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The Renormalization Group Symmetry for Solution of Integral Equations

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Evolution of the renormgroup algorithm and the related renormgroup symmetry, introduced in mathematical physics for solutions of boundary-value problems based on differential equations, is reviewed. We discuss the essential progress made recently in the application of this algorithm to models with integral equations. Several physical illustrations from nonlinear optics and plasma physics demonstrate the potentialities of the algorithm for models with nonlocal terms in the form of the linear solution functionals.

1 Introduction

The notion of the Renormalization Group Symmetry (RGS) was introduced in mathematical physics in the beginning of the nineties of the last century [1, 2] (see also review papers [3, 4]). It marked the proliferation of the Renormalization Group (RG) ideas [5, 6] from Quantum Field Theory (QFT) to the problems of mathematical physics with the goal that was the same as in QFT: to improve the perturbation theory (PT) solutions and to correct the behavior of these solutions in the vicinity of a singularity. However, the form of the practical implementation of the RG ideas in other fields of theoretical and mathematical physics frequently differs from that used in QFT (see, e.g. the review paper [7]). For boundary value problems (b.v.p.) in mathematical physics that are based on differential equations (DEs) quite new RG algorithm was elaborated [1–4, 8]. In order to clarify this difference we recall that the RG method, as formulated by N.N. Bogoliubov and one of the authors [5] for QFT problems, employs the *exact* group property of a solution with the goal to improve the approximate PT solution. One of the well-known forms of this property is a functional equation (which exhibits the group composition law) for the invariant charge in QFT. Revealing of this symmetry (i.e. the solution group property) in every particular case is a tricky procedure in theoretical physics and is the algorithmic deficiency of the method [9].

In becoming to mathematical physics we usually deal with b.v.p. that are based on DEs. Symmetries of these equations can be obtained in a regular way using the Lie group analysis technique. This fact had a profound impact on creating an algorithm that joined the RG ideology from QFT with a regular procedure of constructing symmetries for b.v.p. It was precisely this algorithm that brought to being the notion of the "renormalization group symmetry" (RGS) for b.v.p. solutions. There were two reasons for calling them RGS: firstly, from mathematical point of view the calculational algorithm for these symmetries is very similar to that used in modern group analysis and, secondly, they are applied to PT solutions with the goal usual to QFT, i.e. to improve these PT solutions.

Early in the development of RGS algorithm in mathematical physics it was mainly applied to mathematical problems based on DEs, though formally this algorithm may be used in any

problem that admit the regular procedure for calculating symmetries for the involved equations. In this sense an application area for RGS algorithm can be substantially expanded to include objects (e.g. integral or integro-differential equations) that traditionally were beyond the scope of the classical group analysis. These integral equations may form the basis of a problem, or they may appear as specific objects for applying RGS algorithm for b.v.p. based on DEs. Frequently we are interested not in a solution itself but in some integral solution characteristic, i.e. the solution functional. This characteristic may appear as a result of integrating with respect to any independent variable or on transition to Fourier variables. In this case RGS algorithm may be applied not for improving a particular solution with the subsequent calculation of some integral characteristic of this solution but for improving the functional of a perturbative solution. In other words this algorithm may restore behavior of an integral characteristic without calculating a solution in an explicit form.

In this report along with the description of the modified RGS algorithm we present also several new illustrations which demonstrate its efficiency.

2 RGS algorithm for nonlocal problems

The scheme for constructing and application of RGS to b.v.p. based on DEs was discussed in details in our recent publications (see, e.g. review papers [4, 8]). Here we briefly touch upon basic stages of this scheme, paying special attention to changes that should be introduced to make the RGS algorithm applicable to nonlocal problems.

We start with a mathematical model, defined by a system of $\nu \geq 1$ *differential* and *integral equations* for functions $u = \{u^\alpha\}$, $\alpha = 1, \ldots, m$ of $x = \{x^i\}$, $i = 1, \ldots, n$,

$$[E]: \quad E_\nu(u(x)) = 0, \tag{1}$$

with appropriate boundary (initial) conditions. Nonlocal terms in these equations depend on integrals of u. We suppose also that we know some approximate solution, U^α, expressed, say, in the form of the truncated PT series in powers of some small parameter or a small distance from the boundary where this solution is given.

Then the general scheme for RGS algorithm, which is depicted on Fig. 1, is realized as a sequence of steps which are:

I constructing the specific RG manifold,

II calculating the symmetry group for this RG manifold,

III restricting this group on a particular (usually approximate) b.v.p. solution that gives the desired RGS,

IV using the RGS obtained to calculate the particular analytical b.v.p. solution.

In the next subsections we consider these steps in more detail.

