Proceedings of Institute of Mathematics of NAS of Ukraine 2004, Vol. 50, Part 2, 539–546

Nonlinear Supersymmetry in Quantum Mechanics

Alexandr A. ANDRIANOV †‡ *and Andrey V. SOKOLOV* ‡

† *INFN, Sezione di Bologna, Via Irnerio 46, 40126 Bologna, Italy*
E-mail: *andrianov@bo.infn.it*

‡ *V.A. Fock Institute of Physics, St.-Petersburg State University,*
1 Ulianovskaya Str., Petrodvorets, 198504 St.-Petersburg, Russia
E-mail: *sokolov@mph.phys.spbu.ru*

We study the Nonlinear (Polynomial, N-fold, ...) SUSY algebra in one-dimensional QM. Its structure is determined by the type of conjugation operation (Hermitian conjugation or transposition) and described with the help of the Super-Hamiltonian projection on the zero-mode subspace of a supercharge. We show that the SUSY algebra with transposition symmetry is always polynomial in the Super-Hamiltonian if supercharges represent differential operators of finite order. The appearance of the extended SUSY with several supercharges is analyzed and it is established that no more than two independent supercharges may generate a nonlinear superalgebra. In the case with two independent supercharges we find a nontrivial hidden symmetry operator. It is revealed that wave functions of all Super-Hamiltonian bound states or, in the case of Super-Hamiltonian with periodic potential(s), all periodic wave functions corresponding to boundaries between allowed and forbidden energy bands are zero-modes of the hidden symmetry operator.

1 Introduction

In our report we consider the following topics: (i) Nonlinear SUSY algebras [1–3] with Hermitian conjugation and transposition symmetries of supercharges, (ii) extended SUSY algebra and appearance of antisymmetric symmetry operator (hidden symmetry operator), (iii) "strip-off" problem, (iv) (in)dependence of supercharges, maximal number of coexisting independent supercharges and optimal basis of supercharges, and (v) example: two independent supercharges of 2nd and 1st orders. The new (with respect to [4]) result presented in the paper is a description of properties of antisymmetric symmetry operator for a Hamiltonian with periodic potential. The proofs of presented results and a more complete list of references can be found in [4].

2 Four types of SUSY algebras with complex supercharges

Let h^+ and h^- be components of the matrix one-dimensional Schrödinger operator, a Super-Hamiltonian,

$$H = \begin{pmatrix} h^+ & 0 \\ 0 & h^- \end{pmatrix} = \begin{pmatrix} -\partial^2 + V_1(x) & 0 \\ 0 & -\partial^2 + V_2(x) \end{pmatrix} \equiv -\partial^2 \boldsymbol{I} + \boldsymbol{V}(x),$$

where $\partial \equiv d/dx$. The isospectral connection between Hamiltonians h^+ and h^- is provided by intertwining relations with the help of Crum–Darboux differential operators $q_N^\pm$,

$$h^+ q_N^+ = q_N^+ h^-, \qquad q_N^- h^+ = h^- q_N^-, \qquad q_N^\pm = \sum_{k=0}^{N} w_k^\pm(x)\partial^k, \qquad w_N^\pm = \text{const} \equiv (\mp 1)^N, \quad (1)$$

which, in the framework of SUSY QM, are components of the supercharges,

$$Q = \begin{pmatrix} 0 & q_N^+ \\ 0 & 0 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} 0 & 0 \\ q_N^- & 0 \end{pmatrix}, \qquad Q^2 = \bar{Q}^2 = 0, \qquad [H, Q] = [H, \bar{Q}] = 0.$$

According to known operator q_N^+ we can choose as q_N^- both $(q_N^+)^t$ and $(q_N^+)^\dagger$ and *vice versa*, where ^{t} and † denote transposition and Hermitian conjugation respectively. Thus, if coefficients of q_N^+ are real then we have unique $q_N^- = (q_N^+)^t = (q_N^+)^\dagger$, but if coefficients of q_N^+ are not real then we have two different q_N^- type operators, $(q_N^+)^t$ and $(q_N^+)^\dagger$. Hence, we have in general four different supercharges

$$Q, \qquad \bar{Q} = Q^t, \qquad \bar{Q}_c = Q^\dagger, \qquad Q_c = \bar{Q}_c^t,$$

pairs of which generates four SUSY algebras:

$$\mathcal{A}_1 \longleftrightarrow (\bar{Q}_c, Q), \qquad \mathcal{A}_2 \longleftrightarrow (\bar{Q}, Q), \qquad \mathcal{A}_3 \longleftrightarrow (\bar{Q}_c, Q_c), \qquad \mathcal{A}_4 \longleftrightarrow (\bar{Q}, Q_c).$$

