DGLAP and BFKL Equations in $N = 4$ Supersymmetric Gauge Theory

A.V. KOTIKOV [†], *L.N. LIPATOV* [‡] *and V.N. VELIZHANIN* [‡]

[†] *Univ. Karlsruhe, D-76128 Karlsruhe, Germany and BLThPh, JINR, 141980 Dubna, Russia*
E-mail: *kotikov@thsun1.jinr.ru*

[‡] *Theoretical Physics Department, PNPI, Gatchina, 188300 St. Petersburg, Russia*
E-mail: *lipatov@thd.pnpi.spb.ru, velizh@thd.pnpi.spb.ru*

We show the results for the DGLAP and BFKL evolution equations in the $N = 4$ supersymmetric gauge theory obtained earlier in the next-to-leading approximation. The eigenvalue of the BFKL kernel in this model turns out to be an analytic function of the conformal spin $|n|$. The corresponding kernel for the Bethe–Salpeter equation has the property of the Hermitian separability. The anomalous dimension matrix can be transformed to a triangle form with the use of the similarity transformation for the diagonalization of the anomalous dimension matrix in the leading order. The eigenvalues of these matrices can be expressed in terms of a universal function by an integer shift of its argument. We also investigate in this approximation possible relations between the DGLAP and BFKL equations.

1 Introduction

The Balitsky–Fadin–Kuraev–Lipatov (BFKL) equation [1] and the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equation [2] are used now for a theoretical description of structure functions of the deep-inelastic ep scattering at small values of the Bjorken variable x. The higher-order QCD corrections to the splitting kernels of the DGLAP equation are well known. But the calculation of the next-to-leading order (NLO) corrections to the BFKL kernel was completed recently [3, 4].

In supersymmetric gauge theories the structure of the BFKL and DGLAP equations is simplified significantly. In the case of an extended $N = 4$ SUSY the NLO corrections to the BFKL equation were calculated in ref. [4] for arbitrary values of the conformal spin n. Below these results are presented in the dimensional reduction ($\overline{\text{DR}}$) scheme [5] which does not violate the supersymmetry. The analyticity of the eigenvalue of the BFKL kernel as a function of the conformal spin $|n|$ gives a possibility to relate in the leading logarithmic approximation (LLA) the DGLAP and BFKL equations in this model (see [4]). It was shown [6] that the eigenvalue of the BFKL kernel has the property of Hermitian separability similar to the holomorphic separability.

Let us introduce the unintegrated parton distributions $\varphi_a(x, k_\perp^2)$ (hereafter $a = q, g, \varphi$ for the spinor, vector and scalar particles, respectively) and the (integrated) parton distributions $n_a(x, Q^2)$. In DIS $Q^2 = -q^2$ and $x = Q^2/(2pq)$ are the Bjorken variables, $k_\perp$ is the transverse component of the parton momentum and q and p are the photon and hadron momenta, respectively.

After the Mellin transformation of partonic distributions $n_a(x, Q^2)$ the DGLAP equation can be written as follows [2]

$$\frac{d}{d\ln Q^2}\, n_a\big(j, Q^2\big) = \sum_b \gamma_{ab}(j)\, n_b\big(j, Q^2\big) \qquad \left(n_a\big(j, Q^2\big) = \int_0^1 dx\, x^{j-1} n_a\big(x, Q^2\big)\right).$$

The Mellin moment of the splitting kernel $\gamma_{ab}(j)$ coincides with the anomalous dimension

(AD) matrix for the twist-2 operators[1]. These operators are constructed as bilinear combinations of the fields which describe corresponding partons

$$
\begin{aligned}
O^g_{\mu_1,\dots,\mu_j} &= \hat{S} G_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \cdots D_{\mu_{j-1}} G_{\rho\mu_j}, & \tilde{O}^g_{\mu_1,\dots,\mu_j} &= \hat{S} G_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \cdots D_{\mu_{j-1}} \tilde{G}_{\rho\mu_j}, \\
O^q_{\mu_1,\dots,\mu_j} &= \hat{S} \overline{\Psi} \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_j} \Psi, & \tilde{O}^q_{\mu_1,\dots,\mu_j} &= \hat{S} \overline{\Psi} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \cdots D_{\mu_j} \Psi, \\
O^\varphi_{\mu_1,\dots,\mu_j} &= \hat{S} \bar{\Phi} D_{\mu_1} D_{\mu_2} \cdots D_{\mu_j} \Phi, & & \quad (1)
\end{aligned}
$$

where the spinor Ψ and field tensor $G_{\rho\mu}$ describe gluinos and gluons, respectively. The last expression is constructed from the covariant derivatives D_μ of the scalar field Φ appearing in extended supersymmetric models. The symbol $\hat{S}$ implies a symmetrization of the tensor in the Lorenz indices $\mu_1, \dots, \mu_j$ and a subtraction of its traces.