2.1 Constructing the RG manifold

The key idea of the first step consists in involving in group transformations the parameters and boundary conditions that define the particular solution of the problem. This is achieved by constructing the special RG manifold $\mathcal{RM}$ that we assume to have the form of s differential equations of the k-th order and q nonlocal relations,

$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(k)}) = 0, \qquad \sigma = 1, \ldots, s, \tag{2}$$

$$F_\sigma(z, u, u_{(1)}, \ldots, u_{(r)}, A(u)) = 0, \qquad \sigma = 1 + s, \ldots, q + s. \tag{3}$$

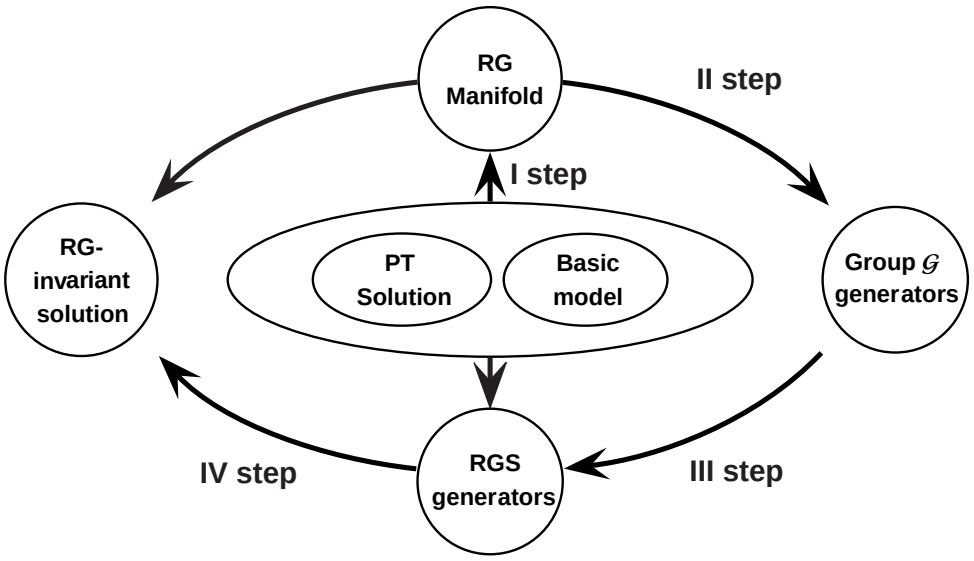

Figure 1. General scheme for construction of RGS.

We use the traditional for the differential algebra notations for independent variables $z = \{z^i\}$, differential variables $u = \{u^\alpha\}$ with the subsequent derivatives $u_{(1)} = \{u^\alpha_i\}$, $u_{(2)} = \{u^\alpha_{ij}\}$, $i, j = 1, \ldots, n + l$. These variables are related by the operator of the total differentiation D_i,

$$u^\alpha_i = D_i\,(u^\alpha)\,, \quad u^\alpha_{ij} = D_j\,(u^\alpha_i)\,D_j D_i\,(u^\alpha)\,, \quad \ldots, \quad D_i = \frac{\partial}{\partial z^i} + u^\alpha_i \frac{\partial}{\partial u^\alpha} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_j} + \cdots\,. \quad (4)$$

The parameters $p = \{p^j\}$, $j = 1, \ldots, l$ are included in $z = \{x, p\}$ and nonlocal variables are given by integral relations

$$A(u) = \int \mathcal{F}(u(z))\mathrm{d}z. \quad (5)$$

Nonlocal relations (3) characterize the principal difference of $\mathcal{RM}$ for nonlocal models as compared to b.v.p. for DEs, where we have the differential RG manifold.

The particular form of the realization of the first step depends both on the form of input equations and the form of boundary (initial) conditions and is inspired by PT solution as well. In general case $\mathcal{RM}$ does not coincide with E. We can indicate several possible routines to the problem, that were formulated for local models (see, e.g., [4]) and remain valid also for nonlocal models:

- One can extend the space of variables involved in the group transformations and in fact is realized in equations (2)–(3), where we add parameters p that enter a solution both via equations and/or via boundary (initial) conditions. Adding derivatives with respect to p extends the space of differential variables. The extension of the space of independent and differentiable variables frequently transforms the basic equations (1) into $\mathcal{RM}$, however the non-locality is preserved.
- Another possibility employs reformulating the boundary conditions in terms of *embedding equations* or *differential constraints*. The key idea here is to treat simultaneously the solution of the b.v.p. as an analytic function of the independent variables and the boundary parameters $b = \{x^i_0, u^\alpha_0\}$. Differentiation with respect to these parameters leads to additional DEs (*embedding equations*) that, together with the basic equations (1), form the $\mathcal{RM}$. In general, different boundary conditions for the same input equations lead to

different embedding equations. In the simplest case embedding equations have the form of evolution first-order partial DEs, however, more complicated situations are possible.

A more general approach is based on reformulating the boundary conditions in terms of *differential constraints* that are compatible with the basic DEs. In particular these differential constraints can arise while calculating the higher order (or Lie–Bäcklund) symmetries for the basic or embedding equations.

