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Four Dimensional Cubic Supersymmetry

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A four dimensional non-trivial extension of the Poincaré algebra different from supersymmetry is explicitly studied. Representation theory is investigated and an invariant Lagrangian is exhibited. Some discussion on the Noether theorem is also given.

1 Introduction

The concept of symmetries is a central tool in the description of physical systems. One of the main questions is, of course, what are the mathematical structures which are useful in describing the laws of physics, and, in particular, useful in particle physics or quantum field theories. Among others, finite-dimensional Lie algebras become essential for the description of space-time symmetries and fundamental interactions. On the other hand, it was the discovery of supersymmetry in relativistic quantum field theory or as a possible non-trivial extension of Poincaré invariance [1] which gave rise to the concept of Lie superalgebras. One natural question one should address is the possibility to weigh up, in relativistic quantum field theory, algebraic structures which are not Lie (super)algebras. *A priori* this should be a difficult task. Indeed, according to the Noether theorem, to all (Noetherian) symmetries correspond a conserved currents. These symmetries are then generated by charges which are expressed in terms of the fields. But, having two kinds of fields, of integer or half-integer spin, which, by spin-statistics theorem will close with commutators or anticommutators, *a priori* one should obtain only Lie and Lie superalgebras. Furthermore, the Coleman–Mandula [2] and the Haag–Lopuszanski–Sohnius [3] theorems state that within the framework of Lie algebras one obtains only the description of space-time and/or internal symmetries, while within Lie superalgebras supersymmetry is the *unique* non-trivial extension of the Poincaré algebra which is possible.

But, if one examines the hypotheses of the above theorems, one sees that it is possible to imagine symmetries which go beyond supersymmetry. Several possibilities have been considered in the literature, the intuitive idea being that the generators of the Poincaré algebra are obtained as an appropriate product of more fundamental additional symmetries. These new generators are in an appropriate representation of the Poincaré algebra.

In this contribution we would like to study one of the possible non-trivial extension of the Poincaré algebra, different from supersymmetry, named fractional supersymmetry (FSUSY) [4–20] and its associated underlying algebraic structure named F-Lie algebras [17,19]. In supersymmetric theories, the extensions of the Poincaré algebra are obtained from a "square root" of the translations, "$QQ \sim P$". In this paper, new algebras are obtained from yet higher order roots. We mainly focus on the simplest alternative where "cubic roots" are involved "$QQQ \sim P$" [20]. It is important to stress that such structures are not Lie (super)algebras and as such escape *a priori* the Coleman–Mandula [2] as well as the Haag–Lopuszanski–Sohnius [3] no-go theorems. Furthermore, as far as we know, no no-go theorem associated to such types of extensions has been considered in the literature. This can open interesting possibilities to search for a field theoretic realization of a non-trivial extension of the Poincaré algebra that *is not the supersymmetric one*.

The aim of this paper is to summarize some results already obtained in [20], *i.e.* to construct explicitly the first field theoretic construction in $(1+3)$ dimensions of an FSUSY with $F=3$, which we will refer to as *cubic supersymmetry*, or 3SUSY. Representation of our algebra leads to fermionic or bosonic multiplets. We find that the fermion multiplets are made of three definite chirality fermions that are degenerate in mass, while the boson multiplets contain Lorentz scalars, vectors and two-forms. A striking feature for the boson multiplets is the compatibility of 3SUSY with gauge symmetry only when the latter is gauge fixed in the usual way. Some discussions on Noether theorem in relation with 3SUSY are also given. The paper is organized as follow. In Section 2 some results on the algebraic extension of the Poincaré algebra are given. In Section 3 representations of the 3SUSY algebra are exhibited. In Section 4, we construct an invariant Lagrangian in the case of the bosonic multiplet. Section 5 is devoted to some discussions on Noether theorem. Finally, some conclusions are given in Section 6.

