

Massive Gauge Field Theory without Higgs Mechanism

Junchen SU

*Center for Theoretical Physics, Department of Physics, Jilin University,
Changchun 130023, People's Republic of China
E-mail: junchens@public.cc.jl.cn*

It is argued that the massive gauge field theory without the Higgs mechanism can well be set up on the gauge-invariance principle based on the viewpoint that a massive gauge field must be viewed as a constrained system and the Lorentz condition, as a constraint, must be introduced from the beginning and imposed on the Yang–Mills Lagrangian. The quantum theory for the massive gauge fields may perfectly be established by the quantization performed in the Hamiltonian or the Lagrangian path-integral formalism by means of the Lagrange undetermined multiplier method and shows good renormalizability and unitarity.

It is the prevailing viewpoint that the massive gauge field theory can not be set up without introducing the Higgs mechanism. The first obstacle is the gauge-non-invariance of the mass term in the massive Yang–Mills Lagrangian for a massive gauge field. On the contrary, we present an argument to show that the conventional viewpoint is not true [1]. In fact, a certain massive gauge field theory can be well established on the basis of gauge-invariance principle without recourse to the Higgs mechanism. The basic ideas are stated in the following.

(1) A massive gauge field must be viewed as a constrained system. In the previous attempt of building up the massive non-Abelian gauge field theory, the massive Yang–Mills Lagrangian density written below was chosen to be the starting point [2, 3]

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a + \frac{1}{2}m^2A^{a\mu}A_{\mu}^a, \quad (1)$$

where A_{μ}^a are the vector potentials for the non-Abelian massive gauge fields,

$$F_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a + gf^{abc}A_{\mu}^bA_{\nu}^c \quad (2)$$

are the field strengths and m is the mass of gauge bosons. The first term in the Lagrangian is the ordinary Yang–Mills Lagrangian which is gauge-invariant under a whole Lie group and used to determine the form of interactions among the gauge fields themselves. The second term in the Lagrangian is the mass term which is not gauge-invariant and only affects the kinematic property of the fields. The above Lagrangian itself was ever considered to give a complete description of the massive gauge field dynamics. This consideration is not correct because the Lagrangian is not only not gauge-invariant, but also contains redundant unphysical degrees of freedom. As one knows, a full vector potential $A^{\mu}(x)$ can be split into two Lorentz-covariant parts: the transverse vector potential $A_T^{\mu}(x)$ and the longitudinal vector potential $A_L^{\mu}(x)$, $A^{\mu}(x) = A_T^{\mu}(x) + A_L^{\mu}(x)$. The transverse vector potential $A_T^{\mu}(x)$ contains three independent spatial components which is sufficient to represent the polarization states of a massive vector boson. Whereas, the longitudinal vector potential A_L^{μ} appears to be a redundant unphysical variable which must be constrained by introducing the Lorentz condition

$$\varphi^a \equiv \partial^{\mu}A_{\mu}^a = 0, \quad (3)$$

whose solution is $A_L^{\mu} = 0$. With this solution, the massive Yang–Mills Lagrangian may be expressed in terms of the independent dynamical variables $A_T^{\mu}(x)$,

$$\mathcal{L} = -\frac{1}{4}F_T^{a\mu\nu}F_{T\mu\nu}^a + \frac{1}{2}m^2A_T^{a\mu}A_{T\mu}^a, \quad (4)$$

which gives a complete description of the massive gauge field dynamics. If we want to represent the dynamics in the whole space of the full vector potential as described by the massive Yang–Mills Lagrangian in equation (1), the massive gauge field must be treated as a constrained system. In this case, the Lorentz condition in equation (3), as a constraint, is necessary to be introduced from the onset and imposed on the Lagrangian in equation (1) so as to guarantee the redundant degrees of freedom to be eliminated from the Lagrangian.