- In the case when the basic equations (1) contain a small parameter μ, the desired *approximate* manifold $\mathcal{RM}$ can be obtained by simplification of the equations (1) and use of "perturbation methods of group analysis" (see Vol. 3, Chapter 2, p. 31 in [10]). The main idea here is to consider a simplified ($\mu = 0$) model, which admits a wider symmetry group in comparison with the case $\mu \neq 0$. When we take the contributions from small μ into account, this symmetry is *inherited* by the equations (1), which results in some additional terms, corrections in powers of μ, in group generators. As a result the same representation arises for the RGS generators that are calculated at the next steps of the algorithm.

The routines mentioned do not exhaust all possibilities for constructing $\mathcal{RM}$, rather they point to different possibilities for the realization of the first step. In every particular case the choice is defined by the model equations (1) and the form of the PT solution.

2.2 Calculation of the admitted group

The next step is the calculation of the most general symmetry group $\mathcal{G}$, admitted by $\mathcal{RM}$ (2)–(3). Precisely here we need the substantial modification of the algorithm as compared to differential RG manifold. Indeed, in application to $\mathcal{RM}$ defined by DEs, e.g. (2), we deal with a *local group of transformations* in the space of differential functions $\mathcal{A}$ that leaves these equations unaltered. The classical group algorithm for finding these transformations employs the Lie's infinitesimal technique that replaces the procedure of solving original, possibly nonlinear, equations by analyzing linear equations for coordinates of group generators, determining equations. In constructing RG symmetries several generalizations of the classical Lie algorithm that are in use in modern group analysis may be employed (see [10–15] and references therein), i.e. approximate symmetries, non-classical and conditional symmetries, non-local symmetries, discrete symmetries, etc.

This situation drastically changes when moving to RG manifold (3) given by integro-differential equations or to solution functionals, introduced by integral terms. The major obstacle for the application of Lie's infinitesimal techniques to integro-differential equations (or infinite systems of differential equations) is that the *frames* (see, e.g. [13]) of these equations are not locally defined in the space of differential functions. In consequence, the crucial idea of splitting of determining equations into over-determined systems, commonly used in the classical Lie group analysis, fails. The next problem in application of the symmetry group obtained is the problem of prolongation of the symmetry group operator on solution functionals that have the form of nonlocal (integral) variables. Below we indicate several routines to the problem and describe common approaches in application to nonlocal objects.

Loosely speaking, different known approaches to calculating symmetry groups for integro-differential equations can be divided into two large groups: indirect and direct methods.

Algorithms of the first group rest on the possibility to replace in any way input nonlocal (integro-differential) equations by a system of differential equations. Then the resulting system of differential equations is analyzed using standard methods of a classical Lie group analysis. Here we point on two different ways of reducing nonlocal equations to differential ones: a) *method of moments* that was realized to calculate Lie point symmetry group for Vlasov–Maxwell equations in plasma theory [16] and for Benney, Vlasov-type and Boltzmann-type kinetic equations [17,18]

and b) *method of boundary-differential equations* that was developed in [19] on basis of the concept of covering and applied to a coagulation kinetic equation.

With the goal to modify the RGS algorithm we employ methods of the second group, namely, direct methods of finding symmetries that were developed in [20–22] and applied to find symmetries of kinetic Boltzmann equation, the equations of motion of viscoelastic medium and Vlasov–Maxwell equations of plasma theory. In this method the utilization of the Lie–Bäcklund group and the canonical group representation is of principal value, i.e. instead of the traditional form for the group generator

$$X = \xi^i \partial_{z^i} + \eta^\alpha \partial_{u^\alpha} + \zeta^\alpha_i \partial_{u^\alpha_i} + \zeta^\alpha_{i_1 i_2} \partial_{u^\alpha_{i_1 i_2}} + \cdots ,$$
$$\zeta^\alpha_i = D_i(\varkappa^\alpha) + \xi^j u^\alpha_{ij}, \qquad \zeta^\alpha_{i_1 i_2} = D_{i_1} D_{i_2}(\varkappa^\alpha) + \xi^j u^\alpha_{j i_1 i_2}, \qquad \varkappa^\alpha = \eta^\alpha - \xi^i u^\alpha_i, \tag{6}$$

with coordinates $\xi^i([z,u])$, $\eta^\alpha([z,u])$, $\zeta^\alpha_i([z,u])$, …, we use the equivalent operator Y that is known as the canonical representation for X,

$$X \sim Y = X - \xi^i D_i = \varkappa^\alpha \partial_{u^\alpha}. \tag{7}$$

It is essential that only dependent variables u^α are involved in the group infinitesimal transformation with the generator (7) and the group parameter a whereas independent variables z^i do not vary

$$u'^\alpha = u^\alpha + a\varkappa^\alpha + O\big(a^2\big), \qquad z'^i = z^i. \tag{8}$$

Hence one can define the local group $\mathcal{G}$ of point transformations as a symmetry group of integro-differential equations (3) iff for any a the functions F_σ do not vary [20]. Differentiating the appropriate invariance condition with respect to the group parameter a and assuming $a \to 0$ gives the determining equations. In contrast to the case of input differential equations these determining equations are in general also integro-differential.

