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Three-Dimensional Integrable and Superintegrable Isospectral Potentials

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As an extension of the intertwining operator idea a constructive method for establishment of families of three-dimensional (super)integrable and isospectral potentials having higher-order dynamical symmetries is developed.

In analogy to integrability concepts of classical mechanics [1–3], a quantum mechanical system described in N dimensional (ND) Euclidean space by a stationary Hamiltonian operator H is called to be completely integrable if there exists a set of $N-1$ (together with H, N) algebraically independent linear operators commuting with H and among each other [4–8]. If there exist k ($0 < k \leq N-1$), additional operators commuting with H it is said to be superintegrable. The superintegrability is said to be minimal if $k = 1$ and maximal if $k = N-1$. Recently, the intertwining operator method has been systematically used in establishing families of $2D$ superintegrable potentials that are at the same time isospectral [9, 10]. In this letter we shall extend this program in constructing $3D$ integrable and superintegrable isospectral potentials.

The method of intertwining is a unified approach widely used in various fields of physics and mathematics [11–20]. To see the usefulness of this method let us consider two Hamiltonian operators $\mathcal{H}_0$ and $\mathcal{H}_1$ that are Hermitian (on some common function space) and intertwined by a linear intertwining operator $\mathcal{L}$ such that $\mathcal{L}\mathcal{H}_0 = \mathcal{H}_1\mathcal{L}$. Two dimension and form independent general properties [14, 15] immediately follow from such a relation; (i) If ψ^0 is an eigenfunction of $\mathcal{H}_0$ with eigenvalue of E^0 then $\psi^1 = \mathcal{L}\psi^0$ is an (unnormalized) eigenfunction of $\mathcal{H}_1$ with the same eigenvalue E^0. That is $\mathcal{L}$ transforms one solvable problem into another. (ii) $\mathcal{L}^\dagger$ intertwines in the other direction $\mathcal{H}_0\mathcal{L}^\dagger = \mathcal{L}^\dagger\mathcal{H}_1$ and this in turn provides two hidden dynamical symmetries of $\mathcal{H}_0$ and $\mathcal{H}_1$ in terms of $\mathcal{L}$: $[\mathcal{H}_0, \mathcal{L}^\dagger\mathcal{L}] = 0 = [\mathcal{H}_1, \mathcal{L}\mathcal{L}^\dagger]$, where † and $[\,,\,]$ stand for Hermitian conjugation and commutator.

In this letter we shall first apply this method to a pair of $3D$ systems described by

$$\mathcal{H}_0 = -\nabla^2 + V_0, \qquad \mathcal{H}_1 = -\nabla^2 + V_1, \tag{1}$$

and take $\mathcal{L}$ to be the following first order linear operator

$$\mathcal{L} = L_0 + \boldsymbol{L}\cdot\boldsymbol{\nabla}. \tag{2}$$

∇^2 is the Laplace operator in the Cartesian coordinates (x_1, x_2, x_3), $\boldsymbol{L} = (L_1, L_2, L_3)$ and "$\cdot$" denotes the usual inner product of $\mathbb{R}^3$. The potentials V_0, V_1 and L_0, $\boldsymbol{L}$ are some functions of coordinates that are to be determined from consistency equations of $\mathcal{L}\mathcal{H}_0 = \mathcal{H}_1\mathcal{L}$ which for (1) and (2) takes the form:

$$[\nabla^2, \boldsymbol{L}\cdot\boldsymbol{\nabla}] = -[\nabla^2, L_0] + [V_0, \boldsymbol{L}\cdot\boldsymbol{\nabla}] + \mathcal{P}\,\mathcal{L}, \tag{3}$$

where $\mathcal{P} = V_1 - V_0$.