On the other hand, the BFKL equation relates the unintegrated gluon distributions with small values of the Bjorken variable x:

$$
\frac{d}{d\ln(1/x)} \varphi_g(x, k_\perp^2) = 2\omega(-k_\perp^2) \varphi_g(x, k_\perp^2) + \int d^2 k'_\perp K(k_\perp, k'_\perp) \varphi_g\big(x, (k'_\perp)^2\big),
$$

where $\omega(-k_\perp^2) < 0$ is the gluon Regge trajectory [1].

The matrix elements of $O^a_{\mu_1,\dots,\mu_j}$ and $\tilde{O}^a_{\mu_1,\dots,\mu_j}$ are related to the moments of the parton distributions $n_a(x, Q^2)$ and $\Delta n_a(x, Q^2)$ in a hadron h in the following way

$$
\begin{aligned}
\int_0^1 dx\, x^{j-1} n_a(x, Q^2) &= \langle h \,|\, \tilde{n}^{\mu_1} \cdots \tilde{n}^{\mu_j} O^a_{\mu_1,\dots,\mu_j} \,|\, h \rangle, & a &= (q, g, \varphi), \\
\int_0^1 dx\, x^{j-1} \Delta n_a(x, Q^2) &= \langle h \,|\, \tilde{n}^{\mu_1} \cdots \tilde{n}^{\mu_j} \tilde{O}^a_{\mu_1,\dots,\mu_j} \,|\, h \rangle, & a &= (q, g). & \quad (2)
\end{aligned}
$$

Here the vector $\tilde{n}^\mu$ is light-like: $\tilde{n}^2 = 0$. Note, that in the deep-inelastic ep scattering we have $\tilde{n}^\mu \sim q^\mu + x p^\mu$.

The quantum numbers appeared in the solution of the BFKL equation being the integer conformal spin $|n|$ and the quantity $1 + \omega$ (ω is an eigenvalue of the BFKL kernel) coincides respectively with the total numbers of transversal and longitudinal Lorentz indices of the tensor $O^a_{\mu_1,\dots,\mu_J}$ with the rank $J = 1 + \omega + |n|$. The corresponding matrix elements can be expressed through the solution of the BFKL equation

$$
\tilde{n}^{\mu_1} \cdots \tilde{n}^{\mu_{1+\omega}} l_\perp^{\mu_2+\omega} \cdots l_\perp^{\mu_j} \langle P \,|\, O^g_{\mu_1,\dots,\mu_j} \,|\, P \rangle \sim \int_0^1 dx\, x^\omega \int d^2 k_\perp \left(\frac{k_\perp}{|k_\perp|} \right)^n \varphi_g(x, k_\perp^2).
$$

It is important that the AD matrices $\gamma_{ab}(j)$ and $\tilde{\gamma}_{ab}(j)$ for the twist-2 operators $O^a_{\mu_1,\dots,\mu_j}$ and $\tilde{O}^a_{\mu_1,\dots,\mu_j}$ do not depend on various projections of indices due to the Lorentz invariance. But generally the Lorentz spin j is less then J. Thus, it looks reasonable to extract some additional information concerning the parton x-distributions satisfying the DGLAP equation from the analogous $k_\perp$-distributions satisfying the BFKL equation.

In LLA the integral kernel of the BFKL equation is the same in all supersymmetric gauge theories. Due to the Möbius invariance in the impact parameter representation the solution of the homogeneous BFKL equation has the form (see [7])

$$
E_{\nu,n}(\overrightarrow{\rho_{10}}, \overrightarrow{\rho_{20}}) \equiv \langle \phi(\overrightarrow{\rho_1}) O_{m,\tilde{m}}(\overrightarrow{\rho_0}) \phi(\overrightarrow{\rho_2}) \rangle = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^m \left(\frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*} \right)^{\tilde{m}},
$$

---

[1] As in ref. [3,4], the anomalous dimensions differ from those used usually in the description of DIS by a factor (-2), i.e. $\gamma_{ab}(j) = (-1/2)\gamma_{ab}^{\text{DIS}}(j)$.

where $m = 1/2 + i\nu + n/2$ and $\widetilde{m} = 1/2 + i\nu - n/2$ are conformal weights related to the eigenvalues of the Casimir operators of the Möbius group. We introduced also the complex variables $\rho_k = x_k + iy_k$ in the transverse subspace and used the notation $\rho_{kl} = \rho_k - \rho_l$.