Certain properties of $\mathcal{A}_{1,4}$ type complex SUSY algebras were considered in [2].

3 Superalgebras with transposition symmetry

The structure of SUSY algebras with real coefficient functions as well as the structure of the $\mathcal{A}_{2,3}$ type complex SUSY algebras is described by the following theorem.

Theorem 1 (on SUSY algebras with $\boldsymbol{T}$-symmetry). *Let us introduce two sets of N linearly independent functions $\phi_n^\pm(x)$ $(n = 1, \ldots, N)$ which represent complete sets of zero-modes of the supercharge components* (1),

$$q_N^\pm \phi_n^\pm = 0, \qquad q_N^- = (q_N^+)^t.$$

Then[1]*:*

1) *the Hamiltonians $h^\pm$ have finite matrix representations when acting on the set of functions $\phi_n^\pm(x)$,*

$$h^\pm \phi_n^\mp = \sum_m S_{nm}^\pm \phi_m^\mp,$$

2) *the SUSY algebra closure with $\bar{Q} = Q^t$ takes the polynomial form,*

$$\left\{Q, Q^t\right\} = \det\left[E\boldsymbol{I} - \boldsymbol{S}^+\right]_{E=H} = \det\left[E\boldsymbol{I} - \boldsymbol{S}^-\right]_{E=H} \equiv \mathcal{P}_N(H), \tag{2}$$

irrespectively on whether the Q type supercharge of order N is unique or there exist several such supercharges for a given Super-Hamiltonian H.

Corollary 1. *From* (2) *it is evident that eigenvalues of $\boldsymbol{S}^+$ and $\boldsymbol{S}^-$ and their corresponding degeneracies coincide.*

4 Several supercharges and appearance of symmetry operators

Let us now examine the case when for the Super-Hamiltonian H there are two different supercharges K and P of the type Q and of the orders N and N_1 respectively,

$$K = \begin{pmatrix} 0 & k_N^+ \\ 0 & 0 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & p_{N_1}^+ \\ 0 & 0 \end{pmatrix}, \tag{3}$$

where k_N^+ and $p_{N_1}^+$ have real coefficient functions and $N > N_1$. In particular, in the case when complex supercharge Q exists, we can choose K and P as $(Q + Q^*)/2$ and $(Q - Q^*)/(2i)$

[1]The first proposition is a necessary condition for the Hamiltonian system to be quasi-exactly solvable and it was investigated recently [5] within the notion of "conditional symmetry".

respectively, where * denotes complex conjugation of coefficient functions. Thus, each of supercharges K and P generates in our case a unique supercharge of type $\bar{Q}$:

$$\bar{K} = K^t = K^\dagger = \begin{pmatrix} 0 & 0 \\ k_N^- & 0 \end{pmatrix}, \qquad k_N^- = (k_N^+)^t,$$
$$\bar{P} = P^t = P^\dagger = \begin{pmatrix} 0 & 0 \\ p_{N_1}^- & 0 \end{pmatrix}, \qquad p_{N_1}^- = (p_{N_1}^+)^t.$$

