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Generation of Glauber Coherent State Superpositions via Unitary Transformations

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A unitary transformation mapping the vacuum state of a harmonic oscillator into a normalized superposition of two arbitrary Glauber coherent states is reported. Interesting features of the transformation are discussed. Generalizations to the superposition of N coherent states are discussed.

1 Introduction

Since their introduction by Glauber in 1963 [1], coherent states have been deeply studied [2] due to the fact that, in some sense, the statistical properties related to the first momenta of their Fock distribution exhibit features interpretable in the realm of classical probability theory. Of course the quantum nature of a coherent state, and ultimately the granularity of the energy of a quantum harmonic oscillator, becomes more and more evident when higher momenta of the Fock distribution are considered. It is of relevance that coherent states of both a mechanical quantum oscillator or a quantized mode of an electromagnetic resonator may be currently proposed in accordance with well-defined protocols. Even more interesting is the fact that they provide the effective possibility of realizing interesting experiments able to bring to the light peculiarities of quantum mechanics. In particular, the very famous *gedanken* Schrödinger Cat State (SCS) experiment [3] consisting in entangling a microscopic quantum object (an atom) with a macroscopic one (a cat) has been realized. No unfortunate feline has been tortured! Trapped ions have been indeed exploited. In fact, such systems provide the possibility of testing fundamental quantum physics aspects giving at experimentalist's disposal compound quantum systems consisting of a few-level atomic part and a tri-dimensional harmonic oscillator describing the center of mass motion of an ion confined into a Paul trap. Suitably acting upon the system with Raman lasers it is possible to engineer the *Schrödinger Cat-like state* [4, 5]

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle\,|-\rangle + |-\alpha\rangle\,|+\rangle\right),$$

where $|\pm\rangle$ indicate two internal atomic states and $|\pm\alpha\rangle$ are two macroscopically well distinguishable harmonic oscillator configurations, i.e. *opposite* Glauber coherent states.

Today the definition of SCS is extended to all those situations wherein two macroscopically distinguishable classical states are quantum coherently simultaneously present [6]. Hence the purely vibrational normalized quantum superposition

$$|\phi\rangle = N(\alpha)\left(|\alpha\rangle + |-\alpha\rangle\right) \tag{1}$$

is referred to as a SCS. Here $N(\alpha)$ is the normalizing constant[1]. It is worth to stress that, apart from semantic questions, the superposition in equation (1) is of relevance in the study of fundamental aspects of quantum mechanics.

[1]Due to the non-orthogonality of coherent states, in most of the cases $N(\alpha)$ is not trivially $\frac{1}{\sqrt{2}}$.

In this paper we present, from the mathematical point of view, the construction of a new unitary operator able to generate coherent superpositions of Glauber coherent states when applied to the vibrational vacuum. In particular, dynamical configurations like that in equation (1) may be generated. The operator, initially related to the generation of two coherent states, will be generalized to the case of more than two such states. It is worth noting that, acting with such an operator upon states different from the vacuum but possessing appropriate spatial symmetries, it is possible to generate a wider class of superpositions of translated wave function.

The unitary operator introduced here may be used to solve elegantly and effectively those variational problems involving superpositions of Glauber coherent states as candidate solution.

2 Generation of coherent state superpositions

The physical system we consider is a harmonic oscillator of frequency ω described by the Hamiltonian

$$\hat{H} = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right),$$

where $\hat{a}$ and $\hat{a}^{\dagger}$ are the annihilation and creation operators respectively. Its eigenstates $|n\rangle$, n being a non-negative integer, describe a spectrum having equidistant levels: $\hat{H}\,|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle$.

It is well known that, applying a displacement operator to the vacuum state, $|0\rangle$, one obtains a so called Glauber coherent state

$$|\alpha\rangle \equiv \hat{D}\left(\alpha\right)|0\rangle \equiv e^{\alpha^{*}\hat{a}-\alpha\hat{a}^{\dagger}}\,|0\rangle\,,$$

where α is a complex number whose real and imaginary parts return the mean values of the position and momentum operators respectively. Observe that, since the vacuum state realizing a non-squeezed minimum position-momentum uncertainty is rigidly translated, the minimal uncertainty is preserved.