(2) The gauge-invariance should generally be required for the action written in the physical subspace. Usually, the gauge-invariance is required to the Lagrangian. From the dynamical viewpoint, as we know, the action is of more essential significance than the Lagrangian. This is why in Mechanics and Field Theory, to investigate the dynamical and symmetric properties of a system, one always starts from the action of the system. Similarly, when we examine the gauge-symmetric property of a field system, in more general, we should also see whether the action of the system is gauge-invariant or not. In particular, for a constrained system such as the massive gauge field, we should see whether or not the action represented by the independent dynamical variables is gauge-invariant. This point of view is familiar to us in the mechanics for constrained systems. Certainly, in some special cases, the Lagrangian itself is gauge-invariant in the physical subspace so that the gauge-invariance of the action is ensured. This situation happens for the massless gauge fields and the massive Abelian gauge field. For the non-Abelian gauge fields, the infinitesimal gauge transformation usually is given by [3]

$$\delta A_\mu^a = D_\mu^{ab} \theta^b, \quad (5)$$

where

$$D_\mu^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A_\mu^c. \quad (6)$$

This gauge transformation is different from the Abelian one in that in the physical subspace defined by the Lorentz condition, i.e., spanned by the transverse vector potential $A_T^{a\mu}$, the fields still undergo nontrivial gauge transformations

$$\delta A_{T\mu}^a = D_{T\mu}^{ab} \theta^b, \quad (7)$$

where

$$D_{T\mu}^{ab} = \delta^{ab} \partial_\mu - g f^{abc} A_{T\mu}^c. \quad (8)$$

Therefore, the mass term in the massive Yang–Mills Lagrangian written in equation (4) is not gauge-invariant. But, the action given by this Lagrangian is invariant with respect to the gauge transformation shown in equations (7) and (8). In fact, noticing the identity: $f^{abc} A_T^{a\mu} A_{T\mu}^b = 0$ and the transversity condition (an identity): $\partial^\mu A_{T\mu}^a = 0$, it is easy to see

$$\delta S = -m^2 \int d^4x \theta^a \partial^\mu A_{T\mu}^a = 0.$$

This shows that the dynamics of massive non-Abelian gauge fields is gauge-invariant. Alternatively, the gauge-invariance may also be seen from the action given by the Lagrangian in equation (1) which is constrained by the Lorentz condition in equation (3). Under the gauge transformation written in equation (5) and (6), noticing the identity $f^{abc} A^{a\mu} A_\mu^b = 0$ and the Lorentz condition, it can be found that

$$\delta S = -m^2 \int d^4x \theta^a \partial^\mu A_\mu^a = 0.$$

This suggests that the massive non-Abelian gauge field theory may also be set up on the basis of gauge-invariance principle.

(3) Only infinitesimal gauge transformations need to be considered in the physical subspace. In examining the gauge invariance of the action for the massive non-Abelian gauge fields, we confine ourself to consider the infinitesimal gauge transformation only. The reason for this arises from the fact that the Lorentz condition limits the gauge transformation only to take place in the vicinity of the unity of the gauge group. In other words, the residual gauge degrees of freedom existing in the physical subspace are characterized by the infinitesimal gauge transformations. This fact was pointed out in the pioneering article by Faddeev and Popov for the quantization of massless non-Abelian gauge fields [4]. Usually, the dynamics of massless gauge fields is described by the Yang–Mills Lagrangian. It is well-known that the Yang–Mills Lagrangian itself is not quantizable, namely, it cannot be used to construct a convergent generating functional of Green’s functions even though it is gauge-invariant with respect to the whole gauge group. This is because the Yang–Mills Lagrangian contains redundant unphysical degrees of freedom and hence is not complete for describing the massless gauge field dynamics unless a suitable constraint such as the Lorentz condition is introduced to eliminate the unphysical degrees of freedom. In the article by Faddeev and Popov, the Lorentz condition is introduced through the following identity [4]