The invariance criterion for F_σ with respect to the admitted group can be expressed in an infinitesimal form using the canonical group operator Y,

$$YF_\sigma\Big|_{[F_\sigma]} = 0, \qquad \sigma = 1+s, \ldots, q+s, \qquad \text{where} \qquad Y \equiv \int \mathrm{d}z \varkappa(z) \frac{\delta}{\delta u(z)}, \tag{9}$$

and the symbol $|_{[F_\sigma]}$ means evaluation on the manifold, generated by (3). Here with the goal to generalize the action of a canonical group operator not only on differential functions but on *functionals* as well we use variational differentiation in the definition of Y [22]. One can verify by direct calculation that the action of Y on any differential function and its derivatives, e.g., u, u_z, … produces the usual result: $Yu = \varkappa$, $Yu_z = D_z(\varkappa)$ and so on. Hence, if $F_\sigma = 0$ describe usual DEs then formulas (9) lead to standard local determining equations, while for $F_\sigma = 0$ having the form of integro-differential equations formulas (9) can be treated as *nonlocal* determining equations as they depend both on local and nonlocal variables.

In order to find solutions of determining equations one can use different approaches, e.g. expanding coordinates of group generator into formal power series and equating coefficients of various powers [20]. However there exists a more traditional way. As we treat local and nonlocal variables in determining equations as independent it is possible to separate these equations into local and nonlocal. The procedure of solving local determining equations is fulfilled in a standard way using Lie algorithm based on splitting the system of over-determined equations with respect to local variables and their derivatives. As a result we get expressions for coordinates of group generator that define the so-called group of *intermediate* symmetry [22]. In the similar manner the solution of nonlocal determining equations is fulfilled using the information borrowed from an intermediate symmetry and by "variational" splitting of nonlocal determining equations using

the procedure of variational differentiation. Therefore, the algorithm of finding symmetries of nonlocal equations appears as an algorithmic procedure that consists of a sequence of several steps:

- defining the set of local group variables,
- constructing determining equations on basis of the infinitesimal criterion of invariance that employs the generalization of the definition of the canonical operator,
- separating determining equations into local and nonlocal,
- solving local determining equations using a standard Lie algorithm,
- solving nonlocal determining equations using the procedure of variational differentiation.

This procedure generalizes the second step of the RGS algorithm in case of the integral or integro-differential manifold (2)–(3). As a result of this step we get infinitesimal operators (group generators) which correspond to the admitted vector field and form a Lie algebra with a general element

$$Y = \sum_j C^j Y_j, \qquad Y_j = \varkappa_j^\alpha \partial_{u^\alpha} \equiv \left(\eta_j^\alpha - \xi_j^i u_i^\alpha\right) \partial_{u^\alpha}, \tag{10}$$

where C^j are arbitrary constants.

To complete we describe the procedure of prolongation of the symmetry group on nonlocal variables, say, in the form of the integral relation (5). To fulfill this procedure one should first rewrite the operator, say, Y, in a canonical form and then formally prolong this operator on the nonlocal variable A

$$Y + \varkappa^A \partial_A \equiv \varkappa \partial_u + \varkappa^A \partial_A. \tag{11}$$

The integral relation between $\varkappa$ and $\varkappa^A$ is obtained by applying the generator (11) to the *definition* of A, i.e. to (5). Substituting the explicit expression for the coordinate $\varkappa$ of the known operator Y and calculating integrals obtained gives the desired coordinate $\varkappa^A$,

$$\varkappa^A = \int \frac{\delta A(u)}{\delta u(z)}\, \varkappa(z)\, \mathrm{d}z \equiv \int \frac{\delta \mathcal{F}(u(z'))}{\delta u(z)}\, \varkappa(z)\, \mathrm{d}z\mathrm{d}z' = \int \mathcal{F}_u \varkappa(z)\, \mathrm{d}z. \tag{12}$$

The group defined by the generators (10) is in general wider than the desired RG, that usually appears as its subgroup. As the RGS is related to a particular b.v.p. solution, it can be revealed by *restricting the admitted group* $\mathcal{G}$ on a manifold defined by this given solution.

2.3 Restriction the group on a solution

Mathematically, this procedure that forms the third step of the RG algorithm appears as checking the vanishing condition for the linear combination of coordinates $\varkappa_j^\alpha$ of the operator (10) on a particular approximate (or exact) b.v.p. solution $U^\alpha(z)$

$$\left\{ \sum_j C^j \varkappa_j^\alpha \equiv \sum_j C^j \left(\eta_j^\alpha - \xi_j^i u_i^\alpha\right) \right\}_{\Big| u^\alpha = U^\alpha(z)} = 0. \tag{13}$$

While the general form of the condition given by (13) is the same for any b.v.p. solution, the specific way of realization of the restriction procedure in every particular case employs a well-defined PT solution for the concrete b.v.p. This is depicted by the additional arrow that links two objects on the general scheme, namely “PT solution” and “RGS generators”.