For the principal series of the unitary representations the quantities ν and n are respectively real and integer numbers. The projection n of the conformal spin $|n|$ can be positive or negative, but the eigenvalue of the BFKL equation in LLA [1]

$$\omega = \omega^0(n, \nu) = 8\bar{a}\left(\Psi(1) - \mathrm{Re}\,\Psi\left(\frac{1}{2} + i\nu + \frac{|n|}{2}\right)\right), \qquad \bar{a} = \frac{g^2 N_c}{16\pi^2} = \frac{\alpha_s N_c}{4\pi} \tag{3}$$

depends only on $|n|$. The Möbius invariance takes place also for the Schrödinger equation describing the composite states of several reggeized gluons [8]. As a consequence of the relation

$$\omega^0(n, \nu) = \omega^0(m) + \omega^0(\widetilde{m}), \qquad \omega^0(m) = 2\bar{a}\big(2\Psi(1) - \Psi(m) - \Psi(1-m)\big).$$

one obtains the property of the holomorphic separability $H = h + h^*$, $[h, h^*] = 0$ for the Hamiltonian of an arbitrary number of reggeized gluons in the multi-colour QCD [9]. In the same limit the BFKL dynamics is completely integrable [10, 11] and the holomorphic Hamiltonian h coincides with the local Hamiltonian for an integrable Heisenberg spin model [12]. Moreover the theory turns out to be invariant under the duality transformation [13]. Presumably the remarkable mathematical properties of the reggeon dynamics in LLA are consequences of the extended $N = 4$ supersymmetry [14]. It was argued also in ref. [14] that generalized DGLAP equations for the matrix elements of quasi-partonic operators [15] for $N = 4$ SUSY are integrable.

The solution of the inhomogeneous BFKL equation in the LLA approximation can be constructed in the impact parameter representation and for $\overrightarrow{\rho_{1'}} \to \overrightarrow{\rho_{2'}}$ we obtain [7]

$$\langle \phi(\overrightarrow{\rho_1})\,\phi(\overrightarrow{\rho_2})\,\phi(\overrightarrow{\rho_{1'}})\,\phi(\overrightarrow{\rho_{2'}})\rangle \sim \sum_n C(\nu_\omega, |n|)\frac{E_{\nu_\omega, |n|}(\overrightarrow{\rho_{11'}}, \overrightarrow{\rho_{21'}})}{\omega'(|n|, \nu_\omega)}|\rho_{1'2'}|^{1+2i\nu_\omega}\left(\frac{\rho_{1'2'}}{\rho_{1'2'}^*}\right)^{|n|/2},$$

where ν_ω is a solution of the equation $\omega = \omega^0(|n|, \nu)$ with $\mathrm{Im}\,\nu_\omega < 0$.

The above asymptotics has a simple interpretation in terms of the Wilson operator-product expansion of two reggeon fields produced the local operator $O_{\nu_\omega, |n|}(\overrightarrow{\rho_{1'}})$ having the transverse dimension $\Gamma_\omega = 1 + i\nu_\omega$ calculated in units of a squared mass. The corresponding tensor has a mixed projections of a gauge-invariant tensor O with $J = 1 + \omega + |n|$ indices. Note, that because Γ_ω is real in the deep-inelastic regime $\rho_{12} \to 0$, the operator $O_{\nu_\omega, |n|}(\overrightarrow{\rho_{1'}})$ belongs to an exceptional series of unitary representations of the Möbius group (see [16]).

