+q)(g_f+p-1)}}{(p-1)!q!}y^1_{b_q}\cdots y^1_{b_1}y^1_{a_p}\cdots y^1_{a_2}f^{la_2\ldots a_p}\partial_{z^l}h^{b_1\ldots b_q}\\&\stackrel{\text{def}}{=}y^1_{a_{p+q}}\cdots y^1_{a_2}[F,H]^{a_2\ldots a_{p+q}},\end{aligned}$$

where

$$d_1z^a\equiv y^a_1=y^1_b\omega^{ba}_0.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H]=-(-1)^{(g_f+p+1)(g_h+q+1)}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)}(-1)^{(g_e+s+1)(g_h+q+1)}[E,[F,H]]=0.$$

6. In the case of the even bracket in superspace as initial one with $\zeta=0$ we obtain

$$\begin{aligned}(F,H)_0&=\frac{(-1)^{(g_{b_1}+\cdots+g_{b_{q-1}})(g_f+g_l)}}{p!(q-1)!}y^0_{b_{q-1}}\cdots y^0_{b_1}y^0_{a_p}\cdots y^0_{a_1}(f^{a_1\ldots a_p}\overleftarrow{\partial}_{z^l})h^{b_1\ldots b_{q-1}l}\\&-\frac{(-1)^{g_l(g_f+g_{a_2}+\cdots+g_{a_p})+g_f(g_{b_1}+\cdots+g_{b_q})}}{(p-1)!q!}y^0_{b_q}\cdots y^0_{b_1}y^0_{a_p}\cdots y^0_{a_2}f^{la_2\ldots a_p}\partial_{z^l}h^{b_1\ldots b_q}\\&\stackrel{\text{def}}{=}y^0_{a_{p+q}}\cdots y^0_{a_2}[F,H]^{a_2\ldots a_{p+q}},\end{aligned}$$

where

$$d_0z^a\equiv y^a_0=y^0_b\omega^{ba}_0.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H]=-(-1)^{g_fg_h}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)}(-1)^{g_eg_h}[E,[F,H]]=0.$$

7. Taking as an initial bracket the odd Poisson bracket in superspace with coordinates z^a, for the case $\zeta = 0$ we have

$$(F,H)_1 = \frac{(-1)^{(g_{b_1}+\cdots+g_{b_{q-1}}+q-1)(g_f+g_l)}}{p!(q-1)!} y^1_{b_{q-1}} \cdots y^1_{b_1} y^1_{a_p} \cdots y^1_{a_1} (f^{a_1\ldots a_p} \overleftarrow{\partial}_{z^l}) h^{b_1\ldots b_{q-1}l}$$
$$- \frac{(-1)^{(g_l+1)(g_f+p+g_{a_2}+\cdots+g_{a_p})+(g_f-1)(g_{b_1}+\cdots+g_{b_q}+q)}}{(p-1)!q!}$$
$$\times y^1_{b_q} \cdots y^1_{b_1} y^1_{a_p} \cdots y^1_{a_2} f^{la_2\ldots a_p} \partial_{z^l} h^{b_1\ldots b_q} \stackrel{\text{def}}{=} y^1_{a_{p+q}} \cdots y^1_{a_2} [F,H]^{a_2\ldots a_{p+q}},$$

where

$$d_0 z^a \equiv y^a_0 = y^1_b \omega^{ba}_1.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{(g_f+1)(g_h+1)}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{(g_e+1)(g_h+1)}[E,[F,H]] = 0.$$

8. At last for the odd Poisson bracket in superspace, taking as an initial one, we obtain in the case $\zeta = 1$

$$(F,H)_0 = (-1)^{(g_{b_1}+\cdots+g_{b_{q-1}})(g_f+p)} \left[\frac{1}{p!(q-1)!} y^0_{b_{q-1}} \cdots y^0_{b_1} y^0_{a_p} \cdots y^0_{a_1} (f^{a_1\ldots a_p} \overleftarrow{\partial}_{z^l}) h^{lb_1\ldots b_{q-1}} \right.$$
$$\left. - \frac{(-1)^{(g_f+p)(g_l+g_{b_q})}}{(p-1)!q!} y^0_{b_q} \cdots y^0_{b_1} y^0_{a_p} \cdots y^0_{a_2} f^{a_2\ldots a_p l} \partial_{z^l} h^{b_1\ldots b_q} \right]$$
$$\stackrel{\text{def}}{=} y^0_{a_{p+q}} \cdots y^0_{a_2} [F,H]^{a_2\ldots a_{p+q}},$$

where

$$d_1 z^a \equiv y^a_1 = y^0_b \omega^{ba}_1.$$

The bracket $[F,H]$ has the following symmetry property

$$[F,H] = -(-1)^{(g_f+p)(g_h+q)}[H,F]$$

and satisfies the Jacobi identity

$$\sum_{(EFH)} (-1)^{(g_e+s)(g_h+q)}[E,[F,H]] = 0.$$

Thus, we see that the formula (13) contains as particular cases quite a number of the Schouten–Nijenhuis type brackets.

4 Conclusion

We give the recipe for the construction from a given Poisson bracket of the definite Grassmann parity another bracket. For this construction we use the exterior differentials with different Grassmann parities. We proved that the resulting Poisson bracket essentially depends on the parity of the exterior differential in spite of these differentials give the same exterior calculus [1]. The recipe leads to the set of different generalizations for the Schouten–Nijenhuis bracket. Thus, we see that the Schouten–Nijenhuis bracket and its possible generalizations are particular cases of the usual Poisson brackets of different Grassmann parities (12). We hope that these generalizations will find their own application for the deformation quantization (see, for example, [7,14]) as well as the usual Schouten–Nijenhuis bracket.

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The Role of Symmetry in the Regularity Properties of Optimal Controls

Delfim F.M. TORRES

Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
E-mail: *delfim@mat.ua.pt*

The role of symmetry is well studied in physics and economics, where many great contributions have been made. With the help of Emmy Noether's celebrated theorems, a unified description of the subject can be given within the mathematical framework of the calculus of variations. It turns out that Noether's principle can be understood as a special application of the Euler–Lagrange differential equations. We claim that this modification of Noether's approach has the advantage to put the role of symmetry on the basis of the calculus of variations, and in a key position to give answers to some fundamental questions. We will illustrate our point with the interplay between the concept of invariance, the theory of optimality, Tonelli existence conditions, and the Lipschitzian regularity of minimizers for the autonomous basic problem of the calculus of variations. We then proceed to the general nonlinear situation, by introducing a concept of symmetry for the problems of optimal control, and extending the results of Emmy Noether to the more general framework of Pontryagin's maximum principle. With such tools, new results regarding Lipschitzian regularity of the minimizing trajectories for optimal control problems with nonlinear dynamics are obtained.

1 Introduction

A natural first step when dealing with an optimal control problem is to apply the existence theorems. For many interesting and important optimal control problems, such as those of the calculus of variations, the existence does not assure the validity of first-order optimality conditions. It may be the case, for example, that the minimizers whose existence is guaranteed by the existence theorem fail to satisfy Pontryagin's maximum principle. Obviously, the study of regularity conditions which close the discrepancy between the hypotheses of existence and necessary optimality theories, is of a crucial importance. This paper contains a survey of the recent results obtained by the author.

We give a new perspective on the results concerning the regularity of the minimizing trajectories. Our claim is that those results are fruit of the existence of certain symmetry properties, and follow from appropriate conservation laws. Such conservation laws are explained by using the recent extensions to optimal control of the classical symmetry theorem of E. Noether.

Emmy Amalie Noether was the one to prove, in 1918, that conservation laws in the calculus of variations are the manifestation of a powerful and universal principle:

> "When a system shows up symmetry with respect to a parameter-transformation, then there exists a conservation law for that system associated with the invariance property".

This assertion comprises all theorems on first integrals known to classical mechanics. Typical application of those results is to lower the order of the differential equations. Here we claim that the conservation laws are also useful for different purposes. We claim that they can be used with success to prove Lipschitzian regularity of the minimizing trajectories.

Noether's principle is so deep and rich, that it can be formalized as a theorem in many different contexts, and in each of such contexts under many different assumptions (see e.g. [12]).

Formulations of Noether's symmetry principle to the more general framework of optimal control have been recently obtained (see e.g. [7, 8, 10]). In this context conservation laws are interpreted as quantities which are preserved along all the Pontryagin extremals. They play here a fundamental role in order to bring together the regularity results in [6, 11].

2 Optimal control and the calculus of variations

The optimal control problem consists in finding a r-vector function $u(\cdot)$, called the control, and the corresponding n-state trajectory $x(\cdot)$, solution of a dynamical system described by n ordinary differential equations under specified boundary conditions, in such a way the pair $(x(\cdot), u(\cdot))$ minimizes a given integral functional:

$$\begin{aligned} &I[x(\cdot), u(\cdot)] = \int_a^b L\left(t, x(t), u(t)\right) \mathrm{d}t \longrightarrow \min, \\ &\dot{x}\left(t\right) = \varphi\left(t, x\left(t\right), u\left(t\right)\right), \qquad (x(a), x(b)) \in \mathcal{F}, \\ &x\left(\cdot\right) \in \mathcal{X}\left([a,b]\,; \mathbb{R}^n\right), \qquad u\left(\cdot\right) \in \mathcal{U}\left([a,b]\,; \Omega \subseteq \mathbb{R}^r\right). \end{aligned} \tag{1}$$

The Lagrangian L and the velocity vector φ are assumed to be smooth functions:

$$C^1 \ni L : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}, \qquad C^1 \ni \varphi : [a,b] \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n.$$

For the particular case when there are no restrictions on the control value, $\Omega = \mathbb{R}^r$, and the control system is just $\varphi(t,x,u) = u$, one obtains the fundamental problem of the calculus of variations ($r = n$),

$$\begin{aligned} &I[x(\cdot), u(\cdot)] = \int_a^b L\left(t, x(t), u(t)\right) \mathrm{d}t \longrightarrow \min, \\ &\dot{x}(t) = u(t), \qquad (x(a), x(b)) \in \mathcal{F}, \qquad x\left(\cdot\right) \in \mathcal{X}\left([a,b]\,; \mathbb{R}^n\right), \end{aligned}$$

which includes all classical mechanics. Clearly, for the optimal control problem to be well defined, it is crucial to specify the class of functions $\mathcal{X}$ and $\mathcal{U}$. This is addressed in the following section.

3 Logical-deductive approach, discrepancy, and bad behavior

The logical-deductive approach for the resolution of an optimization problem is the following:

1. A solution exists for the problem.
2. The necessary optimality conditions are applicable and they identify certain candidates (in the context of the calculus of variations and optimal control called *extremals*).
3. Elimination, if necessary, identifies the solution or solutions.

The historical facts show that mathematics does not always advance using the logical steps: both the calculus of variations and the mathematical theory of optimal control have born from the study of necessary optimality conditions, the existence question being delayed. In the calculus of variations the first obtained necessary conditions were the Euler–Lagrange differential equations, first proved by L. Euler in 1744, while the existence was only addressed two centuries later, in 1911, by L. Tonelli. The mathematical theory of optimal control, the modern face of the calculus of variations, was born with the Pontryagin maximum principle, proved by V.G. Boltyanski, R.V. Gamkrelidze, and L.S. Pontryagin in 1956 [2], while existence was solved three years later, in 1959, by A.F. Filippov [5]. It is very interesting the explanation why the study of necessary conditions have been first in time. It turns out that existence and necessary optimality theories rely on different classes $\mathcal{X}$ and $\mathcal{U}$:

- In the calculus of variations the existence is assured in the class of absolutely continuous functions ($\mathcal{X} = W_{1,1}$), while the arguments which lead to the necessary conditions demand the minimizing trajectory to be at least Lipschitzian ($\mathcal{X} \subseteq W_{1,\infty}$).
- In optimal control the existence is assured in the class of integrable controls ($\mathcal{U} = L_1$), while the classical formulation of the Pontryagin maximum principle assumes that the optimal controls are essentially bounded ($\mathcal{U} \subseteq L_\infty$).

The discrepancy between the hypotheses of existence and optimality theories is not the result of technical details, but has an intrinsic nature. The gap is a reality, even for problems very innocent in aspect. For example, the minimizers predicted by the existence theory can violate the necessary conditions even when the Lagrangian L is a polynomial and φ is linear. The first problem showing this possibility of bad behavior was proposed by J. Ball and V. Mizel in 1985.

Example 1 (cf. [1]). For the Ball–Mizel problem,

$$\int_0^1 \left(\left|x^3 - t^2\right|^2 |u|^{14} + \varepsilon\, |u|^2\right) dt \to \min,$$
$$\dot{x}(t) = u(t), \qquad x(0) = 0, \qquad x(1) = k,$$

it happens:

- All hypotheses of Tonelli's existence theorem are satisfied.
- For some choices of the constants ε and k one has the unique optimal control $u(t) = k\, t^{-1/3}$.
- Pontryagin maximum principle (Euler–Lagrange equation in integral form) is not satisfied since $\dot{\psi}(t) = L_x(t, x(t), \dot{x}(t)) = c\, t^{-4/3}$ is not integrable.

The study of conditions which assure the regularity of the minimizers is important, and excludes occurrence of bad behavior.

4 Regularity of minimizers

A *regularity result* is any assertion assuring that the solution belongs to a class of functions more restrict than the one considered in the formulation of the problem (e.g., $\tilde{x}(\cdot) \in W_{1,\infty}$ instead of $W_{1,1}$; $\tilde{u}(\cdot) \in L_\infty$ instead of L_1). The objective is to find conditions beyond those of the existence theory, assuring that all the minimizers satisfy the standard necessary optimality conditions (e.g. the Pontryagin maximum principle or the Euler–Lagrange differential equations). A profound search in the literature shows that:

- The calculus of variations is very rich in regularity results.
- Regularity results for optimal control problems are a rarity.

In this paper we comment on a new approach to establish regularity properties for the optimal controls. Our approach has its genesis in a very beautiful regularity theorem of 1985, due to F.H. Clarke and R.B. Vinter.

Theorem 1 (cf. [4]). *Under the coercivity and convexity conditions of Tonelli's existence theorem, all the minimizing trajectories $\tilde{x}(\cdot)$ of the* autonomous *fundamental problem of the calculus of variations,*

$$I[x(\cdot), u(\cdot)] = \int_a^b L(x(t), u(t))\, \mathrm{d}t \longrightarrow \min,$$
$$\dot{x}(t) = u(t), \qquad (x(a), x(b)) \in \mathcal{F}, \qquad x(\cdot) \in W_{1,1}([a,b]; \mathbb{R}^n),$$

are Lipschitzian: $\tilde{x}(\cdot) \in W_{1,\infty} \subset W_{1,1}$.

Theorem 1 is not valid for autonomous optimal control problems with dynamics different from $\dot{x}(t) = u(t)$. We also remark that Theorem 1 is only valid in the autonomous situation, that is, in the case the Lagrangian L and the velocity vector φ do not depend explicitly on the time variable t.

The proof of Clarke and Vinter is very technical, and has the focus on the use of nonsmooth analysis. This, in our opinion, hides the crucial point and the natural question to be asked.

5 Autonomous problems . . . why special?

For autonomous problems, the Hamiltonian $H(x, u, \psi_0, \psi) = \psi_0 L(x, u) + \psi \cdot \varphi(x, u)$ is preserved along the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the optimal control problem:

$$H(x(t), u(t), \psi_0, \psi(t)) \equiv \text{const}, \qquad t \in [a, b]. \tag{2}$$

This is a consequence of the Pontryagin maximum principle, and holds generally when admissible controls are bounded: $u(\cdot) \in L_\infty$. The property is important in many different contexts:

- In the calculus of variations it corresponds to the 2^{nd} *Erdmann necessary condition.*
- In classical mechanics has the meaning of *conservation of energy.*
- In economy is interpreted has the *income/wealth law.*

The starting point for our approach is the observation that Theorem 1 is a straightforward consequence of the coercivity hypotheses of the existence theory, if one uses the fact that, for the fundamental autonomous problem of the calculus of variations, (2) is valid in the class of absolutely continuous admissible trajectories ($u(\cdot) \in L_1$): see [3, pp. 61–63].