Consider now the observable

$$\hat{\Pi} = \cos\left(\pi\hat{a}^{\dagger}\hat{a}\right)$$

which is a Hermitian and unitary operator commuting with $\hat{H}$ and satisfying the following relations:

$$\hat{\Pi}\,|n\rangle = (-1)^{n}\,|n\rangle \tag{2}$$

and

$$\hat{\Pi}\hat{a}\hat{\Pi} = -\hat{a}, \qquad \hat{\Pi}\hat{a}^{\dagger}\hat{\Pi} = -\hat{a}^{\dagger}.$$

Due to equation (2), $\hat{\Pi}$ is called Parity operator. Starting from this point we introduce the operator

$$\hat{D}\left(\alpha\right)\Pi.$$

It turns out that, like $\hat{\Pi}$, it is both unitary, being the product of two unitary operators, and Hermitian, as easily verifiable:

$$\left[\hat{D}\left(\alpha\right)\Pi\right]^{\dagger} = \Pi\hat{D}\left(\alpha\right)^{\dagger} = \Pi\hat{D}\left(-\alpha\right) = \Pi\hat{D}\left(-\alpha\right)\Pi^{2} = \hat{D}\left(\alpha\right)\Pi.$$

Moreover, the square of such an operator is the Identity. Indeed

$$\left[\hat{D}\left(\alpha\right)\Pi\right]^{2} = \hat{D}\left(\alpha\right)\Pi\hat{D}\left(\alpha\right)\Pi = \hat{D}\left(\alpha\right)\hat{D}\left(-\alpha\right) = \hat{1}. \tag{3}$$

Taking into account all such properties, we define the following two-parameter unitary operator:

$$\hat{U}(\lambda,\alpha) \equiv e^{i\lambda \hat{D}(\alpha)\Pi}, \tag{4}$$

λ being a real number.

In consideration of equation (3), $\hat{U}(\lambda,\alpha)$ may be cast in the form

$$\begin{aligned} \hat{U}(\lambda,\alpha) &\equiv \sum_{k=0}^{\infty} \frac{\left(i\lambda \hat{D}(\alpha)\Pi\right)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(i\lambda)^{2k}}{(2k)!}\hat{1} + \sum_{k=0}^{\infty} \frac{(i\lambda)^{2k+1}}{(2k+1)!}\hat{D}(\alpha)\Pi \end{aligned} \tag{5}$$

and finally

$$\hat{U}(\lambda,\alpha) = \cos\lambda\,\hat{1} + i\sin\lambda\,\hat{D}(\alpha)\Pi. \tag{6}$$

The action of such an operator on a state we shall refer to as the *state of reference* returns a normalized superposition of the original wave-packet and its rigid translation of $\alpha = x + ip$ in the phase space, x and p being position and momentum expectation value changes. In particular, taking the vibrational vacuum $|0\rangle$ as state of reference one obtains a normalized superposition involving the ground state itself and the coherent state $|\alpha\rangle$:

$$\hat{U}(\lambda,\alpha)|0\rangle = \cos\lambda\,|0\rangle + i\sin\lambda\,|\alpha\rangle\,. \tag{7}$$

It is worth recalling that $|0\rangle$ may be thought of as a coherent state corresponding to the classical situation wherein the harmonically confined particle is *at rest in the origin*. Observe also the peculiarity of the state in equation (7) consisting in the fact that the normalized superposition has two weights expressible as sines and cosine of the same number, i.e. it looks like $|0\rangle$ and $|\alpha\rangle$ are orthogonal[2].