By equating the second power of partial derivatives $\partial_j \equiv \partial/\partial x_j$ in equation (3) (these come only from $[\nabla^2, \boldsymbol{L}\cdot\boldsymbol{\nabla}]$) we obtain

$$\boldsymbol{L} = \boldsymbol{a} - \boldsymbol{b}\times\boldsymbol{r}, \tag{4}$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3)$ are arbitrary constant vectors and "$\times$" stands for the usual cross product of $\mathbb{R}^3$. In terms of vector operators $\boldsymbol{T} = \boldsymbol{\nabla}$, $\boldsymbol{J} = -(\boldsymbol{r} \times \boldsymbol{\nabla})$ whose components close into defining relations of six dimensional Euclidean algebra $e(3)$

$$[T_j, T_k] = 0, \qquad [J_j, J_k] = \epsilon_{jkl} J_l, \qquad [J_j, T_k] = \epsilon_{jkl} T_l, \tag{5}$$

the differential part of $\mathcal{L}$ can be written as

$$\boldsymbol{L} \cdot \boldsymbol{\nabla} = \boldsymbol{a} \cdot \boldsymbol{T} + \boldsymbol{b} \cdot \boldsymbol{J}. \tag{6}$$

T_j's and J_k's generate, respectively, the translation subalgebra $t(3)$ and rotation subalgebra $so(3)$ of $e(3)$ which accepts $\nabla^2 = \boldsymbol{T} \cdot \boldsymbol{T}$ as its Casimir.

The first and zeroth powers of derivatives of (3) gives the following set of consistency equations

$$\boldsymbol{\nabla} L_0 = \mathcal{P} \boldsymbol{L}, \tag{7}$$

$$(-\nabla^2 + \mathcal{P}) L_0 = \boldsymbol{L} \cdot \boldsymbol{\nabla} V_0. \tag{8}$$

Equation (8) is nonlinear and the integrability conditions $\partial_j \partial_k L_0 = \partial_k \partial_j L_0$ for the linear set (7) are equivalent to $\boldsymbol{L} \times \boldsymbol{\nabla} \mathcal{P} = -2\boldsymbol{b}\mathcal{P}$, where we have used the relation $\boldsymbol{\nabla} \times \boldsymbol{L} = -2\boldsymbol{b}$. Scalar multiplication of the both sides of this relation by $\boldsymbol{L}$ implies, for $\mathcal{P} \neq 0$, that

$$\boldsymbol{b} \cdot \boldsymbol{L} = \boldsymbol{a} \cdot \boldsymbol{b} = 0. \tag{9}$$

Equations (7) and (8) must be solved after choosing $\boldsymbol{a}$ and $\boldsymbol{b}$ in accordance with condition (9). In doing that we shall make use of the orbit structure of $e(3)$ under the adjoint action of the Euclidean group $E(3)$ [20]. Under such a transformation, generated by

$$U = e^{\eta_1 J_3} e^{\eta_2 J_1} e^{\eta_3 J_3} e^{\boldsymbol{\xi} \cdot \boldsymbol{T}}$$

$e(3)$ has three orbit types with representatives T_3, J_3 and $\kappa_1 T_3 + \kappa_2 J_3$, where η_i, ξ_j, κ_1 and κ_2 are real group parameters. That is, any element of $e(3)$ can be transformed by a similarity transformation U to one of the orbit representatives. Evidently, such a transformation leaves the intertwining relation form invariant; $\bar{\mathcal{L}} \bar{\mathcal{H}}_0 = \bar{\mathcal{H}}_1 \bar{\mathcal{L}}$, where $\bar{X} = U X U^\dagger$ and $U^{-1} = U^\dagger$ stands for the inverse of $U \in E(3)$. Thus, if we choose the differential part of $\mathcal{L}$ as one of the orbit representatives the so found potentials and L_0 will be unique up to a similarity action of $E(3)$. Noting that only the choices T_3 or J_3 for $\boldsymbol{L} \cdot \boldsymbol{\nabla}$ respect the condition (9) below we shall follow this procedure in two steps.