2 Non trivial extension of the Poincaré algebra

A natural generalization of Lie (super)algebras which is relevant for the algebraic description of FSUSY was defined in [18, 19] and called an F-Lie algebra. An $F-$Lie algebra admits a $\mathbb{Z}_F$-gradation, the zero-graded part being a Lie algebra. An F-fold symmetric product (playing the role of the anticommutator in the case $F = 2$) expresses the zero graded part in terms of the non-zero graded part. The first examples of F-Lie algebras where infinite-dimensional [17]. It was then established that one of these examples of infinite-dimensional algebras was relevant to apply FSUSY on relativistic anyons in $(1 + 2)D$ [16]. Later on, it was shown how to construct finite-dimensional F-Lie algebras with $F > 2$ by an inductive process starting from Lie algebras and Lie superalgebras [19]. Among these families of examples one can identify F-Lie algebras that could generate extensions of the Poincaré algebra. One of these examples is given by

$$\mathfrak{g} = \mathfrak{sp}(4, \mathbb{R}) \oplus \mathrm{ad}\,(\mathfrak{sp}(4, \mathbb{R})) \tag{1}$$

with $\mathrm{ad}\,(\mathfrak{sp}(4, \mathbb{R}))$ the adjoint representation of $\mathfrak{sp}(4, \mathbb{R})$. $\mathfrak{g}_0 = \mathfrak{sp}(4, \mathbb{R})$ is the zero graded part ("bosonic") of $\mathfrak{g}$ and $\mathfrak{g}_1 = \mathrm{ad}\,(\mathfrak{sp}(4, \mathbb{R}))$ is the graded part of the algebra. If we denote J_a, $a = 1, \ldots, 10$ a basis of $\mathfrak{sp}(4, \mathbb{R})$, A_a the corresponding basis for $\mathrm{ad}\,(\mathfrak{sp}(4, \mathbb{R}))$, and $g_{ab} = \mathrm{Tr}\,(A_aA_b)$ the Killing form of $\mathfrak{sp}(4, \mathbb{R})$, then the F-Lie algebra of order 3 $\mathfrak{g}$ reads [19, 20][1]

$$[J_a, J_b] = f_{ab}{}^{c} J_c, \qquad [J_a, A_b] = f_{ab}{}^{c} A_c, \qquad \{A_a, A_b, A_c\} = g_{ab}J_c + g_{ac}J_b + g_{bc}J_a, \tag{2}$$

where $f_{ab}{}^{c}$ are the structure constant of $\mathfrak{sp}(4, \mathbb{R})$ and $\{A_a, A_b, A_c\}$ is given by the symmetric three-fold product $A_aA_bA_c + A_aA_cA_b + A_bA_cA_a + A_bA_aA_c + A_cA_aA_b + A_cA_bA_a$.

Observing that $\mathfrak{so}(1, 3) \subset \mathfrak{so}(2, 3) \cong \mathfrak{sp}(4)$, and that the $(1 + 3)D$ Poincaré algebra is related to $\mathfrak{sp}(4)$ through an Inönü–Wigner contraction, from the F-Lie algebra (2), an extension of the Poincaré algebra can be constructed [20]:

$$\begin{aligned}
&[L_{mn}, L_{pq}] = \eta_{nq}L_{pm} - \eta_{mq}L_{pn} + \eta_{np}L_{mq} - \eta_{mp}L_{nq}, \qquad [L_{mn}, L_p] = \eta_{np}P_m - \eta_{mp}P_n,\\
&[L_{mn}, Q_p] = \eta_{np}Q_m - \eta_{mp}Q_n, \qquad [P_m, Q_n] = 0,\\
&\{Q_m, Q_n, Q_r\} = \eta_{mn}P_r + \eta_{mr}P_n + \eta_{rn}P_m,
\end{aligned} \tag{3}$$

where η_{mn} is the Minkowski metric, L_{mn}, P_m are the Poincaré generators and Q_m are the "supercharges" in the vector representation of $\mathfrak{so}(1, 3)$.

3 Representations

Representations of (3) were also studied in [20]. It turns out that, for algebras defined by cubic relations, the situation is a more difficult task than in usual supersymmetric theories. Indeed,

[1] In addition to these relations, one has also some appropriate Jacobi identities. See [18–20] for details.