The AD $\gamma(j)$ obtained from the BFKL equation in LLA (3) has the poles

$$\Gamma_\omega = 1 + \frac{|n|}{2} - \gamma(j), \qquad \gamma(j)|_{\omega \to 0} = \frac{4\bar{a}}{\omega}. \tag{4}$$

The operator $O_{\nu_\omega, |n|}$ for $|n| = 1, 2, \ldots$ has the twist higher than 2 because its AD γ is singular at $\omega \to 0$. In the paper [4] the analytic continuation of the BFKL AD $\gamma(|n|, \omega)$ to the points $|n| = -r - 1$ $(r = 0, 1, 2, \ldots)$ was suggested to calculate the AD singularities in the twist-2 operators in negative integer $J = 1 + \omega + |n| \to -r$. Because for positive $|n|$ and $\omega \to 0$ the quantity $\gamma(|n|, \omega)$ corresponds to higher twist operators, to obtain γ for the twist-2 operators, one should push $\Delta|n| = |n| + r + 1$ to zero at $\omega \to 0$ sufficiently rapidly $\Delta|n| = C(r)\omega^2$. In LLA the results obtained from the BFKL equation $\gamma(j)|_{j \to -r} = \frac{4\bar{a}}{j+r}$ for $r = -1, 0, 1, \ldots$ coincide with the direct calculation of the eigenvalues of the AD matrix $\gamma_{a,b}$ for $N = 4$ SUSY in accordance with the fact, that only in this theory the eigenvalue of the BFKL equation is an analytic function of $|n|$ [4]. It is important that the AD is the same for all twist-2 operators entering in the $N = 4$ supermultiplet up to a shift of its argument by an integer number, because this

property leads to an integrability of the evolution equations for quasi-partonic operators [15] in the multi-color limit $N_c \to \infty$ (see [14]).

In the NLO approximation for the BFKL equation there is a difficulty related to an appearance of the double-logarithmic (DL) terms leading to triple poles at $j = -r$ for even r. The origin of the DL terms can be understood in a simple way using as an example the process of the forward annihilation of the e^+e^- pair in the $\mu^+\mu^-$ pair in QED [17]. For this process the t-channel partial wave f_ω ($\omega = j$) in the DL approximation can be written as follows $f_\omega = \frac{\gamma^{-1}}{\omega} - \frac{\alpha_{em}}{2\pi\left(\frac{1}{\omega - \gamma} + \frac{1}{\gamma}\right)}$. By expanding perturbatively the position of the pole in γ we shall generate the triple pole term in the AD $\Delta\gamma \sim \alpha_{em}^2/\omega^3$.

It is important to note that the equation for the pole position of f_ω is similar to the BFKL equation in the modified leading logarithmic approximation

$$\omega \sim 2\Psi(1) - \Psi(1 + |n| + \omega - \gamma) - \Psi(\gamma), \qquad |n| = -1, -2, \dots.$$

2   NLO corrections to the BFKL kernel in the $N = 4$ SUSY

Let us introduce the new variables $M = \gamma + \frac{|n|}{2}$, $\widetilde{M} = \gamma - \frac{|n|}{2}$, $\gamma = \frac{1}{2} + i\nu$. Then the eigenvalue relation for the BFKL equation in the $\overline{\text{DR}}$-scheme can be written in the Hermitially separabel form [6]

$$1 = \frac{4\hat{a}}{\omega} \left(2\Psi(1) - \Psi(M) - \Psi(1 - \widetilde{M}) + \hat{a}\left(\phi(M) + \phi(1 - \widetilde{M})\right)\right)$$
$$- 2\hat{a}\left(\rho(M) + \rho(1 - \widetilde{M})\right), \qquad \hat{a} = \overline{a} + \frac{\overline{a}^2}{3}, \qquad \overline{a} = \frac{g^2 N_c}{16\pi^2}, \tag{5}$$

where $\overline{a}$ (see (3)) and $\hat{a}$ are expressed through the Yang–Mills constants g in the $\overline{\text{MS}}$ and $\overline{\text{DR}}$-schemes, respectively, and

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2), \quad \phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi_2(M) + 2\beta'(M)\left(\Psi(1) - \Psi(M)\right). \tag{6}$$

Here

$$\Psi(M) = \frac{\Gamma'(M)}{\Gamma(M)}, \quad \beta'(z) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}}{(l+z)^2}, \quad \Phi_2(M) = \sum_{k=0}^{\infty} \frac{\left(\beta'(k+1) + (-1)^k \Psi'(k+1)\right)}{k + M}.$$