Step I. First we take $\boldsymbol{L} \cdot \boldsymbol{\nabla} = T_3$. In that case equation (7) gives the equations: $\partial_1 L_0 = 0 = \partial_2 L_0$, $2\partial_3 L_0 = \mathcal{P}$ which imply $L_0 = g_1(x_3)$ and $\mathcal{P} = 2g_1'(x_3)$, where g_1 is an arbitrary differentiable function of x_3 (we use prime to denote derivative with respect argument). Then from the nonlinear equation (8) which takes the form $(-\partial_3^2 + \mathcal{P}) L_0 = \partial_3 V_0$, we obtain

$$V_0 = V(x_1, x_2) + V_-, \qquad V_1 = V(x_1, x_2) + V_+, \tag{10}$$

where

$$V_\pm = g_1^2(x_3) \pm g_1'(x_3). \tag{11}$$

Hence, we have found two isospectral Hamiltonians $H_0 = -\nabla^2 + V_0$ and $H_1 = -\nabla^2 + V_1$ that are intertwined by $\mathcal{L}_{10} = g_1(x_3) + T_3$ as follows $\mathcal{L}_{10} H_0 = H_1 \mathcal{L}_{10}$.

Step II. Fixing the form of H_1 found above we shall now intertwine it to an another Hamiltonian H_2 with $\mathcal{L}_{21}$. This will amount to the fact that we have three pairwise intertwined Hamiltonians

$$\mathcal{L}_{10} H_0 = H_1 \mathcal{L}_{10}, \qquad \mathcal{L}_{21} H_1 = H_2 \mathcal{L}_{21}, \qquad \mathcal{L}_{20} H_0 = H_2 \mathcal{L}_{20},$$

where $\mathcal{L}_{20} \equiv \mathcal{L}_{21}\mathcal{L}_{10}$ and the last relation is a direct result of the first two. Moreover, since each Hamiltonian will be double intertwined we will have two additional symmetries for each one.

For $H_2 = -\nabla^2 + V_2$ and $\mathcal{L}_{21} = K_0 + J_3$ the linear set of consistency equations can be read from (7) to be

$$2\partial_1 K_0 = Px_2, \qquad 2\partial_2 K_0 = -Px_1, \qquad \partial_3 K_0 = 0. \tag{12}$$

Now $P = V_2 - V_1$ and solutions are $K_0 = g_2(u)$ and $P = -2g_2'(u)/x_1^2$, where g_2 is a differentiable function of $u = x_2/x_1$. In that case the nonlinear equation is $(-\nabla^2 + P)K_0 = J_3 V_1$ and for the found K_0, P takes the form

$$\partial_u \left[(1+u^2)g_2'(u) + g_2^2(u)\right] = -x_1^2 J_3 V(x_1,x_2). \tag{13}$$

The most general form of V that makes the right-hand side of equation (13) a function of u is

$$V(x_1,x_2) = F(\rho) + \frac{1}{x_1^2}h(u), \tag{14}$$

where F and h are arbitrary functions of $\rho = (x_1^2+x_2^2)^{1/2}$ and u respectively. Note that $J_3F = 0$. Using (14) in (13) we see that g_2 and h have to satisfy the equation

$$\left(1+u^2\right) g_2'(u) + g_2^2(u) = \left(1+u^2\right) h(u). \tag{15}$$

Combining the results of above two steps we get the following triplet of isospectral potentials

$$V_0 = V + V_-, \qquad V_1 = V + V_+, \qquad V_2 = V_1 - \frac{2g_2'(u)}{x_1^2}. \tag{16}$$

$V_\pm$ are given by (11) and g_2, h are any solutions of equation (15). The corresponding symmetry generators are collected as follows

$$\begin{aligned}
X_0 &= \mathcal{L}_{10}^\dagger \mathcal{L}_{10} = V_- - T_3^2,\\
Y_0 &= \mathcal{L}_{20}^\dagger \mathcal{L}_{20} = \left[g_2^2 - (1+u^2)g_2' - J_3^2\right] X_0,\\
X_1 &= \mathcal{L}_{10}\mathcal{L}_{10}^\dagger = V_+ - T_3^2,\\
Y_1 &= \mathcal{L}_{21}^\dagger \mathcal{L}_{21} = g_2^2 - \left(1+u^2\right) g_2' - J_3^2,\\
X_2 &= \mathcal{L}_{21}\mathcal{L}_{21}^\dagger = g_2^2 + (1+u^2)g_2' - J_3^2,\\
Y_2 &= \mathcal{L}_{20}\mathcal{L}_{20}^\dagger = \left[g_2^2 + \left(1+u^2\right) g_2' - J_3^2\right] X_1,
\end{aligned} \tag{17}$$