$$\Delta[A] \int D(g) \delta[\partial^\mu A_\mu^g] = 1,$$

which is inserted into the functional integral representing the vacuum-to-vacuum transition amplitude. After doing this, the authors said that “We must know $\Delta[A]$ is only for the transverse fields and in this case all contributions to the last integral are given in the neighborhood of the unity element of the group”. The delta-functional in the above identity implies $\partial^\mu (A^g)_\mu^a = \partial^\mu A_\mu^a = 0$ which represents the gauge-invariance of the Lorentz condition because the Lorentz condition is required to hold for all the field variables including the ones before and after gauge transformations. In the physical subspace where only the transverse fields are allowed to appear, only infinitesimal gauge transformations around the unity element are permitted and needed to be considered in the course of Faddeev–Popov’s quantization even though the Yang–Mills Lagrangian used is invariant under the whole gauge group. Obviously, there are no reasons of considering the gauge-transformation property of the fields in the region beyond the physical subspace because the fields do not exist in that region. By this point, it can be understood why in the ordinary quantum gauge field theories such as the standard model, the BRST-transformations are all taken to be infinitesimal.

According to the general procedure, the Lorentz condition (3) may be incorporated into the Lagrangian (1) by the Lagrange undetermined multiplier method to give a generalized Lagrangian. In the first order formalism, this Lagrangian is written as [5]

$$\mathcal{L} = \frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a - \frac{1}{2} F^{a\mu\nu} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c) + \frac{1}{2} m^2 A^{a\mu} A_\mu^a + \lambda^a \partial^\mu A_\mu^a, \quad (9)$$

where A_μ^a and $F_{\mu\nu}^a$ are now treated as the mutually independent variables and λ^a are chosen to represent the Lagrange multipliers. Using the canonically conjugate variables defined by

$$\Pi_\mu^a(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}^{a\mu}} = F_{\mu 0}^a + \lambda^a \delta_{\mu 0} = \begin{cases} F_{k0}^a = E_k^a, & \text{if } \mu = k = 1, 2, 3, \\ \lambda^a = -E_0^a, & \text{if } \mu = 0, \end{cases}$$

the Lagrangian in equation (9) may be rewritten in the canonical form

$$\mathcal{L} = E^{a\mu} \dot{A}_\mu^a + A_0^a C^a - E_0^a \varphi^a - \mathcal{H}, \quad (10)$$

where

$$C^a = \partial^\mu E_\mu^a + g f^{abc} A_k^b E^{ck} + m^2 A_0^a, \quad (11)$$

$$\mathcal{H} = \frac{1}{2} (E_k^a)^2 + \frac{1}{4} (F_{ij}^a)^2 + \frac{1}{2} m^2 [(A_0^a)^2 + (A_k^a)^2], \quad (12)$$

here $E_\mu^a = (E_0^a, E_k^a)$ is a Lorentz vector, \mathcal{H} is the Hamiltonian density in which F_{ij}^a are defined in equation (2). In the above, the four-dimensional and the spatial indices are respectively denoted by the Greek and Latin letters. From equation (10), it is clearly seen that the second and third terms are given respectively by incorporating the constraint condition

$$C^a = 0, \quad (13)$$

where C^a was represented in equation (11) and the Lorentz condition in equation (3) into the Lagrangian.

Now, let us first perform the quantization of the massive non-Abelian gauge fields in the Hamiltonian path-integral formalism [5]. In accordance with the general procedure of the quantization, we should first write the generating functional of Green's functions via the independent canonical variables which are now chosen to be the transverse parts of the vectors A_μ^a and E_μ^a

$$Z[J] = \frac{1}{N} \int D(A_T^{a\mu}, E_T^{a\mu}) \exp \left\{ i \int d^4x \left[E_T^{a\mu} \dot{A}_{T\mu}^a - \mathcal{H}^*(A_T^{a\mu}, E_T^{a\mu}) + J_T^{a\mu} A_{T\mu}^a \right] \right\}, \quad (14)$$

where $\mathcal{H}^*(A_T^{a\mu}, E_T^{a\mu})$ is the Hamiltonian which is obtained from the Hamiltonian in equation (12) by replacing the constrained variables $A_L^{a\mu}$ and $E_L^{a\mu}$ with the solutions of equations (3) and (13)

$$\mathcal{H}^*(A_T^{a\mu}, E_T^{a\mu}) = \mathcal{H}(A^{a\mu}, E^{a\mu}) |_{\varphi^a=0, C^a=0}.$$