The existence of two supercharges of type Q (*i.e.* K and P) conventionally implies the extension of SUSY algebra. To close the algebra one has to include all anticommutators between supercharges. Two supercharges K and P generate two Polynomial SUSY,

$$\left\{K, K^\dagger\right\} = \tilde{\mathcal{P}}_N(H), \qquad \left\{P, P^\dagger\right\} = \tilde{\mathcal{P}}_{N_1}(H),$$

which have to be embedded into a $\mathcal{N} = 2$ SUSY algebra. The closure of the extended, $\mathcal{N} = 2$ SUSY algebra is given by

$$\left\{P, K^\dagger\right\} \equiv \mathcal{R} = \begin{pmatrix} p_{N_1}^+ k_N^- & 0 \\ 0 & k_N^- p_{N_1}^+ \end{pmatrix}, \qquad \left\{K, P^\dagger\right\} \equiv \bar{\mathcal{R}} = \begin{pmatrix} k_N^+ p_{N_1}^- & 0 \\ 0 & p_{N_1}^- k_N^+ \end{pmatrix}.$$

Evidently the components of operators $\mathcal{R}$, $\bar{\mathcal{R}} = \mathcal{R}^\dagger = \mathcal{R}^t$ are differential operators of $N + N_1$ order commuting with the Hamiltonians $h^\pm$, hence they form symmetry operators $\mathcal{R}$, $\bar{\mathcal{R}}$ for the Super-Hamiltonian. However, in general, they are not polynomials of the Hamiltonians $h^\pm$ and these symmetries impose certain constraints on potentials.

Let us find the formal relation between the symmetry operators $\mathcal{R}$, $\bar{\mathcal{R}}$ and the Super-Hamiltonian. These operators can be decomposed into a Hermitian and an anti-Hermitian parts,

$$\mathcal{B} \equiv \frac{1}{2}(\mathcal{R} + \bar{\mathcal{R}}) \equiv \begin{pmatrix} b^+ & 0 \\ 0 & b^- \end{pmatrix}, \qquad i\mathcal{E} \equiv \frac{1}{2}(\mathcal{R} - \bar{\mathcal{R}}) \equiv i \begin{pmatrix} e^+ & 0 \\ 0 & e^- \end{pmatrix}.$$

It can be shown that the Hermitian operator $\mathcal{B}$ is a polynomial of the Super-Hamiltonian of the order $N_b \le [(N+N_1)/2] \le N-1$. The second Hermitian symmetry operator $\mathcal{E}$ is antisymmetric, $\mathcal{E}^t = -\mathcal{E}$. Hence, if $\mathcal{E}$ does not vanish identically it is a differential operator of *odd* order and cannot be realized by a polynomial of H. But at the same time

$$\mathcal{E}^2(H) \equiv \frac{1}{4}\left[2(\mathcal{R}\bar{\mathcal{R}} + \bar{\mathcal{R}}\mathcal{R}) - (\mathcal{R} + \bar{\mathcal{R}})^2\right] = \tilde{\mathcal{P}}_N(H)\tilde{\mathcal{P}}_{N_1}(H) - \mathcal{B}^2(H) \equiv \mathcal{P}_e(H) \tag{4}$$

is polynomial of H. Thus the nontrivial operator $\mathcal{E}(H)$ is a non-polynomial function of H — square root of equation (4) in an operator sense.

Closures of SUSY algebras $\mathcal{A}_1$–$\mathcal{A}_4$ can be written in terms of notation relating to real supercharges $K = (Q + Q^*)/2$ and $P = (Q - Q^*)/(2i)$ in the following form:

$$\mathcal{A}_1: \ \{Q, \bar{Q}_c\} = \tilde{\mathcal{P}}_N(H) + \tilde{\mathcal{P}}_{N_1}(H) - 2\mathcal{E}(H),$$
$$\mathcal{A}_2: \ \{Q, \bar{Q}\} = \tilde{\mathcal{P}}_N(H) - \tilde{\mathcal{P}}_{N_1}(H) + i2\mathcal{B}(H),$$
$$\mathcal{A}_3: \ \{Q_c, \bar{Q}_c\} = \tilde{\mathcal{P}}_N(H) - \tilde{\mathcal{P}}_{N_1}(H) - i2\mathcal{B}(H),$$
$$\mathcal{A}_4: \ \{Q_c, \bar{Q}\} = \tilde{\mathcal{P}}_N(H) + \tilde{\mathcal{P}}_{N_1}(H) + 2\mathcal{E}(H),$$
$$\bar{Q}_c = Q^\dagger = K^\dagger - iP^\dagger, \qquad Q_c = Q^* = K - iP, \qquad \bar{Q} = Q^t = K^\dagger + iP^\dagger.$$

One can show that components of $\mathcal{E}(H)$ have the following important properties.