In order to obtain a linear combination of *two arbitrary* Glauber coherent states, it is enough the subsequent action of another displacement operator, $\hat{D}(\beta)$:

$$\hat{D}(\beta)\hat{U}(\lambda,\alpha)|0\rangle = \cos\lambda\,|\beta\rangle + ie^{i\Im(\alpha\beta^*)}\sin\lambda\,|\alpha+\beta\rangle\,. \tag{8}$$

Hence the unitary operator

$$\begin{aligned} \hat{V}(\alpha,\beta,\lambda) &\equiv \hat{D}(\alpha)\hat{U}(\lambda,\beta-\alpha) \\ &= \cos\lambda\hat{D}(\alpha) + ie^{i\Im(\alpha\beta^*)}\sin\lambda\hat{D}(\beta)\hat{\Pi} \end{aligned} \tag{9}$$

applied on the vacuum state generates the normalized superposition

$$\hat{V}(\alpha,\beta,\lambda)|0\rangle = \cos\lambda\,|\alpha\rangle + ie^{i\Im(\alpha\beta^*)}\sin\lambda\,|\beta\rangle\,. \tag{10}$$

Such a result may be easily extended taking as state of reference any eigenfunction of the Parity operator. Indeed, let $|\psi_0\rangle$ be a $\hat{\Pi}$ eigenstate meaning that $\psi_0(x) \equiv \langle x|\,\psi_0\rangle$ is an even or odd function. The action of $\hat{V}(\alpha,\beta,\lambda)$ on such a state generates

$$\hat{V}(\lambda,\alpha,\beta)|\psi_0\rangle = \cos\lambda\hat{D}(\alpha)|\psi_0\rangle + (-1)^{\sigma} ie^{i\Im(\alpha\beta^*)}\sin\lambda\hat{D}(\beta)|\psi_0\rangle\,, \tag{11}$$

σ being the parity of $\psi_0(x)$.

[2] Generally speaking such a circumstance is realized only in correspondence to a specific phase correlation between the two coefficients involved in the coherent superposition. To this end observe that, given the state $|\psi\rangle = c_1|\alpha_1\rangle + c_2 e^{i\phi}|\alpha_2\rangle$ with c_1, c_2, ϕ real numbers, the modulus $\langle\psi|\,\psi\rangle = |c_1|^2 + |c_2|^2 + c_1c_2 e^{-\frac{|\alpha-\beta|^2}{2}}\Re\left\{e^{i\phi}e^{i\Im(\alpha^*\beta)}\right\}$ may be expressed as $\left(c_1^2+c_2^2\right)$ only if $\phi = \frac{\pi}{2} - \Im(\alpha^*\beta) + k\pi$.

In the case when α and β are real and $|\psi_0\rangle$ is a localized wave function centered in the origin, the state in equation (10) turns out to be the superposition of two localized packets centered in $x = \alpha$ and $x = \beta$:

$$\langle x| \hat{V}(\lambda, \alpha, \beta) |\psi_0\rangle = \cos\lambda \psi_0 (x + \alpha) + i \sin\lambda \psi_0 (x + \beta) . \tag{12}$$

3 Superpositions of more than two coherent states

The question of how the number of coherent states in the superposition can be increased immediately arises. To this end the subsequent action of two $\hat{V}$ unitary operators is considered. The relevant transformation is given by

$$\hat{V}^{(2)}(\lambda, \mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \equiv \hat{V}(\lambda, \alpha_1, \alpha_2) \hat{V}(\mu, \alpha_3, \alpha_4) \tag{13}$$

which may be cast in the form

$$\begin{aligned}\hat{V}^{(2)}(\lambda, \mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = &\left[\cos\lambda \hat{D}(\alpha_1) + i e^{\Im\left(\alpha_1 \alpha_2^*\right)} \sin\lambda \hat{D}(\alpha_2) \hat{\Pi}\right] \\ &\times \left[\cos\mu \hat{D}(\alpha_3) + i e^{\Im\left(\alpha_3 \alpha_4^*\right)} \sin\mu \hat{D}(\alpha_4) \hat{\Pi}\right] .\end{aligned} \tag{14}$$