representations of supersymmetry are related to representations theory of the well-known Clifford algebra, while representation theory of FSUSY is related to Clifford algebras of polynomial [21]. To obtain representations of the algebra (3), we rewrite the RHS of the trilinear bracket as $\{Q_m, Q_n, Q_r\} = f_{mnr} = f_{mnr}{}^s P_s$, with $f_{mnr}{}^s = \eta_{mn}\delta_r{}^s + \eta_{mr}\delta_n{}^s + \eta_{rn}\delta_m{}^s$. This substitution shows that the cubic polynomial $f(v^0, v^1, v^2, v^3) = f_{mnr}v^m v^n v^r = 3(v.P)(v.v)$ is associated to the symmetric tensor f_{mnr}. Moreover, the algebra (3) simply means that $f(v) = (v^m Q_m)^3$, as can be verified by developing the cube and identifying all terms, using the trilinear bracket. The generators Q_m, $m = 0, \ldots, 3$, which are associated with the variables v^m, $m = 0, \ldots, 3$, then generate an extension of the Clifford algebra called Clifford algebra of the polynomial f (denoted $\mathcal{C}_f$). This means that the Q's allow to "linearize" f. The general representation theory of $\mathcal{C}_f$ is not known, however, a systematic method to represent $\mathcal{C}_f$ by appropriate matrices has been given [22]. For the algebra (3) one obtains [20]

$$Q_m = \begin{pmatrix} 0 & \Lambda^{1/3}\gamma_m & 0 \\ 0 & 0 & \Lambda^{1/3}\gamma_m \\ \Lambda^{-2/3}P_m & 0 & 0 \end{pmatrix}. \tag{4}$$

with γ_m being the $4D$ Dirac matrices and $P_m = -i\frac{\partial}{\partial x^m}$. It is interesting to notice that because P is dimensionful, a parameter with a mass dimension appears naturally in (4).

The Q being in the vector representation of $\mathfrak{so}(1,3)$, it is easy to see that

$$J_{mn} = \frac{1}{4}(\gamma_m\gamma_n - \gamma_n\gamma_m) + i(x_m P_n - x_n P_m) \tag{5}$$

are the appropriate Lorentz generators acting on Q: $[J_{mn}, Q_r] = \eta_{nr}Q_m - \eta_{mr}Q_m$.

Introducing the $4D$ Dirac matrices in the Weyl representation, $\gamma_m = \begin{pmatrix} 0 & \sigma_m \\ \bar\sigma_m & 0 \end{pmatrix}$, with $\sigma_{m\,\alpha\dot\alpha} = (1, \sigma_i)$, $\bar\sigma_m{}^{\dot\alpha\alpha} = (1, -\sigma_i)$ and σ_i the Pauli matrices (the convention for dotted and undotted indices are those conventionally used in SUSY, – see *e.g.* appendix B of [20]), shows that the representation (4) is reducible and leads to the two inequivalent $6D$ representations:

$$Q_m = \begin{pmatrix} 0 & \Lambda^{1/3}\sigma_m & 0 \\ 0 & 0 & \Lambda^{1/3}\bar\sigma_m \\ \Lambda^{-2/3}P_m & 0 & 0 \end{pmatrix}, \qquad Q_m = \begin{pmatrix} 0 & \Lambda^{1/3}\bar\sigma_m & 0 \\ 0 & 0 & \Lambda^{1/3}\sigma_m \\ \Lambda^{-2/3}P_m & 0 & 0 \end{pmatrix}. \tag{6}$$

In this representation the trilinear part of the algebra (3) is realized as

$$\begin{aligned} &Q_mQ_nQ_r + Q_mQ_rQ_n + Q_nQ_mQ_r + Q_nQ_rQ_m + Q_rQ_mQ_n + Q_rQ_nQ_m \\ &\qquad = \eta_{mn}P_r + \eta_{nr}P_m + \eta_{mr}P_n. \end{aligned} \tag{7}$$

3.1 Fermionic multiplet

As usual, the content of the representation is not only specified by the form of the matrix representation, but also by the behavior of the vacuum under Lorentz transformations. If we denote by Ω the vacuum, which is in some specified representation of the Lorentz algebra, with Σ_{mn} the corresponding Lorentz generators, then J_{mn} given in (5) is replaced by $J_{mn} + \Sigma_{mn}$. In the case, where Ω is a Lorentz scalar, one sees that the multiplet of the representations (6) contains two left-handed and one right-handed fermions for the first matrices, while the multiplet of the representation contains one left-handed and two right-handed fermions for the second matrices. These two multiplets are CPT conjugate. In the first case, if we denote $\mathbf{\Psi} = \begin{pmatrix} \psi_{1\alpha} \\ \bar\psi_2^{\dot\alpha} \\ \psi_{3\alpha} \end{pmatrix}$, then under a 3SUSY transformation we have $\delta_\varepsilon \mathbf{\Psi} = \varepsilon^m Q_m \mathbf{\Psi}$ and we obtain