1. In the case when $h^\pm$ has bound states, wave functions $\Psi_i^\pm$ of all these states are zero-modes of $e^\pm(h^\pm)$ and corresponding energies $E_i^\pm$ are roots of the polynomial $\mathcal{P}_e(E)$:

$$e^\pm(h^\pm)\Psi_i^\pm = e^\pm(E_i^\pm)\Psi_i^\pm = 0, \qquad \mathcal{P}_e(E_i^\pm) = 0. \tag{5}$$

Among solutions of (5) one reveals also a zero-energy state at the bottom of continuum spectrum.

2. In the case, when the potential of $h^{\pm}$ is periodic, all boundaries between allowed and forbidden energy bands $E_i^{\pm}$ are roots of $\mathcal{P}_e(E)$ and corresponding to these boundaries periodic wave functions $\Psi_i^{\pm}$ are zero-modes of $e^{\pm}(h^{\pm})$ (see (5) again).

Thus (5) represents an algebraic equation on bound state energies or on energy-band boundaries of a system possessing two supersymmetries. On the other hand one could find also the solutions of (5) which are not associated to any bound state or to any energy-band boundary. The very appearance of such unphysical solutions is accounted for by the trivial possibility to replicate supercharges by their multiplication on the polynomials of the Super-Hamiltonian and it is discussed below.

5 "Strip-off" problem

The pair of two supersymmetries analyzed before may rigidly determine the class of potentials $V_{1,2}$ contracting the freedom in their choice from a functional one to a parametric one. On the other hand, there exists a trivial possibility when the intertwining operators $k_N^{\pm}$ and $p_{N_1}^{\pm}$ are related by a factor depending on the Hamiltonian,

$$k_N^{\pm} = F(h^{\pm}) p_{N_1}^{\pm} = p_{N_1}^{\pm} F(h^{\mp}),$$

where $F(x)$ is assumed to be a polynomial. Obviously in this case the symmetry operator $\mathcal{E}(H)$ identically vanishes and the appearance of the second supercharge does not result in any restrictions on potentials.

More generally the orders of polynomial superalgebras and some of the roots of associated polynomials may not be involved in determination of the structure of the potentials. In particular, let the operators $k_N^{\pm}$ and $p_{N_1}^{\pm}$ be reducible to some lower-order ones $\tilde{k}_{\tilde{N}}^{\pm}$ and $\tilde{p}_{\tilde{N}_1}^{\pm}$,

$$k_N^{\pm} = F_k(h^{\pm}) \tilde{k}_{\tilde{N}}^{\pm} = \tilde{k}_{\tilde{N}}^{\pm} F_k(h^{\mp}), \qquad p_{N_1}^{\pm} = F_p(h^{\pm}) \tilde{p}_{\tilde{N}_1}^{\pm} = \tilde{p}_{\tilde{N}_1}^{\pm} F_p(h^{\mp}), \tag{6}$$

where $F_k(y)$ and $F_p(y)$ are polynomials. Then evidently the superalgebra generated by $\tilde{k}_{\tilde{N}}^{\pm}$ and $\tilde{p}_{\tilde{N}_1}^{\pm}$ equally well characterizes the Super-Hamiltonian system with the same potentials.

Thus, we have come to the problem of how to discern the nontrivial part of a supercharge and avoid multiple SUSY algebras generated by means of "dressing" (6). It can be systematically performed with the help of the following theorem.

Theorem 2 ("strip-off" theorem). *Let us admit the construction of the Theorem on SUSY algebras with T-symmetry. Then the requirement that Jordan form of the matrix $\boldsymbol{S}^-$ (or $\boldsymbol{S}^+$) contains n pairs (and no more) of Jordan cells with equal eigenvalues λ_l and the sizes ν_l of a smallest cell in the l-th pair is necessary and sufficient to ensure for the intertwining operator q_N^+ (or q_N^-) to be represented in the factorized form:*

$$q_N^{\pm} = \tilde{q}_{\tilde{N}}^{\pm} \prod_{l=1}^{n} (\lambda_l - h^{\mp})^{\nu_l},$$

where $\tilde{q}_{\tilde{N}}^{\pm}$ are intertwining operators of order $\tilde{N} = N - 2\sum_{l=1}^{n} \nu_l$ which cannot be decomposed further on in the product similar to (6) with $F_q(x) \neq$ const.