In the right-hand side of the eigenvalue equation the first contribution corresponds to the singularity at $l = -1$ generated by the pole of the Legendre function $Q_l(x)$ in the kernel at $\omega = 1 + l \to 0$ and the last term appears from the regular part of the Born contribution. Note that M and $1 - \widetilde{M}$ coincide with the anomalous dimensions appearing in the asymptotical expressions for the BFKL kernel in the limits when the gluon virtualities are large: $q_1^2 \to \infty$ and $q_2^2 \to \infty$, respectively. Because $1 - \widetilde{M} = M^*$, the Hermitian separability guarantees the symmetry of ω for the principal series of the unitary representations of the Möbius group to the substitution $\nu \to -\nu$ and the Hermicity of the BFKL Hamiltonian. It turns out, however, that the holomorphic separability corresponding to the symmetry $m \leftrightarrow \tilde{m}$ is violated in the NLO approximation [6].

Analogously to refs. [3, 4] one can calculate the eigenvalues of the BFKL kernel in the case of the non-symmetric choice for the energy normalization s_0 in equation (15) related to the interpretation of the NLO corrections in the framework of the renormalization group. For the scale $s_0 = q^2$ natural for the deep-inelastic scattering we obtain the corresponding eigenvalue equation

$$1 = 4\hat{a}\omega^{-1}\left(2\Psi(1) - \Psi(\gamma) - \Psi(J - \gamma) + \hat{a}\left(\phi(\gamma) + \phi(J - \gamma)\right)\right)$$
$$+ 2\hat{a}\left(\Psi'(\gamma) - \rho(\gamma) + \Psi'(J - \gamma) - \rho(J - \gamma)\right). \tag{7}$$

3 Anomalous dimension matrix in the $N = 4$ SUSY

In LLA AD matrices in the N=4 SUSY have the following form (see [18]):

for tensor twist-2 operators (hereafter $\Psi(j+1) - \Psi(1) \equiv S_1(j)$)

$$\gamma_{gg}^{(0)}(j) = 4\left(-S_1(j-2) - \frac{2}{j} + \frac{1}{j+1} - \frac{1}{j+2}\right), \qquad \gamma_{qg}^{(0)}(j) = 8\left(\frac{1}{j} - \frac{2}{j+1} + \frac{2}{j+2}\right),$$

$$\gamma_{\varphi g}^{(0)}(j) = 12\left(\frac{1}{j+1} - \frac{1}{j+2}\right), \qquad \gamma_{gq}^{(0)}(j) = 2\left(\frac{2}{j-1} - \frac{2}{j} + \frac{1}{j+1}\right), \qquad \gamma_{q\varphi}^{(0)}(j) = \frac{8}{j},$$

$$\gamma_{qq}^{(0)}(j) = 4\left(-S_1(j-1) + \frac{1}{j} - \frac{2}{j+1}\right), \qquad \gamma_{\varphi q}^{(0)}(j) = \frac{6}{j+1},$$

$$\gamma_{\varphi\varphi}^{(0)}(j) = -4S_1(j), \qquad \gamma_{g\varphi}^{(0)}(j) = 4\left(\frac{1}{j-1} - \frac{1}{j}\right), \tag{8}$$

for the pseudo-tensor operators:

$$\tilde{\gamma}_{gg}^{(0)}(j) = 4\left(-S_1(j) - \frac{2}{j+1} + \frac{2}{j}\right), \qquad \tilde{\gamma}_{qq}^{(0)}(j) = 4\left(-S_1(j) + \frac{1}{j+1} - \frac{1}{j}\right),$$

$$\tilde{\gamma}_{qg}^{(0)}(j) = 8\left(-\frac{1}{j} + \frac{2}{j+1}\right), \qquad \tilde{\gamma}_{gq}^{(0)}(j) = 2\left(\frac{2}{j} - \frac{1}{j+1}\right). \tag{9}$$

Note that in the $N = 4$ SUSY multiplet there are twist-2 operators with fermion quantum numbers but their AD are the same as for the bosonic components of the corresponding supermultiplet (cf. ref. [15]). It is possible to construct 5 independent twist-two operators with a multiplicative renormalization. The corresponding parton distribution momenta and their LLA AD have the form [18]:

$$n_I(j) = n_g^j + n_q^j + n_\varphi^j, \qquad\qquad \gamma_I^{(0)}(j) = -4S_1(j-2) \equiv \gamma_+^{(0)}(j),$$

$$n_{II}(j) = -2(j-1)n_g^j + n_q^j + \frac{2}{3}(j+1)n_\varphi^j, \qquad \gamma_{II}^{(0)}(j) = -4S_1(j) \equiv \gamma_0^{(0)}(j),$$