$$\delta_\varepsilon \psi_{1\alpha} = \varepsilon^n \Lambda^{1/3} \sigma_{n\alpha\dot\alpha} \bar\psi_2{}^{\dot\alpha}, \qquad \delta_\varepsilon \bar\psi_2{}^{\dot\alpha} = \varepsilon^n \Lambda^{1/3} \bar\sigma_n{}^{\dot\alpha\alpha} \psi_{3\alpha}, \qquad \delta_\varepsilon \psi_{3\alpha} = \varepsilon^n \Lambda^{-2/3} P_n \psi_{1\alpha} \tag{8}$$

with ε a pure imaginary number.

3.2 Bosonic multiplet

In the previous subsection, we were considering the fundamental representation associated to the matrices (6), say $\mathbf{\Psi} = \begin{pmatrix} \psi_{1\alpha} \\ \bar\psi_2{}^{\dot\alpha} \\ \psi_{3\alpha} \end{pmatrix}$ and $\mathbf{\Psi}' = \begin{pmatrix} \bar\psi'_1{}^{\dot\alpha} \\ \psi'_{2\alpha} \\ \bar\psi'_3{}^{\dot\alpha} \end{pmatrix}$, *i.e.* with the vacuum Ω in the trivial representation of the Lorentz algebra. In this section boson multiplets, will be introduced, corresponding to a vacuum in the spinor representations of the Lorentz algebra. This means that four types of boson multiplets can be introduced: $\mathbf{\Psi} \otimes \Omega^\alpha$, $\mathbf{\Psi}' \otimes \Omega^\alpha$, with Ω^α a left-handed spinor and $\mathbf{\Psi} \otimes \bar\Omega_{\dot\alpha}$, $\mathbf{\Psi}' \otimes \bar\Omega_{\dot\alpha}$ with $\bar\Omega_{\dot\alpha}$ a right-handed spinor.

For the multiplet associated to $\mathbf{\Psi}^\beta = \mathbf{\Psi} \otimes \Omega^\beta$, we have $\mathbf{\Psi}^\beta = \begin{pmatrix} \rho_{1\alpha}{}^\beta \\ \bar\rho_2{}^{\dot\alpha\beta} \\ \rho_{3\alpha}{}^\beta \end{pmatrix}$ and for the one (CPT conjugate of the previous) $\mathbf{\Psi}_{\dot\beta} = \mathbf{\Psi}' \otimes \Omega_{\dot\beta} = \begin{pmatrix} \bar\rho_1{}^{\dot\alpha}{}_{\dot\beta} \\ \rho_{2\alpha\dot\beta} \\ \bar\rho_3{}^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}$. The transformation under 3SUSY is $\delta_\varepsilon \mathbf{\Psi}^\beta = \varepsilon^m Q_m \mathbf{\Psi}^\beta$ with Q_m given in (6) and similarly for $\mathbf{\Psi}_{\dot\beta}$.

Notice that ρ_1, $\bar\rho_1$, $\bar\rho_2$, ρ_2 and ρ_3, $\bar\rho_3$ are not irreducible representations of $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(1,3)$, we therefore define

$$\begin{aligned}
&\rho_1 = \varphi\, I_2 + \frac{1}{2} B_{mn}\, \sigma^{nm}, \qquad \bar\rho_1 = \varphi'\, \bar I_2 + \frac{1}{2} B'_{mn} \bar\sigma^{nm}, \qquad \bar\rho_2 = A^m\, \bar\sigma_m, \qquad \rho_2 = A'^m\, \sigma_m, \\
&\rho_3 = \tilde\varphi\, I_2 + \frac{1}{2} \tilde B_{mn}\, \sigma^{nm}, \qquad \bar\rho_3 = \tilde\varphi'\, \bar I_2 + \frac{1}{2} \tilde B'{}_{mn}\, \bar\sigma^{nm}
\end{aligned} \tag{9}$$