Corollary 2. *Jordan forms of $\boldsymbol{S}^+$ and $\boldsymbol{S}^-$ are identical up to transposition of certain cells.*

6 Optimization of supercharges

Still the stripped-off supercharges $k_N^\pm$ and $p_{N_1}^\pm$ do not necessarily represent an optimal set of them and provide an optimal structure of the symmetry operator $\mathcal{E}(H)$. Let us illustrate it with the help of the following example. One can show that there is the case when intertwining operators of both pairs $k_N^\pm$, $p_{N_1}^\pm$ and $t^\pm \equiv p_{N_1}^\pm k_N^\mp p_{N_1}^\pm$, $p_{N_1}^\pm$ cannot be stripped-off, but at the same time the latter pair is evidently more complex than the former. In addition, the composed of $t^\pm$ and $p_{N_1}^\pm$ symmetry operator $\mathcal{E}_t(H)$ with components,

$$e_t^\pm = \pm i\frac{1}{2}(t^\pm p_{N_1}^\mp - p_{N_1}^\pm t^\mp) = \mp i\frac{1}{2}(k_N^\pm p_{N_1}^\mp - p_{N_1}^\pm k_N^\mp)p_{N_1}^\pm p_{N_1}^\mp = -e^\pm \tilde{\mathcal{P}}_{N_1}(h^\pm)$$

is not optimal because of the superfluous polynomial factor $\tilde{\mathcal{P}}_{N_1}(H)$.

Thus, if there are several SUSY generators we have not only such fundamental problems as (i) to introduce the notion of (in)dependence of intertwining operators and (ii) to find out how many independent intertwining operators can coexist, but also and the problem (iii) to define an optimal basis of intertwining operators.

Let us define the intertwining operators $q_i^\pm$, $i = 1,\ldots,n$ to be dependent if and only if the polynomials $\alpha_i^\pm(y)$ exist such that not all of them are vanishing and

$$\sum_{i=1}^{n} \alpha_i^\pm(h^\pm)q_i^\pm = 0. \tag{7}$$

If the relation (7) results in $\alpha_i^\pm(y) = 0$ for all i the corresponding SUSY generators are independent. Evidently the (in)dependence of q_i^+ entails the (in)dependence of q_i^- and *vice versa*.

The following theorem plays a key role in resolution of how many independent supercharges can commute with a given Super-Hamiltonian.

Theorem 3 (on (in)dependence of supercharges). *Consider two non-trivial intertwining operators $q_i^\pm$, $i = 1, 2$ with transposition symmetry $q_i^+ = (q_i^-)^t$ which in general may have complex coefficients and let us normalize them in accordance to* (1). *Then the stripped-off intertwining operators $\tilde{q}_i^\pm$ coincide if and only if the symmetry operator made of $q_i^\pm$ vanishes, $q_1^+ q_2^- - q_2^+ q_1^- = 0$ (or equivalently $q_1^- q_2^+ - q_2^- q_1^+ = 0$).*

With the help of this Theorem one can show the following:

a) any symmetric (self-transposed) symmetry operator $B^\pm = (B^\pm)^t$, $B^\pm h^\pm = h^\pm B^\pm$ is a polynomial of the Hamiltonian $h^\pm$;

b) any two antisymmetric symmetry operators $e_i^\pm = -(e_i^\pm)^t$, $e_i^\pm h^\pm = h^\pm e_i^\pm$, $i = 1, 2$ are dependent, *i.e.* being stripped off coincide;

c) the maximal number of independent intertwining operators is two.

Using the fact that maximal number of independent intertwining operators is two, one can show the next.

1. In the case when any two intertwining operators are dependent, every intertwining operator q^+ can be represented in the form

$$q^+ = \alpha_q(h^+)p^+,$$

where $\alpha_q(y)$ is polynomial and p^+ is normalized intertwining operator of minimal order. It is evident that p^+ is unique and real.