As mentioned before, equation (3) leads to $A_L^{a\mu} = 0$. Noticing this solution and the decomposition $E^{a\mu}(x) = E_T^{a\mu} + E_L^{a\mu}(x)$, when setting

$$E_L^{a\mu}(x) = \partial_x^\mu Q^a(x), \quad (15)$$

where $Q^a(x)$ is a scalar function, one may get from equation (13) an equation obeyed by the scalar function $Q^a(x)$

$$K^{ab}(x)Q^b(x) = W^a(x), \quad (16)$$

where

$$K^{ab}(x) = \delta^{ab} \square_x - g f^{abc} A_T^c(x) \partial_\mu^x \quad \text{and} \quad W^a(x) = g f^{abc} E_T^{b\mu}(x) A_{T\mu}^c(x) - m^2 A_T^{a0}(x).$$

With the aid of the Green's function $G^{ab}(x-y)$ (the ghost particle propagator) which satisfies the following equation

$$K^{ac}(x)G^{cb}(x-y) = \delta^{ab} \delta^4(x-y)$$

one may find the solution to the equation in equation (16) as follows

$$Q^a(x) = \int d^4y G^{ab}(x-y) W^b(y). \quad (17)$$

From the expressions given in equations (15) and (17), one can see that the $E_L^{a\mu}(x)$ is a complicated functional of the variables $A_T^{a\mu}$ and $E_T^{a\mu}$ so that the Hamiltonian $\mathcal{H}^*(A_T^{a\mu}, E_T^{a\mu})$ is of much more complicated functional structure which is not convenient for constructing the diagrammatic technique in the perturbation theory. Therefore, it is better to express the generating functional in equation (14) in terms of the variables A_μ^a and E_μ^a . For this purpose, it is necessary to insert the following delta-functional into equation (14) [5]

$$\delta[A_L^{a\mu}] \delta[E_L^{a\mu} - E_L^{a\mu}(A_T^{a\mu}, E_T^{a\mu})] = \det M \delta[C^a] \delta[\varphi^a], \quad (18)$$

where M is the matrix whose elements are defined by the Poisson bracket

$$M^{ab}(x, y) = \left\{ C^a(x), \varphi^b(y) \right\} = \int d^4x \left\{ \frac{\delta C^a}{\delta A_\mu^a(x)} \frac{\delta \varphi^b}{\delta E^{a\mu}(x)} - \frac{\delta C^a}{\delta E_\mu^a(x)} \frac{\delta \varphi^b}{\delta A^{a\mu}(x)} \right\} \\ = D_\mu^{ab}(x) \partial_x^\mu \delta^4(x - y).$$

The relation in equation (18) is easily derived from equations (3) and (13) by applying the property of delta-functional. Upon inserting equation (18) into equation (14) and utilizing the Fourier representation of the delta-functional

$$\delta[C^a] = \int D(\eta^a/2\pi) e^{i \int d^4x \eta^a C^a},$$

we have

$$Z[J] = \frac{1}{N} \int D(A_\mu^a, E_\mu^a, \eta^a) \\ \times \det M \delta[\partial^\mu A_\mu^a] \exp \left\{ i \int d^4x [E^{a\mu} \dot{A}_\mu^a + \eta^a C^a - \mathcal{H}(A^{a\mu}, E^{a\mu}) + J^{a\mu} A_\mu^a] \right\}.$$

In the above exponential, there is a E_0^a -related term $E_0^a(\partial_0 A_0^a - \partial_0 \eta^a)$ which permits us to perform the integration over E_0^a , giving a delta-functional

$$\delta[\partial_0 A_0^a - \partial_0 \eta^a] = \det |\partial_0|^{-1} \delta[A_0^a - \eta^a].$$