6 What about the non autonomous problems?

Property (2) admits a generalization for the problems which depend explicitly on the time variable t. If $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ is a Pontryagin extremal, then $H\left(t, x(t), u(t), \psi_0, \psi(t)\right)$ is an absolutely continuous function in t, and satisfies the equality

$$\frac{\mathrm{d}H}{\mathrm{d}t}\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \frac{\partial H}{\partial t}\left(t, x(t), u(t), \psi_0, \psi(t)\right), \tag{3}$$

where on the left-hand side we have the total derivative with respect to t, and on the right-hand side the partial derivative of the Hamiltonian with respect to t. In the calculus of variations (3) corresponds to the classical necessary condition of DuBois–Reymond. We now give a more general formulation.

Theorem 2 (cf. [9]). *If $F\left(t, x, u, \psi_0, \psi\right)$ is a real value function; continuously differentiable with respect to t, x and ψ, for fixed u; $\exists\ G(\cdot) \in L_1\left([a, b]; \mathbb{R}\right)$ such that*

$$\begin{aligned}
&\left\|\nabla_{(t,x,\psi)} F\left(t, x(t), u(s), \psi_0, \psi(t)\right)\right\| \leq G(t) \qquad (s, t \in [a, b]), \\
&F\left(t, x(t), u(t), \psi_0, \psi(t)\right) = \sup_{v \in \Omega} F\left(t, x(t), v, \psi_0, \psi(t)\right),
\end{aligned}$$

then $t \to F\left(t, x(t), u(t), \psi_0, \psi(t)\right)$ is an absolute continuous function and satisfies the equality

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x} \tag{4}$$

along all the extremals.

Choosing $F = H$ in Theorem 2 we obtain equality (3). In the classical context, where no dependence exists on the controls, (4) is the well-known relation $\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \{F, H\}$, where $\{F, H\}$ denotes the Poisson bracket of the functions F and H.

7 Conservation laws in optimal control

Theorem 2 is very useful. From it one obtains a necessary and sufficient condition for a function to be a conservation law.

Definition 1. A quantity $F(t, x, u, \psi_0, \psi)$ which is preserved in t along all the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the optimal control problem (1),

$$F(t, x(t), u(t), \psi_0, \psi(t)) = \text{const}, \tag{5}$$

is called a *first integral*. Equation (5) is the corresponding *conservation law*.

Corollary 1. *Under the conditions of Theorem 2, F is a first integral if, and only if,*

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x} \cdot \frac{\partial H}{\partial \psi} - \frac{\partial F}{\partial \psi} \cdot \frac{\partial H}{\partial x} = 0.$$

Equality (2) is a trivial consequence of Corollary 1:

Example 2. The Hamiltonian H is a first integral if, and only if, $\frac{\partial H}{\partial t} = 0$. This means that the problem must be autonomous in order for the Hamiltonian to be a first integral.

We have reached the point where it is impossible not to think of the celebrated symmetry theorem of Emmy Amalie Noether (1882–1935). In 1918 E. Noether proved that the conservation laws of the calculus of variations are fruit of the symmetries of the problems. For example, the autonomous fundamental problem of the calculus of variations is invariant under time-translations $t \to t + s$. From this invariance it results, from Noether's symmetry theorem, the conservation law $H = \text{const}$.

8 Symmetry in optimal control

To obtain a formulation of the theorem of Emmy Noether in the more general setting of optimal control, we begin by extending the concept of symmetry that one finds in the calculus of variations. We deal with transformations of the optimal control problem depending on time, state, and control variables; and we consider an invariance notion up to addition of gauge terms which, in general, are non-linear with respect to the parameters, time, state, and control variables.

Definition 2. The optimal control problem (1) is said to be invariant under $h^s(t, x, u) = (h^s_t(t, x, u), h^s_x(t, x, u)) \in C^1$ up to $\Phi^s(t, x, u) \in C^1([a, b], \mathbb{R}^n, \Omega; \mathbb{R})$, if $h^0(t, x, u) = (t, x)$ and for all $s = (s_1, \ldots, s_\rho)$, $\|s\| < \varepsilon$, and for all $\beta \in [a, b]$, there exists a $u^s(\cdot) \in L_\infty([a, b]; \Omega)$ such that

$$\begin{aligned}
&\int_{h^s_t(a,x(a),u(a))}^{h^s_t(\beta,x(\beta),u(\beta))} L\left(t^s, h^s_x(t^s, x(t^s), u(t^s)), u^s(t^s)\right) \mathrm{d}t^s \\
&\qquad = \int_a^\beta \left(L(t, x(t), u(t)) + \frac{\mathrm{d}}{\mathrm{d}t} \Phi^s(t, x(t), u(t)) \right) \mathrm{d}t, \\
&\frac{\mathrm{d}}{\mathrm{d}t^s} h^s_x(t^s, x(t^s), u(t^s)) = \varphi(t^s, h^s_x(t^s, x(t^s), u(t^s)), u^s(t^s)).
\end{aligned}$$

Equivalently, one says that h^s is a *symmetry* for problem (1).

Theorem 3 (cf. [7]). *If the optimal control problem* (1) *is invariant under the transformations $(h^s_t(t, x, u), h^s_x(t, x, u))$ up to $\Phi^s(t, x, u)$, in the sense of Definition 2, then the following ρ conservation laws hold ($k = 1, \ldots, \rho$):*

$$\begin{aligned}
&\psi_0 \frac{\partial}{\partial s_k} \Phi^s(t, x(t), u(t))|_{s=0} + \psi(t) \cdot \frac{\partial}{\partial s_k} h^s_x(t, x(t), u(t))|_{s=0} \\
&\qquad - H\left(t, x(t), u(t), \psi_0, \psi(t)\right) \frac{\partial}{\partial s_k} h^s_t(t, x(t), u(t))|_{s=0} = \text{const}.
\end{aligned} \tag{6}$$

Example 3. The autonomous optimal control problem is invariant under $h^s_t = t+s$ and $h^s_x = x$. Theorem 3 implies the conservation law (2).

It is important to note that the conservation laws (6) are valid along all the Pontryagin extremals $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ of the problem: both for normal ($\psi_0 \neq 0$) and abnormal ones ($\psi_0 = 0$). While abnormal extremals are not a possibility for the fundamental problem of the calculus of variations (one can always put the cost multiplier ψ_0 to be -1), they happen to occur frequently in the general situation. The Martinet flat problem of sub-Riemannian geometry, considered in Example 4, is one such case: it admits abnormal minimizers.

For complex situations, the conditions found in [13] help to obtain the invariance transformations. For the present purposes, it is enough to consider the homogeneous case.

Corollary 2. *If there exist constants α, $\beta_1, \ldots, \beta_n$, $\gamma_1, \ldots, \gamma_r \in \mathbb{R}$, such that for all $\lambda > 0$*

$$L\left(\lambda^\alpha t, \lambda^{\beta_1} x_1, \ldots, \lambda^{\beta_n} x_n, \lambda^{\gamma_1} u_1, \ldots, \lambda^{\gamma_r} u_r\right) = \lambda^{-\alpha} L\left(t, x_1, \ldots, x_n, u_1, \ldots, u_r\right),$$
$$\varphi_i\left(\lambda^\alpha t, \lambda^{\beta_1} x_1, \ldots, \lambda^{\beta_n} x_n, \lambda^{\gamma_1} u_1, \ldots, \lambda^{\gamma_r} u_r\right) = \lambda^{\beta_i - \alpha} \varphi_i\left(t, x_1, \ldots, x_n, u_1, \ldots, u_r\right),$$

the following conservation law holds:

$$\sum_{i=1}^{n} \beta_i \psi_i(t) x_i(t) - \alpha H\left(t, x(t), u(t), \psi_0, \psi(t)\right) t \equiv \text{const}.$$

Proof. Conclusion follows from Theorem 3 by choosing $h^s_t = (s+1)^\alpha t$, $h^s_{x_i} = (s+1)^{\beta_i} x_i$, $i = 1, \ldots, n$, and $u^s_k = (s+1)^{\gamma_k} u_k$, $k = 1, \ldots, r$. ■

Example 4. Let us consider the Martinet flat problem of sub-Riemannian geometry:

$$\int_a^b \left(u_1(t)\right)^2 + \left(u_2(t)\right)^2 dt \longrightarrow \min,$$
$$\dot{x_1}(t) = u_1(t), \qquad \dot{x_2}(t) = u_2(t), \qquad \dot{x_3}(t) = \left(x_2(t)\right)^2 u_1(t).$$

With $\alpha = 2$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 3$, $\gamma_1 = \gamma_2 = -1$, we get from Corollary 2 the conservation law $\psi_1 x_1(t) + \psi_2(t) x_2(t) + 3\psi_3 x_3(t) - 2Ht \equiv \text{const}$.

The regularity conditions we are looking for, are obtained with the help of Corollary 2.

9 Symmetry and regularity

The regularity results in [4, 6, 11] are obtained using very different techniques. In this paper we argue that from the point of view of symmetries, all the proofs, which are completely different in detail, can be summarized, in the large, by the same steps. In fact the results follow from the same assumption: that under the typical coercivity conditions of the existence theory, the Lipschitzian regularity conditions for the minimizing trajectories $\tilde{x}(\cdot)$ can be obtained from the applicability conditions of the Pontryagin maximum principle to an equivalent auxiliary problem which possess rich symmetry properties. Our general scheme to prove regularity theorems is summarized in the following steps and ingredients:

1. Reduce the problem to an equivalent form (equivalence in the sense of Elie Cartan, 1908) with a richer set of extremals, and possessing an appropriate symmetry.
2. Establish the relationship between the extremals of the problems (equivalence in the sense of Constantin Carathéodory, 1906).

3. Impose the applicability conditions of the maximum principle to the auxiliary problem, and use the conservation law related to it. From the coercivity condition and the Carathéodory equivalence, conclude that all (normal) minimizing controls of the optimal control problem (1) are essentially bounded, and the respective trajectories are Lipschitzian.

Applying such algorithm with the ideas of R.V. Gamkrelidze and Weierstrass, gives an explanation for the results in [6, 11].

9.1 Regularity from time-homogeneity

Following R.V. Gamkrelidze, we make a reduction of problem (1) to an equivalent autonomous time-optimal problem with the controls taking values in a sphere.

Theorem 4 (cf. [6]). *For the case of control-affine dynamics, $\varphi(t,x,u) = f(t,x) + g(t,x)\cdot u$, the coercivity hypotheses; complete rank r of $g(\cdot,\cdot)$; and the growth condition: there exist constants γ, β, η and μ, with $\gamma > 0$, $\beta < 2$ and $\mu \geq \max\{\beta - 2, -2\}$, such that for all $t \in [a,b]$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$,*

$$(|L_t| + |L_{x^i}| + \|L\,\varphi_t - L_t\,\varphi\| + \|L\,\varphi_{x^i} - L_{x^i}\,\varphi\|)\ \|u\|^{\mu} \leq \gamma\, L^{\beta} + \eta,$$

imply that all the minimizers $\tilde{u}(\cdot)$ of the optimal control problem, which are not abnormal extremal controls, are essentially bounded on $[a,b]$.

9.2 Regularity from control-homogeneity

Writing the problem in a parametric form, an idea as old as Weierstrass, one gets *control homogeneity*. Applying the general scheme, the following result comes out.

Theorem 5 (cf. [11]). *Under the coercivity hypothesis of the existence theorem, the growth conditions: there exist constants $c > 0$ and k such that*

$$\left|\frac{\partial L}{\partial t}\right| \leq c\,|L| + k, \qquad \left\|\frac{\partial L}{\partial x}\right\| \leq c\,|L| + k, \qquad \left\|\frac{\partial \varphi}{\partial t}\right\| \leq c\,\|\varphi\| + k, \qquad \left\|\frac{\partial \varphi_i}{\partial x}\right\| \leq c\,|\varphi_i| + k;$$

imply that all the minimizing controls $\tilde{u}(\cdot)$ of the optimal control problem (1), which are not abnormal extremal controls, are essentially bounded on $[a,b]$.

10 Conclusions

Regularity theorems are very important because they imply that all the minimizers are Pontryagin extremals. New conditions of Lipschitzian regularity of the minimizing trajectories, to a broad class of problems of optimal control, are established using appropriate conservation laws. The results are valid for nonlinear control systems. Even for linear dynamics, for example for the problems of the calculus of variations, one can deal with situations for which the previously known regularity conditions fail (cf. [6]). Our approach does not require the Lagrangian L and the velocity vector φ to be convex functions with respect to the control variables.

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Analytic Transformations between Surfaces with Animations

Vesna VELIČKOVIĆ and Eberhard MALKOWSKY

Department of Mathematics, Faculty of Science and Mathematics, University of Niš, Višegradska 33, 18000 Niš, Serbia and Montenegro
E-mail: *vvesna@BankerInter.net, ema@BankerInter.net*

We give a short description of our own software for geometry and differential geometry and its extensions [4, 1–3, 5–8] and apply it to the visualisation and animation of certain analytic transformations between surfaces.

1 Introduction

The main purpose of our software is to visualise the classical results in differential geometry on PC screens, plotters, printers or any other postscript device, but it also has extensions to physics, chemistry, crystallography and the engineering sciences. To the best of our knowledge, no other comparable, comprehensive software of this kind is available.

The software is *open* which means that its source files are accessible to the users, thus enabling them to apply it in the solutions of their own problems. This makes it extendable and flexible, and applicable to both teaching and research in many fields. In contrast to this, almost all other available graphics packages are *closed*; in general, the area below the user interface is inaccessible and consequently the software cannot be extended beyond the scope of solutions it offers. The software uses *OOP*, *object oriented programming*, and its programming language is *PASCAL*. The software is self-contained in the sense that no graphics package is needed other than PASCAL.

The advantages of PASCAL are the *hierarchy of objects* and the *polymorphy* which is not available in some *OOP languages*. In the hierarchy of objects, a successor inherits all the data, in particular the methods and procedures, of its predecessors. Polymorphy means that virtual methods can be declared, a virtual method can be rewritten with the same name in a successor, and one may have more methods than one with the same name. The development of our software could not have been achieved without OOP.

2 Spherical and pseudo-spherical surfaces of revolution

Let γ be a curve with a parametric representation $\vec{x}(s) = (r(s), 0, h(s))$ where $s \in I \subset \mathbb{R}$ is the arc length along γ, and $r(s) > 0$ on the interval I. Furthermore let $RS(\gamma)$ be the surface of revolution generated by rotating γ about the x^3-axis. Putting $u^1 = s$ and writing u^2 for the angle of rotation, we obtain the following parametric representation for $RS(\gamma)$ on $D = I \times (0, 2\pi)$

$$\vec{x}\big(u^i\big) = \big(r\big(u^1\big)\cos u^2, r\big(u^1\big)\sin u^2, h\big(u^1\big)\big). \tag{1}$$

Omitting the argument u^1, we find that the first and second fundamental coefficients of $RS(\gamma)$ are given by $g_{11} = (r')^2 + (h')^2 = 1$, since u^1 is the arc length along γ, $g_{12} = 0$, $g_{22} = r^2$, $L_{11} = r'h'' - r''h'$, $L_{12} = 0$ and $L_{22} = rh'$. So the Gaussian curvature

$$K = \frac{L}{g} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2} \quad \text{of } RS(\gamma) \text{ is given by} \quad K = \frac{r'h'' - r''h'}{r}.$$

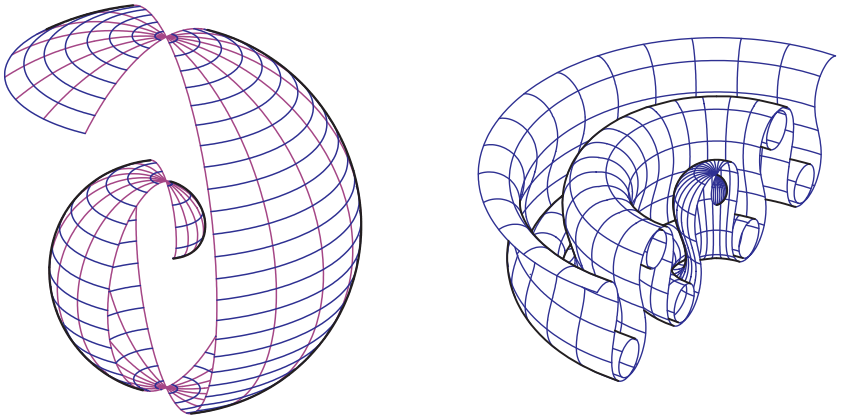

Figure 1. Surfaces of revolution.