where the subscripts of X_j, Y_j indicate the Hamiltonians they belong to. By construction, all these operators are factorized, have even orders (Y_2 and Y_0 are both the fourth order and X_1, Y_1, X_0, X_2 are second order) and $[H_i, X_i] = 0 = [H_i, Y_i]$; $i = 0,1,2$. One can also easily verify that $[X_i, Y_i] = 0$. As we have, together with the Hamiltonians, three independent, pairwise commuting symmetry generators for each Hamiltonian, equation (16) describes 3D integrable isospectral systems. From (17) we also observe that the fourth order symmetries are of the form $Y_0 = Y_1X_0$, $Y_2 = X_2X_1$. This is due to the fact that $\mathcal{L}_{10}$ and $\mathcal{L}_{21}$ are commuting.

We shall now introduce two additional methods which produce minimally superintegrable (having four independent symmetries) subclasses of the above integrable $3D$ systems.

Method I. In this method we shall construct an additional symmetry generator Z_1 of H_1. Since H_1 is intertwined to H_0 and H_2 the existence of such symmetry will provide $Z_0 = \mathcal{L}_{10}^\dagger Z_1 \mathcal{L}_{10}$ and $Z_2 = \mathcal{L}_{21} Z_1 \mathcal{L}_{21}^\dagger$ as new symmetry generators of H_0 and H_2. This method is another general property (independent from the dimension and Hamiltonians) of the intertwining operator idea.

As a simple realization of this method we shall take $Z_1 = \boldsymbol{c}\cdot\boldsymbol{J}$, where $\boldsymbol{c} = (c_1, c_2, c_3)$ is a constant vector. Then the condition $[Z_1, H_1] = 0$ yields the consistency equation $\boldsymbol{c}\cdot\boldsymbol{K} = 0$, where

$$K_1 = x_3\partial_2 V - x_2\partial_3 V_+(x_3), \qquad K_2 = x_3\partial_1 V - x_1\partial_3 V_+(x_3), \qquad K_3 = -J_3[\frac{h(u)}{x_1^2}], \tag{18}$$

and V_+, V are given by (11) and (14). Below three special cases are considered.

Case I. In the case of $Z_1 = J_3$ we have the equation $K_3 = 0$, which yields $h = \beta/(1+u^2)$, where β is an arbitrary constant. Since there is no restriction on F and $V_\pm$, the potentials are given by (16) with $V(\rho) = F(\rho) + \beta\rho^{-2}$. The condition (15) now is as follows

$$(1+u^2)g_2'(u) + g_2^2(u) = \beta. \tag{19}$$

The new symmetry generators of H_0 and H_2 are the following third-order operators

$$Z_0 = \mathcal{L}_{10}^\dagger Z_1 \mathcal{L}_{10} = (g_1 - T_3)J_3(g_1 + T_3) = X_0 J_3,$$
$$Z_2 = \mathcal{L}_{21} Z_1 \mathcal{L}_{21}^\dagger = (g_2 + J_3)J_3(g_2 - J_3).$$

In addition to $[Z_0, H_0] = 0 = [Z_2, H_2]$, we have

$$\begin{aligned}
&[X_1, Z_1] = 0,\\
&[Y_1, Z_1] = 4(1+u^2)g_2 g_2',\\
&[X_0, Z_0] = \mathcal{L}_{10}^\dagger\,[X_1, Z_1]\,\mathcal{L}_{10} = 0,\\
&[Y_0, Z_0] = \mathcal{L}_{10}^\dagger[Y_1, Z_1]X_1\mathcal{L}_{10},\\
&[X_2, Z_2] = \mathcal{L}_{21}\,[Y_1, Z_1]\,\mathcal{L}_{21}^\dagger,\\
&[Y_2, Z_2] = \mathcal{L}_{21}[Y_1, Z_1]X_1\mathcal{L}_{21}^\dagger.
\end{aligned} \tag{20}$$

Provided that $[Y_1, Z_1] \neq 0$, that is $g \neq \text{const}$, we have $[Y_i, Z_i] \neq 0$ for $i = 0, 2$, and $[X_2, Z_2] \neq 0$. Each one of H_i corresponds to a minimally superintegrable system, with four symmetry generators (H_i, X_i, Y_i, Z_i).