The determinant $\det |\partial_0|^{-1}$, as a constant, may be put in the normalization constant N and the delta-functional $\delta[A_0^a - \eta^a]$ will disappear when the integration over η^a is carried out. The integral over E_k^a is of Gaussian-type and hence easily calculated. After these manipulations, we arrive at

$$Z[J] = \frac{1}{N} \int D(A_\mu^a) \det M \delta[\partial^\mu A_\mu^a] \\ \times \exp \left\{ i \int d^4x \left[-\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a + \frac{1}{2} m^2 A^{a\mu} A_\mu^a + J^{a\mu} A_\mu^a \right] \right\}. \tag{19}$$

When employing the familiar expression [4]

$$\det M = \int D(\bar{C}^a, C^a) \exp \left\{ i \int d^4x d^4y \bar{C}^a(x) M^{ab}(x, y) C^b(y) \right\},$$

where $\bar{C}^a(x)$ and $C^a(x)$ are the mutually conjugate ghost field variables and the following limit for the Fresnel functional

$$\delta[\partial^\mu A_\mu^a] = \lim_{\alpha \rightarrow 0} C[\alpha] e^{-\frac{i}{2\alpha} \int d^4x (\partial^\mu A_\mu^a)^2},$$

where $C[\alpha] \sim \prod_x (\frac{i}{2\pi\alpha})^{1/2}$ and supplementing the external source terms for the ghost fields, the generating functional in equation (19) is finally given in the form

$$Z[J, \bar{\xi}, \xi] = \frac{1}{N} \int D(A_\mu^a, \bar{C}^a, C^a) \exp \left\{ i \int d^4x [\mathcal{L}_{\text{eff}} + J^{a\mu} A_\mu^a + \bar{\xi}^a C^a + \bar{C}^a \xi^a] \right\}, \tag{20}$$

where

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a + \frac{1}{2} m^2 A^{a\mu} A_\mu^a - \frac{1}{2\alpha} (\partial^\mu A_\mu^a)^2 - \partial^\mu \bar{C}^a D_\mu^{ab} C^b, \tag{21}$$

which is the effective Lagrangian for the quantized massive non-Abelian gauge field in which the third and fourth terms are the so-called gauge-fixing term and the ghost term respectively. In equation (20), the limit $\alpha \rightarrow 0$ is implied. Certainly, the theory may be given in arbitrary gauges ($\alpha \neq 0$). In this case, as will be seen soon later, the ghost particle will acquire a spurious mass $\mu = \sqrt{\alpha}m$.

To confirm the result of the quantization given above, let us turn to quantize the massive non-Abelian gauge fields in the Lagrangian path-integral formalism. For later convenience, the massive Yang–Mills Lagrangian in equation (1) and the Lorentz constraint condition in equation (3) are respectively generalized to the following forms

$$\mathcal{L}_\lambda = -\frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a + \frac{1}{2}m^2A^{a\mu}A_\mu^a - \frac{1}{2}\alpha(\lambda^a)^2 \quad (22)$$

and

$$\partial^\mu A_\mu^a + \alpha\lambda^a = 0, \quad (23)$$

where $\lambda^a(x)$ are the extra functions which will be identified with the Lagrange multipliers and α is an arbitrary constant playing the role of gauge parameter. According to the general procedure for constrained systems, the constraint condition in equation (23) may be incorporated into the Lagrangian in equation (22) by the Lagrange undetermined multiplier method, giving a generalized Lagrangian

$$\mathcal{L}_\lambda = -\frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a + \frac{1}{2}m^2A^{a\mu}A_\mu^a + \lambda^a\partial^\mu A_\mu^a + \frac{1}{2}\alpha(\lambda^a)^2. \quad (24)$$