Since $(r')^2 + (h')^2 = 1$ implies $r'r'' + h'h'' = 0$, we obtain

$$K = \frac{r'h''h' - r''(h')^2}{r} = -\frac{\big((r')^2 + (h')^2\big)r''}{r} = -\frac{r''}{r},$$

and consequently writing $u = u^1$

$$r''(u) + K(u)r(u) = 0. \tag{2}$$

Surfaces of revolution with constant Gaussian curvature $K > 0$ or $K < 0$ are called *spherical* or *pseudo-spherical surfaces of revolution*, respectively[1].

First, we assume $K = 0$. Then $r = c_1 u + c_2$ with constants c_1 and c_2. If we choose $c_1 = 0$ then $h' = \pm 1$ implies $h = \pm u + d$ with some constant d, and we obtain a circular cylinder. If $c_1 \neq 0$ then $(r')^2 + (h')^2 = 1$ implies $|c_1| \leq 1$. For $|c_1| = 1$, we have $h' \equiv 0$, hence $h \equiv \text{const}$, and we obtain a plane. For $0 < |c_1| < 1$ and a suitable choice of the coordinate system, we have $r = c_1 u$ and $h = d_1 u$ for some constant d_1 with $c_1^2 + d_1^2 = 1$, and we obtain a circular cone.

2.1 Spherical surfaces of revolution

We assume $K > 0$ and put $K = 1/c^2$ for some constant $c > 0$. Then the general solution of the differential equation (2) is $r(u) = \lambda \cdot \cos(u/c + u_0)$ and by a suitable choice of the arc length we may assume $r(u) = \lambda \cos(u/c)$ with $\lambda > 0$. Now $(r')^2 + (h')^2 = 1$ implies

$$h(u) = \pm \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \frac{u}{c}}\, du$$

and we may choose the upper sign without loss of generality. Thus the spherical surfaces of revolution are given by parametric representations (1) with

$$r\big(u^1\big) = \lambda \cos \frac{u^1}{c} \text{ and } h\big(u^1\big) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \sin^2 \frac{u^1}{c}}\, du^1,$$

where $\lambda > 0$, $c = \frac{1}{\sqrt{K}}$ and $K > 0$, the integral for h is a so-called elliptic integral. We obtain three different types of spherical surfaces of revolution corresponding to the cases $\lambda = c$, $\lambda > c$ or $\lambda < c$.

Case 1 $\lambda = c$. Then the surface has a parametric representation

$$\vec{x}\big(u^i\big) = \left(c \cos \frac{u^1}{c} \cos u^2, c \cos \frac{u^1}{c} \sin u^2, c \sin u^1\right).$$

Hence the surface is a sphere with radius c and centre in the origin.

[1]Figures in colour will be available only in electronic version.

Case 2 $\lambda > c$. The corresponding surfaces are called *hyperbolic spherical surfaces of revolution*. Now the integral for h only exists for values of u^1 that satisfy

$$\sin^2 \frac{u^1}{c} \leq \frac{c^2}{\lambda^2}, \text{ that is } u^1 \in I_k = \left[-c \arcsin \frac{c}{\lambda} + k\pi, c \arcsin \frac{c}{\lambda} + k\pi\right] \text{ for } k = 0, \pm 1, \pm 2, \ldots.$$

Every interval I_k defines a region of the surface. The radii of the circles of the u^2-lines are minimal at the end points of the intervals I_k and equal to $r = \sqrt{\lambda^2 - c^2}$, whereas the maximum radius $R = \lambda$ is attained in the middle of each region.

Case 3 $\lambda < c$. The corresponding surfaces are called *elliptic spherical surfaces of revolution*. Now the integral for h exists for all u^1 and the radii r of the circles of the u^2-lines attain all values $r \leq \lambda$.

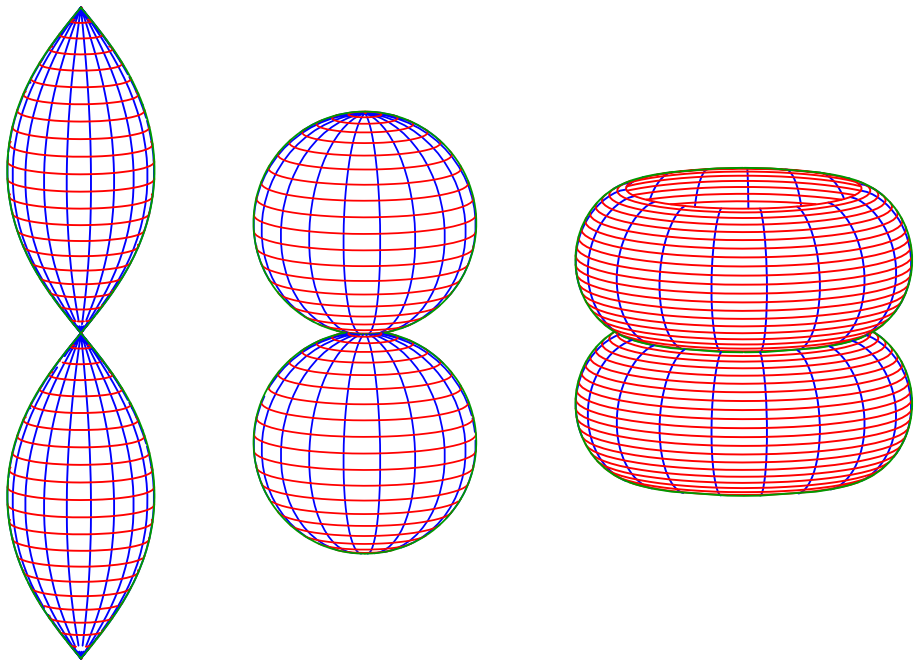

Figure 2. Spherical surfaces of revolution: $c = 1$, $\lambda = 0.6$, elliptic; $c = \lambda = 1$, spheres; $c = 1$, $\lambda = 1.4$, hyperbolic.

2.2 Pseudo-spherical surfaces of revolution

Now we assume $K < 0$ and put $K = -1/c^2$ for some constant $c > 0$. The general solution of the differential equation (2) is

$$r(u) = C_1 \cosh\left(\frac{u}{c}\right) + C_2 \sinh\left(\frac{u}{c}\right) \quad \text{with constants } C_1 \text{ and } C_2. \tag{3}$$

Case 1 $C_1 = -C_2 = \lambda \neq 0$. Then we obtain from (3) $r(u) = \lambda \exp(-u/c)$, and the surface has a parametric representation with

$$r(u^1) = \lambda \exp\left(-\frac{u^1}{c}\right) \quad \text{and} \quad h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2} \exp\left(-\frac{2u^1}{c}\right)}\, du.$$

Such a surface is called *parabolic pseudo-spherical surface of revolution*. The integral for h exists for $|u^1| > c \log(|\lambda|/c)$.

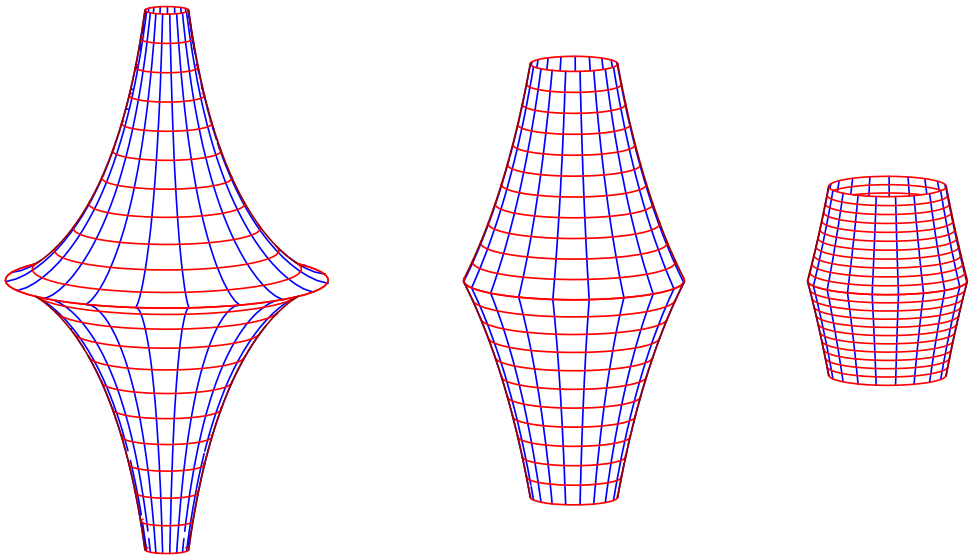

Figure 3. Parabolic pseudo-spherical surfaces of revolution ($\lambda = 1$): $c = 1$; $c = 1.5$; $c = 2$.

Case 2 $C_2 = 0$ and $C_1 = \lambda \neq 0$. Then the surface has a parametric representation with

$$r(u^1) = \lambda \cosh\left(\frac{u^1}{c}\right) \quad \text{and} \quad h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\sinh^2\left(\frac{u^1}{c^2}\right)}\, du^1.$$

Such a surface is called *hyperbolic pseudo-spherical surface of revolution*. The integral for h is an elliptic integral and exists for $|u^1| \leq c \cdot \operatorname{Arsh}(c/|\lambda|) = c \log\left(c/|\lambda| + \sqrt{1 + c^2/\lambda^2}\right)$. The radii r of the circles of the u^2-lines satisfy $\lambda \leq r \leq \sqrt{\lambda^2 + c^2}$.

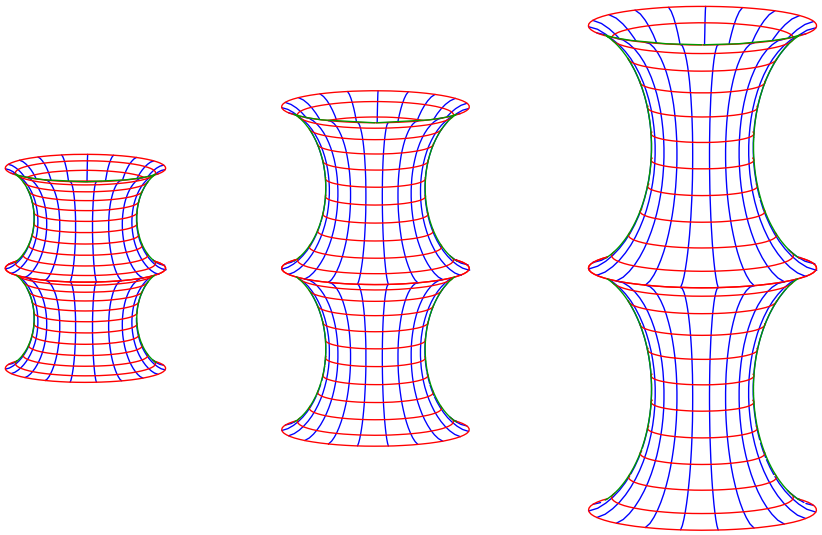

Figure 4. Hyperbolic pseudo-spherical surfaces of revolution ($\lambda = 1$): $c = 1.2$; $c = 1.6$; $c = 2$.

Case 3 $C_1 = 0$ and $C_2 = \lambda \neq 0$. Then the surface has a parametric representation with

$$r(u^1) = \lambda \sinh\left(\frac{u^1}{c}\right) \quad \text{and} \quad h(u^1) = \int \sqrt{1 - \frac{\lambda^2}{c^2}\cosh^2\left(\frac{u^1}{c^2}\right)}\, du^1.$$

Such a surface is called *elliptic pseudo-spherical surface of revolution*. Since $\cosh x \geq 1$ for all x, we must have $|\lambda| \leq c$. The integral for h is elliptic and exists for $\cosh(u^1/c) \leq c/|\lambda|$. The radii r of the circles of the u^2-lines satisfy $0 \leq r \leq \sqrt{c^2 - \lambda^2}$.

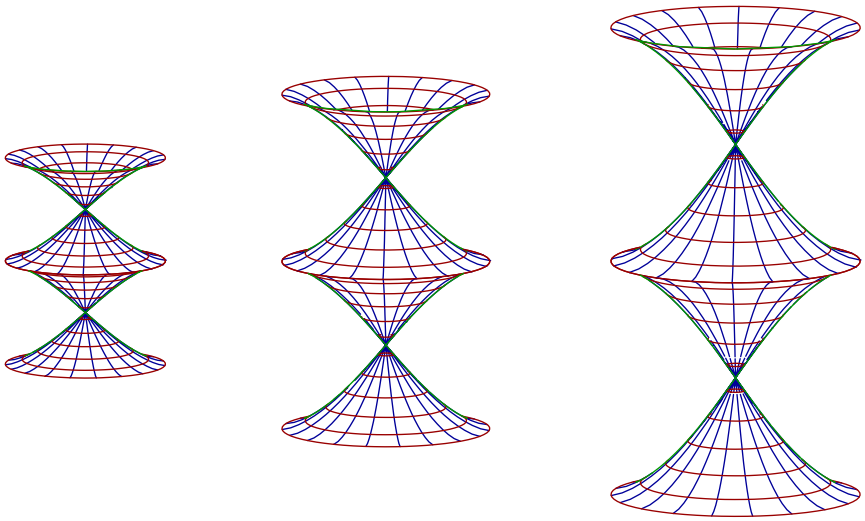

Figure 5. Elliptic pseudo-spherical surfaces of revolution ($\lambda = 1$): $c = 1.35$; $c = 1.55$; $c = 1.75$.

3 Isometric mappings of surfaces of revolution

Now we determine all surfaces of revolution that can be mapped isometrically onto a surface of revolution with non-constant Gaussian curvature.