Case II. If we take $Z_1 = J_2$ then the separable equation $K_2 = 0$ gives

$$V_+ = \frac{1}{2}\alpha x_3^2, \qquad h = \frac{\beta}{1+u^2} + \frac{\gamma}{u^2}, \qquad F = \frac{1}{2}\alpha\rho^2 - \frac{\beta}{\rho^2}, \tag{21}$$

where α, β, γ are arbitrary constants. In that case the potentials given by (16) are restricted to the following forms

$$V_0 = V_1 - 2g_1', \qquad V_1 = \frac{\alpha}{2}r^2 + \frac{\gamma}{x_2^2}, \qquad V_2 = V_1 - \frac{2}{x_1^2}g_2', \tag{22}$$

where $r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$, and g_1, g_2 are any solutions of

$$g_1' + g_1^2 = \frac{1}{2}\alpha x_3^2, \qquad \left(1+u^2\right)g_2' + g_2^2 = \beta + \gamma + \frac{\gamma}{u^2}. \tag{23}$$

It is easy to verify that $[X_1, Z_1]$ and $[Y_1, Z_1]$ are always different from zero. The new third-order symmetry generators of H_0 and H_2 satisfy the commutators given by (20).

Case III. Finally we take $Z_1 = kJ_2 + J_1$. In that case V_+, F are the same as in (21) and

$$h = \frac{\beta}{(1+u^2)} + \frac{\gamma}{(1-ku)^2}.$$

On substituting these solutions into (16), we obtain

$$V_0 = V_1 - 2g_1', \qquad V_1 = \frac{\alpha}{2}r^2 + \frac{\gamma}{x_1^2(1-ku)^2}, \qquad V_2 = V_1 - \frac{2}{x_1^2}g_2', \tag{24}$$

where g_1 is determined from the first equation of (23) and by equation (15), g_2 is any solution of

$$\left(1+u^2\right) g_2' + g_2^2 = \beta + \gamma\frac{1+u^2}{(1-ku)^2}. \tag{25}$$

Other choices of $\boldsymbol{c}$ produce similar potentials.

Method II. By fixing the form of H_1 found in the Step I we shall now intertwine it to another Hamiltonian H_3 with $\mathcal{L}_{31}$. This will provide four pairwise intertwined Hamiltonians with three additional symmetries since each one will be three times intertwined as follows:

$$\begin{aligned}
&\mathcal{L}_{10}H_0 = H_1\mathcal{L}_{10}, \qquad \mathcal{L}_{21}H_1 = H_2\mathcal{L}_{21}, \qquad \mathcal{L}_{20}H_0 = H_2\mathcal{L}_{20},\\
&\mathcal{L}_{30}H_0 = H_3\mathcal{L}_{30}, \qquad \mathcal{L}_{31}H_1 = H_3\mathcal{L}_{31}, \qquad \mathcal{L}_{32}H_2 = H_3\mathcal{L}_{32},
\end{aligned}$$

where $\mathcal{L}_{20} \equiv \mathcal{L}_{21}\mathcal{L}_{10}$, $\mathcal{L}_{30} \equiv \mathcal{L}_{31}\mathcal{L}_{10}$ and $\mathcal{L}_{32} \equiv \mathcal{L}_{31}\mathcal{L}_{21}^{\dagger}$.

Let us take $H_3 = -\nabla^2 + V_3$ and $\mathcal{L}_{31} = M_0 + a_1T_1 + J_2$. From equation (7) the linear set of consistency equations are

$$2\partial_1 M_0 = P(a_1 - x_3), \qquad \partial_2 M_0 = 0, \qquad 2\partial_3 M_0 = Px_1, \tag{26}$$

where $P = V_3 - V_1$ and the solutions are $M_0 = g_3(v)$ and $P = -2g_3'(v)/x_1^2$, where g_3 is a differentiable function of $v = (a_1 - x_3)/x_1$. Now the nonlinear equation (corresponding to equation (8)) is $(-\nabla^2 + P)M_0 = (a_1T_1 + J_2)(V + V_+)$, and for M_0 and P found above this equation takes the form