with I_2 and $\bar I_2$ the two by two identity matrices, σ_{mn} and $\bar\sigma_{mn}$ the Lorentz generators for the two spin representations, A^m and A'^m two vectors, φ, $\tilde\varphi$ and φ', $\tilde\varphi'$ four scalars, B_{mn}, $\tilde B_{mn}$ two self-dual two-forms and $B'{}_{mn}$, $\tilde B'{}_{mn}$ two anti-self-dual two-forms. Then one can show that the transformations read [20]

$$\begin{aligned}
&\delta_\varepsilon \varphi = \Lambda^{1/3} \varepsilon^m A_m, \qquad \delta_\varepsilon \varphi' = \Lambda^{1/3} \varepsilon^m A'_m, \\
&\delta_\varepsilon B_{mn} = -\Lambda^{1/3} \left(\varepsilon_m A_n - \varepsilon_n A_m\right) + \Lambda^{1/3} i \varepsilon_{mnpq} \varepsilon^p A^q, \\
&\delta_\varepsilon B'_{mn} = -\Lambda^{1/3} \left(\varepsilon_m A'_n - \varepsilon_n A'_m\right) - \Lambda^{1/3} i \varepsilon_{mnpq} \varepsilon^p A'^q, \\
&\delta_\varepsilon A_m = \Lambda^{1/3} \left(\varepsilon_m \tilde\varphi + \varepsilon^n \tilde B_{mn}\right), \qquad \delta_\varepsilon A'_m = \Lambda^{1/3} \left(\varepsilon_m \tilde\varphi' + \varepsilon^n \tilde B'_{mn}\right), \\
&\delta_\varepsilon \tilde\varphi = \Lambda^{-2/3} \varepsilon^m P_m \varphi, \qquad \delta_\varepsilon \tilde\varphi' = \Lambda^{-2/3} \varepsilon^m P_m \varphi', \\
&\delta_\varepsilon \tilde B_{mn} = \Lambda^{-2/3} \varepsilon^p P_p B_{mn}, \qquad \delta_\varepsilon \tilde B'_{mn} = \Lambda^{-2/3} \varepsilon^p P_p B'_{mn}.
\end{aligned} \tag{10}$$

The second bosonic multiplet being CPT conjugate to the first one, we have $\left(\bar\rho_1{}^{\dot\alpha}{}_{\dot\beta}\right)^\star = \rho_1{}^\alpha{}_\beta$, $\left(\rho_{2\alpha\dot\beta}\right)^\star = \bar\rho_{2\dot\alpha\beta}$ and $\left(\bar\rho_3{}^{\dot\alpha}{}_{\dot\beta}\right)^\star = \rho_3{}^\alpha{}_\beta$. [$B^\star$, the complex conjugate of B, is not to be confused with ${}^\star B$, the dual of B, see after.] This means, paying attention to the position of the indices, that we have

$$\varphi'^\star = -\varphi, \qquad \tilde\varphi'^\star = -\tilde\varphi, \qquad B'^\star_{mn} = -B_{mn}, \qquad \tilde B'^\star_{mn} = -\tilde B_{mn}, \qquad A'^\star_m = A_m. \tag{11}$$

These relations are compatible with the transformations laws given in (10) since $\varepsilon_n{}^\star = -\varepsilon_n$ and $P_n = -i\frac{\partial}{\partial x^n}$.

4 Invariant action

To construct a real invariant action involving the bosonic multiplets (9), we introduce the real fields

$$\begin{aligned}
&A_- = i\frac{A-A'}{\sqrt{2}}, \qquad A_+ = \frac{A+A'}{\sqrt{2}},\\
&B_- = \frac{B-B'}{\sqrt{2}}, \qquad B_+ = i\frac{B+B'}{\sqrt{2}}, \qquad \tilde{B}_- = \frac{\tilde{B}-\tilde{B}'}{\sqrt{2}}, \qquad \tilde{B}_+ = i\frac{\tilde{B}+\tilde{B}'}{\sqrt{2}},\\
&\varphi_- = \frac{\varphi-\varphi'}{\sqrt{2}}, \qquad \varphi_+ = i\frac{\varphi+\varphi'}{\sqrt{2}}, \qquad \tilde{\varphi}_- = \frac{\tilde{\varphi}-\tilde{\varphi}'}{\sqrt{2}}, \qquad \tilde{\varphi}_+ = i\frac{\tilde{\varphi}+\tilde{\varphi}'}{\sqrt{2}}.
\end{aligned} \tag{12}$$