Since the first and second fundamental coefficients of surfaces of revolution S depend on the parameter u^1 only, the Gaussian curvature is constant along any u^2-line of S. Furthermore the Gaussian curvature of surfaces is invariant under isometric mappings. Since the Gaussian curvature of the given surface of revolution S is not constant, the u^2-lines of the surfaces of revolution S^* that are to be determined have to be mapped onto the u^2-lines of S. Finally, since isometric maps are conformal, the u^1-lines of S^* have to be mapped onto the u^1-lines of S. Let S and S^* be given by parametric representations (1) and

$$\vec{x}\big(u^{*i}\big) = \big(r^*\big(u^{*1}\big)\cos u^{*2}, r^*\big(u^{*1}\big)\sin u^{*2}, h^*\big(u^{*1}\big)\big),$$

where u^1 and u^{*1} are the arc lengths along the u^1-lines of S and the u^{*1}-lines of S^*, respectively, that is $\big(r'\big(u^1\big)\big)^2 + \big(h'\big(u^1\big)\big)^2 = 1$ and $\big(r^{*'}\big(u^{*1}\big)\big)^2 + \big(h^{*'}\big(u^{*1}\big)\big)^2 = 1$. The first fundamental coefficients are $g_{11} = g^*_{11} = 1$, $g_{12} = g^*_{12} = 0$ and $g_{22} = r\big(u^1\big)$ and $g^*_{22} = r^*\big(u^{*1}\big)$, hence the first fundamental forms are

$$(ds)^2 = \big(du^1\big)^2 + r^2\big(u^1\big)\big(du^2\big)^2 \quad \text{and} \quad (ds^*)^2 = \big(du^{*1}\big)^2 + \big(r^*\big(u^{*1}\big)\big)^2\big(du^{*2}\big)^2. \tag{4}$$

Since the u^1- and u^2-lines of S correspond to the u^{*1}- and u^{*2}-lines of S^*, we must have

$$u^{*1}\big(u^1, u^2\big) = u^{*1}\big(u^1\big) \quad \text{and} \quad u^{*2}\big(u^1, u^2\big) = u^{*2}\big(u^2\big).$$

The first fundamental coefficients of isometric surfaces must satisfy $g_{ik} = g^*_{ik}$ for $i, k = 1, 2$. Therefore we obtain from (4)

$$du^{*1} = \pm du^1 \quad \text{and} \quad \frac{r\big(u^1\big)}{r^*\big(u^1\big)} = \pm\frac{du^{*2}}{du^2}. \tag{5}$$

First, the left-hand side of (5) implies $u^{*1} = \pm u^1 + c_0$ where c_0 is a constant. If we choose the same orientation of the u^1 and u^{*1}-lines, we may assume $u^{*1} = u^1$. Then the right hand side of (5) becomes

$$\frac{r\big(u^1\big)}{r^*\big(u^1\big)} = \pm\frac{du^{*2}\big(u^2\big)}{du^2}.$$

Since the left-hand side of this identity only depends on u^1 and the right hand side only on u^2, we must have

$$r^*\big(u^1\big) = c \cdot r\big(u^1\big) \quad \text{and} \quad \frac{du^{*2}}{du^2} = \pm\frac{1}{c} \quad \text{where } c \neq 0 \text{ is a constant.}$$

This implies $u^{*2} = \pm u^2/c + d$, where d is a constant. We may choose $d = 0$ and the upper sign, since d corresponds to a certain rotation about the axis and the sign to a reflection. Thus we obtain $u^{*1} = u^1$ and $u^{*2} = u^2/c$. Since u^{*1} is the arc length along the u^*-lines, it follows that

$$h^{*'}\big(u^{*1}\big) = \sqrt{1 - \big(r^{*'}\big(u^{*1}\big)\big)^2} = \sqrt{1 - c^2\big(r'\big(u^1\big)\big)^2}.$$

Consequently every surface of revolution S^* that can be mapped isometrically onto a given surface of revolution S with non–constant Gaussian curvature is given by a parametric representation

$$\vec{x}^*\big(u^i\big) = \left(cr\big(u^1\big)\cos\frac{u^2}{c}, cr\big(u^1\big)\sin\frac{u^2}{c}, h^*\big(u^1\big)\right) \quad \text{with} \quad h^*\big(u^1\big) = \int\sqrt{1 - c^2\big(r'\big(u^1\big)\big)^2}du^1.$$

If $c = 1$ then $S^* = S$. A continuous change of c corresponds to a continuous deformation. For a given surface of rotation there is a one parametric family of isometric surfaces of revolution.

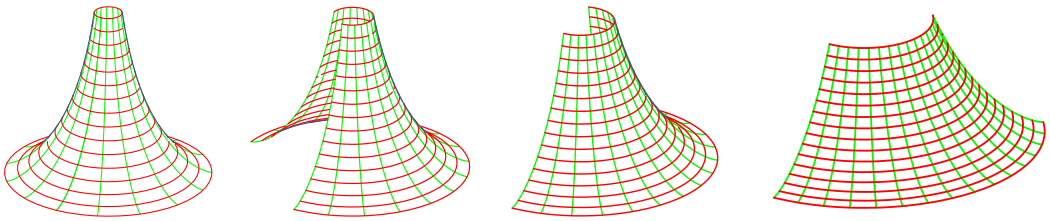

Figure 6. An isometric deformation of a pseudo-sphere.

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Modulational Instability and Multiple Scales Analysis of Davydov's Model

Anca VISINESCU and Dan GRECU

Department of Theoretical Physics, National Institute of Physics and Nuclear Engineering "Horia Hulubei", Bucharest, Romania
E-mail: *avisin@theor1.theory.nipne.ro, dgrecu@theor1.theory.nipne.ro*

The modulational instability (MI) of some discrete nonlinear evolution equations, representing approximations of Davydov's model of α-helix in protein, is studied. In a multiple scales analysis the dominant amplitude usually satisfies the nonlinear Schrödinger equation (NLS), or the Zakharov–Benney equations (ZB), if a long wave-short wave resonance takes place. The MI is studied from a statistical point of view, where a new phenomenon, similar with the Landau damping in plasma physics, can appear for a Lorentzian distribution of the unperturbed Fourier transform of the two-point correlation function.

1 Introduction

Many quasi-one-dimensional molecular systems, especially those of biological interest, are very complicated structures, built from complexes of atoms connected by hydrogen bonds. A typical example is the complex structure of α-helix in protein. It consists of three coupled chains, helically distorted, with hydrogen bonds between the groups of atoms along the chain. It is hard to imagine that a model can be elaborated which could take into account all this complexity. A very simple, but still useful approximation consists in replacing of the three coupled chains by a single straight one. Further from the multitude of the inter- and intra-molecular excitations, only one, corresponding to the amide I oscillations, is taken into account. This is considered to be the "basket" where the energy released in the adenosine triphosphate hydrolysis is stored. The existing dipole-dipole interactions between them provides the mechanism for the transport of these excitonic excitations along the chain. Such an exciton generates a local distortion and a nonlinear coupling between excitons and the acoustic phonon field (describing the harmonic oscillations of the molecules along the chain) appears. The consequence of this nonlinear interaction is generation of robust self-trapped excitations that can transport the energy along the chain. Such a model for storage and transport of energy along the α-helix structure was proposed thirty years ago by Davydov [1–7].

2 Davydov's model

In its simplest form, Davydov's model considers a linear chain of aminoacids – named afterward "molecules" – connected between by hydrogen bonds. If $B_n^+(B_n)$ is the creation (annihilation) operator of the intra-molecular excitation in the n-th cell, the excitonic Hamiltonian can be described by an excitonic Hamiltonian of the form

$$H_{ex} = E_0 \sum B_n^+ B_n - \frac{1}{2} \sum_{mn} J_{mn} B_n^+ B_m,$$

where E_0 is the vibration energy of the $C = O$ group, and J_{mn} the dipole-dipole interaction between two such excitations in the n- and m-th cells. Usually only nearest neighbour interaction

is taken into account ($m = n \pm 1$), but also a long range interaction like Kac–Baker model can be considered.

$$J_{mn} = J\frac{1-r}{2r}e^{-\gamma|m-n|}, \qquad r = e^{-\gamma}.$$

The acoustic mode of vibration of the molecules along the chain are described by harmonic Hamiltonian

$$H_{ph} = \frac{1}{2}\sum_n \left(\frac{1}{M}\hat{p}_n^2 + w(\hat{u}_{n+1} - \hat{u}_n)^2\right).$$

M being the mass of the molecule, and w the elastic spring between two neighbouring molecules. The nonlinearity enters in the interaction Hamiltonian

$$H_{int} = \chi\sum_n (\hat{u}_{n+1} - \hat{u}_n)B_n^+ B_n$$

with χ being the phonon-exciton coupling constant. The total Hamiltonian is

$$H = H_{ex} + H_{\rm ph} + H_{\rm int}.$$

A coherent state approximation is suitable for describing extended localized states in such systems. For the state vector we shall consider (Davydov ansatz)

$$|\Psi(t)\rangle = \sum_n a_n(t)B_n^+ \exp\left[-\frac{1}{\hbar}\sum_{n'}(\beta_{n'}(t)\hat{p}_{n'} - \pi_{n'}(t)\hat{u}_{n'})\right]|0\rangle,$$

where $a_n(t)$, $\beta_n(t)$, $\pi_n(t)$ are time dependent c-numbers, and $|0\rangle$ is the vacuum state both for excitons and phonons. Using the average value of $\hat{H}$ in the classical equations of motion, one gets

$$i\hbar\dot{a}_n = Ea_n - \sum_p J(a_{n+p} + a_{n-p}) + \chi(\beta_{n+1} - \beta_n)a_n,$$
$$M\ddot{\beta}n - w(\beta_{n+1} - 2\beta_n + \beta_{n-1}) = \chi(|a_n|^2 - |a_{n-1}|^2).$$

With introduction of the relative displacement ρ_n by $\rho_n = \beta_{n+1} - \beta_{n-1}$, the previous equations become

$$i\hbar\dot{a}_n = Ea_n - \sum_{p=1} J_p(a_{n+p} + a_{n-p}) + \chi\rho_n a_n,$$
$$M\ddot{\rho}_n - w(\rho_{n+1} - 2\rho_n + \rho_{n-1}) = \chi(|a_{n+1}|^2 - 2|a_n|^2 + |a_{n-1}|^2).$$

3 Multiple scales method

The multiple scales method is an adequate mathematical method to study long-time evolution of this nonlinear system of coupled equations [8,9]. The classical excitonic variable is written as a Taylor expansion in a small parameter ϵ

$$a_n = e^{i(kln-\omega t)}\epsilon^\alpha \sum_j \epsilon^j A_j(\xi, t_2, t_3, \ldots),$$

where the amplitudes A_j depend only on "slow variables" $(\xi, t_2, t_3, \ldots)$, defined as

$$\xi = \epsilon(ln - v_g t), \quad t_2 = \epsilon^2 t, \quad t_3 = \epsilon^3 t, \quad \ldots.$$

As we shall see and explain below in the non-resonant case $\alpha = 0$, the dominant amplitude A_1 will satisfy the well known nonlinear Schrödinger equation (NLS equation), while in the resonant case we have to take $\alpha = \frac{1}{2}$ and the dominant amplitudes A_1, P_1 will satisfy the Zakharov–Benney equations [10, 11]. For the phononic variable ρ_n we assume an expansion of the form

$$\rho_n = \sum_j \epsilon^{j+1} P_j(\xi, t_2, \ldots).$$

Introducing these expansions in the equations of motion for a_n and ρ_n in different orders of ϵ we get:

In order $\epsilon^{1+\alpha}$ we get the dispersion relation

$$\hbar\omega = E - 2\sum_j J_j \cos klj$$

for the linearized system.

In order $\epsilon^{2+\alpha}$ the velocity v_g in the expression of ξ is given by the group velocity (we denote $\omega_n = \frac{1}{n!}\frac{d^n\omega(k)}{dk^n}$)

$$v_g = \omega_1 = \frac{\partial\omega}{\partial k}.$$

The next orders will represent constraints on the amplitudes A_1, P_1, ... namely

$$i\frac{\partial A_1}{\partial t_2} + \omega_2\frac{\partial^2 A_1}{\partial\xi^2} - \frac{\chi}{\hbar}P_1 A_1 = 0 \tag{1}$$

in order $\epsilon^{3+\alpha}$ and

$$(v_g^2 - c_p^2)\frac{\partial^2 P_1}{\partial\xi^2} = \frac{\chi}{M}l^2\frac{\partial^2}{\partial\xi^2}|A_1|^2 \tag{2}$$

in order ϵ^4. Here by $c_p = \sqrt{\frac{w}{M}}l$ we have denoted the sound velocity of the acoustic field.

The non-resonant case will correspond to the situation when $v_g \neq c_p$. Then from (2)

$$P_1 = -\frac{\chi}{w}\frac{1}{1 - \frac{v_g^2}{c_p^2}}|A_1|^2$$

that after substitution into (1) gives

$$i\frac{\partial A_1}{\partial t_2} + \omega_2\frac{\partial^2 A_1}{\partial\xi^2} + \nu|A_1|^2 A_1 = 0, \tag{3}$$

where $\nu = \frac{\chi^2}{\hbar M}\frac{1}{1-\frac{v_g^2}{c_p^2}}$. This is the well-known NLS equation, a completely integrable system. For $\nu > 0$ (focusing case) it has solitonic solutions. This happens if $v_g < c_p$, and consequently the soliton is a subsonic excitation.

The resonant case corresponds to $v_g \simeq c_p$, and the condition can be realized if the group velocity in the optical branch (excitonic branch) is equal to the phase velocity in the acoustic one. In the multiple scales method we have to go to the order ϵ^5 in the phonon variable, and one obtains

$$\frac{\partial P_1}{\partial t_2} = -\frac{\chi}{2Mv_g}l^2\frac{\partial|A_1|^2}{\partial\xi}. \tag{4}$$

The system of equations (1) and (4) represents the completely integrable Zakharov–Benney system [10, 11].

In the non-resonant case ($\alpha = 0$), by going to higher order of approximation (ϵ^4) the following equation is obtained [12–19]

$$i\frac{\partial A_2}{\partial t_2} + \omega_2 \frac{\partial^2 A_2}{\partial \xi^2} + \nu(A_1^2 A_2^* + 2|A_1|^2 A_2) = -i\frac{\partial A_1}{\partial t_3} + i\omega_3 \frac{\partial^3 A_1}{\partial \xi^3}. \tag{5}$$

This is a linear non-homogeneous equation for the next amplitude A_2. In the left-hand side we recognize the linearized NLS equation, and the non-homogeneity in the right hand side depends only on the dominant amplitude A_1. It contains the unknown derivative of A_1 with respect to the next slow time t_3. A secular behaviour of this equation is possible if in the rhs we identify some members of the null space (symmetries) of the linearized NLS equation. It is easily seen that the following symmetry appears in the rhs of (5).

$$\sigma_3 = -\left(\frac{\partial^3 A_1}{\partial \xi^3} + 6c|A_1|^2 \frac{\partial A_1}{\partial \xi}\right), \qquad c = \frac{\nu}{2\omega_2}.$$

The secular behaviour is eliminated if the t_3 dependence of A_1 is determined by the equation

$$-\frac{\partial A_1}{\partial t_3} + \omega_3 \left(\frac{\partial^3 A_1}{\partial \xi^3} + 6c|A_1|^2 \frac{\partial A_1}{\partial \xi}\right) = 0 \tag{6}$$

that is the complex modified KdV equation, the next equation in NLS hierarchy. We remain with

$$i\frac{\partial A_2}{\partial t_2} + \omega_2 \frac{\partial^2 A_2}{\partial \xi^2} + \nu\left(A_1^2 A_2^* + 2|A_1|^2 A_2\right) = -6i\omega_3 c|A_1|^2 \frac{\partial A_2}{\partial \xi} \tag{7}$$

which now is a linear non-homogeneous equation free of secularities [18]. In the Appendix both equations (6) and (7) will be solved when A_1 is given by one-soliton solution.