This Lagrangian is obviously not gauge-invariant. However, for building up a correct gauge field theory, it is necessary to require the action given by the Lagrangian in equation (24) to be invariant with respect to the gauge transformations shown in equations (5) and (6) so as to guarantee the dynamics of the gauge field to be gauge-invariant. By this requirement, noticing the identity $f^{abc}A^{a\mu}A_\mu^b = 0$ and applying the constraint condition in equation (23), we have

$$\delta S_\lambda = -\frac{1}{\alpha} \int d^4x \partial^\nu A_\nu^a(x) \partial^\mu (\mathcal{D}_\mu^{ab}(x)\theta^b(x)) = 0,$$

where

$$\mathcal{D}_\mu^{ab}(x) = \delta^{ab} \frac{\mu^2}{\square_x} \partial_\mu^x + D_\mu^{ab}(x),$$

in which $\mu^2 = \alpha m^2$ and $D_\mu^{ab}(x)$ was defined in equation (6). From equation (23), we see $\frac{1}{\alpha} \partial^\nu A_\nu^a = -\lambda^a \neq 0$. Therefore, to ensure the action to be gauge-invariant, the following constraint condition on the gauge group is necessary to be required

$$\partial_x^\mu (\mathcal{D}_\mu^{ab}(x)\theta^b(x)) = 0. \quad (25)$$

These are the coupled equations satisfied by the parametric functions $\theta^a(x)$ of the gauge group. Since the Jacobian of the equations is not singular, $\det M \neq 0$ where M is the matrix whose elements are

$$M^{ab}(x, y) = \left. \frac{\delta(\partial_x^\mu \mathcal{D}_\mu^{ac}(x)\theta^c(x))}{\delta\theta^b(y)} \right|_{\theta=0} = \delta^{ab}(\square_x + \mu^2)\delta^4(x-y) - gf^{abc}\partial_x^\mu(A_\mu^c(x)\delta^4(x-y)),$$

the above equations are solvable and would give a set of solutions which express the functions $\theta^a(x)$ as functionals of the vector potentials $A_\mu^a(x)$. The constraint conditions in equation (25) may also be incorporated into the Lagrangian in equation (24) by the Lagrange undetermined

multiplier method. In doing this, it is convenient, as usually done, to introduce ghost field variables $C^a(x)$ in such a fashion [3–5]: $\theta^a(x) = \varsigma C^a(x)$, where ς is an infinitesimal Grassmann's number. In accordance with this relation, the constraint condition in equation (25) can be rewritten as

$$\partial^\mu (\mathcal{D}_\mu^{ab} C^b) = 0, \quad (26)$$

which usually is called ghost equation. When this constraint condition is incorporated into the Lagrangian in equation (24) by the Lagrange multiplier method, we obtain a more generalized Lagrangian as follows

$$\mathcal{L}_\lambda = -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a + \frac{1}{2} m^2 A^{a\mu} A_\mu^a + \lambda^a \partial^\mu A_\mu^a + \frac{1}{2} \alpha (\lambda^a)^2 + \bar{C}^a \partial^\mu (\mathcal{D}_\mu^{ab} C^b), \quad (27)$$

where $\bar{C}^a(x)$, acting as Lagrange multipliers, are the new scalar variables conjugate to the ghost variables $C^a(x)$.

At present, we are ready to formulate the quantization of the massive gauge field in the Lagrangian path-integral formalism. As we learn from the Lagrange multiplier method, the dynamical and constrained variables as well as the Lagrange multipliers in equation (27) can all be treated as free ones, varying arbitrarily. Therefore, we are allowed to use this kind of Lagrangian to construct the generating functional of Green's functions

$$\begin{aligned} Z[J^{a\mu}, \bar{\xi}^a, \xi^a] &= \frac{1}{N} \int D(A_\mu^a, \bar{C}^a, C^a, \lambda^a) \\ &\times \exp \left\{ i \int d^4x [\mathcal{L}_\lambda(x) + J^{a\mu}(x) A_\mu^a(x) + \bar{\xi}^a(x) C^a(x) + \bar{C}^a(x) \xi^a(x)] \right\}, \end{aligned} \quad (28)$$

where $D(A_\mu^a, \dots, \lambda^a)$ denotes the functional integration measure, J_μ^a , $\bar{\xi}^a$ and ξ^a are the external sources coupled to the gauge and ghost fields and N is the normalization constant. Looking at the expression of the Lagrangian in equation (27), it is seen that the integral over $\lambda^a(x)$ is of Gaussian-type. Upon completing the calculation of this integral, we finally arrive at