These new fields form now one (reducible) multiplet of 3SUSY, with $^\star B_- = B_+$ ($^\star B_-$ is the dual of B_- [20]). The corresponding two- and three-form field strengths read

$$\begin{aligned}
&F_{\pm mn} = \partial_m A_{\pm n} - \partial_n A_{\pm m},\\
&H_{\pm mnp} = \partial_m B_{\pm np} + \partial_n B_{\pm pm} + \partial_p B_{\pm mn}.
\end{aligned} \tag{13}$$

They are invariant under the gauge transformations

$$\begin{aligned}
&\varphi_\pm \to \varphi_\pm,\\
&A_{\pm m} \to A_{\pm m} + \partial_m \chi_\pm,\\
&B_{\pm mn} \to B_{\pm mn} + (\partial_m \chi_{\pm\, n} - \partial_n \chi_{\pm\, m}),
\end{aligned} \tag{14}$$

where $\chi_\pm$ ($\chi_\pm^m$) are arbitrary scalar (vector) functions (χ_-^m and χ_+^m can still be related in order to preserve the duality relations between B_- and B_+)[2].

In a similar way we introduce the field strength $\tilde{H}_{-mnp}$, $\tilde{H}_{+mnp}$, as well as the dual fields $^\star H_{-m}$, $^\star H_{+m}$, $^\star\tilde{H}_{-m}$, $^\star\tilde{H}_{+m}$ (where $^\star H_m \equiv \frac{1}{6}\varepsilon_{mnpq}H^{npq}$. For instance $^\star\tilde{H}_{-m} = i\partial^n B_{+mn}$). We consider now the following local gauge invariant and zero graded Lagrangian,

$$\begin{aligned}
\mathcal{L} = &\ \partial_m\varphi_-\partial^m\tilde{\varphi}_- - \partial_m\varphi_+\partial^m\tilde{\varphi}_+\\
&-\frac{1}{4}F_{-mn}F_-{}^{mn} + \frac{1}{4}F_{+mn}F_+{}^{mn} - \frac{1}{2}\left(\partial_m A_-{}^m\right)^2 + \frac{1}{2}\left(\partial_m A_+{}^m\right)^2\\
&-\frac{1}{12}H_{-mnp}\tilde{H}_-{}^{mnp} + \frac{1}{12}H_{+mnp}\tilde{H}_+{}^{mnp} + \frac{1}{2}{}^\star H_{-m}{}^\star\tilde{H}_-{}^m - \frac{1}{2}{}^\star H_{+m}{}^\star\tilde{H}_+{}^m.
\end{aligned} \tag{15}$$

By means of (12), a direct calculation shows that (15) is invariant under the transformations (10), up to a surface term. It is interesting to notice that the 3SUSY invariance is compatible with gauge symmetries if the latter are gauged fixed. A usual 't Hooft Feynman gauge fixing term $(-\frac{1}{2}\left(\partial_m A_-{}^m\right)^2 + \frac{1}{2}\left(\partial_m A_+{}^m\right)^2)$ is required for the vector fields. For the two forms, due to the relation $^\star B_- = B_+$ also some gauge fixing terms *à la* 't Hooft Feynman are present. Developing all terms in (15) the Lagrangian can be rewritten *à la* "Fermi-like"

$$\begin{aligned}
\mathcal{L} = &\ \partial_m\varphi_-\partial^m\tilde{\varphi}_- - \frac{1}{2}\partial_m A_{-n}\partial^m A_-{}^n + \frac{1}{4}\partial_m B_{-np}\partial^m\tilde{B}_{-np}\\
&-\partial_m\varphi_+\partial^m\tilde{\varphi}_+ + \frac{1}{2}\partial_m A_{+n}\partial^m A_+{}^n - \frac{1}{4}\partial_m B_{+np}\partial^m\tilde{B}_{+np}
\end{aligned}$$

(a similar Lagrangian appears in the action-at-a-distance formalism for vector and two-forms – see *e.g.* [23]).

[2]Note that the gauge transformations (14) correspond naturally to the zero-, one- and two-form character of the components of the 3SUSY gauge multiplet.