4 Statistical approach of the modulational instability

The modulational instability, also known as the Benjamin–Feir instability [20–23], has proved to be a general process occurring in a large variety of physical situations. The phenomenon is characteristic both for continuous and discrete systems and describes the exponential growth of the amplitude of a (quasi) monochromatic wave propagating in a weakly nonlinear dispersive medium. It is generally believed to be responsible for the formation of robust localized coherent structures in these systems. We shall mainly focus our attention to the NLS equation (3), where $A(X, T)$ is the slowly varying amplitude – considered as a random variable (here X and T are the slow variables of the multiple scales method) and ω_2, ν are the parameters associated with the dispersion (ω_2) and the cubic nonlinearity (ν). One constructs a kinetic equation for the correlation function [24, 25]

$$\rho(X_1, X_2) = \langle A(X_1)A^*(X_2)\rangle,$$
$$\bar{a^2}(X) = \langle A(X)A^*(X)\rangle,$$

namely

$$i\frac{\partial \rho}{\partial t} + \omega_2 \left(\frac{\partial^2}{\partial X_1^2} - \frac{\partial^2}{\partial X_2^2}\right)^2 \rho + 2\nu\big[\bar{a}^2(X_1) - \bar{a}^2(X_2)\big]\rho = 0.$$

Further a Wigner–Moyal transform is used [26, 27]. One introduces the new variables

$$X = \frac{1}{2}(X_1 + X_2), \qquad x = (X_1 - X_2)$$

and the Fourier transform

$$F(k, X, T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} \rho(x, X, T) dx.$$

We obtain

$$\frac{\partial F}{\partial T} + 2\omega_2 k \frac{\partial F}{\partial X} + 4\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} \frac{\partial^{2n+1} F}{\partial k^{2n+1}} \frac{\partial^{2n+1} \bar{a}^2}{\partial X^{2n+1}} = 0.$$

A linear stability analysis is done considering

$$F(k, X, Y) = F_0(k) + \epsilon F_1(k, X, T),$$
$$\bar{a^2}(X, T) = \bar{a_0^2} + \epsilon \bar{a_1^2}(X, T),$$
$$\bar{a_0^2} = \int F_0(k) dk, \qquad \bar{a_1^2} = \int F_1(k, X, T) dk.$$

The linearized equation for F_1 writes

$$\frac{\partial F_1}{\partial T} + 2\omega_2 k \frac{\partial F_1}{\partial X} + 4\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}(2n+1)!} \frac{\partial^{2n+1} F_0}{\partial k^{2n+1}} \frac{\partial^{2n+1} \bar{a_1^2}}{\partial X^{2n+1}} = 0.$$

If a plane wave solution for $F(k, X, T)$ is considered

$$F_1(k, X, T) = f_1(k) e^{i(KX - \Omega T)}.$$

the following integral stability equation is easily found

$$1 + \frac{\nu}{K\omega_2} \int_{-\infty}^{+\infty} \frac{F_0(k + \frac{K}{2}) - F_0(k - \frac{K}{2})}{k - i\lambda} dk = 0. \tag{8}$$

Here we assumed Ω purely imaginary, $\Omega = i\Omega_i$, and $\lambda = \frac{\Omega_i}{2K\omega_2}$. If we assume $\rho_0(x)$ to be an even function of x the previous equation is easily transformed into

$$1 + \frac{i\nu}{K\omega_2} \int_0^x \rho_0(x) e^{-\lambda x} \left(e^{i\frac{K}{2}x} - e^{-i\frac{K}{2}x} \right) dx = 0.$$

Several forms for the initial condition F_0 will be analyzed. As a first example let us consider a δ-spectrum

$$F_0(k) = \bar{a_0^2} \delta(k).$$

It corresponds to a constant initial $\rho_0(x)$. The integration in (8) is straightforward giving

$$\Omega_i = K\omega_2 \left(\frac{4\nu}{\omega_2} \bar{a_0^2} - K^2 \right)^{\frac{1}{2}}.$$

An instability exists if both ω_2 and ν have the same sign (this situation corresponds to the focusing case of the NLS equation) and if $K < 2\left(\frac{\nu}{\omega_2}\bar{a_0^2}\right)^{\frac{1}{2}}$. As $\bar{a_0^2}$ is a small quantity this condition corresponds to a "long wave limit". The first condition on the sign of ω_2 and ν is

a general one, and we assume it to be satisfied in all the cases we shall discuss. Also we can consider $\nu = 2\omega_2 = 1$, a condition which corresponds to reduction of the NLS equation to the canonical form

$$i\frac{\partial A}{\partial T} + \frac{1}{2}\frac{\partial^2 A}{\partial X^2} + |A|^2 A = 0.$$

The next example is a Lorentzian spectrum

$$F_0(k) = \frac{\bar{a_0^2}}{\pi}\frac{p}{p^2 + k^2}, \qquad \rho_0(x) = \bar{a_0^2} e^{-px}.$$

It is straightforward to show that in this case

$$\Omega_i = K\left(\sqrt{2 - \frac{K^2}{4}} - p\right).$$

In this expression it is easily seen that the instability depends also on the correlation length p of the initial distribution, not only of the wave number K. If p is greater than $\sqrt{2 - \frac{K^2}{4}}$ the system becomes stable. This is similar with the well-known phenomenon of Landau damping in plasma physics [28].

5 Conclusions

In summary the main conclusions of the present paper are:

- The multiple scales method is used for the analysis of the 1-D Davydov model. It is shown that in the non-resonant case the NLS equation is obtained while in the resonant case the Zakharov–Benney system is found.
- In a higher order approximation, for the non-resonant case, in order for the **secular behavior** to be eliminated the dominant amplitude **has to satisfy** the next equation in the NLS hierarchy (complex mKdV equation).
- The MI of the NLS equation was studied using a statistical approach. As is seen from the Lorentzian case, the statistical properties have a great influence on the development of MI. If the width of the initial 2-point correlation function is too small (p is too large) then the MI is suppressed.

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Appendix

The equation (6) and (7) will be solved when A_1 is the one-soliton solution. With a suitable scaling of the amplitude and the temporal variable ($T = 2\omega_2 t_2$, $\Psi = \sqrt{\frac{\nu}{2\omega_2}} A_1$) the NLS equation (3) can be written in the canonical form

$$i\frac{\partial \Psi}{\partial T} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial \xi^2} + |\Psi|^2 \Psi = 0. \tag{A.1}$$

Also with scaling the t_3 variable as $\tau = \omega_3 t_3$ the complex mKdV equation (6) writes

$$-\frac{\partial \Psi}{\partial \tau} + \frac{\partial^3 \Psi}{\partial \xi^3} + 6|\Psi|^2 \frac{\partial \Psi}{\partial \xi} = 0.$$

It is convenient to scale the amplitude A_2 in the same way as A_1 ($\phi = \sqrt{\frac{\nu}{2\omega_2}} A_2$) and then equation (7) becomes

$$i\frac{\partial \phi}{\partial T} + \frac{1}{2}\frac{\partial^2 \phi}{\partial \xi^2} + (\Psi^2 \phi^* + 2|\Psi|^2 \phi) = -i\alpha |\Psi|^2 \frac{\partial \phi}{\partial \xi}, \qquad \alpha = 3\frac{\omega_3}{\omega_2}. \tag{A.2}$$

The one-soliton solution of the NLS equation (A.1) is

$$\Psi = 2v\frac{e^{-i\Phi}}{\cosh\theta},$$

where

$$\Phi = 2u\xi + 2(u^2 - v^2)T + \Phi_0,$$
$$\theta = 2v(X - X_0 + 2uT).$$

Here u, v are the real and imaginary parts of the eigenvalue of the spectral problem, and they characterize the velocity and the amplitude of the soliton respectively.

As all the equations in the NLS hierarchy have the same spectral problem, the τ dependence in Ψ can appear only in the initial positions and phases, characterizing the unperturbed solution. For the one-soliton solution, only X_0 and Φ_0 will become dependent on τ. It is easy to show that

$$\frac{d\Phi_0}{d\tau} = 8u(u^2 - 3v^2),$$
$$\frac{dX_0}{d\tau} = 4(3u^2 - v^2)$$

leading to a linear dependence of Φ_0 and X_0 on the slow-time variable τ.

Concerning the solution of (A.2), let us consider the case of the solution at rest ($u = 0$). With the new variable $\rho = \tanh 2v\xi$ the equation (A.2) transforms into the equation for the associated Legendre polynomials, with a non-homogeneity in the right hand side, which can be solved exactly [16].

In the simple case of the one-soliton solution we see that in the next order of approximation the initial phase and position of the soliton will depend on the next slow time variable t_3, the effect which cannot be obtained by a perturbation treatment. Also the second amplitude can be determined exactly, and can be expressed through associate Legendre polynomials.

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On the Localized Invariant Solutions of Some Non-Local Hydrodynamic-Type Models

Vsevolod A. VLADIMIROV and Ekaterina V. KUTAFINA

Faculty of Applied Mathematics, University of Science and Technology,
30 Mickiewicz Ave., 30-059 Kraków, Poland
E-mail: *vsan@rambler.ru, vladimir@mat.agh.edu.pl*

We analyze the conditions, assuring the existence of invariant wave patterns in the non-local hydrodynamic models of structured media.

1 Introduction

This paper deals with a family of invariant solutions of some modelling systems of PDE, taking into account non-local effects. These effects are manifested when an intense pulse loading (impact, explosion etc.) is applied to media, possessing an internal structure on mesoscale. Description of the non-linear waves propagation in such media depends in essential way on the ratio of a characteristic size d of elements of the medium structure to a characteristic length λ of the wave pack. If d/λ is $O(1)$, then the basic concepts of continuum mechanics are not applicable any more, and one should use the description, based, e.g. on the element dynamics methods [1]. The applications of classical continuum mechanics equations are justified in those cases, when $d/\lambda \ll 1$, and the discreteness of the matter could be completely ignored.

The models studied in this work apply when the ratio d/λ is much less that unity and therefore the continual approach is still valid, but it is not as small that we can ignore the presence of the internal structure. As it has been shown in a number of papers (see e.g. [2]), in the long wave approximation the balance equations for mass and momentum retain their classical form, which in the one-dimensional case can be written as follows:

$$u_t + p_x = 0, \qquad \rho_t + \rho^2 u_x = 0, \tag{1}$$

where u is the mass velocity, p is the pressure, ρ is the density, t is the time, x is the mass (Lagrangian) coordinate, lower indices denote partial derivatives with respect to subsequent variables. Thus, the whole information about the presence of structure in this approximation is contained in a dynamic equation of state (DES), which should be incorporated to system (1) in order to make it closed.

Generally speaking, DES take on the form of integral equations [3], linking the thermodynamical flows I_n and generalized thermodynamical forces L_m, causing these flows:

$$I_n = \int_{-\infty}^{t} dt' \int_R dx' K_{nm}(t, t', x, x') L_m(t', x'). \tag{2}$$

Here $K_{nm}(t, t', x, x')$ is a kernel, taking into account nonlocal effects. Function K_{nm} can be calculated, in principle, by solving dynamic problem of structure's elements interaction, however such calculations are extremely difficult. Therefore in practice one uses, as a role, some model kernels, describing well enough the main properties of the non-local effects and, in particular, the fact that these effects vanish rapidly as $|t - t'|$ and $|x - x'|$ grow. This property will enable us to pass from the equations of the integro-differential type to systems of differential equations, including higher-order derivatives.

Thus our goal is to present the modelling systems, associated with the integral DES, which describe the non-local effects connected with structure. We show that, in spite of the fact that the systems of PDE's considered here are not Hamiltonian, they do possess families of invariant solutions, satisfying certain systems of ODE's, being equivalent to the Hamiltonian ones. Due to this property, it becomes possible to state the existence of periodic and soliton-like solutions, which are inherent to the models of structured media and presumably do not occur in hydrodynamic-type models of structureless media. At least, that is true under the assumption that the characteristic velocity of the wave perturbations is a growing function of ρ.

2 Invariant localized solutions in the model with spatial nonlocality

One of the simplest state equation accounting for the effects of spatial nonlocality takes on the form [4]:

$$p = \hat{\sigma} \int_{-\infty}^{+\infty} \rho^n(t,x') \exp\left\{-\left[\left(x-x'\right)/l\right]^2\right\} dx'. \tag{3}$$

Since the kernel $K(x,x') = \exp\left\{-\left[\left(x-x'\right)/l\right]^2\right\}$ extremely quickly approaches zero as $|x-x'|$ grows, we can substitute the function $\rho^n(t,x')$ by its decomposition into the power series:

$$\rho^n(t,x') = \rho^n(t,x) + [\rho^n(t,x)]_x \frac{x'-x}{1!} + [\rho^n(t,x)]_{xx} \frac{(x'-x)^2}{2!} + o\left(|x-x'|^2\right),$$

obtaining this way so called gradient model [4]:

$$p = c_0 \hat{\sigma} \rho^n(t,x) + c_2 \hat{\sigma} [\rho^n(t,x)]_{xx}. \tag{4}$$

Here

$$c_0 = l \int_{-\infty}^{+\infty} e^{-\tau^2} d\tau = l\sqrt{\pi}, \qquad c_2 = \frac{l^3}{2} \int_{-\infty}^{+\infty} \tau^2 e^{-\tau^2} d\tau = \frac{l^3\sqrt{\pi}}{4}.$$

Inserting (4) into the first equation of system (1), we obtain a closed system:

$$\begin{aligned} &u_t + \beta \rho^{\nu+1} \rho_x + \sigma \left[\rho^{\nu+1} \rho_{xxx} + 3(1+\nu)\rho^{\nu} \rho_x \rho_{xx} + \nu(1+\nu)\rho^{\nu-1} \rho_x^3\right] = 0, \\ &\rho_t + \rho^2 u_x = 0, \end{aligned} \tag{5}$$

where $n = \nu + 2$, $\beta = c_0 \hat{\sigma}(\nu+2)$, $\sigma = c_2 \hat{\sigma}(\nu+2)$. Below we analyze a family of invariant travelling wave solutions

$$u = U(\omega), \qquad \rho = R(\omega), \qquad \omega = x - Dt. \tag{6}$$

Inserting the anzatz (6) into the second equation of system (5), we obtain the first integral:

$$U = C_1 - D/R. \tag{7}$$

Asymptotic conditions $\lim\limits_{\omega \to +\infty} U(\omega) = 0$, $\lim\limits_{\omega \to +\infty} R(\omega) = R_1 > 0$, which we assume to take place further on, get the expression $C_1 = D/R_1$. Substituting (6) into the second equation of system (5) and using the formula (7), we obtain a third-order equation for function $R(\omega)$. This equation can also be integrated, and, having done that, we finally get the second order ODE

$$\frac{D^2}{R} + \frac{\beta}{\nu+2} R^{\nu+2} + \sigma \left[R^{\nu+1} \frac{d^2R}{d\omega^2} + (\nu+1) R^{\nu} \left[\frac{dR}{d\omega}\right]^2\right] = E, \tag{8}$$

where

$$E = \frac{D^2}{R_1} + \frac{\beta}{\nu+2} R_1^{\nu+2}, \tag{9}$$

is a constant of integration, defined by the conditions on $+\infty$.

Let us write equation (8) in the form of the first order dynamic system:

$$\begin{aligned} &\frac{dR}{d\omega} = Y, \\ &\frac{dY}{d\omega} = \left(\sigma R^{\nu+2}\right)^{-1} \left\{ ER - \left[D^2 + \frac{\beta}{\nu+2} R^{\nu+3} + \sigma(\nu+1) R^{\nu+1} Y^2 \right] \right\}. \end{aligned} \tag{10}$$

It is evident that all isolated critical points of system (10) are located on the horizontal axis OR. They are determined by solutions of the algebraic equation

$$P(R) = \frac{\beta}{\nu+2} R^{\nu+3} - ER + D^2 = 0. \tag{11}$$

As can be easily seen, one of the roots of equation (11) coincides with R_1. Location of the second real root depends on relations between the parameters. If $\nu + 3 > 1$ and D^2 satisfies inequality

$$D^2 > D_{\rm cr}^2 = \beta R_1^{\nu+3}, \tag{12}$$

then there exists the second critical point $R_2 > R_1$. Moreover, if $\nu > 0$ is a natural number, then the polynomial $P(R)$ has the representation

$$P(R) = (R - R_1)(R - R_2)\Psi(R), \tag{13}$$

where

$$\Psi(R) = \frac{\beta}{(\nu+2)(R_2 - R_1)} \left\{ R^{\nu+1} + R^{\nu}(R_2 - R_1) + \cdots + R\left(R_2^{\nu} - R_1^{\nu}\right) + \left(R_2^{\nu+1} - R_1^{\nu+1}\right) \right\}.$$

Note that $\Psi(R)$ is positive, when $R > 0$. It is a direct consequence of the existence of representation (13) when ν is a natural number. But this is also true for any $\nu > -2$, or, in other words, whenever the function $R^{\nu+3}$ is concave for positive R.