Taking into account properties of $\hat{\Pi}$, one obtains

$$\begin{aligned}\hat{V}^{(2)}(\lambda, \mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = &\cos\lambda \cos\mu\, e^{i\phi_{13}} \hat{D}(\alpha_1 + \alpha_3) + \cos\lambda \sin\mu\, e^{i\phi_{14}} \hat{D}(\alpha_1 + \alpha_4) \\ &+ \sin\lambda \sin\mu\, e^{i\phi_{24}} \hat{D}(\alpha_2 - \alpha_4) + \sin\lambda \cos\mu\, e^{i\phi_{23}} \hat{D}(\alpha_2 - \alpha_3) \hat{\Pi},\end{aligned} \tag{15}$$

where ϕ_{jk} are suitable quantum phases.

The four coherent states obtained when $\hat{V}^{(2)}$ is applied upon $|0\rangle$ are

$$|\alpha_1 + \alpha_3\rangle , \qquad |\alpha_1 + \alpha_4\rangle , \qquad |\alpha_2 - \alpha_3\rangle , \qquad |\alpha_2 - \alpha_4\rangle .$$

Despite appearances, we do not have four independently parametrically changeable states. Indeed, to generate the *arbitrary quadruplet*

$$|c_1\rangle , \qquad |c_2\rangle , \qquad |c_3\rangle , \qquad |c_4\rangle$$

one has to choose the α-parameters satisfying the following equation system:

$$\alpha_1 + \alpha_3 = c_1, \qquad \alpha_1 + \alpha_4 = c_2, \qquad \alpha_2 - \alpha_3 = c_3, \qquad \alpha_2 - \alpha_4 = c_4$$

which, unfortunately, has a vanishing determinant:

$$\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = 0.$$

Nevertheless, the usefulness of the transformation given by equation (15) remains valid, in particular considering the fact that it can be exploited to generate threefold Schrödinger Cat-like states. More in detail

$$\begin{aligned}\hat{V}^{(2)}(\lambda, \mu, 0, 0, 0, \alpha) |0\rangle = &\cos\mu \left(\cos\lambda\, e^{i\phi_{13}} + \sin\lambda\, e^{i\phi_{23}}\right) |0\rangle \\ &+ \cos\lambda \sin\mu\, e^{i\phi_{14}} |\alpha\rangle + \sin\lambda \sin\mu\, e^{i\phi_{24}} |-\alpha\rangle .\end{aligned} \tag{16}$$

In this equation we have three equidistant wave packets corresponding to the same wave function centered in the three positions $x = 0, \pm\Re\{\alpha\}$. A rigid translation of this state is obtained considering $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = 2\alpha$ and $\alpha_4 = \alpha$: $\hat{V}^{(2)}(\lambda, \mu, 0, 2\alpha, 0, \alpha)$. In fact in this case one has a linear combination of $|0\rangle$, $|\alpha\rangle$ and $|2\alpha\rangle$.

The method reported here may be extended to the case of a superposition involving 2^N coherent state just considering

$$\hat{V}^{(N)} \equiv \prod_{k=1}^{N} \hat{V}(\lambda_k, \alpha_k, \beta_k).$$

Of course, like in the case $N = 2$, the number of independently determinable states is smaller than 2^N.

4 Conclusive remarks

In this paper we have presented the construction of a unitary operator, $\hat{V}$ which in some sense behaves like a superposition of phase space translators. Such a circumstance, joint with the well known definition of Glauber coherent states as *displaced vacuum*, has provided an effective and elegant expression for an arbitrary two coherent state superposition. The subsequent application of two such unitary operators produces a four coherent state superposition and so on following the law 2 raised to the number of applications of $\hat{V}$, which is larger than the number of coherent state which may be independently parametrically changed at will.

As a conclusive remark we point out that the possibility of considering a generic linear combination of Glauber states via a unitary transformation of the vibrational vacuum leads to a more elegant and effective way to deal with those variational problems having such wave functions as candidate solutions.

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