$$\begin{aligned} Z[J^{a\mu}, \bar{\xi}^a, \xi^a] &= \frac{1}{N} \int D(A_\mu^a, \bar{C}^a, C^a,) \\ &\times \exp \left\{ i \int d^4x [\mathcal{L}_{\text{eff}}(x) + J^{a\mu}(x) A_\mu^a(x) + \bar{\xi}^a(x) C^a(x) + \bar{C}^a(x) \xi^a(x)] \right\}, \end{aligned} \quad (29)$$

where

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a + \frac{1}{2} m^2 A^{a\mu} A_\mu^a - \frac{1}{2\alpha} (\partial^\mu A_\mu^a)^2 - \partial^\mu \bar{C}^a \mathcal{D}_\mu^{ab} C^b. \quad (30)$$

is the effective Lagrangian given in the general gauges. In the Landau gauge ($\alpha \rightarrow 0$), The Lagrangian in equation (30) just gives the result in equation (21). It has been proved [1] that the above quantization carried out by means of the Lagrange multiplier method is equivalent to the Faddeev–Popov approach of quantization [4].

There are three points we would like to emphasize: (1) In the quantization by the Lagrange multiplier method, the gauge-invariance is always to be required even in the arbitrary gauge. Moreover, it has been found that the action given by the Lagrangian in equation (30) is invariant under a kind of BRST-transformations [6]. Thus, the quantum non-Abelian gauge field theory is set up from beginning to end on the firm basis of gauge-invariance. (2) In the Lagrangian path-integral formalism, as shown before, the quantized result is derived by utilizing the infinitesimal gauge transformations. This result is, as mentioned before, identical to that obtained by the quantization in the Hamiltonian path-integral formalism. In the latter quantization, we only need to calculate the classical Poisson brackets without concerning any gauge transformation.

This fact reveals that to get a correct quantum theory in the Lagrangian path-integral formalism, the infinitesimal gauge transformations are only necessary to be taken into account and thereby confirms the fact that in the physical subspace restricted by the Lorentz condition, only the infinitesimal gauge transformations are possible to exist. (3) From the generating functional shown in equation (29) and (30), one may derive the gauge boson propagator as follows

$$iD_{\mu\nu}^{ab}(k) = -i\delta^{ab} \left\{ \frac{g_{\mu\nu} - k_\mu k_\nu / k^2}{k^2 - m^2 + i\varepsilon} + \frac{\alpha k_\mu k_\nu / k^2}{k^2 - \mu^2 + i\varepsilon} \right\},$$

which is of good renormalizable behavior. In the zero-mass limit, this propagator with the massive Yang–Mills Lagrangian and the generating functional together all go over to the results given in the massless gauge field theory, different from the quantum theory established previously from the massive Yang–Mills Lagrangian alone without any constraint [7–10]. For the previous theory, there occurs a severe contradiction in the zero-mass limit that the massive Yang–Mills Lagrangian in equation (1) is converted to the massless one, but, the propagator is not and of a singular behavior. In particular, the previous theory was shown to be nonrenormalizable [3, 7–9] because the unphysical longitudinal fields and residual gauge degrees of freedom are not excluded from the theory.

Up to the present, we limit ourself to discuss the gauge fields themselves without concerning fermion fields. For the gauge fields, in order to guarantee the mass term in the action to be gauge-invariant, the masses of all gauge bosons are taken to be the same. If fermions are included, obviously, the QCD with massive gluons fulfils this requirement because all the gluons can be considered to have the same mass. Such a QCD, as has been proved, is not only renormalizable, but also unitary [6]. The renormalizability and unitarity are warranted by the fact that the unphysical degrees of freedom in the theory have been removed by the constraint conditions in equations (23) and (26). The gauge-fixing term and the ghost term in equation (30) just play the role of counteracting the unphysical degrees of freedom contained in the massive Yang–Mills Lagrangian as verified by the perturbative calculations [6].

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