Analysis of linearization matrix for the system (10)

$$\hat{M}(R_i, 0) = \begin{bmatrix} 0 & 1 \\ \left(\sigma R_i^{\nu+2}\right)^{-1} \Psi(R_i)(R_j - R_i) & 0 \end{bmatrix}, \qquad i = 1, 2, \qquad j \neq i \tag{14}$$

shows that the critical points $A_1(R_1, 0)$ is a saddle, while the critical point $A_2(R_2, 0)$ is a center. Thus, system (10) has only such critical points that are characteristic to the Hamiltonian system. This circumstance suggests that there could exist a Hamiltonian system equivalent to (10). The Hamiltonian function would help us to make a complete study of the phase portrait of system (10) and find out homoclinic trajectories, which correspond to a soliton-like wave packs, providing that such trajectories do exist.

If we introduce a new independent variable T, obeying the equation $\frac{d}{dT} = \sigma R^{\nu+2} \phi(R, Y) \frac{d}{d\omega}$, then system (10) can be written as

$$\begin{aligned} &\frac{dR}{dT} = \sigma R^{\nu+2} \phi(R, Y), \\ &\frac{dY}{dT} = \phi(R, Y) \left(ER - \left[D^2 + \frac{\beta}{\nu+2} R^{\nu+3} + \sigma(\nu+1) R^{\nu+1} Y^2 \right] \right). \end{aligned} \tag{15}$$

Here $\phi(R,Y)$ is a function, that is to be chosen in such a way that system (15) be Hamiltonian. So we assume the existence of a function $H(R,Y)$, satisfying the system

$$\frac{\partial H}{\partial Y} = \sigma R^{\nu+2}\phi(R,Y),$$
$$\frac{\partial H}{\partial R} = -\phi(R,Y)\left\{ER - \left[D^2 + \frac{\beta}{\nu+2}R^{\nu+3} + \sigma(\nu+1)R^{\nu+1}Y^2\right]\right\}.$$

Equating mixed derivatives of H, we obtain the characteristic system

$$\frac{dR}{\sigma R^{\nu+2}Y} = \frac{dY}{ER - [D^2 + \frac{\beta}{\nu+2}R^{\nu+3} + \sigma(\nu+1)R^{\nu+1}Y^2]} = \frac{d\phi}{\nu\sigma Y R^{\nu+1}\phi}. \tag{16}$$

The general solution of system (16) can be written as follows:

$$\phi = R^{\nu}\Psi(\Omega),$$

where $\Psi(\cdot)$ is an arbitrary function of the variable

$$\Omega = R^{\nu+1}\left\{Y^2 + D^2\sigma^{-1}R^{-(\nu+1)}\ln R + R^2\beta/[\sigma(\nu+2)(\nu+3)] - E\sigma^{-1}R^{-\nu}\right\}.$$

Putting $\phi = 2R^{\nu}$, we can easily restore the Hamiltonian function:

$$H = 2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)} + \sigma Y^2 R^{2(\nu+1)} - 2E\frac{R^{\nu+2}}{\nu+2}. \tag{17}$$

By elementary checking one can get convinced that the function H is constant on phase trajectories of both systems (10) and (15), and since the integrating multiplier $\phi = 2R^{\nu}$, occurring in formula (15) is positive for $R > 0$, then phase trajectories of systems (10) and (15) are almost similar in the right half-plane of the phase plane (R,Y), the only difference between them manifesting in velocities of motion. Thus all the statements concerning the geometry of the phase trajectories of system (15) lying in the right half-plane is applicable to corresponding solutions of system (10).

Using the linear analysis we showed that the critical point $A_2(R_2,0)$ is a center. Stated above relations between systems (10) and (15) enable to conclude that this point does not change when the nonlinear terms are added. This means that the critical point $A_2(R_2,0)$ is surrounded by closed trajectories and hence the source system (5) possesses a one-parameter family of periodic solutions. If the right branches of the separatrices of the saddle $A_1(R_1,0)$ go to infinity (the stable branch W^s when $t \to -\infty$ and the unstable branch W^u when $t \to +\infty$), then the domain of finite periodic motions is unlimited. Another possibility is connected with the existence of homoclinic trajectories in phase space. In this case the domain of periodic solutions is bounded, and the source system, besides periodic solutions, possesses localized soliton-like regimes. To answer the question on which of the above mentioned possibilities is realised in system (10), the behavior of the saddle separatices, lying to the right from the line $R = R_1$ should be analyzed. We obtain the equation for saddle separatices by putting $H = H(R_1,0) = H_1$ in the left hand side of the equation (17) and solving it next with respect to Y:

$$Y = \pm\frac{\sqrt{H_1 + 2E\frac{R^{\nu+2}}{\nu+2} - \left[2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)}\right]}}{\sqrt{\sigma}R^{\nu+1}}. \tag{18}$$

It is evident from equation (18), that incoming and outgoing separatrices are symmetrical with respect to OR axis. Therefore we can restrict our analysis to one of them, e.g. to the upper

separatrix Y_+. First of all, let us note, that in the point$(R_1, 0)$ separatrix Y_+ forms with OR axis a positive angle

$$\alpha = \arctan \sqrt{(R_2 - R_1)\Psi(R_1)/\left(\sigma R_1^{\nu+2}\right)}.$$

The above formula arises from the linear analysis of system (10) in critical point $A_1(R_1, 0)$. So $Y_+(R)$ is increasing when $R - R_1$ is small and positive. On the other hand, function

$$G(R) = H_1 + 2E\frac{R^{\nu+2}}{\nu+2} - \left[2D^2\frac{R^{\nu+1}}{\nu+1} + \frac{\beta}{(\nu+2)^2}R^{2(\nu+2)}\right],$$

standing inside the square root in equation (18), tends to $-\infty$ as $R \to +\infty$, because the coefficient at the highest order monomial $R^{2(\nu+2)}$ is negative, while the index $\nu + 2$ is assumed to be positive. Therefore the function $G(R)$ intersects the open set $R > R_1$ of the OR axis at least once. Let us denote a point of the first intersection by R_3, and let us assume that $R_3 > R_2$. If with this assumption we were able to prove that $Y_\pm(R)$ form the right angle with the OR axis at the point R_3, then we would have the evidence of the continuous homoclinic loop existence.

We begin with the note that $\lim_{R\to R_3^-} G(R) = +0$. Calculating derivative of $G(R)$ we have:

$$G'(R) = -2R^\nu\left(\frac{\beta}{\nu+2}R^{\nu+3} - ER + D^2\right) = -2R^\nu P(R). \tag{19}$$

It follows from the decomposition (13) that $G'(R) < 0$ when $R > R_2$. Therefore

$$\lim_{R\to R_3^-}\frac{dY}{dR} = \frac{RG'(R) - 2(\nu+1)G(R)}{2\sqrt{\sigma G}R^{\nu+2}} = -\infty$$

and we merely have to show that the inequality $R_3 > R_2$ is true. Supposing that the inequalities $R_1 < R_3 < R_2$ take place, we obtain from equations (19), (13) that $\lim\limits_{R\to R_3^-} Y'_+(R) = +\infty$. On the other hand, the function $Y_+(R)$ approaches zero remaining positive as $R \to R_3^-$. But such behavior is impossible for any function, regular on the interval (R_1, R_3). The case $R_3 = R_2$ should also be excluded, because the critical point $A_2(R_2, 0)$ is a center. The result obtained can be formulated as follows.

Theorem 1. *If $\nu > -2$ and $D^2 > \beta R_1^{\nu+3}$, then system* (10) *possesses a one parameter family of periodic solutions, localized around the critical point $A_2\,(R_2, 0)$ in a bounded set* **M**. *The boundary of this set is formed by the homoclinic intersection of separatrices of the saddle point $A_1\,(R_1, 0)$.*

Thus the source system (5) possesses periodic and soliton-like invariant solutions.

3 Invariant wave patterns in the model with spatio-temporal nonlocality

In this section we consider system (1), closed by DES:

$$p = \int_{-\infty}^{t} sin\frac{t-t'}{\tau_1}e^{-\frac{t-t'}{\tau_2}}L[\rho, \rho_x, \rho_{xx}]dt', \tag{20}$$

where

$$L[\rho, \rho_x, \rho_{xx}] = \frac{\beta}{\nu+2}\rho^{\nu+2}(t', x) + \hat{\sigma}[\rho^{\mu+1}(t', x)\rho_x(t', x)]_x.$$

Taking the second derivative of (20) with respect to t, we obtain a higher-order equation, which, together with (1), forms a closed system of the following form:

$$u_t + p_x = 0, \qquad \rho_t + \rho^2 u_x = 0,$$
$$hp_{tt} + \tau p_t + p = \frac{\beta}{\nu+2}\rho^{\nu+2} + \sigma\left[\rho^{\mu+1}\rho_{xx} + (\mu+1)\rho^{\mu}(\rho_x)^2\right], \tag{21}$$

where $h = (\tau_1\tau_2)^2/\left(\tau_1^2+\tau_2^2\right)$, $\tau = 2\tau_1^2\tau_2/\left(\tau_1^2+\tau_2^2\right)$.

As in the previous section, we analyze the family of invariant travelling wave solutions

$$u = U(\omega), \qquad \rho = R(\omega), \qquad p = \Pi(\omega), \qquad \omega = x - Dt. \tag{22}$$

Inserting (22) into (21), we obtain a system of ODE, possessing two integrals. The first one, arising from the balance of mass equation, coincides with (7), while the second one, arising from the momentum equation, is as follows:

$$\Pi = E - D^2/R. \tag{23}$$

If the conditions on $+\infty$ are identical with those from the previous section, then the expression for E is given by the formula (9). Taking advantage of the formulae (7), (23), we can write down the remaining second order equation in the form of a dynamic system:

$$R\Delta\phi\dot{R} = YR\Delta\phi,$$
$$R\Delta\phi\dot{Y} = -\phi\left\{R^2P(R) + Y^2\left[2hD^4 + \sigma(\mu+1)R^{\mu+3}\right] + \tau D^3R^2Y\Delta\right\}, \tag{24}$$

where $\Delta = \sigma R^{\mu+3} - hD^4$ and $P(R)$ is given by the formula (13). Here we introduced an extra function $\phi(R,Y)$ that would help us to make system (24) Hamiltonian.

We postpone with the construction of the Hamiltonian function for system (24) and study its critical points first, for a pure linear analysis shows that the multiplier $\phi(R,Y)$ must have the singularities in the area of interest, unless some parameters nullify. It is evident that all isolated critical points of system (24) satisfy the equation (11). The case we are interested in is again that with two real solutions R_1, R_2, satisfying the inequality $R_2 > R_1$. Therefore we assume that parameter D^2 satisfies the inequality (12). An extra inequality

$$\frac{\beta(\nu+3)}{\nu+2}R_1^{\nu+2} < E < \frac{\beta(\nu+3)}{\nu+2}R_2^{\nu+2}, \tag{25}$$

which will be used later on, is the direct consequence of the inequality (12) and concavity of the function $\beta R^{\nu+3}$.

The linearization matrix of system (24) in a critical point $A_i\,(R_i,0)$, $i=1,2$, takes on the form:

$$\hat{M}(R_i,0) = \begin{bmatrix} 0, & R_i\Delta_i\phi(R_i,0) \\ \phi(R_i,0)R_i\left(3ER_i - 2D^2 - \frac{\beta(\nu+5)}{\nu+2}R_i^{\nu+3}\right), & -\phi(R_i,0)\tau D^3R_i^2\Delta_i \end{bmatrix}, \tag{26}$$

where $\Delta_i = \sigma R_i^{\mu+3} - hD^4$. Thus, the eigenvalues of matrix $\hat{M}(R_i,0)$ satisfy the equation

$$\lambda\left(\lambda + \phi(R_i,0)\tau D^3R_i^2\Delta_i\right) = \phi(R_i,0)^2\Delta_iR_i^3\left[E - (\nu+3)(\nu+2)^{-1}R^{\nu+2}\right]. \tag{27}$$

Considering equation (27) together with the inequalities (25) one can easily conclude that the features of the isolated critical points of system (24) depend in essential way on the sign of Δ_i. If R_1 is placed to the right from the line $\Delta(R)=0$, i.e.

$$R_4 = \left(D^4h/\sigma\right)^{1/(\mu+3)} < R_1 \tag{28}$$

and therefore inequality $\Delta(R_i) > 0$ holds for both $i = 1, 2$, then the eigenvalues corresponding to critical point $A_1\,(R_1, 0)$ are nonzero and have the opposite signs, so this point still is a saddle. The critical point $A_2\,(R_2, 0)$ turns out to be either a stable focus, when τ is sufficiently small, or even a stable node. Anyhow, it is rather impossible to chose a nonsingular multiplier ϕ, unless $\tau = 0$ and the critical point $A_2\,(R_2, 0)$ is a center.

If the inequality $R_4 > R_2$ takes place, then the critical point $A_2\,(R_2, 0)$ is a saddle, while $A_1\,(R_1, 0)$ becomes a focus. The situation is more complicated when the inequalities $R_1 < R_4 < R_2$ hold, but in our attempts to capture the homoclinic trajectories we will rather focus upon the case $R_4 < R_1$, as the only one leading to the localized compressed waves appearance. Before we start to study the Hamiltonian case, let us pay attention to the fact that the inequalities (25), together with the condition $R_4 < R_1$, pose the following restrictions on the invariant wave pack velocity D:

$$\beta R_1^{\nu+3} < D^2 < \sqrt{\sigma h^{-1} R_1^{\mu+3}} \tag{29}$$

As we have mentioned above, it is rather impossible to chose a proper multiplier ϕ, whenever $\tau \neq 0$, because the corresponding term in the RHS of the second equation of system (24) introduces the dissipation. Yet, by proper choice of the parameters τ_1 and τ_2 in equation (20), we can fulfil the relations $\tau \ll 1$ and $h = O(1)$, enabling to drop down the term proportional to τ. For $\tau = 0$ function ϕ can be written, e.g. in the form $\phi = C\Delta R^{-5}$, where C is an arbitrary constant. The system (24) becomes Hamiltonian, if we choose ϕ in accordance with the above formula and introduce new independent variable T, using the relation $R\Delta\phi\frac{d}{dT} = \frac{d}{d\omega}$.

With such a choice of the multiplier, the Hamiltonian function is as follows:

$$H(R, Y) = \frac{\Delta^2}{\sigma R^4} Y^2 + \frac{1}{\sigma R^2}\left\{ D^6 h + \frac{2D^2\sigma R^{\mu+3}}{\mu+1} - \frac{2E\sigma}{\mu+2} R^{\mu+4} + \frac{2\sigma\beta}{(\nu+2)(\nu+\mu+4)} R^{\nu+\mu+6} - 2D^4 hRE - \frac{2D^4 h\beta R}{\nu^2+3\nu+2}\right\}. \tag{30}$$

To show that the closed loop does exist among the solutions of system (24), let us consider one of the separatrices of the saddle point $A_1\,(R_1, 0)$, namely:

$$Y_+(R) = \sqrt{\frac{\tilde{G}(R)}{F(R)}}, \tag{31}$$

where

$$\tilde{G}(R) = H\,(R_1, 0) - H\,(R, 0)\,, \qquad F(R) = \frac{\Delta^2}{R^4\sigma} \tag{32}$$

(note, that both equation (30) and (31), become identical with the corresponding equations of the previous section as we put $h = 0$ and $\nu = \mu$). As it follows from the linear analysis of system (24) in the critical point $A_1\,(R_1, 0)$, the curve $Y_+(R)$ forms with OR axis a positive angle α:

$$\alpha = \arctan\sqrt{{R_1}^2{\Delta_1}^{-1}\left[E - (\nu+3)(\nu+2)^{-1}R_1^{\nu+2}\right]}.$$

In virtue of the function $\tilde{G}(R)$ features, arising from the formulae (30), (32), it should intersect open set $(R_1, +\infty)$ at least once. Let us denote the point of the first intersection by R_3. Repeating the analysis from Section 2, we can show that the saddle separatrix $Y_+(R)$ and the OR axis form the right angle in the point of intersection and the inequalities $R_3 > R_2 > R_1$ take place. So the following statement holds.