Evaluating (13) on a particular b.v.p. solution, $U^\alpha(z)$, transforms the system of DEs for the group invariants into algebraic relations. Firstly, it gives relations between the C^j thus "combining" different coordinates of the group generators Y_j admitted by the $\mathcal{RM}$ (2)–(3). Secondly, it eliminates (partially or entirely) the arbitrariness that may appear in the coordinates ξ^i, η^α in the case of an infinite group $\mathcal{G}$.

Generally, the restriction procedure reduces the dimension of $\mathcal{G}$. Hence, the general element (10) of the group $\mathcal{G}$ after the fulfillment of a restriction procedure is expressed as a linear combination of the new generators R_i with the coordinates $\hat{\varkappa}^i$, $\hat{\eta}^\alpha_j$ and $\hat{\xi}^i_j$

$$Y \;\Rightarrow\; R = \sum_j B^j R_j, \qquad R_j = \hat{\varkappa}^\alpha_j \partial_{u^\alpha} \equiv \left(\hat{\eta}^\alpha_j - \hat{\xi}^i_j u^\alpha_i\right) \partial_{u^\alpha}, \tag{14}$$

where the B^j are arbitrary constants. The set of RGS generators R_i, each containing the desired b.v.p. solution in its invariant manifold, define a group of transformations that we also refer to as a *renormgroup* similar to the variant with $\mathcal{RM}$ defined by DEs.

The above prescribed three steps entirely define a regular algorithm for RGS construction but do not touch on how a b.v.p. solution is found. Hence, one more important, fourth step should be added. It consists of using the RGS generators to find analytical expressions for the new, "improved", solution of the b.v.p.

2.4 Construction of the RG invariant solution

Mathematically, the fourth step makes use of the RG *invariance conditions* that are given by a combined system of (2)–(3) and the vanishing condition for the linear combination of coordinates $\hat{\varkappa}^\alpha_j$ of the canonical operator equivalent to (14),

$$\sum_j B^j \hat{\varkappa}^\alpha_j \equiv \sum_j B^j \left(\hat{\eta}^\alpha_j - \hat{\xi}^i_j u^\alpha_i\right) \;= 0. \tag{15}$$

The necessity of using $\mathcal{RM}$ while calculating RG invariant solutions is depicted by the additional arrow that links these two objects on the general scheme.

One can see that the conditions (15) are akin to (13). However, in contrast to the previous step, the differential variables u in (15) should not be replaced by an approximate expression for the b.v.p. solution $U(z)$, but should be treated as usual dependent variables.

For the *one-parameter Lie point renormgroup*, RG invariance conditions (15) lead to the *first-order partial DE* that gives rise to the so-called *group invariants* (such as invariant couplings in QFT) which arise as solutions of associated characteristic equations. The general solution of the b.v.p. is now expressed in terms of these invariants. On the one hand, this is in direct analogy with the structure of RG invariant solutions in QFT. In the general case of arbitrary RGS the group invariance conditions obtained for a b.v.p. are not necessarily characteristic equations for the Lie point group operator. They may appear in a more complicated form, e.g., as a combination of partial DEs and higher order ordinary DEs. Nevertheless, the general idea of finding solutions to the b.v.p. in terms of RG invariant solutions remains valid.

The description of the fourth step completes the presentation of the RGS algorithm. Next we will present several examples that illustrate the application of the algorithm to various physical systems based on nonlocal models.

3 RGS algorithm in application to nonlocal models

3.1 Laser beam refraction in a nonlinear medium

In the section we demonstrate the application of RGS algorithm to solution functionals using a model that describes the evolution of the laser beam intensity $I(z, x)$ and the eikonal deriva-

tive $v(z,x)$ in a nonlinear medium ($z>0$). These functions obey nonlinear optics equations:

$$v_z + vv_x - \alpha I_x = 0, \qquad I_z + vI_x + Iv_x = \nu Iv/x; \qquad v(0,x) = 0, \qquad I(0,x) = \mathcal{I}(x). \quad (16)$$

where α is the parameter that defines the nonlinear refraction, z and x are the coordinates along and transverse to the beam axis, respectively; $\nu = 1$ for cylindrical beam and $\nu = 0$ for the plane beam geometry. Up to now there does not exist a universal algorithm for solving b.v.p. (16) that is suitable for arbitrary boundary conditions and beam geometry. The application of the RGS algorithm to (16) helps to construct new analytical solutions [23,4] and enables to prolong PT solution which is valid in the vicinity of a medium boundary $z \simeq 0$ up to large values of z, and in particular in the vicinity of a solution singularity $z_{\rm sing}$.