2. In the case when there are two independent intertwining operators, every intertwining operator q^+ can be represented in the form

$$q^+ = \alpha_q(h^+)p^+ + \beta_q(h^+)k^+,$$

where $\alpha_q(y)$ and $\beta_q(y)$ are polynomials, p^+ is normalized intertwining operator of minimal order and k^+ is independent of p^+ normalized intertwining operator of minimal order. It is evident that p^+ is unique and real and that k^+ can be chosen with real coefficients. It can be shown that sum of orders of p^+ and k^+ is odd.

Thus the set of p^+ and k^+ (or only p^+ in the case 1) form an optimal basis of intertwining operators. As all $q^- = (q^+)^t$ the same results are translated to the set of $p^- = (p^+)^t$ and $k^- = (k^+)^t$.

7 Example: $N = 2$, $N_1 = 1$

Let us examine the algebraic structure of the simplest non-linear SUSY with two supercharges K of the 2nd order and P of the 1st order. The supersymmetries generated by K, $\bar{K}$ and P, $\bar{P}$ with components

$$k_2^\pm \equiv \partial^2 \mp 2f(x)\partial + \tilde{b}(x) \mp f'(x), \qquad p_1^\pm \equiv \mp\partial + \chi(x)$$

prescribe that

$$V_{1,2} = \chi^2 \mp \chi' = \mp 2f' + f^2 + \frac{f''}{2f} - \left(\frac{f'}{2f}\right)^2 - \frac{d}{4f^2} - a, \qquad \tilde{b} = f^2 - \frac{f''}{2f} + \left(\frac{f'}{2f}\right)^2 + \frac{d}{4f^2},$$

where χ, f are real functions and a, d are real constants. The related superalgebra closure for K, $\bar{K}$ and P, $\bar{P}$ takes the form,

$$\{K, K^\dagger\} = (H + a)^2 + d, \qquad \{P, P^\dagger\} = H,$$

the latter one clarifies the role of constants a, d.

The compatibility of two supersymmetries is achieved on solutions of the following equations

$$\chi = 2f + \chi_0, \qquad f^2 + \frac{f''}{2f} - \left(\frac{f'}{2f}\right)^2 - \frac{d}{4f^2} - a = \chi^2 = (2f + \chi_0)^2, \tag{8}$$

where χ_0 is an arbitrary real constant. The latter one represents a nonlinear second-order differential equation which solutions are parameterized by two integration constants. Therefore as it was advertised the existence of two SUSY constrains substantially the class of potentials for which they may hold.

Let us use the freedom to redefine the higher-order supercharge $k_2^\pm \to k_2^\pm + \chi_0 p_1^\pm$ for eliminating the constant χ_0 in (8). After this simplification the equation (8) is integrated into the following, first-order one,

$$\chi = 2f, \qquad (f')^2 = 4f^4 + 4af^2 + 4G_0 f - d \equiv \Phi_4(f),$$

where G_0 is a real constant.

The solutions of this equation are elliptic functions which can be easily found in the implicit form,

$$\int_{f_0}^{f(x)} \frac{df}{\sqrt{\Phi_4(f)}} = \pm(x - x_0),$$

where the lower limit of integration f_0 and x_0 are real constants.

It can be shown that they may be nonsingular in three situations.

a) The polynomial $\Phi_4(f)$ has four different real roots $f_1 < f_2 < f_3 < f_4$ and f_0 is chosen between f_2 and f_3. The corresponding potentials are periodic. This case will not be examined here.

b) $\Phi_4(f)$ has three different real roots and the double root $\beta/2$ is either the maximal one or a minimal one,

$$\Phi_4(f) = 4\left(f - \frac{\beta}{2}\right)^2 \left[\left(f + \frac{\beta}{2}\right)^2 - (\beta^2 - \epsilon)\right], \qquad 0 < \epsilon < \beta^2. \tag{9}$$

Then there exists a relation between constants a, d, G_0 in terms of parameters β, ϵ,

$$a = \epsilon - \frac{3\beta^2}{2} < 0, \qquad G_0 = \beta(\beta^2 - \epsilon), \qquad d = \beta^2\left(\frac{3\beta^2}{4} - \epsilon\right). \tag{10}$$

Besides, the constant f_0 is taken between the double root and a nearest simple root.

c) $\Phi_4(f)$ has two different real double roots which correspond in (9), (10) to $G_0 = 0$, $\beta^2 = \epsilon > 0$, $a = -\beta^2/2$, $d = -\beta^4/4$. The constant f_0 is taken between the roots.