Theorem 2. *If* $\nu > -2$, $R_1 > \left(D^4 h/\sigma\right)^{1/(\mu+3)}$ *and* D^2 *satisfies inequalities* (29)*, then system* (24) *possesses a one parameter family of periodic solutions, localized around the critical point* $A_2\left(R_2, 0\right)$ *in a bounded set* $\mathbf{M}$*. The boundary of this set is formed by the homoclinic intersection of separatrices of the saddle point* $A_1\left(R_1, 0\right)$*.*

4 Conclusions

Two non-local models have been presented describing long waves propagation in the media with internal structure. The main result obtained is that the hydrodynamic-type systems, accounting for non-local effects, possess periodic and soliton-like travelling wave solutions. The presence of the above solutions seems to be the direct consequence of the non-local effects, since any local hydrodynamic-type model, i.e. the system of balance equations (1) closed by the functional state equation $p = \Phi(\rho)$, does not possess them, when $d\Phi(\rho)/d\rho$ is a growing function for positive ρ. Numerous counter-examples (given e.g. in [5]) do not satisfy this condition and therefore deliver merely a set of unstable travelling wave solutions.

In order to prove the existence of nonlinear wave patterns we used in this work combination of symmetry reduction and qualitative analysis. Let us stress that a number of solutions with very interesting applications cannot be integrated explicitly, and the role of qualitative analysis is hardly to be overestimated in such situations. Very often they do enable to prove the existence of solutions of specific kind. For example the soliton-like solutions could be extracted by regular methods, providing that the system under consideration is either Hamiltonian, or close to the Hamiltonian [6]. But the preliminary studies of system (5) show [7], that it is not Hamiltonian for any physically justified values of the parameters. Nevertheless, the dynamic system, obtained from (5) by group-theory factorization, occurs to be equivalent to the Hamiltonian one, and therefore it is relatively easy to analyze. System (21), in turn, is a non-conservative system, because of the presence of dissipative term proportional to τ in the third (governing) equation. Therefore soliton-like solutions belonging to the family (22) appear when τ nullifies. Strictly speaking, the passage $\tau \to 0$ is an improper procedure, but the results obtained in the last section might serve as a starting point in looking for the waves patterns in the case when $0 < \tau \ll 1$ and an external force is present in the momentum equation. Analysis of the relaxing hydrodynamic-type system, undertaken in our previous works, shows [8] that the spatial inhomogeneity, associated with the presence of mass force, could compensate the dissipative effects, making possible the wave patterns appearance.

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Geometric Models, Fiber Bundles and Biomedical Applications

V. YATSENKO [†1], *J.C. SACKELLARES* [†2], *P. PARDALOS* [†3], *P. CARNEY* [†4] *and O. PROKOPYEV* [†3]

[†1] *Scientific Foundation of Researchers and Specialists on Molecular Cybernetics and Informatics (FMCI), 03022 Kyiv, Ukraine*
E-mail: *fond@pirogov.kiev.ua*

[†2] *Departments of Neurology and Neuroscience, University of Florida, Gainesville, FL 32611-6595, USA*

[†3] *Center for Applied Optimization and ISE Department and Biomedical Engineering Program, University of Florida, Gainesville, FL 32611-6595, USA*

[†4] *Department of Pediatrics, University of Florida, Gainesville, FL 32611-6595, USA*

In this paper we explore the use of dynamical system techniques, geometric methods and experimental approaches to estimate the characteristics of brain electrical activity. We assume that the diagnostic information is in the form of a nonlinear time series. Then we sequentially apply a geometric approach for characterizing dynamical instability, based on a nonlinear estimation of dynamical characteristics. In addition to characterizing epileptic seizures, we discuss other diagnostically useful topics including the T-index of the short-term maximum Lyapunov exponents (STLmax) among critical sites of the cerebral cortex. Dynamical instability is here related to curvature fluctuation of the manifolds where geodesics are natural motion and is described by means of the Jacoby–Levi-Civita geodesic spread. The methods are illustrated using EEG data previously recorded from transgenic epileptic mouse.

1 Introduction

Epilepsy, among the most common disorders of the nervous system, affects approximately 1 % of the population [1–3]. About 25 % of the patients with epilepsy have seizures that are resistant to medical therapy [4]. Epileptic seizures result from a transient electrical discharge of the brain. These discharges often spread first to ipsilateral, then to contralateral cerebral cortex, thereby disturbing normal brain function. Clinical effects may include motor, sensory, affective, cognitive, automatic and physical symptomatology. The occurrence of an epileptic seizure appears to be unpredictable and the underlying mechanisms that cause seizures to occur are poorly understood. A recent approach in epilepsy research is to investigate the underlying dynamics that account for the physiological disturbances in the epileptic brain [5–7].

One of the most promising approaches to understanding of the dynamics of epilepsy is to determine the dynamical properties of the EEG signal generated by the epileptic brain. The traditional technique used to solve this problem in time series analysis is to fit a linear model to the data, and determine the optimal order (*dimension*) and the *optimal parameters* of the model. A simple generalization is to fit the best nonlinear dynamical model. However, the results of applying this methodology are usually not very illuminating. Without any prior knowledge, any model that we fit is likely to be *ad hoc*. We are more interested in questions as the following: How nonlinear is the time series? How many degrees of freedom does it have?

The present study was undertaken to develop a geometric methodology for estimating dynamical characteristics of an EEG time series and classifying the physiological state (e.g., interseizure state, pre-seizure state, seizure state and post-seizure state) of the human epileptic brain. This can be accomplished by employing a computer based algorithm that continuously estimates multiple dynamical measures from the input EEG time series. In addition, in order to create a modeling technique that incorporates existing information and can be interpreted neurodynamically (robust to noise), we propose a technique based on a combination of *dynamics*, *geometry*, and *fiber bundles*.

2 Geometrization of the brain dynamics

During the past decade, there has been growing evidence of the independence of the two properties of instability and predictability of the human brain dynamics. The generic situation of the brain dynamics is instability of the trajectories in the Lyapunov sense. Nowdays such instability is called intrinsic stochasticity, or chaoticity, of the brain dynamics and is caused by nonlinearity of the equation of motion.

In order to characterize the dynamical instability, we first examined some properties that characterize the EEG signal, including the spectrum of Lyapunov exponents, the energy, geodesics and fiber bundles. We analyze the parameter spaces as well as related quantities of T-index of STLmax, Jacoby–Levi-Civita equation, and modeling of EEG time series from the experimental point of view.

2.1. Spectrum of Lyapunov exponents. The leading Lyapunov exponent [9] λ is defined as:

$$\lambda_1(x_0) = \lim_{t\to\infty}\lim_{\epsilon\to 0}\frac{1}{t}\log\left(\frac{\|\langle y(t)\rangle - \langle x(t)\rangle\|}{\epsilon}\right), \tag{1}$$

where $y(t)$ and $x(t)$ are experimental time series of the dynamical system with two initial conditions $y(0)$ and $x(0)$; $\langle y(t)\rangle$ and $\langle x(t)\rangle$ are ensemble averages; $\|\langle y(t)\rangle - \langle x(t)\rangle\|$ is the Euclidean distance between $\langle y(t)\rangle$ and $\langle x(t)\rangle$; $\epsilon = \|\langle y(0)\rangle - \langle x(0)\rangle\|$. The equation (1) remains well defined in the presence of noise.

To characterize the dynamics from the observed EEG data $x(t)$ we need to find the local Lyapunov exponents using the local Jacobian matrix and the Oseledec matrix. The eigenvalues of the Oseledec matrix tell us how rapidly perturbations to the orbit at point x in phase space grow or shrink in P time steps away from the time of the perturbation. These $\lambda_i(x, P)$ are called the local Lyapunov exponents or the finite time Lyapunov exponents [9, 11].

The Lyapunov exponents provide a coordinate-independent measure of the asymptotic local stability of EEG data. The trajectories of an N dimensional state space have N Lyapunov exponents. These N Lyapunov exponents are often called the Lyapunov spectrum [9]. The qualitative features of the asymptotic local stability properties can be summarized by the sign of each Lyapunov exponent. A positive Lyapunov exponent indicates an unstable direction; a negative exponent indicates a stable direction. Thus $(+,-,-)$ implies a trajectory in three-dimensional state space with one positive Lyapunov exponent. The Lyapunov numbers Λ_i are defined as

$$\Lambda_i \equiv e^{\lambda_i}. \tag{2}$$

The Lyapunov spectrum and the metric entropy are related by Pesin's identity [14]

$$h_\mu = \sum_{i=1}^{r} \text{positive } \lambda_i, \tag{3}$$

where λ_i represents the Lyapunov exponents and r is the maximal number of positive Lyapunov exponents.

2.2. Lyapunov dimension. The Lyapunov dimension is a characteristics related to the spectrum of Lyapunov exponents and predictability of the EEG signal. The hypersurface of dimension m of the EEG signal expands at a rate governed by the sum $\sum_{i=1}^{m} \lambda_i$, where m is the embedding dimension. Kaplan and Yorke [12] have suggested the following definition of the Lyapunov dimension

$$d_L = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}, \tag{4}$$

where k is the largest value such that $\sum_{i=1}^{k} \lambda_i > 0$, and the second term characterizes the fractional part of the dimension. We may anticipate that for an epileptic attractor, the Lyapunov dimension is equal to the information dimension.

2.3. Geometric models and fiber bundles. This approach involves a geometric description of Lyapunov exponents for correcting the nonlinear process that provides adaptive dynamic control.

We separate the Lyapunov exponents into tangent space (fiber bundle) and its functional space. Control involves signal processing, calculation of an information characteristic, measurement of Lyapunov exponents, and feed-back to the system. With more information, we can reduce uncertainty by a certain degree.

We have demonstrated the computational aspects of the proposed geometric approach on the base of different mathematical models in the presence of noise of various origins [10]. We review the EEG signal, and outline a typical application of the geometrical representation: three dimensional reconstruction of Lyapunov exponents and correlation dimension obtained from EEG data.

The novelty in this report is in the representation of the dynamical instability by a Riemannian theory in a way that permits practical applications. We discuss an application of this approach to the development of novel devices for seizure control through electromagnetic feed-back.

2.4. Geometric description of dynamical instability. The actual interest of the Riemannian formulation of dynamics stems from the possibility of studying the instability of the brain dynamics through the instability of geodesics of a suitable manifold, a circumstance that has several advantages.

First of all, a powerful mathematical tool exists to investigate the stability or instability of a geodesic flow: the Jacobi—Levi-Civita (JLC) equation for geodesic spread.

The JLC equation describes covariantly how nearby geodesics locally scatter and is a familiar object in both Riemannian geometry and theoretical physics.

Moreover, the JLC equation relates the stability or instability of a geodesic flow with curvature properties of the ambient manifold, thus opening a wide and largely unexplored field for investigation, as far as physical systems are concerned, of the connections among geometry, topology, and geodesic instability.

Geometrization of the brain dynamics includes the following stages: 1) reconstruction of equations of the epileptic brain from experimental data; 2) realization in local coordinates of a one-parameter group of diffeomorphisms of a manifold M; 3) estimation of the largest Lyapunov exponent; 4) geometrization of the dynamics; 5) geometric description of the dynamical instability; 6) applying Jacobi–Levi-Civita equation for geodesic spread; 7) analytical description of the largest Lyapunov exponent.

2.5. An analytic formula for the largest Lyapunov exponent. By transforming the Jacobi–Levi-Civita equation from geodesic spread into a scalar equation for ψ variable, the original complexity of the JLC equation has been considerably reduced: from a tensor equation we have obtained an effective scalar equation formally representing a stochastic oscillator [13].

Our Lyapunov exponent is defined as

$$\lambda = \lim_{t \longrightarrow \infty} \frac{1}{2t} \ln \frac{\psi^2(t) + \dot{\psi}^2(0)}{\psi^2(0) + \dot{\psi}^2(0)}, \tag{5}$$

where $\psi(t)$ is solution of the equation

$$\frac{d^2\psi}{ds^2} + \Omega(t)\psi = 0, \tag{6}$$

$\Omega(t)$ is a Gaussian stochastic process;

$$\Omega(t) = \langle k_R \rangle_\mu + \mu \frac{1}{\sqrt{N}} \langle \delta^2 K_R \rangle_\mu^{\frac{1}{2}} \eta(t), \tag{7}$$

if the Eisenhart metric is used.

The instability growth rate of ψ measures the instability growth rate of $\|J\|^2$ (geodetic separation field) and thus provides the dynamical instability exponent in our Riemannian framework.

Equation (6) is a scalar equation that, independently of the knowledge of the dynamics, provides a measure of the average degree of instability of the dynamics itself through the behavior of $\psi(s)$. The peculiar properties of a given Hamiltonian system enter (6) through the global geometric properties $\langle k_R \rangle_\mu$ and $\langle \delta^2 K_R \rangle_\mu$ of the ambient Riemannian manifold whose geodesics are natural motions and are sufficient to determine the average degree of chaoticity of the dynamics.

3 Experimental results

In this section, we examine EEGs from an experimental murine model of human epilepsy that has seizures which like humans, are spontaneous, intermittent, and sometimes lethal. This mouse model was the product of genetic engineering in which the H218/AGR16/Edg-5/LP(B2) sphingosine 1-phosphate receptor gene has been disrupted. Beginning on postnatal day 19 through postnatal day 22, H218 deficient mice exhibit generalized seizures which can be easily identified in intracranial EEG recordings [8].

In brief, this experimental model was produced by knocking out the entire protein coding region of the single copy mouse H218 gene through homologous recombination [8]. Southern blot analysis with 3' and 5' probes, as well as PCR analysis confirmed the appropriate location of the mutation in both ES cells and mice following germ line transmission. The appearance and behavior of the newborn H218 gene deficient mice were indistinguishable from that of their control littermates. Postnatal days 19 through 22 H218 deficient mice ($n = 8$) were continuously monitored with video-EEG for electrographic and behavioral seizures. Microelectrodes were placed in both frontal and hippocampal regions, bilaterally (Fig. 1).

Signals were sampled at 200 Hz, using an analog to digital (A/D) converter with 12 bits quantitation, and amplifiers with input range of -2.5 to $+2.5$ mV and frequency range of 0.05 Hz to 70 Hz. These recordings were obtained using a BSMI/Nicolet (Madison, WI) 32 channel video-EEG instrument. Seizures consisted of wild-running fits and/or clonic-tonic movements of the arms and legs. Concomitant EEG changes consisted of continuous bilateral spike and wave discharges.

EEG seizures were the most robust over the frontal electrodes. Following each seizure, EEG background activity was slowed and suppressed by 50 % in comparison to the preictal EEG activity. At a cellular level, whole-cell patch clamp recordings revealed that the loss of H218 leads to a large increase in the excitability of neocortical pyramidal neurons [8]. The loss of H218 did not affect intrinsic membrane properties. However, the H218 neurons displayed significant increases in both the frequency and amplitude of spontaneous excitatory postsynaptic currents (sEPSCs). Under current clamp, spontaneous, paroxysmal depolarizing shift accompanied by bursts of action potentials were observed in 10 out of 14 H218 cells in physiological bath solution [8].