The appearance of a singularity in the solution can be obtained using the reduced description via two *solution functionals*, namely the laser beam intensity, $I^0(z) \equiv I(z,0)$, and the second derivative of the eikonal, $W^0(z) \equiv v_x(z,0)$, on the beam axis, that are formally introduces as

$$I^0(z) = \int \mathrm{d}x \delta(x) I(z,x), \quad W^0(z) = \int \mathrm{d}x \delta(x) v_x(z,x), \quad I^0(0) = 1, \quad W^0(0) = 0. \quad (17)$$

Here we present two RGS generators that describe the evolution of I^0 and W^0 for the cylindrical laser beam ($\nu = 1$) with the parabolic intensity distribution, $\mathcal{I}(x) = 1 - x^2$, and for the plane laser beam ($\nu = 0$) with a "soliton" profile at the boundary, $\mathcal{I}(x) = \cosh^{-2}(x)$,

$$\begin{aligned} R_{\rm par} &= \left(4\alpha I^0 z - \left(1 - 2\alpha z^2\right) I^0_z\right) \partial_{I^0} - \left(2\alpha(1 - 2zW^0) + \left(1 - 2\alpha z^2\right) W^0_z\right) \partial_{W^0}, \\ R_{\rm sol} &= \left(4 - 5I^0 - zI^0_z + 2(I^0 - 1)\frac{I^0 I^0_{zz}}{(I^0_z)^2}\right) \partial_{I^0} \\ &\quad + \left(\frac{I^0_z}{I^0} + z\frac{I^0_{zz}}{I^0} - z\left(\frac{I^0_z}{I^0}\right)^2 - 2(I^0 - 1)\left[\frac{I^0_{zzz}}{(I^0_z)^2} + 2\frac{I^0_z}{(I^0)^2} - 2\frac{(I^0_{zz})^2}{(I^0_z)^3}\right]\right) \partial_{W^0}. \end{aligned} \quad (18)$$

These generators result from the corresponding RGS generators [23,24], prolonged on the functionals (17). Utilizing RG invariance conditions (15) for the generator $R_{\rm par}$ gives the behavior of $I^0(z)$ and $W^0(z)$ for the "parabolic" laser beam

$$I^0 = \frac{1}{1 - 2\alpha z^2}, \qquad W^0 = -\frac{2\alpha z}{1 - 2\alpha z^2}, \qquad z_{\rm sing} = 1/\sqrt{2\alpha}, \quad (19)$$

starting from the boundary of the nonlinear medium up to the point of the solution singularity, $z = z_{\rm sing}$, where both the beam intensity and the eikonal derivative go to infinity. Quite similarly RG invariance conditions (15) for the generator $R_{\rm sol}$ in view of the additional constraint, $(I^0_z/\sqrt{I^0 - 1})_{|z\to 0} = 2\sqrt{\alpha}$, yield the behavior of I^0 and W^0 for the "soliton" laser beam

$$z = \frac{\sqrt{I^0 - 1}}{\sqrt{\alpha} I^0}, \qquad W^0 = -\frac{2\alpha z I^0}{1 - 2\alpha z^2 I^0}, \qquad z_{\rm sing} = 1/2\sqrt{\alpha} \quad (20)$$

from the boundary up to the singularity point $z = z_{\rm sing}$, where $I^0 = 2$. Though the formulas (19), (20) reproduce the results that were obtained in the context of the RGS algorithm for local models [23,24] (see also [25]) we note that the description of the solution singularity in terms of solution functionals (17) lead to invariance conditions in the form of *ordinary* DEs, in contrast to *partial* DEs, that arise from invariance conditions obtained for b.v.p. solutions.

3.2 RGS for a nonlinear dielectric permittivity of plasma

The next example demonstrates the application of RGS algorithm to calculating the nonlinear dielectric permittivity (NDP) tensors of plasma. In nonlinear electrodynamics the material equation, i.e. the relation between the induced current density $\boldsymbol{j}(t,\boldsymbol{r})$ and the electric field $\boldsymbol{E}(t,\boldsymbol{r})$,

is defined by the dependence of the electric induction $\boldsymbol{D}(t,\boldsymbol{r})$ upon the electric field $\boldsymbol{E}$, that frequently is given by a power series in $\boldsymbol{E}$. Moreover, due to the temporal and spatial dispersion the relation between the induced current and the electric field is integral (nonlocal). Hence, the material equation has the form of an integral power series in $\boldsymbol{E}$ and in Fourier variables (denoted by the sign "tilde") is given by:

$$\tilde{D}_s(k) = \tilde{E}_s(k) + i\frac{4\pi}{\omega}\sum_l \tilde{j}_s^{(l)}(k) = \varepsilon_{sa}(k)\tilde{E}_a(k) + \sum_{n=2}^{\infty}\int \mathrm{d}k_1\cdots\mathrm{d}k_n\delta(k-k_1-\cdots-k_n)$$
$$\times\,\varepsilon_{sj_1\cdots j_n}(k_1;\ldots;k_n)\tilde{E}_{j_1}(k_1)\cdots\tilde{E}_{j_n}(k_n), \quad \tilde{\boldsymbol{j}}^{(l)}(k) \backsim O(\tilde{\boldsymbol{E}}^l), \quad (k)\equiv(\omega,\boldsymbol{k}). \tag{21}$$

Comparison of two parts of this expression gives relations between NDP tensors of plasma and the current density $\tilde{\boldsymbol{j}}^{(l)}(k)$ of the given order $l \geqslant 2$.