The corresponding potentials $V_{1,2}$ are well known and in the cases b) and c) are reflectionless, with one bound state at the energy $E_b = (\beta^2 - \epsilon)$ and with the continuum spectrum starting from $E_c = \beta^2$. In particular, in the case b) the potentials coincide in their form and differ only by shift in the coordinate (about the latter phenomenon see [6]),

$$V_{1,2} = \beta^2 - \frac{2\epsilon}{\mathrm{ch}^2\left(\sqrt{\epsilon}(x - x_0^{(1,2)})\right)}, \qquad x^{(1,2)} = x_0 \pm \frac{1}{4\sqrt{\epsilon}} \ln\frac{\beta - \sqrt{\epsilon}}{\beta + \sqrt{\epsilon}},$$

and in the case c) one of the potentials can be chosen constant,

$$V_1 = \beta^2, \qquad V_2 = \beta^2\left(1 - \frac{2}{\mathrm{ch}^2\left(\beta(x - x_0)\right)}\right).$$

For these potentials one can illustrate all the relations of extended SUSY algebra. Thus, in particular, the polynomial symmetry operator $\mathcal{B}(H)$ turns out to be constant, $\mathcal{B}(H) = G_0$, the second symmetry operator reads,

$$\mathcal{E}(H) = i\left[\boldsymbol{I}\,\partial^3 - \left(a\boldsymbol{I} + \frac{3}{2}\boldsymbol{V}(x)\right)\partial - \frac{3}{4}\boldsymbol{V}'(x)\right]$$

and, finally, $\mathcal{E}^2(H)$ takes the form

$$\mathcal{E}^2(H) \equiv \mathcal{P}_e(H) = H\left[(H + a)^2 + d\right] - G_0^2 \equiv (H - E_b)^2(H - E_c).$$

Hence, in considered case both roots of $\mathcal{P}_e(E)$ characterize spectrum of H: one of them (E_b) characterizes the energy of a bound state and another (E_c) characterizes the energy of a state in the bottom of continuum spectrum.

8 More about symmetry operators

Let us suppose that:

a) the Hamiltonian h_0 commutes with an antisymmetric real operator R_0 of order $2n + 1$ which cannot be stripped off;

b) h_0 has n bound states with wave functions Ψ_l and energies E_l, $l = 0, \ldots, n-1$, where $E_{l+1} > E_l$, $l = 0, \ldots, n-2$.

Then one can show that:

a) the (normalized) symmetry operator R_0 can be factorized in the form,

$$R_0 = r_0^t \cdots r_{n-1}^t\, \partial\, r_{n-1} \cdots r_0, \qquad r_l \equiv \partial + \chi_l, \tag{11}$$

with non-singular real superpotentials χ_l such that

$$r_l \cdots r_0 \Psi_l = 0, \qquad l = 0, \ldots, n-1;$$

b) the ladder (dressing chain) relations hold,

$$h_{l+1} r_l = r_l h_l, \qquad l = 0, \ldots, n-1,$$
$$h_l \equiv r_{l-1} r_{l-1}^t + E_{l-1} = r_l^t r_l + E_l, \qquad l = 1, \ldots, n-1,$$
$$h_0 = r_0^t r_0 + E_0, \qquad h_n = r_{n-1} r_{n-1}^t + E_{n-1};$$

c) the antisymmetric symmetry operators arise for each intermediate Hamiltonian,

$$R_l = r_l^t \cdots r_{n-1}^t \,\partial\, r_{n-1} \cdots r_l, \qquad R_n = \partial, \qquad R_l h_l = h_l R_l, \qquad l = 0, \ldots, n.$$

Evidently the Hamiltonian h_n describes a free particle and therefore the Hamiltonian h_0 is intertwined with the Hamiltonian of free particle.