One should note, though, the relative minus signs in front of the kinetic terms of the vector fields in (15) which endanger *a priori* the boundedness from below of the density energy of the "electromagnetic" fields. This difficulty does not have a clear physical interpretation as long as interaction terms have not been included, and necessitates a more careful study of the field manifold associated to the density energy.

5 Noether currents

The 3SUSY algebra we have studied has one main difference from the usual Lie (super)algebra: it does not close through quadratic, but rather cubic, relations. Moreover, it might be possible that some usual results of Lie (super)algebra do not apply straightforwardly. One example is the Noether currents and their associated algebra. This interesting point was studied in [20].

In this section we would like, however, to construct explicitly the Noether current. Using (15) and (10) we obtain

$$\begin{aligned} J_{mn} = &-i\Lambda^{1/3}(A_{-n}\partial_m\tilde{\varphi}_- - \partial_m A_{-n}\tilde{\varphi}_-) - i\Lambda^{1/3}(A_{+n}\partial_m\tilde{\varphi}_+ - \partial_m A_{+n}\tilde{\varphi}_+) \\ &+ i\Lambda^{1/3}(\tilde{B}_{-rn}\partial_m A_-^r - \tilde{B}_{-rn}\partial_m A_-^r) + i\Lambda^{1/3}(\tilde{B}_{+rn}\partial_m A_+^r - \tilde{B}_{+rn}\partial_m A_+^r) \\ &- i\Lambda^{-2/3}(\partial_m\varphi_-\partial_n\varphi_- - \partial_m\varphi_+\partial_n\varphi_+) - \frac{i}{4}\Lambda^{-2/3}(\partial_m B_-^{rs}\partial_n B_{-rs} - \partial_m B_+^{rs}\partial_n B_{+rs}), \end{aligned} \tag{16}$$

which is, due to the equations of motion, conserved $\partial_m J^{mn} = 0$ (up to a surface term). Then, the conserved charges are obtained as usual $\hat{Q}_m = \int d^3x J_{0m}$. Introducing the conjugate momentum Π of the fields Ψ (where Ψ is one of the fields of Section 4): $\Pi = \frac{\delta\mathcal{L}}{\delta\partial_0\Psi}$ after the usual quantization (equal-time commutation relations) one easily obtains

$$\delta_\varepsilon\Psi = \left[\varepsilon_n\hat{Q}^n, \Psi\right] \qquad \left(\delta_{Q_m}\Psi = \left[\hat{Q}^m, \Psi\right]\right). \tag{17}$$

In particular, this means that the algebra (3) is realized through

$$\begin{aligned} (\delta_{Q_m}.\delta_{Q_n}.\delta_{Q_r} + \text{perm})\,\Psi &= \left[\hat{Q}_m, \left[\hat{Q}_n, \left[\hat{Q}_r, \Psi\right]\right]\right] + \text{perm} \\ &= \eta_{mn}\left[\hat{P}_r, \Psi\right] + \eta_{mr}\left[\hat{P}_n, \Psi\right] + \eta_{rn}\left[\hat{P}_m, \Psi\right] = (\eta_{mn}\delta_{P_r} + \eta_{mr}\delta_{P_n} + \eta_{rn}\delta_{P_m})\,\Psi \end{aligned} \tag{18}$$

with $\hat{P}$ the generators of the Poincaré translations. Indeed, starting with the abstract algebra (3), we can represent it by some matrices as in Section 3 (see *e.g.* (4)). In this case the product of two transformations will be given by $\delta_n\delta_m\Psi = Q_nQ_m\Psi$ and the algebra will be realized as in (7). But, we can also represent (3) with commutators (17) acting on some Hilbert space, thus the product of two transformations will be given by $\delta_n\delta_m\Psi = \left[\hat{Q}_n, \left[\hat{Q}_m, \Psi\right]\right]$ leading to the realization (18) of the algebra (3). For a more general discussion one can see [20].

6 Conclusion

In this paper we have studied some four-dimensional realizations of 3SUSY, a non-trivial extension of the Poincaré algebra different from supersymmetry. Representation theory was explicitly constructed. Then, an invariant Lagrangian involving bosonic fields was given (some Lagrangian involving fermionic fields was also considered in [20]). The next step will be to construct an interacting theory.

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