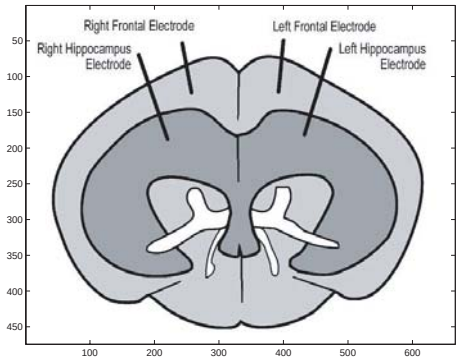

Figure 1. Schematic diagram of the depth electrode placement. Ictal discharge begins as spike and wave discharges in bilateral frontal and hippocampal electrodes.

Our experimental studies utilized the T-index of a homozygous and a wild mouse from an electrode site overlying the seizure focus (Figs. 2–6).

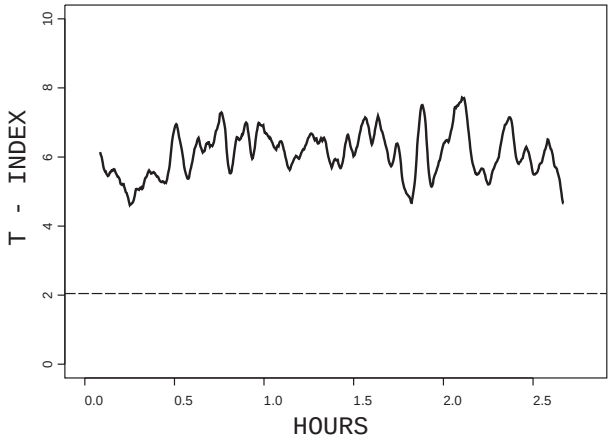

Figure 2. T-Index profiles in H218+/+ littermate. An example of continuous 3 hr trace obtained from a 21 day old littermate control illustrating a persistently higher average T-index. Note that the average T-index does not decline below the critical value of 2.045 (represented by the discontinuous horizontal line). No seizures occurred in littermates. T-index plot in postnatal day 21 control mouse.

Time-resolved analysis of intracranial electroencephalogram (EEG) signals recorded in a transgenic epileptic mouse indicates marked changes in spatiotemporal dynamics, often beginning several minutes prior to seizure onset. The spatiotemporal dynamics of this preictal (before seizure) transition differ markedly from that of the interictal (between seizure) period. If interpreted as a loss of complexity in the brain electrical activity of the mouse, these changes could reflect continuous increase synchronization between pathological discharging neurons and alow one to study seizures-generating mechanisms in a seizure-prone brain. Time-resolved analysis of neuronal activity recorded in a seizure prone mouse indicates marked changes in nonlinear

dynamical characteristics for up to several minutes prior to seizure onset in comparison to the interictal seizure state.

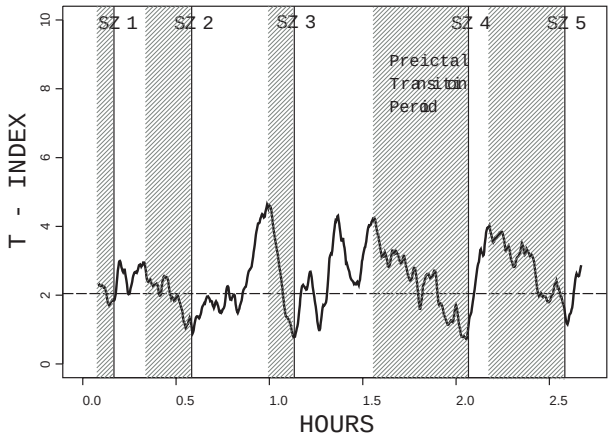

Figure 3. T-Index profiles in H218 deficient mice. Five seizures (SZ 1-5) are depicted during 3 hrs of continuous EEG recording in a 21 day old H218 deficient mouse. Lines indicate the time of the EEG seizure onset. The gray area represents the pre-seizure time period. Note that the average T-index falls below the threshold value of 2.045 several min prior to an EEG seizure.

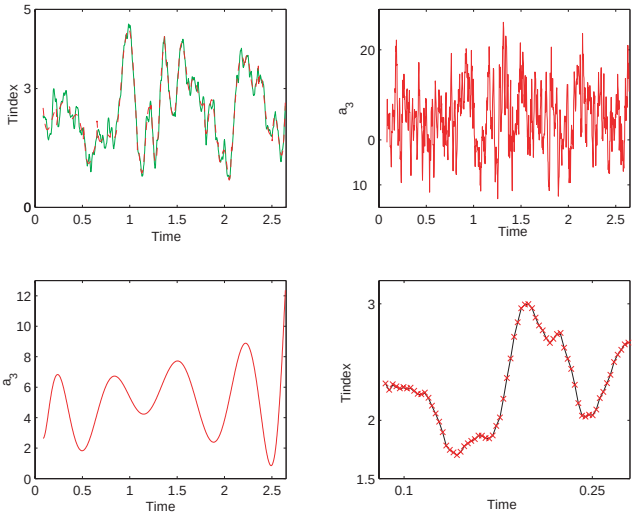

Figure 4. Local prediction T-index based on reconstructed three-dimensional model. The top two plots illustrate the T-index created from the H218 deficient mice and the corresponding parameter a_3 as function of time; the bottom two plots demonstrate the polynomial approximation of the parameter a_3 and estimated and predicted (criss-cross) T-index.

4 Conclusion

We have proposed a geometric approach to studying of the physiological disturbances that occur in human epilepsy. Under reasonable hypotheses, which obviously restrict the validity of the geometric approach, this paper provides the possibility for numerical computation of the state changes in the EEG signals using the largest Lyapunov exponent and the combination of the curvature of the underlying manifold and the geodesics. These geodesics flows may have very specific hidden symmetries, mathematically defined through Killing tensor field.

The development of an appropriate model of the epileptic brain will provide opportunities to investigate the effects of various approaches to a dynamical control. This would be an essential first step in the development of novel treatments, based upon the geometric theory of nonlinear

dynamics. Clearly, further work is needed to carefully probe experimentally observed dynamics of the epileptic brain, and to clarify the bifurcation and self-organization structures that display complex probability distribution functions.

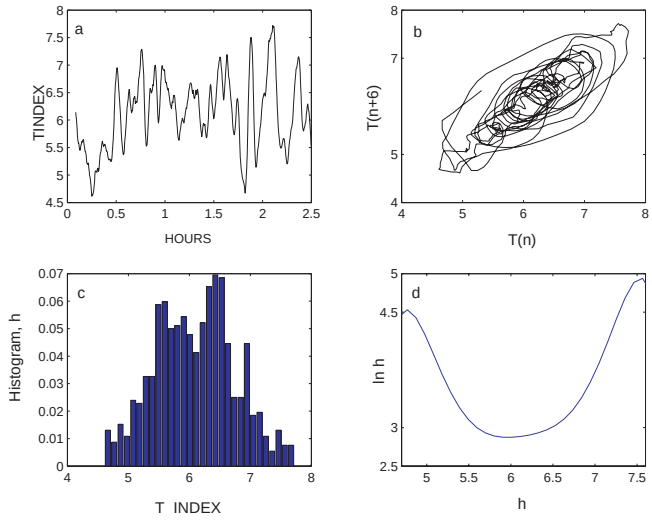

Figure 5. The top two plots illustrate the T-index created from the H218+/+ littermate deficient mice and the corresponding phase space; the bottom two plots demonstrate the histogram obtained from T-index and the and $\ln f$.

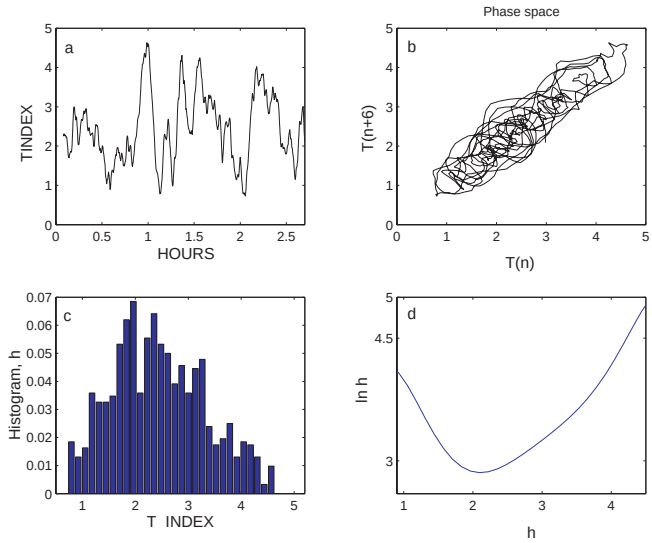

Figure 6. The top two plots illustrate the T-index created from the H218 deficient mice and the corresponding phase space; the bottom two plots demonstrate the histogram obtained from T-index and the and $\ln f$.

We estimate the T-index which reflects the complexity of the motion of the brain considered. Our results suggest that neuronal oscillatory behavior in the H218 deficient brain reflects neuronal network synchrony between frontal and hippocampal brain sites. Importantly, the observed oscillatory neuronal behavior occurs in anticipation of an impending seizure. The main factors that contribute to the occurrence of such synchronous oscillations in neuronal networks may include the intrinsic properties of the neurons, the structure of the interconnectivity between the networks elements, the synaptic processes that subserve specific inputs and feedback/feedforward loops, and the modulating influence from general or local neurotransmitters.

The hybrid time series analysis of ongoing EEG signals may be used to extract features of the signal which are characteristic of the preictal (before seizures) transition. It could be

demonstrated that the synchronization phenomena of the preictal state differ clearly from that found during seizures-free interval under various conditions. Both synchronization duration and strength are of a sufficient magnitude to open a time frame that provides possibilities for pharmacological or electro-physiologic interventions in the pre-ictal period. It remains to be established whether different methods of nonlinear time series analysis can provide information that will lead to understanding of the mechanisms underlying epileptogenesis.

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Conclusion

Now when we have completed preparation of these three large tomes we can consider the Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" to be over. We cannot expect all the readers to have read all parts of the Proceedings but we believe that they present a good picture of the state-of-art research on various symmetry-related problems.

We endorse the current tendency for facilitation of free access to research results, and we believe that publications of research should serve first of all to scientific community. In this framework we will place all papers included into these Proceedings at the conference web-page: `www.imath.kiev.ua/~appmath/conf.html` with free access. Parallel electronic publication have a special benefit for the papers included into these Proceedings having colour pictures and animations that are available only in the electronic form.

Free access to proceedings of the previous conferences is also available at the conference page. Our page has also pictures from the 2003 and previous conferences, together with links to home pages of the participants with more photos. A unique information asset of our web site is the page of late Professor Wilhelm Fushchych that contains downloadable version of majority of his papers and books (and is to include all his papers in nearest future), and many photos.

Now our web-page has a most complete collection of links to home pages of people from all over the world whose research is related to symmetry and other exact methods in differential equations: `http://www.imath.kiev.ua/~appmath/links.html` (please send us new links if we have omitted somebody). Many of these people have their papers available at their pages, and we call everybody to follow their example! At the same page you can also find links to lists of forthcoming conferences and books on the subject.

We were very glad to see that all our efforts in the previous year led to a successful meeting. We would like to share the secret of this success that is very simple – all the work was done by the people who there interested in having this conference. We had almost no funds and external support, but unlimited enthusiasm of the local organising committee. We would like to thank here once more to IAMP and STCU for partial support of publication of these Proceedings, and to the publishers that provided books and journals for the exhibition during the conference and donated them to the library of the Institute of Mathematics.

We would like to thank all the participants for their contribution to the success of the conference and to these Proceedings, and for providing inspiration to the organisers. We also would like to list the talks and posters that were presented at the conference but not represented in these Proceedings:

D. Abrarov, "Gauge symmetries of the Euler–Poisson equations"
L. Badowski, "From separable geodesic motion to bi-Hamiltonian dispersionless chains"
O. Batsula, "Conformal invariance, geometry-matter duality and number theory"
Yu. Berest, "Representation theory of rational Cherednik algebras"
Yu. Bespalov, "Categories beyond globes and cubes"
L. Chekhov, "Continuous genus limit of quantum Teichmuller spaces"
O. Hentosh, "Nonlocal invariant reductions of supersymmetric hierarchies"
L. Hoevenaars, "On the WDVV equations"
P. Holod, "Integrable equations for nonlinear Alfven waves propagation"
K. Ilyenko, "Minimal null two-surfaces in 4D Lorentzian space-times"
V. Inozemtsev, "Symmetries of 1D Hubbard model with variable range exchange"
M. Kassem, "Solution of Burgers equation using the characteristic function method"
M. Koca, "Quaternionic roots of SO(8), SO(9), F4 and the related Weyl groups"
D. Korotkin, "Solution of matrix Riemann–Hilbert problem with quasi-permutation monodromy matrices"

A. Lopatin, "Some issues of qualitative theory of differential equations: group theoretical approach"
O. Maspfuhl, "Linearized Poisson geometry and gauge fields"
R. Matsyuk, "Poisson structures for Mathisson equation"
A. Mikhailov, "Symmetry approach in differential and symbolic representations"
P. Morozov, "New results and unsolved problems in Liouville classification of integrable Hamiltonian systems"
O. Mul, "On vibrations in the dynamical systems of controlled machine units with discrete parameters"
D. Nikshych, "On fusion categories"
V. Olyeynik, "Maxwell–Dirac–Yang–Mills–Einstein equations: the integrability conditions for the particle-like models with spherically symmetric gravitational fields"
S. Pakuliak, "Integral presentations for the universal R-matrices in quantum current algebras"
A. Penskoi, "Discrete matrix Riccati equations with superposition formulas"
I. Pirozhenko, "Heat kernel expansion for the dielectric like spectral problems"
A. Piryatinski, "Non-adiabatic dynamics in electron phonon coupled system"
O. Prylypko, "Symmetry operators for the Schrödinger equation"
M. Sait-Ametov, "Constructions of non-Gaussian measures on the space of distributions over the field of p-adic numbers"
M. Savina, "On a correspondence between $N = 2$ mKdV and $N = 2$ sin-Gordon equations"
J. Sławianowski, "Problems of affine symmetry in physics"
V. Stogniy, "Symmetry properties of two-dimensional Fokker–Planck equations"
V. Taranov, "Symmetries of the kinetic plasma theory"
P. Tempesta, "Symmetries of linear difference equations via umbral calculus"
V. Tkachuk, "Supersymmetric approach for generating quasi-exactly solvable potentials"
T. Tsuchida, "Classification of polynomial integrable systems with one scalar and one vector unknown"
V. Tsukanov, "Collective motions in nucleon"
N. Weaver, "Operator algebras associated to the Klein–Gordon position representation"
T. Wolf, "Quadratic Hamiltonians on $e(3)$ with 3rd and 4th degree first integrals"
A. Zhalij, "On integrable three-dimensional quantum systems in magnetic fields"
A. Zolotaryuk, "Countable sets of self-adjoint extensions of the Schrödinger operator with point dipole interaction"
K. Zuev, "Non-commutative integrability of geodesic flows on homogeneous spaces"

The abstracts of these talks are available at the conference web-page.

To conclude, we invite all researchers working in the symmetry-related fields to take part in the Sixth Conference "Symmetry in Nonlinear Mathematical Physics" that is to be held on June 20–26, 2005.

EDITORS

March, 2004

Наукове видання

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Математика та її застосування

Том 50

Праці
П'ятої міжнародної конференції

Симетрія в нелінійній математичній фізиці

Частина 3

Редактори	*А.Г. Нікітін, В.М. Бойко, Р.О. Попович, І.А. Єгорченко*
Комп'ютерний оригінал-макет	*В.М. Бойко, Р.О. Попович*
Художнє оформлення	*Р.О. Попович, Г.В. Попович*
Коректори	*Н.М. Іванова, О.В. Магда*

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тел.: (044) 234–63–22, E-mail: appmath@imath.kiev.ua
web-page: www.imath.kiev.ua/~appmath/

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