$$\partial_v\left[\left(1+v^2\right)g_3'(v) + g_3^2(v)\right] = -x_1^2(a_1 - x_3)\partial_1\left[F(\rho) + \frac{h(u)}{x_1^2}\right] - x_1^3\partial_3V_+(x_3), \tag{27}$$

where V_+ is given by equations (11) and we made use of equation (14). It is not hard to see that the most general forms of F, h and V_+ which make the right-hand side of equation (27) only a function of v are as follows

$$F = \mu\rho^2 - \frac{\gamma}{\rho^2} + \lambda, \qquad h = \frac{\gamma}{1+u^2} + \frac{\mu_2}{u^2} + \mu_1, \qquad V_+ = \mu(a_1 - x_3)^2 + \frac{\mu_3}{(a_1 - x_3)^2},$$

where μ, γ, μ_i; $i = 1, 2, 3$ are arbitrary constants. On substituting these in (11), (15) and (27), we see that g_1, g_2 and g_3 have to satisfy the following Riccati equations

$$\begin{aligned}
&g_1'(x_3) + g_1^2(x_3) = \mu(a_1 - x_3)^2 + \frac{\mu_3}{(a_1 - x_3)^2},\\
&\left(1+u^2\right)g_2'(u) + g_2^2(u) = \gamma + \mu_2 + \mu_1\left(1+u^2\right) + \mu_2\frac{1}{u^2},\\
&\left(1+v^2\right)g_3'(v) + g_3^2(v) = \mu_1 v^2 + \frac{\mu_3}{v^2} + \lambda_1.
\end{aligned} \tag{28}$$

Provided that g_1, g_2, g_3 are any solutions of these equations the potentials are

$$V_0 = V_1 - 2g_1'(x_3), \qquad V_1 = V + V_+, \qquad V_2 = V_1 - \frac{2g_2'(u)}{x_1^2}, \qquad V_3 = V_1 - \frac{2g_3'(v)}{x_1^2}, \tag{29}$$

where

$$V = \mu\left(x_1^2 + x_2^2\right) + \left(\frac{\mu_1}{x_1^2}\right) + \left(\frac{\mu_2}{x_2^2}\right) + \lambda,$$

and λ_1, λ are arbitrary constants. The old symmetry generators X_i, Y_i, $i = 0, 1, 2$ are given by (17) and the new ones are as follows

$$Z_0 = \mathcal{L}_{30}^\dagger \mathcal{L}_{30}, \qquad Z_1 = \mathcal{L}_{31}^\dagger \mathcal{L}_{31}, \qquad Z_2 = \mathcal{L}_{32}^\dagger \mathcal{L}_{32},$$
$$X_3 = \mathcal{L}_{31} \mathcal{L}_{31}^\dagger, \qquad Y_3 = \mathcal{L}_{30} \mathcal{L}_{30}^\dagger, \qquad Z_3 = \mathcal{L}_{32} \mathcal{L}_{32}^\dagger. \tag{30}$$

By construction, all generators commute with the corresponding Hamiltonians, $(Y_0, Z_0, Y_2, Z_2, Y_3, Z_3)$ are of the order four (in derivative, or equivalently, in the generators of $e(3)$ algebra) and the remaining ones are the second order. H_i's are minimally superintegrable since $[X_i, Y_i] = 0$, $[X_i, Z_i] \neq 0$ and $[Y_i, Z_i] \neq 0$, for $i = 0, 1, 2$. But, because of the relations $[X_3, Y_3] \neq 0$, $[X_3, Z_3] \neq 0$ and $[Y_3, Z_3] \neq 0$ the new Hamiltonian H_3 is not even integrable. However, by applying the method I we have introduced above to the potentials given by (29), H_3 (and the other Hamiltonians) can be made maximally superintegrable (another way which may produce different potentials is to use the intertwining method once again). Finally we would like to emphasise that a hierarchy of potentials can be constructed by first linearizing the Riccati equations found above and then using the solutions of resulting $1D$ Schrödinger equations [10].

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