In hot plasma NDP tensors are usually obtained by iterating Vlasov–Maxwell equations with the stationary and homogeneous background distribution function $f_0(\boldsymbol{v})$, while in cold plasma we derive NDP tensors from collisionless hydrodynamic equations. It is generally taken that expressions for NDP tensors in hot plasma are of more general type, hence "cold" expressions follow from them in the particular case $f_0(\boldsymbol{v}) = \delta(\boldsymbol{v})$. The use of RGS gives the method of constructing "hot" expression from the "cold" ones [2, 8].

The key idea here is to express the current density $\tilde{\boldsymbol{j}}^{(l)}(k)$ in hot plasma as a convolution of a partial current density $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$ with the equilibrium distribution function $f_0(\boldsymbol{w})$, depending upon the Lagrangian velocity $\boldsymbol{w}$,

$$\tilde{\boldsymbol{j}}^{(l)}(k) = \int \mathrm{d}\boldsymbol{w} f_0(\boldsymbol{w})\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w}). \tag{22}$$

Then the procedure of constructing NDP tensors in hot plasma is realized as a sequence of the following steps: a) calculating $\tilde{\boldsymbol{j}}^{(l)}(k,0)$ in cold plasma using collisionless hydrodynamic equations, b) constructing $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$ from $\tilde{\boldsymbol{j}}^{(l)}(k,0)$ for arbitrary $\boldsymbol{w}\neq 0$, c) integrating $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$ over $\boldsymbol{w}$ with the "weight" function $f_0(\boldsymbol{w})$ to get the desired expression for $\tilde{\boldsymbol{j}}^{(l)}(k)$ in hot plasma, d) making use of $\tilde{\boldsymbol{j}}^{(l)}(k)$ to calculate NDP tensor in hot plasma. It is essential that the step b), i.e. the transition from the "cold" expression for $\tilde{\boldsymbol{j}}^{(l)}(k,0)$ to the "hot" expression $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$, is a group transformation that is defined by the corresponding RGS generator. It is constructed from the Lie point group, admitted by the plasma kinetic equations with Lagrangian velocity. For example, in non-relativistic plasma the RGS generator appears as a linear combination of the generator of Galilean transformations, prolonged on Fourier variables, and translations along the Lagrangian velocity $\boldsymbol{w}$,

$$\boldsymbol{R}_{NDP} = \boldsymbol{k}\partial_\omega + \partial_{\boldsymbol{w}} - \frac{1}{c}\big[\tilde{\boldsymbol{B}},\partial_{\tilde{\boldsymbol{E}}}\big] + \tilde{\varrho}\partial_{\tilde{\boldsymbol{j}}}. \tag{23}$$

The finite transformations defined by this three-dimensional RG have the form[1]

$$\omega = \omega' + \boldsymbol{k}'\boldsymbol{w}, \qquad (\beta_{is}/\omega)\tilde{E}_s = (1/\omega')\tilde{E}'_i, \qquad \tilde{\varrho} = \tilde{\varrho}', \qquad \tilde{j}_i = \beta_{si}\tilde{j}'_s,$$
$$\boldsymbol{k} = \boldsymbol{k}', \qquad \tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}' = \big(c/\omega'\big)\,[\boldsymbol{k}',\tilde{\boldsymbol{E}}'], \qquad \beta_{is} = \delta_{is} + k_i w_s/(\omega - \boldsymbol{k}\boldsymbol{w}), \tag{24}$$

and relate $\tilde{\boldsymbol{j}}^{(l)}(k,0)$ to $\tilde{\boldsymbol{j}}^{(l)}(k,\boldsymbol{w})$ for arbitrary $\boldsymbol{w}\neq 0$. For example, in first order, $l=1$, this procedure gives

$$\tilde{j}_a^{(1)}(k,0) = i\frac{e^2 n_{e0}}{m\omega}\tilde{E}_a(k) \quad \Leftrightarrow \quad \tilde{j}_c{}^{(1)}(k,\boldsymbol{w}) = i\frac{e^2 n_{e0}}{m\omega}\beta_{sc}\beta_{sa}\tilde{E}_a(k), \tag{25}$$