$$n_{III}(j) = -\frac{j-1}{j+2}n_g^j + n_q^j - \frac{j+1}{j}n_\varphi^j, \qquad \gamma_{III}^{(0)}(j) = -4S_1(j+2) \equiv \gamma_-^{(0)}(j),$$

$$n_{IV}(j) = 2\Delta n_g^j + \Delta n_q^j, \qquad\qquad \gamma_{IV}^{(0)}(j) = -4S_1(j-1) \equiv \tilde{\gamma}_+^{(0)}(j),$$

$$n_V(j) = -(j-1)\Delta n_g^j + \frac{j+2}{2}\Delta n_q^j, \qquad \gamma_V^{(0)}(j) = -4S_1(j+1) \equiv \tilde{\gamma}_-^{(0)}(j).$$

Thus, we have one supermultiplet of operators with the same AD $\gamma^{\mathrm{LLA}}(j)$ proportional to $\Psi(1) - \Psi(j-1)$. The momenta of the corresponding linear combinations of the parton distributions can be obtained from the above expressions by an appropriate shift of their argument j to obtain this universal anomalous dimension $\gamma^{\mathrm{LLA}}(j)$. Moreover, the coefficients in these linear combinations for $N = 4$ SUSY can be found from the super-conformal invariance (cf. ref. [15]). However, in two-loop approximation these coefficients are slightly renormalized [19] due to the breaking of the conformal invariance [20]. In the paper [6] using some plausible arguments an universal AD in two-loops for $N = 4$ SUSY in the $\overline{\mathrm{DR}}$-scheme was suggested. Other AD are obtained by an integer shift of its arguments. These results were justified by a direct calculation of the AD matrix in ref. [19][2]. With the use of the basis for the multiplicatively renormalizable operators obtained in LLA in [6] one can transform this matrix to a triangle form. The diagonal elements of the triangle matrix are expressed in terms of the universal AD $\gamma(j)$ by an appropriate integer shift of its argument:

$$\gamma(j) = -4\hat{a}\big(S_1(j-2) + 4\hat{a}Q(j-2)\big), \tag{10}$$

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[2]Note that in [19] the results have been obtained in $\overline{\mathrm{DR}}$-scheme but with the $\overline{\mathrm{MS}}$ coupling constant $\overline{a}$ that leads to the additional term $-1/3S_1(j)$ in the NLO correction $Q(j)$.

where

$$Q(j) = S_{-2,1}(j) - S_1(j)\big(S_2(j) + S_{-2}(j)\big) - \big(S_3(j) + S_{-3}(j)\big)/2,$$

$$S_k(n) = \sum_{i=1}^{n} \frac{1}{i^k}, \qquad \zeta(k) = \sum_{i=1}^{\infty} \frac{1}{i^k}, \qquad S_{-k}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{i^k}, \qquad S_{-k,l}(n) = \sum_{i=1}^{n} \frac{(-1)^i}{i^k} S_l(i).$$

The analytical continuation of functions $\gamma_{ab}^{(1)}$ $(a, b = g, q, \varphi)$ and $\tilde{\gamma}_{ab}^{(1)}$ $(a, b = g, q)$ to the complex values of j can be done analogously to refs. [21, 6].

4   Relation between the DGLAP and BFKL equations

As we have discussed already above, in the case of $N = 4$ SUSY the BFKL eigenvalue is analytic in $|n|$ and one can continue the AD to the negative values of $|n|$. It gives a possibility to find the AD singular contributions in the twist-2 operators not only at $j = 1$ but also at other integer points $j = 0, -1, -2, \ldots$. As it was discussed already in the Introduction, in the Born approximation we obtain $\gamma = 4\hat{a}(\Psi(1) - \Psi(j - 1))$, which coincides with the result of the direct calculations (see [14, 18]). Thus, in the case of $N = 4$ the BFKL equation presumably contains the information sufficient for restoring the kernel of the DGLAP equation. Below we investigate the relation between these equations in the NLO approximation.