The described result can be illustrated with the help of the case b) of Section 7, where each component of the symmetry operator $\mathcal{E}(H)$ can be represented in the canonical factorized form (11),

$$e^{\pm} = i\left[\partial^3 - \left(a + \frac{3}{2}V_{1,2}\right)\partial - \frac{3}{4}V_{1,2}'\right] = -i\left(-\partial - \frac{\Psi_{01,2}'}{\Psi_{01,2}}\right)\partial\left(\partial - \frac{\Psi_{01,2}'}{\Psi_{01,2}}\right),$$

by means of the bound-state wave functions, $\Psi_{01,2} = C\sqrt{V_{1,2} - \beta^2}$.

Acknowledgments

One of us (A.S.) is grateful to Organizing Committee of the Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" for hospitality. Our work was supported by Grant RFBR 02-01-00499.

[1] Andrianov A.A., Ioffe M.V. and Spiridonov V.P., Higher derivative supersymmetry and the Witten index, *Phys. Lett. A*, 1993, V.174, 273–279; hep-th/9303005;
Veselov A.P. and Shabat A.B., Dressing chain and spectral theory of Schrödinger operator, *Funk. Anal. i ego Pril.*, 1993, V.27, N 2, 1–21;
Andrianov A.A., Cannata F., Dedonder J.P. and Ioffe M.V., Second order derivative supersymmetry, q deformations and scattering problem, *Int. J. Mod. Phys. A*, 1995, V.10, 2683–2702; hep-th/9404061;
Andrianov A.A., Ioffe M.V. and Nishnianidze D.N., Polynomial SUSY in quantum mechanics and second derivative Darboux transformations, *Phys. Lett. A*, 1995, V.201, N 2–3, 103–110;
Andrianov A.A., Ioffe M.V. and Nishnianidze D.N., Polynomial supersymmetry and dynamical symmetries in quantum mechanics, *Theor. Math. Phys.*, 1995, V.104, 1129–1140;
Bagrov V.G. and Samsonov B.F., Darboux transformation of Schrödinger equation, *Phys. Part. Nucl.*, 1997, V.28, N 4, 374–397;
Fernández D.J., SUSUSY quantum mechanics, *Int. J. Mod. Phys. A*, 1997, V.12, 171–176;
Bagchi B., Ganguli A., Bhaumik D. and Mitra A., Higher derivative supersymmetry, a modified Crum–Darboux transformation and coherent state, *Mod. Phys. Lett. A*, 1999, V.14, 27–34;
Andrianov A.A., Cannata F., Dedonder J.P. and Ioffe M.V., SUSY quantum mechanics with complex superpotentials and real energy spectra, *Int. J. Mod. Phys. A*, 1999, V.14, 2675–2688; quant-ph/9806019.

[2] Aoyama H., Sato M. and Tanaka T., $\mathcal{N}$-fold supersymmetry in quantum mechanics: general formalism, *Nucl. Phys. B*, 2001, V.619, 105–127; quant-ph/0106037.

[3] Klishevich S.M. and Plyushchay M.S., Nonlinear supersymmetry, quantum anomaly and quasiexactly solvable systems. *Nucl. Phys. B*, 2001, V.606, 583–612; hep-th/0012023.

[4] Andrianov A.A. and Sokolov A.V., Nonlinear supersymmetry in quantum mechanics: algebraic properties and differential representation, *Nucl. Phys. B*, 2003, V.660, 25–50; hep-th/0301062.

[5] Zhdanov R.Z., Conditional symmetry and spectrum of the one-dimensional Schrödinger equation, *J. Math. Phys.*, 1996, V.37, N 7, 3198–3217; hep-th/9606028.
Fushchych W.I. and Nikitin A.G., Higher symmetries and exact solutions of linear and nonlinear Schrödinger equation, *J. Math. Phys.*, 1997, V.38, N 11, 5944–5959.

[6] Fernández D.J., Mielnik B., Rosas-Ortiz O. and Samsonov B.F., The phenomenon of Darboux displacements, *Phys. Lett. A*, 2002, V.294, 168–174; quant-ph/0302204.