[1]The variables with "primes" are referred to cold plasma limit, i.e. $\boldsymbol{w}=0$.

where e is the electron charge and n_{e0} is the equlibrium density of electrons. By substituting these expressions into (22) and then in (21) we get the one-to-one correspondence between the *scalar* dielectric permittivity in cold plasma and the corresponding *tensor* in hot plasma

$$\varepsilon(k) = 1 - \frac{4\pi e^2 n_{e0}}{m\omega^2} \quad \Leftrightarrow \quad \varepsilon_{ab}(k) = \delta_{ab} - \frac{4\pi e^2 n_{e0}}{m\omega^2} \int \mathrm{d}\boldsymbol{w} f_0(\boldsymbol{w}) \beta_{sa} \beta_{sb}. \tag{26}$$

The generalization of this result for the arbitrary order l is straightforward [8].

3.3 RGS algorithm for the problem of a plasma bunch expansion

The last example deals with application of RGS algorithm to the problem of ion acceleration in an expanding collisionless plasma bunch heated by an ultra short laser pulse. Kinetic treatment of this problem is based on a solution to the Cauchy problem for the coupled Vlasov equations for electrons and ions in a self-consistent electric field. The use of the RGS algorithm here enables to find a general class of kinetic solutions [26–28] for initial conditions of practical interest for laser-plasma interactions which includes a two-temperature Maxwellian electron distribution function (DF), a super-Gaussian electron DF, and a plasma with ion multi-species.

Let us consider an expansion of a plasma slab, which is inhomogeneous in x. The basic system includes kinetic equations for particle DFs f^α,

$$f^\alpha_t + v f^\alpha_x + (e_\alpha/m_\alpha) E(t,x) f^\alpha_v = 0, \qquad f^\alpha\big|_{t=0} = f^\alpha_0(x,v), \tag{27}$$

with the additional quasi-neutrality conditions

$$\int \mathrm{d}v \sum_\alpha e_\alpha f^\alpha = 0, \qquad \int \mathrm{d}v v \sum_\alpha e_\alpha f^\alpha = 0. \tag{28}$$

The solution to the initial-value problem is found perturbatively for $t \to 0$ and is continued in time using the RGS algorithm. The RGS generator is a linear combination of time translations and the projective operator, which arises by virtue of nonlocal relations (28),

$$R_{\text{bunch}} = \big(1 + \Omega^2 t^2\big)\partial_t + \Omega^2 t x \partial_x + \Omega^2 (x - vt)\partial_v. \tag{29}$$

This operator is the only one which selects the spatially symmetric initial DFs with zero mean velocity. The value Ω can be treated as the ratio of the ion acoustic velocity to the gradient length L_0. One can easily find DFs which are invariants of the RG transformation, however for practical applications rough integral characteristics, such as partial ion density, $n_q(t,r)$, might be more useful,

$$n^q(t,x) = \int_{-\infty}^{\infty} \mathrm{d}v f^q(t,x,v). \tag{30}$$

Treating $n^q(t,x)$ as the linear functional of f^q we prolong (29) to get the following RG generator

$$R_{\text{density}} = \big(1 + \Omega^2 t^2\big)\partial_t + \Omega^2 t x \partial_x - \Omega^2 t n^q \partial_{n^q}. \tag{31}$$

Finite group transformations defined by this operator lead to the ion density distribution in terms of a universal function, defined by the initial DFs and in view of the quasi-neutrality conditions (28),

$$n^q = \frac{1}{\sqrt{1+\Omega^2 t^2}} \mathcal{N}_q \left(\frac{x}{\sqrt{1+\Omega^2 t^2}} \right), \qquad \mathcal{N}_q = \int_{-\infty}^{\infty} \mathrm{d}v f^q_0. \tag{32}$$

Therefore, in order to find density distribution we need not know the explicit form of the particle DFs. This example certainly does not exhaust all possible applications of RGS algorithm to plasma dynamics, rather it points to some feasible fields of application (see, e.g. [28]).

4 Conclusion

In this review we have described the expansion of possibilities of the RGS algorithm to various nonlocal models in mathematical physics. In the modern form this algorithm is now applicable both to models based on differential equations and to models that also include nonlocal terms, e.g. integro-differential and integral equations.

The new formulation of the algorithm preserves the general scheme as a sequence of four steps, that were employed for differential models, however the form of their implementation now takes into account the peculiarities introduced by nonlocal terms. Several routines to overcome the difficulties with calculating the symmetry group for nonlocal models are indicated and the procedure of prolongation of the symmetry on to solution functionals is described. The efficiency of the algorithm is illustrated by various physical applications.

We hope that this report serves as an illustration of the universality of the RGS algorithm in application both to models based on DEs and nonlocal (integral) equations. The possibility to prolong the symmetry onto solution functionals enables to investigate the behavior of solution characteristics even in the case when the explicit form of the solution is not known. The results presented gives the promise for the progress in this field and can give rise to many potential applications.

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