Let us start with an investigation of AD singularities obtained from the DGLAP equation. By presenting the Lorentz spin j as $\omega - r$, where $r = -1, 0, 1, \ldots$ and pushing $\omega \to 0$ we can calculate the singular behavior of the universal anomalous dimension $\gamma(j)$

$$\gamma(j) = 4\hat{a}\left[\frac{1}{\omega} - S_1(r+1) + O(\omega)\right] \tag{11}$$

$$+ (4\hat{a})^2 \begin{cases} \dfrac{1}{\omega^3} - 2S_1(r+1)\dfrac{1}{\omega^2} - (\zeta(2) + S_2(r+1))\dfrac{1}{\omega} + O(\omega^0) & \text{if } r = 2m, \\[2ex] S_2(r+1)\dfrac{1}{\omega} + O(\omega^0) & \text{if } r = 2m+1. \end{cases}$$

Let us consider initially the BFKL equation in a modified LLA, i.e. when $\omega^{\mathrm{MLLA}} = 4\,\hat{a}\big(2\Psi(1) - \Psi(\gamma) - \Psi(J - \gamma)\big)$. In the limit $J = 1 + |n| + \omega \to -r + \omega$ by inverting this equation one can obtain

$$\gamma = \frac{4\hat{a}}{\omega} + (4\hat{a})^2 \left[\frac{1}{\omega^3} - S_1(r)\frac{1}{\omega^2} - (\zeta(2) + S_2(r))\frac{1}{\omega}\right] + O(\hat{a}^3), \tag{12}$$

i.e. this result coincides after the shift $r \to r + 1$ with the singular part of the corresponding DGLAP result for the even values of r with the exception of the coefficient in the front of $\hat{a}^2/\omega^2$.

In a general case the next-to-leading corrections contain the divergencies at $|n| \to -r - 1$. Their appearance is related to the presence of the double-logarithms. Indeed the eigenvalue relation for the Bethe–Salpeter equation can be written near $\gamma = 0$ and $J \simeq j = -r + \omega$ in the form

$$1 = \frac{4\hat{a}}{\gamma(\omega - \gamma)} + O(\hat{a}^2).$$

Because the first contribution in the right-hand side contains additional singularities in comparison with the pole $1/\omega$ in the physical case of the positive $|n|$, we should subtract from the correction $O(\hat{a}^2)$ the terms appeared in its first iteration:

$$(4\hat{a})^2 \frac{1}{\gamma^2(\omega - \gamma)^2} \simeq (4\hat{a})^2 \left(\frac{1}{\gamma^2\omega^2} + \frac{2}{\gamma\omega^3}\right).$$

This subtraction leads to the final result for even r, which is in an agreement with the fact that the double-logarithms in the universal AD $\gamma(j)$ exist at even negative j. For the odd r the divergency $\hat{a}^2/(\gamma\omega^3)$ is absent in accordance with the absence in $\gamma(j)$ of the DL terms $\sim \hat{a}^2/\omega^3$ at odd negative j. For a more accurate comparison of the singularities of the BFKL and DGLAP equations in two-loop approximation one needs to calculate in the BFKL kernel non-singular terms at $j \to -r$.

5   Conclusion

Above we reviewed the LLA and NLL results for the eigenvalue of the kernels of the BFKL and DGLAP equations in the $N = 4$ supersymmetric gauge theory and constructed the operators with a multiplicative renormalization [6]. These AD can be obtained from the universal AD $\gamma(j)$ by a shift of its argument $j \to j + k$. The NLO corrections to the AD matrix were found with the use of the plausible arguments [6] and by the direct methods [19].

Note that recently the LLA AD in this theory for large $\alpha_s N_c$ were constructed in ref. [22] in the limit $j \to \infty$ from the superstring model with the use of the Maldacena correspondence [23]. Also in $N = 4$ SUSY at large $\alpha_s N_c$ the Pomeron coincides with the graviton [24]. It will be interesting to obtain these results directly from the DGLAP and BFKL equation. Already in the perturbation theory as it was demonstrated above, the BFKL dynamics has remarkable properties: analyticity in the conformal spin $|n|$, Möbius invariance, holomorphic (and Hermitian) separability and integrability in a generalized LLA. On the other hand, for the DGLAP dynamics the AD for all twist-2 operators are proportional in LLA to the function $\Psi(1) - \Psi(j - 1)$ up to an integer shift of its arguments, which corresponds to the eigenvalue of a pair Hamiltonian in the integrable Heisenberg spin model [4, 25]. The investigation of the $N = 4$ supersymmetric model should be continued in the perturbation theory and for large $\alpha_s N_c$ because it is helpful for understanding of QCD.

Acknowledgements

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